SECOND-ORDER DIFFERENCE-QUADRATURE APPROACH ON GRADED MESHES FOR FRACTIONAL LAPLACIAN VIA RIESZ FRACTIONAL DERIVATIVE

MINGHUA CHEN AND JIANXING HAN

Abstract. This paper deals with the *integral-differential* version of the fractional Laplacian via the Riesz fractional derivative on graded meshes. The high-order numerical analysis presents significant challenges in the past decades, partly because the solution to the problem has a weak singularity at the boundary, and the model equation can involve a singular source term. In such cases, many prevalent numerical methods may suffer from a severe order reduction. To fill in this gap, we combine the finite difference method and numerical quadrature, called difference-quadrature method, to approximate the *differential* and *integral* operator of the fractional Laplacian on graded meshes, respectively. We design a grid mapping function and a natural-skew ordering to handle local truncation errors, and construct an appropriate right-preconditioner for the resulting matrix algebraic equation. By utilizing the Hölder regularity of the data, we prove that the proposed scheme is second-order convergence on graded meshes even if the source term is hypersingular. Numerical experiments illustrate the theoretical results.

1. Introduction

Fractional Laplacian is a powerful tool in modeling phenomena for anomalous diffusion, which appears naturally in the α-stable Lévy process instead of the standard Brownian motion [1, 3, 18, 36, 25]. It can be found in many applications, such as porous media flow [12], image processing [17], biophysics [2]. In this work, we study a second-order difference-quadrature (DQ) scheme on graded meshes for the integral-differential version of the fractional Laplacian

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \mathbb{R} \setminus \Omega. \end{cases} \tag{1.1}$$

Here $(-\Delta)^{\frac{\alpha}{2}}$ is the integral-differential fractional Laplacian, in terms of the Riesz (left and right Riemann-Liouville) fractional derivative [1, 21, 24, 34], defined by

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{d^2}{dx^2} I^{2-\alpha} u(x) \quad \text{with} \quad 1 < \alpha < 2. \tag{1.2}$$

Note that the Riesz fractional integration can be realized in the form of the Riesz potential [33, eq. (1.30)], namely,

$$I^{2-\alpha} u(x) = \int_\Omega K(x-y) u(y) dy \quad \text{with} \quad K(x) = \frac{|x|^{1-\alpha}}{2\cos((2-\alpha)\pi/2)\Gamma(2-\alpha)}. \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary 26A33, 26A30, 65L20.
Key words and phrases. Fractional Laplacian via Riesz fractional derivative, finite difference-quadrature method, singular source term, stability and convergence, graded meshes.

The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ can be defined in several equivalent ways [22, 24] on the whole space $\mathbb{R}^n$. For example, it can be defined as a pseudo-differential operator via the Fourier transform

$$\mathcal{F}[(-\Delta)^{\frac{\alpha}{2}} u](\xi) = |\xi|^{\alpha} \mathcal{F}[u](\xi),$$

or in terms of the hypersingular integral operator

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy. \tag{1.4}$$

A key challenge in dealing with the fractional Laplacian arises from the fact that typical solutions u exhibit a weak singularity at the boundary. For example, the exact (Getoor) solution [18, 21, 26] is

$$u(x) = \frac{2^{-\alpha}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)} \left[(x-a)(b-x)\right]^{\frac{\alpha}{2}},$$

when Ω is a bounded interval $(a,b) \subset \mathbb{R}$ and $f \equiv 1$. Moreover, the model equation (1.1) can involve a singular/hypersingualr source term, even if the exact solution u is absolutely continuous [30, 31, 40]. This leads to a severe order reduction for many numerical methods.

Among various techniques for approximating *integral* version of the fractional Laplacian (1.4), numerical quadrature with piecewise linear polynomials (collocation) is the simplest, since it only need a single integration and are much simpler to implement on a computer. In [21], Huang and Oberman first proposed a quadrature-based finite difference method for solving the one-dimensional (1D) integral fractional Laplacian. The method yields a numerical solution with an accuracy of $\mathcal{O}\left(h^{2-\alpha}\right)$ in the discrete $L^{\infty}(\mathbb{R}^n)$ norm, provided that the solution is sufficiently smooth. However, this accuracy reduces to $\mathcal{O}\left(h^{\alpha/2}\right)$ in the case $f \equiv 1$, leading to a boundary singularity in u. Inspired by [21], $\mathcal{O}\left(|\log h| h^{2-\alpha/2}\right)$ convergence for $0 < \alpha < 2$ and $\mathcal{O}\left(h^{\alpha}\right)$ for $\alpha \leq 4/3$, respectively, is proved [20] in the discrete $L^{\infty}(\mathbb{R}^n)$ norm on graded meshes for $n = 1, 2$ by means of a discrete barrier function. Recently, $\mathcal{O}\left(h^{2-\alpha}\right)$ convergence for $0 < \alpha < 1$ is given in [9] by collocation method on graded meshes, where it remains to be proved for $1 < \alpha < 2$. It seems that achieving a second-order accurate scheme using piecewise linear polynomials collocation method for fractional Laplacian (1.4) with $1 < \alpha < 2$ is not an easy task.

Nevertheless, there are already some important progress for numerically solving *integral-differential* version of the fractional Laplacian (1.2) with $1 < \alpha < 2$ via the Riesz (left and right Riemann-Liouville) fractional derivative. Take, for example, the finite difference method [5, 15, 28, 29, 38, 8, 7, 6, 32, 35, 39], finite element method [4, 16, 14], and spectral method [11, 13, 37]. However, these methods may suffer from a severe order reduction when the exact solution has a weak singularity at the boundary and the source term is singular.

How to design the second-order convergence for the model (1.1) under the low regularity solution has not been addressed in the literature. In this work, we combine the finite difference method and numerical quadrature, called the difference-quadrature (DQ) method, to approximate the *differential* and *integral* operator of the fractional Laplacian (1.2) on graded meshes. The DQ method was proposed by the authors for solving Riesz fractional diffusion equations on uniform mesh [10, 11] when the solution is sufficiently smooth with $u \in C^4(\bar{\Omega})$. However, the numerical analysis [10, 11] under the low regularity solution presents significant challenges in the past decades. To fill in this gap, we design a grid mapping function and a natural-skew ordering to handle local truncation errors, and construct an appropriate right-preconditioner for the resulting matrix algebraic equation. By utilizing the Hölder regularity of the data, we prove that the proposed scheme is second-order convergence on graded meshes even if the source term is hypersingular.

2. The main results

In this section, we describe the difference-quadrature scheme on graded meshes for fractional Laplacian (1.1) via the Riesz fractional derivative and state our main results about the convergence rate of the numerical solutions.

2.1. **Difference-quadrature scheme.** To keep the expressions simple below we assume we are on the interval $\Omega = (0, 2T)$, but everything can be shifted to an arbitrary interval (a, b). Partition Ω by the graded mesh

$$\pi_h : 0 = x_0 < x_1 < x_2 < \cdots < x_{2N-1} < x_{2N} = 2T,$$

where we set

$$x_j = \begin{cases} T\left(\frac{j}{N}\right)^r & \text{for } j = 0, 1, ..., N, \\ 2T - T\left(\frac{2N-j}{N}\right)^r & \text{for } j = N+1, N+2, ..., 2N, \end{cases} \tag{2.1}$$

with a bounded grading exponent $r \geq 1$. When $r > 1$, the mesh points are clustered near $x = 0$ and $x = 2T$.

Set $h_j = x_j - x_{j-1}$ for $j = 1, 2, ..., 2N$ and define $h := \frac{1}{N}$. Let S_h be the space of globally continuous piecewise linear functions on the mesh π_h that vanish at $x = 0, 2T$. In this space, we choose as a basis the standard hat functions

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}) & \text{for } x_{j-1} \leq x \leq x_j, \\ \frac{1}{h_{j+1}}(x_{j+1} - x) & \text{for } x_j \leq x \leq x_{j+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Then, define the piecewise linear interpolant of the true solution u to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x). \tag{2.3}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x), \tag{2.4}$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

$$-D_h^\alpha u_h(x_i) := -D_h^2 I^{2-\alpha} u_h(x_i) = f(x_i) =: f_i. \tag{2.5}$$

Here the approximation of second order derivatives can be found by interpolating by a quadratic function and differentiating twice [23, eq. (1.14)]

$$D_h^2 u(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} u(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}} u(x_{i+1}) \right). \tag{2.6}$$

Moreover, the Riesz fractional derivatives in (1.2) can be approximated by

$$-D_h^\alpha u_h(x_i) = -D_h^2 I^{2-\alpha} \sum_{j=1}^{2N-1} u_j \phi_j(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u_j. \tag{2.7}$$

We have replaced $-\frac{d^2}{dx^2} I^{2-\alpha} u(x_i) = f(x_i)$ in (1.2) by $-D_h^\alpha u_h(x_i) = f(x_i)$ in (2.5), with truncation error

$$\tau_i := -D_h^\alpha \Pi_h u(x_i) - f(x_i) \quad \text{for} \quad i = 1, 2, ..., 2N-1, \tag{2.8}$$

where

$$-D_h^\alpha \Pi_h u(x_i) = -\sum_{j=1}^{2N-1} D_h^\alpha \phi_j(x_i)\, u(x_j) = \sum_{j=1}^{2N-1} a_{ij} u(x_j). \tag{2.9}$$

The discrete equation (2.5) can be written in matrix form

$$AU = F, \tag{2.10}$$

where the coefficient matrix A and the vectors U and F are defined by $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ and $F = (f_1, \cdots, f_{2N-1})^T$. In particular, the coefficient a_{ij} can be explicitly expressed as

$$\begin{aligned} a_{ij} &= -D_h^2 I^{2-\alpha} \phi_j(x_i) \\ &= -\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \end{aligned} \tag{2.11}$$

with the quadrature coefficients

$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha} \phi_j(x_i) \\ &= \frac{\kappa_\alpha}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_j} + \frac{1}{h_{j+1}} \right) |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right), \end{aligned}$$

and $\kappa_\alpha = \frac{1}{2\cos((2-\alpha)\pi/2)} = -\frac{1}{2\cos(\alpha\pi/2)} > 0$.

2.2. **Regularity of the true solution.** For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous [19, p. 52] with exponent β' in Ω. Here, $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

For convenience, we define

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) = \begin{cases} x & 0 < x \le T, \\ 2T - x & T < x < 2T, \end{cases} \tag{2.12}$$

and $\delta(x, y) = \min\{\delta(x), \delta(y)\}$. To bound the derivatives of u, we introduce the following δ-dependent Hölder norms.

Definition 2.1 (δ-dependent Hölder norms [27])**.** For any $\beta > 0$, write $\beta = k + \beta'$, where k is an integer and $\beta' \in (0, 1]$. Given $\sigma \ge -\beta$, define the seminorm

$$|w|_\beta^{(\sigma)} = \sup_{x,y\in\Omega} \left(\delta(x,y)^{\beta+\sigma} \frac{|w^{(k)}(x) - w^{(k)}(y)|}{|x-y|^{\beta'}} \right).$$

For $\sigma > -1$, we also define the norm $\|\cdot\|_\beta^{(\sigma)}$ as follows: in case that $\sigma \geq 0$,

$$\|w\|_\beta^{(\sigma)} = \sum_{l=0}^{k} \sup_{x\in\Omega} \left(\delta(x)^{l+\sigma}|w^{(l)}(x)|\right) + |w|_\beta^{(\sigma)},$$

while for $-1 < \sigma < 0$,

$$\|w\|_\beta^{(\sigma)} = \|w\|_{C^{-\sigma}(\bar\Omega)} + \sum_{l=1}^{k} \sup_{x\in\Omega} \left(\delta(x)^{l+\sigma}|w^{(l)}(x)|\right) + |w|_\beta^{(\sigma)}.$$

Lemma 2.2. [27, pp. 276-277] *Let $f \in L^\infty(\Omega)$ and u be a solution of (1.1). Then, $u \in C^{\alpha/2}(\mathbb{R})$ and $u/\delta^{\alpha/2} \in C^\sigma(\bar\Omega)$ for some $\sigma \in (0, 1-\alpha/2)$, $\alpha \in (1,2)$ with*

$$\|u\|_{C^{\alpha/2}(\mathbb{R})} \leq C\|f\|_{L^\infty(\Omega)} \quad \text{and} \quad \|u/\delta^{\alpha/2}\|_{C^\sigma(\bar\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$$

for some positive constant $C = C(\Omega, \alpha)$.

