Collocation method for the Riesz fractional Laplacian problem on Graded meshes

Chao Min, Minghua Chen*

School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P.R. China

Abstract

The nonlocal diffusion problem is an important model, which is used to describe a large number of scientific phenomena. In this paper, a nonlocal diffusion problem with weakly singular solution is studied by piecewise linear polynomial collocation method and L_1 scheme on graded meshes. The singularity of solution leads to a decline in the accuracy for many numerical methods, so piecewise linear collocation method and L_1 scheme on graded meshes are considered to handle the singularity of solution. Through rigorous theoretical derivation, we prove the stability and convergence of this method. Finally, some numerical examples are given to confirm our theoretical results.

Keywords: nonlocal problem; collocation method; L_1 scheme; graded meshes

1. Introduction

In this paper, an error estimate of piecewise linear polynomial collocation method and L_1 scheme has been studied on graded meshes for a time-dependent nonlocal diffusion problem which is posed in [1, 2, 3] like

$$\frac{\partial u}{\partial t} + \int_{\Omega} (u(y) - u(x)) J(x, y)\, dy = f(x, t) \quad (x, t) \in \Omega \times [0, T],$$

here $\Omega \subseteq R$ and the kernel $J(x, y) : \Omega \times \Omega \mapsto R$ is a nonnegative symmetric.

In recently, the above nonlocal diffusion problem has attracted researchers' attention. Many researches show that this nonlocal model can be used to describe numerous scientific phenomena in various fields, like image processing, biology, mathematical finances, anomalous superdiffusion, more details can be seen from[1, 2]. And the existence and uniqueness of (1.2) is proved in the monograph [2, Theorem 2.9].

There are many choices for the kernel $J(x, y)$, for example, the constant kernel, fractional Laplacian kernel or weakly singular kernels [2, 4, 5, 6, 3, 7, 8]

$$J(x, y) \sim \frac{1}{|y - x|^{1+2s}} \tag{1.1}$$

where $s \in (-\frac{1}{2}, 1)$.

In this paper, we mainly study the situation of $s \in \left(-\frac{1}{2}, 0\right)$. More precisely, we will focus on the following nonlocal diffusion problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} D_t^{\alpha} u(x,t) + L[u](x,t) = f(x,t) & \text{for } x \in \Omega, \\ u(x, 0) = \phi(x) & \text{for } x \in \Omega, \\ u(x, t) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{1.2}$$

Here $\Omega = (a, b)$ and the nonlocal operater with weakly singular kernel and the Caputo derivative are defined, respectively, by

$$L[v](x) = \int_a^b \frac{v(x) - v(y)}{|x - y|^{\gamma}}\, dy, \quad \text{for } 0 < \gamma < 1,$$

$$D_t^{\alpha} \nu(t) = \int_0^t \frac{\nu'(s)}{(t - s)^{\alpha}}\, ds, \quad \text{for } 0 < \alpha < 1.$$

From the view of biology, the homogeneous Dirichlet boundary condition means that Ω is surrounded by a hostile environment, and any one who jumps out of the Ω dies immediately, such as fish out of water.

For convenience, we define $\mathcal{N}[u] := D_t^{\alpha} u(x,t) + L[u](x,t)$.

*Corresponding author. E-mail: minch21@lzu.edu.cn; chenmh@lzu.edu.cn

The important step of the error analysis for problem (1.2) is to study the above nonlocal operator $L[u]$. It is worth noting that there are some significant differences between $L[u]$ and the second kind Fredholm integral operator

$$\lambda(x)\,u(x) - \int_a^b \frac{u(x)}{|y-x|^{\gamma}}\,dy, \quad x \in (a,b) \quad \text{and} \quad 0 < \gamma < 1, \tag{1.3}$$

where the $\lambda(x)$ is a nonzero complex constant for space. For instance, the inverse operators of the second kind Fredholm integral operators are uniformly bounded [9, 10]. However, the inverses of the nonlocal operators $L[u]$ are unbounded [11, Lemma 3.5]. On the other hand, the convergence rate $\mathcal{O}\left(h^2\right)$ of a piecewise linear collocation method is proved for the second kind Fredholm integral equation (1.3) with smooth solutions [9], and it is only convergence rate of $\mathcal{O}(h)$ for the nonlocal diffusion equation $L[u](x) = f(x)$,even solutions $u(x) \in C^4(\Omega)$ [12].

Moreover, the following integrals with weakly singular kernels have also been widespreadly studied among the researchers:

$$\int_a^b \frac{v(y)}{|x-y|^{\gamma}}\,dy, \quad x \in (a,b) \quad \text{for } 0 < \gamma < 1. \tag{1.4}$$

If (1.4) multiplied by $\frac{1}{\Gamma(1-\gamma)}$, it is regarded as the Riemann–Liouville fractional integrals [13, 14] and Weyl fractional integrals [15].

In various kinds of numerical methods for solving integral equation, piecewise polynomial collocation method is the simplest [6], because only a single integration needs to be computed. Piecewise polynomial collocation method for the integral (1.4) has made significant progress. In [10], for the weakly singular integral (1.4), Atkinson K.E. proved that the order of convergence is $\mathcal{O}\left(h^2\right)$ for the piecewise linear collocation method. For integrals with hypersingular kernels ($\gamma \geq 1$), it was first researched by Linz.P [16]. Since the literature [16], the Hadamard finite-part integral has been used to studied the hypersingular integral [17, 6, 18, 19, 20].

Up to now, various numerical methods for the nonlocal diffusion problem have been studied by many researchers. In [21, 4], the peridynamic models (nonlocal problems with a horizon parameter) were studied by finite element method and finite difference method, respectively, and the second order of convergence was rigorously proved for the smooth solution. I. Graham studied a collocation method for solving the two dimensional Fredholm integral equations of the second kind under ideal assumption [22]. Zhang et al. [6] used piecewise linear collocation method to study hypersingular nonlocal diffusion problem, and proofed that the convergence rate is $\mathcal{O}(h^{2-2s})$ for solutions which belong to $C^2(\bar{\Omega})$. Later, Cao et al. [11] developed the fast and high-order numerical method to solve the time-dependent nonlocal diffusion problem based on the piecewise quadratic polynomial collocation method, and proofed the convergence rate $\mathcal{O}(h^2 + \tau^2)$ for smooth solutions. And Chen et al.[12] found that weakly singularity kernels lead to a lower accuracy, which reduced to $\mathcal{O}(h)$ for solving nonlocal boundary value problem $L[u](x) = f(x)$ with $u(x) = g(x), \quad x \in \partial\Omega$ by piecewise linear collocation method, even though the solution is smooth.

As the above mentioned, many advances have been made in numerical methods for nonlocal diffusion problems. But unfortunately, those studies have restrictions for the regularity of solutions. What will happen if there is a weak singularity solution for (1.2) ? In general, there exists weakly singularity for the solution of integral equation with weakly singular kernel (1.1). For example, fractional Laplacian problems [23], time-fractional diffusion equation [24], integral equation with highly oscillatory [25] and the second kind Fredholm integral equation with singular kernel [26, 9, 27, 28, 7].

This gap inspires us to consider solving nonlocal diffusion problem (1.2) with weakly singular solutions on the boundary layer by the piecewise linear collocation method and graded meshes . Here the regularity similar to [9, Theorem 4.2.2][26] is given. In the rest of this paper, we will assume the solutions of (1.2) satisfy the following properties:

Proposition 1.1. *$u(x,t) \in C^{\alpha}\left([0,T]; C^{\sigma}\left(\bar{\Omega}\right) \bigcap C^2(\Omega)\right)$ with $\sigma \in (0,+\infty)$, and $u(x,t)$ satisfy the following properties*

$$\left|\frac{\partial^{\ell}}{\partial x^{\ell}} u(x,t)\right| \leq C[(x-a)(b-x)]^{\frac{\sigma}{2}-\ell} \quad \textit{for } \ell = 0,1,2$$

and

$$\left|\frac{\partial^{\omega}}{\partial t^{\omega}} u(x,t)\right| \leq C t^{\alpha-\omega} \quad \textit{for } \omega = 0,1,2,$$

for $x \in \Omega, \quad t \in [0,T]$.

The structure of this paper is as follows. In section 2, the graded meshes are introduced. And temporal dimension and spatial dimension are discretized by the Crank-Nicolson method and the piecewise linear collocation method, respectively. Later in section 3, the properties of stiff matrix of the discrete schemes is analyzed. And the local truncation error and the convergence rate are proved in the section **??**. Finally, in section 7, some numerical results are given to confirm conclusions which are proofed at section **??**.

2. Piecewise linear collocation method and Graded meshes

2.1. Graded meshes

Let the partition π_h with the interval $\Omega = (a, b)$

$$\pi_h : a = x_0 < x_1 < x_2 < \cdots < x_{2N-1} < x_{2N} = b,$$

here the grid sizes $h_j := x_j - x_{j-1}$.

In this paper, we focus on symmetric Graded meshes as

$$x_j = \begin{cases} a + \frac{b-a}{2}\left(\frac{j}{N}\right)^r & j = 0, 1, \ldots, N \\ b - \frac{b-a}{2}\left(2 - \frac{j}{N}\right)^r & j = N+1, N+2, \ldots, 2N \end{cases} \tag{2.1}$$

with the grading exponent $r \geq 1$.

