AN α-ROBUST DIFFERENCE QUADRATURE METHOD FOR SPACE-TIME CAPUTO-RIESZ FRACTIONAL DIFFUSION EQUATION *

JIANXING HAN

Abstract. This is an example SIAM LaTeX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, LaTeX

MSC codes. 68Q25, 68R10, 68U05

1. Introduction.

We study

$$\frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u = f(x,t), \quad x \in \Omega, t \in (0,T]. \tag{1.1}$$

Implicity scheme: Let $\tau = \frac{T}{M}$, $U^n, F^n \in \mathbb{R}^{2N-1}$,

$$\frac{U^{n+1} - U^n}{\tau} + AU^{n+1} = F^{n+1}. \tag{1.2}$$

Then $E^n = U^n - \hat{U}^n \in \mathbb{R}^{2N-1}$,

$$(I + \tau A)E^{n+1} = E^n + \tau R^{n+1}. \tag{1.3}$$

We will prove the convergence of this scheme.

2. Property of A.

LEMMA 2.1. *The stiffness matrix A has the following properties:*
1. *The eigenvalues of A are positive real numbers.*
2. *A is positive definite, which means that the eigenvalues of $\frac{A+A^T}{2}$ are positive.*
3. *The eigenvectors of A are orthogonal in space where $\langle u, v\rangle := uHv$, where $H := diag\left(\frac{h_i+h_{i+1}}{2}\right)$.*
4. *$(I + \tau A)^{-1} > O$ for any $\tau > 0$.*

Proof. Since

$$A = H^{-1}D = H^{-1/2}H^{-1/2}DH^{-1/2}H^{1/2}, \tag{2.1}$$

where $H^{-1/2}DH^{-1/2}$ is symmetric positive definite, $H^{-1/2}DH^{-1/2} = U\Lambda U^T$. Thus,

$$A = H^{-1/2}U\Lambda U^T H^{1/2} = (H^{-1/2}U)\Lambda(H^{-1/2}U)^{-1}. \tag{2.2}$$

The eigenvectors of A form an orthogonal basis of the Hilbert space defined by $\langle u, v\rangle := uHv$. Let $v_i = H^{-1/2}u_i$ be an eigenvector of A with eigenvalue λ_i. ∎

We need to prove $\lambda_1 > c$ for some positive constant c.

---

*Submitted to the editors DATE.

Funding: This work was funded by the Fog Research Institute under contract no. FRI-454.

3. Regularity.

Lemma 3.1. *If*

$$\left|\frac{\partial^l u}{\partial x^l}\right| \le \delta(x)^{\alpha/2-l}, \quad x\in\Omega, \tag{3.1}$$

then

$$\left|-\frac{d^l}{dx^l} I^{2-\alpha} u\right| \le C\frac{\kappa_\alpha(\alpha-1)}{\Gamma(3-\alpha)}\delta(x)^{-\alpha/2-(l-2)}, \quad l=1,2,3,4, \quad x\in\Omega, \tag{3.2}$$

where C is independent of α.

Proof.

$$\begin{aligned}\frac{d^l}{dx^l} I^{2-\alpha} u = \frac{\kappa_\alpha}{\Gamma(2-\alpha)} \Bigg(&(1-\alpha)\cdots(1-(l-1)-\alpha)\left(\int_0^{x/2} u(y)(x-y)^{1-l-\alpha}dy + (-1)^l \int_{T+x/2}^{2T} u(y)(y-x)^{1-l-\alpha}dy\right)\\ &+\sum_{k=1}^{l-1}(1-\alpha)\cdots(1-(k-1)-\alpha)\left(\frac{\partial^{l-1-k}}{\partial x^{l-1-k}}u(x/2)\left(\frac{x}{2}\right)^{1-k-\alpha} - (-1)^k\frac{\partial^{l-1-k}}{\partial x^{l-1-k}}u(T+x/2)\left(T+\frac{x}{2}\right)^{1-k-\alpha}\right)\\ &+(1-\alpha)\left(\int_{x/2}^{x}\frac{\partial^{l-1}}{\partial x^{l-1}}u(y)(x-y)^{-\alpha} - \int_x^{T+x/2}\frac{\partial^{l-1}}{\partial x^{l-1}}u(y)(y-x)^{-\alpha}\right)\Bigg)\end{aligned} \tag{3.3}$$

