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Superlinearly convergent algorithms for the two-dimensional space–time Caputo–Riesz fractional diffusion equation

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ABSTRACT

In this paper, we discuss the space–time Caputo–Riesz fractional diffusion equation with variable coefficients on a finite domain. The finite difference schemes for this equation are provided. We theoretically prove and numerically verify that the implicit finite difference scheme is unconditionally stable (the explicit scheme is conditionally stable with the stability condition $\frac{\tau^\gamma}{(\Delta x)^\alpha}+\frac{\tau^\gamma}{(\Delta y)^\beta}<C$) and 2nd order convergent in space direction, and $(2-\gamma)$th order convergent in time direction, where $\gamma\in(0,1]$.

1. Introduction

Nowadays, fractional calculus has become popular in both the science and engineering societies. There are several nonequivalent definitions of fractional derivatives [9–11]. The Caputo derivative is most often used for time fractional derivative, and the Riemann–Liouville derivative and Grünwald–Letnikov derivative, the two fractional derivatives being equivalent if the functions performed are regular enough, are most frequently used for space fractional derivative. Some important progress has been made for numerically solving this kind of fractional PDEs by finite difference methods, e.g., see [1,6,7,12,13,15–17,19].

Another space fractional derivative having a vast majority of applications is the symmetric fractional derivative, namely the Riesz fractional derivative, e.g., see [8,18]. Zhuang and Liu et al. consider a variable-order fractional advection-diffusion equation with a nonlinear source term on a finite domain [20]. Jiang et al. analytically discuss the space–time fractional advection-diffusion equation with Riesz fractional derivative as space fractional derivative [4]. Based on the shifted Grünwald approximation strategy and the method of lines, Yang, Liu, and Turner numerically study the Riesz space fractional PDEs with two different fractional orders $1<\alpha\leqslant 2$ and $0<\beta<1$ [14]. The explicit finite difference scheme for fractional Fokker–Planck equation with Riesz fractional derivative is discussed in [3]. With the desire of obtaining 2nd order convergence in the space discretization, here we further discuss the finite difference approximations for two-dimensional space–time Caputo–Riesz fractional diffusion equation with variable coefficients in a finite domain, namely,

$$\begin{cases} {}_0^C D_t^\gamma u(x,y,t)=c(x,y,t)\dfrac{\partial^\alpha u(x,y,t)}{\partial|x|^\alpha}+d(x,y,t)\dfrac{\partial^\beta u(x,y,t)}{\partial|y|^\beta}+f(x,y,t),\\ u(x,y,0)=u_0(x,y),\quad \text{for } (x,y)\in\Omega,\\ u(x,y,t)=0,\quad \text{for } (x,y,t)\in\partial\Omega\times[0,T], \end{cases}\tag{1.1}$$

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in the domain $\Omega = (x_L, x_R) \times (y_L, y_R), 0 < t \leqslant T$, with the orders of the Riesz fractional derivative $1 < \alpha, \beta \leqslant 2$ and the order of the Caputo fractional operator $0 < \gamma \leqslant 1$; the function $f(x, y, t)$ is a source term; and the variable coefficients $c(x, y, t) \geqslant 0, d(x, y, t) \geqslant 0$. The Riesz fractional derivative for $n \in \mathbb{N}$, $n-1 \leqslant \nu < n$, in a finite interval $x_L \leqslant x \leqslant x_R$ is defined as [11]

$$\frac{\partial^\nu u(x,y,t)}{\partial |x|^\nu} = -\kappa_\nu \left({}_{x_L}D_x^\nu + {}_xD_{x_R}^\nu\right)u(x,y,t), \tag{1.2}$$

where the coefficient $\kappa_\nu = \frac{1}{2\cos(\nu\pi/2)}$, and

$${}_{x_L}D_x^\nu u(x,y,t) = \frac{1}{\Gamma(n-\nu)}\frac{\partial^n}{\partial x^n}\int_{x_L}^{x}(x-\xi)^{n-\nu-1}u(\xi,y,t)\,d\xi, \tag{1.3}$$

$${}_xD_{x_R}^\nu u(x,y,t) = \frac{(-1)^n}{\Gamma(n-\nu)}\frac{\partial^n}{\partial x^n}\int_{x}^{x_R}(\xi-x)^{n-\nu-1}u(\xi,y,t)\,d\xi, \tag{1.4}$$

are the left and right Riemann–Liouville space fractional derivatives, respectively. The Caputo fractional derivative of order $\gamma \in (0,1]$ is defined by [9,10]

$${}_0^CD_t^\gamma u(x,y,t) = \begin{cases} \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\partial u(x,y,\eta)}{\partial \eta}(t-\eta)^{-\gamma}\,d\eta, & 0<\gamma<1, \\ \frac{\partial u(x,y,t)}{\partial t}, & \gamma = 1. \end{cases} \tag{1.5}$$

For the 2nd order discretization of the Riemann–Liouville fractional derivatives (1.3) and (1.4), it has been detailedly discussed in [2] being the sequel of [12]. This paper applies the discretization scheme to Riesz fractional derivative. The implicit and explicit finite difference schemes are designed. We theoretically prove that the implicit finite difference scheme is unconditionally stable and the stability condition of the explicit scheme is $\frac{\tau^\gamma}{(\Delta x)^\alpha} + \frac{\tau^\gamma}{(\Delta y)^\beta} < C$, confirming the general conclusions on the stability conditions of the explicit schemes for fractional PDEs [3]. The desired 2nd order convergence in space and $(2-\gamma)$th order convergence in time of both implicit and explicit schemes are theoretically proved and numerically verified.

The outline of this paper is as follows. In Section 2, we introduce the approximation of the Caputo fractional derivative and the 2nd order finite difference discretizations for the Riesz fractional derivatives, and derive the full discretization schemes of (1.1). In Sections 3 and 4, the stability and convergence of the provided implicit and explicit finite difference schemes are analyzed, respectively. To show the effectiveness of the schemes, we perform the numerical experiments to verify the theoretical results in Section 5. Finally, we conclude the paper with some remarks in the last section.

2. Derivation of the finite difference scheme

We use two subsections to derive the full discretization schemes of (1.1). The first subsection introduces the approximation of the Caputo fractional derivative and the 2nd order finite difference discretizations for the Riesz fractional derivatives in a finite domain. The second subsection gives the full discretization scheme (implicit scheme and explicit schemes) to the one-dimensional case of (1.1) and (1.1) itself, respectively.

2.1. Discretizations for the Caputo and Riesz fractional derivatives

Take the mesh points $x_i = x_L + i\Delta x, i = 0, 1, \ldots, N_x$, $y_j = y_L + j\Delta y, j = 0, 1, \ldots, N_y$ and $t_k = k\tau, k = 0, 1, \ldots, N_t$, where $\Delta x = (x_R - x_L)/N_x$, $\Delta y = (y_R - y_L)/N_y$, $\tau = T/N_t$, i.e., Δx and Δy are the uniform space stepsizes in the corresponding directions, τ the time stepsize. For $\nu \in (1,2)$, the left and right Riemann–Liouville space fractional derivatives (1.3) and (1.4) have the 2nd order approximation operators $\delta_{\nu,+x}u_{i,j}^k$ and $\delta_{\nu,-x}u_{i,j}^k$, respectively, given in a finite domain [2,12], where $u_{i,j}^k$ denotes the approximated value of $u(x_i, y_j, t_k)$.

The approximation operator of (1.3) is defined by [2,12]

$$\delta_{\nu,+x}u_{i,j}^k := \frac{1}{\Gamma(4-\nu)(\Delta x)^\nu}\sum_{m=0}^{i+1}u_{m,j}^k p_{i,m}^\nu, \tag{2.1}$$

and there exists

$${}_{x_L}D_x^\nu u(x,y,t) = \delta_{\nu,+x}u_{i,j}^k + \mathcal{O}(\Delta x)^2, \tag{2.2}$$

where

$$p_{i,m}^{\nu}=\begin{cases} a_{i-1,m}-2a_{i,m}+a_{i+1,m}, & m\leqslant i-1,\\ -2a_{i,i}+a_{i+1,i}, & m=i,\\ a_{i+1,i+1}, & m=i+1,\\ 0, & m>i+1,\end{cases} \tag{2.3}$$

and

$$a_{j,m}=\begin{cases} (j-1)^{3-\nu}-j^{2-\nu}(j-3+\nu), & m=0,\\ (j-m+1)^{3-\nu}-2(j-m)^{3-\nu}+(j-m-1)^{3-\nu}, & 1\leqslant m\leqslant j-1,\\ 1, & m=j,\end{cases}$$

with $j=i-1,i,i+1$.

Analogously, the approximation operator of (1.4) is described as [2]

$$\delta_{\nu,-x}u_{i,j}^{k}:=\frac{1}{\Gamma(4-\nu)(\Delta x)^{\nu}}\sum_{m=i-1}^{N_x}u_{m,j}^{k}q_{i,m}^{\nu}, \tag{2.4}$$

and it holds that

$${}_{x}D_{x_R}^{\nu}u(x,y,t)=\delta_{\nu,-x}u_{i,j}^{k}+\mathcal{O}(\Delta x)^2, \tag{2.5}$$

with

$$q_{i,m}^{\nu}=\begin{cases} 0, & m<i-1,\\ b_{i-1,i-1}, & m=i-1,\\ -2b_{i,i}+b_{i-1,i}, & m=i,\\ b_{i-1,m}-2b_{i,m}+b_{i+1,m}, & i+1\leqslant m\leqslant N_x,\end{cases} \tag{2.6}$$

and

$$b_{j,m}=\begin{cases} 1, & m=j,\\ (m-j+1)^{3-\alpha}-2(m-j)^{3-\alpha}+(m-j-1)^{3-\alpha}, & j+1\leqslant m\leqslant N_x-1,\\ (3-\alpha-N_x+j)(N_x-j)^{2-\alpha}+(N_x-j-1)^{3-\alpha}, & m=N_x,\end{cases}$$

with $j=i-1,i,i+1$.

Combining (2.2) and (2.5), we obtain the approximation operator of the Riesz fractional derivative

$$\begin{aligned}\left.\frac{\partial^{\nu}u(x,y_j,t_k)}{\partial|x|^{\nu}}\right|_{x=x_i} &= -\kappa_{\nu}\left({}_{x_L}D_{x}^{\nu}+{}_{x}D_{x_R}^{\nu}\right)u(x,y_j,t_k)\big|_{x=x_i}\\ &= -\kappa_{\nu}(\delta_{\nu,+x}+\delta_{\nu,-x})u_{i,j}^{k}+\mathcal{O}(\Delta x)^2\\ &= \frac{-\kappa_{\nu}}{\Gamma(4-\nu)\Delta x^{\nu}}\sum_{m=0}^{N_x}\left(p_{i,m}^{\nu}+q_{i,m}^{\nu}\right)u_{m,j}^{k}+\mathcal{O}(\Delta x)^2\\ &:= \frac{-\kappa_{\nu}}{\Gamma(4-\nu)\Delta x^{\nu}}\sum_{m=0}^{N_x}g_{i,m}^{\nu}u_{m,j}^{k}+\mathcal{O}(\Delta x)^2,\end{aligned} \tag{2.7}$$

where

$$g_{i,m}^{\nu}=\begin{cases} p_{i,m}^{\nu}, & m<i-1,\\ p_{i,i-1}^{\nu}+q_{i,i-1}^{\nu}, & m=i-1,\\ p_{i,i}^{\nu}+q_{i,i}^{\nu}, & m=i,\\ p_{i,i+1}^{\nu}+q_{i,i+1}^{\nu}, & m=i+1,\\ q_{i,m}^{\nu}, & m>i+1.\end{cases} \tag{2.8}$$

Taking $\nu=2$, both Eqs. (2.2) and (2.5) reduce to the following form

$$\frac{\partial^2u(x_i,y,t)}{\partial x^2}=\frac{u(x_{i+1},y,t)-2u(x_i,y,t)+u(x_{i-1},y,t)}{(\Delta x)^2}+\mathcal{O}(\Delta x)^2.$$

The Caputo derivative in the time direction is discretized as [5]

$$
\begin{aligned}
{}_0^C D_t^{\gamma} u(x_i, y_j, t)\big|_{t=t_{k+1}} &= \frac{1}{\Gamma(1-\gamma)} \sum_{s=0}^{k} \int_{t_s}^{t_{s+1}} \frac{\partial u(x_i, y_j, \eta)}{\partial \eta} \frac{d\eta}{(t_{k+1}-\eta)^{\gamma}} \\
&= \frac{1}{\Gamma(1-\gamma)} \sum_{s=0}^{k} \frac{u(x_i, y_j, t_{s+1}) - u(x_i, y_j, t_s)}{\tau} \int_{t_s}^{t_{s+1}} \frac{d\eta}{(t_{k+1}-\eta)^{\gamma}} + \mathcal{O}\big(\tau^{2-\gamma}\big) \\
&= \frac{1}{\Gamma(2-\gamma)} \sum_{s=0}^{k} l_s \frac{u(x_i, y_j, t_{k+1-s}) - u(x_i, y_j, t_{k-s})}{\tau^{\gamma}} + \mathcal{O}\big(\tau^{2-\gamma}\big),
\end{aligned}
\tag{2.9}
$$

where $\gamma \in (0,1)$, $l_s = (s+1)^{1-\gamma} - s^{1-\gamma}$.

When $0 < \gamma < 1$, the time Caputo fractional derivative uses the information of the classical derivatives at all previous time levels (non-Markovian process). If $\gamma = 1$, then $l_0 = 1$, $l_s = 0, s > 0$, it can be seen that by taking the limit $\gamma \to 1$ in (2.9), which gives the following equation

$$
\frac{\partial u(x, y, t_{k+1})}{\partial t} = \frac{u(x, y, t_{k+1}) - u(x, y, t_k)}{\tau} + \mathcal{O}(\tau).
$$

Similarly, it is easy to get the one-dimensional case of (2.1)–(2.9).

Remark 2.1. (See [2].) Denoting $\tilde{U}^n = [u_{1,j}^n, u_{2,j}^n, \ldots, u_{N_x-1,j}^n]^{\mathrm{T}}$, $j = 0, 1, \ldots, N_y$, and rewriting (2.1) and (2.4) as matrix forms $\delta_{\alpha,+x}\tilde{U}^n = \tilde{A}\tilde{U}^n + b_1$ and $\delta_{\alpha,-x}\tilde{U}^n = \tilde{B}\tilde{U}^n + b_2$, respectively, then there exists $\tilde{A} = \tilde{B}^{\mathrm{T}}$.