Remark 2.1. *There is also some literatures ,such as [24, 29], that use single-sided Graded meshes, as following:*

$$x_j = a + (b-a)\left(\frac{j}{2N}\right)^r, \quad j = 0, 1, \cdots, 2N,$$

or

$$x_j = b - (b-a)\left(1 - \frac{j}{2N}\right)^r, \quad j = 0, 1, \cdots, 2N.$$

where the grading exponent $r \geq 1$.

2.2. Piecewise linear basis function

Let S^h be the space of continuous piecewise-linear polynomials defined with respect to the partition π_h and choose the standard hat functions as a basis which we denote as $\{\phi_j(x)\}_0^{2N}$. Namely, the piecewise linear basis function is defined by

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j], \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}], \\ 0, & \text{otherwise.} \end{cases}$$

with $j = 1, \cdots, 2N-1$, and

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0}, & x \in [x_0, x_1], \\ 0, & \text{otherwise.} \end{cases} \qquad \phi_{2N}(x) = \begin{cases} \dfrac{x - x_{2N-1}}{x_{2N} - x_{2N-1}}, & x \in [x_{2N-1}, x_{2N}], \\ 0, & \text{otherwise.} \end{cases}$$

2.3. L_1 discretization for temporal dimension

We take classical L_1 method to approximate the temporal dimension on graded meshes. Let M is a positive integer. Set $\tau = \frac{T}{M}$ and $t_j = T\left(\frac{j}{M}\right)^{r_1}$ for $j = 0, 1, \cdots, M$, then $\tau_j := t_j - t_{j-1}$ for $j = 1, 2, \cdots, M$.

$$\begin{aligned} D_t^\alpha u(x, t_m) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_m} \frac{\partial u(x,s)}{\partial s} (t_m - s)^{-\alpha}\, ds \\ &= \sum_{k=1}^{m} \frac{1}{\Gamma(1-\alpha)} \int_{x_{k-1}}^{x_k} \frac{\partial u(x,s)}{\partial s} (t_m - s)^{-\alpha}\, ds \\ &= \sum_{k=1}^{m} \frac{1}{\Gamma(1-\alpha)} \frac{u(x, t_k) - u(x, t_{k-1})}{\tau_k} \int_{x_{k-1}}^{x_k} (t_m - s)^{-\alpha}\, ds + R_m^t \\ &= \sum_{k=1}^{m} \frac{1}{\Gamma(2-\alpha)} \left(u(x, t_k) - u(x, t_{k-1})\right) \frac{(t_m - x_{k-1})^{1-\alpha} - (t_m - x_k)^{1-\alpha}}{\tau_k} + R_m^t \end{aligned}$$

where the R_m^t is the Local truncation error for the temporal dimension.

Therefore, we obtain the corresponding L_1 discrete formula

$$D_M^{\alpha} u\left(x, t_m\right)=\sum_{k=1}^m \frac{1}{\Gamma(2-\alpha)}\left(u\left(x, t_k\right)-u\left(x, t_{k-1}\right)\right) \frac{\left(t_m-x_{k-1}\right)^{1-\alpha}-\left(t_m-x_k\right)^{1-\alpha}}{\tau_k}$$

Therefore, using L_1 approximation, we obtain

$$\mathcal{N}_\tau[u](x, t_m)=f(x, t_m)+R_m^t, \tag{2.2}$$

where

$$\mathcal{N}_\tau[u](x, t_m)=D_M^{\alpha} u\left(x, t_m\right)+L[u]\left(x, t_m\right).$$

Theorem 2.1. *[24, Theorem 5.2] $u(x,t)$ satisfy Proposition 1.1, then there exist a constant C such that*

$$\left|R_m^t\right|=\left|D_t^{\alpha} u\left(x, t_m\right)-D_M^{\alpha} u\left(x, t_m\right)\right| \leq C m^{-\min \{2-\alpha, r_1 \alpha\}}. \tag{2.3}$$

2.4. Fully discrete scheme

The solution of (1.2) is approximated by piecewise linear polynomial on graded meshes (2.1) with grading exponent r_2 in space. Substituting the piecewise linear interpolation (since $u(x,t)=0, x \in \partial\Omega$)

$$\Pi_h u(x, t_m)=\sum_{k=1}^{2N-1} u\left(x_k, t_m\right) \phi_k(x)$$

into (2.2) yield approximation for the value of (2.2) at the nodal points x_i given by

$$\mathcal{N}_\tau[\Pi_h u](x_i, t_m)=f(x_i, t_m)-R_i^m+R_m^t \tag{2.4}$$

where the local truncation error for the space dimension is

$$R_i^m:=L[\Pi_h u-u](x_i, t_m)=\int_a^b \frac{u(y, t_m)-\Pi_h u(y, t_m)}{|x_i-y|^\gamma}\, dy. \tag{2.5}$$

Denoting $d_{m,k}:=\frac{(t_m-t_{k-1})^{1-\alpha}-(t_m-t_k)^{1-\alpha}}{\tau_k}$, the above equation (2.4) can be rewrited as

$$\frac{d_{m,m}}{\Gamma(2-\alpha)} u_i^m+\sum_{j=1}^{2N-1} \int_a^b \frac{\phi_j(x_i)-\phi_j(y)}{|x_i-y|^\gamma} u_j^m-\frac{\sum_{k=1}^{m-1}\left(d_{m,k}-d_{m,k+1}\right) u_i^k-d_{m,1} u_i^0}{\Gamma(2-\alpha)}=f_i^m-R_i^m+R_m^t \tag{2.6}$$

where $u_i^m:=u\left(x_i, t_m\right)$ and $f\left(x_i, t_m\right):=f_i^m$.

Further, we take U_i^m as the approximation of u_i^m, to get the follow discrete scheme:

$$\frac{d_{m,m}}{\Gamma(2-\alpha)} U_i^m+\sum_{j=1}^{2N-1} \int_a^b \frac{\phi_j(x_i)-\phi_j(y)}{|x_i-y|^\gamma} U_j^m=f_i^m-\frac{\sum_{k=1}^{m-1}\left(d_{m,k}-d_{m,k+1}\right) U_i^k-d_{m,1} U_i^0}{\Gamma(2-\alpha)}. \tag{2.7}$$

This system of equations has the matrix form

$$\left(\frac{d_{m,m}}{\Gamma(2-\alpha)} I+A\right) U^m=F^m-\frac{\sum_{k=1}^{m-1}\left(d_{m,k}-d_{m,k+1}\right) U^k-d_{m,1} U^0}{\Gamma(2-\alpha)},\ m=1,2,\cdots,M, \tag{2.8}$$

where the coefficient matrix A and the matrix of the grid function are defined by

$$A=(a_{i,j}) \in \mathbb{R}^{(2N-1)\times(2N-1)},\ U^k=(U_1^k,\ldots,U_{2N-1}^k)^T,\ F^m=(f_1^m,\ldots,f_{2N-1}^m)^T,$$

with $a_{i,j}=\int_a^b \frac{\phi_j(x_i)-\phi_j(y)}{|x_i-y|^\gamma}\, dy$.

3. Explicit expression of $a_{i,j}$

In this section, we will show the explicit expression and properties of stiff matrix A.

For $j = i$, we have

$$a_{i,i} = \int_a^b \frac{dy}{|x_i - y|^\gamma} \phi_i(x_i) - \int_{x_{i-1}}^{x_{i+1}} \phi_i(y) |x_i - y|^{-\gamma} \, dy$$
$$= \int_a^b \frac{dy}{|x_i - y|^\gamma} - \int_{x_{i-1}}^{x_i} \frac{y - x_{i-1}}{h_i} (x_i - y)^{-\gamma} \, dy + \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - y}{h_i} (y - x_i)^{-\gamma} \, dy$$
$$= \frac{1}{1-\gamma} \left[(x_i - a)^{1-\gamma} + (b - x_i)^{1-\gamma} \right] - \frac{1}{(1-\gamma)(2-\gamma)} \left(h_i^{1-\gamma} + h_{i+1}^{1-\gamma} \right).$$

For $j > i$, we have

$$a_{i,j} = -\int_{x_{j-1}}^{x_{j+1}} \frac{\phi_j}{(y - x_i)^\gamma} \, dy$$
$$= -\frac{1}{1-\gamma} \int_{x_{j-1}}^{x_j} \phi_j(y) \, d(y - x_i)^{1-\gamma} - \frac{1}{1-\gamma} \int_{x_j}^{x_{j+1}} \phi_j(y) \, d(y - x_i)^{1-\gamma}$$
$$= -\frac{1}{(1-\gamma)(2-\gamma)} \left[\frac{(x_{j+1} - x_i)^{2-\gamma}}{h_{j+1}} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} (x_j - x_i)^{2-\gamma} + \frac{(x_{j-1} - x_i)^{2-\gamma}}{h_j} \right].$$

Using the same way, we can get the case of $j < i$ as following:

$$a_{i,j} = -\frac{1}{(1-\gamma)(2-\gamma)} \left[\frac{(x_i - x_{j+1})^{2-\gamma}}{h_{j+1}} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} (x_i - x_j)^{2-\gamma} + \frac{(x_i - x_{j-1})^{2-\gamma}}{h_j} \right].$$

Therefore, we have the following result:

