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A second-order numerical method for two-dimensional two-sided space fractional convection diffusion equation

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ABSTRACT

Space fractional convection diffusion equation describes physical phenomena where particles or energy (or other physical quantities) are transferred inside a physical system due to two processes: convection and superdiffusion. In this paper, we discuss the practical alternating directions implicit method to solve the two-dimensional two-sided space fractional convection diffusion equation on a finite domain. We theoretically prove and numerically verify that the presented finite difference scheme is unconditionally von Neumann stable and second order convergent in both space and time directions.

1. Introduction

Relaxing the restriction of the boundedness of the second moments, fractional derivatives naturally appear to characterize anomalous diffusion [1,15], usually the time fractional derivative $(0 < \gamma < 1)$ is used for describing the subdiffusion and space fractional derivative $(1 < \alpha < 2)$ for the superdiffusion. More often, diffusion corresponds to a power law, $\langle x^2(t)\rangle - \langle x(t)\rangle^2 \sim Dt^{2\gamma/\alpha}$, where D is the diffusion coefficient and t the elapsed time. In a classical diffusion process, $2\gamma/\alpha = 1$. If $2\gamma/\alpha < 1$, the phenomenon is called subdiffusion. If $2\gamma/\alpha > 1$, the particles undergo superdiffusion. For $\gamma = 1, \alpha = 2$, the position probability density of the particles satisfies the classical diffusion equation with first order time derivative and second order space derivative; for $\gamma = 1, 1 < \alpha < 2$, the position probability density satisfies the space fractional diffusion equation with first order time derivative and αth order space derivative, we focus on this kind of equation in this paper.

The space fractional advection diffusion equation describes the physical phenomena involving two physical processes: convection and superdiffusion, i.e., for the equation, besides the αth order space fractional derivative term there exists the classical first order space derivative. This paper focuses on the numerical algorithm of the following two-dimensional two-sided space fractional convection diffusion equation

$$\begin{aligned}\frac{\partial u(x,y,t)}{\partial t} &= d_+(x,y)\,{}_{x_L}D_x^{\alpha}u(x,y,t) + d_-(x,y)\,{}_xD_{x_R}^{\alpha}u(x,y,t) + e_+(x,y)\,{}_{y_L}D_y^{\beta}u(x,y,t) + e_-(x,y)\,{}_yD_{y_R}^{\beta}u(x,y,t)\\ &\quad + g(x,y)u_x(x,y,t) + h(x,y)u_y(x,y,t) + s(x,y,t),\end{aligned} \tag{1.1}$$

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where $(x,y) \in \Omega = (x_L,x_R) \times (y_L,y_R)$, $0 < t \leqslant T$, the fractional orders $1 < \alpha, \beta < 2$; the function $s(x,y,t)$ is a source term; and the diffusion coefficients $d_+(x,y) \geqslant 0, d_-(x,y) \geqslant 0, e_+(x,y) \geqslant 0$ and $e_-(x,y) \geqslant 0$. The initial and boundary conditions are, respectively, taken as

$$u(x,y,0) = u_0(x,y) \quad \text{for } (x,y) \in \Omega \tag{1.2}$$

and

$$u(x,y,t)|_{\partial\Omega} = B(x,y,t). \tag{1.3}$$

The left and right Riemann–Liouville fractional derivatives of order μ ($0 \leqslant n-1 \leqslant \mu < n$ and n is an integer) are, respectively, defined by [6,16,17]

$${}_{x_L}D_x^{\mu}u(x) = D^n[{}_{x_L}D_x^{-(n-\mu)}u(x)] \tag{1.4}$$

and

$${}_{x}D_{x_R}^{\mu}u(x) = E^n[{}_{x}D_{x_R}^{-(n-\mu)}u(x)], \tag{1.5}$$

where

$${}_{x_L}D_x^{-\mu}u(x) = \frac{1}{\Gamma(\mu)}\int_{x_L}^{x}(x-\xi)^{\mu-1}u(\xi)d\xi, \quad \mu > 0,$$

$${}_{x}D_{x_R}^{-\mu}u(x) = \frac{1}{\Gamma(\mu)}\int_{x}^{x_R}(\xi-x)^{\mu-1}u(\xi)d\xi, \quad \mu > 0$$

and

$$D = \frac{d}{dx}, \qquad E \equiv -D = -\frac{d}{dx}.$$

The Grünwald–Letnikov definitions for the left and right fractional derivatives are, respectively, given as

$${}_{x_L}^{G}D_x^{\mu}u(x) = \lim_{M_+\to\infty}\frac{1}{h_+^{\mu}}\sum_{i=0}^{M_+}(-1)^i\binom{\mu}{i}u(x-ih_+) \tag{1.6}$$

and

$${}_{x}^{G}D_{x_R}^{\mu}u(x) = \lim_{M_-\to\infty}\frac{1}{h_-^{\mu}}\sum_{i=0}^{M_-}(-1)^i\binom{\mu}{i}u(x+ih_-), \tag{1.7}$$

where $h_+ = (x-x_L)/M_+$, $h_- = (x_R-x)/M_-$, and M_+ and M_- are positive integers. The Riemann–Liouville and Grünwald–Letnikov derivatives are equivalent [9] under the assumptions that the function performed are sufficiently smooth.

Using the formulae (1.6) and (1.7) to discretize the space fractional derivatives is a nature idea for designing the numerical schemes. Unfortunately, this usually leads to unstable finite difference schemes for time dependent problems with μth order space derivative, $1 < \mu < 2$, but Meerschaert et al. successfully circumvent this difficulties by modifying the formulae (1.6) and (1.7) to obtain the so-called shifted Grüwald formulae [12]. Based on these shifted formulae, Meerschaert et al. did a series of works for numerically solving space fractional diffusion equations [13,14,21,22], in particular, the second order accuracy is obtained by Richardson extrapolation in [21]. Also using the shifted Grüwald formulae, Liu et al. provide the explicit and implicit difference methods for the space–time fractional advection–diffusion equation [11]. Applying the L_1 and L_2 approximations, Yang et al. discuss the finite difference methods for fractional partial differential equations with Riesz space fractional derivatives [23]. Sousa proposes another way to approximate the fractional Caputo derivatives [18,19], which can obtain second order accuracy. More recently Sousa and Li further discuss using the similar idea to discretize Riemann–Liouville fractional derivatives in infinite domain [20]. For the discretization of the time fractional derivatives, usually the different challenges will be met [3,4,8,10]. Here we will combine the alternating directions implicit (ADI) method with Crank–Nicolson scheme to design the finite difference scheme for the two-sided two-dimensional space fractional advection diffusion equation (1.1). The numerical scheme will be theoretically proven and numerically verified to be unconditionally von Neumann stable and second order convergent.

The paper is organized as follows. In Section 2, we derive the linear spline approximation to the right Riemann–Liouville fractional derivative, and the full discretization of (1.1) is presented, where the Crank–Nicolson scheme and the alternating directions implicit method are combined together. Section 3 does the detailed theoretical analyses for the consistency and stability of the given schemes. To show the effectiveness of the algorithm, we perform the numerical experiments to verify the theoretical results in Section 4. Finally, we conclude the paper with some remarks in the last section.

2. Discretization schemes

We use three subsections to derive the full discretization of (1.1). Since the linear spline approximation for the left Riemann–Liouville fractional derivative in the infinite interval can be easily got from [20], we further derive the linear spline approximation for the right Riemann–Liouville fractional derivative and make some remarks on the relationship between the discretization schemes of left and right Riemann–Liouville fractional derivatives in the first subsection. Then in the second subsection, we present the scheme for the one dimensional case of (1.1). The third subsection detailedly provides the full discrete scheme of the two-dimensional two-sided space fractional convection diffusion equation.

2.1. Discretizations for the left and right Riemann–Liouville fractional derivatives

Let the mesh points $x_i = x_L + i\Delta x, i = 0, 1, \ldots, N_x$, where $\Delta x = (x_R - x_L)/N_x$ is the uniform space step. Taking $\alpha \in (1,2)$ in the left Riemann–Liouville fractional derivative (1.4), its approximation operator $\delta'_{\alpha,+x}u_i^n$ has second order accuracy in a bounded domain (proved in next section), where u_i^n denotes the approximated value of $u(x_i, t_n)$ and the left fractional approximation operator, which can be obtained by truncating its infinite version [20], is defined as

$$\delta'_{\alpha,+x}u_i^n := \frac{1}{\Gamma(4-\alpha)\Delta x^\alpha}\sum_{k=0}^{i+1} u_k^n p_{i,k}^\alpha, \tag{2.1}$$

where

$$p_{i,k}^\alpha = \begin{cases} a_{i-1,k} - 2a_{i,k} + a_{i+1,k}, & k \leqslant i-1, \\ -2a_{i,i} + a_{i+1,i}, & k = i, \\ a_{i+1,i+1}, & k = i+1, \\ 0, & k > i+1 \end{cases} \tag{2.2}$$

and

$$a_{i,k} = \begin{cases} (i-1)^{3-\alpha} - i^{2-\alpha}(i-3+\alpha), & k = 0, \\ (i-k+1)^{3-\alpha} - 2(i-k)^{3-\alpha} + (i-k-1)^{3-\alpha}, & 1 \leqslant k \leqslant i-1, \\ 1, & k = i. \end{cases}$$

Here in the finite interval $x_L < x < x_R$ with $\alpha \in (1,2)$, we further derive the linear spline approximation for the right Riemann–Liouville fractional derivative defined by

$${}_xD_{x_R}^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^{x_R} u(\xi,t)(\xi - x)^{1-\alpha}d\xi. \tag{2.3}$$