In particular, this result says that if $f \in L^\infty(\Omega)$, then

$$|u(x)| \leq C\delta(x)^{\alpha/2} \quad \text{for all } x \in \bar\Omega. \tag{2.13}$$

Lemma 2.3. [27, Proposition 1.4] *Let $\beta > 0$ be such that neither β nor $\beta+\alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R})$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right)$$

for some positive constant $C = C(\Omega, \alpha, \beta)$.

By definition of δ-dependent Hölder norms, we have following results obviously.

Lemma 2.4. *Let $\beta = 4-\alpha+\gamma$ with $0 < \gamma < \alpha-1$. Assume that $f \in L^\infty(\Omega)\cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and u be a solution of (1.1). Then*

$$|u^{(l)}(x)| \leq C\delta(x)^{\alpha/2-l} \quad \text{for } x \in \Omega \text{ and } l = 0,1,2,3,4,$$
$$|f^{(l)}(x)| \leq C\delta(x)^{-\alpha/2-l} \text{ for } x \in \Omega \text{ and } l = 0,1,2,$$

for some positive constant $C = C(\Omega, \alpha, \beta, f)$.

Proof. Our hypotheses imply that $2 < \beta < 3$, and $4 < \beta+\alpha < 5$. By Lemma 2.3, we have

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right).$$

By Definition 2.1 and Lemma 2.2, it yields

$$\sum_{l=1}^{4} \sup_{x\in\Omega} \left(\delta(x)^{l-\alpha/2}\left|u^{(l)}(x)\right|\right) \leq C\left(\|f\|_{L^\infty(\Omega)} + \|f\|_\beta^{(\alpha/2)}\right),$$

which is desired result $l = 1,2,3,4$. The case $l = 0$ is covered by (2.13).

The second inequality can be obtained by Definition 2.1, namely,

$$\sum_{l=0}^{2} \sup_{x\in\Omega} \left(\delta(x)^{l+\alpha/2}|f^{(l)}(x)|\right) \leq \|f\|_\beta^{(\alpha/2)}.$$

The proof is completed. □

2.3. **Main results.** The main results of this paper consist of the following theorems, which will be proved in Section 3 and Section 4, respectively.

Theorem 2.5 (Local Truncation Error)**.** *Let $\alpha \in (1,2)$ and $f \in L^{\infty}(\Omega) \cap C^{\beta}(\Omega)$ be such that $\|f\|_{\beta}^{(\alpha/2)} < \infty$, where $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$. Then,*

$$\begin{aligned} |\tau_i| &= |-D_h^{\alpha}\Pi_h u(x_i) - f(x_i)| \\ &\leq Ch^{\min\{\frac{r\alpha}{2},2\}}\delta(x_i)^{-\alpha} + C(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha} \end{aligned}$$

for some positive constant $C = C(\Omega, \alpha, \beta, r, f)$.

Theorem 2.6 (Global Error)**.** *Let $\alpha \in (1,2)$ and $f \in L^{\infty}(\Omega) \cap C^{\beta}(\Omega)$ be such that $\|f\|_{\beta}^{(\alpha/2)} < \infty$, where $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$. Let u_i be the approximate solution of $u(x_i)$ computed by the discretization scheme* (2.5)*. Then,*

$$\max_{1 \leq i \leq 2N-1} |u_i - u(x_i)| \leq Ch^{\min\{\frac{r\alpha}{2},2\}}$$

for some positive constant $C = C(\Omega, \alpha, \beta, r, f)$.

3. Local Truncation Error

For convenience, we use the notation $\simeq$, where $x \simeq y$ means that $C_1 x \leq y \leq C_2 x$ for some positive constants C_1 and C_2 independent of h.

For $1 \leq j \leq 2N$, we define the combination of adjacent grid points as

$$y_j^{\theta} = (1-\theta)x_{j-1} + \theta x_j, \quad \theta \in (0,1). \tag{3.1}$$

Then, using the definition of grid points $\{x_j\}$ in (2.1), it follows that

Lemma 3.1. *Let $h = \frac{1}{N}$ and $\delta(x_j)$ be defined by* (2.12)*. Then we have*

$$\begin{aligned} &h_j \simeq h_{j+1} \simeq h\delta(x_j)^{1-1/r}, \quad 1 \leq j \leq 2N-1, \\ &\delta(x_j) \simeq \delta(x_{j+1}) \simeq \delta(y_{j+1}^{\theta}), \ \ 1 \leq j \leq 2N-2. \end{aligned}$$

We next give a detailed analysis of the local truncation error.

3.1. **Proof of Theorem 2.5.** The local truncation error (2.8) can be expressed by

$$\begin{aligned} \tau_i &= -D_h^2 I^{2-\alpha}\Pi_h u(x_i) + \frac{d^2}{dx^2} I^{2-\alpha} u(x_i) \\ &= D_h^2 I^{2-\alpha}\left(u - \Pi_h u\right)(x_i) - \left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i). \end{aligned} \tag{3.2}$$

We estimate each component of this partition.

Theorem 3.2. *There exists a constant C such that*

$$\left|\left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i)\right| \leq Ch^2\delta(x_i)^{-\alpha/2-2/r}. \tag{3.3}$$

Proof. Since $f \in C^2(\Omega)$ and $-\frac{d^2}{dx^2} I^{2-\alpha} u(x) = f(x)$ for $x \in \Omega$, it implies $I^{2-\alpha}u \in C^4(\Omega)$. From Lemma A.1 in Appendix A, we have for $1 \leq i \leq 2N-1$,

$$\begin{aligned} -\left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i) &= \frac{h_{i+1} - h_i}{3} f'(x_i) \\ &+ \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} f''(y)\frac{(y - x_{i-1})^3}{3!}dy + \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f''(y)\frac{(y - x_{i+1})^3}{3!}dy\right). \end{aligned}$$

According to Lemmas 2.4, B.1 and B.2, the desired result is obtained. □

Now we consider the first term of the local truncation error in (3.2), which we denote for simplicity

$$R_i := D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i), \quad 1 \le i \le 2N-1. \tag{3.4}$$

We have derived the following results concerning the estimation of R_i including Theorems 3.3 and 3.4, which will be demonstrated in Subsection 3.3.

Theorem 3.3. *For $1 \le i < N/2$, there exists a constant C such that*

$$|R_i| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

Theorem 3.4. *For $N/2 \le i \le N$, there exists a constant C such that*

$$|R_i| \le C(r-1)h^2(T - x_i + h_N)^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

Remark 3.5. And for $N < i \le 2N-1$, observe first that the mesh (2.1) is symmetric about $x = T$ (i.e., $x = x_i$ is a mesh point if and only if $x = 2T - x_i = x_{2N-i}$ is a mesh point), and the a priori derivative bounds of Lemma 2.4 are also symmetric about $x = T$. But the locations of the mesh points and these bounds on derivatives are the only ingredients used in the analysis of the case $1 \le i \le N$. Thus, one can define $\tilde{u}(x) = u(2T - x)$, and now, the truncation error of $u(x)$ at $x = x_i$ for $i = N+1, N+2, ..., 2N-1$ is exactly the same as the truncation error of $\tilde{u}(x)$ at $x = x_i$ for $i = N-1, N-2, ...1$, which can be handled in exactly the same way as the truncation error analysis of $u(x)$ for $i = 1, 2, ..., N-1$. Transforming back via $x \mapsto 2T - x$, we get the desired result for $i = N+1, N+2, ..., 2N-1$. This technique will be used several times.

Combine Theorems 3.2 to 3.4 and remark 3.5, and for $1 \le i \le N$, we have

$$\begin{gathered} h^2 x_i^{-\alpha/2-2/r} \le T^{\alpha/2-2/r} h^{\min\{\frac{r\alpha}{2},2\}} x_i^{-\alpha}, \\ h^{r\alpha/2+r} x_i^{-1-\alpha} \le T^{-1} h^{r\alpha/2} x_i^{-\alpha}, \\ h^r x_i^{-1}\ln(i) = T^{-1}\frac{\ln(i)}{i^r} \le T^{-1}, \quad h^r\ln(N) = \frac{\ln(N)}{N^r} \le 1, \end{gathered}$$

the proof of Theorem 2.5 completed.

3.2. **Grid mapping functions.** In this subsection, we offer an overview of the framework for estimating R_i, where we introduce the *natural-skew ordering* and *grid mapping functions*.

From (1.3) and (3.4), we know that

$$I^{2-\alpha}(u - \Pi_h u)(x_i) = \sum_{j=1}^{2N} \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) K(x_i - y) dy = \sum_{j=1}^{2N} T_{ij} \tag{3.5}$$

with

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) K(x_i - y) dy, \quad i = 0, \cdots, 2N, \ j = 1, \cdots, 2N. \tag{3.6}$$

To estimate R_i more precisely, we define the *vertical difference quotients* of T_{ij}

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right), \tag{3.7}$$

and the *skew difference quotients* of T_{ij}

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j-1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right). \tag{3.8}$$

From (3.4), (3.5) and (3.6), we have

$$R_1 = \sum_{j=1}^{3} V_{1,j} + \sum_{j=4}^{2N} V_{1,j} \quad \text{and} \quad R_2 = \sum_{j=1}^{4} V_{2,j} + \sum_{j=5}^{2N} V_{2,j}. \tag{3.9}$$

Moreover, using (3.4)-(3.8), we can express R_i based on the natural-skew ordering, as shown in Figure 1:

$$R_i = I_1 + I_2 + I_3 + I_4 + I_5 \quad \text{for} \quad 3 \le i \le N. \tag{3.10}$$

Here,

$$I_1 = \sum_{j=1}^{k-1} V_{ij}, \quad I_3 = \sum_{j=k+1}^{m-1} S_{ij}, \quad I_5 = \sum_{j=m+1}^{2N} V_{ij} \quad \text{for} \quad k = \lceil i/2 \rceil,$$

and

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right),$$
$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,m} \right)$$

with

$$m = \begin{cases} 2i, & 3 \le i < N/2, \\ 2N - \lceil N/2 \rceil + 1, & N/2 \le i \le N. \end{cases} \tag{3.11}$$

Noted that I_1 and I_5 along with V_{ij} as defined in (3.7), represent natural (vertical) ordering, while I_3, along with S_{ij} as defined in (3.8), represents skew ordering, which is referred to as natural-skew ordering here.

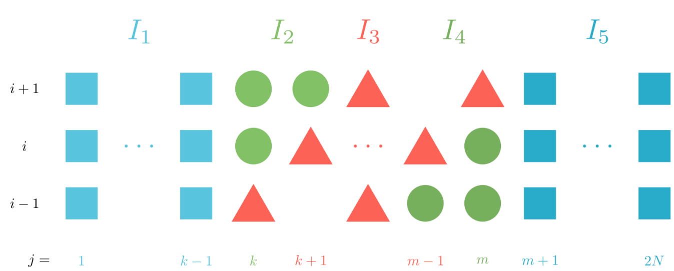

FIGURE 1. Natural-Skew ordering of R_i.

The complexity in estimating S_{ij} in (3.8) lies in the fact that the integral domains for $T_{i-1,j-1}, T_{i,j}$ and $T_{i+1,j+1}$ are distinct. We first normalize T_{ij} to the unit interval.

Lemma 3.6. *For any $y \in (x_{j-1}, x_j)$, there exits*

$$\begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y))K(x_i - y)dy \\ &= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta))K(x_i - y_j^\theta)h_j d\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) K(x_i - y_j^\theta) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} h_j^4 K(x_i - y_j^\theta) \left(\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)\right) d\theta \end{aligned}$$

with $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

Proof. By (3.6) and Lemma A.2, the desired result is obtained. $\square$

To estimate the local truncation error more concisely, we construct the following grid mapping functions.