Lemma 3.1. *The entries of the stiffness matrix $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$ with $\gamma \in (0,1)$ can be explicitly computed by*

$$a_{ij} = \frac{1}{1-\gamma} \left[(x_i - a)^{1-\gamma} + (b - x_i)^{1-\gamma} \right] \cdot \delta_{i,j} - \frac{1}{(1-\gamma)(2-\gamma)} C_j D_j^i$$

with

$$C_j = \left(\frac{1}{h_j}, -\frac{1}{h_j} - \frac{1}{h_{j+1}}, \frac{1}{h_{j+1}} \right), \quad D_j^i = \begin{pmatrix} |x_{j-1} - x_i|^{2-\gamma} \\ |x_j - x_i|^{2-\gamma} \\ |x_{j+1} - x_i|^{2-\gamma} \end{pmatrix},$$

and $\delta_{i,j} = \begin{cases} 1, & \textit{if } i = j \\ 0, & \textit{otherwise.} \end{cases}$

Lemma 3.2. *Let A be defined by (2.8), then A is a strictly diagonally dominant matrix with positive entries on the diagonal and nonpositive off-diagonal entries.*

Proof. It is clearly to see that from Lemma 3.1

$$a_{ii} = \frac{1}{1-\gamma} \left[(x_i - a)^{1-\gamma} + (b - x_i)^{1-\gamma} \right] - \int_{x_{i-1}}^{x_{i+1}} \frac{\phi_i(y)}{|x_i - y|^\gamma} \, dy$$
$$= \frac{1}{1-\gamma} \left[(x_i - a)^{1-\gamma} + (b - x_i)^{1-\gamma} \right] - \frac{1}{(1-\gamma)(2-\gamma)} \left(h_i^{1-\gamma} + h_{i+1}^{1-\gamma} \right) > 0.$$

For $j \neq i$,

$$a_{i,j} = -\frac{1}{(1-\gamma)(2-\gamma)} \left[\frac{|x_{j+1} - x_i|^{2-\gamma}}{h_{j+1}} - \left(\frac{1}{h_{j+1}} + \frac{1}{h_j} \right) |x_j - x_i|^{2-\gamma} + \frac{|x_{j-1} - x_i|^{2-\gamma}}{h_j} \right] < 0,$$

since $x^{2-\gamma}$ is a concave function for $0 < x$ and $0 < \gamma < 1$.

For $1 \le i \le 2N-1$, using the Taylor expansion we can figure out that

$$\begin{aligned}\sum_{j=1}^{2N-1} a_{i,j} &= \frac{1}{1-\gamma}\left[(x_i-a)^{1-\gamma}+(b-x_i)^{1-\gamma}\right] - \sum_{j=1}^{2N-1}\int_{x_{j-1}}^{x_{j+1}} \frac{\phi_j\left(y\right)}{\left|x_i-y\right|^{\alpha-1}}\,dy \\ &= \frac{1}{1-\gamma}\left[(x_i-a)^{1-\gamma}+(b-x_i)^{1-\gamma}\right] - \frac{1}{\left(1-\gamma\right)\left(2-\gamma\right)} \\ &\qquad \left[\frac{\left(x_i-x_0\right)^{2-\gamma}-\left(x_i-x_1\right)^{2-\gamma}}{h_1} + \frac{\left(x_{2N}-x_i\right)^{2-\gamma}-\left(x_{2N-1}-x_i\right)^{2-\gamma}}{h_{2N}}\right] \\ &= \frac{1}{1-\gamma}\left[(x_i-a)^{1-\gamma}+(b-x_i)^{1-\gamma}\right] - \frac{1}{1-\gamma}\left[\left(x_i-\xi_1\right)^{1-\gamma}+\left(\xi_2-x_i\right)^{1-\gamma}\right] > 0,\end{aligned}$$

where $\xi_1 \in (x_0, x_1)$ and $\xi_2 \in (x_{2N-1}, x_{2N})$. $\square$

Remark 3.1. *As the above mentioned, the matrix A is a M-matrix. However, it is interesting to note that this property is determined by the basis function, has nothing to do with the space grid.*

4. Local truncation error R_i^m

For notational convenience, without loss of generality we take $\Omega = (0, 2\mathcal{D})$ and rewrite (2.1) as

$$x_j = \begin{cases} \mathcal{D}\left(\frac{j}{N}\right)^{r_2} & \text{for } j = 0, 1, \ldots, N, \\ 2\mathcal{D} - \mathcal{D}\left(2-\frac{j}{N}\right)^{r_2} & \text{for } j = N+1, N+2, \ldots, 2N. \end{cases} \tag{4.1}$$

Set $h_j := x_j - x_{j-1}$ for $j = 1, \ldots, 2N$. From the mean value theorem and the definition of $\{x_j\}$, it follows that

$$h_j \le \begin{cases} \mathcal{D}N^{-r_2} r_2 j^{r_2-1} \le CN^{-r_2} j^{r_2-1} & \text{for } j = 1, \ldots, N, \\ CN^{-r_2}(2N+1-j)^{r_2-1} & \text{for } j = N+1, \ldots, 2N; \end{cases} \tag{4.2}$$

we shall use this inequality many times. In particular, $h_j \le CN^{-1}$ for all j.

Notation. Above and throughout the rest of the paper, C denotes a generic constant that is independent of N and of any index such as i or j. For any $s \in \mathbb{R}$, $\lceil s \rceil$ denotes the smallest integer that is not less than s.

We begin with a detailed analysis of the local truncation error $R_i^j := L[\Pi_h u - u](x_i, t_j)$ defined in (2.5). As the definition of (2.2) and (2.4), we have

$$R_i^j = \sum_{k=1}^{2N} \int_{x_{k-1}}^{x_k} \frac{u\left(y, t_j\right) - \Pi_h u\left(y, t_j\right)}{\left|x_i - y\right|^{\gamma}}\, dy = \sum_{k=1}^{2N} \mathcal{T}_{i,k}^j \tag{4.3}$$

with

$$\mathcal{T}_{i,k}^j := \int_{x_{k-1}}^{x_k} \frac{u\left(y, t_j\right) - \Pi_h u\left(y, t_j\right)}{\left|x_i - y\right|^{\gamma}}\, dy \tag{4.4}$$

By a standard error estimate for linear interpolation, since $u \in C^2(0, 2T)$ in Proposition 1.1 one has

$$\left|\mathcal{T}_{i,k}^j\right| \le Ch_k^2 \left(\max_{s\in[x_{k-1},x_k]} |u_{xx}(s, t_j)|\right) \int_{x_{k-1}}^{x_k} \left|x_i - y\right|^{-\gamma} dy, \text{ for } k \ne 1, i, i+1, 2N. \tag{4.5}$$

Lemma 4.1. *There exists a constant C such that*

$$\sum_{k=1}^{i} |\mathcal{T}_{i,k}^j| \le CN^{-\min\left\{r_2\left(\frac{\sigma+2}{2}-\gamma\right),\, 2\right\}}$$

for all $i \in \{1, \ldots, N\}$.

Proof. Let $i \in \{1, \ldots, N\}$ be arbitrary but fixed. Consider separately the cases $k = 1 = i,\ k = 1 < i,\ 1 < k = i$ and $1 < k < i$.

Case $k = i = 1$:

$$\begin{aligned}
\left|\mathcal{T}_{1,1}^{j}\right| &= \left|\int_{x_0}^{x_1} \frac{u_x\left(y,t_j\right) - \frac{u(x_1,t_j)-u(x_0,t_j)}{x_1-x_0}}{\left|x_1-y\right|^{\gamma-1}}\, dy\right| \\
&\leq C\left[\int_{x_0}^{x_1} y^{\frac{\sigma}{2}-1}\left(x_1-y\right)^{1-\gamma}\, dy + \frac{x_1^{\frac{\sigma}{2}}}{h_1}\int_{x_0}^{x_1}\left(x_1-y\right)^{1-\gamma} dy\right] \\
&\leq C x_1^{\frac{\sigma+2}{2}-\gamma} \leq N^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)}.
\end{aligned} \tag{4.6}$$

Case $1 = k < i$:

$$\begin{aligned}
\left|\mathcal{T}_{i,1}^{j}\right| &= \left|\int_{x_0}^{x_1}\left(u\left(y\right)-\Pi_h u\left(y\right)\right)\left(x_i-y\right)^{-\gamma}\, dy\right| \\
&\leq C\left|\int_{x_0}^{x_1}\left(u\left(y,t_j\right)-\Pi_h u\left(y,t_j\right)\right)\, dy\right|\left(x_i-x_1\right)^{-\gamma} \\
&\leq C\left(\int_{x_0}^{x_1} y^{\frac{\sigma}{2}}\, dy + x_1^{\frac{\sigma}{2}} h_1\right) x_i^{-\gamma} = x_1^{\frac{\sigma+2}{2}} x_i^{-\gamma} \\
&\leq C N^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{-r_2\gamma}.
\end{aligned} \tag{4.7}$$

Case $1 < k \leq \left\lceil \frac{i}{2} \right\rceil$: From (4.5), we can get

$$\begin{aligned}
\sum_{k=2}^{\lceil i/2\rceil}\left|\mathcal{T}_{i,k}^{j}\right| &\leq \sum_{k=2}^{\lceil i/2\rceil} C h_k^3 x_{k-1}^{\frac{\sigma}{2}-2}\left(x_i-x_k\right)^{-\gamma} \\
&\leq C\sum_{k=2}^{\lceil i/2\rceil}\left(N^{-r_2}k^{r_2-1}\right)^3\left(N^{-r_2}k^{r_2}\right)^{\frac{\sigma}{2}-2}\left(N^{-r_2}i^{r_2}\right)^{-\gamma} \\
&\leq C N^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{-r_2\gamma}\sum_{k=2}^{\lceil i/2\rceil} k^{r_2\frac{\sigma+2}{2}-3}
\end{aligned} \tag{4.8}$$

