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An α-Robust Semidiscrete Finite Element Method for a Fokker–Planck Initial-Boundary Value Problem with Variable-Order Fractional Time Derivative

Kim-Ngan Le[1] · **Martin Stynes**[2]

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Abstract

A time-fractional initial-boundary value problem of Fokker–Planck type is considered on the space-time domain $\Omega \times [0, T]$, where Ω is an open bounded domain in $\mathbb{R}^d$ for some $d \geq 1$, and the order $\alpha(x)$ of the Riemann-Liouville fractional derivative may vary in space with $1/2 < \alpha(x) < 1$ for all x. Such problems appear naturally in the formulation of certain continuous-time random walk models. Uniqueness of any solution u of the problem is proved under reasonable hypotheses. A semidiscrete numerical method, using finite elements in space to yield a solution $u_h(t)$, is constructed. Error estimates for $\|(u-u_h)(t)\|_{L^2(\Omega)}$ and $\int_0^t \left|\partial_t^{1-\alpha}(u-u_h)(s)\right|_1^2 ds$ are proved for each $t \in [0, T]$ under the assumptions that the following quantities are finite: $\|u(\cdot, 0)\|_{H^2(\Omega)}$, $|u(\cdot, t)|_{H^1(\Omega)}$ for each t, and $\int_0^t [\|u(\cdot, t)\|^2_{H^2(\Omega)} + |\partial_t^{1-\alpha} u|^2_{H^2(\Omega)}]$, where $u(x, t)$ is the unknown solution. Furthermore, these error estimates are α-robust: they do not fail when $\alpha \to 1$, the classical Fokker–Planck problem. Sharper results are obtained for the special case where the drift term of the problem is not present (which is of interest in certain applications).

Keywords Variable-order fractional derivative · Fokker–Planck equation · α-robust

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✉ Martin Stynes
m.stynes@csrc.ac.cn

Kim-Ngan Le
ngan.le@monash.edu

[1] School of Mathematics, Monash University, Melbourne, VIC 3800, Australia

[2] Applied and Computational Mathematics Division, Beijing Computational Science Research Center, Beijing 100193, China

1 Introduction

In this paper, we consider a time-fractional initial-boundary value problem of Fokker–Planck type where the order of the fractional derivative may vary in space. This mathematical problem appears naturally in the formulation of continuous-time random walk (CTRW) models of anomalous diffusion [14]; it is pointed out there [14, p.452] that "The situation where [the fractional order] varies in space is of interest in physics since the strength of a trapping effect may vary throughout a disordered medium". The problem was derived in [14] as the continuum limit of a CTRW where the spatially variable time-scaling behaviour of the mean square displacement is modelled by a spatially varying $\alpha(x)$-stable Lévy noise in the waiting time probability density function. See also [7] for references to other applications where the order of the fractional derivative varies in space.

Let Ω be an open bounded domain in $\mathbb{R}^d$ for some $d \geq 1$, with closure $\bar{\Omega}$ and boundary $\partial\Omega$. Let $\partial_t^{1-\alpha}$ denote the standard Riemann–Liouville fractional derivative operator defined by $\partial_t^{1-\alpha} u = (J^\alpha u)_t$, where J^α is the Riemann–Liouville fractional integral operator of order α, viz.,

$$J^\alpha u(t) = \int_0^t \omega_\alpha(t-s)u(s)\,ds = (\omega_\alpha * u)(t) \quad \text{where} \quad \omega_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)} \text{ for } \beta > 0.$$

We investigate the problem specified by [14, (3.12)]; written in conservative form, this initial-boundary problem is

$$\begin{aligned} u_t(x,t) - \nabla \cdot \left(a\, \nabla \partial_t^{1-\alpha(x)} u\right)(x,t) + \nabla \cdot (\mathbf{b}\, \partial_t^{1-\alpha(x)} u)(x,t) & \\ = f(x,t) \text{ for } (x,t) \in \Omega \times (0,T), & \qquad (1.1a) \\ u(x,0) = u_0(x) \text{ for } x \in \Omega, & \qquad (1.1b) \\ u(x,t) = 0 \text{ for } x \in \partial\Omega \text{ and } 0 < t \leq T. & \qquad (1.1c) \end{aligned}$$

Here a and $\mathbf{b}$ are the generalized diffusivity and drift, which are allowed to vary in space and time. We assume that $a \in C^1(\bar{\Omega} \times [0,T])$, $\mathbf{b} \in \left(C^1(\bar{\Omega} \times [0,T])\right)^d$, and that a satisfies the standard ellipticity condition

$$0 < a_0 \leq a(x,t) \leq a_1 \quad \forall (x,t) \in \bar{\Omega} \times [0,T] \text{ for some constants } a_0 \text{ and } a_1. \qquad (1.2)$$

Assume also that $f(\cdot,t) \in L^2(\Omega)$ for each $t \in [0,T]$.

The striking feature of problem (1.1), which distinguishes it from almost all previous numerical analyses of time-fractional initial-boundary value problems, is that we allow the order α of the fractional time derivative in (1.1a) to vary in space, viz., $\alpha = \alpha(x)$. To be precise, we assume that

$$\alpha(\cdot) \in C^1(\bar{\Omega}) \text{ with } \frac{1}{2} < \alpha_0 \leq \alpha(\cdot) \leq \alpha_1 < 1 \qquad (1.3)$$

for some constants α_0 and α_1. Naturally this includes the case of constant $\alpha \in (0,1)$ that has been considered in many papers. Allowing α to vary introduces significant complications in the analysis.

Remark 1.1 Our analysis involves the term $\int_0^t [\omega_{\alpha(x)}(s)]^2\,ds$ for each $x \in \Omega$; see the function $z(t)$ of Lemma 3.2. The inequality $1/2 < \alpha_0$ is needed to ensure that this integral is finite. This condition on α_0 is not excessively restrictive because (1.1) is often regarded as a variant of the classical case $\alpha \equiv 1$. The special case of constant α is considered in [5], where the condition $1/2 < \alpha$ is again needed.

Existence and uniqueness of a weak solution to a closely related initial-value problem with spatially-dependent order of fractional derivative is shown in [7]. At this time there is no theory guaranteeing existence of a solution to (1.1), but from modelling considerations [14, Section 3] one can argue on physical grounds that a solution does exist. Uniqueness of the solution of (1.1) (in a certain class of possible solutions) will be proved in Lemma 2.4.

Existence, uniqueness and regularity of the special case of (1.1) where a and α are constants is investigated in [9,11,12]; see also [5,10]. From these sources one can expect that a typical solution u of (1.1) has a weak singularity at the initial time $t = 0$: for each fixed $x \in \Omega$, near $t = 0$ one has

$$u(x,t) \approx u_0(x) + Kt^{\alpha} \quad \text{and} \quad u_t(x,t) \approx \alpha K t^{\alpha-1} \tag{1.4}$$

for some quantity K that depends on x and on the data of the problem.

To solve (1.1) numerically, we shall construct and analyse a semidiscrete method, using finite elements in space. The analysis of this semidiscrete method is already rather involved, so we postpone consideration of a fully discrete method to a later investigation.

We assume that u_0 lies in $H^2(\Omega) \cap H_0^1(\Omega)$, in the standard Sobolev space notation (see below). Our analysis will require $|u(t)|_1$ and $\int_0^t \left[\|u_t(s)\|_2^2 + \left|\partial_t^{1-\alpha} u(s)\right|_2^2\right] ds$ to be finite for each t. Recalling (1.4), our assumption (1.3) fits naturally with this integral requirement.