For a fixed time t, denote

$$\mathcal{I}_\alpha(x) = \frac{1}{\Gamma(2-\alpha)}\int_x^{x_R} u(\xi,t)(\xi - x)^{1-\alpha}d\xi, \tag{2.4}$$

then

$${}_xD_{x_R}^\alpha u(x,t) = \frac{\partial^2}{\partial x^2}\mathcal{I}_\alpha(x) \tag{2.5}$$

and we can do the following approximation at x_i,

$$\frac{\partial^2}{\partial x^2}\mathcal{I}_\alpha(x_i) \simeq \frac{1}{\Delta x^2}[\mathcal{I}_\alpha(x_{i-1}) - 2\mathcal{I}_\alpha(x_i) + \mathcal{I}_\alpha(x_{i+1})], \quad 1 \leqslant i \leqslant N_x - 1. \tag{2.6}$$

For each x_i, take

$$\mathcal{I}_\alpha(x_i) \simeq I_\alpha(x_i) = \frac{1}{\Gamma(2-\alpha)}\int_{x_i}^{x_R} S_i(\xi)(\xi - x_i)^{1-\alpha}d\xi, \tag{2.7}$$

where the spline $S_i(\xi)$ is defined by

$$S_i(\xi) = \sum_{k=i}^{N_x} u(x_k,t)s_{i,k}(\xi), \tag{2.8}$$

with $s_{i,k}(\xi)$, in every subinterval $[x_{k-1}, x_{k+1}]$, for $i+1 \leqslant k \leqslant N_x - 1$, given as

$$s_{i,k}(\xi) = \begin{cases} \frac{\xi - x_{k-1}}{x_k - x_{k-1}}, & x_{k-1} \leqslant \xi \leqslant x_k, \\ \frac{x_{k+1} - \xi}{x_{k+1} - x_k}, & x_k \leqslant \xi \leqslant x_{k+1}, \\ 0, & \text{otherwise} \end{cases}$$

and for $k = i$ and $k = N_x, s_{i,k}(\xi)$ taken as

$$s_{i,i}(\xi) = \begin{cases} \frac{x_{i+1} - \xi}{x_{i+1} - x_i}, & x_i \leqslant \xi \leqslant x_{i+1}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$s_{i,N_x}(\xi) = \begin{cases} \frac{\xi - x_{N_x-1}}{x_{N_x} - x_{N_x-1}}, & x_{N_x-1} \leqslant \xi \leqslant x_{N_x}, \\ 0, & \text{otherwise.} \end{cases}$$

According to (2.7) and (2.8), we have

$$\begin{aligned} \mathcal{I}_\alpha(x_i) \simeq I_\alpha(x_i) &= \frac{1}{\Gamma(2-\alpha)} u(x_i,t) \int_{x_i}^{x_{i+1}} s_{i,i}(\xi)(\xi - x_i)^{1-\alpha} d\xi + \frac{1}{\Gamma(2-\alpha)} \sum_{k=i+1}^{N_x-1} u(x_k,t) \int_{x_{k-1}}^{x_{k+1}} s_{i,k}(\xi)(\xi - x_i)^{1-\alpha} d\xi \\ &\quad + \frac{1}{\Gamma(2-\alpha)} u(x_{N_x},t) \int_{x_{N_x-1}}^{x_{N_x}} s_{i,N_x}(\xi)(\xi - x_i)^{1-\alpha} d\xi \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)} \int_{x_i}^{x_{i+1}} s_{i,i}(\xi)(\xi - x_i)^{1-\alpha} d\xi = \frac{\Delta x^{2-\alpha}}{\Gamma(4-\alpha)} b_{i,i}, \\ &\frac{1}{\Gamma(2-\alpha)} \int_{x_{k-1}}^{x_{k+1}} s_{i,k}(\xi)(\xi - x_i)^{1-\alpha} d\xi = \frac{\Delta x^{2-\alpha}}{\Gamma(4-\alpha)} b_{i,k}, \\ &\frac{1}{\Gamma(2-\alpha)} \int_{x_{N_x-1}}^{x_{N_x}} s_{i,N_x}(\xi)(\xi - x_i)^{1-\alpha} d\xi = \frac{\Delta x^{2-\alpha}}{\Gamma(4-\alpha)} b_{i,N_x}, \end{aligned} \tag{2.10}$$

where

$$b_{i,k} = \begin{cases} 1, & k = i, \\ (k-i+1)^{3-\alpha} - 2(k-i)^{3-\alpha} + (k-i-1)^{3-\alpha}, & i+1 \leqslant k \leqslant N_x - 1, \\ (3-\alpha-N_x+i)(N_x-i)^{2-\alpha} + (N_x-i-1)^{3-\alpha}, & k = N_x. \end{cases} \tag{2.11}$$

Then

$$\mathcal{I}_\alpha(x_i) \simeq I_\alpha(x_i) = \frac{\Delta x^{2-\alpha}}{\Gamma(4-\alpha)} \sum_{k=i}^{N_x} u(x_k,t) b_{i,k} \tag{2.12}$$

and (2.6) can be written as

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \mathcal{I}_\alpha(x_i) &\simeq \frac{1}{\Delta x^2} [\mathcal{I}_\alpha(x_{i-1}) - 2\mathcal{I}_\alpha(x_i) + \mathcal{I}_\alpha(x_{i+1})] \simeq \frac{1}{\Gamma(4-\alpha)\Delta x^\alpha} \left[\sum_{k=i-1}^{N_x} u(x_k,t) b_{i-1,k} - 2\sum_{k=i}^{N_x} u(x_k,t) b_{i,k} + \sum_{k=i+1}^{N_x} u(x_k,t) b_{i+1,k} \right] \\ &= \frac{1}{\Gamma(4-\alpha)\Delta x^\alpha} \sum_{k=i-1}^{N_x} u(x_k,t) q_{i,k}^\alpha, \end{aligned} \tag{2.13}$$

where

$$q_{i,k}^\alpha = \begin{cases} 0, & k < i-1, \\ b_{i-1,i-1}, & k = i-1, \\ -2b_{i,i} + b_{i-1,i}, & k = i, \\ b_{i-1,k} - 2b_{i,k} + b_{i+1,k}, & i+1 \leqslant k \leqslant N_x. \end{cases} \tag{2.14}$$

Denoting u_i^n as the approximated value of $u(x_i, t_n)$, we can define the right fractional approximation operator as

$$\delta'_{\alpha,-x} u_i^n := \frac{1}{\Gamma(4-\alpha)\Delta x^\alpha} \sum_{k=i-1}^{N_x} u_k^n q_{i,k}^\alpha, \tag{2.15}$$

which has second order accuracy for approximating (2.3) (proved in the next section).

Remark 2.1. Denoting $\widetilde{U}^n = [u_1^n, u_2^n, \ldots, u_{N_x-1}^n]^{\mathrm{T}}$, and rewriting (2.1) and (2.15) as matrix forms $\delta'_{\alpha,+x}\widetilde{U}^n = \widetilde{A}\widetilde{U}^n$ and $\delta'_{\alpha,-x}\widetilde{U}^n = \widetilde{B}\widetilde{U}^n$, respectively, then there exists $\widetilde{A} = \widetilde{B}^{\mathrm{T}}$, i.e., the two matrices are transposes.

2.2. Numerical scheme for one-dimensional fractional convection diffusion equation

We now examine the full discretization scheme to the one-dimensional two-sided fractional convection diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = d_+(x)_{x_L}D_x^{\alpha}u(x,t) + d_-(x)_xD_{x_R}^{\alpha}u(x,t) + g(x)u_x(x,t) + s(x,t). \tag{2.16}$$

In the time direction, we use the Crank–Nicolson scheme. The central difference formula, left fractional approximation operator (2.1), and right fractional approximation operator (2.15) are respectively used to discretize the classical first order space derivative, left Riemann–Liouville fractional derivative, and right Riemann–Liouville fractional derivative. Taking the uniform time step Δt and space step Δx, and setting $d_{+,i} = d_+(x_i), d_{-,i} = d_-(x_i), g_i = g(x_i)$, and $s_i^{n+1/2} = s(x_i, t_{n+1/2})$, where $t_{n+1/2} = (t_n + t_{n+1})/2$, the full discretization of (2.16) has the following form

$$\begin{aligned}\frac{u_i^{n+1} - u_i^n}{\Delta t} = {}& \frac{1}{\Gamma(4-\alpha)\Delta x^{\alpha}}\left[\sum_{k=0}^{i+1} p_{i,k}^{\alpha} d_{+,i}\frac{u_k^{n+1} + u_k^n}{2} + \sum_{k=i-1}^{N_x} q_{i,k}^{\alpha} d_{-,i}\frac{u_k^{n+1} + u_k^n}{2}\right] + \frac{g_i}{2\Delta x}\left(\frac{u_{i+1}^{n+1} + u_{i+1}^n}{2} - \frac{u_{i-1}^{n+1} + u_{i-1}^n}{2}\right)\\ &+ s_i^{n+1/2}.\end{aligned} \tag{2.17}$$