From (4.8), using well-known convergence properties of the series $\sum_{j=2}^{\infty} j^{\mu}$ for constant $\mu \in \mathbb{R}$, we get

$$\begin{aligned}
\sum_{k=2}^{\lceil i/2\rceil}\left|\mathcal{T}_{i,k}^{j}\right| &\leq C N^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{-r_2\gamma}\sum_{k=2}^{\lceil i/2\rceil} k^{r_2\frac{\sigma+2}{2}-3} \\
&\leq \begin{cases} C N^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{-r_2\gamma} & \text{if } r_2\frac{\sigma+2}{2} < 2, \\ C N^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{-r_2\gamma}\ln i & \text{if } r_2\frac{\sigma+2}{2} = 2, \\ C N^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{r_2\left(\frac{\sigma+2}{2}-\gamma\right)-2} & \text{if } r_2\frac{\sigma+2}{2} > 2, \end{cases}
\end{aligned} \tag{4.9}$$

Case $\left\lceil \frac{i}{2} \right\rceil < k < i$: From (4.5), we can get

$$\begin{aligned}
\sum_{k=\lceil i/2\rceil+1}^{i-1}\left|\mathcal{T}_{i,k}^{j}\right| &\leq \sum_{k=\lceil i/2\rceil+1}^{i-1} C h_i^2 x_{\lceil \frac{i}{2}\rceil}^{\frac{\sigma}{2}-2}\int_{x_{k-1}}^{x_k}\left(x_i-y\right)^{-\gamma}\, dy \\
&= C h_i^2 x_{\lceil \frac{i}{2}\rceil}^{\frac{\sigma}{2}-2}\int_{x_{\lceil i/2\rceil}}^{x_{i-1}}\left(x_i-y\right)^{-\gamma}\, dy \\
&\leq C h_i^2 x_i^{\frac{\sigma}{2}-2}\left(x_i^{1-\gamma}-h_i^{1-\gamma}\right) \\
&\leq C\left(h_i^2 x_i^{\frac{\sigma-2}{2}-\gamma} - h_i^{3-\gamma}x_i^{\frac{\sigma}{2}-2}\right) \\
&\leq C N^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{r_2\left(\frac{\sigma+2}{2}-\gamma\right)-2}.
\end{aligned} \tag{4.10}$$

Case $1 < k = i$

$$\begin{aligned} \left|\mathcal{T}_{i,i}^{j}\right| &\le Ch_i^2 x_{i-1}^{\frac{\sigma}{2}-2} \int_{x_{i-1}}^{x_i} (x_i - y)^{-\gamma}\, dy \\ &\le Ch_i^{3-\gamma} x_i^{\frac{\sigma}{2}-2} \\ &\le CN^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{r_2\left(\frac{\sigma+2}{2}-\gamma\right)-(3-\gamma)}. \end{aligned} \tag{4.11}$$

To finish the proof, observe that adding the bounds (4.6), (4.7), (4.9), (4.10) and (4.11) will bound $\sum_{k=1}^{i} |T_{i,k}^{j}|$. The desired result follows easily.

□

Lemma 4.2. *There exists a constant C such that*

$$\sum_{k=i+1}^{N} |\mathcal{T}_{i,k}^{j}| \le \begin{cases} CN^{-2}\ln N, & \text{if} \quad r_2\left(\frac{\sigma+2}{2}-\gamma\right) = 2 \\ CN^{-\min\left\{r_2\left(\frac{\sigma+2}{2}-\gamma\right),\, 2\right\}}, & \text{otherwise.} \end{cases}$$

for all $i \in \{1, \dots, N-1\}$.

Proof. **Case** $k = i+1$

$$\begin{aligned} \left|\mathcal{T}_{i,i+1}^{j}\right| &\le Ch_{i+1}^2 x_i^{\frac{\sigma}{2}-2} \int_{x_i}^{x_{i+1}} (y - x_i)^{-\gamma}\, dy \\ &\le Ch_{i+1}^{3-\gamma} x_i^{\frac{\sigma}{2}-2} \\ &\le CN^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{r_2\left(\frac{\sigma+2}{2}-\gamma\right)-(3-\gamma)}. \end{aligned} \tag{4.12}$$

Case $i < k \le N$: Letting $K = \min\{2i, N\}$, then we have

$$\begin{aligned} \sum_{k=i+2}^{K} \left|\mathcal{T}_{i,k}^{j}\right| &\le \sum_{k=i+2}^{K} Ch_k^2 x_{k-1}^{\frac{\sigma}{2}-2} \int_{x_{k-1}}^{x_k} (y - x_i)^{-\gamma} \\ &\le Ch_i^2 x_i^{\frac{\sigma}{2}-2} \sum_{k=i+2}^{K} \int_{x_{k-1}}^{x_k} (y - x_i)^{-\gamma} \\ &\le Ch_i^2 x_i^{\frac{\sigma}{2}-2} \left[(x_K - x_i)^{1-\gamma} - (x_{i+1} - x_i)^{1-\gamma}\right] \\ &\le C\left(h_i^2 x_i^{\frac{\sigma-2}{2}-\gamma} - h_i^{3-\gamma} x_i^{\frac{\sigma}{2}-2}\right) \\ &\le CN^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{r_2\left(\frac{\sigma+2}{2}-\gamma\right)-2}. \end{aligned} \tag{4.13}$$

For the special case $K = 2i < N$, we have

$$\begin{aligned} \sum_{j=K+1}^{N} \left|\mathcal{T}_{i,k}^{j}\right| &\le C \sum_{k=K+1}^{N} h_k^2 x_{k-1}^{\frac{\sigma}{2}-2} \int_{x_{k-1}}^{x_k} (y - x_i)^{-\gamma}\, dy \\ &\le C \sum_{k=K+1}^{N} h_k^3 x_{k-1}^{\frac{\sigma}{2}-2} (x_{k-1} - x_i)^{-\gamma} \\ &\le C \sum_{k=K+1}^{N} h_k^3 x_{k-1}^{\frac{\sigma}{2}-2} x_{k-1}^{-\gamma} \\ &\le C \sum_{k=K+1}^{N} \left(N^{-r} k^{r-1}\right)^3 \left(N^{-r} k^{r}\right)^{\frac{\sigma}{2}-2-\gamma} = C \sum_{k=K+1}^{N} \left(N^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} k^{r_2\left(\frac{\sigma+2}{2}-\gamma\right)-3}\right) \\ &\le \begin{cases} CN^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} i^{r_2\left(\frac{\sigma+2}{2}-\gamma\right)-2} & \text{if } r_2\left(\frac{\sigma+2}{2}-\gamma\right) < 2, \\ CN^{-r_2\left(\frac{\sigma+2}{2}-\gamma\right)} \ln N & \text{if } r_2\left(\frac{\sigma+2}{2}-\gamma\right) = 2, \\ CN^{-2} & \text{if } r_2\left(\frac{\sigma+2}{2}-\gamma\right) > 2, \end{cases} \end{aligned} \tag{4.14}$$

Adding (4.12), (4.13) and (4.14) gives the desired result. □

Lemma 4.3. *There exists a constant C such that*

$$\sum_{k=N+1}^{2N} |\mathcal{T}_{i,k}^{j}| \leq CN^{-\min\left\{r\left(\frac{\sigma+2}{2}-\gamma\right),\,2\right\}}$$

for all $i \in \{1, \ldots, N\}$.

Proof. **Case** $k = N+1$: For the special case $k = N+1$

$$\begin{aligned}\left|\mathcal{T}_{i,N+1}^{j}\right| &\leq Ch_{N+1}^{3}\left(2T - x_{N+1}\right)^{\frac{\sigma}{2}-2}\left(x_N - x_i\right)^{-\gamma} \\ &\leq Ch_{N-1}^{3-\gamma} = CN^{-(3-\gamma)}.\end{aligned} \tag{4.15}$$

Case $N+1 < k \leq \left\lceil \frac{3N}{2} \right\rceil$

$$\begin{aligned}\sum_{k=N+2}^{\lceil 3N/2 \rceil}\left|\mathcal{T}_{i,k}^{j}\right| &\leq Ch_k^2\left(2\mathcal{D} - x_k\right)^{\frac{\sigma}{2}-2}\int_{x_{k-1}}^{x_k}\left(y - x_i\right)^{-\gamma}\,dy \\ &\leq CN^{-2}\int_{x_{N+1}}^{x_{\lceil\ 3N/2\rceil}}\left(y - x_i\right)^{-\gamma}\,dy \\ &= CN^{-2}\left[\left(x_{\lceil\frac{3N}{2}\rceil} - x_i\right)^{1-\gamma} - \left(x_{N+1} - x_i\right)^{1-\gamma}\right] \\ &\leq CN^{-2}.\end{aligned} \tag{4.16}$$

