A SUPERLINEAR METHOD FOR SPACE-TIME FRACTIONAL DIFFUSION EQUATION WITH THE LOW REGULARITY SOLUTION*

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Key words. example, LaTeX

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1. Introduction.

We study $\gamma \in (0,1)$, $\alpha \in (1,2)$ and $\Omega = (0, 2L)$.

$$D_t^{\gamma} u + (-\Delta)^{\frac{\alpha}{2}} u = f(x,t), \quad x \in \Omega, t \in (0,T]. \tag{1.1}$$

where

$$D_t^{\gamma} u(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial u(x,s)}{\partial s}(t-s)^{-\gamma} ds \tag{1.2}$$

$$(-\Delta)^{\frac{\alpha}{2}} u(x,t) = \frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)} \int_0^{2L} u(y,t)|x-y|^{1-\alpha} dy \tag{1.3}$$

where $\gamma \in (0,1)$, $\alpha \in (1,2)$.

2. Regularity of the solution.

For the space-time fractional diffusion equation, it was first assumed that the solution regularity satisfies [3, 4, 1, 2]

$$\left| \frac{\partial^l u}{\partial t^l}(x,t) \right| \le C(1 + t^{\gamma - l}) \quad for \quad l = 0,1,2, \tag{2.1a}$$

$$\left| \frac{\partial^l u}{\partial x^l}(x,t) \right| \le C\delta(x)^{\alpha/2 - l} \quad for \quad l = 0,1,2,3,4, \tag{2.1b}$$

$$\left| \frac{\partial^l}{\partial x^l}(-\Delta)^{\alpha/2} u(x,t) \right| \le C\delta(x)^{-\alpha/2 - l} \quad for \quad l = 0,1,2, \tag{2.1c}$$

for all $(x,t) \in (0,2L) \times (0,T]$.

Remark 2.1. (2.1c) can be derived from (2.1b) by

$$\begin{aligned} I^{2-\alpha} u(x,t) = & \int_0^{x/2} + \int_{L+x/2}^{2L} u(y,t) \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ & + \int_0^{x/2} \left(u(x-z,t) + u(x+z,t) \right) \frac{z^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ & + \int_{x+x/2}^{L+x/2} u(y,t) \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)} dy \end{aligned}$$

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3. Numerical scheme.

3.1. Discretisation of $(-\Delta)^{\frac{\alpha}{2}}$ on Graded Mesh. We discretize the $(-\Delta)^{\alpha/2}$ on a graded mesh.

$$-D_M^{\alpha}u(x_m,t_n) = -D_M^2 I^{2-\alpha}\Pi_M u(x_m,t_n) \tag{3.1}$$

The discrete operator can be written in matrix form

$$A = H^{-1}D, \quad \text{with} \quad H = \operatorname{diag}\left(\frac{h_i+h_{i+1}}{2}\right) \text{ and } D_{ij} = \frac{\kappa_\alpha}{\Gamma(4-\alpha)}C_i K_{ij} C_j^T \tag{3.2}$$

$C_j := \left(\frac{1}{h_j}, -\frac{1}{h_j} - \frac{1}{h_{j+1}}, \frac{1}{h_{j+1}}\right)$ and

$$K_{ij} := \begin{pmatrix} |x_{i-1}-x_{j-1}|^{3-\alpha} & |x_{i-1}-x_j|^{3-\alpha} & |x_{i-1}-x_{j+1}|^{3-\alpha} \\ |x_i-x_{j-1}|^{3-\alpha} & |x_i-x_j|^{3-\alpha} & |x_i-x_{j+1}|^{3-\alpha} \\ |x_{i+1}-x_{j-1}|^{3-\alpha} & |x_{i+1}-x_j|^{3-\alpha} & |x_{i+1}-x_{j+1}|^{3-\alpha} \end{pmatrix}.$$

LEMMA 3.1. *The stiffness matrix A has the following properties:*

1. *The eigenvalues of $A = H^{-1}D$ are positive real numbers. Inparticular, D is symmetric positive definite.*
2. *The eigenvectors of A are orthogonal in space where $\langle u, v\rangle := uHv$, where $H := diag\left(\frac{h_i+h_{i+1}}{2}\right)$.*
3. *$(I+\tau A)^{-1} > O$ for any $\tau > 0$, and $A^{-1} > O$.*