Similar to (2.1) and (2.15), we define

$$\begin{aligned}
D''_{\alpha,x}u_i^n &:= \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}g_i;\\
\delta''_{\alpha,+x}u_i^n &:= \frac{d_{+,i}}{\Gamma(4-\alpha)\Delta x^{\alpha}}\sum_{k=0}^{i+1} u_k^n p_{i,k}^{\alpha};\\
\delta''_{\alpha,-x}u_i^n &:= \frac{d_{-,i}}{\Gamma(4-\alpha)\Delta x^{\alpha}}\sum_{k=i-1}^{N_x} u_k^n q_{i,k}^{\alpha},
\end{aligned} \tag{2.18}$$

then (2.17) can be expressed as

$$\left[1 - \frac{\Delta t}{2}\left(\delta''_{\alpha,+x} + \delta''_{\alpha,-x} + D''_{\alpha,x}\right)\right]u_i^{n+1} = \left[1 + \frac{\Delta t}{2}\left(\delta''_{\alpha,+x} + \delta''_{\alpha,-x} + D''_{\alpha,x}\right)\right]u_i^n + s_i^{n+1/2}\Delta t, \tag{2.19}$$

for $i = 1, 2, \ldots, N_x - 1$, associated with the boundary conditions u_0^n and $u_{N_x}^n$. Putting $\xi_i = \frac{\Delta t}{2\Gamma(4-\alpha)\Delta x^{\alpha}}d_{+,i}$, $\eta_i = \frac{\Delta t}{2\Gamma(4-\alpha)\Delta x^{\alpha}}d_{-,i}$, and $\gamma_i = \frac{\Delta t}{4\Delta x}g_i$, the system of equations given by (2.17) takes the form

$$(I - A)U^{n+1} = (I + A)U^n + \Delta t S^{n+1/2}, \tag{2.20}$$

where I is the identity matrix, and

$$U^n = [u_0^n, u_1^n, u_2^n, \ldots, u_{N_x}^n]^{\mathrm{T}}, \quad S^{n+1/2} = [s_0^{n+1/2}, s_1^{n+1/2}, s_2^{n+1/2}, \ldots, s_{N_x}^{n+1/2}]^{\mathrm{T}}$$

and the matrix entries $A_{i,j}$ for $i = 1, \ldots, N_x - 1$ and $j = 1, \ldots, N_x - 1$ are defined by

$$A_{i,j} = \begin{cases} \xi_i p_{i,i}^{\alpha} + \eta_i q_{i,i}^{\alpha}, & j = i,\\ \xi_i p_{i,i-1}^{\alpha} + \eta_i q_{i,i-1}^{\alpha} + \gamma_i, & j = i-1,\\ \xi_i p_{i,i+1}^{\alpha} + \eta_i q_{i,i+1}^{\alpha} - \gamma_i, & j = i+1,\\ \xi_i p_{i,j}^{\alpha}, & j < i+1,\\ \eta_i q_{i,j}^{\alpha}, & j > i+1 \end{cases} \tag{2.21}$$

and $A_{0,0} = 1, A_{0,i} = 0$ for $i = 1, \ldots, N_x; A_{N_x,N_x} = 1$ and $A_{N_x,i} = 0$ for $i = 0, \ldots, N_x - 1$.

2.3. ADI scheme for two-dimensional two-sided fractional convection diffusion equation

Under the direction of discretizing the one-dimensional case of (1.1) in the last subsection, we use ADI [5,7] to numerically solve (1.1). First we introduce and list the denotations (some of them already given above) that will be used in the following:

$$\begin{aligned}
&D''_{\alpha,x}u^n_{i,j} := \frac{u^n_{i+1,j}-u^n_{i-1,j}}{2\Delta x}g_{i,j}; \quad D''_{\beta,y}u^n_{i,j} := \frac{u^n_{i,j+1}-u^n_{i,j-1}}{2\Delta y}h_{i,j};\\
&\delta''_{\alpha,+x}u^n_{i,j} := \frac{d_{+,i,j}}{\Gamma(4-\alpha)\Delta x^\alpha}\sum_{k=0}^{i+1}u^n_{k,j}p^\alpha_{i,k}; \quad \delta''_{\beta,+y}u^n_{i,j} := \frac{e_{+,i,j}}{\Gamma(4-\beta)\Delta y^\alpha}\sum_{k=0}^{j+1}u^n_{i,k}p^\beta_{j,k};\\
&\delta''_{\alpha,-x}u^n_{i,j} := \frac{d_{-,i,j}}{\Gamma(4-\alpha)\Delta x^\alpha}\sum_{k=i-1}^{N_x}u^n_{k,j}q^\alpha_{i,k}; \quad \delta''_{\beta,-y}u^n_{i,j} := \frac{e_{-,i,j}}{\Gamma(4-\beta)\Delta y^\alpha}\sum_{k=j-1}^{N_y}u^n_{i,k}q^\beta_{j,k}.
\end{aligned} \tag{2.22}$$

Analogously we still use the Crank–Nicolson scheme to do the discretization in time direction. Taking $u^n_{i,j}$ as the approximated value of $u(x_i,y_j,t_n)$, $d_{+,i,j}=d_+(x_i,y_j)$, $d_{-,i,j}=d_-(x_i,y_j)$, $e_{+,i,j}=e_+(x_i,y_j)$, $e_{-,i,j}=e_-(x_i,y_j)$, $g_{i,j}=g(x_i,y_j)$, $h_{i,j}=h(x_i,y_j)$, $t_{n+1/2}=(t_n+t_{n+1})/2$, $s^{n+1/2}_{i,j}=s(x_i,y_j,t_{n+1/2})$, $\Delta x=(x_R-x_L)/N_x$, and $\Delta y=(y_R-y_L)/N_y$, for the uniform space steps $\Delta x,\Delta y$ and time step Δt, the resulting discretization of (1.1) can be written as

$$\begin{aligned}
\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\Delta t} &= \frac{1}{\Gamma(4-\alpha)\Delta x^\alpha}\left[\sum_{k=0}^{i+1}p^\alpha_{i,k}d_{+,i,j}\frac{u^{n+1}_{k,j}+u^n_{k,j}}{2}+\sum_{k=i-1}^{N_x}q^\alpha_{i,k}d_{-,i,j}\frac{u^{n+1}_{k,j}+u^n_{k,j}}{2}\right]\\
&+\frac{1}{\Gamma(4-\beta)\Delta y^\beta}\left[\sum_{k=0}^{j+1}p^\beta_{j,k}e_{+,i,j}\frac{u^{n+1}_{i,k}+u^n_{i,k}}{2}+\sum_{k=j-1}^{N_y}q^\beta_{j,k}e_{-,i,j}\frac{u^{n+1}_{i,k}+u^n_{i,k}}{2}\right]\\
&+\frac{g_{i,j}}{2\Delta x}\left(\frac{u^{n+1}_{i+1,j}+u^n_{i+1,j}}{2}-\frac{u^{n+1}_{i-1,j}+u^n_{i-1,j}}{2}\right)+\frac{h_{i,j}}{2\Delta y}\left(\frac{u^{n+1}_{i,j+1}+u^n_{i,j+1}}{2}-\frac{u^{n+1}_{i,j-1}+u^n_{i,j-1}}{2}\right)+s^{n+1/2}_{i,j}.
\end{aligned} \tag{2.23}$$

Using the notations (2.22), we have

$$\begin{aligned}
&\left[1-\frac{\Delta t}{2}\left(\delta''_{\alpha,+x}+\delta''_{\alpha,-x}+D''_{\alpha,x}\right)-\frac{\Delta t}{2}\left(\delta''_{\beta,+y}+\delta''_{\beta,-y}+D''_{\beta,y}\right)\right]u^{n+1}_{i,j}\\
&\quad=\left[1+\frac{\Delta t}{2}\left(\delta''_{\alpha,+x}+\delta''_{\alpha,-x}+D''_{\alpha,x}\right)+\frac{\Delta t}{2}\left(\delta''_{\beta,+y}+\delta''_{\beta,-y}+D''_{\beta,y}\right)\right]u^n_{i,j}+s^{n+1/2}_{i,j}\Delta t.
\end{aligned} \tag{2.24}$$

Further define

$$\begin{aligned}
&\delta_{\alpha,x} := \delta''_{\alpha,+x}+\delta''_{\alpha,-x}+D''_{\alpha,x};\\
&\delta_{\beta,y} := \delta''_{\beta,+y}+\delta''_{\beta,-y}+D''_{\beta,y},
\end{aligned} \tag{2.25}$$

thus, (2.24) may be rewritten as

$$\left(1-\frac{\Delta t}{2}\delta_{\alpha,x}-\frac{\Delta t}{2}\delta_{\beta,y}\right)u^{n+1}_{i,j} = \left(1+\frac{\Delta t}{2}\delta_{\alpha,x}+\frac{\Delta t}{2}\delta_{\beta,y}\right)u^n_{i,j}+s^{n+1/2}_{i,j}\Delta t. \tag{2.26}$$

For the two-dimensional two-sided fractional convection diffusion equation (1.1), the relevant perturbation of (2.26) is of the form

$$\left(1-\frac{\Delta t}{2}\delta_{\alpha,x}\right)\left(1-\frac{\Delta t}{2}\delta_{\beta,y}\right)u^{n+1}_{i,j} = \left(1+\frac{\Delta t}{2}\delta_{\alpha,x}\right)\left(1+\frac{\Delta t}{2}\delta_{\beta,y}\right)u^n_{i,j}+s^{n+1/2}_{i,j}\Delta t. \tag{2.27}$$

The scheme (2.27) differs from (2.26) by a perturbation equals to [21]

$$\frac{(\Delta t)^2}{4}\delta_{\alpha,x}\delta_{\beta,y}(u^{n+1}_{i,j}-u^n_{i,j}),$$

which may be deduced by distributing the operator products in (2.27). Since $(u^{n+1}_{i,j}-u^n_{i,j})$ is an $\mathcal{O}(\Delta t)$ term, it follows that the perturbation contributes an $\mathcal{O}((\Delta t)^2)$ error component to the truncation error of (2.23). Thus, the scheme (2.27) has a truncation error also $\mathcal{O}((\Delta x)^2)+\mathcal{O}((\Delta y)^2)+\mathcal{O}((\Delta t)^2)$.