Case $\left\lceil \frac{3N}{2} \right\rceil < k < 2N$

$$\begin{aligned}\sum_{k=\lceil 3N/2\rceil+1}^{2N-1}\left|\mathcal{T}_{i,k}^{j}\right| &\leq C\sum_{k=\lceil 3N/2\rceil+1}^{2N-1} h_k^2\left(2\mathcal{D} - x_k\right)^{\frac{\sigma}{2}-2}\int_{x_{k-1}}^{x_k}\left(y - x_i\right)^{-\gamma}\,dy \\ &\leq C\sum_{k=\lceil 3N/2\rceil+1}^{2N-1} h_k^3\left(2\mathcal{D} - x_k\right)^{\frac{\sigma}{2}-2} \\ &\leq C\sum_{k=\lceil 3N/2\rceil+1}^{2N-1}\left(N^{-r}\left(2N+1-k\right)^{r-1}\right)^3\left(N^{-r}\left(2N-k\right)^r\right)^{\frac{\sigma}{2}-2} \\ &\leq C\sum_{q=1}^{\lceil N/2\rceil+1} N^{-r\frac{\sigma+2}{2}}q^{r\frac{\sigma+2}{2}-3} \\ &\leq \begin{cases} CN^{-r_2\frac{\sigma+2}{2}} & \text{if } r_2\frac{\sigma+2}{2} < 2, \\ CN^{-r_2\frac{\sigma+2}{2}}\ln N & \text{if } r_2\frac{\sigma+2}{2} = 2, \\ CN^{-2} & \text{if } r_2\frac{\sigma+2}{2} > 2. \end{cases}\end{aligned} \tag{4.17}$$

Case $k = 2N$

$$\begin{aligned}\left|\mathcal{T}_{i,2N}^{j}\right| &\leq \left|\int_{x_{2N-1}}^{x_{2N}}\frac{u\left(y\right) - \Pi_h u\left(y\right)}{\left(y - x_i\right)^{\gamma}}\,dy\right| \\ &\leq C\left|\int_{x_{2N-1}}^{x_{2N}}\left(2\mathcal{D} - y\right)^{\frac{\sigma}{2}}\,dy\right| + \frac{\left|u\left(x_{2N-1}\right)\right|}{h_{2N}}\int_{x_{2N-1}}^{x_{2N}}\left(x_{2N} - y\right)\,dy \\ &\leq Cx_1^{\frac{\sigma+2}{2}} \leq CN^{-r_2\frac{\sigma+2}{2}}\end{aligned} \tag{4.18}$$

Adding (4.15), (4.16), (4.17) and (4.18) gives the desired result. ∎

Lemma 4.4. *[Truncation error bound] Let R_i^j be defined by (2.5). Then there exists a constant C such that*

$$|R_i^j| = |L[u - \Pi_h u](x_i, t_j)| \leq \begin{cases} CN^{-2}\ln N, & \textit{if} \quad r_2\left(\frac{\sigma+2}{2} - \gamma\right) = 2 \\ CN^{-\min\left\{r_2\left(\frac{\sigma+2}{2}-\gamma\right),\,2\right\}}, & \textit{otherwise.} \end{cases}$$

for $i \in 1, 2, \cdots, 2N$

Proof. For $i = 1, 2 \ldots, N$, this result is an immediate consequence of the definition of $T_{i,k}^j$ and Lemmas 4.1–4.3.

For $i = N+1, N+2, \ldots, 2N-1$, observe first that the mesh (4.1) is symmetric about $x = T$ (i.e., $x = x_i$ is a mesh point if and only if $x = 2T - x_i = x_{2N-i}$ is a mesh point), and the a priori derivative bounds of proposition 1.1 are also symmetric about $x = T$. But the locations of the mesh points and these bounds on derivatives are the only ingredients used in the analysis of the case $i = 1, 2 \ldots, N$. Thus one can define $\tilde{u}(x) = u(2T - x)$, and now the truncation error of $u(x)$ at $x = x_i$ for $i = N+1, N+2, \ldots, 2N-1$ is exactly the same as the truncation error of $\tilde{u}(x)$ at $x = x_i$ for $i = N-1, N-2, \ldots 1$, which can be handled in exactly the same way as the truncation error analysis of $u(x)$ for $i = 1, 2, \ldots, N-1$ in Lemmas 4.1–4.3. Transforming back via $x \mapsto 2T - x$, we get the desired result for $i = N+1, N+2, \ldots, 2N-1$. $\square$

5. Stability of the scheme (2.8)

In this section, we will construct the stability of the scheme (2.8).And our stability analysis will be proofed in the following general framework.

$$\begin{cases} \mathcal{N}_M^N v_i^m = g_i^m, & \text{for } 1 \le i \le 2N-1, \quad 1 \le m \le M \\ v_0^m = v_{2N}^m = 0, & \text{for } 1 \le m \le M \\ v_i^0 = v^0(x_i), & \text{for } 1 \le i \le 2N-1, \end{cases} \tag{5.1}$$

here the function $v^0(x)$ is given and

$$\mathcal{N}_M^N v_i^m := \frac{d_{m,m}}{\Gamma(2-\alpha)} v_i^m + \sum_{j=1}^{2N-1} a_{i,j} v_j^m + \frac{\sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) v_i^k - d_{m,1} U_i^0}{\Gamma(2-\alpha)}$$

Lemma 5.1. *The solution of (5.1) satisfies*

$$\|v^m\| \le \tau_m^\alpha \left[\Gamma(2-\alpha) \|g^m\|_\infty + d_{m,1} \|v^0\|_\infty + \sum_{k=1}^{m-1} (d_{m,k+1} - d_{m,k}) \|v^k\|_\infty \right]$$

Proof. For a fix $m \in \{1, 2, \cdots, M\}$, select i_0 so that $|v_{i_0}^m| = \max_{1 \le i \le 2N-1} |v_i^m|$. Substituting $v_{i_0}^m$ into the formula (5.1) yields

$$\frac{d_{m,m}}{\Gamma(2-\alpha)} v_{i_0}^m + \sum_{j=1}^{2N-1} a_{i_0,j} v_j^m + \frac{\sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) v_{i_0}^k - d_{m,1} v_{i_0}^0}{\Gamma(2-\alpha)} = g_{i_0}^m.$$

Since Lemma 3.2, we can get

$$\begin{aligned} \left(\frac{d_{m,m}}{\Gamma(2-\alpha)} + a_{i_0,i_0} \right) |v_{i_0}^m| &\le \sum_{\substack{j=1, \\ j \ne i_0}}^{2N-1} |a_{i_0,j}| |v_j^m| + \left| \frac{\sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) v_{i_0}^k - d_{m,1} v_{i_0}^0}{\Gamma(2-\alpha)} + g_{i_0}^m \right| \\ &\le a_{i_0,i_0} |v_{i_0}^m| + \left| \frac{\sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) v_{i_0}^k - d_{m,1} v_{i_0}^0}{\Gamma(2-\alpha)} + g_{i_0}^m \right| \end{aligned} \tag{5.2}$$

Since $d_{m,k} = \frac{(t_m - t_{k-1})^{1-\alpha} - (t_m - t_k)^{1-\alpha}}{\tau_k}$, it can be figured out by Taylor expansion that

$$(1-\alpha)(t_m - t_{k-1})^{-\alpha} \le d_{m,k} \le (1-\alpha)(t_m - t_k)^{-\alpha} \le d_{m,k+1} \le (1-\alpha)(t_m - t_{k+1})^{-\alpha}.$$

Substituting the above equation into (5.2), we can get

$$\|v^m\|_\infty \le \tau_m^\alpha \left[\Gamma(2-\alpha) \|g^m\| + d_{m,1} \|v^0\|_\infty + \sum_{k=1}^{m-1} (d_{m,k+1} - d_{m,k}) \|v^k\|_\infty \right].$$

$\square$

Here, we define $\theta_{m,j}$, $1 \le m \le M$ and $1 \le j \le m-1$ as

$$\theta_{m,m} = 1, \qquad \theta_{m,j} = \sum_{k=j}^{m-1} \tau_k^\alpha (d_{m,k+1} - d_{m,k}) \theta_{k,j}. \tag{5.3}$$

Then, we can conclude the following conclusion

Theorem 5.1 (Stability analysis). *The solution of (5.1) satisfies*

$$\|v^m\|_\infty \le \|v^0\|_\infty + \Gamma\left(2-\alpha\right)\tau_m^\alpha \sum_{j=1}^{m} \theta_{m,j}\|g^j\|_\infty, \qquad \text{for} 1 \le m \le M.$$

Proof. We will now prove this lemma by induction.