2.2. Implicit and explicit difference schemes for 1D

Consider the one-dimensional space–time Caputo–Riesz fractional diffusion equation

$$
{}_0^C D_t^{\gamma} u(x, t) = c(x, t) \frac{\partial^{\alpha} u(x, t)}{\partial |x|^{\alpha}} + f(x, t).
\tag{2.10}
$$

Using the one-dimensional case of (2.1)–(2.9), we can write (2.10) as

$$
\begin{aligned}
&\frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{s=0}^{k} l_s \big[u(x_i, t_{k+1-s}) - u(x_i, t_{k-s})\big] \\
&\quad = -\frac{\kappa_{\alpha} c(x_i, t_{k+1})}{\Gamma(4-\alpha)(\Delta x)^{\alpha}} \sum_{m=0}^{N_x} g_{i,m}^{\alpha} u(x_m, t_{k+1}) + f(x_i, t_{k+1}) + \mathcal{O}\big(\tau^{2-\gamma} + (\Delta x)^2\big),
\end{aligned}
$$

where $i = 0, 1, \ldots, N_x$, $k = 0, 1, \ldots, N_t$. Assuming that $c_i^k = c(x_i, t_k)$, $f_i^k = f(x_i, t_k)$, $\omega_{i,k+1} = -\frac{\Gamma(2-\gamma)\tau^{\gamma}\kappa_{\alpha} c_i^{k+1}}{\Gamma(4-\alpha)(\Delta x)^{\alpha}}$ and $\mu = \Gamma(2-\gamma)\tau^{\gamma}$, we have

$$
u(x_i, t_{k+1}) = u(x_i, t_k) - \sum_{s=1}^{k} l_s \big[u(x_i, t_{k+1-s}) - u(x_i, t_{k-s})\big] + \omega_{i,k+1} \sum_{m=0}^{N_x} g_{i,m}^{\alpha} u(x_m, t_{k+1}) + \mu f(x_i, t_{k+1}) + R_i^{k+1},
\tag{2.11}
$$

where $|R_i^{k+1}| \leqslant C\tau^{\gamma}(\tau^{2-\gamma} + (\Delta x)^2)$.

Therefore, the implicit difference scheme of (2.10) has the following form

$$
u_i^{k+1} = u_i^k - \sum_{s=1}^{k} l_s \big(u_i^{k+1-s} - u_i^{k-s}\big) + \omega_{i,k+1} \sum_{m=0}^{N_x} g_{i,m}^{\alpha} u_m^{k+1} + \mu f_i^{k+1},
\tag{2.12}
$$

and it can be rewritten as

$$
\begin{aligned}
&\big(1 - \omega_{i,1} g_{i,i}^{\alpha}\big) u_i^1 - \omega_{i,1} \sum_{m=0, m\neq i}^{N_x} g_{i,m}^{\alpha} u_m^1 = u_i^0 + \mu f_i^1, \quad k = 0, \\
&\big(1 - \omega_{i,k+1} g_{i,i}^{\alpha}\big) u_i^{k+1} - \omega_{i,k+1} \sum_{m=0, m\neq i}^{N_x} g_{i,m}^{\alpha} u_m^{k+1} = \sum_{s=0}^{k-1} (l_s - l_{s+1}) u_i^{k-s} + l_k u_i^0 + \mu f_i^{k+1}, \quad k > 0.
\end{aligned}
\tag{2.13}
$$

Taking $\gamma = 1$, thus the implicit difference scheme (2.13) reduces to the following equation

$$\left(1 - \omega_{i,k+1} g^{\alpha}_{i,i}\right) u^{k+1}_i - \omega_{i,k+1} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i,m} u^{k+1}_m = u^k_i + \mu f^{k+1}_i, \quad k \geqslant 0.$$

Analogously, the explicit difference scheme of (2.10) is

$$u^1_i = \left(1 + \sigma_{i,0} g^{\alpha}_{i,i}\right) u^0_i + \sigma_{i,0} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i,m} u^0_m + \mu f^0_i, \quad k = 0,$$

$$u^{k+1}_i = \left(1 - l_1 + \sigma_{i,k} g^{\alpha}_{i,i}\right) u^k_i + \sum_{s=1}^{k-1} (l_s - l_{s+1}) u^{k-s}_i + l_k u^0_i + \sigma_{i,k} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i,m} u^k_m + \mu f^k_i, \quad k > 0, \tag{2.14}$$

where $\sigma_{i,k} = -\frac{\Gamma(2-\gamma)\tau^{\gamma}\kappa_\alpha c^k_i}{\Gamma(4-\alpha)(\Delta x)^{\alpha}} > 0$.

2.3. Implicit and explicit difference schemes for 2D

We examine the two-dimensional space–time Caputo–Riesz fractional diffusion equation (1.1). According to (2.1)–(2.9), then (1.1) can be recast as

$$\begin{aligned}
&\frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \sum_{s=0}^{k} l_s \left[u(x_i, y_j, t_{k+1-s}) - u(x_i, y_j, t_{k-s})\right] \\
&\quad = -\frac{\kappa_\alpha c(x_i, y_j, t_{k+1})}{\Gamma(4-\alpha)(\Delta x)^{\alpha}} \sum_{m=0}^{N_x} g^{\alpha}_{i,m} u(x_m, y_j, t_{k+1}) - \frac{\kappa_\beta d(x_i, y_j, t_{k+1})}{\Gamma(4-\beta)(\Delta y)^{\beta}} \sum_{m=0}^{N_y} g^{\beta}_{j,m} u(x_i, y_m, t_{k+1}) \\
&\qquad + f(x_i, y_j, t_{k+1}) + \mathcal{O}\left(\tau^{2-\gamma} + (\Delta x)^2 + (\Delta y)^2\right),
\end{aligned}$$

where $i = 0, 1, \ldots, N_x$, $j = 0, 1, \ldots, N_y$, $k = 0, 1, \ldots, N_t$. Denoting $c^k_{i,j} = c(x_i, y_j, t_k)$, $d^k_{i,j} = d(x_i, y_j, t_k)$, $f^k_{i,j} = f(x_i, y_j, t_k)$ and

$$\omega'_{i,j,k+1} = -\frac{\Gamma(2-\gamma)\tau^{\gamma}\kappa_\alpha c^{k+1}_{i,j}}{\Gamma(4-\alpha)(\Delta x)^{\alpha}}; \qquad \omega''_{i,j,k+1} = -\frac{\Gamma(2-\gamma)\tau^{\gamma}\kappa_\beta d^{k+1}_{i,j}}{\Gamma(4-\beta)(\Delta y)^{\beta}};$$

and $\mu = \Gamma(2-\gamma)\tau^{\gamma}$, we obtain

$$\begin{aligned}
u(x_i, y_j, t_{k+1}) &= u(x_i, y_j, t_k) - \sum_{s=1}^{k} l_s \left[u(x_i, y_j, t_{k+1-s}) - u(x_i, y_j, t_{k-s})\right] + \omega'_{i,j,k+1} \sum_{m=0}^{N_x} g^{\alpha}_{i,m} u(x_m, y_j, t_{k+1}) \\
&\quad + \omega''_{i,j,k+1} \sum_{m=0}^{N_y} g^{\beta}_{j,m} u(x_i, y_m, t_{k+1}) + \mu f(x_i, y_j, t_{k+1}) + R^{k+1}_{i,j},
\end{aligned} \tag{2.15}$$

where $|R^{k+1}_{i,j}| \leqslant C\tau^{\gamma}(\tau^{2-\gamma} + (\Delta x)^2 + (\Delta y)^2)$.

Then we obtain the full discretization implicit difference scheme of (1.1) as

$$u^{k+1}_{i,j} = u^k_{i,j} - \sum_{s=1}^{k} l_s \left(u^{k+1-s}_{i,j} - u^{k-s}_{i,j}\right) + \omega'_{i,j,k+1} \sum_{m=0}^{N_x} g^{\alpha}_{i,m} u^{k+1}_{m,j} + \omega''_{i,j,k+1} \sum_{m=0}^{N_y} g^{\beta}_{j,m} u^{k+1}_{i,m} + \mu f^{k+1}_{i,j}, \tag{2.16}$$

and Eq. (2.16) can be rewritten as

$$\begin{aligned}
&\left(1 - \omega'_{i,j,1} g^{\alpha}_{i,i} - \omega''_{i,j,1} g^{\beta}_{j,j}\right) u^1_{i,j} - \omega'_{i,j,1} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i,m} u^1_{m,j} - \omega''_{i,j,1} \sum_{m=0,m\neq j}^{N_y} g^{\beta}_{j,m} u^1_{i,m} = u^0_{i,j} + \mu f^1_{i,j}, \quad k = 0, \\
&\left(1 - \omega'_{i,j,k+1} g^{\alpha}_{i,i} - \omega''_{i,j,k+1} g^{\beta}_{j,j}\right) u^{k+1}_{i,j} - \omega'_{i,j,k+1} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i,m} u^{k+1}_{m,j} - \omega''_{i,j,k+1} \sum_{m=0,m\neq j}^{N_y} g^{\beta}_{j,m} u^{k+1}_{i,m} \\
&\quad = \sum_{s=0}^{k-1} (l_s - l_{s+1}) u^{k-s}_{i,j} + l_k u^0_{i,j} + \mu f^{k+1}_{i,j}, \quad k > 0.
\end{aligned} \tag{2.17}$$

When $\gamma = 1$, Eq. (2.17) becomes

$$\left(1 - \omega'_{i,j,k+1} g^{\alpha}_{i,i} - \omega''_{i,j,k+1} g^{\beta}_{j,j}\right) u^{k+1}_{i,j} - \omega'_{i,j,k+1} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i,m} u^{k+1}_{m,j} - \omega''_{i,j,k+1} \sum_{m=0,m\neq j}^{N_y} g^{\beta}_{j,m} u^{k+1}_{i,m}$$
$$= u^k_{i,j} + \mu f^{k+1}_{i,j}, \quad k \geqslant 0.$$

Let $\mathbf{u}^k = [\mathbf{u}^k_0, \mathbf{u}^k_1, \ldots, \mathbf{u}^k_{N_x}]^T$, $\mathbf{f}^k = [\mathbf{f}^k_0, \mathbf{f}^k_1, \ldots, \mathbf{f}^k_{N_x}]^T$, where $\mathbf{u}^k_i = [u^k_{i,0}, u^k_{i,1}, \ldots, u^k_{i,N_y}]^T$ and $\mathbf{f}^k_i = [f^k_{i,0}, f^k_{i,1}, \ldots, f^k_{i,N_y}]^T$. Then (2.17) can be written in the matrix form

$$A\mathbf{u}^1 = \mathbf{u}^0 + \mu \mathbf{f}^1, \quad k = 0,$$
$$A\mathbf{u}^{k+1} = \sum_{s=0}^{k-1} (l_s - l_{s+1}) \mathbf{u}^{k-s} + l_k \mathbf{u}^0 + \mu \mathbf{f}^{k+1}, \quad k > 0, \tag{2.18}$$

where A is an $(N_x N_y) \times (N_x N_y)$ coefficient matrix.

Similarily, the explicit difference scheme of (1.1) can be expressed as

$$u^1_{i,j} = \left(1 + \sigma'_{i,j,0} g^{\alpha}_{i,i} + \sigma''_{i,j,0} g^{\beta}_{j,j}\right) u^0_{i,j} + \sigma'_{i,j,0} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i,m} u^0_{m,j} + \sigma''_{i,j,0} \sum_{m=0,m\neq j}^{N_y} g^{\beta}_{j,m} u^0_{i,m} + \mu f^0_{i,j}, \quad k = 0,$$
$$u^{k+1}_{i,j} = \left(1 - l_1 + \sigma'_{i,j,k} g^{\alpha}_{i,i} + \sigma''_{i,j,k} g^{\beta}_{j,j}\right) u^k_{i,j} + \sum_{s=1}^{k-1} (l_s - l_{s+1}) u^{k-s}_{i,j} + l_k u^0_{i,j} + \sigma'_{i,j,k} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i,m} u^k_{m,j}$$
$$+ \sigma''_{i,j,k} \sum_{m=0,m\neq j}^{N_y} g^{\beta}_{j,m} u^k_{i,m} + \mu f^k_{i,j}, \quad k > 0, \tag{2.19}$$

where $\sigma'_{i,j,k} = -\frac{\Gamma(2-\gamma)\tau^{\gamma}\kappa_{\alpha} c^k_{i,j}}{\Gamma(4-\alpha)(\Delta x)^{\alpha}}$, $\sigma''_{i,j,k} = -\frac{\Gamma(2-\gamma)\tau^{\gamma}\kappa_{\beta} d^k_{i,j}}{\Gamma(4-\beta)(\Delta y)^{\beta}}$.

3. Stability analysis

Now we perform the detailed stability analysis for the implicit and explicit schemes (2.13) and (2.14) of the one-dimensional case (2.10), and the implicit and explicit schemes (2.17) and (2.19) of the two-dimensional case (1.1). First we introduce two lemmas on the properties of the coefficients of the discretized fractional operators.

Lemma 3.1. *(See [5,6].) Let $\gamma \in (0, 1)$, then coefficients l_s defined in (2.9) satisfy*

(1) $l_s > 0, \quad s = 0, 1, \ldots, k.$

(2) $1 = l_0 > l_1 \cdots > l_k, \quad l_k \to 0 \quad \text{as} \quad k \to \infty.$

(3) $C_1 k^{\gamma} \leqslant (l_k)^{-1} \leqslant C_2 k^{\gamma}, \quad$ *where C_1 and C_2 are constants.*

(4) $\sum_{s=0}^{k} (l_s - l_{s+1}) + l_{k+1} = (1 - l_1) + \sum_{s=1}^{k-1} (l_s - l_{s+1}) + l_k = 1.$

Lemma 3.2. *The coefficients $g^{\nu}_{i,m}$, $\nu \in (1, 2]$ defined in (2.8) satisfy*

(1) $g^{\nu}_{i,i} < 0, \quad g^{\nu}_{i,m} > 0 \ (m \neq i);$

(2) $\sum_{m=0}^{N_x} g^{\nu}_{i,m} < 0 \quad \text{and} \quad -g^{\nu}_{i,i} > \sum_{m=0,m\neq i}^{N_x} g^{\nu}_{i,m}.$

Proof. According to [2,12], it is easy to verify that $g^{\nu}_{i,i+1} = g^{\nu}_{i,i-1} = 7 - 2^{5-\nu} + 3^{3-\nu} > 0$, $g^{\nu}_{i,i} = -8 + 2^{4-\nu} < 0$, and other $g^{\nu}_{i,m}$ are positive. And letting u be a constant in (2.7), we can easily derive that $\sum_{m=0}^{N_x} g^{\nu}_{i,m} < 0$, then $-g^{\nu}_{i,i} > \sum_{m=0,m\neq i}^{N_x} g^{\nu}_{i,m}$. $\square$

Next, we use two subsections to strictly prove that both the implicit schemes (2.13) and (2.17) are unconditionally stable; the explicit scheme (2.14) is stable under the condition $\frac{\tau^\gamma}{(\Delta x)^\alpha} < C$ and the explicit scheme (2.19) is stable under the condition $\frac{\tau^\gamma}{(\Delta x)^\alpha} + \frac{\tau^\gamma}{(\Delta y)^\beta} < C$.