The system of equations defined by (2.27) may be solved by the following ADI (Peaceman–Rachford type) scheme [21]:

$$\left(1-\frac{\Delta t}{2}\delta_{\alpha,x}\right)u^*_{i,j} = \left(1+\frac{\Delta t}{2}\delta_{\beta,y}\right)u^n_{i,j}+\frac{\Delta t}{2}s^{n+1/2}_{i,j}; \tag{2.28}$$

$$\left(1-\frac{\Delta t}{2}\delta_{\beta,y}\right)u^{n+1}_{i,j} = \left(1+\frac{\Delta t}{2}\delta_{\alpha,x}\right)u^*_{i,j}+\frac{\Delta t}{2}s^{n+1/2}_{i,j}, \tag{2.29}$$

where $u^*_{i,j}$ is an intermediate solution. For maintaining the consistency, we need to carefully specify the boundary conditions of $u^*_{i,j}$. Subtracting (2.29) from (2.28), we obtain

$$2u^*_{i,j} = \left(1-\frac{\Delta t}{2}\delta_{\beta,y}\right)u^{n+1}_{i,j}+\left(1+\frac{\Delta t}{2}\delta_{\beta,y}\right)u^n_{i,j}, \tag{2.30}$$

then the boundary conditions for $u^*_{i,j}$ ($i = 0$ and $i = N_x$ with $j = 1, \dots, N_y - 1$) can be given as

$$\begin{aligned} u^*_{0,j} &= \frac{1}{2}\left[\left(1 - \frac{\Delta t}{2}\delta_{\beta,y}\right)u^{n+1}_{0,j} + \left(1 + \frac{\Delta t}{2}\delta_{\beta,y}\right)u^n_{0,j}\right] = \frac{1}{2}\left[\left(1 - \frac{\Delta t}{2}\delta_{\beta,y}\right)B^{n+1}_{0,j} + \left(1 + \frac{\Delta t}{2}\delta_{\beta,y}\right)B^n_{0,j}\right]; \\ u^*_{N_x,j} &= \frac{1}{2}\left[\left(1 - \frac{\Delta t}{2}\delta_{\beta,y}\right)u^{n+1}_{N_x,j} + \left(1 + \frac{\Delta t}{2}\delta_{\beta,y}\right)u^n_{N_x,j}\right] = \frac{1}{2}\left[\left(1 - \frac{\Delta t}{2}\delta_{\beta,y}\right)B^{n+1}_{N_x,j} + \left(1 + \frac{\Delta t}{2}\delta_{\beta,y}\right)B^n_{N_x,j}\right]. \end{aligned} \tag{2.31}$$

The corresponding procedure is executed as follows:

(1) First for every fixed $y = y_k$ ($k = 1, \dots, N_y - 1$), solving a set of $N_x - 1$ equations defined by (2.28) at the mesh points $x_i, i = 1, \dots, N_x - 1$, to get $u^*_{i,k}$.
(2) Next alternating the spatial direction, and for each fixed $x = x_k$ ($k = 1, \dots, N_x - 1$) solving a set of $N_y - 1$ equations defined by (2.29) at the points $y_j, j = 1, \dots, N_y - 1$, to obtain $u^{n+1}_{k,j}$.

3. Convergence and stability analysis

We show the convergence for one-dimensional and two-dimensional two-sided fractional convection diffusion equation by proving the consistency and stability (according to Lax's equivalence theorem).

3.1. Convergence and stability for one dimensional two-sided fractional convection diffusion equation

Lemma 3.1. *Let $u \in C^4(\bar{\Omega})$ and $\xi \in [x_k, x_{k+1}]$, then we have*

$$u(\xi) - S_i(\xi) = \theta(1-\theta)\frac{(\Delta x)^2}{2!}u''(\xi) + \frac{\theta(1-\theta)(1-2\theta)(\Delta x)^3}{3!}u'''(\xi) + \mathcal{O}\left((\Delta x)^4\right),$$

where $\Delta x = x_{k+1} - x_k, \theta = (\xi - x_k)/\Delta x$.

Proof. For any $\xi \in [x_k, x_{k+1}]$, using Taylor series expansion, there exist η_1 and η_2 in $[x_k, x_{k+1}]$ such that

$$u(x_k) = u(\xi) - (\xi - x_k)u'(\xi) + \frac{(\xi - x_k)^2}{2!}u''(\xi) - \frac{(\xi - x_k)^3}{3!}u'''(\xi) + \frac{(\xi - x_k)^4}{4!}u''''(\eta_1);$$

$$u(x_{k+1}) = u(\xi) + (x_{k+1} - \xi)u'(\xi) + \frac{(x_{k+1} - \xi)^2}{2!}u''(\xi) + \frac{(x_{k+1} - \xi)^3}{3!}u'''(\xi) + \frac{(x_{k+1} - \xi)^4}{4!}u''''(\eta_2),$$

thus

$$\begin{aligned} u(\xi) - S_i(\xi) &= u(\xi) - \frac{x_{k+1} - \xi}{\Delta x}u(x_k) - \frac{\xi - x_k}{\Delta x}u(x_{k+1}) \\ &= \frac{(x_{k+1} - \xi)(\xi - x_k)}{2!}u''(\xi) + \frac{(x_{k+1} - \xi)(\xi - x_k)(x_{k+1} + x_k - 2\xi)}{3!}u'''(\xi) + \frac{(x_{k+1} - \xi)(\xi - x_k)^4}{4!\Delta x}u''''(\eta_1) \\ &\quad + \frac{(\xi - x_k)(x_{k+1} - \xi)^4}{4!\Delta x}u''''(\eta_2). \end{aligned}$$

Since $\xi \in [x_k, x_{k+1}]$, we can write $\xi = x_k + \theta\Delta x, 0 \leqslant \theta \leqslant 1$, then

$$u(\xi) - S_i(\xi) = \frac{\theta(1-\theta)(\Delta x)^2}{2!}u''(\xi) + \frac{\theta(1-\theta)(1-2\theta)(\Delta x)^3}{3!}u'''(\xi) + \frac{\theta(1-\theta)(\Delta x)^4}{4!}(\theta^3 u''''(\eta_1) + (1-\theta)^3 u''''(\eta_2)).$$

□

Lemma 3.2 ([6,16]). *$D^n u(x)$ be continuous in the interval $[x_L, x_R]$ and $0 \leqslant n - 1 \leqslant \mu < n$, then for $x_L < x < x_R$ the following holds*

$${}_{x_L}D^{\mu}_x u(x) = D^n[{}_{x_L}D^{-(n-\mu)}_x u(x)] = {}_{x_L}D^{-(n-\mu)}_x [D^n u(x)] + \sum_{k=0}^{n-1}\frac{(x - x_L)^{k-\mu}}{\Gamma(-\mu + k + 1)}D^k u(x_L).$$

Lemma 3.3. *Let $D^n u(x)$ be continuous in the interval $[x_L, x_R]$ and $0 \leqslant n - 1 \leqslant \mu < n$, then for $x_L < x < x_R$ the following holds*

$${}_xD^{\mu}_{x_R} u(x) = E^n[{}_xD^{-(n-\mu)}_{x_R} u(x)] = {}_xD^{-(n-\mu)}_{x_R}[E^n u(x)] + \sum_{k=0}^{n-1}\frac{(x_R - x)^{k-\mu}}{\Gamma(-\mu + k + 1)}E^k u(x_R).$$

Proof. Since ${}_xD_{x_R}^{-\mu}u(x) = \frac{1}{\Gamma(\mu)}\int_x^{x_R}(\xi - x)^{\mu-1}u(\xi)d\xi, \mu > 0$, it can be obtained

$$D[{}_xD_{x_R}^{-\mu}u(x)] = \frac{1}{\Gamma(\mu)}\int_x^{x_R}\frac{d(\xi-x)^{\mu-1}}{dx}u(\xi)d\xi = -\frac{1}{\Gamma(\mu)}\int_x^{x_R}u(\xi)d(\xi-x)^{\mu-1} = {}_xD_{x_R}^{-\mu}[Du(x)] - \frac{(x_R-x)^{\mu-1}}{\Gamma(\mu)}u(x_R),$$

then

$$E[{}_xD_{x_R}^{-\mu}u(x)] = {}_xD_{x_R}^{-\mu}[Eu(x)] + \frac{(x_R-x)^{\mu-1}}{\Gamma(\mu)}u(x_R). \tag{3.1}$$

Using E to differentiate both sides of (3.1) leads to

$$E^2[{}_xD_{x_R}^{-\mu}u(x)] = E\{E[{}_xD_{x_R}^{-\mu}u(x)]\} = E\{{}_xD_{x_R}^{-\mu}[Eu(x)]\} + \frac{(x_R-x)^{\mu-2}}{\Gamma(\mu-1)}u(x_R). \tag{3.2}$$

Replacing u with Eu in (3.1), then (3.2) can be rewritten as

$$E^2[{}_xD_{x_R}^{-\mu}u(x)] = {}_xD_{x_R}^{-\mu}[E^2u(x)] + \frac{(x_R-x)^{\mu-1}}{\Gamma(\mu)}Eu(x_R) + \frac{(x_R-x)^{\mu-2}}{\Gamma(\mu-1)}u(x_R). \tag{3.3}$$

Repeated iterations establish the desired result. $\square$

Theorem 3.4. *Let $u \in C^4(\bar{\Omega})$ and ${}_xD_{x_R}^{\alpha-2}u \in C^4(\bar{\Omega})$. Then ${}_xD_{x_R}^{\alpha}u(x_i,t_n) = \delta'_{\alpha,-x}u_i^n + \mathcal{O}\left((\Delta x)^2\right)$, where $\alpha \in (1,2)$.*

Proof. In the following we use $u(x_i)$ to denote $u(x_i,t_n)$. Since t_n is fixed and ${}_xD_{x_R}^{\alpha-2}u \in C^4(\bar{\Omega})$, then the following holds

$$ {}_xD_{x_R}^{\alpha}u(x_i) = \frac{\partial^2}{\partial x^2}\mathcal{I}_\alpha(x_i) = \frac{1}{(\Delta x)^2}[\mathcal{I}_\alpha(x_{i-1}) - 2\mathcal{I}_\alpha(x_i) + \mathcal{I}_\alpha(x_{i+1})] + \mathcal{O}\left((\Delta x)^2\right).$$

Denoting $\epsilon(x_i)$ as the error satisfying

$$\mathcal{I}_\alpha(x_{i-1}) - 2\mathcal{I}_\alpha(x_i) + \mathcal{I}_\alpha(x_{i+1}) = I_\alpha(x_{i-1}) - 2I_\alpha(x_i) + I_\alpha(x_{i+1}) + \epsilon(x_i),$$

then there exists

$$ {}_xD_{x_R}^{\alpha}u(x_i) = \frac{1}{(\Delta x)^2}[I_\alpha(x_{i-1}) - 2I_\alpha(x_i) + I_\alpha(x_{i+1})] + \frac{1}{(\Delta x)^2}\epsilon(x_i) + \mathcal{O}\left((\Delta x)^2\right),$$

i.e.,

$$ {}_xD_{x_R}^{\alpha}u(x_i) = \delta'_{\alpha,-x}u_i^n + \frac{1}{(\Delta x)^2}\epsilon(x_i) + \mathcal{O}\left((\Delta x)^2\right).$$

