AN α-ROBUST DIFFERENCE QUADRATURE METHOD FOR SPACE-TIME CAPUTO-RIESZ FRACTIONAL DIFFUSION EQUATION *

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Key words. example, LaTeX

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1. Introduction.

We study

$$\frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u = f(x,t), \quad x \in \Omega, t \in (0,T]. \tag{1.1}$$

Implicity scheme: Let $\tau = \frac{T}{M}$, $U^n, F^n \in \mathbb{R}^{2N-1}$,

$$\frac{U^{n+1} - U^n}{\tau} + AU^{n+1} = F^{n+1}. \tag{1.2}$$

Then $E^n = U^n - \hat{U}^n \in \mathbb{R}^{2N-1}$,

$$(I + \tau A)E^{n+1} = E^n + \tau R^{n+1}. \tag{1.3}$$

We will prove the convergence of this scheme.

2. Property of A.

LEMMA 2.1. *The stiffness matrix A has the following properties:*

1. *The eigenvalues of A are positive real numbers.*
2. *A is positive definite, which means that the eigenvalues of $\frac{A+A^T}{2}$ are positive.*
3. *The eigenvectors of A are orthogonal in space where $\langle u, v\rangle := uHv$, where $H := diag\left(\frac{h_i+h_{i+1}}{2}\right)$.*
4. *$(I + \tau A)^{-1} > O$ for any $\tau > 0$.*

Proof. Since

$$A = H^{-1}D = H^{-1/2}H^{-1/2}DH^{-1/2}H^{1/2}, \tag{2.1}$$

where $H^{-1/2}DH^{-1/2}$ is symmetric positive definite, $H^{-1/2}DH^{-1/2} = U\Lambda U^T$. Thus,

$$A = H^{-1/2}U\Lambda U^T H^{1/2} = (H^{-1/2}U)\Lambda(H^{-1/2}U)^{-1}. \tag{2.2}$$

The eigenvectors of A form an orthogonal basis of the Hilbert space defined by $\langle u, v\rangle := uHv$. Let $v_i = H^{-1/2}u_i$ be an eigenvector of A with eigenvalue λ_i. ∎

We need to prove $\lambda_1 > c$ for some positive constant c.

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3. Regularity.

LEMMA 3.1. *If*

$$\left|\frac{\partial^l u}{\partial x^l}\right| \le \delta(x)^{\alpha/2-l}, \quad x \in \Omega, \tag{3.1}$$

then

$$\left|-\frac{d^l}{dx^l} I^{2-\alpha} u\right| \le C \frac{\kappa_\alpha(\alpha-1)}{\Gamma(3-\alpha)} \delta(x)^{-\alpha/2-(l-2)}, \quad l = 1,2,3,4, \quad x \in \Omega, \tag{3.2}$$

where C is independent of α.

Proof.

$$\begin{aligned}\frac{d^l}{dx^l} I^{2-\alpha} u = \frac{\kappa_\alpha}{\Gamma(2-\alpha)} \Bigg(& (1-\alpha)\cdots(1-(l-1)-\alpha)\left(\int_0^{x/2} u(y)(x-y)^{1-l-\alpha} dy + (-1)^l \int_{T+x/2}^{2T} u(y)(y-x)^{1-l-\alpha} dy\right) \\ & + \sum_{k=1}^{l-1} (1-\alpha)\cdots(1-(k-1)-\alpha)\left(\frac{\partial^{l-1-k}}{\partial x^{l-1-k}} u(x/2)\left(\frac{x}{2}\right)^{1-k-\alpha} - (-1)^k \frac{\partial^{l-1-k}}{\partial x^{l-1-k}} u(T+x/2)\left(T+\frac{x}{2}\right)^{1-k-\alpha}\right) \\ & + (1-\alpha)\left(\int_{x/2}^{x} \frac{\partial^{l-1}}{\partial x^{l-1}} u(y)(x-y)^{-\alpha} - \int_x^{T+x/2} \frac{\partial^{l-1}}{\partial x^{l-1}} u(y)(y-x)^{-\alpha}\right)\Bigg)\end{aligned} \tag{3.3}$$

Set $x \le T$, we have

$$\left|\int_0^{x/2} u(y)(x-y)^{1-l-\alpha} dy\right| \le 2^{\alpha+l-1} \int_0^{x/2} y^{\alpha/2} x^{1-l-\alpha} dy \simeq x^{-\alpha/2-(l-2)}$$

$$\frac{\partial^{l-1-k}}{\partial x^{l-1-k}} u(x/2)\left(\frac{x}{2}\right)^{1-k-\alpha} \lesssim x^{-\alpha/2-(l-2)}$$

$$\begin{aligned}&\int_{x/2}^{x} \frac{\partial^{l-1}}{\partial x^{l-1}} u(y)(x-y)^{-\alpha} - \int_x^{T+x/2} \frac{\partial^{l-1}}{\partial x^{l-1}} u(y)(y-x)^{-\alpha} \\ &\quad = \int_0^{x/2} \left(\frac{\partial^{l-1}}{\partial x^{l-1}} u(x-z) - \frac{\partial^{l-1}}{\partial x^{l-1}} u(x+z)\right) z^{-\alpha} dz - \int_{x+x/2}^{T+x/2} \frac{\partial^{l-1}}{\partial x^{l-1}} u(y)(y-x)^{-\alpha} dy\end{aligned}$$