3.1. The stability of the numerical methods in 1D

Theorem 3.3. *The implicit difference scheme (2.13) of the one-dimensional space–time Caputo–Riesz fractional diffusion equation (2.10) with $0 < \gamma \leqslant 1$, $1 < \alpha \leqslant 2$ is unconditionally stable.*

Proof. Let $\widetilde{u_i^k}$ $(i = 0, 1, \ldots, N_x;\ \ k = 0, 1, \ldots, N_t)$ be the approximate solution of u_i^k, which is the exact solution of the implicit scheme (2.13). Putting $\epsilon_i^k = \widetilde{u_i^k} - u_i^k$, then from (2.13) we obtain the following perturbation equation

$$\begin{aligned}
&\left(1 - \omega_{i,1} g_{i,i}^\alpha\right)\epsilon_i^1 - \omega_{i,1} \sum_{m=0,m\neq i}^{N_x} g_{i,m}^\alpha \epsilon_m^1 = \epsilon_i^0, \quad k = 0,\\
&\left(1 - \omega_{i,k+1} g_{i,i}^\alpha\right)\epsilon_i^{k+1} - \omega_{i,k+1} \sum_{m=0,m\neq i}^{N_x} g_{i,m}^\alpha \epsilon_m^{k+1} = \sum_{s=0}^{k-1}(l_s - l_{s+1})\epsilon_i^{k-s} + l_k\epsilon_i^0, \quad k > 0.
\end{aligned} \tag{3.1}$$

When $\gamma = 1$, Eq. (3.1) can be written as

$$\left(1 - \omega_{i,k+1} g_{i,i}^\alpha\right)\epsilon_i^{k+1} - \omega_{i,k+1} \sum_{m=0,m\neq i}^{N_x} g_{i,m}^\alpha \epsilon_m^{k+1} = \epsilon_i^k, \quad k \geqslant 0.$$

Denoting $E^k = [\epsilon_0^k, \epsilon_1^k, \ldots, \epsilon_{N_x}^k]$ and $\|E^k\|_\infty = \max_{0\leqslant i\leqslant N_x} |\epsilon_i^k|$, then we use the mathematical induction to prove the unconditional stability. For $k = 0$, supposing $|\epsilon_{i_0}^1| = \|E^1\|_\infty = \max_{0\leqslant i\leqslant N_x} |\epsilon_i^1|$, according to Lemma 3.2, we get

$$\begin{aligned}
\left\|E^1\right\|_\infty &= \left|\epsilon_{i_0}^1\right| \leqslant \left|\epsilon_{i_0}^1\right| - \omega_{i_0,1}\sum_{m=0}^{N_x} g_{i_0,m}^\alpha\left|\epsilon_{i_0}^1\right| = \left|\epsilon_{i_0}^1\right| - \omega_{i_0,1} g_{i_0,i_0}^\alpha\left|\epsilon_{i_0}^1\right| - \omega_{i_0,1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^\alpha\left|\epsilon_{i_0}^1\right|\\
&\leqslant \left(1 - \omega_{i_0,1} g_{i_0,i_0}^\alpha\right)\left|\epsilon_{i_0}^1\right| - \omega_{i_0,1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^\alpha\left|\epsilon_m^1\right| \leqslant \left|\left(1 - \omega_{i_0,1} g_{i_0,i_0}^\alpha\right)\epsilon_{i_0}^1 - \omega_{i_0,1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^\alpha \epsilon_m^1\right|\\
&= \left|\epsilon_{i_0}^0\right| \leqslant \left\|E^0\right\|_\infty.
\end{aligned}$$

Assuming $\|E^{\tilde{k}}\|_\infty \leqslant \|E^0\|_\infty, \tilde{k} = 1, 2, \ldots, k$, and $|\epsilon_{i_0}^{k+1}| = \|E^{k+1}\|_\infty = \max_{0\leqslant i\leqslant N_x} |\epsilon_i^{k+1}|$, there exists

$$\begin{aligned}
\left\|E^{k+1}\right\|_\infty &= \left|\epsilon_{i_0}^{k+1}\right| \leqslant \left|\epsilon_{i_0}^{k+1}\right| - \omega_{i_0,k+1}\sum_{m=0}^{N_x} g_{i_0,m}^\alpha\left|\epsilon_{i_0}^{k+1}\right|\\
&= \left|\epsilon_{i_0}^{k+1}\right| - \omega_{i_0,k+1} g_{i_0,i_0}^\alpha\left|\epsilon_{i_0}^{k+1}\right| - \omega_{i_0,k+1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^\alpha\left|\epsilon_{i_0}^{k+1}\right|\\
&\leqslant \left(1 - \omega_{i_0,k+1} g_{i_0,i_0}^\alpha\right)\left|\epsilon_{i_0}^{k+1}\right| - \omega_{i_0,k+1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^\alpha\left|\epsilon_m^{k+1}\right|\\
&\leqslant \left|\left(1 - \omega_{i_0,k+1} g_{i_0,i_0}^\alpha\right)\epsilon_{i_0}^{k+1} - \omega_{i_0,k+1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^\alpha \epsilon_m^{k+1}\right|\\
&= \left|\sum_{s=0}^{k-1}(l_s - l_{s+1})\epsilon_{i_0}^{k-s} + l_k\epsilon_{i_0}^0\right|\\
&= \left|(1 - l_1)\epsilon_{i_0}^k + l_k\epsilon_{i_0}^0 + \sum_{s=1}^{k-1}(l_s - l_{s+1})\epsilon_{i_0}^{k-s}\right|.
\end{aligned}$$

From Lemma 3.1, we obtain

$$\begin{aligned}\|E^{k+1}\|_\infty &\leqslant (1-l_1)\|E^k\|_\infty + l_k\|E^0\|_\infty + \sum_{s=1}^{k-1}(l_s - l_{s+1})\|E^{k-s}\|_\infty \\ &\leqslant (1-l_1)\|E^0\|_\infty + l_k\|E^0\|_\infty + \sum_{s=1}^{k-1}(l_s - l_{s+1})\|E^0\|_\infty \\ &\leqslant \|E^0\|_\infty.\end{aligned}$$

When $\gamma = 1$, using similar idea, we can prove

$$\|E^{k+1}\|_\infty \leqslant \left|(1-\omega_{i_0,k+1}g^\alpha_{i_0,i_0})\epsilon^{k+1}_{i_0} - \omega_{i_0,k+1}\sum_{m=0,m\neq i_0}^{N_x} g^\alpha_{i_0,m}\epsilon^{k+1}_m\right| = |\epsilon^k_{i_0}| \leqslant \|E^0\|_\infty. \qquad \square$$

Theorem 3.4. *If*

$$0 < \frac{\tau^\gamma}{(\Delta x)^\alpha} \leqslant -\frac{\Gamma(4-\alpha)(1-2^{-\gamma})}{4\kappa_\alpha C_{\max}\Gamma(2-\gamma)(1-2^{1-\alpha})}, \quad \textit{where } C_{\max} = \max_{0\leqslant i\leqslant N_x, 0\leqslant k\leqslant N_t} c(x_i, t_k),$$

then the explicit difference scheme (2.14) of the one-dimensional space–time Caputo–Riesz fractional diffusion equation (2.10) with $0 < \gamma \leqslant 1$, $1 < \alpha \leqslant 2$ is stable.

Proof. Under the above conditions, we obtain $0 < -\sigma_{i,k}g^\alpha_{i,i} \leqslant 2 - 2^{1-\gamma}$ and $1 - l_1 + \sigma_{i,k}g^\alpha_{i,i} \geqslant 0$. Assuming that $\widetilde{u}^k_i$ ($i = 0, 1, \ldots, N_x$; $k = 0, 1, \ldots, N_t$) be the approximate solution of u^k_i, which is the exact solution of the explicit scheme (2.14). Therefore the error $\epsilon^k_i = \widetilde{u}^k_i - u^k_i$ satisfies

$$\begin{aligned}&\epsilon^1_i = (1+\sigma_{i,0}g^\alpha_{i,i})\epsilon^0_i + \sigma_{i,0}\sum_{m=0,m\neq i}^{N_x} g^\alpha_{i,m}\epsilon^0_m, \quad k = 0, \\ &\epsilon^{k+1}_i = (1 - l_1 + \sigma_{i,k}g^\alpha_{i,i})\epsilon^k_i + \sum_{s=1}^{k-1}(l_s - l_{s+1})\epsilon^{k-s}_i + l_k\epsilon^0_i + \sigma_{i,k}\sum_{m=0,m\neq i}^{N_x} g^\alpha_{i,m}\epsilon^k_m, \quad k > 0.\end{aligned} \tag{3.2}$$

When $\gamma = 1$, Eq. (3.2) becomes

$$\epsilon^{k+1}_i = (1+\sigma_{i,k}g^\alpha_{i,i})\epsilon^k_i + \sigma_{i,k}\sum_{m=0,m\neq i}^{N_x} g^\alpha_{i,m}\epsilon^k_m, \quad k \geqslant 0.$$

Denoting $E^k = [\epsilon^k_0, \epsilon^k_1, \ldots, \epsilon^k_{N_x}]$ and $\|E^k\|_\infty = \max_{0\leqslant i\leqslant N_x}|\epsilon^k_i|$, we use mathematical induction to prove the conditional stability. For $k = 0$, supposing $|\epsilon^1_{i_0}| = \|E^1\|_\infty = \max_{0\leqslant i\leqslant N_x}|\epsilon^1_i|$, according to Lemma 3.2, we get

$$\begin{aligned}\|E^1\|_\infty = |\epsilon^1_{i_0}| &= \left|(1+\sigma_{i_0,0}g^\alpha_{i_0,i_0})\epsilon^0_{i_0} + \sigma_{i_0,0}\sum_{m=0,m\neq i_0}^{N_x} g^\alpha_{i_0,m}\epsilon^0_m\right| \\ &\leqslant (1+\sigma_{i_0,0}g^\alpha_{i_0,i_0})|\epsilon^0_{i_0}| + \sigma_{i_0,0}\sum_{m=0,m\neq i_0}^{N_x} g^\alpha_{i_0,m}|\epsilon^0_m| \\ &\leqslant (1+\sigma_{i_0,0}g^\alpha_{i_0,i_0})\|E^0\|_\infty + \sigma_{i_0,0}\sum_{m=0,m\neq i_0}^{N_x} g^\alpha_{i_0,m}\|E^0\|_\infty \\ &= \|E^0\|_\infty + \sigma_{i_0,0}\sum_{m=0}^{N_x} g^\alpha_{i_0,m}\|E^0\|_\infty \leqslant \|E^0\|_\infty.\end{aligned}$$

Assuming $\|E^{\tilde{k}}\|_\infty \leqslant \|E^0\|_\infty$, $\tilde{k} = 1, 2, \ldots, k$, and denoting $|\epsilon^{k+1}_{i_0}| = \|E^{k+1}\|_\infty = \max_{0\leqslant i\leqslant N_x}|\epsilon^{k+1}_i|$, from Lemma 3.1, there exists

$$\begin{aligned}
\|E^{k+1}\|_\infty = |\epsilon_{i_0}^{k+1}| &= \left|(1-l_1+\sigma_{i_0,k}g_{i_0,i_0}^{\alpha})\epsilon_{i_0}^{k}+\sum_{s=1}^{k-1}(l_s-l_{s+1})\epsilon_{i_0}^{k-s}+l_k\epsilon_{i_0}^{0}+\sigma_{i_0,k}\sum_{m=0,m\neq i_0}^{N_x}g_{i_0,m}^{\alpha}\epsilon_m^k\right| \\
&\leqslant (1-l_1+\sigma_{i_0,k}g_{i_0,i_0}^{\alpha})|\epsilon_{i_0}^{k}|+\sum_{s=1}^{k-1}(l_s-l_{s+1})|\epsilon_{i_0}^{k-s}|+l_k|\epsilon_{i_0}^{0}|+\sigma_{i_0,k}\sum_{m=0,m\neq i_0}^{N_x}g_{i_0,m}^{\alpha}|\epsilon_m^k| \\
&\leqslant (1-l_1)\|E^k\|_\infty+\sum_{s=1}^{k-1}(l_s-l_{s+1})\|E^{k-s}\|_\infty+l_k\|E^0\|_\infty+\sigma_{i_0,k}\sum_{m=0}^{N_x}g_{i_0,m}^{\alpha}\|E^k\|_\infty \\
&\leqslant (1-l_1)\|E^0\|_\infty+\sum_{s=1}^{k-1}(l_s-l_{s+1})\|E^0\|_\infty+l_k\|E^0\|_\infty \\
&= \left(1-l_1+\sum_{s=1}^{k-1}(l_s-l_{s+1})+l_k\right)\|E^0\|_\infty=\|E^0\|_\infty.
\end{aligned}$$

If $\gamma=1$, using similar method, we can prove

$$\begin{aligned}
\|E^{k+1}\|_\infty = |\epsilon_{i_0}^{k+1}| &= \left|(1+\sigma_{i_0,k}g_{i_0,i_0}^{\alpha})\epsilon_{i_0}^{k}+\sigma_{i_0,k}\sum_{m=0,m\neq i_0}^{N_x}g_{i_0,m}^{\alpha}\epsilon_m^k\right| \\
&\leqslant (1+\sigma_{i_0,k}g_{i_0,i_0}^{\alpha})|\epsilon_{i_0}^{k}|+\sigma_{i_0,k}\sum_{m=0,m\neq i_0}^{N_x}g_{i_0,m}^{\alpha}|\epsilon_m^k| \\
&\leqslant (1+\sigma_{i_0,k}g_{i_0,i_0}^{\alpha})\|E^k\|_\infty+\sigma_{i_0,k}\sum_{m=0,m\neq i_0}^{N_x}g_{i_0,m}^{\alpha}\|E^k\|_\infty \\
&= \|E^k\|_\infty+\sigma_{i_0,k}\sum_{m=0}^{N_x}g_{i_0,m}^{\alpha}\|E^k\|_\infty\leqslant\|E^k\|_\infty\leqslant\|E^0\|_\infty. \qquad \square
\end{aligned}$$

Remark 3.4. Let

$$\lambda_c=-\frac{\Gamma(4-\alpha)(1-2^{-\gamma})}{4\kappa_\alpha C_{\max}\Gamma(2-\gamma)(1-2^{1-\alpha})}-\frac{\tau^\gamma}{(\Delta x)^\alpha}, \tag{3.3}$$

then the explicit difference scheme (2.14) is stable if and only if $\lambda_c\geqslant 0$.