Next we calculate the error $\epsilon(x_i)$. Extending the definition of $u(x)$ from $[x_L,x_R]$ to $[x_L,x_R+\Delta x]$ with the extended part of $u(x)$ being the one given in Theorem 5.1 of Appendix, then

$$\begin{aligned}
\epsilon(x_i) &= \int_{x_{i-1}}^{x_R}(u(\xi)-S_{i-1}(\xi))(\xi-x_{i-1})^{1-\alpha}d\xi - 2\int_{x_i}^{x_R}(u(\xi)-S_i(\xi))(\xi-x_i)^{1-\alpha}d\xi + \int_{x_{i+1}}^{x_R}(u(\xi)-S_{i+1}(\xi))(\xi-x_{i+1})^{1-\alpha}d\xi \\
&= \int_{x_i}^{x_R+\Delta x}(u(\xi-\Delta x)-S_{i-1}(\xi-\Delta x))(\xi-x_i)^{1-\alpha}d\xi - 2\int_{x_i}^{x_R}(u(\xi)-S_i(\xi))(\xi-x_i)^{1-\alpha}d\xi + \int_{x_i}^{x_R-\Delta x}(u(\xi+\Delta x)-S_{i+1}(\xi+\Delta x))(\xi-x_i)^{1-\alpha}d\xi \\
&= \sum_{k=i}^{N_x-1}\int_{x_k}^{x_{k+1}}(u(\xi-\Delta x)-S_{i-1}(\xi-\Delta x))(\xi-x_i)^{1-\alpha}d\xi - 2\sum_{k=i}^{N_x-1}\int_{x_k}^{x_{k+1}}(u(\xi)-S_i(\xi))(\xi-x_i)^{1-\alpha}d\xi \\
&\quad + \sum_{k=i}^{N_x-1}\int_{x_k}^{x_{k+1}}(u(\xi+\Delta x)-S_{i+1}(\xi+\Delta x))(\xi-x_i)^{1-\alpha}d\xi + C(x_i),
\end{aligned}$$

where

$$C(x_i) = \int_{x_R-\Delta x}^{x_R}(u(\xi)-S_{i-1}(\xi))(\xi-x_{i-1})^{1-\alpha}d\xi - \int_{x_R}^{x_R+\Delta x}(u(\xi)-S_{i+1}(\xi))(\xi-x_{i+1})^{1-\alpha}d\xi$$

and $C(x_i) = 0$. Using Taylor series expansion and denoting $\theta = (\xi - x_k)/\Delta x$, from Lemma 3.1 we get

$$\epsilon(x_i) = \sum_{k=i}^{N_x-1}\int_{x_k}^{x_{k+1}}(\xi-x_i)^{1-\alpha}\left[\frac{\theta(1-\theta)(\Delta x)^2}{2!}(u''(\xi-\Delta x)-2u''(\xi)+u''(\xi+\Delta x))\right.$$
$$\left.+\frac{\theta(1-\theta)(1-2\theta)(\Delta x)^3}{3!}(u'''(\xi-\Delta x)-2u'''(\xi)+u'''(\xi+\Delta x))+\mathcal{O}\left((\Delta x)^4\right)\right]d\xi.$$

Further using the first mean value theorem for integration and Taylor series expansion, there exist $\tilde{\eta}^k$, $\tilde{\eta}_1^k, \tilde{\eta}_2^k, \tilde{\eta}_3^k$, and $\tilde{\eta}_4^k$ such that

$$|\epsilon(x_i)| = \left|\sum_{k=i}^{N_x-1}\left\{\left[\frac{\theta(1-\theta)(\Delta x)^2}{2!}(u''(\tilde{\eta}^k-\Delta x)-2u''(\tilde{\eta}^k)+u''(\tilde{\eta}^k+\Delta x)\right)\right.\right.$$
$$\left.\left.+\frac{\theta(1-\theta)(1-2\theta)}{3!}\times(\Delta x)^3(u'''(\tilde{\eta}^k-\Delta x)-2u'''(\tilde{\eta}^k)+u'''(\tilde{\eta}^k+\Delta x))+\mathcal{O}\left((\Delta x)^4\right)\right]\int_{x_k}^{x_{k+1}}(\xi-x_i)^{1-\alpha}d\xi\right\}\right|$$
$$=\left|\sum_{k=i}^{N_x-1}\left\{\left[\frac{\theta(1-\theta)(\Delta x)^2}{2!}\times\frac{(\Delta x)^2}{2!}(u''''(\tilde{\eta}_1^k)+u''''(\tilde{\eta}_2^k))+\frac{\theta(1-\theta)(1-2\theta)(\Delta x)^4}{3!}\times(u''''(\tilde{\eta}_3^k)-u''''(\tilde{\eta}_4^k))+\mathcal{O}\left((\Delta x)^4\right)\right]\int_{x_k}^{x_{k+1}}(\xi-x_i)^{1-\alpha}d\xi\right\}\right|$$
$$\leqslant |u''''(\tilde{\eta}_0)|\left[\frac{\theta(1-\theta)(\Delta x)^4}{2}+\frac{\theta(1-\theta)(1-2\theta)(\Delta x)^4}{3}+\mathcal{O}\left((\Delta x)^4\right)\right]\sum_{k=i}^{N_x-1}\int_{x_k}^{x_{k+1}}(\xi-x_i)^{1-\alpha}d\xi$$
$$=|u''''(\tilde{\eta}_0)|\cdot\left[\frac{\theta(1-\theta)(\Delta x)^4}{2}+\frac{\theta(1-\theta)(1-2\theta)(\Delta x)^4}{3}+\mathcal{O}\left((\Delta x)^4\right)\right]\frac{1}{2-\alpha}(x_R-x_i)^{2-\alpha},$$

where $|u''''(\tilde{\eta}_0)| = \max_{i\leqslant k\leqslant N_x-1,\,1\leqslant j\leqslant 4}|u''''(\tilde{\eta}_j^k)|$. Therefore $\epsilon(x_i)/(\Delta x)^2 = \mathcal{O}\left((\Delta x)^2\right)$, the desired result is proved. □

Remark 3.1. From Lemma 3.3, it can be noted that $u\in C^4(\bar{\Omega})$ and ${}_xD_{x_R}^{\alpha-2}u\in C^4(\bar{\Omega})$ if and only if $u\in C^4(\bar{\Omega})$ and $u^{(k)}(x_R)=0, k=0,1,2,3$. At the same time, if we do the zero extension to u at $[x_R, x_R+\Delta x]$, then $u\in C^3[x_L, x_R+\Delta x]$ not belongs to $C^4[x_L, x_R+\Delta x]$.

By almost the same procedure as the proof of Theorem 3.4, we can prove the following result for left Riemann–Liouville fractional derivative.

Theorem 3.5. *Let $u\in C^4(\bar{\Omega})$ and ${}_{x_L}D_x^{\alpha-2}u\in C^4(\bar{\Omega})$, then ${}_{x_L}D_x^{\alpha}u(x_i,t_n) = \delta'_{\alpha,+,x}u_i^n+\mathcal{O}\left((\Delta x)^2\right)$, where $\alpha\in(1,2)$.*

Remark 3.2. From Lemma 3.2, it can be noted that $u\in C^4(\bar{\Omega})$ and ${}_{x_L}D_x^{\alpha-2}u\in C^4(\bar{\Omega})$ if and only if $u\in C^4(\bar{\Omega})$ and $u^{(k)}(x_L)=0, k=0,1,2,3$.

Example 1. To numerically verify the truncation error given in Theorem 3.4 in a bounded domain, consider the function $u(x)=\sin((1-x)^4), x\in\Omega=(0,1)$; and it is easy to check that $u^{(4)}(x)|_{\partial\Omega}\neq 0$. From Lemmas 3.2 and 3.3, the following holds

$$f(x) = {}_xD_{x_R}^{\alpha}u(x) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^{x_R}(\xi-x)^{1-\alpha}u(\xi)d\xi = \frac{1}{\Gamma(2-\alpha)}\int_x^{x_R}(\xi-x)^{1-\alpha}\frac{\partial^2 u(\xi)}{\partial\xi^2}d\xi, \tag{3.4}$$

then by the fractional predictor–corrector algorithm given in the Introduction and Preliminary sections of [3], we can numerically obtain the value of $f(x)$ at anywhere of the considered rectangle domain with any desired accuracy.

Example 2. To numerically verify the truncation error given in Theorem 3.5 in a bounded domain, consider the function $u(x)=\sin(x^4), x\in\Omega=(0,1)$; and it is easy to check that $u^{(4)}(x)|_{\partial\Omega}\neq 0$. Denoting $F(x)={}_{x_L}D_x^{\alpha}u(x)$, again by the algorithm given in [3], we can numerically obtain the value of $F(x)$ at anywhere of the considered rectangle domain with any desired accuracy. After performing the computation, we find that the numerical results are completely the same as Table 1, so we do not list them again.

Table 1
The l_∞ norm is used to measure the numerical errors and convergent orders for the scheme (2.15).

Δx	$\alpha=1.1$	Rate	$\alpha=1.5$	Rate	$\alpha=1.9$	Rate
1/50	1.0303e−002		2.4122e−002		3.8358e−002	
1/100	2.7832e−003	1.8882	6.4914e−003	1.8937	1.0322e−002	1.8938
1/200	7.2217e−004	1.9464	1.6846e−003	1.9461	2.7038e−003	1.9327
1/400	1.8379e−004	1.9743	4.2937e−004	1.9721	6.9889e−004	1.9518

The numerical results of Example 1 (Table 1) and Example 2 verify Theorem 3.4 and Theorem 3.5, respectively, and they show that the truncation errors are second order. In fact, watch carefully the numerical results, which also confirm Remark 2.1 in some sense.