Proof. Since

$$A = H^{-1}D = H^{-1/2}H^{-1/2}DH^{-1/2}H^{1/2}, \tag{3.3}$$

where $H^{-1/2}DH^{-1/2}$ is symmetric positive definite, $H^{-1/2}DH^{-1/2} = U\Lambda U^T$. Thus,

$$A = H^{-1/2}U\Lambda U^T H^{1/2} = (H^{-1/2}U)\Lambda(H^{-1/2}U)^{-1}. \tag{3.4}$$

The eigenvectors of A form an orthogonal basis of the Hilbert space defined by $\langle u, v\rangle := uHv$. Let $v_i = H^{-1/2}u_i$ be an eigenvector of A with eigenvalue λ_i. ∎

3.2. Discretisation of D_t^γ on a General Mesh. Consider the temporal mesh $0 = t_0 < t_1 < t_2 < \ < t_M = T$. Set $\tau_j := t_j - t_{j-1}$ for $j = 1, ..., M$.

On this mesh, we discretise $D_t^\gamma v$ for $v \in C[0,T] \cap C^3(0,T]$.

$$\begin{aligned} D_N^\gamma u(x,t_n) &= \sum_{k=1}^{n} \frac{1}{\Gamma(2-\gamma)}\left(u(x,t_k) - u(x,t_{k-1})\right)\frac{(t_n-t_{k-1})^{1-\gamma} - (t_n-t_k)^{1-\gamma}}{\tau_k} \\ &= d_{n,n}u(x,t_n) - \sum_{k=1}^{n-1}(d_{n,k+1} - d_{n,k})u(x,t_k) - d_{n,1}u(x,t_0), \end{aligned} \tag{3.5}$$

where

$$d_{n,k} = \frac{(t_n-t_{k-1})^{1-\gamma} - (t_n-t_k)^{1-\gamma}}{\Gamma(2-\gamma)\tau_k} \quad \text{for} \quad 1 \le k \le n \quad \text{and} \quad d_{n,0} = 0, \tag{3.6}$$

$d_{n,n} = \frac{\tau_n^{-\gamma}}{\Gamma(2-\gamma)}, \quad d_{n,k+1} \ge d_{n,k}.$

The final scheme is

$$D_N^\gamma U^n + AU^n = F^n \tag{3.7}$$

4. truncation error.

THEOREM 4.1. *[4]*

$$|R_t^n| := |D_N^\gamma u(x_m,t_n) - D_t^\gamma u(x_m,t_n)| \le Cn^{\min\{2-\gamma, r_t\gamma\}}.$$

THEOREM 4.2.

$$\begin{aligned}|R_x^n| &:= \left| -D_M^\alpha u(x_m,t_n) - (-\Delta)^{\alpha/2} u(x_m,t_n) \right| \\ &\le CM^{-\min\{r_x\frac{\alpha}{2},2\}}(x_i^{-\alpha} + (2L-x_i)^{-\alpha}) + C(r-1)M^{-2}(L-\delta(x_i)+1/M)^{1-\alpha} \\ &=: R_x.\end{aligned}$$

Proof. Replace the requrements of f by $(-\Delta)^{\alpha/2}u$. ∎

5. Convergence. Numerical scheme:

$$D_N^\gamma U^n + AU^n = F^n \tag{5.1}$$

We have

$$(d_{n,n}I + A)\,E^n = \sum_{k=1}^{n-1}(d_{n,k+1} - d_{n,k})E^k + d_{n,1}E^0 + R^n \tag{5.2}$$

Define the matrices $\Theta_{n,j}$, for $n = 1, 2..., N$ and $j = 0, 1, 2, ..., n-1$ by

$$\Theta_{n,n} = (d_{n,n} + A)^{-1}, \quad \Theta_{0,0} = I, \quad \Theta_{n,j} = \sum_{k=j}^{n-1}(d_{n,k+1} - d_{n,k})\Theta_{n,n}\Theta_{k,j}. \tag{5.3}$$

Observe that $\Theta_{n,j} > O$ for all n, j.

