复几何

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本课程参考以下教材：

1. Demailly: Complex analytic and differential geometry.
2. Huybrechts: Complex geometry: an introduction.
3. Morrow, Kodaira: Complex manifolds.
4. Grauert, Remmert: Coherent analytic sheaves.
5. Hormander: An introduction to complex analysis in several variables.
6. Griffiths, Harris: Principles of algebraic geometry.

在五道口也要红专并进、理实交融呀 ∼

目录

第 1 章　多复变函数

1.1　多元全纯函数

首先快速回顾单复变函数的知识。我们通常用 Ω 来表示 $\mathbb{C}$ 的开子集，$z = x + iy$ 为 $\mathbb{C}$ 的坐标。对于 $z \in \mathbb{C}$ 以及实数 $R > 0$，我们令

$$\mathbb{D}(z, R) := \{w \in \mathbb{C} | \, |w - z| < R\}$$

为以 z 为圆心 R 为半径的开圆盘。

此外，我们有如下常用记号：

$$\begin{cases} \mathrm{d}z := \mathrm{d}x + i\mathrm{d}y \\ \mathrm{d}\bar{z} := \mathrm{d}x - i\mathrm{d}y \end{cases} \qquad \begin{cases} \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{cases}$$

对于函数 $f : \Omega \to \mathbb{C}$，称 f 是**全纯**（holomorphic）的，若在 Ω 中成立

$$\bar{\partial} f := \frac{\partial f}{\partial \bar{z}} \mathrm{d}\bar{z} = 0$$

我们知道，f 是全纯的当且仅当 f 在 Ω 处处能够局部地展开为收敛幂级数。

对于 $\mathbb{C}$ 中的紧致集 K，称函数 $f : K \to \mathbb{C}$ 是全纯的，如果存在 K 的开邻域 $\Omega \supseteq K$，使得 f 可延拓为 Ω 上的全纯函数。

单复变函数论中有如下重要结果：

定理 1.1.1. （柯西积分公式）设 $\mathbb{D} \subseteq \mathbb{C}$ 为 $\mathbb{C}$ 中的开圆盘，$f : \mathbb{D} \to \mathbb{C}$ 为 $\mathbb{D}$ 上的全纯函数，且在 $\partial\mathbb{D}$ 连续，则对于任意 $w \in \mathbb{D}$，成立

$$f(w) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(z)}{z - w} \mathrm{d}z$$

此定理能推导出单变量全纯函数理论的“almost everything”. 这里不再赘述。

我们开始考虑多变量全纯函数。

定义 1.1.2. 设 $\Omega \subseteq \mathbb{C}^n$ 为 $\mathbb{C}^n$ 的开子集，函数 $f : \Omega \to \mathbb{C}$ 称为（多变量）**全纯函数**，如果满足以下条件：

（*1*）f 是连续函数；

（*2*）对任意 $1 \le j \le n$，以及任意固定的 $z_1, ..., z_{j-1}; z_{j+1}, ..., z_n \in \mathbb{C}$，关于 z_j 的单变量函数

$$z_j \mapsto f(z_1, ..., z_{j-1}; z_j; z_{j+1}, ..., z_n)$$

是（单变量）全纯函数。

事实上，如果该定义中的（2）成立，那么能推出（1）成立，也就是说此定义中的（1）可以去掉。其证明比较复杂，我们承认之。

记号 1.1.3. 对于 $\mathbb{C}^n$ 的开子集 Ω，我们记

$$\mathcal{O}(\Omega) := \{f : \Omega \to \mathbb{C} | f\text{是 } \Omega \text{ 上的全纯函数}\}$$

容易知道 $\mathcal{O}(\Omega)$ 有显然的 $\mathbb{C}$-代数结构。

本节将说明，多变量全纯函数具有一些与单变量全纯函数类似的性质。

记号 1.1.4. 对于 $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$ 以及 $R = (R_1, R_2, ..., R_n) \in \mathbb{R}^n$，并且 $R_j > 0$ $(\forall 1 \le j \le n)$，则我们记

$$\mathbb{D}(z, R) := \mathbb{D}(z_1, R_1) \times \mathbb{D}(z_2, R_2) \times \cdots \times \mathbb{D}(z_n, R_n)$$