At the first, for $m = 1$, we have the conclusion, from Lemma 5.1

$$\|v^1\|_\infty \le \|v^0\|_\infty + \Gamma\left(2-\alpha\right)\tau_1^\alpha \|g^1\|_\infty.$$

Assuming that this Lemma is valid for $n = 1, \cdots, m-1$, then we can figure out that

$$\begin{aligned}
\|v^m\|_\infty &\le \tau_m^\alpha \left[\Gamma\left(2-\alpha\right)\|g^m\| + d_{m,1}\|v^0\|_\infty + \sum_{k=1}^{m-1}\left(d_{m,k+1} - d_{m,k}\right)\|v^k\|_\infty\right] \\
&\le \tau_m^\alpha \left\{\Gamma\left(2-\alpha\right)\|g^m\| + d_{m,1}\|v^0\|_\infty + \sum_{k=1}^{m-1}\left(d_{m,k+1} - d_{m,k}\right)\left[\|v^0\|_\infty + \Gamma\left(2-\alpha\right)\tau_k^\alpha \sum_{j=1}^{k}\theta_{k,j}\|g^j\|_\infty\right]\right\} \\
&= \tau_m^\alpha \left\{\Gamma\left(2-\alpha\right)\|g^m\| + d_{m,m}\|v^0\|_\infty + \Gamma\left(2-\alpha\right)\sum_{k=1}^{m-1}\left(d_{m,k+1} - d_{m,k}\right)\tau_k^\alpha \sum_{j=1}^{k}\theta_{k,j}\|g^j\|_\infty\right\} \\
&= \|v^0\|_\infty + \Gamma(2-\alpha)\tau_m^\alpha \left[\|g^m\|_\infty + \sum_{j=1}^{m-1}\sum_{k=j}^{m-1}\left(d_{m,k+1} - d_{m,k}\right)\tau_k^\alpha\theta_{k,j}\|g^j\|_\infty\right] \\
&= \|v^0\|_\infty + \Gamma(2-\alpha)\tau_m^\alpha \left[\|g^m\|_\infty + \sum_{j=1}^{m-1}\theta_{m,j}\|g^j\|_\infty\right].
\end{aligned}$$

□

6. Convergence analysis

We can now prove the global convergence of our collocation method, though the following Lemma.

Lemma 6.1. *For any nonnegative real number $\beta \le r_t\alpha$, there exists*

$$\tau_m^\alpha \sum_{j=1}^{m} j^{-\beta}\theta_{m,j} \le \frac{T^\alpha M^{-\beta}}{1-\alpha}, \quad 1 \le m \le M. \tag{6.1}$$

Proof. Here, the induction will be employed to prove this lemma.

At the first, for $m = 1$, we have

$$\tau_1^\alpha 1^{-\beta}\theta_{1,1} = \tau_1^\alpha \le \frac{T^\alpha M^{-\beta}}{1-\alpha},$$

since $\tau_1 = TM^{-r}$ and $\theta_{1,1} = 1$.

Then, assuming (6.1) is valid for $k = 1, \cdots, m-1 (1 \le m \le M)$, we can figure out that, by the definition of (5.3)

$$\begin{aligned}
\tau_m^\alpha \sum_{j=1}^{m} j^{-\beta}\theta_{m,j} &= \tau_m^\alpha m^{-\beta} + \tau_m^\alpha \sum_{j=1}^{m-1} j^{-\beta} \sum_{k=j}^{m-1} \tau_k^\alpha\left(d_{m,k+1} - d_{m,k}\right)\theta_{k,j} \\
&= \tau_m^\alpha m^{-\beta} + \tau_m^\alpha \sum_{k=1}^{m-1}\left(d_{m,k+1} - d_{m,k}\right)\tau_k^\alpha \sum_{j=1}^{k} j^{-\beta}\theta_{k,j} \\
&\le \tau_m^\alpha m^{-\beta} + \tau_m^\alpha \sum_{k=1}^{m-1}\left(d_{m,k+1} - d_{m,k}\right)\frac{T^\alpha M^{-\beta}}{1-\alpha} \\
&= \tau_m^\alpha m^{-\beta} + \tau_m^\alpha\left(d_{m,m} - d_{m,1}\right)\frac{T^\alpha M^{-\beta}}{1-\alpha}.
\end{aligned}$$

On the other hand, using the mean value theorem, we have

$$
\begin{aligned}
d_{m,1}\frac{T^{\alpha}M^{-\beta}}{1-\alpha} &= \frac{(t_m-t_0)^{1-\alpha}-(t_m-t_1)^{1-\alpha}}{\tau_1}\frac{T^{\alpha}M^{-\beta}}{1-\alpha}\\
&=(1-\alpha)\left(t_m-\zeta\right)^{-\alpha}\frac{T^{\alpha}M^{-\beta}}{1-\alpha}\\
&\geq t_m^{-\alpha}T^{\alpha}M^{-\beta}=m^{-\beta}\left(\frac{M}{m}\right)^{r_t\alpha-\beta}\\
&\geq m^{-\beta}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tau_m^{\alpha}\sum_{j=1}^{m}j^{-\beta}\theta_{m,j} &\leq \frac{T^{\alpha}M^{-\beta}}{1-\alpha}+\tau_m^{\alpha}\left[m^{-\beta}-d_{m,1}\frac{T^{\alpha}M^{-\beta}}{1-\alpha}\right]\\
&\leq \frac{T^{\alpha}M^{-\beta}}{1-\alpha}.
\end{aligned}
$$

□

Theorem 6.1 (Convergence analysis)**.** *Let U_i^j be the approximate solution of $u(x_i,t_j)$ computed by the discretization scheme (2.8) . Then*

$$
\max_{1\leq i\leq 2N-1}|U_i^j-u_i^j|\leq\begin{cases} C\left(N^{-2}\ln N+M^{-\min\{r_1\alpha,2-\alpha\}}\right), & \text{if}\quad r_2\left(\frac{\sigma+2}{2}-\gamma\right)=2\\ C\left(N^{-\min\left\{r_2\left(\frac{\sigma+2}{2}-\gamma\right),\,2\right\}}+M^{-\min\{r_1\alpha,2-\alpha\}}\right), & \text{otherwise.}\end{cases}
$$

Proof. Denoting $e_i^j=\bar{U}_i^j-u_i^j$, with $e_i^0=0$, $i=1,2,\cdots,2N-1$, $j=0,1,\cdots,M$ and $E_j=\left(e_1^j,e_2^j,\cdots,e_{2N-1}^j\right)^T$, from (2.6) and (2.7), we can obtain

$$
\mathcal{N}_M^N\left[e_i^m\right]=R_i^m-R_m^t \tag{6.2}
$$

and

$$
\left|\mathcal{N}_M^N\left[U_i^m-u_i^m\right]\right|=\left|R_m^t-R_i^m\right|\leq C\left(m^{-\min\{2-\alpha,r_t\alpha\}}+|R_i^m|\right) \tag{6.3}
$$

Therefore,

$$
\begin{aligned}
\|E^m\|_{\infty} &\leq \|E^0\|_{\infty}+\Gamma(2-\alpha)\tau_m^{\alpha}\sum_{j=1}^{m}\theta_{m,j}\|R^j+R_j^t\|_{\infty}\\
&\leq \Gamma(2-\alpha)\tau_m^{\alpha}\sum_{j=1}^{m}\theta_{m,j}\left(\|R_h^j\|_{\infty}+\|R_j^t\|_{\infty}\right)\\
&\leq \Gamma(2-\alpha)\tau_m^{\alpha}\sum_{j=1}^{m}\theta_{m,j}\left(\|R_h^j\|_{\infty}+j^{-\min\{2-\alpha,r_t\alpha\}}\right)\\
&\leq \begin{cases} C\left(N^{-2}\ln N+M^{-\min\{r_1\alpha,2-\alpha\}}\right), & \text{if}\quad r_2\left(\frac{\sigma+2}{2}-\gamma\right)=2\\ C\left(N^{-\min\left\{r_2\left(\frac{\sigma+2}{2}-\gamma\right),\,2\right\}}+M^{-\min\{r_1\alpha,2-\alpha\}}\right), & \text{otherwise.}\end{cases}
\end{aligned}
$$

by the Lemma 6.1 for $\beta=0$ and $\beta=\min\{2-\alpha,r\alpha\}$. □

7. Numerical Example

In this section, we take $\Omega=(0,1)$, and use piecewise linear collocation method and L_1 scheme to solve (1.2) on graded meshes which have grading exponents r_1 and r_2 for temporal direction and spatial direction, respectively. In here, we measure the error of numerical method by the maximum norm:

$$
\mathrm{Error}_N^M=\max_{\substack{0\leq i\leq 2N\\ 0\leq j\leq M}}\left|u\left(x_i,t_j\right)-U_i^j\right|.
$$

And the corresponding rate of convergence of Error_N^M is computed by

$$\mathrm{Rate}_N^M := \log_2\left(\frac{\mathrm{Error}_N^M}{\mathrm{Error}_{2N}^{2M}}\right)$$

7.1. Smoother

Example 7.1. *We take $u\left(x,t\right)=t^{\alpha}x^{2}\left(1-x\right)^{2}$ as the analytical solution of (1.2). And the corresponding source function is*

$$f\left(x,t\right)=\Gamma(1+\alpha)x^2(1-x)^2+\frac{1}{1-\gamma}(x^{1-\gamma}+(1-x)^{1-\gamma})u\left(x,t\right)$$
$$+t^{\alpha}\left[\frac{2x^{3-\gamma}+2\left(1-x\right)^{3-\gamma}}{\left(1-\gamma\right)\left(2-\gamma\right)\left(3-\gamma\right)}-\frac{12x^{4-\gamma}+12\left(1-x\right)^{4-\gamma}}{\left(1-\gamma\right)\left(2-\gamma\right)\left(3-\gamma\right)\left(4-\gamma\right)}+\frac{24x^{5-\gamma}+24\left(1-x\right)^{5-\gamma}}{\left(1-\gamma\right)\left(2-\gamma\right)\left(3-\gamma\right)\left(4-\gamma\right)\left(5-\gamma\right)}\right].$$

It is clear that corresponding σ is 4 for this $u(x,t)$. From theorem 6.1, the optimal convergence order $\mathcal{O}\left(N^{-2}+M^{-\min\{r_1\alpha,2-\alpha\}}\right)$ can be obtained at graded meshes (including uniform meshes). The convergence order of $\mathcal{O}\left(N^{-2}+M^{-\min\{r_1\alpha,2-\alpha\}}\right)$ is shown in Figures 1 - 4. It exactly agree with Theorem 6.1.