LEMMA 5.1.

$$\begin{aligned}E^n &= \sum_{j=1}^{n}\Theta_{n,j}R^j + \Theta_{n,0}E^0 \\ &= \sum_{j=1}^{n}\Theta_{n,j}R_t^j + \sum_{j=1}^{n}\Theta_{n,j}R_x^j + \Theta_{n,0}E^0\end{aligned} \tag{5.4}$$

Our stability result will be presented in a general framework. Assume that

$$\mathcal{L}_{M,N}v_m^n = g_m^n \quad \text{for} \quad 1 \le m \le 2M-1, \quad 1 \le n \le N, \tag{5.5}$$

with $v_0^n = v_{2M}^n = 0$ for $0 \le n \le N$ and v_m^0 given for $0 \le m \le 2M$.

Here the discrete operator is $\mathcal{L}_{M,N}v_m^n = D_N^\gamma v_m^n - D_M^\alpha v_m^n$.

LEMMA 5.2. *The solution of the discrete problem* (5.5) *satisfies*

$$\|v^n\|_\infty \le d_{n,n}^{-1}\left[\|g^n\|_\infty + d_{n,1}\|v^0\|_\infty + \sum_{k=1}^{n-1}(d_{n,k+1} - d_{n,k})\,\|v^k\|_\infty\right]$$

for $n = 1, 2, ..., N$.

Proof. Fix $n \in \{1,2,...,N\}$. Choose i_0 such that $|v_{i_0}^n| = \|v^n\|_\infty$. Then, it yields

$$d_{n,n} v_{i_0} + \sum_{j=1}^{2M-1} a_{i_0,j} v_j^n = g_{i_0}^n + \sum_{k=1}^{n-1} \left(d_{n,k+1} - d_{n,k}\right) v_{i_0}^k + d_{n,1} v_{i_0}^0.$$

Hence, by $a_{i_0,i_0} > 0$ and the choice of i_0, one obtains

$$\left(d_{n,n} + a_{i_0,i_0}\right) |v_{i_0}^n| \leq \sum_{j \neq i_0} |a_{i_0,j}||v_j^n| + |g_{i_0}^n| + \sum_{k=1}^{n-1} \left(d_{n,k+1} - d_{n,k}\right) |v_{i_0}^k| + d_{n,1}|v_{i_0}^0|.$$

Since A is strcitly diagonally dominant, we can get

$$d_{n,n}\|v^n\|_\infty \leq \|g\|_\infty + d_{n,1}\|v^0\|_\infty + \sum_{k=1}^{n-1} \left(d_{n,k+1} - d_{n,k}\right) \|v^k\|_\infty.$$

The proof is completed. ∎

Theorem 5.3.

$$\left| \sum_{j=1}^{n} \Theta_{n,j} R_t^j \right| \leq C T^\gamma N^{-\min\{2-\gamma, r_t\gamma\}}. \tag{5.6}$$

Proof. Let $v^n = \sum_{j=1}^n \Theta_{n,j} R_t^j$. Then, we can check that v_n satisfies (5.5)

$$\mathcal{L}_{M,N}\ v^n = R_t^n.$$

According to Theorem 4.1 and Lemma 5.2 and the proof of [4, Theorem 5.3], one has ∎

$$\|v^n\|_\infty \leq C T^\gamma N^{-\min\{2-\gamma, r_t\gamma\}}.$$

Lemma 5.4. *For $n = 1,2,...,N$, one has*

$$\sum_{j=1}^{n} \Theta_{n,j} < A^{-1} \tag{5.7}$$

Proof. Use induction on n. When $n = 1$, then $\sum_{j=1}^1 \Theta_{1,j} = \Theta_{1,1} < A^{-1}$. Next, assume that (5.7) holds for $k = 1,2,...,m-1 (2 \leq m \leq N)$. We want to prove (5.7) for $n = m$. Invoking (5.3) and interchanging the order of summation,

$$\begin{aligned}
\sum_{j=1}^{m} \Theta_{m,j} &= \Theta_{m,m} + \sum_{j=1}^{m-1} \sum_{k=j}^{m-1} (d_{m,k+1} - d_{m,k}) \Theta_{m,m} \Theta_{k,j} \\
&= \Theta_{m,m} + \sum_{k=1}^{m-1} (d_{m,k+1} - d_{m,k}) \Theta_{m,m} \sum_{j=1}^{k} \Theta_{k,j} \\
&\leq \Theta_{m,m} + \sum_{k=1}^{m-1} (d_{m,k+1} - d_{m,k}) \Theta_{m,m} A^{-1} \\
&= \Theta_{m,m} + (d_{m,m} - d_{m,1}) \Theta_{m,m} A^{-1} \\
&= A^{-1} - d_{m,1} \Theta_{m,m} A^{-1} < A^{-1}
\end{aligned}$$