Definition 3.7. For $1 \le i, j \le 2N-1$, we define the grid mapping functions

$$y_{i,j}(x) = \begin{cases} (x^{1/r} + Z_{j-i})^r & i < N, j < N, \\ \dfrac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N, \\ 2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N, \\ \left(\dfrac{Z_1}{h_N}(x - x_N) + Z_j\right)^r & i = N, j < N, \\ x & i = N, j = N, \\ 2T - \left(\dfrac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j}\right)^r & i = N, j > N, \\ (Z_{2N+j-i} - (2T-x)^{1/r})^r & i > N, j < N, \\ \dfrac{Z_{2N-j} - (2T-x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N, \\ 2T - ((2T-x)^{1/r} - Z_{j-i})^r & i > N, j > N \end{cases} \tag{3.12}$$

with $Z_j := T^{1/r}\frac{j}{N}$.

Let us further define

$$h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x), \tag{3.13}$$

$$y_{i,j}^\theta(x) = (1-\theta)y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0,1), \tag{3.14}$$

$$P_{i,j}^\theta(x) = (h_{i,j}(x))^3 K(x - y_{i,j}^\theta(x)) u''(y_{i,j}^\theta(x)), \tag{3.15}$$

$$Q_{i,j,l}^\theta(x) = (h_{i,j}(x))^l K(x - y_{i,j}^\theta(x)), \quad l = 3, 4. \tag{3.16}$$

Then, we can check that

$$\begin{aligned} &y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1}, \\ &h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1}, \\ &y_{i,j}^\theta(x_{i-1}) = y_{j-1}^\theta, \quad y_{i,j}^\theta(x_i) = y_j^\theta, \quad y_{i,j}^\theta(x_{i+1}) = y_{j+1}^\theta. \end{aligned} \tag{3.17}$$

Now, we can rewrite T_{ij} by (3.15) in (3.6) as

$$
\begin{aligned}
T_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} P_{i,j}^{\theta}(x_i) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{i,j,4}^{\theta}(x_i) \left[\theta^2 u'''(\eta_{j,1}^{\theta}) - (1-\theta)^2 u'''(\eta_{j,2}^{\theta})\right] d\theta.
\end{aligned} \tag{3.18}
$$

From (2.6), (3.8) and (3.18), for $1 \le i \le 2N-1$, $2 \le j \le 2N-1$, we have

$$
\begin{aligned}
S_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i,j}^{\theta}(x_i) d\theta \\
& + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j,4}^{\theta}(x_{i+1}) u'''(\eta_{j+1,1}^{\theta}) - Q_{i,j,4}^{\theta}(x_i) u'''(\eta_{j,1}^{\theta})}{h_{i+1}} \right) d\theta \\
& - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j,4}^{\theta}(x_i) u'''(\eta_{j,1}^{\theta}) - Q_{i,j,4}^{\theta}(x_{i-1}) u'''(\eta_{j-1,1}^{\theta})}{h_i} \right) d\theta \\
& - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j,4}^{\theta}(x_{i+1}) u'''(\eta_{j+1,2}^{\theta}) - Q_{i,j,4}^{\theta}(x_i) u'''(\eta_{j,2}^{\theta})}{h_{i+1}} \right) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j,4}^{\theta}(x_i) u'''(\eta_{j,2}^{\theta}) - Q_{i,j,4}^{\theta}(x_{i-1}) u'''(\eta_{j-1,2}^{\theta})}{h_i} \right) d\theta.
\end{aligned} \tag{3.19}
$$

The derivatives of the grid mapping functions are calculated as follows.

Lemma 3.8. *For $1 \le i, j \le 2N-1$, there exist,*

$$
y_{i,j}'(x) = \begin{cases}
y_{i,j}^{1-1/r}(x) x^{1/r-1}, & i < N, j < N, \\
\dfrac{h_N}{rZ_1} x^{1/r-1}, & i < N, j = N, \\
(2T - y_{i,j}(x))^{1-1/r} x^{1/r-1}, & i < N, j > N, \\
y_{i,j}^{1-1/r}(x) \dfrac{rZ_1}{h_N}, & i = N, j < N, \\
1 & i = N, j = N,
\end{cases}
$$

and

$$
y_{i,j}''(x) = \frac{1-r}{r} \begin{cases}
y_{i,j}^{1-2/r}(x) x^{1/r-2} Z_{j-i}, & i < N, j < N, \\
\dfrac{h_N}{rZ_1} x^{1/r-2}, & i < N, j = N, \\
(2T - y_{i,j}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i}, & i < N, j > N, \\
-y_{i,j}^{1-2/r}(x) \left(\dfrac{rZ_1}{h_N}\right)^2, & i = N, j < N, \\
0, & i = N, j = N.
\end{cases}
$$

Proof. The desired results can be obtained by Definition 3.7 directly. □

The following lemmas about the grid mapping functions will be used in next subsection. They are proved in Appendix C.

Lemma 3.9. *For any $\xi \in (x_{i-1}, x_{i+1})$, $2 \le i, j \le 2N-2$, there exist*

$$\xi \simeq x_i, \quad \delta(y_{i,j}(\xi)) \simeq \delta(x_j), \quad h_{i,j}(\xi) \simeq h_j,$$
$$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|, \quad |y_{i,j-1}(\xi) - \xi| \simeq |x_{j-1} - x_i|,$$
$$|y^\theta_{i,j}(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta|y_{i,j}(\xi) - \xi| \simeq |y^\theta_j - x_i|.$$

Lemma 3.10. *For any $\xi \in (x_{i-1}, x_{i+1})$, $2 \le i \le N, 2 \le j \le 2N-2$, there exist*

$$|h'_{i,j}(\xi)| \le C(r-1)Z_1 x_i^{1/r-1}\delta(x_j)^{1-2/r} \le C(r-1)h_j x_i^{1/r-1}\delta(x_j)^{-1/r},$$
$$|(y_{i,j}(\xi) - \xi)'| \le C x_i^{-1}|x_j - x_i|.$$

Lemma 3.11. *For any $\xi \in (x_{i-1}, x_{i+1})$, $2 \le i \le N, 2 \le j \le 2N-2$, there exist*

$$|y''_{i,j}(\xi)| \le C(r-1)\begin{cases} x_j^{-1/r}x_i^{1/r-2}|x_j - x_i|, & i < N, j < N, \\ x_N^{1-1/r}x_i^{1/r-2}, & i < N, j = N, \\ \delta(x_j)^{1-2/r}x_i^{1/r-2}x_N^{1/r}, & i < N, j > N, \\ \delta(x_j)^{1-2/r}x_N^{2/r-2}, & i = N, j \ne N, \\ 0 & i = N, j = N. \end{cases}$$

For $2 \le i \le N, 3 \le j \le 2N-2$, there exist

$$|h''_{i,j}(\xi)| \le C(r-1)\begin{cases} Z_1 x_i^{1/r-2}x_j^{-2/r}(|x_j - x_i| + x_j), & i < N, j < N, \\ x_i^{1/r-2}x_N^{1-1/r}, & i < N, j = N, N+1, \\ Z_1 x_i^{1/r-2}\delta(x_j)^{1-3/r}x_N^{1/r}, & i < N, j > N+1, \\ Z_1 x_N^{2/r-2}\delta(x_j)^{1-3/r}, & i = N, j \ne N, N+1, \\ x_N^{-1}, & i = N, j = N, N+1. \end{cases}$$

Lemma 3.12. *Let $P^\theta_{i,j}(x_i)$ be defined by (3.15) and the difference quotient operator D_h^2 be defined by (2.6). Then we have*
Case 1. *For $3 \le i < N, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, there exists*

$$|D_h^2 P^\theta_{i,j}(x_i)| \le C h_j^3 |y^\theta_j - x_i|^{1-\alpha} x_i^{\alpha/2-4}.$$

Case 2. *For $N/2 \le i \le N$, $j = N, N+1$, there exists*

$$|D_h^2 P^\theta_{i,j}(\xi)| \le C h_j^3 |y^\theta_j - x_i|^{1-\alpha} + C(r-1)h_j^2\Big(|y^\theta_j - x_i|^{1-\alpha} + h_j|y^\theta_j - x_i|^{-\alpha}\Big).$$

Case 3. *For $N/2 \le i \le N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil$, there exists*

$$|D_h^2 P^\theta_{i,j}(\xi)| \le C h_j^3 \Big(|y^\theta_j - x_i|^{1-\alpha} + (r-1)|y^\theta_j - x_i|^{-\alpha}\Big).$$

Lemma 3.13. *Let $Q^\theta_{i,j,l}(x_i)$ be defined by (3.16). Then we have for $2 \le i \le N$, $2 \le j \le 2N-2$, $l = 3, 4$, there exist*

$$\left|\frac{Q^\theta_{i,j,l}(x_{i+1})u^{(l-1)}(\eta^\theta_{j+1}) - Q^\theta_{i,j,l}(x_i)u^{(l-1)}(\eta^\theta_j)}{h_{i+1}}\right|$$
$$\le C h_j^l |y^\theta_j - x_i|^{1-\alpha} x_i^{-1}\delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r} + \delta(x_j)^{1/r}),$$

and

$$\left|\frac{Q^\theta_{i,j,l}(x_i)u^{(l-1)}(\eta^\theta_j) - Q^\theta_{i,j,l}(x_{i-1})u^{(l-1)}(\eta^\theta_{j-1})}{h_i}\right|$$
$$\le C h_j^l |y^\theta_j - x_i|^{1-\alpha} x_i^{-1}\delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r} + \delta(x_j)^{1/r})$$

with $\eta_j^\theta \in (x_{j-1}, x_j)$.

3.3. **Error analysis of R_i.** In this subsection, we estimate the first term of the local truncation error R_i in (3.4) through (3.9) and (3.10). We denote

$$K_y(x) := K(x-y) = \frac{\kappa_\alpha}{\Gamma(2-\alpha)}|x-y|^{1-\alpha}, \quad 1<\alpha<2, \tag{3.20}$$

where the kernel function $K(x)$ is given in (1.3) and κ_α is given in (2.11).

Lemma 3.14. *Let* $I_5 = \sum_{j=m+1}^{2N} V_{ij}$ *be defined by* (3.10). *Then we have*
Case 1. *For* $1 \le i < N/2$ *and* $m = \max\{2i, 3\}$, *there exists*

$$\sum_{j=m+1}^{2N} |V_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0,\\ Ch^2\ln(N), & \alpha/2-2/r+1=0,\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0.\end{cases}$$

Case 2. *For* $N/2 \le i \le N$ *and* $m = 2N - \lceil \frac{N}{2} \rceil + 1$, *there exists*

$$\sum_{j=m+1}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2-2/r+1>0,\\ Ch^2\ln(N), & \alpha/2-2/r+1=0,\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0.\end{cases}$$

Proof. For $1 \le i < N/2$, $m+1 \le j \le 2N$ with $m = \max\{2i,3\}$, using (3.6), (3.7), (3.20), Lemmas A.3 and B.3, we have

$$\begin{aligned} |V_{ij}| &= \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y))\, D_h^2 K_y(x_i) dy \right| \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} \delta(y)^{\alpha/2-2/r} |x_i - y|^{-1-\alpha} dy. \end{aligned}$$

Since $y \ge x_{j-1} \ge x_{2i}$, $y - x_i \simeq y$, and $x_i \simeq x_{2i}$, it yields

$$\begin{aligned} \sum_{j=m+1}^{N} |V_{ij}| &\le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\le Ch^2 x_i^{-\alpha/2-2/r}. \end{aligned}$$

On the other hand, since $y - x_i \simeq T$ if $y \ge x_N = T$, there exist

$$\begin{aligned} \sum_{j=N+1}^{2N-1} |V_{ij}| &\le CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\ &\le \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\, h^2, & \alpha/2-2/r+1>0,\\ CrT^{-1-\alpha}h^2\ln(N), & \alpha/2-2/r+1=0,\\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\, h^{r\alpha/2+r}, & \alpha/2-2/r+1<0.\end{cases} \end{aligned}$$

Finally, by Lemma A.4, one has

$$|V_{i,2N}| \le CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}.$$

Then, the desired result in Case 1 is obtained. We can similarly prove for Case 2, the details are omitted here. $\square$

Immediately, we can calculate R_1, R_2 from (3.9).