称为以 z 为中心，R 为半径的**多圆柱**（*polydisk*）。

对于多圆柱 $\mathbb{D}(z, R)$，我们记

$$\Gamma(z, R) := \partial\mathbb{D}(z_1, R_1) \times \partial\mathbb{D}(z_2, R_2) \times \cdots \times \partial\mathbb{D}(z_n, R_n)$$

称为 $\mathbb{D}(z, R)$ 的**特征边界**（*distinguished boundary*）。

特别注意特征边界 $\Gamma(z, R)$ 并不等于该多圆柱的边界 $\partial\mathbb{D}(z, R)$.

定理 1.1.5. （多变量全纯函数的柯西积分公式）

设 $f:\overline{\mathbb{D}(z,R)}\to\mathbb{C}$ 为全纯函数，则对任意的 $w\in\mathbb{D}(z,R)$，成立

$$f(w)=\frac{1}{(2\pi i)^n}\int_{\Gamma(z,R)}\frac{f(\xi)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_n}{(\xi_1-w_1)(\xi_2-w_2)\cdots(\xi_n-w_n)}$$

证明. 由多变量全纯函数的定义，反复使用单变量全纯函数的柯西积分公式即可。这是容易的。 □

与单复变函数完全类似，我们也有泰勒展开：

推论 1.1.6. （多元全纯函数的泰勒展开公式）

对于 $f\in\mathcal{O}(\Omega)$，其中 $\Omega\subseteq\mathbb{C}^n$ 为开子集，则对于任何多圆柱 $\mathbb{D}(z_0,R)$，如果 $\overline{\mathbb{D}(z_0,R)}\subseteq\Omega$，则对于任意 $w\in\mathbb{D}(z_0,R)$，成立

$$f(w)=\sum_{\alpha\in\mathbb{N}^n}a_\alpha(w-z_0)^\alpha$$

其中

$$a_\alpha=\frac{1}{(2\pi i)^n}\int_{\Gamma(z_0,R)}\frac{f(z)}{(z-z_0)^{\alpha+1}}\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_n=\frac{f^{(\alpha)}(z_0)}{\alpha!}$$

注意这里的 α 为多重指标，即 $\alpha=(\alpha_1,...,\alpha_n)$，其中每个 α_i 都为非负整数。我们记

$$\begin{aligned}
z^\alpha &:= z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n}\\
\alpha! &:= \alpha_1!\alpha_2!\cdots\alpha_n!\\
f^{(\alpha)} &:= (\partial_{z_1})^{\alpha_1}(\partial_{z_2})^{\alpha_2}\cdots(\partial_{z_n})^{\alpha_n}f\\
\alpha+1 &:= (\alpha_1+1,\alpha_2+1,...,\alpha_n+1)
\end{aligned}$$

其中 $z=(z_1,...,z_n)\in\mathbb{C}^n$，$f$ 为 n 元全纯函数。

证明. 与单复变函数的情形完全类似，可由柯西积分公式得到。 □

定理 1.1.7. （柯西不等式）对于 $\mathbb{C}^n$ 的开子集 Ω，若 $f\in\mathcal{O}(\Omega)$，多圆柱 $\overline{\mathbb{D}(z_0,R)}\subseteq\Omega$，则对任意多重指标 $\alpha\in\mathbb{N}^n$，成立

$$\left|f^{(\alpha)}(z_0)\right|\le\frac{\alpha!}{R^\alpha}\sup_{z\in\Gamma(z_0,R)}|f(z)|$$