Set $x \le T$, we have

$$\left|\int_0^{x/2} u(y)(x-y)^{1-l-\alpha}dy\right| \le 2^{\alpha+l-1}\int_0^{x/2} y^{\alpha/2}x^{1-l-\alpha}dy \simeq x^{-\alpha/2-(l-2)}$$

$$\frac{\partial^{l-1-k}}{\partial x^{l-1-k}}u(x/2)\left(\frac{x}{2}\right)^{1-k-\alpha} \lesssim x^{-\alpha/2-(l-2)}$$

$$\begin{aligned}&\int_{x/2}^{x}\frac{\partial^{l-1}}{\partial x^{l-1}}u(y)(x-y)^{-\alpha} - \int_x^{T+x/2}\frac{\partial^{l-1}}{\partial x^{l-1}}u(y)(y-x)^{-\alpha}\\ &\quad= \int_0^{x/2}\left(\frac{\partial^{l-1}}{\partial x^{l-1}}u(x-z) - \frac{\partial^{l-1}}{\partial x^{l-1}}u(x+z)\right)z^{-\alpha}dz - \int_{x+x/2}^{T+x/2}\frac{\partial^{l-1}}{\partial x^{l-1}}u(y)(y-x)^{-\alpha}dy\end{aligned}$$

While

$$\begin{aligned}&\left|\int_0^{x/2}\left(\frac{\partial^{l-1}}{\partial x^{l-1}}u(x-z) - \frac{\partial^{l-1}}{\partial x^{l-1}}u(x+z)\right)z^{-\alpha}dz\right|\\ &\quad= 2\int_0^{x/2}\frac{\partial^l}{\partial x^l}u(\xi)z^{1-\alpha}dz \le \left(\frac{x}{2}\right)^{\alpha/2-l} 2\int_0^{x/2} z^{1-\alpha}dz \simeq \frac{1}{2-\alpha}x^{-\alpha/2-(l-2)}\end{aligned}$$

□

$$\begin{aligned}&\int_{x+x/2}^{T+x/2}\frac{\partial^{l-1}}{\partial x^{l-1}}u(y)(y-x)^{-\alpha}dy\\ &\quad\le \int_{x+x/2}^{T+x/2} y^{\alpha/2-(l-1)}y^{-\alpha}dy \lesssim \begin{cases}\frac{1}{2-\alpha}T^{-\alpha/2-(l-2)}, & l=1,\\ \frac{1}{\alpha/2+l-1}x^{-\alpha/2-(l-2)}, & l=2,3,4.\end{cases}\end{aligned}$$

4. truncation error.

5. Convergence.

$$
\begin{aligned}
E^n &= (I+\tau A)^{-1}E^{n-1} + (I+\tau A)^{-1}\tau R^n \\
&= (I+\tau A)^{-n}E^0 + \sum_{k=1}^{n}(I+\tau A)^{-k}\tau R^{n-k+1}
\end{aligned} \tag{5.1}
$$

Assume that $|R^n| \le Ch^{\min\{r\alpha/2,2\}}(x_i^{-\alpha} + (2T-x_i)^{-\alpha}) + C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha} + C\tau^?$

$$
\begin{aligned}
(I+\tau A)^{-k}\tau R^{n-k+1} &= (\tau A)(I+\tau A)^{-k}(\tau A)^{-1}\tau R^{n-k+1} \\
&= (\tau A)(I+\tau A)^{-k}(A^{-1}R^{n-k+1})
\end{aligned} \tag{5.2}
$$

Suppose that

$$
\begin{aligned}
|R^n| &\le |R| \\
&:= Ch^{\min\{r\alpha/2,2\}}(x_i^{-\alpha} + (2T-x_i)^{-\alpha}) \\
&\quad + C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha} + C\tau^?
\end{aligned} \tag{5.3}
$$

Since $0 < A^{-1}R \le Ch^{\min}$,

$$
|(I+\tau A)^{-k}\tau R^{n-k+1}| \le (I+\tau A)^{-k}\tau R = \tau A(1+\tau A)^{-k}A^{-1}R \tag{5.4}
$$

Then

$$
\begin{aligned}
|E^n| &\le |(I+\tau A)^{-n}E^0| + \sum_{k=1}^{n}\tau A(1+\tau A)^{-k}A^{-1}R \\
&= |(I+\tau A)^{-n}E^0| + \left(I-(I+\tau A)^{-n}\right)A^{-1}R.
\end{aligned} \tag{5.5}
$$