Lemma 3.15. *For $i = 1, 2$, we have*

$$|R_i| \le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

Proof. According to (3.9), Lemmas 3.14 and B.4, the desired result is obtained. □

For R_i with $3 \le i \le N$, the terms $\{I_1, I_2, I_3, I_4\}$ in (3.10) remain to be estimated.

Lemma 3.16. *Let $I_1 = \sum_{j=1}^{k-1} V_{ij}$ be defined by* (3.10)*. Then we have, for $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$,*

$$\sum_{j=1}^{k-1} |V_{ij}| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

Proof. According to (3.7), Lemmas A.4 and B.3, it yields

$$|V_{i1}| \le C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2+1} x_i^{-1-\alpha} = T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}.$$

Using Lemma A.3, Lemma B.3 and $y \le x_{k-1} < 2^{-r} x_i$, $x_i - y \simeq x_i$, we have

$$|V_{ij}| \le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} x_i^{-1-\alpha} dy, \quad 2 \le j \le k-1,$$

and

$$\sum_{j=2}^{k-1} |V_{ij}| \le Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy.$$

Moreover we can check that

$$\int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0, \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0, \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

The proof is completed. □

Subsequently, we turn our attention to $I_3 = \sum_{j=k+1}^{m-1} S_{ij}$ with $m = 2i$ for $3 \le i < N/2$ and $m = 2N - \lceil N/2 \rceil + 1$ for $N/2 \le i \le N$ in (3.11).

Lemma 3.17. *Let $I_3 = \sum_{j=k+1}^{m-1} S_{ij}$ be defined by* (3.10)*. Then we have*
Case 1. *For $N/2 \le i \le N$, $m = 2N - \lceil N/2 \rceil + 1$, there exist*

$$|S_{ij}| \le C(h^3 + (r-1)h^2)(T - x_i + h_N)^{1-\alpha}, \quad j = N, N+1,$$

and

$$\sum_{j=N+2}^{m-1} |S_{ij}| \le Ch^2 + C(r-1)h^2 (T - x_i + h_N)^{1-\alpha}.$$

Case 2. *For* $3 \le i \le N-1$, $k = \lceil \frac{i}{2} \rceil$, *there exist*

$$\sum_{j=k+1}^{\min\{m-1,N-1\}} |S_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r},$$

and

$$\sum_{j=\lceil \frac{N}{2} \rceil+1}^{N-1} |S_{Nj}| \le Ch^2 + C(r-1)h^2 h_N^{1-\alpha}.$$

Proof. Case 1: From (3.19), using $\theta(1-\theta)h_j \le |y_j^\theta - x_i|$, Lemmas 3.1, 3.12 and 3.13, it yields

$$|S_{ij}| \le C(h_j^3 + (r-1)h_j^2) \int_0^1 |y_j^\theta - x_i|^{1-\alpha} d\theta, \quad j = N, N+1$$

with

$$\int_0^1 |y_j^\theta - x_i|^{1-\alpha} dy \simeq (|x_j - x_i| + h_N)^{1-\alpha}.$$

On the other hand, for $j \ge N+2$, $x_i \simeq x_j \simeq T$, we have

$$\begin{aligned} |S_{ij}| &\le Ch_j^2 \int_0^1 \left(|y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha}\right) h_j d\theta \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy. \end{aligned}$$

It implies that

$$\begin{aligned} \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| &= Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy \\ &\le Ch^2 \left(T^{2-\alpha} + (r-1)(T - x_i + h_N)^{1-\alpha}\right). \end{aligned}$$

Case 2: for $3 \le i \le N-1$, $k+1 \le j \le \min\{m-1, N-1\}$, using Lemmas 3.1, 3.12 and 3.13, $x_i \simeq x_j$ and $h_i \simeq h_j$, we have

$$\begin{aligned} |S_{ij}| &\le Ch_j^2 x_i^{\alpha/2-4} \int_0^1 |y_j^\theta - x_i|^{1-\alpha} h_j d\theta \\ &= Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy, \end{aligned}$$

and

$$\begin{aligned} \sum_{k+1}^{\min\{2i-1,N-1\}} |S_{ij}| &\le Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_k}^{x_{\min\{2i-1,N-1\}}} |y - x_i|^{1-\alpha} dy \\ &\le Ch^2 x_i^{\alpha/2-2-2/r} x_i^{2-\alpha} = Ch^2 x_i^{-\alpha/2-2/r}. \end{aligned}$$

We can similarly prove the last inequality by Case 1. The proof is completed. □

Finally, we focus our error analysis on the terms I_2 and I_4.

Lemma 3.18. *Let* I_2, I_4 *be defined by* (3.10)*. Then we have*
Case 1. *For* $3 \le i \le N$, $k = \lceil \frac{i}{2} \rceil$, *there exists*

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r}.$$

Case 2. *For $3 \le i < N/2$, $m = 2i$, there exists*

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,2i} + T_{i-1,2i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r}.$$

Case 3. *For $N/2 \le i \le N$, $m = N - \lceil \frac{N}{2} \rceil + 1$, there exists*

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,m} + T_{i-1,m-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,m} \right) \le Ch^2.$$

Proof. Since

$$\begin{aligned} &\frac{1}{h_{i+1}} \left(T_{i+1,k} + T_{i+1,k+1}\right) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \\ &= \frac{1}{h_{i+1}} \left(T_{i+1,k} - T_{i,k}\right) + \frac{1}{h_{i+1}} \left(T_{i+1,k+1} - T_{i,k}\right) + \left(\frac{1}{h_{i+1}} - \frac{1}{h_i} \right) T_{i,k}. \end{aligned} \tag{3.21}$$

According to $x_i - x_k \simeq x_i \simeq x_k$, Lemmas 3.1, A.3 and B.3, we have

$$\begin{aligned} \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) D_h K_y(x_i) dy \\ &\le Ch_k^2 x_k^{\alpha/2-2}\, h_k |x_i - x_k|^{-\alpha} \le Ch^2 x_i^{-\alpha/2-2/r} h_k. \end{aligned}$$

From Lemmas 3.6 and A.2 and (3.16), we can obtain

$$\frac{1}{h_{i+1}} \left(T_{i+1,k+1} - T_{i,k}\right) = \int_0^1 \frac{\theta(\theta-1)}{2} \frac{Q^\theta_{i,k;3}(x_{i+1}) u''(\eta^\theta_{k+1}) - Q^\theta_{i,k;3}(x_i) u''(\eta^\theta_k)}{h_{i+1}} d\theta$$

with $\eta^\theta_k \in (x_{k-1}, x_k)$ and $\eta^\theta_{k+1} \in (x_k, x_{k+1})$. Using Lemmas 3.1 and 3.13, we have

$$\frac{1}{h_{i+1}} |T_{i+1,k+1} - T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} h_k.$$

For the third term in (3.21), using $h_i \simeq h_k$, Lemmas 3.1, A.3 and B.1, it yields

$$\begin{aligned} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\le C(r-1) h_i^{-2} h^2 x_i^{1-2/r}\, h_k^3 x_k^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\ &\le C(r-1) h^2 x_i^{-\alpha/2-2/r} h_k. \end{aligned}$$

Then, the desired result of Case 1 is obtained. The Case 2 and Case 3 for I_4 can be similarly proven as the way in Case 1; the details are omitted here. □

Proof of Theorem 3.3. For $1 \le i < N/2$ with $m = 2i$ in (3.10), combining Lemma 3.15, Lemma 3.16, Cases 1 and 2 of Lemma 3.18, Case 2 of Lemma 3.17 and Case 1 of Lemma 3.14, the proof is completed. □

Proof of Theorem 3.4. For $N/2 \le i \le N$ with $m = 2N - \lceil N/2 \rceil + 1$ in (3.10), we split I_3 as

$$I_3 = \sum_{j=k+1}^{m-1} S_{ij} = \sum_{j=k+1}^{N-1} S_{ij} + (S_{iN} + S_{i,N+1}) + \sum_{j=N+2}^{m-1} S_{ij}. \tag{3.22}$$

According to Lemma 3.16, Cases 1 and 3 of Lemma 3.18, Lemma 3.17 and Case 2 of Lemma 3.14, the desired result is obtained. □

4. Convergence analysis

We can now prove our main convergence result for Theorem 2.6.

4.1. **Some properties of the stiffness matrix.** In this subsection, we show some properties of the stiffness matrix A defined by (2.10) and construct an appropriate right-preconditioner for the resulting matrix algebraic equation.

Lemma 4.1. *The stiffness matrix A defined by* (2.10) *is strictly diagonally dominant, with positive entries on the main diagonal and negative off-diagonal entries. In particular, there exists a constant C_A such that*

$$\sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha})$$

with $C_A = \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)} 2^{-r\alpha}$.

Proof. Let $C_j := \left(\frac{1}{h_j}, -\frac{1}{h_j} - \frac{1}{h_{j+1}}, \frac{1}{h_{j+1}}\right)$ and

$$D_{ij} := \begin{pmatrix} |x_{i-1} - x_{j-1}|^{3-\alpha} & |x_{i-1} - x_j|^{3-\alpha} & |x_{i-1} - x_{j+1}|^{3-\alpha} \\ |x_i - x_{j-1}|^{3-\alpha} & |x_i - x_j|^{3-\alpha} & |x_i - x_{j+1}|^{3-\alpha} \\ |x_{i+1} - x_{j-1}|^{3-\alpha} & |x_{i+1} - x_j|^{3-\alpha} & |x_{i+1} - x_{j+1}|^{3-\alpha} \end{pmatrix}.$$

From (2.11), we have

$$a_{ij} = \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{2}{h_i + h_{i+1}} C_i D_{ij} C_j^T$$

with $\text{sign}(a_{ij}) = \text{sign}(a_{ji})$. For $i = j$, there exists

$$a_{ii} = \frac{\kappa_\alpha}{\Gamma(4-\alpha)} \frac{4}{h_i h_{i+1}} \left(h_i^{2-\alpha} + h_{i+1}^{2-\alpha} - (h_i + h_{i+1})^{2-\alpha}\right) > 0,$$

where we use $1 + t^\theta > (1+t)^\theta$ with $t = \frac{h_{i+1}}{h_i}$ for $\theta \in (0,1)$.