证明. 与单复变函数的情形完全类似。利用多元泰勒展开（推论1.1.6）即可。 □

推论 1.1.8. 设 $\Omega \subseteq \mathbb{C}^n$ 为**连通**开集，$f \in \mathcal{O}(\Omega)$ 满足 $\forall 1 \leq k \leq n$，$\frac{\partial f}{\partial z_k}$ 在 Ω 上恒为 0，则 f 在 Ω 上为常值函数。

推论 1.1.9. （刘维尔定理）设 $f \in \mathcal{O}(\mathbb{C}^n)$，并且满足

$$|f(z)| \leq A(1+|z|)^B$$

其中 A, B 为正实数，那么 f 必为次数不超过 B 的多项式函数。

这些性质于单变量全纯函数雷同，证明也是类似的。

推论 1.1.10. （*Montel* 定理）

设 Ω 为 $\mathbb{C}^n$ 的开子集，则 $\mathcal{O}(\Omega)$ 中的任何局部一致有界的全纯函数列都存在一致收敛的子列。

证明. 仍类似于单复变全纯函数的情形。使用柯西积分公式，再配合 Arzela-Ascoli 定理即可。从略。 □

现在，简单介绍一些复的微分形式。对于 $\mathbb{C}^n$，记其复坐标为 $(z_1, z_2, ..., z_n)$；视 $\mathbb{C}^n$ 为 $2n$ 维实线性空间，

$$z_k = x_k + iy_k$$

从而引入

$$\begin{aligned} \mathrm{d}z_k &= \mathrm{d}x_k + i\mathrm{d}y_k \qquad (1,0)\text{形式} \\ \mathrm{d}\bar{z}_k &= \mathrm{d}x_k - i\mathrm{d}y_k \qquad (0,1)\text{形式} \end{aligned}$$

定义 1.1.11. （(p,q)-形式）

设 Ω 为 $\mathbb{C}^n$ 的非空开集，则形如

$$u(z) = \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ}(z) \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

的光滑张量场称为 (p,q)-形式。记 Ω 上的 (p,q)-形式之全体为 $C^\infty_{p,q}(\Omega)$.

这里的 I, J 为多重指标。“光滑”指的是系数函数 a_{IJ} 为 Ω 上的光滑复值函数。另外，显然 $(0,0)$-形式即为光滑函数；$C^\infty_{p,q}(\Omega)$ 具有显然的复线性空间结构，事实上还是 $C^\infty(\Omega)$-模。

记号 1.1.12. （$\bar\partial$-算子）*定义算子*

$$\bar\partial : C^\infty_{p,q}(\Omega) \to C^\infty_{p,q+1}(\Omega)$$

如下：对于 (p,q)-形式

$$u := \sum_{\substack{|I|=p\\|J|=q}} a_{IJ}\mathrm{d}z_I \wedge \mathrm{d}\bar z_J$$

则

$$\bar\partial u = \sum_{\substack{|I|=p\\|J|=q}} \sum_{k=1}^{n} \frac{\partial a_{IJ}}{\partial \bar z_k}\mathrm{d}\bar z_k \wedge \mathrm{d}z_I \wedge \mathrm{d}\bar z_J$$

类似地，也有

$$\partial : C^\infty_{p,q}(\Omega) \to C^\infty_{p+1,q}(\Omega)$$

它们与外微分算子 d 满足关系

$$\mathrm{d} = \partial + \bar\partial$$

由 $\mathrm{d}^2 = 0$，易知

$$\partial^2 = 0, \quad \bar\partial^2 = 0, \quad \partial\bar\partial + \bar\partial\partial = 0$$

以下事实显然成立：

引理 1.1.13. *对于区域 Ω 上的光滑函数 $f \in C^\infty(\Omega)$，则 f 全纯当且仅当 $\bar\partial f = 0$.*

注记 1.1.14. （*Dolbeault 上同调*）*对于 $\Omega \subseteq \mathbb{C}^n$，注意 $\bar\partial^2 = 0$，从而对任意 $p \geq 0$，有上链复形 $C^\infty_{p,\bullet}(\Omega)$：*

$$\cdots \to C^\infty_{p,q-1}(\Omega) \xrightarrow{\bar\partial} C^\infty_{p,q}(\Omega) \xrightarrow{\bar\partial} C^\infty_{p,q+1}(\Omega) \to \cdots$$