Lemma 3.6. *The coefficients $q_{i,k}^{\alpha}$ with $1 < \alpha < 2$ in (2.14) satisfy $\sum_{k=i-1}^{N_x} q_{i,k}^{\alpha} < 0, 1 \leqslant i \leqslant N_x - 1$.*

Proof. Taking $u(x,t) \equiv 1$, from (2.7, 2.12, 2.13), we have

$$\sum_{k=i-1}^{N_x} q_{i,k}^{\alpha} = \sum_{k=i-1}^{N_x} u(x_k,t) q_{i,k}^{\alpha} = I_\alpha(x_{i-1}) - 2I_\alpha(x_i) + I_\alpha(x_{i+1}),$$

since $u(x,t) \equiv 1$ implies its linear interpolation function $S_i(x) = u(x,t) \equiv 1$.

Then

$$\begin{aligned}
I_\alpha(x_{i-1}) - 2I_\alpha(x_i) + I_\alpha(x_{i+1}) &= \frac{1}{\Gamma(2-\alpha)}\left[\int_{x_{i-1}}^{x_R} (\xi - x_{i-1})^{1-\alpha} d\xi - 2\int_{x_i}^{x_R} (\xi - x_i)^{1-\alpha} d\xi + \int_{x_{i+1}}^{x_R} (\xi - x_{i+1})^{1-\alpha} d\xi\right] \\
&= \frac{1}{\Gamma(3-\alpha)}\left((x_R - x_{i-1})^{2-\alpha} - 2(x_R - x_i)^{2-\alpha} + (x_R - x_{i+1})^{2-\alpha}\right) \\
&= \frac{1}{\Gamma(3-\alpha)}\left((a+\Delta x)^{2-\alpha} - 2a^{2-\alpha} + (a-\Delta x)^{2-\alpha}\right) \\
&= \frac{1}{\Gamma(3-\alpha)}\left[\left((a+\Delta x)^{2-\alpha} - a^{2-\alpha}\right) - \left(a^{2-\alpha} - (a-\Delta x)^{2-\alpha}\right)\right] \\
&= \frac{1}{(2-\alpha)\Gamma(3-\alpha)}\left(\int_a^{a+\Delta x} x^{1-\alpha} dx - \int_{a-\Delta x}^{a} x^{1-\alpha} dx\right),
\end{aligned}$$

where $a = x_R - x_i > 0$; obviously, $x^{1-\alpha}$ is a decreasing function, so $I_\alpha(x_{i-1}) - 2I_\alpha(x_i) + I_\alpha(x_{i+1}) < 0$. $\square$

Using similar method, we can prove.

Lemma 3.7. *The coefficients $p_{i,k}^{\alpha}$ with $1 < \alpha < 2$ in (2.2) satisfy $\sum_{k=0}^{i+1} p_{i,k}^{\alpha} < 0, 1 \leqslant i \leqslant N_x - 1$.*

Theorem 3.8. *The scheme (2.19) of the fractional convection diffusion equation (2.16) with constant coefficients and $1 < \alpha < 2$ is unconditionally von Neumann stable.*

Proof. Let $\widetilde{u_j^n}$ be the approximate solution of u_j^n, which is the exact solution of the scheme (2.19). Setting $\varepsilon_j^n = \widetilde{u_j^n} - u_j^n, 1 \leqslant j \leqslant N_x - 1$, and denoting

$$\delta_{\alpha,x}\varepsilon_j^n = \left(\delta''_{\alpha,+x} + \delta''_{\alpha,-x} + D''_{\alpha,x}\right)\varepsilon_j^n, \tag{3.5}$$

then from (2.19) we get the following perturbation equation

$$\left(1 - \frac{\Delta t}{2}\delta_{\alpha,x}\right)\varepsilon_j^{n+1} = \left(1 + \frac{\Delta t}{2}\delta_{\alpha,x}\right)\varepsilon_j^n, \tag{3.6}$$

with zero boundary conditions, i.e., $\varepsilon_0^n = \varepsilon_{N_x}^n = 0$. Then we can use the Von Neumann analysis or Fourier method [5] to do the stability analysis. Putting $\xi_j = \frac{\Delta t}{2\Gamma(4-\alpha)\Delta x^\alpha} d_{+,j}, \eta_j = \frac{\Delta t}{2\Gamma(4-\alpha)\Delta x^\alpha} d_{-,j}, \gamma_j = \frac{\Delta t}{4\Delta x} g_j$, and assuming

$$\varepsilon_j^n = \frac{1}{\sqrt{2\pi}}\widehat{\varepsilon}^n(\omega)e^{i\omega x_j},$$

then (3.6) leads to

$$\widehat{\varepsilon}^{n+1}(\omega)(1 - \lambda_x) = \widehat{\varepsilon}^n(\omega)(1 + \lambda_x),$$

where

$$\lambda_x = \xi_j \sum_{k=0}^{j+1} p_{j,k}^{\alpha} e^{i\omega(k-j)\Delta x} + \eta_j \sum_{k=j-1}^{N_x} q_{j,k}^{\alpha} e^{i\omega(k-j)\Delta x} + \gamma_j\left(e^{i\omega\Delta x} - e^{-i\omega\Delta x}\right)$$

and the amplification factor is

$$\widehat{Q}(\omega) = (1 - \lambda_x)^{-1}(1 + \lambda_x).$$

Next we prove that the real part of λ_x is negative, i.e.,

$$\sum_{k=0}^{j+1} p_{j,k}^{\alpha}\cos(\omega(k-j)\Delta x) < 0 \quad \text{and} \quad \sum_{k=j-1}^{N_x} q_{j,k}^{\alpha}\cos(\omega(k-j)\Delta x) < 0,$$

then it implies that $|\widehat{Q}(\omega)| = |\frac{1+\lambda_x}{1-\lambda_x}| < 1$. The terms can be rearranged as

$$\sum_{k=0}^{j+1} p_{j,k}^{\alpha}\cos(\omega(k-j)\Delta x) = (p_{j,j-1}^{\alpha} + p_{j,j+1}^{\alpha})\cos(\omega\Delta x) + p_{j,j}^{\alpha} + \sum_{k=0}^{j-2} p_{j,k}^{\alpha}\cos(\omega(k-j)\Delta x),$$

$$\sum_{k=j-1}^{N_x} q_{j,k}^{\alpha}\cos(\omega(k-j)\Delta x) = (q_{j,j-1}^{\alpha} + q_{j,j+1}^{\alpha})\cos(\omega\Delta x) + q_{j,j}^{\alpha} + \sum_{k=j+2}^{N_x} q_{j,k}^{\alpha}\cos(\omega(k-j)\Delta x).$$

From Theorem 5.2 of Appendix, we know $p_{j,j-1}^{\alpha} + p_{j,j+1}^{\alpha} \geqslant 0, p_{j,k}^{\alpha} \geqslant 0$ for $k \leqslant j-2$, and $q_{j,j-1}^{\alpha} + q_{j,j+1}^{\alpha} \geqslant 0, q_{j,k}^{\alpha} \geqslant 0$ for $k \geqslant j+2$, then

$$\sum_{k=0}^{j+1} p_{j,k}^{\alpha}\cos(\omega(k-j)\Delta x) \leqslant (p_{j,j-1}^{\alpha} + p_{j,j+1}^{\alpha}) + p_{j,j}^{\alpha} + \sum_{k=0}^{j-2} p_{j,k}^{\alpha} \leqslant \sum_{k=0}^{j+1} p_{j,k}^{\alpha} < 0,$$

$$\sum_{k=j-1}^{N_x} q_{j,k}^{\alpha}\cos(\omega(k-j)\Delta x) \leqslant (q_{j,j-1}^{\alpha} + q_{j,j+1}^{\alpha}) + q_{j,j}^{\alpha} + \sum_{k=j+2}^{N_x} q_{j,k}^{\alpha} \leqslant \sum_{k=j-1}^{N_x} q_{j,k}^{\alpha} < 0,$$

where Lemma 3.6 and 3.7 are used. □

3.2. Convergence and stability for two-dimensional two-sided fractional convection diffusion equation

Theorem 3.9. *The ADI-CN scheme, defined by (2.27) with constant coefficients, is unconditionally von Neumann stable for $1 < \alpha, \beta < 2$.*

Proof. Let $\widetilde{u_{j,m}^n}$ be the approximate solution of $u_{j,m}^n$, which is the exact solution of the scheme (2.27). Setting $\varepsilon_{j,m}^n = \widetilde{u_{j,m}^n} - u_{j,m}^n, 1 \leqslant j \leqslant N_x - 1$, and $1 \leqslant m \leqslant N_y - 1$, then from (2.27) we get the following perturbation equation

$$\left(1 - \frac{\Delta t}{2}\delta_{\alpha,x}\right)\left(1 - \frac{\Delta t}{2}\delta_{\beta,y}\right)\varepsilon_{j,m}^{n+1} = \left(1 + \frac{\Delta t}{2}\delta_{\alpha,x}\right)\left(1 + \frac{\Delta t}{2}\delta_{\beta,y}\right)\varepsilon_{j,m}^{n}. \tag{3.7}$$