For $j = i-1$, we can check that

$$\begin{aligned} C_i D_{i,i-1} C_{i-1}^T = \frac{1}{h_{i-1} h_i h_{i+1}} \Big(& h_{i+1} h_{i-1}^{3-\alpha} - (h_i + h_{i+1})(h_{i-1} + h_i)^{3-\alpha} \\ & + h_i (h_{i-1} + h_i + h_{i-1})^{3-\alpha} + (h_{i-1} + h_i)(h_i + h_{i+1}) h_i^{2-\alpha} \\ & - (h_{i-1} + h_i)(h_i + h_{i+1})^{3-\alpha} + h_{i-1} h_{i+1} h_i^{2-\alpha} + h_{i-1} h_{i+1}^{3-\alpha} \Big). \end{aligned}$$

Let $s = \frac{h_{i-1}}{h_i}$ and $t = \frac{h_{i+1}}{h_i}$. Then by Lemma B.6, we have

$$\begin{aligned} C_i D_{i,i-1} C_{i-1}^T = \frac{h_i^{3-\alpha}}{h_{i-1} h_{i+1}} \Big(& st(1 + s^{2-\alpha} + t^{2-\alpha}) + (1+s+t)^{3-\alpha} \\ & - (1+s)(1+t)\left((1+s)^{2-\alpha} + (1+t)^{2-\alpha} - 1\right) \Big) > 0, \end{aligned}$$

which implies that

$$a_{i,i-1} = \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{2}{h_i + h_{i+1}} C_i D_{i,i-1} C_{i-1}^T < 0.$$

For $|i - j| \geq 2$, $x_{i+1} - y$, $x_i - y$ and $x_{i-1} - y$ have the same sign (> 0 or < 0) for $y \in (x_{j-1}, x_{j+1})$, it yields

$$\frac{h_i}{h_i + h_{i+1}} |x_{i+1} - y| + \frac{h_{i+1}}{h_i + h_{i+1}} |x_{i-1} - y| = |x_i - y|.$$

Since $x^{1-\alpha}$ is a convex function for $\alpha \in (1,2)$, by Jensen's inequality, we have

$$\frac{h_i}{h_i+h_{i+1}}|x_{i+1}-y|^{1-\alpha}+\frac{h_{i+1}}{h_i+h_{i+1}}|x_{i-1}-y|^{1-\alpha}>|x_i-y|^{1-\alpha},$$

which implies that $D_h^2 K_y(x_i)>0$ by (2.6) and (3.20). Thus, from (2.11), we get

$$a_{ij}=-D_h^2 I^{2-\alpha}\phi_j(x_i)=-\int_{x_{j-1}}^{x_{j+1}}\phi_j(y)D_h^2 K_y(x_i)dy<0.$$

We next prove that the stiffness matrix A defined by (2.10) is strictly diagonally dominant. For the quadrature coefficients $\tilde{a}_{ij}$ in (2.11), we calculate that

$$\sum_{j=1}^{2N-1}\tilde{a}_{ij}=g_0(x_i)+g_{2N}(x_i)$$

with

$$\begin{aligned} g_0(x)&=\frac{-\kappa_\alpha}{\Gamma(4-\alpha)}\frac{|x-x_0|^{3-\alpha}-|x-x_1|^{3-\alpha}}{h_1},\\ g_{2N}(x)&=\frac{-\kappa_\alpha}{\Gamma(4-\alpha)}\frac{|x_{2N}-x|^{3-\alpha}-|x_{2N-1}-x|^{3-\alpha}}{h_{2N}}. \end{aligned}$$

It implies that

$$\sum_{j=1}^{2N-1}a_{ij}=D_h^2 g_0(x_i)+D_h^2 g_{2N}(x_i).$$

For $i=1$, there exists

$$\begin{aligned} D_h^2 g_0(x_1)&=\frac{2}{h_1+h_2}\left(\frac{1}{h_2}g_0(x_2)-(\frac{1}{h_1}+\frac{1}{h_2})g_0(x_1)+\frac{1}{h_1}g_0(x_0)\right)\\ &=\frac{2\kappa_\alpha}{\Gamma(4-\alpha)}\frac{1+(2^r-1)^{3-\alpha}+2(2^r-1)-(2^r)^{3-\alpha}}{2^r(2^r-1)}x_1^{-\alpha}\\ &\geq\frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}2^{-r\alpha}x_1^{-\alpha}, \end{aligned}$$

since $h(t)=2\left(1+(t-1)^{3-\alpha}+2(t-1)-t^{3-\alpha}\right)-(3-\alpha)(2-\alpha)(\alpha-1)(t^{2-\alpha}-t^{1-\alpha})$ is a increasing function for $t=2^r\geq 1$ and $h(1)=0$.

For $i\geq 2$, using Lemma A.1, it leads to

$$\begin{aligned} D_h^2 g_0(x_i)&=g_0''(\xi),\quad \xi\in(x_{i-1},x_{i+1})\\ &=-\kappa_\alpha\frac{|\xi-x_0|^{1-\alpha}-|\xi-x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1}\\ &=\frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}|\xi-\eta|^{-\alpha},\quad \eta\in[x_0,x_1]\\ &\geq\frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}x_{i+1}^{-\alpha}\geq\frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}2^{-r\alpha}x_i^{-\alpha}. \end{aligned}$$

Then we have $D_h^2 g_0(x_i)\geq C_A x_i^{-\alpha}$ with $C_A=\frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}2^{-r\alpha}$ for $i\geq 1$. We can similarly prove $D_h^2 g_{2N}(x_i)\geq C_A(2T-x_i)^{-\alpha}$. The proof is completed. □

Let us first introduce the quasi-preconditioner

$$G = \mathrm{diag}(\delta(x_1), ..., \delta(x_{2N-1})), \tag{4.1}$$

where $\delta(x)$ is defined by (2.12). Then we have

Lemma 4.2. *Let $\tilde{B} := AG$ and A be defined by (2.10). Then the matrix $\tilde{B} = (\tilde{b}_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$ has positive entries on the main diagonal and negative off-diagonal entries. In particular, there exist constants $C_{\tilde{B}}, C_B$ such that*

$$\sum_{j=1}^{2N-1} \tilde{b}_{ij} \geq C_B(T - \delta(x_i) + h_N)^{1-\alpha} - C_{\tilde{B}}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}).$$

with $C_B = \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}$, $C_{\tilde{B}} = \frac{\kappa_\alpha}{\Gamma(2-\alpha)} 2^{r(\alpha-1)}$.

Proof. From (2.11) and (4.1), it yields

$$\tilde{b}_{ij} = a_{ij}\delta(x_j) = -\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})\tilde{a}_{i,j} + \frac{1}{h_i}\tilde{a}_{i-1,j}\right)\delta(x_j).$$

Since $\delta(x) \equiv \Pi_h\delta(x) = \sum_{j=1}^{2N-1}\phi_j(x)\delta(x_j)$ by (2.3) and (2.12), from the definition of the quadrature coefficients $\tilde{a}_{ij}$ in (2.11), we have

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij}\delta(x_j) = \sum_{j=1}^{2N-1} I^{2-\alpha}\phi_j(x_i)\delta(x_j) = I^{2-\alpha}\delta(x_i) = -p(x_i) + q(x_i)$$

with

$$p(x) = \frac{2\kappa_\alpha}{\Gamma(4-\alpha)}|T - x|^{3-\alpha} \quad \text{and} \quad q(x) = \frac{\kappa_\alpha}{\Gamma(4-\alpha)}\left(x^{3-\alpha} + (2T - x)^{3-\alpha}\right).$$

Thus, we have

$$\sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} a_{ij}\delta(x_j) = D_h^2 p(x_i) - D_h^2 q(x_i).$$

For $i \neq N$, by Lemma A.1, it leads to

$$\begin{aligned} D_h^2 p(x_i) &= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}|T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\ &\geq C_B(T - \delta(x_i) + h_N)^{1-\alpha} \text{ with } C_B = \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}, \end{aligned}$$

and for $i = N$, it yields

$$D_h^2 p(x_N) = \frac{4\kappa_\alpha}{\Gamma(4-\alpha)h_N^2}h_N^{3-\alpha} \geq C_B(T - \delta(x_N) + h_N)^{1-\alpha}.$$

We can similarly prove the following inequality.

$$D_h^2 q(x_i) \leq C_{\tilde{B}}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1$$

The proof is completed. □

Noted that $\tilde{B} = AG$ in Lemma 4.2 is not diagonally dominant, e.g., $\sum_{j=1}^{2N-1}\tilde{b}_{ij} < 0$ if x_i is near the boundary. We introduce the preconditioner $\lambda I + \mu G$ as following.

Lemma 4.3. *Let $B := A(\lambda I+\mu G)$ with $\lambda = 1+2^{r(\alpha-1)}T$, $\mu = (\alpha-1)2^{-r\alpha-1}$. Then the matrix $B = (b_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$ is strictly diagonally dominant, with positive entries on the main diagonal and negative off-diagonal entries. In particular, there exists*

$$\sum_{j=1}^{2N-1} b_{ij} \geq C_A \left((x_i^{-\alpha} + (2T-x_i)^{-\alpha}) + (T-\delta(x_i)+h_N)^{1-\alpha}\right).$$

Proof. From Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\sum_{j=1}^{2N-1} b_{ij} &= \sum_{j=1}^{2N-1}\left(\lambda a_{ij} + \mu \tilde{b}_{ij}\right)\\ &\geq \lambda C_A\left(x_i^{-\alpha} + (2T-x_i)^{-\alpha}\right) - \mu C_{\tilde{B}} 2T\left(x_i^{-\alpha} + (2T-x_i)^{-\alpha}\right)\\ &\quad + \mu C_B\left(T-\delta(x_i)+h_N\right)^{1-\alpha},\end{aligned}$$

with $\lambda = 1 + 2TC_{\tilde{B}}/C_B = 1 + 2^{r(\alpha-1)}T$ and $\mu = C_A/C_B = (\alpha-1)2^{-r\alpha-1}$. The proof is completed. □

4.2. **Proof of Theorem 2.6.** Let $\epsilon_i = u(x_i) - u_i$ with $\epsilon_0 = \epsilon_{2N} = 0$. Subtracting (2.7) from (2.9), we get

$$A\epsilon = \tau, \tag{4.2}$$

where $\epsilon = [\epsilon_1, \epsilon_2, ..., \epsilon_{2N-1}]^T$ and $\tau = [\tau_1, \tau_2, ..., \tau_{2N-1}]^T$ with τ_i in (2.8).

Let $\lambda I + \mu G$ be the right-preconditioner and $B = A(\lambda I + \mu G)$ defined in Lemma 4.3. Then we can rewrite (4.2) as

$$B(\lambda I + \mu G)^{-1}\epsilon = \tau, \quad \text{i.e.} \quad \sum_{j=1}^{2N-1} b_{ij}\frac{\epsilon_j}{\lambda+\mu\delta(x_j)} = \tau_i.$$

Assume that

$$\left|\frac{\epsilon_{i_0}}{\lambda+\mu\delta(x_{i_0})}\right| = \max_{1\leq j\leq 2N-1}\left|\frac{\epsilon_j}{\lambda+\mu\delta(x_j)}\right|.$$

From Lemma 4.3 with $b_{ii} > 0$ and $b_{ij} < 0, i \neq j$, it yields

$$\begin{aligned}|\tau_{i_0}| &= \left|\sum_{j=1}^{2N-1} b_{i_0,j}\frac{\epsilon_j}{\lambda+\mu\delta(x_j)}\right|\\ &\geq b_{i_0,i_0}\left|\frac{\epsilon_{i_0}}{\lambda+\mu\delta(x_{i_0})}\right| - \sum_{j\neq i_0}|b_{i_0,j}|\left|\frac{\epsilon_j}{\lambda+\mu\delta(x_j)}\right|\\ &\geq b_{i_0,i_0}\left|\frac{\epsilon_{i_0}}{\lambda+\mu\delta(x_{i_0})}\right| - \sum_{j\neq i_0}|b_{i_0,j}|\left|\frac{\epsilon_{i_0}}{\lambda+\mu\delta(x_{i_0})}\right|\\ &= \sum_{j=1}^{2N-1} b_{i_0,j}\left|\frac{\epsilon_{i_0}}{\lambda+\mu\delta(x_{i_0})}\right|.\end{aligned}$$

According to the above inequality, Theorem 2.5 and Lemma 4.3, we have

$$\left|\frac{\epsilon_i}{\lambda+\mu\delta(x_i)}\right| \leq \left|\frac{\epsilon_{i_0}}{\lambda+\mu\delta(x_{i_0})}\right| \leq \frac{|\tau_{i_0}|}{\sum_{j=1}^{2N-1} b_{i_0,j}} \leq Ch^{\min\{\frac{r\alpha}{2},2\}} + C(r-1)h^2.$$

Since $\lambda+\mu\delta(x_i)\le\lambda+\mu T$, we can derive

$$|\epsilon_i|\le C(\lambda+\mu T)h^{\min\{\frac{r\alpha}{2},2\}}\le C\left(1+(2^{r(\alpha-1)}+(\alpha-1)2^{-r\alpha-1})T\right)h^{\min\{\frac{r\alpha}{2},2\}}.$$

The proof is completed.

