AN α-ROBUST DIFFERENCE QUADRATURE METHOD FOR SPACE-TIME CAPUTO-RIESZ FRACTIONAL DIFFUSION EQUATION *

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Key words. example, LaTeX

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1. Introduction.

We study

$$\frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u = f(x,t), \quad x \in \Omega, t \in (0,T]. \tag{1.1}$$

Implicity scheme: Let $\tau = \frac{T}{M}$, $U^n, F^n \in \mathbb{R}^{2N-1}$,

$$\frac{U^{n+1} - U^n}{\tau} + AU^{n+1} = F^{n+1}. \tag{1.2}$$

Then

$$(I + \tau A)E^{n+1} = E^n + \tau R^{n+1}. \tag{1.3}$$

We will prove the convergence of this scheme.

2. Property of A.

LEMMA 2.1. *The stiffness matrix A has the following properties:*
1. *The eigenvalues of A are positive real numbers.*
2. *The eigenvectors of A form an orthogonal basis of the space of solutions.*
3. *The eigenvectors of A are orthogonal in space where $\langle u, v\rangle := uHv$.*

Proof. Since

$$A = H^{-1}D, \tag{2.1}$$

where $H := \operatorname{diag}\left(\frac{h_i + h_{i+1}}{2}\right)$ and D is symmetric positive definite, $H^{1/2}AH^{-1/2} = H^{-1/2}DH^{-1/2}$ is symmetric positive definite. Thus, the eigenvalues of A are positive real numbers.

The eigenvectors of A form an orthogonal basis of the space of solutions. Let v_i be an eigenvector of A with eigenvalue λ_i. Then,

$$\lambda_i v_i H v_j = v_i H A v_j = v_i D v_j = \lambda_j v_i H v_j,$$

Thus $v_i H v_j = 0$ for $i \neq j$. ∎

We need to prove $\lambda_1 > c$ for some positive constant c.

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3. Convergence. We prove the shceme in convergence in the meaning of L^2 norm. Both $\|v\|_2^2 := \frac{T}{N}\sum_{i=1}^{2N-1} v_i^2$ and $\|v\|_{2,h}^2 := \sum_{i=1}^{2N-1} v_i^2 \frac{h_i+h_{i+1}}{2}$ are bounded.

$$\begin{aligned} E^n &= (I+\tau A)^{-1}E^{n-1} + (I+\tau A)^{-1}\tau R^n \\ &= (I+\tau A)^{-n}E^0 + \sum_{k=1}^{n}(I+\tau A)^{-k}\tau R^{n-k+1} \end{aligned} \tag{3.1}$$

Assume that $|R^n| \leq Ch^{\min\{r\alpha/2,2\}}(x_i^{-\alpha} + (2T-x_i)^{-\alpha}) + C(r-1)h^2(T-\delta(x_i) + h_N)^{1-\alpha} + C\tau^?$

$$\begin{aligned} (I+\tau A)^{-k}\tau R^{n-k+1} &= (\tau A)(I+\tau A)^{-k}(\tau A)^{-1}\tau R^{n-k+1} \\ &= (\tau A)(I+\tau A)^{-k}(A^{-1}R^{n-k+1}) \end{aligned} \tag{3.2}$$

Since $A^{-1}R^{n-k+1}$ is bounded, $\|A^{-1}R^{n-k+1}\|_2$ is bounded. Depose it by the basis v_i, we have

$$\|(I+\tau A)^{-k}\tau R^{n-k+1}\|_2 \leq \frac{\tau\lambda_i}{(1+\tau\lambda_i)^k}\|A^{-1}R^{n-k+1}\|_2 \tag{3.3}$$

Then

$$\begin{aligned} \|E^n\|_2 &\leq \|E_0\|_2 + \sum_{k=1}^{n}\frac{\tau\lambda_1}{(1+\tau\lambda_1)^k}\|A^{-1}R^{n-k+1}\|_2 \\ &\leq \|E_0\|_2 + \max_k \|A^{-1}R^{n-k+1}\|_2 \end{aligned} \tag{3.4}$$

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REFERENCES

[1] G. H. Golub and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, 4th ed., 2013.