Similarly, putting $\xi_j = \frac{\Delta t}{2\Gamma(4-\alpha)\Delta x^{\alpha}} d_{+,j}, \eta_j = \frac{\Delta t}{2\Gamma(4-\alpha)\Delta x^{\alpha}} d_{-,j}, \gamma_j = \frac{\Delta t}{4\Delta x} g_j$ and $\tilde{\xi}_m = \frac{\Delta t}{2\Gamma(4-\beta)\Delta y^{\beta}} e_{+,m}, \quad \tilde{\eta}_m = \frac{\Delta t}{2\Gamma(4-\beta)\Delta y^{\beta}} e_{-,m}, \tilde{\gamma}_m = \frac{\Delta t}{4\Delta y} h_m$, assuming

$$\varepsilon_{j,m}^n = \frac{1}{\sqrt{2\pi}}\tilde{\varepsilon}^n(\omega)e^{i\omega(x_j+y_m)},$$

then (3.7) leads to

$$\tilde{\varepsilon}^{n+1}(\omega)(1-\lambda_x)(1-\lambda_y) = \tilde{\varepsilon}^n(\omega)(1+\lambda_x)(1+\lambda_y),$$

where

$$\lambda_x = \xi_j\sum_{k=0}^{j+1} p_{j,k}^{\alpha}e^{i\omega(k-j)\Delta x} + \eta_j\sum_{k=j-1}^{N_x} q_{j,k}^{\alpha}e^{i\omega(k-j)\Delta x} + \gamma_j\left(e^{i\omega\Delta x} - e^{-i\omega\Delta x}\right);$$

$$\lambda_y = \tilde{\xi}_m\sum_{k=0}^{m+1} p_{m,k}^{\beta}e^{i\omega(k-m)\Delta y} + \tilde{\eta}_m\sum_{k=m-1}^{N_y} q_{m,k}^{\beta}e^{i\omega(k-m)\Delta y} + \tilde{\gamma}_m\left(e^{i\omega\Delta y} - e^{-i\omega\Delta y}\right)$$

and the amplification factor is

$$\widetilde{Q}(\omega) = (1-\lambda_x)^{-1}(1-\lambda_y)^{-1}(1+\lambda_x)(1+\lambda_y).$$

Similar to the proof of Theorem 3.8, we can again check that

$$\sum_{k=0}^{j+1} p_{j,k}^{\alpha} \cos(\omega(k-j)\Delta x) < 0 \quad \text{and} \quad \sum_{k=j-1}^{N_x} q_{j,k}^{\alpha} \cos(\omega(k-j)\Delta x) < 0;$$

$$\sum_{k=0}^{m+1} p_{m,k}^{\beta} \cos(\omega(k-m)\Delta y) < 0 \quad \text{and} \quad \sum_{k=m-1}^{N_y} q_{m,k}^{\beta} \cos(\omega(k-m)\Delta y) < 0$$

and it means that $|\widetilde{Q}(\omega)| = \left|\frac{(1+\lambda_x)(1+\lambda_y)}{(1-\lambda_x)(1-\lambda_y)}\right| < 1$. Then we conclude that the scheme (2.27) is unconditionally von Neumann stable. $\square$

Remark 3.3. Since this paper focuses on the unconditional von Neumann stability, that the coefficients of the equation are constant is required. The unconditional stability of the scheme with variable coefficients is also possible. Its proof can follow the ideas of the proof of Lemma 2.5, Theorem 2.9 and 4.5 in [2].

4. Numerical results

In this section, we numerically verify the above theoretical results including convergence rates and numerical stability. And the l_∞ norm is used to measure the numerical errors.

4.1. Numerical results for one-dimensional two-sided fractional convection diffusion equation

Let us consider the one-dimensional fractional convection diffusion equation (2.16), where $0 < x < 1$ and $0 < t \le 1$, with the coefficient functions

$$d_+(x) = \Gamma(3-\alpha)x^{\alpha}, \quad d_-(x) = \Gamma(3-\alpha)(2-x)^{\alpha}, \quad \text{and} \quad g(x) = \frac{1}{4}x.$$

Take the exact solution of the equation as $u(x,t) = e^{-t}\sin((2x)^4)\sin((2-2x)^4)$, then the corresponding initial and boundary conditions are, respectively, $u(x,0) = \sin((2x)^4)\sin((2-2x)^4)$ and $u(0,t) = u(1,t) = 0$; and the forcing function

$$\begin{aligned} s(x,t) = {} & -e^{-t}\sin((2x)^4)\sin((2-2x)^4) - e^{-t}d_+(x)\,{}_{x_L}D_x^{\alpha}\sin((2x)^4)\sin((2-2x)^4) - d_-(x)\,{}_xD_{x_R}^{\alpha}\sin((2x)^4)\sin((2-2x)^4) \\ & - 64e^{-t}g(x)(x^3\cos((2x)^4)\sin((2-2x)^4) - (1-x)^3\sin((2x)^4)\cos((2-2x)^4)), \end{aligned}$$

by the algorithm given in [3], we can numerically obtain the value of $s(x,t)$ at anywhere of the considered rectangle domain with any desired accuracy.

In Table 2, we show the scheme (2.19) is second order convergent in both space and time directions.

4.2. Numerical results for two-dimensional two-sided fractional convection diffusion equation

Consider the two-dimensional two-sided fractional convection diffusion equation (1.1), on a finite domain $0 < x < 1, 0 < y < 1, 0 < t \leqslant 1$, and with the coefficients

$$d_+(x,y) = \Gamma(3-\alpha)x^{\alpha}, \quad d_-(x,y) = \Gamma(3-\alpha)(2-x)^{\alpha}, \quad g(x,y) = \frac{1}{4}x,$$

$$e_+(x,y) = \Gamma(3-\beta)y^{\beta}, \quad e_-(x,y) = \Gamma(3-\beta)(2-y)^{\beta}, \quad h(x,y) = \frac{1}{4}y$$

and the initial condition $u(x,y,0) = \sin((2x)^4)\sin((2-2x)^4)\sin((2y)^2)\sin((2-2y)^2)$ and the Dirichlet boundary conditions on the rectangle in the form $u(0,y,t) = u(x,0,t) = 0$ and $u(1,y,t) = u(x,1,t) = 0$ for all $t \geqslant 0$. The exact solution to this two-dimensional two-sided fractional convection diffusion equation is

Table 2
The maximum errors and convergent orders for the scheme (2.19) of the one-dimensional two-side fractional convection diffusion equation (2.16) at $t = 1$ and $\Delta t = \Delta x$.

$\Delta t,\ \Delta x$	$\alpha = 1.1$	Rate	$\alpha = 1.5$	Rate	$\alpha = 1.9$	Rate
1/50	2.1180e−003		1.9815e−003		1.3809e−003	
1/100	5.2688e−004	2.0072	5.0092e−004	1.9839	3.5593e−004	1.9559
1/200	1.3174e−004	1.9997	1.2649e−004	1.9856	9.1681e−005	1.9569
1/400	3.2913e−005	2.0010	3.1851e−005	1.9896	2.3523e−005	1.9625

Table 3
The maximum errors and convergent orders for the scheme (2.27)–(2.31) of the two-dimensional two-sided fractional convection diffusion equation (1.1) at $t = 1$ and $\Delta t = \Delta x = \Delta y$.

$\Delta t, \Delta x, \Delta y$	$\alpha = 1.1, \beta = 1.1$	Rate	$\alpha = 1.6, \beta = 1.4$	Rate	$\alpha = 1.9, \beta = 1.9$	Rate
1/25	9.5946e−003		8.5313e−003		1.0232e−002	
1/50	2.3956e−003	2.0018	2.1729e−003	1.9731	2.6207e−003	1.9650
1/100	5.9582e−004	2.0075	5.5244e−004	1.9757	6.6155e−004	1.9860
1/200	1.4915e−004	1.9981	1.3959e−004	1.9847	1.6796e−004	1.9778

Table 4
The maximum errors and convergent orders for the scheme (2.27)–(2.31) of the two-dimensional one-sided fractional convection diffusion equation (4.1) at $t = 1$ and $\Delta t = \Delta x = \Delta y$.

$\Delta t, \Delta x, \Delta y$	$\alpha = 1.1, \beta = 1.1$	Rate	$\alpha = 1.6, \beta = 1.4$	Rate	$\alpha = 1.9, \beta = 1.9$	Rate
1/25	1.1435e−003		5.0896e−004		2.6381e−004	
1/50	2.8953e−004	1.9817	1.3592e−004	1.9048	6.9390e−005	1.9267
1/100	6.8091e−005	2.0882	3.4877e−005	1.9624	1.8064e−005	1.9416
1/200	1.5950e−005	2.0939	8.8502e−006	1.9785	4.6728e−006	1.9507

$$u(x,y,t) = e^{-t}\sin((2x)^4)\sin((2-2x)^4)\sin((2y)^4)\sin((2-2y)^4).$$

By the algorithm given in [3] and above conditions, it is easy to obtain the forcing function $s(x,y,t)$ at anywhere of the considered rectangle domain with any desired accuracy.

Table 3 also shows the maximum error, at time $t = 1$ and $\Delta t = \Delta x = \Delta y$, between the exact analytical value and the numerical value obtained by applying the ADI-CN scheme (2.27)–(2.31), and the scheme is second order convergent and this is in agreement with the order of the truncation error.

4.3. Numerical results for two-dimensional one-sided fractional convection diffusion equation

Considering the two-dimensional two-sided fractional convection diffusion equation (1.1), and taking the coefficients functions as

$$d_+(x,y) = 1, \quad d_-(x,y) = 0, \quad g(x,y) = 1,$$

$$e_+(x,y) = 1, \quad e_-(x,y) = 0, \quad h(x,y) = 1,$$

then it becomes the two-dimensional one-sided fractional convection diffusion equation,

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha} + \frac{\partial^\beta u(x,y,t)}{\partial y^\beta} + \frac{\partial u(x,y,t)}{\partial x} + \frac{\partial u(x,y,t)}{\partial y} + s(x,y,t), \tag{4.1}$$

where $0 < x < 1, 0 < y < 1, 0 < t \leqslant 1$, and the initial condition $u(x,y,0) = \sin(x^4)\sin(y^4)$ and the Dirichlet boundary conditions on the rectangle in the simple form $u(0,y,t) = u(x,0,t) = 0, u(1,y,t) = e^{-t}\sin(1)\sin(y^4)$ and $u(x,1,t) = e^{-t}\sin(1)\sin(x^4)$ for all $t > 0$. The exact value to this two-dimensional one-sided fractional convection diffusion equation is

$$u(x,y,t) = e^{-t}\sin(x^4)\sin(y^4).$$

By the algorithm given in [3] and above conditions, it is easy to obtain the forcing function $s(x,y,t)$ at anywhere of the considered rectangle domain with any desired accuracy.