As well as the complication of a varying order of fractional derivative, there is a second substantial obstacle that our analysis seeks to overcome. For constant α, the hypothesis $\alpha < 1$ is commonly used in analyses of time-fractional problems. If one considers what happens as (constant) $\alpha \to 1^-$, one finds [2] that the solution of the fractional problem approaches the solution of the classical parabolic problem with $\alpha = 1$, pointwise uniformly on $\bar{\Omega} \times [0, T]$. Nevertheless, the convergence analyses of the vast majority of numerical methods for constant-α problems resembling (1.1) have error bounds that blow up as $\alpha \to 1^-$, as is discussed at length in [2], even though in practice the methods perform satisfactorily as $\alpha \to 1^-$. It is clearly desirable to prove numerical error estimates that do not exhibit this unnatural and imprecise blow-up behaviour; such error bounds are called *α-robust* in [2].

Our error analysis is designed to be α-robust. Thus, throughout the paper we are careful to prove inequalities that remain satisfactory if one allows $\alpha_1 \to 1^-$ for the upper limit α_1 in (1.3). Our final error bound (Theorem 4.3) remains finite when this limit is taken. This higher standard of error estimation — which, we repeat, is not attained by the majority of numerical analyses of time-fractional problems — means that we must devise new and special estimates to achieve it.

This paper is structured as follows. Section 2 derives several technical inequalities that will be used later. In Sect. 3 we construct our semidiscrete finite element method and prove a fundamental stability estimate (Lemma 3.2) for it. Then in Sect. 4 we obtain our error estimate for the numerical method (Theorem 4.3). The special case when $\mathbf{b} \equiv 0$ (i.e., the drift term vanishes) is dealt with in Corollary 4.4.

Notation. Throughout the paper, we often suppress the spatial variables and write v or $v(t)$ instead of $v(t, \cdot)$ for various functions v. Similarly, we often write α instead of $\alpha(x)$.

Let $\|\cdot\|$ denote the $L^2(\Omega)$ norm defined by $\|v\|^2 = \langle v, v\rangle$, where $\langle \cdot, \cdot \rangle$ is the $L^2(\Omega)$ inner product. For integer $r \geq 1$, we use the standard Sobolev space $H^r(\Omega)$ with its associated norm $\|\cdot\|_r$ and seminorm $|\cdot|_r$. As usual, $H_0^1(\Omega)$ denotes the subspace of $H^1(\Omega)$ comprising those functions whose traces vanish in $\partial\Omega$.

We use C to denote a generic constant that depends on the data $\Omega, a, \mathbf{b}, \alpha, f, u_0$ and T of problem (1.1) but is independent of any parameter describing our discretisation. Thus the constant C can take different values in different places throughout the paper. We also use subscripted quantities C_i and C_i^t that depend on the data of (1.1); the C_i have a defined and fixed value, while the C_i^t depend on the value of t at each stage of our analysis but are otherwise constant.

All these C, C_i and C_i^t depend on α but are α_1-robust: each C_i approaches a finite nonzero limit as $\alpha_1 \to 1^-$, and for each fixed t, each C_i^t approaches a finite nonzero limit as $\alpha_1 \to 1^-$.

2 Technical Preliminaries

This section derives several inequalities that will be used in our analysis and gives a proof that any solution of (1.1) is unique.

Note that in Lemmas 2.1 and 2.2–2.7, the multiplier of the right-hand-side integral (e.g., $T^{1-\beta}/\Gamma(2-\beta)$ in Lemma 2.3) approaches a finite nonzero limit as $\beta \to 1^-$ and $\alpha_1 \to 1^-$; this observation is needed when verifying that our convergence analysis in Sects. 3 and 4 is α-robust.

Lemma 2.1 [5, Lemma 2.2] *Let $\beta \in (1/2, 1)$ be constant. Assume that $v \in L^2(0, T)$. Then*

$$\int_0^t \left(vJ^\beta v\right)(s)\, ds \ge \frac{1}{2} J^{1-\beta}\left[\left(J^\beta v(t)\right)^2\right] \text{ for } 0 \le t \le T.$$

Lemma 2.2 *Assume that $\int_\Omega J^\alpha v(x,t)\, dx$ and $\int_\Omega J^{\alpha_0}|v(x,t)|\, dx$ are finite for $0 \le t \le T$. Then*

$$\int_\Omega J^\alpha v(x,t)\, dx \le \frac{\Gamma(\alpha_0)\, \max\{1, t^{\alpha_1-\alpha_0}\}}{\Gamma(\alpha_1)} \int_\Omega J^{\alpha_0}|v(x,t)|\, dx \text{ for } 0 \le t \le T.$$

Proof First,

$$\int_\Omega J^\alpha v(x,t)\, dx = \int_\Omega \int_0^t \omega_\alpha(t-s) v(x,s)\, ds\, dx.$$

If $0 \le t-s \le 1$, then by (1.3) and standard properties of $\Gamma(\cdot)$ we have

$$0 < \omega_\alpha(t-s) = \frac{(t-s)^{\alpha(x)-1}}{\Gamma(\alpha(x))} \le \frac{(t-s)^{\alpha_0-1}}{\Gamma(\alpha_1)} = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)}\, \omega_{\alpha_0}(t-s).$$

If $t-s > 1$, then similarly

$$0 < \omega_\alpha(t-s) \le \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} \le \frac{\Gamma(\alpha_0) t^{\alpha_1-\alpha_0}}{\Gamma(\alpha_1)}\, \omega_{\alpha_0}(t-s).$$

□

Lemma 2.3 *Let v be absolutely continuous on $[0, T]$. Let $\beta \in (0, 1)$ be constant. Then*

$$|v(t) - v(0)|^2 \le \frac{t^{1-\beta}}{\Gamma(2-\beta)} \left(J^{1-\beta}\left[(J^\beta v_t)^2\right]\right)(t) \text{ for } 0 \le t \le T.$$

Proof The identity $v(t) - v(0) = J^1 v_t(t) = J^{1-\beta}(J^\beta v_t)(t)$ implies that

$$\begin{aligned} |v(t) - v(0)|^2 &= \left|J^{1-\beta}(J^\beta v_t)(t)\right|^2 \\ &= \left[\int_0^t \omega_{1-\beta}(t-s)(J^\beta v_t)(s)\, ds\right]^2 \end{aligned}$$

$$\leq \left[\int_0^t \omega_{1-\beta}(t-s)\,ds\right]\left[\int_0^t \omega_{1-\beta}(t-s)(J^\beta v_t)^2(s)\,ds\right]$$
$$= \frac{t^{1-\beta}}{\Gamma(2-\beta)}\ \left(J^{1-\beta}\big[(J^\beta v_t)^2\big]\right)(t),$$

where we used a Cauchy–Schwarz inequality. □

These first results enable us to prove in the next lemma that any solution of (1.1) is unique, under a mild assumption on the solution's regularity. Although we assumed earlier that $\mathbf{b} \in \left(C^1(\bar{\Omega} \times [0,T])\right)^d$, for the proof of Lemma 2.4 one needs only $\|\mathbf{b}\|_{L^\infty(\Omega\times(0,T))} < \infty$.

Lemma 2.4 *Any solution of* (1.1) *is unique in the class of solutions* $u(x,t)$ *that satisfy* (1.1) *pointwise with* $u(x,\cdot)$ *absolutely continuous on* $[0,T]$ *for each* x.