While

$$\begin{aligned}&\left|\int_0^{x/2} \left(\frac{\partial^{l-1}}{\partial x^{l-1}} u(x-z) - \frac{\partial^{l-1}}{\partial x^{l-1}} u(x+z)\right) z^{-\alpha} dz\right| \\ &\quad = 2\int_0^{x/2} \frac{\partial^l}{\partial x^l} u(\xi) z^{1-\alpha} dz \le \left(\frac{x}{2}\right)^{\alpha/2-l} 2\int_0^{x/2} z^{1-\alpha} dz \simeq \frac{1}{2-\alpha} x^{-\alpha/2-(l-2)}\end{aligned}$$

$\square$

$$\begin{aligned}&\int_{x+x/2}^{T+x/2} \frac{\partial^{l-1}}{\partial x^{l-1}} u(y)(y-x)^{-\alpha} dy \\ &\quad \le \int_{x+x/2}^{T+x/2} y^{\alpha/2-(l-1)} y^{-\alpha} dy \lesssim \begin{cases} \frac{1}{2-\alpha} T^{-\alpha/2-(l-2)}, & l = 1, \\ \frac{1}{\alpha/2+l-1} x^{-\alpha/2-(l-2)}, & l = 2,3,4. \end{cases}\end{aligned}$$

4. truncation error.

5. Convergence.

$$\begin{aligned} E^n &= (I+\tau A)^{-1}E^{n-1} + (I+\tau A)^{-1}\tau R^n \\ &= (I+\tau A)^{-n}E^0 + \sum_{k=1}^{n}(I+\tau A)^{-k}\tau R^{n-k+1} \end{aligned} \tag{5.1}$$

Assume that $|R^n| \le Ch^{\min\{r\alpha/2,2\}}(x_i^{-\alpha} + (2T-x_i)^{-\alpha}) + C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha} + C\tau^?$

$$\begin{aligned} (I+\tau A)^{-k}\tau R^{n-k+1} &= (\tau A)(I+\tau A)^{-k}(\tau A)^{-1}\tau R^{n-k+1} \\ &= (\tau A)(I+\tau A)^{-k}(A^{-1}R^{n-k+1}) \end{aligned} \tag{5.2}$$

Suppose that

$$\begin{aligned} |R^n| &\le |R| \\ &:= Ch^{\min\{r\alpha/2,2\}}(x_i^{-\alpha} + (2T-x_i)^{-\alpha}) \\ &\quad + C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha} + C\tau^? \end{aligned} \tag{5.3}$$

Since $0 < A^{-1}R \le Ch^{\min}$,

$$|(I+\tau A)^{-k}\tau R^{n-k+1}| \le (I+\tau A)^{-k}\tau R = \tau A(1+\tau A)^{-k}A^{-1}R \tag{5.4}$$

Then

$$\begin{aligned} |E^n| &\le |(I+\tau A)^{-n}E^0| + \sum_{k=1}^{n}\tau A(1+\tau A)^{-k}A^{-1}R \\ &= |(I+\tau A)^{-n}E^0| + \left(I-(I+\tau A)^{-n}\right)A^{-1}R. \end{aligned} \tag{5.5}$$

Since A is diagonally dominant, $\|(I+\tau A)^{-1}E\|_\infty \le \|E\|_\infty$, we have

$$\|E^n\|_\infty \le \|E_0\|_\infty + 2\|A^{-1}R\|_\infty. \tag{5.6}$$

Lemma 5.1. *$A^{-1}R$ is bounded by $C\left(h^{\min\{r\alpha/2,2\}} + \tau^?\right)$, where C is a constant independent of h, α.*

6. Caputo-Riesz. $\gamma \in (0,1)$

$$D_t^\gamma u + (-\Delta)^{\frac{\alpha}{2}}u = f(x,t), \quad x\in\Omega, t\in(0,T]. \tag{6.1}$$

where

$$D_t^\gamma u(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\partial u(x,s)}{\partial s}(t-s)^{-\gamma}ds \tag{6.2}$$

$$\begin{aligned} D_N^\gamma u(x,t_n) &= \sum_{k=1}^{n}\frac{1}{\Gamma(2-\gamma)}\left(u(x,t_k)-u(x,t_{k-1})\right)\frac{(t_n-t_{k-1})^{1-\gamma}-(t_n-t_k)^{1-\gamma}}{\tau_k} \\ &= \frac{d_{n,n}}{\Gamma(2-\gamma)}u(x,t_n) - \sum_{k=1}^{n-1}\frac{d_{n,k+1}-d_{n,k}}{\Gamma(2-\gamma)}u(x,t_k) - \frac{d_{n,1}}{\Gamma(2-\gamma)}u(x,t_0), \end{aligned} \tag{6.3}$$

where

$$d_{n,k} = \frac{(t_n - t_{k-1})^{1-\gamma} - (t_n - t_k)^{1-\gamma}}{\tau_k} \tag{6.4}$$

$d_{n,n} = \tau_n^{-\gamma}, \quad d_{n,k+1} \geq d_{n,k}$.

Numerical scheme:

$$D_N^{\gamma} U^n + AU^n = F^n \tag{6.5}$$

We have

$$\left(\frac{d_{n,n}}{\Gamma(2-\gamma)} I + A \right) E^n = \sum_{k=1}^{n-1} \frac{d_{n,k+1} - d_{n,k}}{\Gamma(2-\gamma)} E^k + \frac{d_{n,1}}{\Gamma(2-\gamma)} E^0 + R^n \tag{6.6}$$

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REFERENCES

[1] G. H. Golub and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, 4th ed., 2013.