Since A is diagonally dominant, $\|(I+\tau A)^{-1}E\|_\infty \le \|E\|_\infty$, we have

$$
\|E^n\|_\infty \le \|E_0\|_\infty + 2\|A^{-1}R\|_\infty. \tag{5.6}
$$

Lemma 5.1. *$A^{-1}R$ is bounded by $C\left(h^{\min\{r\alpha/2,2\}} + \tau^?\right)$, where C is a constant independent of h, α.*

6. Caputo-Riesz. $\gamma \in (0,1)$

$$
D_t^\gamma u + (-\Delta)^{\frac{\alpha}{2}}u = f(x,t), \quad x\in\Omega, t\in(0,T]. \tag{6.1}
$$

where

$$
D_t^\gamma u(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\partial u(x,s)}{\partial s}(t-s)^{-\gamma}ds \tag{6.2}
$$

(6.3)

$$
\begin{aligned}
D_N^\gamma u(x,t_n) &= \sum_{k=1}^{n}\frac{1}{\Gamma(2-\gamma)}\left(u(x,t_k)-u(x,t_{k-1})\right)\frac{(t_n-t_{k-1})^{1-\gamma}-(t_n-t_k)^{1-\gamma}}{\tau_k} \\
&= d_{n,n}u(x,t_n) - \sum_{k=1}^{n-1}(d_{n,k+1}-d_{n,k})u(x,t_k) - d_{n,1}u(x,t_0),
\end{aligned}
$$

where

$$d_{n,k} = \frac{(t_n - t_{k-1})^{1-\gamma} - (t_n - t_k)^{1-\gamma}}{\Gamma(2-\gamma)\tau_k} \quad \text{for} \quad 1 \le k \le n \quad \text{and} \quad d_{n,0} = 0, \tag{6.4}$$

$d_{n,n} = \frac{\tau_n^{-\gamma}}{\Gamma(2-\gamma)}$, $\quad d_{n,k+1} \ge d_{n,k}$.

Numerical scheme:

$$D_N^\gamma U^n + AU^n = F^n \tag{6.5}$$

We have

$$(d_{n,n}I + A)\, E^n = \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k}) E^k + d_{n,1} E^0 + R^n \tag{6.6}$$

Define the matrices $\Theta_{n,j}$, for $n = 1, 2..., N$ and $j = 0, 1, 2, ..., n-1$ by

$$\Theta_{n,n} = (d_{n,n} + A)^{-1}, \quad \Theta_{0,0} = I, \quad \Theta_{n,j} = \sum_{k=j}^{n-1} (d_{n,k+1} - d_{n,k}) \Theta_{n,n} \Theta_{k,j}. \tag{6.7}$$

Observe that $\Theta_{n,j} > 0$ for all n, j.

Lemma 6.1.

$$E^n = \sum_{j=1}^{n} \Theta_{n,j} R^j + \Theta_{n,0} E^0 \tag{6.8}$$

Lemma 6.2. *Let the parameter β satisfy $\beta \le r_t \gamma$. Then for $n = 1, 2, ..., N$, one has*

$$\sum_{j=1}^{n} \Theta_{n,j} \le A^{-1} \tag{6.9}$$

Proof. Use induction on n. When $n = 1$, then $\sum_{j=1}^{1} \Theta_{1,j} = \Theta_{1,1}$. Next, assume that (6.9) holds for $k = 1, 2, ..., m-1 (2 \le m \le N)$. We want to prove (6.9) for $n = m$. Invoking (6.7) and interchanging the order of summation,

∎

$$\begin{aligned}
\sum_{j=1}^{m} \Theta_{m,j} &= \Theta_{m,m} + \sum_{j=1}^{m-1} \sum_{k=j}^{m-1} (d_{m,k+1} - d_{m,k}) \Theta_{m,m} \Theta_{k,j} \\
&= \Theta_{m,m} + \sum_{k=1}^{m-1} (d_{m,k+1} - d_{m,k}) \Theta_{m,m} \sum_{j=1}^{k} \Theta_{k,j} \\
&\le \Theta_{m,m} + \sum_{k=1}^{m-1} (d_{m,k+1} - d_{m,k}) \Theta_{m,m} A^{-1} \\
&= \Theta_{m,m} + (d_{m,m} - d_{m,1}) \Theta_{m,m} A^{-1} \\
&= A^{-1} - d_{m,1} \Theta_{m,m} A^{-1} \le A^{-1}
\end{aligned}$$

Acknowledgments. We would like to acknowledge the assistance of volunteers in putting together this example manuscript and supplement.

REFERENCES