Proof To prove uniqueness, we show that $v \equiv 0$ is the only solution of the problem with zero data:

$$v_t(x,t) - \nabla\cdot\left(a\,\nabla\partial_t^{1-\alpha(x)}v\right)(x,t) + \nabla\cdot(\mathbf{b}\,\partial_t^{1-\alpha(x)}v)(x,t)$$
$$= 0 \text{ for } (x,t)\in\Omega\times(0,T), \tag{2.1a}$$
$$v(x,0) = 0 \text{ for } x\in\Omega, \tag{2.1b}$$
$$v(x,t) = 0 \text{ for } x\in\partial\Omega \text{ and } 0 < t \leq T. \tag{2.1c}$$

Let $t\in(0,T]$ be arbitrary. Multiply (2.1a) by $\partial_t^{1-\alpha(x)}v$, then integrate over $\Omega\times[0,t]$. After two integrations by parts in space that use (2.1c), one has

$$\int_\Omega\int_0^t v_t\partial_t^{1-\alpha(x)}v\,dt\,dx + \int_0^t\int_\Omega a\left|\partial_t^{1-\alpha(x)}v\right|_1^2\,dx\,dt$$
$$+\int_0^t\int_\Omega \nabla(\partial_t^{1-\alpha(x)}v)\cdot\mathbf{b}\,\partial_t^{1-\alpha(x)}v\,dx\,dt = 0. \tag{2.2}$$

But the homogenous initial condition (2.1b) implies that the Riemann-Liouville derivative is equivalent to a Caputo derivative, i.e., $\partial_t^{1-\alpha(x)}v = J^{\alpha(x)}v_t$, so

$$\int_0^t v_t\partial_t^{1-\alpha(x)}v\,dt = \int_0^t v_t J^{\alpha(x)}v_t\,dt \geq J^{1-\alpha(x)}\left[\left(J^{\alpha(x)}v_t\right)^2\right] \text{ for each } x$$

by Lemma 2.1. Inserting this inequality in (2.2) and recalling (1.3), we get

$$\int_\Omega J^{1-\alpha(x)}\left[\left(J^{\alpha(x)}v_t\right)^2\right](t)\,dx + \alpha_0\int_0^t \|\nabla(\partial_t^{1-\alpha}v(s))\|^2\,ds$$
$$\leq -\int_0^t\int_\Omega \nabla(\partial_t^{1-\alpha(x)}v)\cdot\mathbf{b}\,\partial_t^{1-\alpha(x)}v\,dx\,dt$$
$$\leq \frac{\alpha_0}{2}\int_0^t \|\nabla(\partial_t^{1-\alpha}v(s))\|^2\,ds + \frac{\|\mathbf{b}\|^2_{L^\infty(\Omega\times(0,T))}}{2\alpha_0}\int_0^t\int_\Omega\left(J^{\alpha(x)}v_t\right)^2(s)\,dx\,ds,$$

by Young's inequality. Hence

$$\int_\Omega J^{1-\alpha(x)}\left(J^{\alpha(x)}v_t\right)^2(t)\,dx \leq \frac{\|\mathbf{b}\|^2_{L^\infty(\Omega\times(0,T))}}{2\alpha_0}\int_\Omega J^{\alpha(x)}\,J^{1-\alpha(x)}\left[\left(J^{\alpha(x)}v_t\right)^2\right](s)\,dx\,ds$$
$$\leq C\,J^{\alpha_0}\left(\int_\Omega J^{1-\alpha(x)}\left[\left(J^{\alpha(x)}v_t\right)^2\right]dx\right)(t),$$

using Lemma 2.2. Then a fractional Gronwall inequality [4, Theorem 3.1] yields

$$\int_\Omega J^{1-\alpha(x)}\Big[\big(J^{\alpha(x)}v_t\big)^2\Big](t)\,dx \le 0 \quad \forall t\in(0,T],$$

which implies that for almost all $x\in\Omega$ one has $J^{1-\alpha(x)}\Big[\big(J^{\alpha(x)}v_t\big)^2\Big](t)=0$ for all $t\in(0,T]$. Now Lemma 2.3, the absolute continuity of v and $v(\cdot,0)=0$ imply that for almost all $x\in\Omega$ one has $v(x,t)=0$ for all $t\in[0,T]$. □

Set $\Gamma_m := \min_{r>0}\Gamma(r)\approx 0.8856$.

Lemma 2.5 *Let $v\in L^2(\Omega\times(0,T))$. Then*

$$\int_0^t \|J^\alpha v(t)\|^2\,dt \le \frac{\max\left\{t^{2\alpha_0},t^{2\alpha_1}\right\}}{\Gamma_m^2}\int_0^t \|v(t)\|^2\,dt \ \ \textit{for } 0\le t\le T.$$

Proof One has

$$\begin{aligned}\int_0^t \|J^\alpha v(t)\|^2\,dt &= \int_0^t\int_\Omega (\omega_\alpha * v)^2(t)\,dx\,ds = \int_\Omega \|\omega_\alpha * v\|^2_{L^2(0,t)}\,dx\\ &\le \int_\Omega \|\omega_\alpha\|^2_{L^1(0,t)}\|v\|^2_{L^2(0,t)}\,dx\end{aligned}$$

by Young's inequality for convolutions. But for each $x\in\Omega$,

$$\|\omega_\alpha\|_{L^1(0,t)} = \int_0^t \frac{s^{\alpha(x)-1}}{\Gamma(\alpha(x))}\,ds = \frac{t^{\alpha(x)}}{\Gamma(\alpha(x)+1)} \le \frac{\max\{t^{\alpha_0},t^{\alpha_1}\}}{\Gamma_m},$$

and the result follows. □

Lemma 2.6 *Let $v\in L^2(\Omega\times(0,T))$. Then*

$$\int_0^t \|(\nabla\omega_\alpha * v)(s)\|^2\,ds \le (C_1^t)^2\int_0^t \|v(s)\|^2\,ds \ \ \textit{for } 0\le t\le T,$$

where

$$C_1^t := \frac{\|\nabla\alpha\|_{L^\infty(\Omega)}\max\{t^{\alpha_0},t^{\alpha_1}\}}{\Gamma_m}\left[\ln t + \left\|\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right\|_{L^\infty(\Omega)} + \frac{1}{\alpha_0}\right].$$

Proof One has

$$\int_0^t \|\nabla\omega_\alpha * v(s)\|^2\,ds \le \int_\Omega \|\nabla\omega_\alpha\|^2_{L^1(0,t)}\|v\|^2_{L^2(0,t)}\,dx \tag{2.3}$$

by Young's inequality for convolutions. Then, writing α instead of $\alpha(x)$, for each x we get

$$\begin{aligned}\|\nabla\omega_\alpha\|_{L^1(0,t)} &= \int_0^t (\nabla\alpha)\,\omega_\alpha(s)\left[\ln s - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right]ds\\ &= \nabla\alpha\left[-\int_0^t \frac{\omega_{\alpha+1}(s)}{s}\,ds + \big(\omega_{\alpha+1}(s)\ln s\big)\Big|_{s=0}^t - \omega_{\alpha+1}(t)\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right]\\ &= \nabla\alpha\left[\omega_{\alpha+1}(t)\ln t - \int_0^t \frac{s^{\alpha-1}(s)}{\Gamma(\alpha+1)}\,ds - \omega_{\alpha+1}(t)\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right]\\ &= (\nabla\alpha)\,\omega_{\alpha+1}(t)\left[\ln t - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{1}{\alpha}\right]\end{aligned}$$

$$\leq \frac{\|\nabla\alpha\|_{L^\infty(\Omega)} \max\{t^{\alpha_0}, t^{\alpha_1}\}}{\Gamma_m} \left[\ln t + \left\| \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \right\|_{L^\infty(\Omega)} + \frac{1}{\alpha_0} \right].$$

where we integrated by parts near the start of the calculation. The lemma now follows from (2.3) and the definition of C_1^t. □

Note that $C_1^t \to 0$ as $t \to 0$.