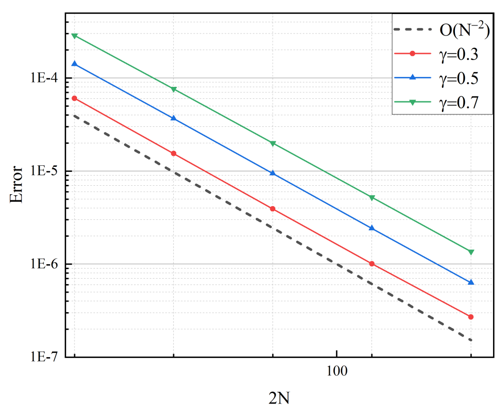

Fig 1: Numerical result with $\alpha=0.5$, $r_1=\frac{2-\alpha}{\alpha}$, $r_2=1$, $M=10000$

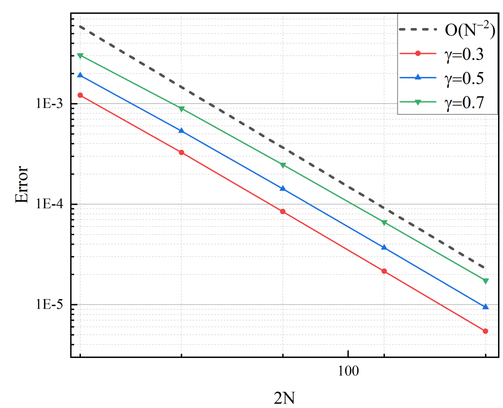

Fig 2: Numerical result with $\alpha=0.5$,$r_1=\frac{2-\alpha}{\alpha}$, $r_2=3$, $M=10000$

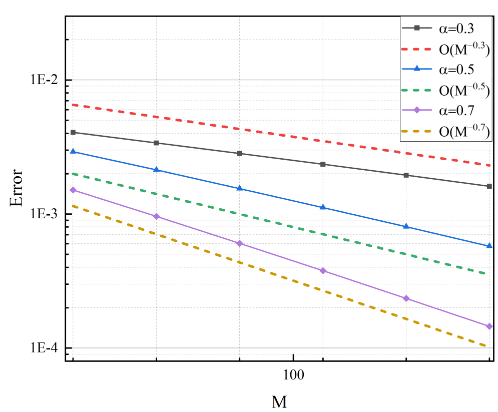

Fig 3: Numerical result with $\gamma=0.5$, $r_1=1$, $r_2=1$, $2N=1000$

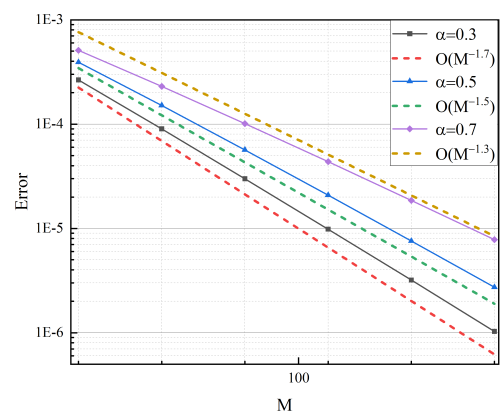

Fig 4: Numerical result with $\gamma=0.5$, $r_1=\frac{2-\alpha}{\alpha}$, $r_2=1$, $2N=1000$

7.2. Low Regularity

Example 7.2. *For convenience letting $\sigma=\gamma+1$, then taking*

$$u(x,t)=\frac{2^{-\sigma}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{\sigma}{2}\right)\Gamma\left(\frac{1+\sigma}{2}\right)}t^{\alpha}x^{\frac{\sigma}{2}}\left(1-x\right)^{\frac{\sigma}{2}}.$$

as the analytical solution of (1.2).

Setting $C_U = \frac{2^{-\sigma}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{\sigma}{2}\right)\Gamma\left(\frac{1+\sigma}{2}\right)}$, $C_R = \frac{1}{2\cos\left(\frac{\sigma\pi}{2}\right)\Gamma(2-\sigma)}$, $a_2 = C_U B\left(2-\frac{\sigma}{2}, 1+\frac{\sigma}{2}\right)$ *and* $a_1 = C_U B\left(1+\frac{\sigma}{2}, 2-\frac{\sigma}{2}\right) - \frac{1}{2C_R} - a_2$, *then we can figure out the corresponding source is*

$$f(x,t) = \frac{2^{-\sigma}\Gamma\left(\frac{1}{2}\right)\Gamma(1+\alpha)}{\Gamma\left(1+\frac{\sigma}{2}\right)\Gamma\left(\frac{1+\sigma}{2}\right)} x^{\frac{\sigma}{2}}(1-x)^{\frac{\sigma}{2}} + \frac{1}{1-\gamma}(x^{1-\gamma} + (1-x)^{1-\gamma})u(x,t) - t^{\alpha}\left(\frac{1}{2C_R}x^2 - a_1 x - a_2\right).$$

It is clear that $|\frac{\partial^{\iota}}{\partial x^{\iota}}u(x,t)| \to +\infty$ as $x \to \partial\Omega$ for any $t \in (0,T]$ and $|\frac{\partial^{\iota}}{\partial t^{\iota}}u(x,t)| \to +\infty$ as $t \to 0$ for any $x \in (0,1)$ where $\iota \geq 1$. This property leads to a lower accurate for numerical method on uniform meshes. Therefore, it is necessary to use Graded meshes in calculation.

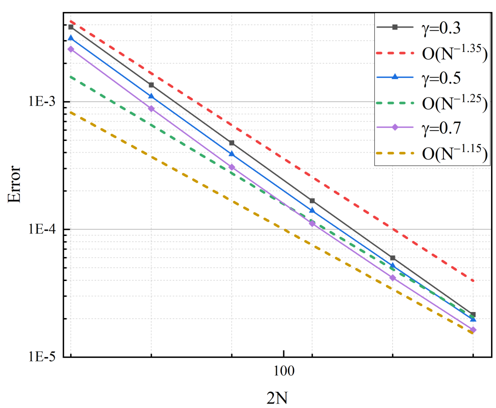

Fig 5: Numerical result with $\alpha = 0.5$, $r_1 = \frac{2-\alpha}{\alpha}$, $r_2 = 1$, $M = 10000$

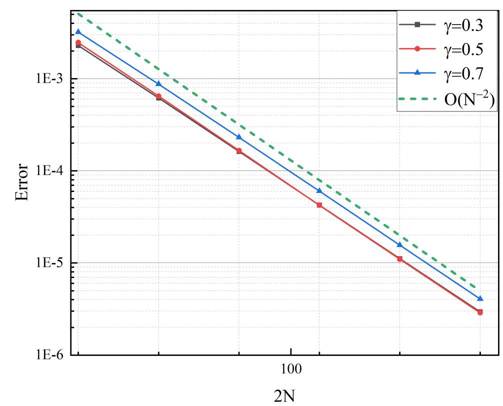

Fig 6: Numerical result with $\alpha = 0.5$,$r_1 = \frac{2-\alpha}{\alpha}$, $r_2 = \frac{4}{\sigma+2-2\gamma}$, $M = 10000$

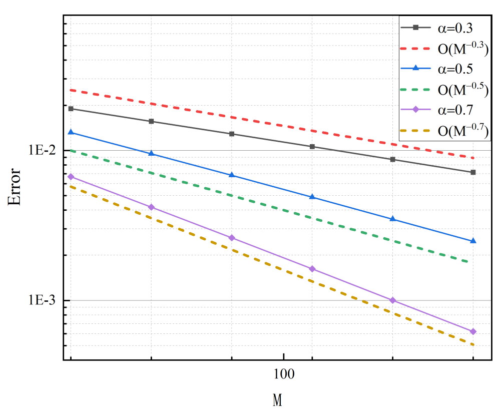

Fig 7: Numerical result with $\gamma = 0.5$, $r_1 = 1$, $r_2 = \frac{8}{5}$, $2N = 1000$

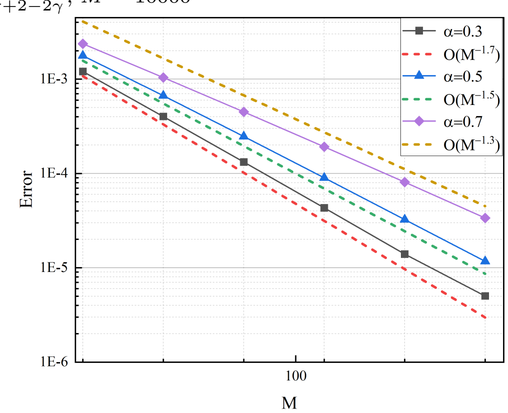

Fig 8: Numerical result with $\gamma = 0.5$, $r_1 = \frac{2-\alpha}{\alpha}$, $r_2 = \frac{8}{5}$, $2N = 1000$

Figures 5 - 8 agree with the Theorem 6.1.