Remark 4.4. Let $B=A(\lambda I+\mu G)$ with $\mu=0$ in the proof of Theorem 2.6, which means that there is no preconditioning. Thus, from Theorem 2.5 and Lemma 4.1, we can only prove

$$|\epsilon_i|\le Ch^{\min\{\frac{r\alpha}{2},2\}}+C(r-1)h^{3-\alpha}\le Ch^{\min\{\frac{r\alpha}{2},3-\alpha\}},\quad 1<\alpha<2,$$

which may suffer from a severe order reduction.

Remark 4.5 (singular source term). From Lemma 2.4, it follows that the source term $|f(x)|\le C\delta(x)^{-\alpha/2}$ could potentially be singular. Suppose that the bound of Lemma 2.4 is replaced by the more general weaker regularity bound, i.e.,

$$\begin{aligned}&|u^{(l)}(x)|\le C\delta(x)^{\sigma-l}\quad \text{for } x\in\Omega \text{ and } l=0,1,2,3,4,\\ &|f^{(l)}(x)|\le C\delta(x)^{\sigma-\alpha-l} \text{ for } x\in\Omega \text{ and } l=0,1,2,\end{aligned}$$

where $\sigma\in(0,\frac{\alpha}{2}]$ is fixed.

Similar to the performer in Theorems 2.5 and 2.6, it is easy to check the local truncation error

$$\begin{aligned}|\tau_i|&=|-D_h^\alpha\Pi_h u(x_i)-f(x_i)|\\ &\le Ch^{\min\{r\sigma,2\}}\delta(x_i)^{-\alpha}+C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha},\end{aligned}$$

and the global error

$$\max_{1\le i\le 2N-1}|u_i-u(x_i)|\le Ch^{\min\{r\sigma,2\}}.$$

5. Numerical experiments

We use the difference-quadrature scheme (2.10) to solve the fractional Laplacian boundary value problem (1.1) with both regular and singular source terms on the interval $\Omega=(0,1)$.

5.1. **Regular source term.**

If $f\equiv 1$, the exact (Getoor) solution [18, 21, 26] of the problem (1.1) is

$$u(x)=\frac{2^{-\alpha}\Gamma(\frac{1}{2})}{\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{1+\alpha}{2})}\left[x(1-x)\right]^{\frac{\alpha}{2}},\quad x\in\Omega.$$

In the numerical experiments of this example, we measure the numerical errors by using the maximum nodal error (i.e., the discrete L^∞ norm):

$$E^N:=\max_{0\le i\le 2N}|u(x_i)-u_i|.$$

The rate of convergence of E^N is computed in the usual way, viz.,

$$Rate^N=\log_2\left(\frac{E^{N/2}}{E^N}\right).$$

Tables 1 and 2 show that the difference-quadrature method (2.10) has convergence order $\mathcal{O}(h^{\min\{\frac{r\alpha}{2},2\}})$, which agrees exactly with Theorem 2.6.

TABLE 1. $r = 1$: maximum nodal errors showing convergence rate $\mathcal{O}(h^{\frac{\alpha}{2}})$

α \ N	100	200	400	800
1.2	1.1269e-3	7.4281e-4	4.8986e-4	3.2311e-4
		0.6013	0.6006	0.6003
1.5	2.4996e-4	1.4876e-4	8.8489e-5	5.2627e-5
		0.7487	0.7494	0.7497
1.8	2.7320e-5	1.4829e-5	7.9970e-6	4.2989e-6
		0.8815	0.8909	0.8955

TABLE 2. $r = \frac{4}{\alpha}$: maximum nodal errors showing convergence rate $\mathcal{O}(h^2)$

α \ N	100	200	400	800
1.2	4.1583e-5	1.0628e-5	2.6919e-6	6.7824e-7
		1.9682	1.9811	1.9888
1.5	2.0681e-5	5.3790e-5	1.3824e-6	3.5239e-7
		1.9429	1.9601	1.9720
1.8	7.6424e-6	2.0649e-6	5.5008e-7	1.4495e-7
		1.8880	1.9083	1.9240

5.2. **Singular source term.** We take the singular source term $f(x) = x^{\sigma-\alpha}$, $\sigma \in (0, \frac{\alpha}{2}]$ with $\sigma = 0.4$ in (1.1). Since the analytic solution is unknown, the convergence rate of the numerical results is computed by

$$Rate^N = \log_2\left(\frac{E^{N/2}}{E^N}\right) \quad \text{with} \quad E^N = \max_{0\le i\le N} |u_i^{N/2} - u_{2i}^N|.$$

TABLE 3. $r = 1$:maximum nodal errors showing convergence rate $\mathcal{O}(h^{\sigma})$

α \ N	100	200	400	800
1.2	2.9193e-2	2.2619e-2	1.7435e-2	1.3395e-2
		0.3681	0.3755	0.3804
1.5	4.0497e-2	3.1068e-2	2.3717e-2	1.8057e-2
		0.3824	0.3895	0.3934
1.8	5.6776e-2	4.3468e-2	3.3112e-2	2.5161e-2
		0.3853	0.3926	0.3962

TABLE 4. $r = \frac{2}{\sigma}$:maximum nodal errors showing convergence rate $\mathcal{O}(h^2)$

α \ N	100	200	400	800
1.2	2.7820e-3	6.9631e-4	1.7418e-4	4.3557e-5
		1.9983	1.9992	1.9996
1.5	3.1742e-3	8.0150e-4	2.0223e-4	5.0947e-5
		1.9856	1.9867	1.9889
1.8	5.0311e-3	1.3190e-3	3.4157e-4	8.7691e-5
		1.9315	1.9492	1.9617

Tables 3 and 4 show that the difference-quadrature method (2.10) has convergence order $\mathcal{O}(h^{\min\{r\sigma,2\}})$, which agrees with Remark 4.5.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant No. 12471381 and Science Fund for Distinguished Young Scholars of Gansu Province under Grant No. 23JRRA1020.

Appendix A. Approximations of difference and interpolation

In this appendix, we provide some approximations for the second-order difference quotients D_h^2 and the interpolation error $u(x) - \Pi_h u(x)$.

Lemma A.1. *Let D_h^2 be the difference quotient operator defined by* (2.6)*. If $g(x) \in C(\bar{\Omega}) \cap C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$, $i = 1, 2, ..., 2N-1$, such that*

$$D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}).$$

Moreover, if $g(x) \in C(\bar{\Omega}) \cap C^4(\Omega)$, then we have

$$\begin{aligned} &D_h^2 g(x_i) = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right). \end{aligned}$$

Proof. By Taylor series expansion, we obtain

$$\begin{aligned} g(x_{i-1}) &= g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i), \\ g(x_{i+1}) &= g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1}). \end{aligned}$$

From (2.6) and the above equations, it yields

$$\begin{aligned} D_h^2 g(x_i) &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \left(g(x_{i+1}) - g(x_i) \right) + \frac{1}{h_i} \left(g(x_{i-1}) - g(x_i) \right) \right) \\ &= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2). \end{aligned}$$

According to intermediate value theorem, there exists some $\xi \in [\xi_1, \xi_2]$ such that

$$D_h^2 g(x_i) = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi).$$

The second equation can also be derived in a similar manner through Taylor expansion. The proof is completed. □

Lemma A.2. *Let $y_j^\theta = (1-\theta) x_{j-1} + \theta x_j, \theta \in (0,1)$ with $2 \leq j \leq 2N-1$. Then we have*

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j),$$

and

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 \left(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2) \right)$$

with $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$.

Proof. Using Taylor series expansion, we get

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta),$$
$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j),$$

which implies that

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta) u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2]. \end{aligned}$$

The second equation can be similarly obtained by

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1),$$
$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

with $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. The proof is completed. □

Lemma A.3. *For any $y \in (x_{j-1}, x_j)$, $2 \le j \le 2N-1$, there exists*

$$|u(y) - \Pi_h u(y)| \le h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \le C h^2 \delta(y)^{\alpha/2 - 2/r}.$$

Proof. According to Lemmas 2.4, 3.1 and A.2, the desired result is obtained. □

Lemma A.4. *For any $x \in [x_{j-1}, x_j]$, $1 \le j \le 2N$, there exists*

$$|u(x) - \Pi_h u(x)| = \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y) dy \right| \le \int_{x_{j-1}}^{x_j} |u'(y)| dy.$$

In particular, there exist,

$$|u(x) - \Pi_h u(x)| \le C \frac{2}{\alpha} h_1^{\alpha/2}, \quad x \in (0, x_1) \cup (x_{2N-1}, 2T).$$

Proof. From the definition of $\Pi_h u(x)$ in (2.3) and using $u(x) = u(x_i) + \int_{x_i}^{x} u'(y) dy$, the proof is completed. □

Appendix B. Bound estimates

Set $h_i = x_i - x_{i-1}$ for $j = 1, 2, ..., 2N$ and define $h := \frac{1}{N}$. The following bounds are needed in several places.

Lemma B.1. *For $i = 1, 2, \cdots, 2N-1$, there exists a constant C such that*

$$|h_{i+1} - h_i| \le C(r-1) h^2 \delta(x_i)^{1-2/r}, \quad r \ge 1.$$

Proof. According to the definition of $h_i = x_i - x_{i-1}$ as defined in (2.1), we obtain

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1, \\ 0, & i = N, \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1. \end{cases}$$

Since $(i+1)^r - 2i^r + (i-1)^r \simeq r(r-1) i^{r-2}$ for $i \ge 1$, the desired result is obtained. □

Lemma B.2. *For $1 \leq i \leq 2N-1$, there exists a constant C such that*

$$\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y-x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y-x_{i+1})^3}{3!} dy \right|$$
$$\leq Ch^2 \delta(x_i)^{-\alpha/2-2/r}.$$

Proof. By Lemma 2.4, for $1 \leq i \leq 2N-1$, we have

$$\left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y-x_{i-1})^3}{3!} dy \right| \leq \frac{\|f\|_\beta^{(\alpha/2)}}{3!} \int_{x_{i-1}}^{x_i} \delta(y)^{-\alpha/2-2} (y-x_{i-1})^3 dy.$$

For $i = 1$, we get

$$\int_{x_{i-1}}^{x_i} \delta(y)^{-\alpha/2-2} (y-x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2} dy = \frac{1}{2-\alpha/2} x_1^{2-\alpha/2} \simeq x_1^{-\alpha/2-2} h_1^4.$$

For $2 \leq i \leq 2N-1$, by Lemma 3.1, we have

$$\int_{x_{i-1}}^{x_i} \delta(y)^{-\alpha/2-2} (y-x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} \delta(x_i)^{-\alpha/2-2} (y-x_{i-1})^3 dy \simeq \delta(x_i)^{-\alpha/2-2} h_i^4$$

The desired result is obtained. □

Lemma B.3. *For all $1 \leq i \leq 2N-1$, $1 \leq j \leq 2N$, $y \in (x_{j-1}, x_j)$, there exist*

$$|D_h K_y(x_i)| \simeq |x_i - y|^{-\alpha} \quad \text{if} \quad [x_{j-1}, x_j] \cap [x_i, x_{i+1}] = \varnothing,$$
$$D_h^2 K_y(x_i) \simeq |x_i - y|^{-1-\alpha} \quad \text{if} \quad [x_{j-1}, x_j] \cap [x_{i-1}, x_{i+1}] = \varnothing.$$

Proof. Since $x_{i-1} - y$, $x_i - y$ and $x_{i+1} - y$ have the same sign, using Lemma A.1 and $K_y(x) = \frac{\kappa_\alpha}{\Gamma(2-\alpha)} |x-y|^{1-\alpha}$ in (3.20), it yields

$$|D_h K_y(x_i)| = \frac{\kappa_\alpha (\alpha - 1)}{\Gamma(2-\alpha)} |\xi - y|^{-\alpha}, \quad \xi \in (x_i, x_{i+1}),$$
$$D_h^2 K_y(x_i) = \frac{\kappa_\alpha \alpha (\alpha - 1)}{\Gamma(2-\alpha)} |\xi - y|^{-1-\alpha}, \quad \xi \in (x_{i-1}, x_{i+1}).$$