Lemma 2.7 *For each $\alpha = \alpha(x)$, one has*

$$\int_0^t \omega_\alpha(t)^2 (\ln t)^2 \, dt \leq C_2^t,$$

where

$$C_2^t := \frac{\max\left\{t^{2\alpha_0 - 1}, t^{2\alpha_1 - 1}\right\}}{\Gamma_m^2} \left[\frac{2}{(2\alpha_0 - 1)^3} - \frac{2\ln t}{(2\alpha_0 - 1)^2} + \frac{(\ln t)^2}{2\alpha_0 - 1} \right].$$

Proof Two integrations by parts yield

$$\int_0^t \omega_\alpha(s)^2 (\ln s)^2 \, ds = \frac{t^{2\alpha - 1}}{\Gamma(\alpha)^2} \left[\frac{2}{(2\alpha - 1)^3} - \frac{2\ln t}{(2\alpha - 1)^2} + \frac{(\ln t)^2}{2\alpha - 1} \right].$$

The result now follows using $\alpha(x) \geq \alpha_0 > 1/2$. □

Note that $C_2^t \to 0$ as $t \to 0$.

3 A Semidiscrete Finite Element Method

We construct a semidiscrete approximation of the problem (1.1), where finite elements are used to discretise in space.

A weak form of (1.1a) is: for all functions $\phi \in H^1(\Omega)$,

$$\langle u_t, \phi \rangle + \left\langle a \, \nabla \left(\partial_t^{1-\alpha} u \right), \nabla\phi \right\rangle - \left\langle \mathbf{b} \, \partial_t^{1-\alpha} u, \nabla\phi \right\rangle = \langle f, \phi \rangle \text{ for each } t \in (0, T). \tag{3.1}$$

It is this weak form that we discretise, but one needs to do this carefully to take account of the space-varying derivative $\partial_t^{1-\alpha} u$ while remaining in the finite element space.

Let $S_h \subset H_0^1(\Omega)$ be the finite element space of globally continuous piecewise linear polynomials on a quasiuniform mesh, and vanishing on $\partial\Omega$.

Let $\Pi_h : L^2(\Omega) \to S_h$ be the $L^2(\Omega)$ orthogonal projector into S_h: for each $\psi \in L^2(\Omega)$ one has $\Pi_h \psi \in S_h$ with

$$\langle \Pi_h \psi, \phi_h \rangle = \langle \psi, \phi_h \rangle \text{ for all } \phi_h \in S_h.$$

Our spatial discretisation of (3.1) is: for all $t \in [0, T]$, find $u_h(t) \in S_h$ such that for all $\phi_h \in S_h$ one has

$$\begin{aligned} \langle u_{ht}, \phi_h \rangle + \left\langle a \, \nabla \Pi_h \left(\partial_t^{1-\alpha} u_h \right), \nabla\phi_h \right\rangle - \left\langle \mathbf{b} \, \Pi_h \left(\partial_t^{1-\alpha} u_h \right), \nabla\phi_h \right\rangle & \\ = \langle f, \phi_h \rangle + \langle g, \nabla\phi_h \rangle \text{ for each } t \in (0, T), & \quad (3.2a) \\ u_h(0) = u_{0h}, & \quad (3.2b) \end{aligned}$$

where $u_{ht} := (u_h)_t$, the initial value $u_{0h} \in S_h$ will be chosen later based on our analysis, and we have introduced an additional term $\langle g, \nabla\phi_h \rangle$ that will be useful in the subsequent theoretical analysis (the actual semidiscretisation of (3.1) is given by taking $g \equiv 0$).

To prove existence and uniqueness of solution to (3.2a), we first define the linear operator $A_h(t) : L^2(\Omega) \to S_h$ by

$$\langle A_h(t)\psi, \phi_h\rangle := \langle a(\cdot,t)\,\nabla(\Pi_h\psi), \nabla\phi_h\rangle - \langle \mathbf{b}(\cdot,t)\,\Pi_h\psi, \nabla\phi_h\rangle \text{ for all } \psi \in L^2(\Omega),\ \phi_h \in S_h,$$

and define $f_h(t) \in S_h$ by

$$\langle f_h(t), \phi_h\rangle := \langle f, \phi_h\rangle + \langle g, \nabla\phi_h\rangle \text{ for all } \phi_h \in S_h.$$

Furthermore, define $A_{ht}(t) : L^2(\Omega) \to S_h$ by

$$\langle A_{ht}(t)\psi, \phi_h\rangle := \langle a_t(\cdot,t)\,\nabla(\Pi_h\psi), \nabla\phi_h\rangle - \langle \mathbf{b}_t(\cdot,t)\,\Pi_h\psi, \nabla\phi_h\rangle \text{ for all } \psi \in L^2(\Omega),\ \phi_h \in S_h.$$

Then equation (3.2a) is equivalent to

$$u_{ht} + A_h(\partial_t^{1-\alpha} u_h) = f_h(t) \text{ for } t \in (0,T]. \tag{3.3}$$

The next lemma demonstrates existence and uniqueness of u_h satisfying (3.2a)–(3.2b).

Lemma 3.1 *For any $u_{0h} \in S_h$, there exists a unique solution $u_h : [0,T] \to S_h$ of (3.4) that lies in $C[0,T]$ and satisfies (3.2a) for each $t \in (0,T]$.*

Proof Formally integrating the equation (3.3) with respect to t, and noting that

$$\int_0^t A_h(s)(\partial_t^{1-\alpha} u_h(s))\,ds = A_h(t)(J^\alpha u_h(t)) - \int_0^t A_{ht}(s)(J^\alpha u_h(s))\,ds,$$

we obtain the Volterra integral equation

$$u_h(t) + \int_0^t K_h(t,s)u_h(s)\,ds = u_h(0) + \int_0^t f_h(s)\,ds \tag{3.4}$$

where the weakly singular kernel $K_h(t,s) : S_h \to S_h$ is defined by

$$K_h(t,s)u_h(s) := A_h(t)(\omega_\alpha(t-s)u_h(s)) - \int_s^t A_{ht}(\tau)\,(\omega_\alpha(\tau-s)u_h(s))\,d\tau.$$

Note that

$$\|K_h(t,s)\| \le C^t\left[\|\omega_\alpha(t-s)\|_{L^\infty(\Omega)} + \int_s^t \|\omega_\alpha(\tau-s)\|_{L^\infty(\Omega)}\,d\tau\right],$$

where $\|\cdot\|$ denotes the matrix norm in $\mathbb{R}^{N\times N}$ induced by the Euclidean norm on $\mathbb{R}^N$ with $N := \dim S_h$, the constant C^t depends on the mesh as well as on the data of (1.1). For any given mesh, this constant is fixed. Thus, by imitating the arguments in the proof of [8, Theorem 3.1], one can verify that $K_h(t,s)$ satisfies all the assumptions of [1, Corollary 2.3], which yields existence and uniqueness of a solution $u_h : [0,T] \to S_h$ to (3.4) that lies in $C[0,T]$. But [13, Theorem 5] then shows that $u_{ht}(t)$ exists for each $t \in (0,T)$. We can therefore differentiate (3.4) with respect to t, obtaining a solution $u_h(t)$ of (3.2).