Example 7.3. *When* $\gamma = \frac{1}{2}$, *we can choose the appropriate* $f(x,t)$ *such that*

$$u(x,t) = t^{\alpha}\left(x^{\frac{1}{2}} + (1-x)^{\frac{1}{2}} - 1\right)$$

is the analytical solution of (1.2), which has the similar properties to example 7.2. It should to note that the corresponding σ *is* 1.

8. Conclusion

In recent years, nonlocal diffusion problem has attracted a lot of scholars' attention. In many fields of study, such as image processing, biology, ecology, and anomalous diffusion, the model has been used to describe various scientific phenomena. The numerical calculation methods for this model have also made a lot of progress, such as finite difference method[21], finite element method[4, 21], collocation method [6, 11, 12], Newton-Cotes rules[20, 17, 19, 18] and multigrid method[4].

In this paper, we consider a collocation method for the nonlocal diffusion problem (1.2) with weakly singular solution. We constructed and analysed a piecewise linear collocation method and L_1 scheme for the nonlocal diffusion problem (1.2) on the graded meshes. In here, we rigorously proofed the stiff matrix is a M-matrix and this property depends on the piecewise linear polynomial and is independent of the grid. Using properties of matrices on graded meshes, we proved the stability of the proposed numerical method. Later, the local truncation error is estimated under graded meshes (4.1) and regularity conditions (1.1). Through the local truncation error and properties of stiff matrix, we theoretically deduce the convergence order of the above numerical method is $\mathcal{O}\left(N^{-\min\left\{r_2\left(\frac{\sigma+2}{2}-\gamma\right),\,2\right\}}+M^{-\min\{r_1\alpha,2-\alpha\}}\right)$. Finally, some numerical examples are used to confirm the convergence.

References

[1] Q. Du, M. Gunzburger, R. B. Lehoucq, K. Zhou, Analysis and approximation of nonlocal diffusion problems with volume constraints, SIAM Rev. 54 (2012) 667–696. URL: `https://doi.org/10.1137/110833294`. doi:10.1137/110833294.

[2] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J. J. Toledo-Melero, Nonlocal diffusion problems, volume 165 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010. URL: `https://doi.org/10.1090/surv/165`. doi:10.1090/surv/165.

[3] S. A. Silling, Reformulation of elasticity theory for discontinuities and long-range forces, J. Mech. Phys. Solids 48 (2000) 175–209. URL: `https://doi.org/10.1016/S0022-5096(99)00029-0`. doi:10.1016/S0022-5096(99)00029-0.

[4] M. Chen, W. Deng, Convergence analysis of a multigrid method for a nonlocal model, SIAM J. Matrix Anal. Appl. 38 (2017) 869–890. URL: `https://doi.org/10.1137/16M1076642`. doi:10.1137/16M1076642.

[5] Q. Du, M. Gunzburger, R. B. Lehoucq, K. Zhou, Analysis and approximation of nonlocal diffusion problems with volume constraints, SIAM Rev. 54 (2012) 667–696. URL: `https://doi.org/10.1137/110833294`. doi:10.1137/110833294.

[6] X. Zhang, M. Gunzburger, L. Ju, Nodal-type collocation methods for hypersingular integral equations and nonlocal diffusion problems, Comput. Methods Appl. Mech. Engrg. 299 (2016) 401–420. URL: `https://doi.org/10.1016/j.cma.2015.11.008`. doi:10.1016/j.cma.2015.11.008.

[7] Y. Yang, Z. Tang, Y. Huang, Numerical solutions for Fredholm integral equations of the second kind with weakly singular kernel using spectral collocation method, Appl. Math. Comput. 349 (2019) 314–324. URL: `https://doi.org/10.1016/j.amc.2018.12.035`. doi:10.1016/j.amc.2018.12.035.

[8] H. Brunner, Collocation methods for Volterra integral and related functional differential equations, volume 15 of *Cambridge Monographs on Applied and Computational Mathematics*, Cambridge University Press, Cambridge, 2004. URL: `https://doi.org/10.1017/CBO9780511543234`. doi:10.1017/CBO9780511543234.

[9] K. E. Atkinson, The numerical solution of integral equations of the second kind, Cambridge monographs on applied and computational mathematics 4, reissue ed., Cambridge University Press, 2009. URL: `http://gen.lib.rus.ec/book/index.php?md5=090763d9bd0216680cf125989c68d5f8`.

[10] K. E. Atkinson, The numerical solution of Fredholm integral equations of the second kind, SIAM J. Numer. Anal. 4 (1967) 337–348. URL: `https://doi.org/10.1137/0704029`. doi:10.1137/0704029.

[11] R. Cao, M. Chen, M. K. Ng, Y.-J. Wu, Fast and high-order accuracy numerical methods for time-dependent nonlocal problems in $\mathbb{R}^2$, J. Sci. Comput. 84 (2020) Paper No. 8, 31. URL: `https://doi.org/10.1007/s10915-020-01260-7`. doi:10.1007/s10915-020-01260-7.

[12] M. Chen, W. Qi, J. Shi, J. Wu, A sharp error estimate of piecewise polynomial collocation for nonlocal problems with weakly singular kernels, IMA J. Numer. Anal. 41 (2021) 3145–3174. URL: `https://doi.org/10.1093/imanum/draa054`. doi:10.1093/imanum/draa054.

[13] Z.-z. Sun, G.-h. Gao, Fractional differential equations—finite difference methods, De Gruyter, Berlin; Science Press Beijing, [Beijing], [2020] ©2020. URL: `https://doi.org/10.1515/9783110616064`. doi:10.1515/9783110616064.

[14] O. P. Agrawal, Fractional variational calculus in terms of Riesz fractional derivatives, J. Phys. A 40 (2007) 6287–6303. URL: `https://doi.org/10.1088/1751-8113/40/24/003`. doi:10.1088/1751-8113/40/24/003.

[15] K. S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1993.

[16] P. Linz, On the approximate computation of certain strongly singular integrals, Computing 35 (1985) 345–353. URL: https://doi.org/10.1007/BF02240199. doi:10.1007/BF02240199.

[17] Y. Gao, H. Feng, H. Tian, L. Ju, X. Zhang, Nodal-type Newton-Cotes rules for fractional hypersingular integrals, East Asian J. Appl. Math. 8 (2018) 697–714. URL: https://doi.org/10.4208/eajam.270418.190818. doi:10.4208/eajam.270418.190818.

[18] B. Li, W. Sun, Newton-Cotes rules for Hadamard finite-part integrals on an interval, IMA J. Numer. Anal. 30 (2010) 1235–1255. URL: https://doi.org/10.1093/imanum/drp011. doi:10.1093/imanum/drp011.

[19] J. Wu, Y. Lü, A superconvergence result for the second-order Newton-Cotes formula for certain finite-part integrals, IMA J. Numer. Anal. 25 (2005) 253–263. URL: https://doi.org/10.1093/imanum/drh025. doi:10.1093/imanum/drh025.

[20] J. Wu, W. Sun, The superconvergence of Newton-Cotes rules for the Hadamard finite-part integral on an interval, Numer. Math. 109 (2008) 143–165. URL: https://doi.org/10.1007/s00211-007-0125-7. doi:10.1007/s00211-007-0125-7.

[21] X. Tian, Q. Du, Analysis and comparison of different approximations to nonlocal diffusion and linear peridynamic equations, SIAM J. Numer. Anal. 51 (2013) 3458–3482. URL: https://doi.org/10.1137/13091631X. doi:10.1137/13091631X, author name corrected by publisher.

[22] I. G. Graham, Collocation methods for two-dimensional weakly singular integral equations, J. Austral. Math. Soc. Ser. B 22 (1980/81) 456–473. URL: https://doi.org/10.1017/S0334270000002800. doi:10.1017/S0334270000002800.

[23] X. Ros-Oton, J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, J. Math. Pures Appl. (9) 101 (2014) 275–302. URL: https://doi.org/10.1016/j.matpur.2013.06.003. doi:10.1016/j.matpur.2013.06.003.

[24] M. Stynes, E. O'Riordan, J. L. Gracia, Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation, SIAM J. Numer. Anal. 55 (2017) 1057–1079. URL: https://doi.org/10.1137/16M1082329. doi:10.1137/16M1082329.

[25] Y. Ma, Y. Xu, Computing highly oscillatory integrals, Math. Comp. 87 (2018) 309–345. URL: https://doi.org/10.1090/mcom/3214. doi:10.1090/mcom/3214.

[26] C. Schneider, Regularity of the solution to a class of weakly singular Fredholm integral equations of the second kind, Integral Equations Operator Theory 2 (1979) 62–68. URL: https://doi.org/10.1007/BF01729361. doi:10.1007/BF01729361.

[27] G. R. Richter, On weakly singular Fredholm integral equations with displacement kernels, J. Math. Anal. Appl. 55 (1976) 32–42. URL: https://doi.org/10.1016/0022-247X(76)90275-4. doi:10.1016/0022-247X(76)90275-4.

[28] J. Pitkäranta, On the differential properties of solutions to Fredholm equations with weakly singular kernels, J. Inst. Math. Appl. 24 (1979) 109–119.

[29] H. Chen, C. Sheng, L.-L. Wang, On explicit form of the FEM stiffness matrix for the integral fractional Laplacian on non-uniform meshes, Appl. Math. Lett. 113 (2021) Paper No. 106864, 8. URL: https://doi.org/10.1016/j.aml.2020.106864. doi:10.1016/j.aml.2020.106864.