称上同调群

$$H^{p,q}(\Omega) := H^q(C^\infty_{p,\bullet}(\Omega), \bar\partial)$$

为区域 Ω 的 ***Dolbeault* 上同调群**。

类似于外微分 d 的 de-Rham 上同调群，Dolbeault 上同调群与 Ω 的拓扑联系密切。例如，以下定理十分重要，我们先陈述，以后再证明：

引理 1.1.15. *(Dolbeault-Grothendieck 引理)*
设 $\mathbb{D} \subseteq \mathbb{C}^n$ 为多圆柱，则对于任意 $p,q \geq 0$，

$$H^{p,q}(\mathbb{D}) = 0$$

不难发现它与 de Rham 上同调的 Poincare 引理有些类似。

1.2 解析延拓与 Hartogs 现象

上一节介绍了多复变函数的一些“普通的”（与单变量类似）性质，本节开始介绍多复变函数的一些独特性质。

引理 1.2.1. 设 $f \in C_c^\infty(\mathbb{C})$ 为复平面上的紧支光滑函数，则对任意 $z \in \mathbb{C}$，成立

$$\frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\partial f/\partial\overline{\tau}}{\tau - z}\mathrm{d}\tau \wedge \mathrm{d}\overline{\tau} = f(z)$$

证明. 基本的微积分练习。考虑换元 $\tau = z + re^{i\theta}$，则易知

$$\begin{aligned}
\mathrm{d}\tau \wedge \mathrm{d}\overline{\tau} &= -2ir\mathrm{d}r \wedge \mathrm{d}\theta \\
\frac{\partial r}{\partial\overline{\tau}} &= \frac{1}{2}e^{i\theta} \\
\frac{\partial\theta}{\partial\overline{\tau}} &= -\frac{1}{2ir}e^{i\theta}
\end{aligned}$$

因此有

$$\begin{aligned}
\frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\partial f/\partial\overline{\tau}}{\tau - z}\mathrm{d}\tau \wedge \mathrm{d}\overline{\tau} &= \frac{-1}{2\pi}\int_0^\infty \mathrm{d}r \int_0^{2\pi} \left(-\frac{1}{ir}\frac{\partial f}{\partial\theta}(z + re^{i\theta})\right)\mathrm{d}\theta \\
&\quad + \frac{-1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int_0^\infty \left(\frac{\partial f}{\partial r}(z + re^{i\theta})\right)\mathrm{d}r \\
&= 0 + \frac{-1}{2\pi}\int_0^{2\pi} -f(z)\mathrm{d}\theta \\
&= f(z)
\end{aligned}$$

证毕。 □

引理 1.2.2. （简单版本的 $\bar{\partial}$-引理）

设 $n \geq 2$，$\varphi \in C^{\infty}_{0,1}(\mathbb{C}^n)$ 为具有紧支集的光滑 $(0,1)$-形式，且 $\bar{\partial}\varphi = 0$，则存在 $\mathbb{C}^n$ 上的紧支光滑函数 g，使得

$$\bar{\partial}g = \varphi$$

证明. 记光滑 $(0,1)$-形式 φ 为

$$\varphi = \sum_{k=1}^{n} \varphi_k(z_1, ..., z_n) \mathrm{d}\bar{z}_k$$

则

$$\bar{\partial}\varphi \;=\; \sum_{k,l} \frac{\partial \varphi_k}{\partial \bar{z}_l} \mathrm{d}\bar{z}_l \wedge \mathrm{d}\bar{z}_k = \sum_{1 \leq l < k \leq n} \left(\frac{\partial \varphi_k}{\partial \bar{z}_l} - \frac{\partial \varphi_l}{\partial \bar{z}_k} \right) \mathrm{d}\bar{z}_l \wedge \mathrm{d}\bar{z}_k$$