This solution is unique, as otherwise integration of (3.2) would yield more than one solution of (3.4), which as we showed above is impossible. □

Set $v_h = u_h - u_{0h}$. Then, since

$$\partial_t^{1-\alpha(x)} 1 = \frac{t^{\alpha(x)-1}}{\Gamma(\alpha(x))} = \omega_{\alpha(x)}(t),$$

the semidiscretisation (3.2) is equivalent to

$$\langle v_{ht}, \phi_h \rangle + \left\langle A_h(\partial_t^{1-\alpha} v_h), \phi_h \right\rangle = \langle f, \phi_h \rangle + \langle g, \nabla \phi_h \rangle - \langle A_h(u_{0h}\omega_\alpha), \phi_h \rangle \tag{3.5a}$$

$$v_h(0) = 0. \tag{3.5b}$$

We shall use the standard Poincaré inequality

$$\|\psi\|^2 \le C_P \|\nabla\psi\|^2 \ \text{ for all } \psi \in H_0^1(\Omega), \tag{3.6}$$

where C_P denotes the Poincaré constant for Ω.

For each constant $\beta \in (0, 1)$, the Mittag-Leffler function $E_\beta(y)$ is defined by

$$E_\beta(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k\beta + 1)} \ \text{ for } y \in \mathbb{R}.$$

Some useful properties of this function are described in [3].

Set

$$z(t) = \int_0^t \left[\|f\|^2 + \|g\|^2 + |u_{0h}\omega_\alpha|_1^2 + |\mathbf{b}\, u_{0h}\omega_\alpha|_1^2 \right] ds \ \text{ for } 0 \le t \le T.$$

The regularity assumptions on α, $\mathbf{b}$ and u_0 in Sect. 1 and the hypothesis $1/2 < \alpha_0$ in (1.3) ensure that the terms $\int_0^t \left[|u_{0h}\omega_\alpha|_1^2 + |\mathbf{b}\, u_{0h}\omega_\alpha|_1^2 \right] ds$ in $z(t)$ are finite.

Lemma 3.2 (Stability of the semidiscretisation) *Let v_h satisfy (3.5). There exist constants C_3, C_4 and C_5 such that*

$$\|v_h(t)\|^2 \le C_3 z(t) E_{\alpha_0}(C_4 t^{\alpha_0}) \tag{3.7}$$

and

$$\int_0^t |\Pi_h \partial_t^{1-\alpha} v_h(s)|_1^2 \, ds \le C_5 z(t) \left[1 + E_{\alpha_0}(C_4 t^{\alpha_0})\, \omega_{1+\alpha_0}(t) \right] \tag{3.8}$$

for $0 \le t \le T$.

Proof Observe first that for any $t \in (0, T]$ and any $\psi, w \in L^2(\Omega)$, one has

$$\langle A_h(t)(w), \Pi_h w \rangle \ge a_0 \|\nabla \Pi_h w\|^2 - \langle \mathbf{b}\Pi_h w, \nabla \Pi_h w \rangle,$$

$$\langle A_h(t)(\psi), \Pi_h w \rangle \le \frac{a_0}{2} \|\nabla \Pi_h w\|^2 + \frac{a_1^2}{a_0} \|\nabla \Pi_h \psi\|^2 + \frac{1}{a_0} \|\mathbf{b}\Pi_h w\|^2.$$

Taking $\phi_h = \Pi_h \partial_t^{1-\alpha} v_h$ in (3.5a) and using the above properties of A_h with $\psi = u_{0h}\omega_\alpha$ and $w = \partial_t^{1-\alpha} v_h$, we deduce that

$$\begin{aligned}
&\langle v_{ht}, \Pi_h \partial_t^{1-\alpha} v_h \rangle + a_0 \left\| \nabla \left(\Pi_h \partial_t^{1-\alpha} v_h \right) \right\|^2 \\
&\quad \le \langle f, \Pi_h \partial_t^{1-\alpha} v_h \rangle + \left\langle g, \nabla \left(\Pi_h \partial_t^{1-\alpha} v_h \right) \right\rangle - \left\langle A_h(u_{0h}\omega_\alpha), \Pi_h(\partial_t^{1-\alpha} v_h) \right\rangle \\
&\qquad + \langle \mathbf{b}\Pi_h(\partial_t^{1-\alpha} v_h), \nabla \Pi_h(\partial_t^{1-\alpha} v_h) \rangle \\
&\quad \le \frac{2C_P}{a_0} \|f\|^2 + \frac{a_0}{8C_P} \|\Pi_h \partial_t^{1-\alpha} v_h\|^2 + \frac{2}{a_0} \|g\|^2 + \frac{3a_0}{4} \left\| \nabla \left(\Pi_h \partial_t^{1-\alpha} v_h \right) \right\|^2 \\
&\qquad + \frac{a_1^2}{a_0} \|\nabla(u_{0h}\omega_\alpha)\|^2 + \frac{1}{a_0} \|\nabla\left(\mathbf{b}\Pi_h(u_{0h}\omega_\alpha)\right)\|^2 + \frac{2}{a_0} \left\| \mathbf{b}\Pi_h(\partial_t^{1-\alpha} v_h) \right\|^2 \\
&\quad \le C \left[\|f\|^2 + \|g\|^2 + |u_{0h}\omega_\alpha|_1^2 + |\mathbf{b}\, u_{0h}\omega_\alpha|_1^2 \right]
\end{aligned}$$

$$+\frac{7a_0}{8}\left\|\nabla\left(\Pi_h\partial_t^{1-\alpha}v_h\right)\right\|^2+\frac{2}{a_0}\|\mathbf{b}\|^2_{L^\infty(\Omega\times(0,T))}\left\|\partial_t^{1-\alpha}v_h\right\|^2,$$

where C is some constant; here we used the Poincaré inequality (3.6) and the stability in $H^1(\Omega)$ of the $L^2(\Omega)$ projector Π_h on our quasiuniform mesh (see (4.2) below) to get $\|\nabla(\mathbf{b}\Pi_h(u_{0h}\omega_\alpha))\|\le C\,|\mathbf{b}\,u_{0h}\omega_\alpha|_1$ for some C. Rearranging, we obtain

$$\begin{aligned}&\langle v_{ht},\Pi_h\partial_t^{1-\alpha}v_h\rangle+\frac{a_0}{8}\left|\Pi_h\partial_t^{1-\alpha}v_h\right|_1^2\\&\quad\le C\left[\|f\|^2+\|g\|^2+|u_{0h}\omega_\alpha\|^2+|\mathbf{b}\,u_{0h}\omega_\alpha|_1^2\right]+\frac{2}{a_0}\|\mathbf{b}\|^2_{L^\infty(\Omega\times(0,T))}\left\|\partial_t^{1-\alpha}v_h\right\|^2\end{aligned}\tag{3.9}$$

for each $t\in(0,T]$.