3.2. The stability of the numerical methods in 2D

Theorem 3.5. *The implicit difference scheme (2.17) of the two-dimensional space–time Caputo–Riesz fractional diffusion equation (1.1) with $0<\gamma\leqslant 1$, $1<\alpha,\beta\leqslant 2$ is unconditionally stable.*

Proof. Let $\widetilde{u_{i,j}^k}$ $(i=0,1,\ldots,N_x;\ \ j=0,1,\ldots,N_y; k=0,1,\ldots,N_t)$ be the approximate solution of $u_{i,j}^k$, which is the exact solution of the implicit scheme (2.17). Denoting that $\epsilon_{i,j}^k=\widetilde{u_{i,j}^k}-u_{i,j}^k$, from (2.17) we get the following perturbation equation

$$\begin{aligned}
&(1-\omega'_{i,j,k+1}g_{i,i}^{\alpha}-\omega''_{i,j,k+1}g_{j,j}^{\beta})\epsilon_{i,j}^1-\omega'_{i,j,k+1}\sum_{m=0,m\neq i}^{N_x}g_{i,m}^{\alpha}\epsilon_{m,j}^1-\omega''_{i,j,k+1}\sum_{m=0,m\neq j}^{N_y}g_{j,m}^{\beta}\epsilon_{i,m}^1 \\
&\quad=\epsilon_{i,j}^0, \quad k=0, \\
&(1-\omega'_{i,j,k+1}g_{i,i}^{\alpha}-\omega''_{i,j,k+1}g_{j,j}^{\beta})\epsilon_{i,j}^{k+1}-\omega'_{i,j,k+1}\sum_{m=0,m\neq i}^{N_x}g_{i,m}^{\alpha}\epsilon_{m,j}^{k+1}-\omega''_{i,j,k+1}\sum_{m=0,m\neq j}^{N_y}g_{j,m}^{\beta}\epsilon_{i,m}^{k+1} \\
&\quad=\sum_{s=0}^{k-1}(l_s-l_{s+1})\epsilon_{i,j}^{k-s}+l_k\epsilon_{i,j}^0, \quad k>0.
\end{aligned} \tag{3.4}$$

When $\gamma = 1$, Eq. (3.4) becomes

$$\left(1-\omega'_{i,j,k+1}g^{\alpha}_{i,i}-\omega''_{i,j,k+1}g^{\beta}_{j,j}\right)\epsilon^{k+1}_{i,j}-\omega'_{i,j,k+1}\sum_{m=0,m\neq i}^{N_x}g^{\alpha}_{i,m}\epsilon^{k+1}_{m,j}-\omega''_{i,j,k+1}\sum_{m=0,m\neq j}^{N_y}g^{\beta}_{j,m}\epsilon^{k+1}_{i,m}=\epsilon^{k}_{i,j},\quad k\geqslant 0.$$

Denote $\mathbf{u}^k=[\mathbf{u}^k_0,\mathbf{u}^k_1,\ldots,\mathbf{u}^k_{N_x}]^T$, $\mathbf{u}^k_i=[\epsilon^k_{i,0},\epsilon^k_{i,1},\ldots,\epsilon^k_{i,N_y}]^T$ and $\|\mathbf{E}^k\|_\infty=\max_{0\leqslant i\leqslant N_x,0\leqslant j\leqslant N_y}|\epsilon^k_{i,j}|$. We prove the results by mathematical induction. For $k=0$, supposing $|\epsilon^1_{i_0,j_0}|=\|\mathbf{E}^1\|_\infty=\max_{0\leqslant i\leqslant N_x,0\leqslant j\leqslant N_y}|\epsilon^1_{i,j}|$, there exists

$$\begin{aligned}
\|\mathbf{E}^1\|_\infty&=|\epsilon^1_{i_0,j_0}|\leqslant|\epsilon^1_{i_0,j_0}|-\omega'_{i_0,j_0,1}\sum_{m=0}^{N_x}g^{\alpha}_{i_0,m}|\epsilon^1_{i_0,j_0}|-\omega''_{i_0,j_0,1}\sum_{m=0}^{N_y}g^{\beta}_{j_0,m}|\epsilon^1_{i_0,j_0}|\\
&=|\epsilon^1_{i_0,j_0}|-\omega'_{i_0,j_0,1}g^{\alpha}_{i_0,i_0}|\epsilon^1_{i_0,j_0}|-\omega'_{i_0,j_0,1}\sum_{m=0,m\neq i_0}^{N_x}g^{\alpha}_{i_0,m}|\epsilon^1_{i_0,j_0}|\\
&\quad-\omega''_{i_0,j_0,1}g^{\beta}_{j_0,j_0}|\epsilon^1_{i_0,j_0}|-\omega''_{i_0,j_0,1}\sum_{m=0,m\neq j_0}^{N_y}g^{\beta}_{j_0,m}|\epsilon^1_{i_0,j_0}|\\
&\leqslant\left(1-\omega'_{i_0,j_0,1}g^{\alpha}_{i_0,i_0}-\omega''_{i_0,j_0,1}g^{\beta}_{j_0,j_0}\right)|\epsilon^1_{i_0,j_0}|-\omega'_{i_0,j_0,1}\sum_{m=0,m\neq i_0}^{N_x}g^{\alpha}_{i_0,m}|\epsilon^1_{m,j_0}|-\omega''_{i_0,j_0,1}\sum_{m=0,m\neq j_0}^{N_y}g^{\beta}_{j_0,m}|\epsilon^1_{i_0,m}|\\
&\leqslant\left|\left(1-\omega'_{i_0,j_0,1}g^{\alpha}_{i_0,i_0}-\omega''_{i_0,j_0,1}g^{\beta}_{j_0,j_0}\right)\epsilon^1_{i_0,j_0}-\omega'_{i_0,j_0,1}\sum_{m=0,m\neq i_0}^{N_x}g^{\alpha}_{i_0,m}\epsilon^1_{m,j_0}-\omega''_{i_0,j_0,1}\sum_{m=0,m\neq j_0}^{N_y}g^{\beta}_{j_0,m}\epsilon^1_{i_0,m}\right|\\
&=|\epsilon^0_{i_0,j_0}|\leqslant\|\mathbf{E}^0\|_\infty.
\end{aligned}$$

Supposing $\|\mathbf{E}^{\tilde{k}}\|_\infty\leqslant\|\mathbf{E}^0\|_\infty,\tilde{k}=1,2,\ldots,k$, and $|\epsilon^{k+1}_{i_0,j_0}|=\|\mathbf{E}^{k+1}\|_\infty=\max_{0\leqslant i\leqslant N_x,0\leqslant j\leqslant N_y}|\epsilon^{k+1}_{i,j}|$, then

$$\begin{aligned}
\|\mathbf{E}^{k+1}\|_\infty&=|\epsilon^{k+1}_{i_0,j_0}|\leqslant|\epsilon^{k+1}_{i_0,j_0}|-\omega'_{i_0,j_0,k+1}\sum_{m=0}^{N_x}g^{\alpha}_{i_0,m}|\epsilon^{k+1}_{i_0,j_0}|-\omega''_{i_0,j_0,k+1}\sum_{m=0}^{N_y}g^{\beta}_{j_0,m}|\epsilon^{k+1}_{i_0,j_0}|\\
&=|\epsilon^{k+1}_{i_0,j_0}|-\omega'_{i_0,j_0,k+1}g^{\alpha}_{i_0,i_0}|\epsilon^{k+1}_{i_0,j_0}|-\omega'_{i_0,j_0,k+1}\sum_{m=0,m\neq i_0}^{N_x}g^{\alpha}_{i_0,m}|\epsilon^{k+1}_{i_0,j_0}|\\
&\quad-\omega''_{i_0,j_0,k+1}g^{\beta}_{j_0,j_0}|\epsilon^{k+1}_{i_0,j_0}|-\omega''_{i_0,j_0,k+1}\sum_{m=0,m\neq j_0}^{N_y}g^{\beta}_{j_0,m}|\epsilon^{k+1}_{i_0,j_0}|\\
&\leqslant\left(1-\omega'_{i_0,j_0,k+1}g^{\alpha}_{i_0,i_0}-\omega''_{i_0,j_0,k+1}g^{\beta}_{j_0,j_0}\right)|\epsilon^{k+1}_{i_0,j_0}|\\
&\quad-\omega'_{i_0,j_0,k+1}\sum_{m=0,m\neq i_0}^{N_x}g^{\alpha}_{i_0,m}|\epsilon^{k+1}_{m,j_0}|-\omega''_{i_0,j_0,k+1}\sum_{m=0,m\neq j_0}^{N_y}g^{\beta}_{j_0,m}|\epsilon^{k+1}_{i_0,m}|\\
&\leqslant\left|\left(1-\omega'_{i_0,j_0,k+1}g^{\alpha}_{i_0,i_0}-\omega''_{i_0,j_0,k+1}g^{\beta}_{j_0,j_0}\right)\epsilon^{k+1}_{i_0,j_0}-\omega'_{i_0,j_0,k+1}\sum_{m=0,m\neq i_0}^{N_x}g^{\alpha}_{i_0,m}\epsilon^{k+1}_{m,j_0}-\omega''_{i_0,j_0,k+1}\sum_{m=0,m\neq j_0}^{N_y}g^{\beta}_{j_0,m}\epsilon^{k+1}_{i_0,m}\right|\\
&=\left|\sum_{s=0}^{k-1}(l_s-l_{s+1})\epsilon^{k-s}_{i_0,j_0}+l_k\epsilon^0_{i_0,j_0}\right|=\left|(1-l_1)\epsilon^k_{i_0,j_0}+l_k\epsilon^0_{i_0,j_0}+\sum_{s=1}^{k-1}(l_s-l_{s+1})\epsilon^{k-s}_{i_0,j_0}\right|.
\end{aligned}$$

By Lemma 3.1, we get

$$\begin{aligned}
\|\mathbf{E}^{k+1}\|_\infty&\leqslant(1-l_1)\|\mathbf{E}^k\|_\infty+l_k\|\mathbf{E}^0\|_\infty+\sum_{s=1}^{k-1}(l_s-l_{s+1})\|\mathbf{E}^{k-s}\|_\infty\\
&\leqslant(1-l_1)\|\mathbf{E}^0\|_\infty+l_k\|\mathbf{E}^0\|_\infty+\sum_{s=1}^{k-1}(l_s-l_{s+1})\|\mathbf{E}^0\|_\infty\leqslant\|\mathbf{E}^0\|_\infty.
\end{aligned}$$

If $\gamma = 1$, we obtain

$$\begin{aligned}\|\mathbf{E}^{k+1}\|_\infty &\leqslant \bigg|\big(1-\omega'_{i_0,j_0,k+1}g^{\alpha}_{i_0,i_0}-\omega''_{i_0,j_0,k+1}g^{\beta}_{j_0,j_0}\big)\epsilon^{k+1}_{i_0,j_0} \\ &\quad -\omega'_{i_0,j_0,k+1}\sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i_0,m}\epsilon^{k+1}_{m,j_0}-\omega''_{i_0,j_0,k+1}\sum_{m=0,m\neq j}^{N_y} g^{\beta}_{j_0,m}\epsilon^{k+1}_{i_0,m}\bigg| \\ &= \big|\epsilon^{k}_{i_0,j_0}\big| \leqslant \|\mathbf{E}^0\|_\infty. \qquad \square\end{aligned}$$

Theorem 3.6. *If* $0<\frac{\tau^\gamma}{(\Delta x)^\alpha}+\frac{\tau^\gamma}{(\Delta y)^\beta}\leqslant\frac{1-2^{-\gamma}}{\Gamma(2-\gamma)\mathbf{C}_{\max}}$, *where*

$$\mathbf{C}_{\max}=\max_{0\leqslant i\leqslant N_x,0\leqslant j\leqslant N_y,0\leqslant k\leqslant N_t}\left\{-\frac{\kappa_\alpha(4-2^{3-\alpha})c^k_{i,j}}{\Gamma(4-\alpha)},-\frac{\kappa_\beta(4-2^{3-\beta})d^k_{i,j}}{\Gamma(4-\beta)}\right\},$$

then the explicit difference scheme (2.19) of the two-dimensional space–time Caputo–Riesz fractional diffusion equation (1.1) with $0<\gamma\leqslant 1$, $1<\alpha,\beta\leqslant 2$ *is stable.*

Proof. Let $\widetilde{u^k_{i,j}}$ $(i=0,1,\ldots,N_x;\ \ j=0,1,\ldots,N_y;\ \ k=0,1,\ldots,N_t)$ be the approximate solution of $u^k_{i,j}$, which is the exact solution of the explicit scheme (2.19). Denoting $\epsilon^k_{i,j}=\widetilde{u^k_{i,j}}-u^k_{i,j}$ and using (2.19), we obtain the following perturbation form

$$\begin{aligned}\epsilon^1_{i,j}&=\big(1+\sigma'_{i,j,0}g^{\alpha}_{i,i}+\sigma''_{i,j,0}g^{\beta}_{j,j}\big)\epsilon^0_{i,j}+\sigma'_{i,j,0}\sum_{m=0,m\neq i}^{N_x}g^{\alpha}_{i,m}\epsilon^0_{m,j}+\sigma''_{i,j,0}\sum_{m=0,m\neq j}^{N_y}g^{\beta}_{j,m}\epsilon^0_{i,m},\quad k=0,\\ \epsilon^{k+1}_{i,j}&=\big(1-l_1+\sigma'_{i,j,k}g^{\alpha}_{i,i}+\sigma''_{i,j,k}g^{\beta}_{j,j}\big)\epsilon^k_{i,j}+\sum_{s=1}^{k-1}(l_s-l_{s+1})\epsilon^{k-s}_{i,j}+l_k\epsilon^0_{i,j}\\ &\quad+\sigma'_{i,j,k}\sum_{m=0,m\neq i}^{N_x}g^{\alpha}_{i,m}\epsilon^k_{m,j}+\sigma''_{i,j,k}\sum_{m=0,m\neq j}^{N_y}g^{\beta}_{j,m}\epsilon^k_{i,m},\quad k>0.\end{aligned}\tag{3.5}$$