从而由 $\bar{\partial}\varphi = 0$ 可得对任意 $k \neq l$，

$$\frac{\partial \varphi_k}{\partial \bar{z}_l} = \frac{\partial \varphi_l}{\partial \bar{z}_k}$$

考虑如下的 $\mathbb{C}^n$ 上的函数 ψ：对于 $z = (z_1, ..., z_n) \in \mathbb{C}^n$，

$$\psi(z) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\varphi_1(\tau; z_2, ..., z_n)}{\tau - z_1} \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}$$

由 φ_1 的紧支性易知 ψ 为 $\mathbb{C}^n$ 上的光滑函数。对于 $1 < k \leq n$，有

$$\begin{aligned}
\frac{\partial \psi(z)}{\partial \bar{z}_k} &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_1}{\partial \bar{z}_k}(\tau; z_2, ..., z_n)}{\tau - z_1} \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau} \\
&= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_k}{\partial \bar{\tau}}(\tau; z_2, ..., z_n)}{\tau - z_1} \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau} \\
&= \varphi_k(z)
\end{aligned}$$

上式对 $k = 1$ 显然也成立。因此 $\bar{\partial}\psi = \varphi$.

最后还需要证明 ψ 是紧支的。由于 φ 紧支，存在足够大的 $R > 0$，使得

$$\operatorname{supp} \varphi \subseteq \mathbb{D}(0, R)$$

因此任意取定 $z \in \mathbb{C}^n$，使得 z 的分量 $z_2, z_3, ..., z_n$ 之中至少有一个模长大于 R，则由 ψ 的定义式直接得到 $\psi(z) = 0$. (注意：这一步严重依赖 $n \geq 2$！) 也就是说，存在 $z \notin \mathbb{D}(0, R)$ 使得 $\psi = 0$ 在 z 的某邻域内都成立。另一方面，由于 $\bar{\partial}\psi = \varphi$ 且 $\operatorname{supp} \varphi \subseteq \mathbb{D}(0, R)$，从而 ψ 在 $\mathbb{D}(0, \mathbb{R})$ 外部全纯，因此由解析延拓唯一性，ψ 在 $\mathbb{D}(0, R)$ 外部恒为零，因此 ψ 紧支。 □

此引理在单复变 $n = 1$ 的情形**不成立**：

例子 1.2.3. 设 $\varphi_1 \in C_0^\infty(\mathbb{C})$ 为复平面上的紧支光滑函数，并且

$$\iint_{\mathbb{C}} \varphi_1(z) \neq 0$$

考虑 $\mathbb{C}$ 上的 $(0,1)$-形式 $\varphi = \varphi_1(z)\mathrm{d}\bar{z}$，则 $\bar{\partial}\varphi = 0$ 是平凡的，但**不存在**紧支光滑函数 ψ 使得 $\bar{\partial}\psi = \varphi$.

证明. 若存在紧支光滑函数 ψ 使得 $\bar{\partial}\psi = \varphi$，则 $\frac{\partial\psi}{\partial\bar{z}} = \varphi_1$. 于是

$$0 \neq \iint_{\mathbb{C}} \varphi_1(z)\mathrm{d}z \wedge \mathrm{d}\bar{z} = \iint_{\mathbb{C}} \frac{\partial\psi}{\partial\bar{z}}\mathrm{d}z \wedge \mathrm{d}\bar{z} = 0$$