The definition of Π_h and Lemma 2.1 yield

$$\int_0^t\langle v_{ht},\Pi_h\partial_t^{1-\alpha}v_h\rangle\,ds=\int_0^t\langle v_{ht},J^\alpha v_{ht}\rangle\,ds\ge\frac{1}{2}\int_0^t\int_\Omega\omega_{1-\alpha}(t-s)\left(J^\alpha v_{ht}(x,s)\right)^2\,dx\,ds.$$

Hence, after integrating (3.9) from 0 to t and recalling the definition of z, we get

$$\begin{aligned}&\frac{1}{2}\int_0^t\int_\Omega\omega_{1-\alpha}(t-s)\left(J^\alpha v_{ht}(x,s)\right)^2\,dx\,ds+\frac{a_0}{8}\int_0^t\left|\Pi_h\partial_t^{1-\alpha}v_h\right|_1^2(s)\,ds\\&\quad\le Cz(t)+C\int_0^t\left\|\partial_t^{1-\alpha}v_h(s)\right\|^2\,ds\quad\text{for }0\le t\le T.\end{aligned}\tag{3.10}$$

But

$$\begin{aligned}\int_0^t\left\|\partial_t^{1-\alpha}v_h(s)\right\|^2\,ds&=\int_0^t\left\|J^\alpha v_{ht}(s)\right\|^2\,ds\\&=\int_0^t\int_\Omega\left[J^\alpha v_{ht}(x,s)\right]^2\,dx\,ds\\&=\int_\Omega\int_0^t\left[J^\alpha v_{ht}(x,s)\right]^2\,ds\,dx\\&=\int_\Omega\left(J^\alpha J^{1-\alpha}\left[\left(J^\alpha v_{ht}\right)^2\right]\right)(x,t)\,dx.\end{aligned}\tag{3.11}$$

Using this identity in (3.10) and discarding the nonnegative second term on the left-hand side, for $0\le t\le T$ we obtain

$$\begin{aligned}\int_\Omega J^{1-\alpha}\left[\left(J^\alpha v_{ht}\right)^2\right](x,t)\,dx&\le Cz(t)+C\int_\Omega\left(J^\alpha J^{1-\alpha}\left[\left(J^\alpha v_{ht}\right)^2\right]\right)(x,t)\,dx\\&\le Cz(t)+C_4\int_\Omega\left(J^{\alpha_0}J^{1-\alpha}\left[\left(J^\alpha v_{ht}\right)^2\right]\right)(x,t)\,dx\\&=Cz(t)+C_4J^{\alpha_0}\int_\Omega J^{1-\alpha}\left[\left(J^\alpha v_{ht}\right)^2\right](x,t)\,dx,\end{aligned}$$

where we used Lemma 2.2 and C_4 is some fixed constant. A fractional Gronwall inequality [4, Theorem 3.1] now yields the bound

$$\int_\Omega J^{1-\alpha}\left[\left(J^\alpha v_{ht}\right)^2\right](x,t)\,dx\le Cz(t)E_{\alpha_0}(C_4t^{\alpha_0})\quad\text{for }0\le t\le T.\tag{3.12}$$

The estimate (3.7) follows from (3.12) using Lemma 2.3.

Finally, by (3.10), (3.11) and Lemma 2.2 we have

$$\begin{aligned}\int_0^t |\Pi_h \partial_t^{1-\alpha} v_h(s)|_1^2\, ds &\le Cz(t) + C\int_\Omega \left(J^\alpha J^{1-\alpha}\Big[\big(J^\alpha v_{ht}\big)^2\Big]\right)(x,t)\,dx \\ &\le Cz(t) + CJ^{\alpha_0}\int_\Omega J^{1-\alpha}\Big[\big(J^\alpha v_{ht}\big)^2\Big](x,t)\,dx \\ &\le Cz(t) + C\int_0^t \omega_{\alpha_0}(t-s)z(s)E_{\alpha_0}(C_4 s^{\alpha_0})\,ds \\ &\le Cz(t) + Cz(t)E_{\alpha_0}(C_4 t^{\alpha_0})\int_0^t \omega_{\alpha_0}(t-s)\,ds \\ &= Cz(t) + Cz(t)E_{\alpha_0}(C_4 t^{\alpha_0})\,\omega_{1+\alpha_0}(t),\end{aligned}$$

where we used (3.12) and the obvious property that $z(s)$ and $E_{\alpha_0}(C_4 s^{\alpha_0})$ are increasing functions of s. This proves (3.8). □

Lemma 3.2 implies that for each choice of u_{0h}, the solution v_h of (3.5) is unique.

In [14] it is pointed out that in certain applications of this initial-boundary value problem, the drift term $\nabla\cdot(\mathbf{b}\,\partial_t^{1-\alpha(x)}u)$ of (1.1) does not appear. The next corollary addresses this special case.

Corollary 3.3 *Suppose that* $\mathbf{b}\equiv 0$. *Then there exists a constant* C *such that*

$$\|v_h(t)\|^2 + \int_0^t \left|\Pi_h\partial_t^{1-\alpha}v_h(s)\right|_1^2\, ds \le C\int_0^t \left[\|f\|^2 + \|g\|^2 + |u_{0h}\omega_\alpha|_1^2\right] ds$$

for $0\le t\le T$.

Proof We now have $\mathbf{b}\equiv 0$ in (3.9). Consequently (3.10) simplifies to

$$\begin{aligned}\frac{1}{2}\int_\Omega J^{1-\alpha}\left(J^\alpha v_{ht}(x,t)\right)^2\,dx + \frac{a_0}{8}\int_0^t \left|\Pi_h\partial_t^{1-\alpha}v_h\right|_1^2(s)\,ds \\ \le C\int_0^t \left[\|f\|^2 + \|g\|^2 + |u_{0h}\omega_\alpha|_1^2\right] ds.\end{aligned}$$

The bound on $\left|\Pi_h\partial_t^{1-\alpha}v_h(s)\right|_1$ is immediate, and the bound on $\|v_h(t)\|$ follows from Lemma 2.3. □

4 Error Estimate for Semidiscrete Solution

To obtain an error estimate for the semidiscrete solution, we imitate the classical finite element analysis of parabolic PDEs.

Define the Ritz projector $R_h : H_0^1(\Omega)\to S_h$ for each $w\in H_0^1(\Omega)$ by

$$R_h w\in S_h \quad\text{and}\quad \langle\nabla(R_h w),\nabla\phi_h\rangle = \langle\nabla w,\nabla\phi_h\rangle \quad\text{for all } \phi_h\in S_h.$$

Recall that our finite element space S_h consists of piecewise linears on a quasiuniform mesh. We assume that there exists a constant C such that

$$\|w - R_h w\| + h|w - R_h w|_1 \le Ch^r|w|_r \quad\text{for } r\in\{1,2\}; \tag{4.1}$$

this holds true, for example, if $\partial\Omega$ is smooth [15, Lemma 1.1]. In more than one place we shall use the easily-proved property $R_h u_t = (R_h u)_t$.

We use R_h to give a quick proof of the stability of Π_h with respect to the $H^1(\Omega)$ seminorm — a property that was used in the proof of Lemma 3.2 and will be used again below. The quasiuniformity of the mesh means that standard inverse inequalities hold on it. Thus for any $v \in H^1(\Omega)$, for some constants C (independent of v) one has

$$\begin{aligned}
|\Pi_h v|_1 &\le |R_h v|_1 + |\Pi_h v - R_h v|_1 = |R_h v|_1 + |\Pi_h(v - R_h v)|_1 \\
&\le |R_h v|_1 + Ch^{-1}\|\Pi_h(v - R_h v)\| \\
&\le |v|_1 + Ch^{-1}\|v - R_h v\| \\
&\le C|v|_1
\end{aligned} \tag{4.2}$$

by (4.1).

Let $\theta_h = u_h - R_h u$ and $\rho_h = u - R_h u$, where we suppress the dependence on t.

Lemma 4.1 *Assume that $\int_0^T \left[|u_t(s)|_1^2 + |u_t(s)|_2^2\right] ds$ is finite. Then for $0 < t \le T$, one has*

$$\int_0^t \|\nabla(\rho_h(0)\omega_\alpha)(s)\|^2\, ds \le C_6^t \|\rho_h(0)\|_1^2 \tag{4.3}$$

and

$$\int_0^t \left|\partial_t^{1-\alpha}\rho_h(s)\right|_1^2 ds \le C_7^t h^2 \left\{\int_0^t \left[|u_t(s)|_1^2 + |u_t(s)|_2^2\right] ds + \|u(0)\|_2^2\right\}, \tag{4.4}$$

where C_6^t and C_7^t depend on t, with $\lim_{t\to 0} C_6^t = \lim_{t\to 0} C_7^t = 0$.