Moreover, from $|\xi - y| \simeq |x_i - y|$, the desired result is obtained. □

Lemma B.4. *There exists a constant C such that*

$$\sum_{j=1}^{3} V_{1j} \leq Ch^2 x_1^{-\alpha/2-2/r} \quad \text{and} \quad \sum_{j=1}^{4} V_{2j} \leq Ch^2 x_2^{-\alpha/2-2/r}.$$

Proof. According Lemma A.4, Lemma A.3, (3.6), (3.7), it implies,

$$T_{ij} \leq C x_1^{2-\alpha/2} \simeq h_1^2 \, h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2 \, h^2 x_2^{-\alpha/2-2/r} \quad \text{for} \quad 0 \leq i \leq 3, 1 \leq j \leq 4.$$

The proof is completed. □

Lemma B.5. *Let $a, b > 0$, $\theta \in [0, 1]$. Then we have*

$$b^{1-\theta} |a^\theta - b^\theta| \leq |a - b|.$$

Proof. Since $|t^\theta - 1| \leq |t - 1|$ with $t = \frac{a}{b} > 0$, the proof is completed. □

Lemma B.6. *Let $x > 0, y \geq 1$ with $\alpha \in (1, 2)$. Then we have*

$$f(x,y) = xy(1 + x^{2-\alpha} + y^{2-\alpha}) + (1 + x + y)^{3-\alpha}$$
$$- (1+x)(1+y)\left((1+x)^{2-\alpha} + (1+y)^{2-\alpha} - 1\right) > 0.$$

Proof. The first and second derivatives of $f(x,y)$ with respect to x are

$$\begin{aligned}\partial_x f(x,y) =&(3-\alpha)\left[x^{2-\alpha}y+(1+x+y)^{2-\alpha}-(1+x)^{2-\alpha}(1+y)\right]\\&+1+2y+y^{3-\alpha}-(1+y)^{3-\alpha},\\ \partial_x^2 f(x,y) =& (3-\alpha)(2-\alpha)\left(yx^{1-\alpha}+(1+x+y)^{1-\alpha}-(1+y)(1+x)^{1-\alpha}\right).\end{aligned}$$

Since $\frac{y}{1+y}x+\frac{1}{1+y}(1+x+y)=1+x$ and $x^{1-\alpha}$ is convex for $x>0$, using Jensen's inequality, it yields

$$\frac{y}{1+y}x^{1-\alpha}+\frac{1}{1+y}(1+x+y)^{1-\alpha}>(1+x)^{1-\alpha},$$

which implies $\partial_x^2 f(x,y)>0$ and $\partial_x f(x,y)>\partial_x f(0,y)$.

Since $\partial_x f(0,y)>0$ for $y\geq 1$, we have $f(x,y)>f(0,y)=0$. The proof is completed. □

Appendix C. Proofs for grid mapping functions

In this appendix, we provide the proofs of Lemmas 3.9-3.13 in subsection 3.2.

Proof of Lemma 3.9. The first two approximations can be derived from (2.1) and (3.17) with $2\leq i,j\leq 2N-2$.

We next prove $|y_{i,j}(\xi)-\xi|\simeq|x_j-x_i|$. From (3.12), we have $y_{i,j}(\xi)-\xi=0$ if $i=j$. Without loss of generality, if $i<j$, then $y_{i,j}(\xi)-\xi\leq x_{j+1}-x_{i-1}\simeq x_j-x_i$. Since the second derivatives of $|y_{i,j}(\xi)-\xi|$ is less than zero by Lemma 3.8, which implies it is concave. Thus, $|y_{i,j}(\xi)-\xi|\geq\min\{x_{j-1}-x_{i-1},x_{j+1}-x_{i+1}\}\simeq|x_j-x_i|$.

From (3.12), (3.13), (3.17), using the approximation above, there exists

$$h_{i,j}(\xi)=y_{i,j}(\xi)-y_{i,j-1}(\xi)=y_{j-1,j}(y_{i,j-1}(\xi))-y_{i,j-1}(\xi)\simeq x_j-x_{j-1}=h_j.$$

The final estimate can be obtained since $y_{i,j-1}(\xi)-\xi$, $y_{i,j}(\xi)-\xi$ have the same sign (≥ 0 or ≤ 0). □

Proof of Lemma 3.10. From (3.13) and Lemma 3.8, we can see that

$$\begin{aligned}&h'_{i,j}(x)=y'_{i,j}(x)-y'_{i,j-1}(x)\\&=\begin{cases}x^{1/r-1}\left(y_{i,j}^{1-1/r}(x)-y_{i,j-1}^{1-1/r}(x)\right), & i<N,j<N,\\ x^{1/r-1}\left(\dfrac{h_N}{rZ_1}-y_{i,N-1}^{1-1/r}(x)\right), & i<N,j=N,\\ x^{1/r-1}\left((2T-y_{i,N+1}(x))^{1-1/r}-\dfrac{h_N}{rZ_1}\right), & i<N,j=N+1,\\ x^{1/r-1}\left((2T-y_{i,j}(x))^{1-1/r}-(2T-y_{i,j-1}(x))^{1-1/r}\right), & i<N,j>N+1,\\ \dfrac{rZ_1}{h_N}\left(y_{N,j}^{1-1/r}(x)-y_{N,j-1}^{1-1/r}(x)\right), & i=N,j<N,\\ \dfrac{rZ_1}{h_N}\left(\dfrac{h_N}{rZ_1}-y_{N,N-1}^{1-1/r}(x)\right), & i=N,j=N.\end{cases}\end{aligned}$$

For $2\leq i\leq N$, $2\leq j<N$, it yields

$$\begin{aligned}y_{i,j}^{1-1/r}(\xi)-y_{i,j-1}^{1-1/r}(\xi)&\leq x_{j+1}^{1-1/r}-x_{j-2}^{1-1/r}\\&=T^{1-1/r}N^{1-r}\left((j+1)^{r-1}-(j-2)^{r-1}\right)\\&\leq CT^{1-1/r}(r-1)N^{1-r}j^{r-2}=C(r-1)Z_1x_j^{1-2/r}.\end{aligned}\tag{C.1}$$

For $2 \le i \le N$, $j = N$, since

$$\frac{h_N}{rZ_1} = T^{1-1/r}\frac{1-(1-h)^r}{rh} = \eta^{1-1/r} \simeq x_N^{1-1/r}, \quad \eta \in (x_{N-1}, x_N), \tag{C.2}$$

we have

$$\left|\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(\xi)\right| \le x_N^{1-1/r} - x_{N-2}^{1-1/r} \simeq (r-1)Z_1 x_N^{1-2/r}.$$

For $2 \le i \le N$, $N+1 \le j \le 2N-2$, it can be checked

$$|h'_{i,j}(\xi)| \le C(r-1)Z_1\xi^{1/r-1}(2T - x_j)^{1-2/r}.$$

Combine with Lemmas 3.1 and 3.9, the first inequality is obtained.

On the other hand, from (3.12), we have $|y_{i,j}(x) - x| = \mathrm{sign}(j-i)(y_{i,j}(x) - x)$ and $(y_{i,j}(x) - x)' = y'_{i,j}(x) - 1$.

For $2 \le i < N$, $2 \le j < N$, by Lemmas 3.9 and B.5, we have

$$\xi^{1/r}|y_{i,j}^{1-1/r}(\xi) - \xi^{1-1/r}| \le |y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|.$$

For $2 \le i < N$, $j = N$, using (C.2) and Lemma B.5, it yields

$$\begin{aligned} \eta^{1/r}\left|\frac{h_N}{rZ_1} - \xi^{1-1/r}\right| &\le |\eta - \xi|, \quad \eta \in (x_{N-1}, x_N) \\ &\le |x_N - x_i| + |h_N| + |h_{i+1}| \le 3|x_N - x_i|. \end{aligned} \tag{C.3}$$

For $2 \le i < N$, $N < j \le 2N-2$, from Lemma B.5, one has

$$\begin{aligned} \xi^{1/r}|(2T - y_{i,j}(\xi))^{1-1/r} - \xi^{1-1/r}| &\le |2T - y_{i,j}(\xi) - \xi| \\ &\le |2T - x_j - x_i| + |y_{i,j}(\xi) - x_j| + |\xi - x_i| \le |2T - x_j - x_i| + 2h_N \\ &\le |x_j - T| + |T - x_i| + 2h_N \le 2|x_j - x_i|. \end{aligned}$$

Similar to proof of (C.3), we have

$$\eta^{1/r}\left|y_{N,j}^{1-1/r}(\xi) - \frac{h_N}{rZ_1}\right| \le C|x_j - x_N|, \quad i = N, j < N,$$

$$\eta^{1/r}\left|(2T - y_{N,j}(\xi))^{1-1/r} - \frac{h_N}{rZ_1}\right| \le C|x_j - x_N|, \quad i = N, j > N,$$

and $y_{N,N}(x) - x \equiv 0$.

Thus, using Lemmas 3.8 and 3.9, the second inequality is obtained. □

Proof of Lemma 3.11. By Definition 3.7, Lemma B.5 and (C.2), there exist

$$\begin{aligned} &x_j^{1-1/r}|Z_{j-i}| = x_j^{1-1/r}|x_j^{1/r} - x_i^{1/r}| \le |x_j - x_i|, \quad i < N, j < N, \\ &Z_{2N-j+i} \le Z_{2N} = 2T^{1/r}, \qquad i < N, j > N, \\ &\frac{h_N}{rZ_1} \simeq x_N^{1-1/r}, \qquad i = N, 2 \le j \le 2N-2. \end{aligned} \tag{C.4}$$

Combine with Lemmas 3.8 and 3.9, the first inequality is obtained.

From (3.13), it yields $h''_{i,j}(x) = y''_{i,j}(x) - y''_{i,j-1}(x)$. For $3 \le j \le 2N-2$, we have

$$\begin{aligned} |y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)| &\simeq (r-2)Z_1 x_j^{1-3/r}, \\ |(2T - y_{i,j}(\xi))^{1-2/r} - (2T - y_{i,j-1}(\xi))^{1-2/r}| &\simeq (r-2)Z_1(2T - x_j)^{1-3/r}, \end{aligned} \tag{C.5}$$

which can be similarly proven as (C.1).

For $2 \le i < N$, $3 \le j < N$, it yields

$$y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1} = \left(y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)\right) Z_{j-i} + y_{i,j-1}^{1-2/r}(\xi)Z_1.$$

Combine with (C.4) and (C.5), we get

$$|y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1}| \le CZ_1 \left(|r-2|x_j^{-2/r}|x_j - x_i| + x_j^{1-2/r}\right).$$

For $2 \le i < N$, $j = N, N+1$, it leads to

$$|h''_{i,j}(x)| \le |y''_{i,j}(x)| + |y''_{i,j-1}(x)| \le C(r-1)x_i^{1/r-2}x_N^{1-1/r}.$$

For $2 \le i < N$, $j > N+1$, from Lemma 3.1 and (C.5), we have

$$\begin{aligned}
&\left|\delta(y_{i,j}(\xi))^{1-2/r}Z_{2N-(j-i)} - \delta(y_{i,j-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}\right| \\
&= \left|\left(\delta(y_{i,j}(\xi))^{1-2/r} - \delta(y_{i,j-1}(\xi))^{1-2/r}\right) Z_{2N-(j-i)} - \delta(y_{i,j-1}(\xi))^{1-2/r}Z_1\right| \\
&\le CZ_1 \left(|r-2|\delta(x_j)^{1-3/r}x_N^{1/r} + \delta(x_j)^{1-2/r}\right) \le CZ_1\delta(x_j)^{1-3/r}x_N^{1/r}.
\end{aligned}$$

For $i = N$, one has

$$|h''_{N,j}(\xi)| = \begin{cases} |y''_{N,N-1}(\xi)|, & j = N, \\ |y''_{N,N+1}(\xi)|, & j = N+1 \end{cases} \le Cx_N^{-1}.$$

For $i = N$, $j \ne N, N+1$, using (C.5), Lemmas 3.8 and 3.9, the second inequality is obtained. □