The proof is completed. ∎

THEOREM 5.5.

$$\left|\sum_{j=1}^{n}\Theta_{n,j}R_x^j\right| \leq CM^{-\min\{r_x\alpha/2,2\}} \tag{5.8}$$

Proof. Since $\Theta_{n,j} > O$, we have

$$\left|\sum_{j=1}^{n}\Theta_{n,j}R_x^j\right| \leq \sum_{j=1}^{n}\Theta_{n,j}|R_x^j| \leq \sum_{j=1}^{n}\Theta_{n,j}|R_x| < A^{-1}R_x \tag{5.9}$$

Since $A^{-1}R_x$ is bounded by $CM^{-\min\{r_x\alpha/2,2\}}$, the proof is completed. ∎

THEOREM 5.6.

$$|E_N| \leq C\left(N^{-\min\{2-\gamma,r_t\gamma\}} + M^{-\min\{r_x\frac{\alpha}{2},2\}}\right) \tag{5.10}$$

Proof. According to Theorems 5.3 and 5.5, the desired result is obtained. ∎

One-order.

$$\frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}u = f(x,t), \quad x \in \Omega, t \in (0,T]. \tag{5.11}$$

scheme: Let $\tau = \frac{T}{M}$, $U^n, F^n \in \mathbb{R}^{2N-1}$,

$$\frac{U^{n+1}-U^n}{\tau} + AU^{n+1} = F^{n+1}. \tag{5.12}$$

Then $E^n = U^n - \hat{U}^n \in \mathbb{R}^{2N-1}$,

$$(I+\tau A)E^{n+1} = E^n + \tau R^{n+1}. \tag{5.13}$$

$$\begin{aligned} E^n &= (I+\tau A)^{-1}E^{n-1} + (I+\tau A)^{-1}\tau R^n \\ &= (I+\tau A)^{-n}E^0 + \sum_{k=1}^{n}(I+\tau A)^{-k}\tau R^{n-k+1} \end{aligned} \tag{5.14}$$

$$\begin{aligned} (I+\tau A)^{-k}\tau R^{n-k+1} &= (\tau A)(I+\tau A)^{-k}(\tau A)^{-1}\tau R^{n-k+1} \\ &= (\tau A)(I+\tau A)^{-k}(A^{-1}R^{n-k+1}) \end{aligned} \tag{5.15}$$

Suppose that

$$\begin{aligned} &|R^n| \leq |R| \\ &\quad := Ch^{\min\{r\alpha/2,2\}}(x_i^{-\alpha} + (2T-x_i)^{-\alpha}) \\ &\qquad + C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha} + C\tau^? \end{aligned} \tag{5.16}$$

Since $0 < A^{-1}R \leq Ch^{\min}$,

$$|(I+\tau A)^{-k}\tau R^{n-k+1}| \leq (I+\tau A)^{-k}\tau R = \tau A(1+\tau A)^{-k}A^{-1}R \tag{5.17}$$

Then

$$
\begin{aligned}
(5.18) \qquad |E^n| &\le |(I+\tau A)^{-n}E^0| + \sum_{k=1}^{n} \tau A(1+\tau A)^{-k}A^{-1}R \\
&= |(I+\tau A)^{-n}E^0| + \left(I - (I+\tau A)^{-n}\right) A^{-1}R.
\end{aligned}
$$

Since A is diagonally dominant, $\|(I+\tau A)^{-1}E\|_\infty \le \|E\|_\infty$, we have

$$
(5.19) \qquad \|E^n\|_\infty \le \|E_0\|_\infty + \|A^{-1}R\|_\infty.
$$

LEMMA 5.7. *$A^{-1}R$ is bounded by $C\left(h^{\min\{r\alpha/2,2\}} + \tau^?\right)$, where C is a constant independent of h, α.*

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