Proof First,

$$\begin{aligned}
\int_0^t \|\nabla(\rho_h(0)\omega_\alpha)(s)\|^2\, ds &= \int_0^t \|(\omega_\alpha \nabla\rho_h(0))(s) + (\rho_h(0)\nabla\omega_\alpha)(s)\|^2\, ds \\
&\le 2\int_0^t \left[\|(\omega_\alpha\nabla\rho_h(0))(s)\|^2 + \|(\rho_h(0)\nabla\omega_\alpha)(s)\|^2\right] ds.
\end{aligned} \tag{4.5}$$

Now

$$\begin{aligned}
\int_0^t \|(\omega_\alpha\nabla\rho_h(0))(s)\|^2\, ds &= \int_\Omega (\nabla\rho_h(0))^2 \int_0^t \omega_\alpha(s)^2\, ds\, dx \\
&= \int_\Omega (\nabla\rho_h(0))^2 \frac{t^{2\alpha(x)-1}}{[2\alpha(x)-1]\Gamma(\alpha(x))^2}\, dx \\
&\le |\rho_h(0)|_1^2 \frac{\max\{t^{2\alpha_0-1}, t^{2\alpha_1-1}\}}{2\alpha_0 - 1},
\end{aligned} \tag{4.6}$$

where we used $\Gamma(\alpha(x)) > 1$. Recalling Lemma 2.7,

$$\begin{aligned}
&\int_0^t \|(\rho_h(0)\nabla\omega_\alpha)(s)\|^2\, ds \\
&\le 2\int_\Omega (\rho_h(0))^2 \int_0^t |\alpha'(x)|^2 \omega_\alpha(s)^2 \left[(\ln s)^2 + \left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right)^2\right] ds\, dx \\
&\le 2\|\rho_h(0))\|^2 \|\alpha'\|_{L^\infty(\Omega)}^2 \left[C_2^t + \left\|\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right\|_{L^\infty(\Omega)} \frac{\max\{t^{2\alpha_0-1}, t^{2\alpha_1-1}\}}{2\alpha_0-1}\right],
\end{aligned} \tag{4.7}$$

again using $\Gamma(\alpha(x)) > 1$. The first result (4.3) follows from (4.5)–(4.7).

Next, the well-known relationship between the Riemann-Liouville and Caputo derivatives [3, Lemma 3.4] yields

$$
\begin{aligned}
&\int_0^t \left|\partial_t^{1-\alpha}\rho_h(s)\right|_1^2\,ds = \int_0^t \left|J^\alpha \rho_{ht}(s) + \rho_h(0)\omega_\alpha(s)\right|_1^2\,ds\\
&= \int_0^t \left\| J^\alpha(\nabla\rho_{ht})(s) + \int_0^s [\nabla\omega_\alpha(s-\eta)]\rho_{ht}(\eta)\,d\eta + \nabla(\rho_h(0)\omega_\alpha)(s)\right\|^2 ds\\
&\le 3\int_0^t \left[\left\|J^\alpha(\nabla\rho_{ht})(s)\right\|^2 + \left\|\int_0^s [\nabla\omega_\alpha(s-\eta)]\rho_{ht}(\eta)\,d\eta\right\|^2 + \|\nabla(\rho_h(0)\omega_\alpha)(s)\|^2\right] ds.\\
&\le \frac{3\max\left\{t^{2\alpha_0}, t^{2\alpha_1}\right\}}{\Gamma_m^2}\int_0^t \|\nabla\rho_{ht}(s)\|^2\,ds + 3(C_1^t)^2\int_0^t \|\rho_{ht}(s)\|^2\,ds + 3C_6(t)\|\rho_h(0)\|_1^2\\
&\le C_7^t h^2\left\{\int_0^t \left[|u_t(s)|_1^2 + |u_t(s)|_2^2\right] ds \;+ \|u(0)\|_2^2\right\},
\end{aligned}
$$

for some C_7^t, where we invoked (4.3) and Lemmas 2.5 and 2.6, then appealed to (4.1). □

Remark 4.2 If one has $\int_0^t |u_t(s)|_3^2 \le C$, then the proof of Lemma 4.1 shows that the bound (4.4) can be improved to $\int_0^t \left|\partial_t^{1-\alpha}\rho_h(s)\right|_1^2 ds \le Ch^4$.

We now choose the initial value in our discretisation in order to prove the following error estimate for our method.

Theorem 4.3 *Assume that $\int_0^T \left[|u_t(s)|_1^2 + |u_t(s)|_2^2\right] ds$ is finite. Choose $u_{0h} = R_h u_0$. Then there exist constants C such that*

$$
\begin{aligned}
&\|u(t) - u_h(t)\|^2 \le Ch^2|u(t)|_1^2\\
&\quad + Ch^2\left\{\|u(0)\|_2^2 + \int_0^t \left[|u_t(s)|_1^2 + |u_t(s)|_2^2 + |\partial_t^{1-\alpha}u(s)|_2^2\right] ds\right\} E_{\alpha_0}(C_4 t^{\alpha_0}) \qquad (4.8)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \left|\partial_t^{1-\alpha}(u-u_h)(s)\right|_1^2 ds \le Ch^2|u(t)|_1^2\\
&\quad + Ch^2\left\{\|u(0)\|_2^2 + \int_0^t \left[|u_t(s)|_1^2 + |u_t(s)|_2^2 + |\partial_t^{1-\alpha}u(s)|_2^2\right] ds\right\}\\
&\quad \times\left[1 + E_{\alpha_0}(C_4 t^{\alpha_0})\omega_{1+\alpha_0}(t)\right] \qquad (4.9)
\end{aligned}
$$

for $0 \le t \le T$. Furthermore, these error estimates are α-robust.

Proof From (3.1) and (3.2a) (with $g = 0$), for each $t \in (0, T]$ we get

$$
\begin{aligned}
\langle\theta_{ht}, \phi_h\rangle + \left\langle A_h(\partial_t^{1-\alpha}\theta_h), \phi_h\right\rangle &= \langle u_{ht}, \phi_h\rangle + \left\langle A_h(\partial_t^{1-\alpha}u_h), \phi_h\right\rangle\\
&\quad - \langle R_h u_t, \phi_h\rangle - \left\langle A_h(\partial_t^{1-\alpha}R_h u), \phi_h\right\rangle\\
&= \langle f, \phi_h\rangle - \langle R_h u_t, \phi_h\rangle - \left\langle A_h(\partial_t^{1-\alpha}R_h u), \phi_h\right\rangle\\
&= \langle u_t, \phi_h\rangle + \left\langle a\nabla\left(\partial_t^{1-\alpha}u\right), \nabla\phi_h\right\rangle + \left\langle \mathbf{b}\,\partial_t^{1-\alpha}u, \nabla\phi_h\right\rangle\\
&\quad - \langle R_h u_t, \phi_h\rangle - \left\langle A_h(\partial_t^{1-\alpha}R_h u), \phi_h\right\rangle\\
&= \langle u_t - R_h u_t, \phi_h\rangle
\end{aligned}
$$

$$+ \left\langle a\nabla\left(\partial_t^{1-\alpha}u\right) - a\nabla\Pi_h\left(\partial_t^{1-\alpha}R_h u\right), \nabla\phi_h\right\rangle$$
$$+ \left\langle \mathbf{b}\,\partial_t^{1-\alpha}u - \mathbf{b}\,\Pi_h\left(\partial_t^{1-\alpha}R_h u\right), \nabla\phi_h\right\rangle.$$