When $\gamma=1$, Eq. (3.5) can be rewritten as

$$\epsilon^{k+1}_{i,j}=\big(1+\sigma'_{i,j,k}g^{\alpha}_{i,i}+\sigma''_{i,j,k}g^{\beta}_{j,j}\big)\epsilon^k_{i,j}+\sigma'_{i,j,k}\sum_{m=0,m\neq i}^{N_x}g^{\alpha}_{i,m}\epsilon^k_{m,j}+\sigma''_{i,j,k}\sum_{m=0,m\neq j}^{N_y}g^{\beta}_{j,m}\epsilon^k_{i,m},\quad k\geqslant 0.$$

Using the mathematical induction, we can prove the desired result. Under the conditions of the theorem, there exists $1+\sigma'_{i_0,j_0,k}g^{\alpha}_{i_0,i_0}+\sigma''_{i_0,j_0,k}g^{\beta}_{j_0,j_0}>0$. Let $\mathbf{u}^k=[\mathbf{u}^k_0,\mathbf{u}^k_1,\ldots,\mathbf{u}^k_{N_x}]^T$, where $\mathbf{u}^k_i=[\epsilon^k_{i,0},\epsilon^k_{i,1},\ldots,\epsilon^k_{i,N_y}]^T$ and $\|\mathbf{E}^k\|_\infty=\max_{0\leqslant i\leqslant N_x,0\leqslant j\leqslant N_y}|\epsilon^k_{i,j}|$. For $k=0$, supposing $|\epsilon^1_{i_0,j_0}|=\|\mathbf{E}^1\|_\infty=\max_{0\leqslant i\leqslant N_x,0\leqslant j\leqslant N_y}|\epsilon^1_{i,j}|$, we obtain

$$\begin{aligned}\|\mathbf{E}^1\|_\infty&=\big|\epsilon^1_{i_0,j_0}\big|\\ &=\bigg|\big(1+\sigma'_{i_0,j_0,0}g^{\alpha}_{i_0,i_0}+\sigma''_{i_0,j_0,0}g^{\beta}_{j_0,j_0}\big)\epsilon^0_{i_0,j_0}+\sigma'_{i_0,j_0,0}\sum_{m=0,m\neq i_0}^{N_x}g^{\alpha}_{i_0,m}\epsilon^0_{m,j_0}+\sigma''_{i_0,j_0,0}\sum_{m=0,m\neq j_0}^{N_y}g^{\beta}_{j_0,m}\epsilon^0_{i_0,m}\bigg|\\ &\leqslant\big(1+\sigma'_{i_0,j_0,0}g^{\alpha}_{i_0,i_0}+\sigma''_{i_0,j_0,0}g^{\beta}_{j_0,j_0}\big)\big|\epsilon^0_{i_0,j_0}\big|+\sigma'_{i_0,j_0,0}\sum_{m=0,m\neq i_0}^{N_x}g^{\alpha}_{i_0,m}\big|\epsilon^0_{m,j_0}\big|+\sigma''_{i_0,j_0,0}\sum_{m=0,m\neq j_0}^{N_y}g^{\beta}_{j_0,m}\big|\epsilon^0_{i_0,m}\big|\\ &\leqslant\big(1+\sigma'_{i_0,j_0,0}g^{\alpha}_{i_0,i_0}+\sigma''_{i_0,j_0,0}g^{\beta}_{j_0,j_0}\big)\|\mathbf{E}^0\|_\infty+\sigma'_{i_0,j_0,0}\sum_{m=0,m\neq i_0}^{N_x}g^{\alpha}_{i_0,m}\|\mathbf{E}^0\|_\infty+\sigma''_{i_0,j_0,0}\sum_{m=0,m\neq j_0}^{N_y}g^{\beta}_{j_0,m}\|\mathbf{E}^0\|_\infty\\ &=\|\mathbf{E}^0\|_\infty+\sigma'_{i_0,j_0,0}\sum_{m=0}^{N_x}g^{\alpha}_{i_0,m}\|\mathbf{E}^0\|_\infty+\sigma''_{i_0,j_0,0}\sum_{m=0}^{N_y}g^{\beta}_{j_0,m}\|\mathbf{E}^0\|_\infty\leqslant\|\mathbf{E}^0\|_\infty.\end{aligned}$$

Assuming $\|\mathbf{E}^{\tilde{k}}\|_\infty \leqslant \|\mathbf{E}^0\|_\infty$, $\tilde{k}=1,2,\ldots,k$, and $|\epsilon_{i_0,j_0}^{k+1}| = \|\mathbf{E}^{k+1}\|_\infty = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y} |\epsilon_{i,j}^{k+1}|$, we get

$$
\begin{aligned}
\|\mathbf{E}^{k+1}\|_\infty = |\epsilon_{i_0,j_0}^{k+1}| &= \Bigg|\big(1-l_1+\sigma'_{i_0,j_0,k}g^\alpha_{i_0,i_0}+\sigma''_{i_0,j_0,k}g^\beta_{j_0,j_0}\big)\epsilon^k_{i_0,j_0}+\sum_{s=1}^{k-1}(l_s-l_{s+1})\epsilon^{k-s}_{i_0,j_0}+l_k\epsilon^0_{i_0,j_0}\\
&\quad+\sigma'_{i_0,j_0,k}\sum_{m=0,m\neq i_0}^{N_x}g^\alpha_{i_0,m}\epsilon^k_{m,j_0}+\sigma''_{i_0,j_0,k}\sum_{m=0,m\neq j_0}^{N_y}g^\beta_{j_0,m}\epsilon^k_{i_0,m}\Bigg|\\
&\leqslant \big(1-l_1+\sigma'_{i_0,j_0,k}g^\alpha_{i_0,i_0}+\sigma''_{i_0,j_0,k}g^\beta_{j_0,j_0}\big)\big|\epsilon^k_{i_0,j_0}\big|+\sum_{s=1}^{k-1}(l_s-l_{s+1})\big|\epsilon^{k-s}_{i_0,j_0}\big|+l_k\big|\epsilon^0_{i_0,j_0}\big|\\
&\quad+\sigma'_{i_0,j_0,k}\sum_{m=0,m\neq i_0}^{N_x}g^\alpha_{i_0,m}\big|\epsilon^k_{m,j_0}\big|+\sigma''_{i_0,j_0,k}\sum_{m=0,m\neq j_0}^{N_y}g^\beta_{j_0,m}\big|\epsilon^k_{i_0,m}\big|\\
&\leqslant (1-l_1)\big\|\mathbf{E}^k\big\|_\infty+\sum_{s=1}^{k-1}(l_s-l_{s+1})\big\|\mathbf{E}^{k-s}\big\|_\infty+l_k\big\|\mathbf{E}^0\big\|_\infty\\
&\quad+\sigma'_{i_0,j_0,k}\sum_{m=0}^{N_x}g^\alpha_{i_0,m}\big\|\mathbf{E}^k\big\|_\infty+\sigma''_{i_0,j_0,k}\sum_{m=0}^{N_y}g^\beta_{j_0,m}\big\|\mathbf{E}^k\big\|_\infty\\
&\leqslant (1-l_1)\big\|\mathbf{E}^0\big\|_\infty+\sum_{s=1}^{k-1}(l_s-l_{s+1})\big\|\mathbf{E}^0\big\|_\infty+l_k\big\|\mathbf{E}^0\big\|_\infty\\
&=\left(1-l_1+\sum_{s=1}^{k-1}(l_s-l_{s+1})+l_k\right)\big\|\mathbf{E}^0\big\|_\infty=\big\|\mathbf{E}^0\big\|_\infty.
\end{aligned}
$$

When $\gamma=1$, it is easy to check that

$$
\begin{aligned}
\|\mathbf{E}^{k+1}\|_\infty &= |\epsilon^{k+1}_{i_0,j_0}|\\
&=\Bigg|\big(1+\sigma'_{i_0,j_0,k}g^\alpha_{i_0,i_0}+\sigma''_{i_0,j_0,k}g^\beta_{j_0,j_0}\big)\epsilon^k_{i_0,j_0}+\sigma'_{i_0,j_0,k}\sum_{m=0,m\neq i_0}^{N_x}g^\alpha_{i_0,m}\epsilon^k_{m,j_0}+\sigma''_{i_0,j_0,k}\sum_{m=0,m\neq j_0}^{N_y}g^\beta_{j_0,m}\epsilon^k_{i_0,m}\Bigg|\\
&\leqslant\big(1+\sigma'_{i_0,j_0,k}g^\alpha_{i_0,i_0}+\sigma''_{i_0,j_0,k}g^\beta_{j_0,j_0}\big)\big|\epsilon^k_{i_0,j_0}\big|+\sigma'_{i_0,j_0,k}\sum_{m=0,m\neq i_0}^{N_x}g^\alpha_{i_0,m}\big|\epsilon^k_{m,j_0}\big|+\sigma''_{i_0,j_0,k}\sum_{m=0,m\neq j_0}^{N_y}g^\beta_{j_0,m}\big|\epsilon^k_{i_0,m}\big|\\
&\leqslant\big(1+\sigma'_{i_0,j_0,k}g^\alpha_{i_0,i_0}+\sigma''_{i_0,j_0,k}g^\beta_{j_0,j_0}\big)\big\|\mathbf{E}^k\big\|_\infty+\sigma'_{i_0,j_0,k}\sum_{m=0,m\neq i_0}^{N_x}g^\alpha_{i_0,m}\big\|\mathbf{E}^k\big\|_\infty+\sigma''_{i_0,j_0,k}\sum_{m=0,m\neq j_0}^{N_y}g^\beta_{j_0,m}\big\|\mathbf{E}^k\big\|_\infty\\
&=\big\|\mathbf{E}^k\big\|_\infty+\sigma'_{i_0,j_0,k}\sum_{m=0}^{N_x}g^\alpha_{i_0,m}\big\|\mathbf{E}^k\big\|_\infty+\sigma''_{i_0,j_0,k}\sum_{m=0}^{N_y}g^\beta_{j_0,m}\big\|\mathbf{E}^k\big\|_\infty\leqslant\big\|\mathbf{E}^k\big\|_\infty\leqslant\big\|\mathbf{E}^0\big\|_\infty. \qquad \square
\end{aligned}
$$

4. Convergence analysis

We use two subsections to prove that the global truncation error of the schemes (2.13) and (2.14) used to solve (2.10) is $\mathcal{O}(\tau^{2-\gamma}+(\Delta x)^2)$, and the global truncation error of the schemes (2.17) and (2.19) used to solve (1.1) is $\mathcal{O}(\tau^{2-\gamma}+(\Delta x)^2+(\Delta y)^2)$.

4.1. The convergence of the numerical methods in 1D

Theorem 4.1. *Let u_i^k be the approximation solution of $u(x_i,t_k)$ computed by use of the implicit difference scheme (2.13), then there is a positive constant C such that*

$$\big|u(x_i,t_k)-u_i^k\big|\leqslant C\big(\tau^{2-\gamma}+(\Delta x)^2\big),\quad i=0,1,\ldots,N_x;\ k=0,1,\ldots,N_t.$$

Proof. Let $u(x_i,t_k)$ be the exact solution of (2.10) at the mesh point (x_i,t_k). Define $\varepsilon_i^k = u(x_i,t_k) - u_i^k$, and $e^k = [\varepsilon_0^k, \varepsilon_1^k, \dots, \varepsilon_{N_x}^k]$. Subtracting (2.11) from (2.12) and using $e^0 = 0$, we obtain

$$\begin{aligned}
&\left(1-\omega_{i,1} g_{i,i}^{\alpha}\right)\varepsilon_i^1 - \omega_{i,1}\sum_{m=0,m\neq i}^{N_x} g_{i,m}^{\alpha}\varepsilon_m^1 = R_i^1, \quad k=0,\\
&\left(1-\omega_{i,k+1} g_{i,i}^{\alpha}\right)\varepsilon_i^{k+1} - \omega_{i,k+1}\sum_{m=0,m\neq i}^{N_x} g_{i,m}^{\alpha}\varepsilon_m^{k+1} = \sum_{s=0}^{k-1}(l_s - l_{s+1})\varepsilon_i^{k-s} + R_i^{k+1}, \quad k>0.
\end{aligned} \tag{4.1}$$

When $\gamma = 1$, Eq. (4.1) can be written as

$$\begin{aligned}
&\left(1-\omega_{i,1} g_{i,i}^{\alpha}\right)\varepsilon_i^1 - \omega_{i,1}\sum_{m=0,m\neq i}^{N_x} g_{i,m}^{\alpha}\varepsilon_m^1 = R_i^1, \quad k=0,\\
&\left(1-\omega_{i,k+1} g_{i,i}^{\alpha}\right)\varepsilon_i^{k+1} - \omega_{i,k+1}\sum_{m=0,m\neq i}^{N_x} g_{i,m}^{\alpha}\varepsilon_m^{k+1} = \varepsilon_i^k + R_i^{k+1}, \quad k>0.
\end{aligned} \tag{4.2}$$

Denoting that $\|e^k\|_\infty = \max_{0\leqslant i\leqslant N_x}|\varepsilon_i^k|$ and $R_{\max} = \max_{0\leqslant i\leqslant N_x, 0\leqslant k\leqslant N_t}|R_i^k|$, the desired result can be proved by using mathematical induction.

