AN α-ROBUST DIFFERENCE QUADRATURE METHOD FOR SPACE-TIME CAPUTO-RIESZ FRACTIONAL DIFFUSION EQUATION *

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Key words. example, LaTeX

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1. Introduction.

We study

$$\frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u = f(x,t), \quad x \in \Omega, t \in (0,T]. \tag{1.1}$$

Implicity scheme: Let $\tau = \frac{T}{M}$, $U^n, F^n \in \mathbb{R}^{2N-1}$,

$$\frac{U^{n+1} - U^n}{\tau} + AU^{n+1} = F^{n+1}. \tag{1.2}$$

Then

$$(I + \tau A)E^{n+1} = E^n + \tau R^{n+1}. \tag{1.3}$$

We will prove the convergence of this scheme.

2. Property of A.

LEMMA 2.1. *The stiffness matrix A has the following properties:*

1. *The eigenvalues of A are positive real numbers.*
2. *A is positive definite, which means that the eigenvalues of $\frac{A+A^T}{2}$ are positive.*
3. *The eigenvectors of A are orthogonal in space where $\langle u, v\rangle := uHv$, where $H := diag\left(\frac{h_i+h_{i+1}}{2}\right)$.*

Proof. Since

$$A = H^{-1}D = H^{-1/2}H^{-1/2}DH^{-1/2}H^{1/2}, \tag{2.1}$$

where $H^{-1/2}DH^{-1/2}$ is symmetric positive definite, $H^{-1/2}DH^{-1/2} = U\Lambda U^T$. Thus,

$$A = H^{-1/2}U\Lambda U^T H^{1/2} = (H^{-1/2}U)\Lambda(H^{-1/2}U)^{-1}. \tag{2.2}$$

The eigenvectors of A form an orthogonal basis of the Hilbert space defined by $\langle u, v\rangle := uHv$. Let $v_i = H^{-1/2}u_i$ be an eigenvector of A with eigenvalue λ_i. ∎

We need to prove $\lambda_1 > c$ for some positive constant c.

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3. Convergence. We prove the shceme in convergence in the meaning of L^2 norm. $\|v\|_{2,h} := v_i^T H v_i$ is bounded.

$$\begin{aligned} E^n &= (I+\tau A)^{-1}E^{n-1} + (I+\tau A)^{-1}\tau R^n \\ &= (I+\tau A)^{-n}E^0 + \sum_{k=1}^{n}(I+\tau A)^{-k}\tau R^{n-k+1} \end{aligned} \tag{3.1}$$

Assume that $|R^n| \leq Ch^{\min\{r\alpha/2,2\}}(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha} + C\tau^?$

$$\begin{aligned} (I+\tau A)^{-k}\tau R^{n-k+1} &= (\tau A)(I+\tau A)^{-k}(\tau A)^{-1}\tau R^{n-k+1} \\ &= (\tau A)(I+\tau A)^{-k}(A^{-1}R^{n-k+1}) \end{aligned} \tag{3.2}$$

Since $A^{-1}R^{n-k+1}$ is bounded, $\|A^{-1}R^{n-k+1}\|_2$ is bounded. Depose it by the basis v_i, we have

$$\|(I+\tau A)^{-k}\tau R^{n-k+1}\|_2 \leq \frac{\tau\lambda_i}{(1+\tau\lambda_i)^k}\|A^{-1}R^{n-k+1}\|_2 \tag{3.3}$$

Then

$$\begin{aligned} \|E^n\|_2 &\leq \|E_0\|_2 + \sum_{k=1}^{n}\frac{\tau\lambda_1}{(1+\tau\lambda_1)^k}\|A^{-1}R^{n-k+1}\|_2 \\ &\leq \|E_0\|_2 + \max_k \|A^{-1}R^{n-k+1}\|_2 \end{aligned} \tag{3.4}$$

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REFERENCES

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