Proof of Lemma 3.12. According to $|y_{i,j}^\theta(\xi) - \xi| = \mathrm{sign}(j-i-1+\theta)(y_{i,j}^\theta(\xi) - \xi)$ with $\theta \in (0,1)$, Lemma A.1, (1.3) and (3.15), we have

$$D_h^2 P_{i,j}^\theta(x_i) = {P_{i,j}^\theta}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}).$$

From Lemmas 2.4, 3.1 and 3.8 to 3.11, and regarding the selection process of i, j within Case 1-3, it turns out that

$$h_{i,j}(\xi) \le Ch_j, \quad |h'_{i,j}(\xi)| \le C(r-1)h_jx_i^{-1},$$

$$|y_{i,j}^\theta(\xi) - \xi| \le C|y_j^\theta - x_i|, \quad \left|(y_{i.j}^\theta(\xi) - x_i)'\right| \le C|y_j^\theta - x_i|x_i^{-1},$$

$$\left|u''(y_{i,j}^\theta(\xi))\right| \le Cx_i^{\alpha/2-2}, \quad \left|\left(u''(y_{i,j}^\theta(\xi))\right)'\right| \le Cx_i^{\alpha/2-3}, \quad \left|\left(u''(y_{i,j}^\theta(\xi))\right)''\right| \le Cx_i^{\alpha/2-4}.$$

By Lemma 3.11, we have

$$\begin{aligned}
&|h''_{i,j}(\xi)| \le C(r-1)h_jx_i^{-2}, && \left|(y_{i,j}^\theta(\xi) - x_i)''\right| \le C(r-1)|y_j^\theta - x_i|x_i^{-2}, && \text{for Case 1,} \\
&|h''_{i,j}(\xi)| \le C(r-1), && \left|(y_{i,j}^\theta(\xi) - x_i)''\right| \le C(r-1), && \text{for Case 2,} \\
&|h''_{i,j}(\xi)| \le C(r-1)h_j, && \left|(y_{i,j}^\theta(\xi) - x_i)''\right| \le C(r-1), && \text{for Case 3.}
\end{aligned}$$

Using Leibniz formula and chain rules, the desired results are obtained. □

Proof of Lemma 3.13. Since

$$\begin{aligned}
&\frac{Q_{i,j,l}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{i,j,l}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\
&\quad = \frac{Q_{i,j,l}^\theta(x_{i+1}) - Q_{i,j,l}^\theta(x_i)}{h_{i+1}}u'''(\eta_{j+1}^\theta) + Q_{i,j,l}^\theta(x_i)\frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}}.
\end{aligned}$$

Using mean value theorem, it yields

$$\frac{Q^{\theta}_{i,j,l}(x_{i+1}) - Q^{\theta}_{i,j,l}(x_i)}{h_{i+1}} = {Q^{\theta}_{i,j,l}}'(\xi), \quad \xi \in (x_i, x_{i+1}).$$

From (3.16), Lemmas 3.1, 3.9 and 3.10, Leibniz formula and chain rule, we have

$$|{Q^{\theta}_{i,j,l}}'(\xi)| \leq C h_j^l |y_j^{\theta} - x_i|^{1-\alpha} (x_i^{-1} + x_i^{1/r-1}\delta(x_j)^{-1/r}),$$

$$Q^{\theta}_{i,j,l}(x_i) = C h_j^l |y_j^{\theta} - x_i|^{1-\alpha}.$$

According to Lemmas 2.4 and 3.1, it implies

$$|u^{(l-1)}(\eta^{\theta}_{j+1})| \leq C(\eta^{\theta}_{j+1})^{\alpha/2-l+1} \simeq \delta(x_j)^{\alpha/2-l+1},$$

and

$$\begin{aligned}\frac{|u^{(l-1)}(\eta^{\theta}_{j+1}) - u^{(l-1)}(\eta^{\theta}_{j})|}{h_{i+1}} &= |u^{(l)}(\eta)| \frac{\eta^{\theta}_{j+1} - \eta^{\theta}_{j}}{h_{i+1}}, \quad \eta \in (x_{j-1}, x_{j+1}) \\ &\leq C\delta(\eta)^{\alpha/2-l} \frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\delta(\eta)^{\alpha/2-l} \frac{h_{j+1} + h_j}{h_{i+1}} \\ &\simeq x_i^{1/r-1} \delta(x_j)^{\alpha/2-l+1-1/r}.\end{aligned}$$

Thus, the first inequality is obtained. The second one can be similarly proven as the way provided above. □

References

[1] Gabriel Acosta, Juan Pablo Borthagaray, Oscar Bruno, and Martín Maas. Regularity theory and high order numerical methods for the (1D)-fractional Laplacian. *Math. Comp.*, 87(312):1821–1857, 2018. 1, 1

[2] Fuensanta Andreu-Vaillo, José M. Mazón, Julio D. Rossi, and J. Julián Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010. 1

[3] Jean Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. 1

[4] Weiping Bu, Yifa Tang, and Jiye Yang. Galerkin finite element method for two-dimensional Riesz space fractional diffusion equations. *J. Comput. Phys.*, 276:26–38, 2014. 1

[5] Cem Çelik and Melda Duman. Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative. *J. Comput. Phys.*, 231(4):1743–1750, 2012. 1

[6] Minghua Chen and Weihua Deng. Fourth order accurate scheme for the space fractional diffusion equations. *SIAM J. Numer. Anal.*, 52(3):1418–1438, 2014. 1

[7] Minghua Chen and Weihua Deng. A second-order numerical method for two-dimensional two-sided space fractional convection diffusion equation. *Appl. Math. Model.*, 38(13):3244–3259, 2014. 1

[8] Minghua Chen and Weihua Deng. High order algorithms for the fractional substantial diffusion equation with truncated Lévy flights. *SIAM J. Sci. Comput.*, 37(2):A890–A917, 2015. 1

[9] Minghua Chen, Weihua Deng, Chao Min, Jiankang Shi, and Martin Stynes. Error analysis of a collocation method on graded meshes for a fractional Laplacian problem. *Adv. Comput. Math.*, 50(3):Paper No. 49, 27, 2024. 1

[10] Minghua Chen, Weihua Deng, and Yujiang Wu. Superlinearly convergent algorithms for the two-dimensional space-time Caputo-Riesz fractional diffusion equation. *Appl. Numer. Math.*, 70:22–41, 2013. 1

[11] Minghua Chen, Yantao Wang, Xiao Cheng, and Weihua Deng. Second-order LOD multigrid method for multidimensional Riesz fractional diffusion equation. *BIT*, 54(3):623–647, 2014. 1

[12] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez. A fractional porous medium equation. *Adv. Math.*, 226(2):1378–1409, 2011. 1

[13] Beichuan Deng, Zhimin Zhang, and Xuan Zhao. Superconvergence points for the spectral interpolation of Riesz fractional derivatives. *J. Sci. Comput.*, 81(3):1577–1601, 2019. 1
[14] Weihua Deng. Finite element method for the space and time fractional Fokker-Planck equation. *SIAM J. Numer. Anal.*, 47(1):204–226, 2008/09. 1
[15] Hengfei Ding and Changpin Li. High-order algorithms for Riesz derivative and their applications (IV). *Fract. Calc. Appl. Anal.*, 22(6):1537–1560, 2019. 1
[16] V. J. Ervin, N. Heuer, and J. P. Roop. Regularity of the solution to 1-D fractional order diffusion equations. *Math. Comp.*, 87(313):2273–2294, 2018. 1
[17] Paolo Gatto and Jan S. Hesthaven. Numerical approximation of the fractional Laplacian via *hp*-finite elements, with an application to image denoising. *J. Sci. Comput.*, 65(1):249–270, 2015. 1
[18] R. K. Getoor. First passage times for symmetric stable processes in space. *Trans. Amer. Math. Soc.*, 101:75–90, 1961. 1, 1, 5.1
[19] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977. 2.2
[20] Rubing Han and Shuonan Wu. A monotone discretization for integral fractional Laplacian on bounded Lipschitz domains: pointwise error estimates under Hölder regularity. *SIAM J. Numer. Anal.*, 60(6):3052–3077, 2022. 1
[21] Yanghong Huang and Adam Oberman. Numerical methods for the fractional Laplacian: a finite difference–quadrature approach. *SIAM J. Numer. Anal.*, 52(6):3056–3084, 2014. 1, 1, 5.1
[22] Mateusz Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.*, 20(1):7–51, 2017. 1
[23] Randall J. LeVeque. *Finite difference methods for ordinary and partial differential equations*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007. Steady-state and time-dependent problems. 2.1
[24] Anna Lischke, Guofei Pang, Mamikon Gulian, and et al. What is the fractional Laplacian? A comparative review with new results. *J. Comput. Phys.*, 404:109009, 62, 2020. 1, 1
[25] Ralf Metzler and Joseph Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, 339(1):77, 2000. 1
[26] Xavier Ros-Oton and Joaquim Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9)*, 101(3):275–302, 2014. 1, 5.1
[27] Xavier Ros-Oton and Joaquim Serra. The dirichlet problem for the fractional laplacian: Regularity up to the boundary. *Journal de Mathématiques Pures et Appliquées*, 101(3):275–302, 2014. 2.1, 2.2, 2.3
[28] Zi-Hang She, Cheng-Xue Lao, Hong Yang, and Fu-Rong Lin. Banded preconditioners for Riesz space fractional diffusion equations. *J. Sci. Comput.*, 86(3):Paper No. 31, 22, 2021. 1
[29] S. Shen, F. Liu, V. Anh, and I. Turner. The fundamental solution and numerical solution of the Riesz fractional advection-dispersion equation. *IMA J. Appl. Math.*, 73(6):850–872, 2008. 1
[30] Jiankang Shi and Minghua Chen. High-order BDF convolution quadrature for subdiffusion models with a singular source term. *SIAM J. Numer. Anal.*, 61(6):2559–2579, 2023. 1
[31] Jiankang Shi, Minghua Chen, and Jianxiong Cao. High-order BDF convolution quadrature for fractional evolution equations with hyper-singular source term. *J. Sci. Comput.*, 101(1):Paper No. 9, 22, 2024. 1
[32] Ercília Sousa and Can Li. A weighted finite difference method for the fractional diffusion equation based on the Riemann-Liouville derivative. *Appl. Numer. Math.*, 90:22–37, 2015. 1
[33] Vasily Tarasov. *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*. Springer Berlin, Heidelberg, 10 2011. 1
[34] Vasily E. Tarasov. *Fractional dynamics*. Nonlinear Physical Science. Springer, Heidelberg; Higher Education Press, Beijing, 2010. Applications of fractional calculus to dynamics of particles, fields and media. 1
[35] WenYi Tian, Han Zhou, and Weihua Deng. A class of second order difference approximations for solving space fractional diffusion equations. *Math. Comp.*, 84(294):1703–1727, 2015. 1
[36] Enrico Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA*, (49):33–44, 2009. 1

[37] Nan Wang, Zhiping Mao, Chengming Huang, and George Em Karniadakis. A spectral penalty method for two-sided fractional differential equations with general boundary conditions. *SIAM J. Sci. Comput.*, 41(3):A1840–A1866, 2019. 1

[38] Q. Yang, F. Liu, and I. Turner. Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. *Appl. Math. Model.*, 34(1):200–218, 2010. 1

[39] Fanhai Zeng, Fawang Liu, Changpin Li, Kevin Burrage, Ian Turner, and V. Anh. A Crank-Nicolson ADI spectral method for a two-dimensional Riesz space fractional nonlinear reaction-diffusion equation. *SIAM J. Numer. Anal.*, 52(6):2599–2622, 2014. 1

[40] Han Zhou and Wenyi Tian. Two time-stepping schemes for sub-diffusion equations with singular source terms. *J. Sci. Comput.*, 92(2):Paper No. 70, 28, 2022. 1

School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P.R. China

Email address: `chenmh@lzu.edu.cn`

School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P.R. China

Email address: `hanjx2023@lzu.edu.cn`