(1) Case $0<\gamma<1$: For $k=0$, supposing $|\varepsilon_{i_0}^1| = \|e^1\|_\infty = \max_{0\leqslant i\leqslant N_x}|\varepsilon_i^1|$, according to Lemma 3.2, we get

$$\begin{aligned}
\|e^1\|_\infty = |\varepsilon_{i_0}^1| &\leqslant |\varepsilon_{i_0}^1| - \omega_{i_0,1}\sum_{m=0}^{N_x} g_{i_0,m}^{\alpha}|\varepsilon_{i_0}^1|\\
&= |\varepsilon_{i_0}^1| - \omega_{i_0,1} g_{i_0,i_0}^{\alpha}|\varepsilon_{i_0}^1| - \omega_{i_0,1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}|\varepsilon_{i_0}^1|\\
&\leqslant \left(1-\omega_{i_0,1} g_{i_0,i_0}^{\alpha}\right)|\varepsilon_{i_0}^1| - \omega_{i_0,1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}|\varepsilon_m^1|\\
&\leqslant \left|\left(1-\omega_{i_0,1} g_{i_0,i_0}^{\alpha}\right)\varepsilon_{i_0}^1 - \omega_{i_0,1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}\varepsilon_m^1\right|\\
&= |R_{i_0}^1| \leqslant R_{\max} = l_0^{-1}R_{\max}.
\end{aligned}$$

Supposing $\|e^{\tilde{k}}\|_\infty \leqslant l_{\tilde{k}-1}^{-1}R_{\max}$, $\tilde{k} = 1,2,\dots,k$, and $|\varepsilon_{i_0}^{k+1}| = \|e^{k+1}\|_\infty = \max_{0\leqslant i\leqslant N_x}|\varepsilon_i^{k+1}|$, according to the result of Lemma 3.1, $l_{\tilde{k}}^{-1} \leqslant l_k^{-1}$, $\tilde{k} = 0,1,\dots,k$, therefore, $\|e^{\tilde{k}}\|_\infty \leqslant l_k^{-1}R_{\max}$, $\tilde{k} = 1,2,\dots,k$. Then we have

$$\begin{aligned}
\|e^{k+1}\|_\infty = |\varepsilon_{i_0}^{k+1}| &\leqslant |\varepsilon_{i_0}^{k+1}| - \omega_{i_0,k+1}\sum_{m=0}^{N_x} g_{i_0,m}^{\alpha}|\varepsilon_{i_0}^{k+1}|\\
&= |\varepsilon_{i_0}^{k+1}| - \omega_{i_0,k+1} g_{i_0,i_0}^{\alpha}|\varepsilon_{i_0}^{k+1}| - \omega_{i_0,k+1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}|\varepsilon_{i_0}^{k+1}|\\
&\leqslant \left(1-\omega_{i_0,k+1} g_{i_0,i_0}^{\alpha}\right)|\varepsilon_{i_0}^{k+1}| - \omega_{i_0,k+1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}|\varepsilon_m^{k+1}|\\
&\leqslant \left|\left(1-\omega_{i_0,k+1} g_{i_0,i_0}^{\alpha}\right)\varepsilon_{i_0}^{k+1} - \omega_{i_0,k+1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}\varepsilon_m^{k+1}\right|\\
&= \left|\sum_{s=0}^{k-1}(l_s - l_{s+1})\varepsilon_{i_0}^{k-s} + R_{i_0}^{k+1}\right|\\
&= \left|(1-l_1)\varepsilon_{i_0}^k + \sum_{s=1}^{k-1}(l_s - l_{s+1})\varepsilon_{i_0}^{k-s} + R_{i_0}^{k+1}\right|.
\end{aligned}$$

From Lemma 3.1, we obtain

$$\begin{aligned}\|e^{k+1}\|_\infty &\leqslant (1-l_1)\|e^k\|_\infty + \sum_{s=1}^{k-1}(l_s - l_{s+1})\|e^{k-s}\|_\infty + |R_{i_0}^{k+1}| \\ &\leqslant l_k^{-1}\left(1 - l_1 + \sum_{s=1}^{k-1}(l_s - l_{s+1}) + l_k\right)R_{\max} \\ &= l_k^{-1}R_{\max} \leqslant Ck^\gamma R_{\max} = C(k\tau)^\gamma\left(\tau^{2-\gamma} + (\Delta x)^2\right) \\ &\leqslant CT^\gamma\left(\tau^{2-\gamma} + (\Delta x)^2\right).\end{aligned}$$

(2) Case $\gamma = 1$: Using similar idea leads to

$$\begin{aligned}\|e^{k+1}\|_\infty &\leqslant \left|(1 - \omega_{i_0,k+1}g_{i_0,i_0}^\alpha)\varepsilon_{i_0}^{k+1} - \omega_{i_0,k+1}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^\alpha \varepsilon_m^{k+1}\right| = |\varepsilon_{i_0}^k + R_{i_0}^{k+1}| \\ &\leqslant \|e^k\|_\infty + |R_{i_0}^{k+1}| \leqslant (k+1)R_{\max} \leqslant C(k+1)\tau\left(\tau + (\Delta x)^2\right) \\ &\leqslant C'\left(\tau + (\Delta x)^2\right) = C'\left(\tau^{2-\gamma} + (\Delta x)^2\right). \qquad \square\end{aligned}$$

Theorem 4.2. *Let u_i^k be the approximation solution of $u(x_i, t_k)$ computed by use of the explicit difference scheme (2.19). If $0 < \frac{\tau^\gamma}{(\Delta x)^\alpha} \leqslant -\frac{\Gamma(4-\alpha)(1-2^{-\gamma})}{4\kappa_\alpha C_{\max}\Gamma(2-\gamma)(1-2^{1-\alpha})}$, where $C_{\max} = \max_{0\leqslant i\leqslant N_x, 0\leqslant k\leqslant N_t} c(x_i, t_k)$, then there is a positive constant C such that*

$$|u(x_i, t_k) - u_i^k| \leqslant C\left(\tau^{2-\gamma} + (\Delta x)^2\right), \quad i = 0, 1, \ldots, N_x;\ k = 0, 1, \ldots, N_t.$$

Proof. Define $\varepsilon_i^k = u(x_i, t_k) - u_i^k$ and $R_{\max} = \max_{0\leqslant i\leqslant N_x, 0\leqslant k\leqslant N_t}|R_i^k|$. Analogously, we have

$$\begin{aligned}&\epsilon_i^1 = R_i^1, \quad k = 0, \\ &\varepsilon_i^{k+1} = \left(1 - l_1 + \sigma_{i,k}g_{i,i}^\alpha\right)\varepsilon_i^k + \sum_{s=1}^{k-1}(l_s - l_{s+1})\epsilon_i^{k-s} + \sigma_{i,k}\sum_{m=0,m\neq i}^{N_x} g_{i,m}^\alpha \varepsilon_m^k + R_i^{k+1}, \quad k > 0.\end{aligned} \tag{4.3}$$

When $\gamma = 1$, Eq. (4.3) can be rewrote as

$$\begin{aligned}&\epsilon_i^1 = R_i^1, \quad k = 0, \\ &\varepsilon_i^{k+1} = \left(1 + \sigma_{i,k}g_{i,i}^\alpha\right)\varepsilon_i^k + \sigma_{i,k}\sum_{m=0,m\neq i}^{N_x} g_{i,m}^\alpha \varepsilon_m^k + R_i^{k+1}, \quad k > 0.\end{aligned} \tag{4.4}$$

Denoting that $e^k = [\varepsilon_0^k, \varepsilon_1^k, \ldots, \varepsilon_{N_x}^k]$ and $\|e^k\|_\infty = \max_{0\leqslant i\leqslant N_x}|\epsilon_i^k|$, we use mathematical induction to prove the desired result.

(1) Case $0 < \gamma < 1$: For $k = 0$, supposing $|\varepsilon_{i_0}^1| = \|e^1\|_\infty = \max_{0\leqslant i\leqslant N_x}|\varepsilon_i^1|$, we have

$$\|e^1\|_\infty = |\varepsilon_{i_0}^1| = |R_{i_0}^1| \leqslant R_{\max} = l_0^{-1}R_{\max}.$$

Assuming $\|e^{\tilde{k}}\|_\infty \leqslant l_{\tilde{k}-1}^{-1}R_{\max}$, $\tilde{k} = 1, 2, \ldots, k$, and $|\varepsilon_{i_0}^{k+1}| = \|e^{k+1}\|_\infty = \max_{0\leqslant i\leqslant N_x}|\varepsilon_i^{k+1}|$, using $l_{\tilde{k}}^{-1} \leqslant l_k^{-1}$, $\tilde{k} = 0, 1, \ldots, k$, therefore, $\|e^{\tilde{k}}\|_\infty \leqslant l_k^{-1}R_{\max}$, $\tilde{k} = 1, 2, \ldots, k$. Then we get

$$\begin{aligned}\|e^{k+1}\|_\infty = |\varepsilon_{i_0}^{k+1}| &= \left|\left(1 - l_1 + \sigma_{i_0,k}g_{i_0,i_0}^\alpha\right)\varepsilon_{i_0}^k + \sum_{s=1}^{k-1}(l_s - l_{s+1})\varepsilon_{i_0}^{k-s} + \sigma_{i_0,k}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^\alpha \varepsilon_m^k + R_{i_0}^{k+1}\right| \\ &\leqslant \left(1 - l_1 + \sigma_{i_0,k}g_{i_0,i_0}^\alpha\right)|\varepsilon_{i_0}^k| + \sum_{s=1}^{k-1}(l_s - l_{s+1})|\varepsilon_{i_0}^{k-s}| + \sigma_{i_0,k}\sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^\alpha|\varepsilon_m^k| + R_{\max} \\ &\leqslant (1-l_1)\|e^k\|_\infty + \sum_{s=1}^{k-1}(l_s - l_{s+1})\|e^{k-s}\|_\infty + \sigma_{i_0,k}\sum_{m=0}^{N_x} g_{i_0,m}^\alpha\|e^k\|_\infty + R_{\max} \\ &\leqslant l_k^{-1}\left(1 - l_1 + \sum_{s=1}^{k-1}(l_s - l_{s+1}) + l_k\right)R_{\max} = l_k^{-1}R_{\max} \leqslant C\left(\tau^{2-\gamma} + (\Delta x)^2\right).\end{aligned}$$

(2) Case $\gamma = 1$: Analogously, we have

$$\begin{aligned}\|e^{k+1}\|_\infty &= |\varepsilon_{i_0}^{k+1}| \leqslant (1+\sigma_{i_0,k} g_{i_0,i_0}^{\alpha})|\varepsilon_{i_0}^{k}| + \sigma_{i_0,k} \sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha} |\varepsilon_m^k| + R_{\max} \\ &\leqslant \|e^k\|_\infty + \sigma_{i_0,k} \sum_{m=0}^{N_x} g_{i_0,m}^{\alpha} \|e^k\|_\infty + R_{\max} \leqslant \|e^k\|_\infty + R_{\max} \leqslant C(\tau^{2-\gamma} + (\Delta x)^2). \qquad \square\end{aligned}$$

4.2. The convergence of the numerical methods in 2D

Theorem 4.3. *Let $u_{i,j}^k$ be the approximation solution of $u(x_i, y_j, t_k)$ computed by use of the implicit difference scheme (2.17), then there is a positive constant C such that*

$$|u(x_i, y_j, t_k) - u_{i,j}^k| \leqslant C(\tau^{2-\gamma} + (\Delta x)^2 + (\Delta y)^2),$$

where $i = 0, 1, \ldots, N_x$; $j = 0, 1, \ldots, N_y$; $k = 0, 1, \ldots, N_t$.

Proof. Defining $\varepsilon_{i,j}^k = u(x_i, y_j, t_k) - u_{i,j}^k$, $\mathbf{R}_{\max} = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y, 0\leqslant k\leqslant N_t} |R_{i_j}^k|$, and subtracting (2.15) from (2.16), we obtain

$$\begin{aligned}&(1 - \omega'_{i,j,k+1} g_{i,i}^{\alpha} - \omega''_{i,j,k+1} g_{j,j}^{\beta})\varepsilon_{i,j}^1 - \omega'_{i,j,k+1} \sum_{m=0,m\neq i}^{N_x} g_{i,m}^{\alpha} \varepsilon_{m,j}^1 - \omega''_{i,j,k+1} \sum_{m=0,m\neq j}^{N_y} g_{j,m}^{\beta} \varepsilon_{i,m}^1 = R_{i,j}^1, \quad k=0, \\ &(1 - \omega'_{i,j,k+1} g_{i,i}^{\alpha} - \omega''_{i,j,k+1} g_{j,j}^{\beta})\varepsilon_{i,j}^{k+1} - \omega'_{i,j,k+1} \sum_{m=0,m\neq i}^{N_x} g_{i,m}^{\alpha} \varepsilon_{m,j}^{k+1} - \omega''_{i,j,k+1} \sum_{m=0,m\neq j}^{N_y} g_{j,m}^{\beta} \varepsilon_{i,m}^{k+1} \\ &\quad = \sum_{s=0}^{k-1} (l_s - l_{s+1}) \varepsilon_{i,j}^{k-s} + R_{i,j}^{k+1}, \quad k>0.\end{aligned} \tag{4.5}$$

When $\gamma = 1$, Eq. (4.5) can be written as

$$\begin{aligned}&(1 - \omega'_{i,j,k+1} g_{i,i}^{\alpha} - \omega''_{i,j,k+1} g_{j,j}^{\beta})\varepsilon_{i,j}^{k+1} - \omega'_{i,j,k+1} \sum_{m=0,m\neq i}^{N_x} g_{i,m}^{\alpha} \varepsilon_{m,j}^{k+1} - \omega''_{i,j,k+1} \sum_{m=0,m\neq j}^{N_y} g_{j,m}^{\beta} \varepsilon_{i,m}^{k+1} = \varepsilon_{i,j}^k + R_{i,j}^{k+1}, \\ &\quad k \geqslant 0.\end{aligned}$$

Denoting $\mathbf{u}^k = [\mathbf{u}_0^k, \mathbf{u}_1^k, \ldots, \mathbf{u}_{N_x}^k]^T$, $\mathbf{u}_i^k = [\varepsilon_{i,0}^k, \varepsilon_{i,1}^k, \ldots, \varepsilon_{i,N_y}^k]^T$ and $\|\mathbf{e}^k\|_\infty = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y} |\varepsilon_{i,j}^k|$, we prove the desired result by mathematical induction.

