A SECOND-ORDER METHOD FOR SPACE-TIME FRACTIONAL DIFFUSION EQUATION WITH LOW REGULAR SOLUTION*

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Key words. example, LaTeX

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1. Introduction.

We study

$$\frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u = f(x,t), \quad x \in \Omega, t \in (0,T]. \tag{1.1}$$

$$D_t^{\gamma} u + (-\Delta)^{\frac{\alpha}{2}} u = f(x,t), \quad x \in \Omega, t \in (0,T]. \tag{1.2}$$

where

$$D_t^{\gamma} u(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial u(x,s)}{\partial s} (t-s)^{-\gamma} ds \tag{1.3}$$

$$(-\Delta)^{\frac{\alpha}{2}} u(x,t) = \frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)} \int_0^{2L} u(y,t)|x-y|^{1-\alpha} dy \tag{1.4}$$

where $\gamma \in (0,1)$, $\alpha \in (1,2)$.

2. Regularity of the solution.

For the space-time fractional diffusion equation, it was first assumed that the solution regularity satisfies

$$\left|\frac{\partial^l u}{\partial t^l}(x,t)\right| \le C(1+t^{\gamma-l}) \quad for \quad l = 0,1,2, \tag{2.1a}$$

$$\left|\frac{\partial^l}{\partial x^l}(-\Delta)^{\alpha/2} u(x,t)\right| \le C\delta(x)^{-\alpha/2-l} \quad for \quad l = 0,1,2, \tag{2.1b}$$

$$\left|\frac{\partial^l u}{\partial x^l}(x,t)\right| \le C\delta(x)^{\alpha/2-l} \quad for \quad l = 0,1,2,3,4, \tag{2.1c}$$

for all $(x,t) \in (0,2L) \times (0,T]$.

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Remark 2.1. (2.1b) can be derived from (2.1a) by

$$\begin{aligned} I^{2-\alpha}u(x,t) = \int_0^{x/2} + \int_{L+x/2}^{2L} u(y,t)\frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\ + \int_0^{x/2} (u(x-z,t)+u(x+z,t))\frac{z^{1-\alpha}}{\Gamma(2-\alpha)}dy \\ + \int_{x+x/2}^{L+x/2} u(y,t)\frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)}dy \end{aligned}$$

3. Numerical scheme.

3.1. Discretisation of $(-\Delta)^{\frac{\alpha}{2}}$ on Graded Mesh.

3.2. Discretisation of D_t^γ on a General Mesh.

Consider the temporal mesh $0 = t0 < t1 < t2 < \ < t_M = T$. Set $\tau_j := t_j - t_{j-1}$ for $j = 1, ..., M$. [1]

On this mesh, we discretise $D_t^\gamma v$ for $v \in C[0,T] \cap C^3(0,T]$.

$$\begin{aligned} \delta_t^\gamma v(t_{k+\sigma}) &= \sum_{j=0}^{k} g_{k,j}\left(v(t_{j+1} - v(t_j))\right) \\ &= g_{k,k}v(t_{k+1}) - \sum_{j=1}^{k}\left(g_{k,j} - g_{k,j-1}\right)v(t_j) - g_{k,0}v(t_0) \end{aligned} \tag{3.1}$$

$$\delta_t^\gamma U^{k+1} = g_{k,k}U^{k+1} - \sum_{j=1}^{k}\left(g_{k,j} - g_{k,j-1}\right)U^j - g_{k,0}U^0 \tag{3.2}$$

Implicity scheme: Let $\tau = \frac{T}{M}$, $U^n, F^n \in \mathbb{R}^{2N-1}$,

$$\frac{U^{n+1} - U^n}{\tau} + AU^{n+1} = F^{n+1}. \tag{3.3}$$

Then $E^n = U^n - \hat{U}^n \in \mathbb{R}^{2N-1}$,

$$(I + \tau A)E^{n+1} = E^n + \tau R^{n+1}. \tag{3.4}$$

$$\delta_t^\gamma U^{n+1} + AU^{n+1} = F^{n+1} \tag{3.5}$$

$$\begin{aligned} E^n &= (I + \tau A)^{-1}E^{n-1} + (I + \tau A)^{-1}\tau R^n \\ &= (I + \tau A)^{-n}E^0 + \sum_{k=1}^{n}(I + \tau A)^{-k}\tau R^{n-k+1} \end{aligned} \tag{3.6}$$

4. Property of A and $g_{k,j}$.

LEMMA 4.1. *The stiffness matrix A has the following properties:*

1. *The eigenvalues of A are positive real numbers.*
2. *A is positive definite, which means that the eigenvalues of $\frac{A+A^T}{2}$ are positive.*

3. *The eigenvectors of A are orthogonal in space where $\langle u, v\rangle := uHv$, where $H := diag\left(\frac{h_i+h_{i+1}}{2}\right)$.*
4. *$(I+\tau A)^{-1} > O$ for any $\tau > 0$.*

Proof. Since

$$A = H^{-1}D = H^{-1/2}H^{-1/2}DH^{-1/2}H^{1/2}, \tag{4.1}$$

where $H^{-1/2}DH^{-1/2}$ is symmetric positive definite, $H^{-1/2}DH^{-1/2} = U\Lambda U^T$. Thus,

$$A = H^{-1/2}U\Lambda U^T H^{1/2} = (H^{-1/2}U)\Lambda(H^{-1/2}U)^{-1}. \tag{4.2}$$

The eigenvectors of A form an orthogonal basis of the Hilbert space defined by $\langle u, v\rangle := uHv$. Let $v_i = H^{-1/2}u_i$ be an eigenvector of A with eigenvalue λ_i. □

We need to prove $\lambda_1 > c$ for some positive constant c.