Table 4 shows the maximum error, at time $t = 1$ and $\Delta t = \Delta x = \Delta y$, between the exact analytical value and the numerical value obtained by applying the ADI-CN scheme (2.27)–(2.31), and the scheme is second order convergent and it corresponds to the order of the truncation error.

5. Conclusions

This work provides the second-order efficient numerical scheme for the two-dimensional two-sided fractional advection diffusion equation on a finite domain. Both the convergent order and numerical stability of the scheme are theoretically proved and numerically verified. This paper can be considered as the sequel of the works [12–14,18–22].

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Appendix

Theorem 5.1. *For any given $u(x) \in C^4[x_L, x_R]$, there exists an extension of $u(x)$ defined on $[x_R, x_R + \Delta x]$, such that the extended $u(x) \in C^4[x_L, x_R + \Delta x]$ and*

$$\int_{x_R-\Delta x}^{x_R} (u(\xi) - S_{i-1}(\xi))(\xi - x_{i-1})^{1-\alpha} d\xi - \int_{x_R}^{x_R+\Delta x} (u(\xi) - S_{i+1}(\xi))(\xi - x_{i+1})^{1-\alpha} d\xi = 0, \tag{5.1}$$

where $S_{i-1}(x)$ is the linear interpolation function of $u(x)$ on the interval $[x_R - \Delta x, x_R]$, and $S_{i+1}(x)$ the linear interpolation function of the extended $u(x)$ on the interval $[x_R, x_R + \Delta x]$.

Proof. Denoting $a = x_R, b = x_R + \Delta x$, and $d_1 = \int_{x_R-\Delta x}^{x_R} (u(\xi) - S_{i-1}(\xi))(\xi - x_{i-1})^{1-\alpha} d\xi$, then the equality (5.1) means to find the extension of $u(x)$ on the interval $[a, b]$ such that $\int_a^b (u(\xi) - S_{i+1}(\xi))(\xi - x_{i+1})^{1-\alpha} d\xi = d_1$. Taking $u(b) = u(a)$, then $S_{i+1}(x) \equiv u(a)$ on the interval $[a, b]$ since $S_{i+1}(x)$ is the linear interpolation of $u(x)$. Now we need to prove that there exists $V(x) = u(x) - S_{i+1}(x)$ on the interval $[a, b]$ such that $V(a) = 0, V(b) = 0$, $V'(a^+) = u'(a^-) = d_2, V''(a^+) = u''(a^-) = d_3$, $V'''(a^+) = u'''(a^-) = d_4$, and $V''''(a^+) = u''''(a^-) = d_5$.

Suppose that $V(x)$ is composed by $\{1, x, x^2, x^3, x^4, x^5, x^6\}$, i.e.,

$$V(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6.$$

Using the first mean value theorem for integration, there exists $\eta \in (a, b)$, such that

$$\int_a^b V(\xi)(\xi - x_{i+1})^{1-\alpha} d\xi = V(\eta) \int_a^b (\xi - x_{i+1})^{1-\alpha} d\xi = d_1$$

and it can be rewritten as

$$c_0 + c_1\eta + c_2\eta^2 + c_3\eta^3 + c_4\eta^4 + c_5\eta^5 + c_6\eta^6 = \left(\int_a^b (\xi - x_{i+1})^{1-\alpha} d\xi\right)^{-1} d_1 =: d_1$$

(again denoted by d_1, and it is easy to check that $\int_a^b (\xi - x_{i+1})^{1-\alpha} d\xi \neq 0$). Combining all the above requirements, we obtain the linear algebraic equations $Ac = d$, where $c = [c_0, c_1, c_2, c_3, c_4, c_5, c_6]^T$, $d = [0, 0, d_1, d_2, d_3, d_4, d_5]^T$, and the determinant of the coefficient matrix A is

$$det(A) = \begin{vmatrix} 1 & a & a^2 & a^3 & a^4 & a^5 & a^6 \\ 1 & b & b^2 & b^3 & b^4 & b^5 & b^6 \\ 1 & \eta & \eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 \\ 0 & 1 & 2a & 3a^2 & 4a^3 & 5a^4 & 6a^5 \\ 0 & 0 & 2 & 6a & 12a^2 & 20a^3 & 30a^4 \\ 0 & 0 & 0 & 6 & 24a & 60a^2 & 120a^3 \\ 0 & 0 & 0 & 0 & 24 & 120a & 360a^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 & a^4 & a^5 & a^6 \\ 0 & b-a & b^2-a^2 & b^3-a^3 & b^4-a^4 & b^5-a^5 & b^6-a^6 \\ 0 & \eta-a & \eta^2-a^2 & \eta^3-a^3 & \eta^4-a^4 & \eta^5-a^5 & \eta^6-a^6 \\ 0 & 1 & 2a & 3a^2 & 4a^3 & 5a^4 & 6a^5 \\ 0 & 0 & 2 & 6a & 12a^2 & 20a^3 & 30a^4 \\ 0 & 0 & 0 & 6 & 24a & 60a^2 & 120a^3 \\ 0 & 0 & 0 & 0 & 24 & 120a & 360a^2 \end{vmatrix}.$$

Using the formula $a^n - b^n = (a - b)(a^{n-1} + ba^{n-2} + b^2a^{n-3} + \ldots + b^{n-2}a + b^{n-1})$, taking out the common factors $\eta - a$ and $b - a$, and repeating the steps, finally we get

$$det(A) = 288(\eta-a)^5(b-a)^5\begin{vmatrix} 1 & a & a^2 & a^3 & a^4 & a^5 & a^6 \\ 0 & 1 & 2a & 3a^2 & 4a^3 & 5a^4 & 6a^5 \\ 0 & 0 & 1 & 3a & 6a^2 & 10a^3 & 15a^4 \\ 0 & 0 & 0 & 1 & 4a & 10a^2 & 20a^3 \\ 0 & 0 & 0 & 0 & 1 & 5a & 15a^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & b+5a \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta-b \end{vmatrix} = 288(\eta-b)(\eta-a)^5(b-a)^5.$$

Since $\eta \neq a, \eta \neq b$, and $a \neq b$, we get $det(A) \neq 0$. Then with the given basis functions $\{1, x, x^2, x^3, x^4, x^5, x^6\}$, there exists a unique extended part of $u(x)$, which is $V(x) + u(a)$ with $x \in [x_R, x_R + \Delta x]$. The extended $u(x) \in C^4[x_L, x_R + \Delta x]$ and (5.1) holds. □

Theorem 5.2. *For the coefficients $q^\alpha_{i,k}$ defined in (2.14) and $p^\alpha_{i,k}$ defined by (2.2), the following hold: $q^\alpha_{i,k} > 0$ for $k \geqslant i+2, p^\alpha_{i,k} > 0$ for $k \leqslant i-2$, and $p^\alpha_{i,i+1} = q^\alpha_{i,i-1} = 1$, $p^\alpha_{i,i} = q^\alpha_{i,i} = -4 + 2^{3-\alpha}, p^\alpha_{i,i-1} = q^\alpha_{i,i+1} = 6 - 2^{5-\alpha} + 3^{3-\alpha}$.*

Proof. Similar to the proof given in [20] for the left Riemann–Liouville fractional derivative, we have $q^\alpha_{i,k} = 0, k < i-1, q^\alpha_{i,i-1} = 1, q^\alpha_{i,i} = -4 + 2^{3-\alpha}, q^\alpha_{i,i+1} = 6 - 2^{5-\alpha} + 3^{3-\alpha}$, and $q^\alpha_{i,k} = (k-i+2)^{3-\alpha} - 4(k-i+1)^{3-\alpha} + 6(k-i)^{3-\alpha} - 4(k-i-1)^{3-\alpha} + (k-i-2)^{3-\alpha}, k \geqslant i+2$. For $q^\alpha_{i,k}, k \geqslant i+2$, denote $m = k-i \geqslant 2$, then

$$\begin{aligned} q^\alpha_{i,i+m} &= (m+2)^{3-\alpha} - 4(m+1)^{3-\alpha} + 6m^{3-\alpha} - 4(m-1)^{3-\alpha} + (m-2)^{3-\alpha} \\ &= m^{3-\alpha}\left[6 + \left(1+\frac{2}{m}\right)^{3-\alpha} - 4\left(1+\frac{1}{m}\right)^{3-\alpha} - 4\left(1-\frac{1}{m}\right)^{3-\alpha} + \left(1-\frac{2}{m}\right)^{3-\alpha}\right] \\ &= m^{3-\alpha}\left\{6 + \sum_{k=0}^{\infty}\binom{3-\alpha}{k}\left[\left(\frac{2}{m}\right)^k - 4\left(\frac{1}{m}\right)^k - 4\left(\frac{-1}{m}\right)^k + \left(\frac{-2}{m}\right)^k\right]\right\} \\ &= m^{3-\alpha}\left\{\sum_{k=4}^{\infty}\binom{3-\alpha}{k}\left[\left(\frac{2}{m}\right)^k - 4\left(\frac{1}{m}\right)^k - 4\left(\frac{-1}{m}\right)^k + \left(\frac{-2}{m}\right)^k\right]\right\} = \frac{1}{m^{\alpha-1}}\left[\frac{(3-\alpha)(2-\alpha)(1-\alpha)(-\alpha)}{m^2} + \cdots\right]. \end{aligned}$$

Denoting $w_k = \left(\frac{2}{m}\right)^k - 4\left(\frac{1}{m}\right)^k - 4\left(\frac{-1}{m}\right)^k + \left(\frac{-2}{m}\right)^k, k \geqslant 4$, it can be checked that the terms $w_k = 0$ when k is odd and $w_k > 0$ when k is even, then we get that $q^\alpha_{i,i+m} > 0$. According to Remark 2.1, all the corresponding properties of $p^\alpha_{i,k}$ are obtained. □

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