(1) Case $0 < \gamma < 1$: For $k = 0$, supposing $|\varepsilon_{i_0,j_0}^1| = \|\mathbf{e}^1\|_\infty = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y} |\varepsilon_{i,j}^1|$, we have

$$\begin{aligned}\|\mathbf{e}^1\|_\infty &= |\varepsilon_{i_0,j_0}^1| \leqslant |\varepsilon_{i_0,j_0}^1| - \omega'_{i_0,j_0,1} \sum_{m=0}^{N_x} g_{i_0,m}^{\alpha} |\varepsilon_{i_0,j_0}^1| - \omega''_{i_0,j_0,1} \sum_{m=0}^{N_y} g_{j_0,m}^{\beta} |\varepsilon_{i_0,j_0}^1| \\ &\leqslant \left| (1 - \omega'_{i_0,j_0,1} g_{i_0,i_0}^{\alpha} - \omega''_{i_0,j_0,1} g_{j_0,j_0}^{\beta}) \varepsilon_{i_0,j_0}^1 - \omega'_{i_0,j_0,1} \sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha} \varepsilon_{m,j_0}^1 - \omega''_{i_0,j_0,1} \sum_{m=0,m\neq j_0}^{N_y} g_{j_0,m}^{\beta} \varepsilon_{i_0,m}^1 \right| \\ &= |R_{i_0,j_0}^1| \leqslant \mathbf{R}_{\max} = l_0^{-1} \mathbf{R}_{\max}.\end{aligned}$$

Assuming $\|e^{\tilde{k}}\|_\infty \leqslant l_{\tilde{k}-1}^{-1} \mathbf{R}_{\max}$, $\tilde{k} = 1, 2, \ldots, k$, and $|\varepsilon_{i_0,j_0}^{k+1}| = \|\mathbf{e}^{k+1}\|_\infty = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y} |\epsilon_{i,j}^{k+1}|$, using $l_{\tilde{k}}^{-1} \leqslant l_k^{-1}$, $\tilde{k} = 0, 1, \ldots, k$, therefore, $\|\mathbf{e}^{\tilde{k}}\|_\infty \leqslant l_k^{-1} \mathbf{R}_{\max}$, $\tilde{k} = 1, 2, \ldots, k$. Then we have

$$\begin{aligned}\|\mathbf{e}^{k+1}\|_\infty &= |\varepsilon_{i_0,j_0}^{k+1}| \leqslant |\varepsilon_{i_0,j_0}^{k+1}| - \omega'_{i_0,j_0,k+1} \sum_{m=0}^{N_x} g_{i_0,m}^{\alpha} |\epsilon_{i_0,j_0}^{k+1}| - \omega''_{i_0,j_0,k+1} \sum_{m=0}^{N_y} g_{j_0,m}^{\beta} |\varepsilon_{i_0,j_0}^{k+1}| \\ &\leqslant (1 - \omega'_{i_0,j_0,k+1} g_{i_0,i_0}^{\alpha} - \omega''_{i_0,j_0,k+1} g_{j_0,j_0}^{\beta}) |\varepsilon_{i_0,j_0}^{k+1}|\end{aligned}$$

$$
\begin{aligned}
&- \omega'_{i_0,j_0,k+1} \sum_{m=0,m\neq i_0}^{N_x} g^{\alpha}_{i_0,m} \big|\varepsilon^{k+1}_{m,j_0}\big| - \omega''_{i_0,j_0,k+1} \sum_{m=0,m\neq j_0}^{N_y} g^{\beta}_{j_0,m} \big|\varepsilon^{k+1}_{i_0,m}\big| \\
&\leqslant \bigg| \big(1 - \omega'_{i_0,j_0,k+1} g^{\alpha}_{i_0,i_0} - \omega''_{i_0,j_0,k+1} g^{\beta}_{j_0,j_0}\big) \varepsilon^{k+1}_{i_0,j_0} \\
&\quad - \omega'_{i_0,j_0,k+1} \sum_{m=0,m\neq i_0}^{N_x} g^{\alpha}_{i_0,m} \varepsilon^{k+1}_{m,j_0} - \omega''_{i_0,j_0,k+1} \sum_{m=0,m\neq j_0}^{N_y} g^{\beta}_{j_0,m} \varepsilon^{k+1}_{i_0,m} \bigg| \\
&= \bigg| \sum_{s=0}^{k-1} (l_s - l_{s+1}) \varepsilon^{k-s}_{i_0,j_0} + R^{k+1}_{i_0,j_0} \bigg| \\
&\leqslant \sum_{s=0}^{k-1} (l_s - l_{s+1}) \big\| \mathbf{e}^{k-s} \big\|_\infty + \mathbf{R}_{\max} \\
&\leqslant l_k^{-1} \Bigg(1 - l_1 + \sum_{s=1}^{k-1} (l_s - l_{s+1}) + l_k \Bigg) \mathbf{R}_{\max} \\
&= l_k^{-1} \mathbf{R}_{\max} \leqslant C\big(\tau^{2-\gamma} + (\Delta x)^2 + (\Delta y)^2\big).
\end{aligned}
$$

(2) Case $\gamma = 1$: Using similar idea leads to

$$
\begin{aligned}
\big\| \mathbf{e}^{k+1} \big\|_\infty &\leqslant \bigg| \big(1 - \omega'_{i_0,j_0,k+1} g^{\alpha}_{i_0,i_0} - \omega''_{i_0,j_0,k+1} g^{\beta}_{j_0,j_0}\big) \varepsilon^{k+1}_{i_0,j_0} \\
&\quad - \omega'_{i_0,j_0,k+1} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i_0,m} \varepsilon^{k+1}_{m,j_0} - \omega''_{i_0,j_0,k+1} \sum_{m=0,m\neq j}^{N_y} g^{\beta}_{j_0,m} \varepsilon^{k+1}_{i_0,m} \bigg| \\
&= \big| \varepsilon^{k}_{i_0,j_0} + R^{k+1}_{i_0,j_0} \big| \leqslant \big\| \mathbf{e}^k \big\|_\infty + \big| R^{k+1}_{i_0,j_0} \big| \leqslant (k+1) \mathbf{R}_{\max} \\
&\leqslant C\big(\tau^{2-\gamma} + (\Delta x)^2 + (\Delta y)^2\big). \qquad \square
\end{aligned}
$$

Theorem 4.4. *Let $u^k_{i,j}$ be the approximation solution of $u(x_i, y_j, t_k)$ computed by use of the explicit difference scheme (2.19). If $0 < \frac{\tau^\gamma}{(\Delta x)^\alpha} + \frac{\tau^\gamma}{(\Delta y)^\beta} \leqslant \frac{1-2^{-\gamma}}{\Gamma(2-\gamma)\mathbf{C}_{\max}}$, where*

$$
\mathbf{C}_{\max} = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y, 0\leqslant k\leqslant N_t} \left\{ -\frac{\kappa_\alpha (4 - 2^{3-\alpha}) c^k_{i,j}}{\Gamma(4-\alpha)}, -\frac{\kappa_\beta (4 - 2^{3-\beta}) d^k_{i,j}}{\Gamma(4-\beta)} \right\},
$$

then there is a positive constant C such that

$$
\big| u(x_i, y_j, t_k) - u^k_{i,j} \big| \leqslant C\big(\tau^{2-\gamma} + (\Delta x)^2 + (\Delta y)^2\big),
$$

where $i = 0, 1, \ldots, N_x$; $j = 0, 1, \ldots, N_y$; $k = 0, 1, \ldots, N_t$.

Proof. Define $\varepsilon^k_{i,j} = u(x_i, y_j, t_k) - u^k_{i,j}$ and $\mathbf{R}_{\max} = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y, 0\leqslant k\leqslant N_t} |R^k_{i,j}|$. Similarly, we obtain the following form

$$
\begin{aligned}
&\varepsilon^1_{i,j} = R^1_{i,j}, \quad k = 0, \\
&\varepsilon^{k+1}_{i,j} = \big(1 - l_1 + \sigma'_{i,j,k} g^{\alpha}_{i,i} + \sigma''_{i,j,k} g^{\beta}_{j,j}\big) \varepsilon^k_{i,j} + \sum_{s=1}^{k-1} (l_s - l_{s+1}) \varepsilon^{k-s}_{i,j} \\
&\qquad + \sigma'_{i,j,k} \sum_{m=0,m\neq i}^{N_x} g^{\alpha}_{i,m} \varepsilon^k_{m,j} + \sigma''_{i,j,k} \sum_{m=0,m\neq j}^{N_y} g^{\beta}_{j,m} \varepsilon^k_{i,m} + R^{k+1}_{i,j}, \quad k > 0.
\end{aligned} \tag{4.6}
$$

When $\gamma = 1$, Eq. (4.6) becomes

$$\varepsilon_{i,j}^{1} = R_{i,j}^{1}, \quad k = 0,$$

$$\epsilon_{i,j}^{k+1} = \left(1 + \sigma'_{i,j,k} g_{i,i}^{\alpha} + \sigma''_{i,j,k} g_{j,j}^{\beta}\right)\varepsilon_{i,j}^{k} + \sigma'_{i,j,k} \sum_{m=0,m\neq i}^{N_x} g_{i,m}^{\alpha}\varepsilon_{m,j}^{k} + \sigma''_{i,j,k} \sum_{m=0,m\neq j}^{N_y} g_{j,m}^{\beta}\varepsilon_{i,m}^{k} + R_{i,j}^{k+1}, \quad k \geqslant 0.$$

Using mathematical induction, we can obtain the desired result. Let $\mathbf{u}^k = [\mathbf{u}_0^k, \mathbf{u}_1^k, \ldots, \mathbf{u}_{N_x}^k]^T$, where $\mathbf{u}_i^k = [\varepsilon_{i,0}^k, \varepsilon_{i,1}^k, \ldots, \varepsilon_{i,N_y}^k]^T$ and $\|\mathbf{e}^k\|_\infty = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y} |\varepsilon_{i,j}^k|$.

(1) Case $0 < \gamma < 1$: For $k = 0$, supposing $|\varepsilon_{i_0,j_0}^1| = \|\mathbf{e}^1\|_\infty = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y} |\varepsilon_{i,j}^1|$, we obtain

$$\left\|\mathbf{e}^1\right\|_\infty = \left|\varepsilon_{i_0,j_0}^1\right| = \left|R_{i_0,j_0}^1\right| \leqslant \mathbf{R}_{\max} = l_0^{-1}\mathbf{R}_{\max}.$$

Assuming $\|e^{\tilde{k}}\|_\infty \leqslant l_{\tilde{k}-1}^{-1}\mathbf{R}_{\max}$, $\tilde{k} = 1, 2, \ldots, k$, and $|\varepsilon_{i_0,j_0}^{k+1}| = \|\mathbf{e}^{k+1}\|_\infty = \max_{0\leqslant i\leqslant N_x, 0\leqslant j\leqslant N_y} |\epsilon_{i,j}^{k+1}|$, using $l_{\tilde{k}}^{-1} \leqslant l_k^{-1}$, $\tilde{k} = 0, 1, \ldots, k$, therefore, $\|\mathbf{e}^{\tilde{k}}\|_\infty \leqslant l_k^{-1}\mathbf{R}_{\max}$, $\tilde{k} = 1, 2, \ldots, k$. Then we have

$$\begin{aligned}
\left\|\mathbf{e}^{k+1}\right\|_\infty &= \left|\varepsilon_{i_0,j_0}^{k+1}\right| \\
&= \Bigg|\left(1 - l_1 + \sigma'_{i_0,j_0,k} g_{i_0,i_0}^{\alpha} + \sigma''_{i_0,j_0,k} g_{j_0,j_0}^{\beta}\right)\varepsilon_{i_0,j_0}^{k} + \sum_{s=1}^{k-1}(l_s - l_{s+1})\varepsilon_{i_0,j_0}^{k-s} \\
&\quad + \sigma'_{i_0,j_0,k} \sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}\varepsilon_{m,j_0}^{k} + \sigma''_{i_0,j_0,k} \sum_{m=0,m\neq j_0}^{N_y} g_{j_0,m}^{\beta}\varepsilon_{i_0,m}^{k} + R_{i,j}^{k+1}\Bigg| \\
&\leqslant \left(1 - l_1 + \sigma'_{i_0,j_0,k} g_{i_0,i_0}^{\alpha} + \sigma''_{i_0,j_0,k} g_{j_0,j_0}^{\beta}\right)\left|\varepsilon_{i_0,j_0}^{k}\right| + \sum_{s=1}^{k-1}(l_s - l_{s+1})\left|\varepsilon_{i_0,j_0}^{k-s}\right| \\
&\quad + \sigma'_{i_0,j_0,k} \sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}\left|\varepsilon_{m,j_0}^{k}\right| + \sigma''_{i_0,j_0,k} \sum_{m=0,m\neq j_0}^{N_y} g_{j_0,m}^{\beta}\left|\varepsilon_{i_0,m}^{k}\right| + \mathbf{R}_{\max} \\
&\leqslant (1 - l_1)\left\|\mathbf{e}^k\right\|_\infty + \sum_{s=1}^{k-1}(l_s - l_{s+1})\left\|\mathbf{e}^{k-s}\right\|_\infty + \\
&\quad + \sigma'_{i_0,j_0,k} \sum_{m=0}^{N_x} g_{i_0,m}^{\alpha}\left\|\mathbf{e}^k\right\|_\infty + \sigma''_{i_0,j_0,k} \sum_{m=0}^{N_y} g_{j_0,m}^{\beta}\left\|\mathbf{e}^k\right\|_\infty + \mathbf{R}_{\max} \\
&\leqslant l_k^{-1}\left(1 - l_1 + \sum_{s=1}^{k-1}(l_s - l_{s+1}) + l_k\right)\mathbf{R}_{\max} \\
&= l_k^{-1}\mathbf{R}_{\max} \leqslant C\left(\tau^{2-\gamma} + (\Delta x)^2 + (\Delta y)^2\right).
\end{aligned}$$

(2) Case $\gamma = 1$: It is easy to check that

$$\begin{aligned}
\left\|\mathbf{e}^{k+1}\right\|_\infty &= \left|\varepsilon_{i_0,j_0}^{k+1}\right| \\
&\leqslant \left(1 + \sigma'_{i_0,j_0,k} g_{i_0,i_0}^{\alpha} + \sigma''_{i_0,j_0,k} g_{j_0,j_0}^{\beta}\right)\left|\varepsilon_{i_0,j_0}^{k}\right| \\
&\quad + \sigma'_{i_0,j_0,k} \sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}\left|\varepsilon_{m,j_0}^{k}\right| + \sigma''_{i_0,j_0,k} \sum_{m=0,m\neq j_0}^{N_y} g_{j_0,m}^{\beta}\left|\varepsilon_{i_0,m}^{k}\right| + \mathbf{R}_{\max} \\
&\leqslant \left(1 + \sigma'_{i_0,j_0,k} g_{i_0,i_0}^{\alpha} + \sigma''_{i_0,j_0,k} g_{j_0,j_0}^{\beta}\right)\left\|\mathbf{e}^k\right\|_\infty \\
&\quad + \sigma'_{i_0,j_0,k} \sum_{m=0,m\neq i_0}^{N_x} g_{i_0,m}^{\alpha}\left\|\mathbf{e}^k\right\|_\infty + \sigma''_{i_0,j_0,k} \sum_{m=0,m\neq j_0}^{N_y} g_{j_0,m}^{\beta}\left\|\mathbf{e}^k\right\|_\infty + \mathbf{R}_{\max} \\
&\leqslant \left\|\mathbf{e}^k\right\|_\infty + \mathbf{R}_{\max} \leqslant (k+1)\mathbf{R}_{\max} \\
&\leqslant C\left(\tau^{2-\gamma} + (\Delta x)^2 + (\Delta y)^2\right). \qquad \square
\end{aligned}$$

Table 1
The maximum errors and convergence rates for the implicit scheme (2.13) of the one-dimensional space–time Caputo–Riesz fractional diffusion equation (2.10) with variable coefficient $c(x,t) = x^{\alpha}t^{1-\gamma}$ at $t = 1$, and the time and space stepsizes are equal, i.e, $\tau = \Delta x$.