5. truncation error.

6. Convergence.

Assume that $|R^n| \le Ch^{\min\{r\alpha/2,2\}}(x_i^{-\alpha}+(2T-x_i)^{-\alpha})+C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha}+C\tau^?$

$$\begin{aligned}(I+\tau A)^{-k}\tau R^{n-k+1} &= (\tau A)(I+\tau A)^{-k}(\tau A)^{-1}\tau R^{n-k+1}\\ &= (\tau A)(I+\tau A)^{-k}(A^{-1}R^{n-k+1})\end{aligned} \tag{6.1}$$

Suppose that

$$\begin{aligned}|R^n| \le |R|&\\ &:= Ch^{\min\{r\alpha/2,2\}}(x_i^{-\alpha}+(2T-x_i)^{-\alpha})\\ &\quad + C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha}+C\tau^?\end{aligned} \tag{6.2}$$

Since $0 < A^{-1}R \le Ch^{\min}$,

$$|(I+\tau A)^{-k}\tau R^{n-k+1}| \le (I+\tau A)^{-k}\tau R = \tau A(1+\tau A)^{-k}A^{-1}R \tag{6.3}$$

Then

$$\begin{aligned}|E^n| &\le |(I+\tau A)^{-n}E^0| + \sum_{k=1}^{n}\tau A(1+\tau A)^{-k}A^{-1}R\\ &= |(I+\tau A)^{-n}E^0| + \left(I-(I+\tau A)^{-n}\right)A^{-1}R.\end{aligned} \tag{6.4}$$

Since A is diagonally dominant, $\|(I+\tau A)^{-1}E\|_\infty \le \|E\|_\infty$, we have

$$\|E^n\|_\infty \le \|E_0\|_\infty + 2\|A^{-1}R\|_\infty. \tag{6.5}$$

Lemma 6.1. *$A^{-1}R$ is bounded by $C\left(h^{\min\{r\alpha/2,2\}}+\tau^?\right)$, where C is a constant independent of h, α.*

7. Caputo-Riesz. $\gamma \in (0,1)$

(7.1)
$$\begin{aligned} D_N^\gamma u(x,t_n) &= \sum_{k=1}^{n} \frac{1}{\Gamma(2-\gamma)} \left(u(x,t_k) - u(x,t_{k-1})\right) \frac{(t_n - t_{k-1})^{1-\gamma} - (t_n - t_k)^{1-\gamma}}{\tau_k} \\ &= d_{n,n} u(x,t_n) - \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k}) u(x,t_k) - d_{n,1} u(x,t_0), \end{aligned}$$

where

$$d_{n,k} = \frac{(t_n - t_{k-1})^{1-\gamma} - (t_n - t_k)^{1-\gamma}}{\Gamma(2-\gamma)\tau_k} \quad \text{for} \quad 1 \le k \le n \quad \text{and} \quad d_{n,0} = 0, \tag{7.2}$$

$d_{n,n} = \frac{\tau_n^{-\gamma}}{\Gamma(2-\gamma)}$, $\quad d_{n,k+1} \ge d_{n,k}$.

Numerical scheme:

$$D_N^\gamma U^n + AU^n = F^n \tag{7.3}$$

We have

$$\left(d_{n,n} I + A\right) E^n = \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k}) E^k + d_{n,1} E^0 + R^n \tag{7.4}$$

Define the matrices $\Theta_{n,j}$, for $n = 1, 2..., N$ and $j = 0, 1, 2, ..., n-1$ by

$$\Theta_{n,n} = (d_{n,n} + A)^{-1}, \quad \Theta_{0,0} = I, \quad \Theta_{n,j} = \sum_{k=j}^{n-1} (d_{n,k+1} - d_{n,k}) \Theta_{n,n} \Theta_{k,j}. \tag{7.5}$$

Observe that $\Theta_{n,j} > O$ for all n, j.

Lemma 7.1.

$$E^n = \sum_{j=1}^{n} \Theta_{n,j} R^j + \Theta_{n,0} E^0 \tag{7.6}$$

Lemma 7.2. *Let the parameter β satisfy $\beta \le r_t \gamma$. Then for $n = 1, 2, ..., N$, one has*

$$\sum_{j=1}^{n} \Theta_{n,j} < A^{-1} \tag{7.7}$$

Proof. Use induction on n. When $n = 1$, then $\sum_{j=1}^{1} \Theta_{1,j} = \Theta_{1,1} < A^{-1}$. Next, assume that (7.7) holds for $k = 1, 2, ..., m-1 (2 \le m \le N)$. We want to prove (7.7) for $n = m$. Invoking (7.5) and interchanging the order of summation,

$$\begin{aligned}
\sum_{j=1}^{m} \Theta_{m,j} &= \Theta_{m,m} + \sum_{j=1}^{m-1} \sum_{k=j}^{m-1} (d_{m,k+1} - d_{m,k}) \Theta_{m,m} \Theta_{k,j} \\
&= \Theta_{m,m} + \sum_{k=1}^{m-1} (d_{m,k+1} - d_{m,k}) \Theta_{m,m} \sum_{j=1}^{k} \Theta_{k,j} \\
&\leq \Theta_{m,m} + \sum_{k=1}^{m-1} (d_{m,k+1} - d_{m,k}) \Theta_{m,m} A^{-1} \\
&= \Theta_{m,m} + (d_{m,m} - d_{m,1}) \Theta_{m,m} A^{-1} \\
&= A^{-1} - d_{m,1} \Theta_{m,m} A^{-1} < A^{-1}
\end{aligned}$$

□

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