That is,

$$\langle\theta_{ht}, \phi_h\rangle + \left\langle A_h(\partial_t^{1-\alpha}\theta_h), \phi_h\right\rangle = \langle\tilde{f}, \phi_h\rangle + \langle\tilde{g}, \nabla\phi_h\rangle \tag{4.10}$$

where

$$\tilde{f} := u_t - R_h u_t$$
$$\text{and } \tilde{g} := a\nabla\left(\partial_t^{1-\alpha}u\right) - a\nabla\Pi_h\left(\partial_t^{1-\alpha}R_h u\right) + \mathbf{b}\left(\partial_t^{1-\alpha}u - \Pi_h\left(\partial_t^{1-\alpha}R_h u\right)\right),$$

with $\theta_h(0) = 0$ because of our choice of u_{0h}. Equation (4.10) is of the form (3.5a) with u_{0h} replaced by zero. Thus we can apply Lemma 3.2 to (4.10), obtaining

$$\|\theta_h(t)\|^2 \le C_3\left[\int_0^t\left(\|\tilde{f}\|^2 + \|\tilde{g}\|^2\right)ds\right]E_{\alpha_0}(C_4 t^{\alpha_0}) \tag{4.11}$$

and

$$\int_0^t\left|\partial_t^{1-\alpha}\theta_h(s)\right|_1^2 ds \le C_5\left[\int_0^t\left(\|\tilde{f}\|^2 + \|\tilde{g}\|^2\right)ds\right]\left[1 + E_{\alpha_0}(C_4 t^{\alpha_0})\omega_{1+\alpha_0}(t)\right]. \tag{4.12}$$

Now (4.1) gives

$$\|(u_t - R_h u_t)(s)\| + h\,|(u_t - R_h u_t)(s)|_1 \le Ch^r|u_t|_r \text{ for } r \in \{1, 2\}$$

and

$$\begin{aligned}\left|\partial_t^{1-\alpha}u - \Pi_h(\partial_t^{1-\alpha}R_h u)\right|_1 &\le \left|\partial_t^{1-\alpha}u - \Pi_h(\partial_t^{1-\alpha}u)\right|_1 \\ &\quad + \left|\Pi_h(\partial_t^{1-\alpha}u) - \Pi_h(\partial_t^{1-\alpha}R_h u)\right|_1 \\ &\le Ch\left|\partial_t^{1-\alpha}u\right|_2 + C\left|\partial_t^{1-\alpha}(u - R_h u)\right|_1,\end{aligned}$$

where we used the $H^1(\Omega)$-stability of the $L^2(\Omega)$ projector Π_h expressed in (4.2). Similarly, one gets

$$\left\|\mathbf{b}\left(\partial_t^{1-\alpha}u - \Pi_h\left(\partial_t^{1-\alpha}R_h u\right)\right)\right\| \le Ch\left\|\partial_t^{1-\alpha}u\right\|_1 + C\left\|\partial_t^{1-\alpha}(u - R_h u)\right\|.$$

Substituting these bounds into (4.11) and recalling that $\rho_h = u - R_h u$, we get

$$\|\theta_h(t)\|^2 \le C\left\{h^2\int_0^t\left[|u_t(s)|_1^2 + \|\partial_t^{1-\alpha}u(s)\|_2^2\right]ds + \int_0^t\left\|\partial_t^{1-\alpha}\rho_h(s)\right\|_1^2 ds\right\}E_{\alpha_0}(C_4 t^{\alpha_0}).$$

Hence, recalling (4.4) from Lemma 4.1, we obtain

$$\|\theta_h(t)\|^2 \le Ch^2\left\{\|u(0)\|_2^2 + \int_0^t\left[|u_t(s)|_1^2 + |u_t(s)|_2^2 + \|\partial_t^{1-\alpha}u(s)\|_2^2\right]ds\right\}E_{\alpha_0}(C_4 t^{\alpha_0}). \tag{4.13}$$

Finally, for each $t \in [0, T]$ one has

$$\|(u - u_h)(t)\|^2 = \|\rho_h(t) - \theta_h(t)\|^2 \le 2(\|\rho_h(t)\|^2 + \|\theta_h(t)\|^2);$$

consequently (4.8) follows by combining (4.1) and (4.13).

A similar calculation bounds $\int_0^t|\Pi_h\partial_t^{1-\alpha}(u - u_h)(s)|_1^2\,ds$: starting from (4.12), instead of (4.13) one gets

$$\int_0^t\left|\partial_t^{1-\alpha}\theta_h(s)\right|_1^2 ds \le Ch^2\left\{\|u(0)\|_2^2 + \int_0^t\left[|u_t(s)|_1^2 + |u_t(s)|_2^2 + \|\partial_t^{1-\alpha}u(s)\|_2^2\right]ds\right\}$$

$$\times \left[1 + E_{\alpha_0}(C_4 t^{\alpha_0})\omega_{1+\alpha_0}(t)\right].$$

Then

$$\int_0^t \left|\partial_t^{1-\alpha}(u-u_h)(s)\right|_1^2 ds \le 2\left[\int_0^t \left|\partial_t^{1-\alpha}\rho_h(s)\right|_1^2 ds + \int_0^t \left|\partial_t^{1-\alpha}\theta_h(s)\right|_1^2 ds\right]$$

and (4.4) yield (4.9).

By inspection one finds that the constants C and C_4 in these error bounds approach finite values as $\alpha_1 \to 1^-$, so (4.8) and (4.9) are α-robust. □

When the drift term $\nabla \cdot (\mathbf{b}\, \partial_t^{1-\alpha(x)} u)$ of (1.1) is not present, which is the case in certain physical applications [14], we have the following simplification of Theorem 4.3.

Corollary 4.4 *Suppose that* $\mathbf{b} \equiv 0$. *Then there exists a constant* C *such that*

$$\begin{aligned}\|(u-u_h)(t)\|^2 &+ \int_0^t \left|\partial_t^{1-\alpha}(u-u_h)(s)\right|_1^2 ds \\ &\le Ch^2 \left\{|u(t)|_1^2 + \|u(0)\|_2^2 + \int_0^t \left[|u_t(s)|_1^2 + |u_t(s)|_2^2 + |\partial_t^{1-\alpha} u(s)|_2^2\right] ds\right\}\end{aligned}$$

for $0 \le t \le T$. *This error estimate is* α*-robust.*

Proof In the proof of Theorem 4.3, appeal now to Corollary 3.3 instead of Lemma 3.2, so that (4.11) and (4.12) are replaced by

$$\|\theta_h(t)\|^2 + \int_0^t \left|\partial_t^{1-\alpha}\theta_h(s)\right|_1^2 ds \le C\left[\int_0^t \left(\|\tilde f\|^2 + \|\tilde g\|^2\right) ds\right]$$

with $\tilde f$ as before but $\tilde g = a\nabla\left(\partial_t^{1-\alpha} u\right) - a\nabla\Pi_h\left(\partial_t^{1-\alpha} R_h u\right)$. One can now follow the remaining part of the proof of Theorem 4.3 to complete the argument. □

The $O(h)$ bounds on $\left\{\int_0^t \left|\partial_t^{1-\alpha}(u-u_h)(s)\right|_1^2 ds\right\}^{1/2}$ that are derived in Theorem 4.3 and Corollary 4.4 are of optimal order, and moreover are α-robust. One could extend our analysis from piecewise linears to piecewise polynomials of higher degree and obtain the analogous optimal-order bound by assuming increased spatial regularity of the solution u, but this assumption is restrictive since u can have at most two more orders of spatial smoothness than the initial data u_0; see [6, Theorem 2.1] and the discussion that follows it.

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