$\tau, \Delta x$	$\alpha = 1.2, \gamma = 0.9$	Rate	$\alpha = 1.2, \gamma = 0.5$	Rate	$\alpha = 1.2, \gamma = 0.1$	Rate
1/20	2.0662e−003		5.1756e−004		3.7968e−004	
1/40	9.2468e−004	1.1600	1.5488e−004	1.7405	1.2018e−004	1.6596
1/80	4.2215e−004	1.1312	4.7975e−005	1.6908	3.3481e−005	1.8438
1/160	1.9472e−004	1.1164	1.5284e−005	1.6503	8.8287e−006	1.9231
	$\alpha = 1.9, \gamma = 0.9$	Rate	$\alpha = 1.9, \gamma = 0.5$	Rate	$\alpha = 1.9, \gamma = 0.1$	Rate
1/20	1.7987e−003		5.2505e−004		4.1966e−004	
1/40	7.9226e−004	1.1829	1.5410e−004	1.7686	1.0723e−004	1.9686
1/80	3.5837e−004	1.1445	4.6422e−005	1.7309	2.7361e−005	1.9705
1/160	1.6437e−004	1.1245	1.4379e−005	1.6908	6.9726e−006	1.9724

Table 2
The maximum errors and convergent rates for the implicit scheme (2.13) of the one-dimensional space–time Caputo–Riesz fractional diffusion equation (2.10) at $t = 1$ with variable coefficient $c(x,t) = x^{\alpha}t^{1-\gamma}$, and $\tau = (\Delta x)^{\frac{2}{2-\gamma}}$, where $\gamma = 0.9$.

$\tau, \Delta x$	$\alpha = 1.9$	Rate	$\alpha = 1.5$	Rate	$\alpha = 0.1$	Rate
1/10	1.4585e−003		1.4703e−003		1.0341e−003	
1/20	3.7387e−004	1.9638	3.7098e−004	1.9867	2.5866e−004	1.9992
1/40	9.3868e−005	1.9939	9.2446e−005	2.0047	6.4141e−005	2.0117
1/80	2.3715e−005	1.9848	2.3080e−005	2.0020	1.6006e−005	2.0027

Remark 4.4. Let $\nu \in (0,1)$, there exist ${}_{x_L}D_x^{\nu}u = D_{x\,x_L}D_x^{-(1-\nu)}u = D^2{}_{x_L}D_x^{-(2-\nu)}u$, and ${}_xD_{x_R}^{\nu}u = -D_x\,{}_xD_{x_R}^{-(1-\nu)}u = D^2{}_xD_{x_R}^{-(2-\nu)}u$ [9]. So after carefully dealing with the boundary conditions, all the above presented numerical schemes still work well when the order of space fractional derivatives $\alpha \in (0,1)$ and/or $\beta \in (0,1)$. And the convergence rates remain, and all the theoretical analyses are valid. Some of the numerical results are also given in Table 2.

5. Numerical results

In this section, we numerically verify the above theoretical results including convergence rates and numerical stability. And the l_∞ norm is used to measure the numerical errors.

5.1. Numerical results for 1D

Consider the one-dimensional space–time Caputo–Riesz fractional diffusion equation (2.10), on a finite domain $0 < x < 1$, $0 < t \leqslant 1$, with the coefficient $c(x,t) = x^{\alpha}t^{1-\gamma}$, the forcing function

$$f(x,t) = \frac{1}{2}\Gamma(3+\gamma)t^2x^2(x-1)^2 + \frac{t^3x^{\alpha}}{\cos(\alpha\pi/2)}\left[\frac{x^{2-\alpha}+(1-x)^{2-\alpha}}{\Gamma(3-\alpha)} - 6\frac{x^{3-\alpha}+(1-x)^{3-\alpha}}{\Gamma(4-\alpha)} + 12\frac{x^{4-\alpha}+(1-x)^{4-\alpha}}{\Gamma(5-\alpha)}\right],$$

the initial condition $u(x,0) = 0$, and the boundary conditions $u(0,t) = u(1,t) = 0$. This fractional diffusion equation has the exact value $u(x,t) = t^{2+\gamma}x^2(1-x)^2$, which may be confirmed by applying the fractional differential equations

$$ {}_{x_L}D_x^{\nu}(x-x_L)^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+1-\nu)}(x-x_L)^{\upsilon-\nu},$$
$$ {}_xD_{x_R}^{\nu}(x_R-x)^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+1-\nu)}(x_R-x)^{\upsilon-\nu}.$$

Table 1 shows the maximum errors, at time $t = 1$ with $\tau = \Delta x$, between the exact analytical values and the numerical values obtained by applying the implicit scheme (2.13). Since the scheme has the global truncation error $\mathcal{O}(\tau^{2-\gamma} + (\Delta x)^2)$, the convergent rate should be $2-\gamma$ being confirmed by the numerical results.

Table 2 shows the maximum errors at time $t = 1$, and the time and space stepsizes are taken as $\tau = (\Delta x)^{\frac{2}{2-\gamma}}$. The numerical results confirm the 2nd order convergence in space directions. In particular, the numerical results when $\alpha = 0.1 \in (0,1)$ are also presented, which confirm the statement of Remark 4.4.

Table 3 displays the maximum errors of the explicit scheme (2.14), and confirms the desired convergent order $\mathcal{O}(\tau^{2-\gamma} + (\Delta x)^2)$. The numerical results given in Table 4 illustrate that the theoretical sufficient stability conditions are close to the practical stability.

Table 3
The maximum errors and convergent rates for the explicit scheme (2.14) of the one-dimensional space–time Caputo–Riesz fractional diffusion equation (2.10) at $t=1$ with variable coefficient $c(x,t)=x^{\alpha}t^{1-\gamma}$, and $\tau=(2\Delta x/5)^{\frac{2}{2-\gamma}}$, where $\gamma=0.9$.

$\tau, \Delta x$	$\alpha=1.2$	Rate	$\alpha=1.5$	Rate	$\alpha=1.8$	Rate
1/10	1.1598e−003		1.3067e−003		1.4064e−003	
1/20	3.8377e−004	1.5956	3.3951e−004	1.9443	3.6767e−004	1.9355
1/40	1.2023e−004	1.6745	8.7302e−005	1.9594	9.5608e−005	1.9432
1/80	3.3426e−005	1.8467	2.3518e−005	1.8923	2.4867e−005	1.9429
1/160	8.8149e−006	1.9230	6.6253e−005	1.8047	6.4702e−006	1.9424

Table 4
The maximum errors and convergent rates for the explicit scheme (2.14) of the one-dimensional space–time Caputo–Riesz fractional diffusion equation (2.10) at $t=1$ with variable coefficient $c(x,t)=x^{\alpha}t^{1-\gamma}$, and $\tau=(\Delta x)^{\frac{2}{2-\gamma}}$, where λ_c is defined by (3.3) and $\Delta x=1/20$.

	$\gamma=0.9$	The error	State	$\gamma=0.5$	The error	State
$\alpha=1.1$	$\lambda_c=0.8407$	6.5060e−004	Convergence	$\lambda_c=-2.9571$	2.1320e+003	Divergence
$\alpha=1.3$	$\lambda_c=0.5460$	6.9114e−004	Convergence	$\lambda_c=-6.0506$	4.4080e+026	Divergence
$\alpha=1.5$	$\lambda_c=0.1182$	7.3271e−004	Convergence	$\lambda_c=-11.6089$	9.7339e+046	Divergence
$\alpha=1.7$	$\lambda_c=-0.5504$	2.0755e+005	Divergence	$\lambda_c=-21.6534$	3.7536e+067	Divergence
$\alpha=1.9$	$\lambda_c=-1.6597$	1.3518e+124	Divergence	$\lambda_c=-39.8670$	8.0772e+087	Divergence

Table 5
The maximum errors and convergence rates for the implicit scheme (2.17) of the two-dimensional space–time Caputo–Riesz fractional diffusion equation (1.1) with variable coefficients $c(x,y,t)=x^{\alpha}y^{\beta}t^{1-\gamma}$ and $d(x,y,t)=x^{\beta}y^{\alpha}t^{1-\gamma}$, where $\tau=(\Delta x)^{\frac{2}{2-\gamma}}=(\Delta y)^{\frac{2}{2-\gamma}}$, $\gamma=0.9$ and $T_{\text{end}}=0.5$.

$\tau, \Delta x, \Delta y$	$\alpha=1.2, \beta=1.3$	Rate	$\alpha=1.5, \beta=1.5$	Rate	$\alpha=1.8, \beta=1.7$	Rate
1/10	1.5773e−005		1.5704e−005		1.5275e−005	
1/20	3.9576e−006	1.9948	3.9535e−006	1.9899	3.8568e−006	1.9858
1/40	9.9013e−007	1.9989	9.9100e−007	1.9962	9.6703e−007	1.9958
1/60	4.3820e−007	2.0105	4.3695e−007	2.0196	4.2043e−007	2.0543

Table 6
The maximum errors and convergence rates for the explicit scheme (2.19) of the two-dimensional space–time Caputo–Riesz fractional diffusion equation (1.1) with variable coefficients $c(x,y,t)=x^{\alpha}y^{\beta}t^{1-\gamma}$ and $d(x,y,t)=x^{\beta}y^{\alpha}t^{1-\gamma}$, where $\tau=(\Delta x)^{\frac{2}{2-\gamma}}=(\Delta y)^{\frac{2}{2-\gamma}}$, $\gamma=0.9$ and $T_{\text{end}}=0.05$.

$\tau, \Delta x, \Delta y$	$\alpha=1.2, \beta=1.3$	Rate	$\alpha=1.5, \beta=1.5$	Rate	$\alpha=1.8, \beta=1.7$	Rate
1/10	1.9418e−008		1.9389e−008		1.9289e−008	
1/20	4.8847e−009	1.9910	4.8794e−009	1.9905	4.8559e−009	1.9900
1/40	1.2235e−009	1.9972	1.2226e−009	1.9967	1.2172e−009	1.9962
1/60	5.4399e−010	1.9991	5.4369e−010	1.9967	5.4141e−010	1.9980

5.2. Numerical results for 2D

We further examine the two-dimensional space–time Caputo–Riesz fractional diffusion equation (1.1), on a finite domain $0<x<1$, $0<t\leqslant T_{\text{end}}$, with the variable coefficients $c(x,y,t)=x^{\alpha}y^{\beta}t^{1-\gamma}$, $d(x,y,t)=x^{\beta}y^{\alpha}t^{1-\gamma}$, the forcing function

$$\begin{aligned}
f(x,y,t)={}&\frac{1}{2}\Gamma(3+\gamma)t^2x^2(x-1)^2y^2(y-1)^2\\
&+\frac{t^3x^{\alpha}y^{2+\beta}(y-1)^2}{\cos(\alpha\pi/2)}\left[\frac{x^{2-\alpha}+(1-x)^{2-\alpha}}{\Gamma(3-\alpha)}-6\frac{x^{3-\alpha}+(1-x)^{3-\alpha}}{\Gamma(4-\alpha)}+12\frac{x^{4-\alpha}+(1-x)^{4-\alpha}}{\Gamma(5-\alpha)}\right]\\
&+\frac{t^3x^{2+\beta}(x-1)^2y^{\alpha}}{\cos(\beta\pi/2)}\left[\frac{y^{2-\beta}+(1-y)^{2-\beta}}{\Gamma(3-\beta)}-6\frac{y^{3-\beta}+(1-y)^{3-\beta}}{\Gamma(4-\beta)}+12\frac{y^{4-\beta}+(1-y)^{4-\beta}}{\Gamma(5-\beta)}\right],
\end{aligned}$$

the initial condition $u_0(x,y)=0$, and the homogeneous boundary conditions. It has the exact solution

$$u(x,y,t)=t^{2+\gamma}x^2(1-x)^2y^2(1-y)^2.$$

Table 5 and Table 6 show the maximum errors of the implicit scheme (2.17) at time $T_{\text{end}}=0.5$ and the explicit scheme (2.19) at time $T_{\text{end}}=0.05$, respectively, both with $\tau=(\Delta x)^{\frac{2}{2-\gamma}}=(\Delta y)^{\frac{2}{2-\gamma}}$, and the numerical results confirm that they have the global truncation error $\mathcal{O}(\tau^{2-\gamma}+(\Delta x)^2+(\Delta y)^2)$.

6. Conclusions

This paper discusses the explicit and implicit finite difference schemes for the space–time Caputo–Riesz fractional PDEs with variable coefficients, and the order of time fractional derivative belongs to $(0, 1)$ and the order of space fractional derivatives locate in $(1, 2)$. Both of the schemes have $(2 - \gamma)$th order convergence rate in time and 2nd order convergent rate in space. The detailed numerical stability analysis and error estimates are presented, and the extensive numerical experiments are performed, which confirm the theoretical results. In particular, the numerical schemes still work well for the space–time Caputo–Riesz fractional PDEs with the order of its space derivatives belongs to $(0, 1)$, and all the theoretical analyses are still valid. The explicit scheme is easy to program, but it has restriction on the rate between the time and space stepsizes. So from the practical point of view, the explicit scheme is preferred if the order of the time derivative γ and the order of space derivatives (α and β) are both close to 1. Otherwise, the implicit scheme may be a better choice.

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