产生矛盾。 □

以下是多复变函数解析延拓的令人惊讶的性质，它与单复变函数有本质不同：

定理 1.2.4. （*Hartogs* 现象）

设 $\Omega \subseteq \mathbb{C}^n$ 为开集 $(n \geq 2)$，$K \subset\subset \Omega$ 且为 $\mathbb{C}^n$ 的紧子集，则对任意的 $f \in \mathcal{O}(\Omega \setminus K)$，都存在解析延拓 $F \in \mathcal{O}(\Omega)$，使得

$$F|_{\Omega\setminus K} = f$$

证明. 取 K 与 Ω 直接的截断函数 $\psi \in C_0^\infty(\mathbb{C}^n)$，使得 $0 \leq \psi \leq 1$，

$$K \subset\subset \operatorname{supp}\psi \subset\subset \Omega$$

并且 $\psi|_K \equiv 1$. 考虑

$$\widetilde{f} := (1-\psi)f$$

则 $\widetilde{f}$ 在整个 Ω 上都有定义。注意

$$\bar{\partial}\widetilde{f} = -(\bar{\partial}\psi)f + (1-\psi)\bar{\partial}f$$

易知 $\operatorname{supp}\bar{\partial}\widetilde{f} \subseteq \operatorname{supp}\psi$. 于是由引理1.2.2，存在光滑函数 v，使得 $\operatorname{supp} v \subseteq \psi$，并且 $\bar{\partial}v = \bar{\partial}\widetilde{f}$，从而考虑函数

$$F := (1-\psi)f - v$$

则 $\bar{\partial}F = 0$，从而 $F \in \mathcal{O}(\Omega)$. 又因为易知

$$F = f \quad (\forall z \in \Omega \setminus \operatorname{supp}\psi)$$

从而由解析延拓唯一性，有 $F_{\Omega\setminus K} = f$. □

关于解析延拓，再介绍如下结果：

引理 1.2.5. (*Hartogs figure*)

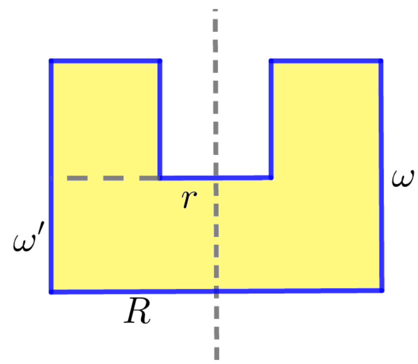
对于 $n>1$，正实数 $0\le r<R$，以及 $\mathbb{C}^{n-1}$ 的开子集 $\omega'\subseteq\omega$，其中 ω 是连通的。记 $\mathbb{C}^n$ 的开子集

$$\Omega:=((\mathbb{D}(0,R)\setminus\mathbb{D}(0,r))\times\omega)\cup\left(\mathbb{D}(0,R)\times\omega'\right)$$

其中 $\mathbb{D}(0,r)$ 与 $\mathbb{D}(0,R)$ 分别为 $\mathbb{C}$ 上的以原点为中心，r,R 为半径的开圆盘。则任意 $f\in\mathcal{O}(\Omega)$ 都可以（唯一地）解析延拓至

$$\widetilde{\Omega}:=\mathbb{D}(0,R)\times\omega$$

如此的区域 Ω 称之为"**Hartogs figure**"。Ω 的几何图像大致如下：



图：Hartogs figure 示意

证明. 容易知道

$$\Omega=\left\{(z_1,\widetilde{z})\in\mathbb{C}\times\mathbb{C}^{n-1}\middle|r<|z_1|<R,\widetilde{z}\in\omega\text{或者}|z_1|\le r,\widetilde{z}\in\omega'\right\}$$

对于 $f\in\mathcal{O}(\Omega)$，定义 $\widetilde{\Omega}$ 上的函数

$$\widetilde{f}(z_1,\widetilde{z}):=\frac{1}{2\pi i}\int_{|w|=\rho}\frac{f(w,\widetilde{z})}{z_1-w}\mathrm{d}w$$

其中 ρ 为满足 $\max\{r,|z_1|\}<\rho<R$ 的任意实数。则易知如此定义的 $\widetilde{f}$ 为 f 在 $\widetilde{\Omega}$ 上的解析延拓。 □

定理 1.2.6. (*Riemann* 延拓定理)

考虑 $\mathbb{C}^n$ 中的多圆柱 $\mathbb{D}(0,R)$，其中 $n\ge 2$，$R\in\mathbb{R}^n_+$。对任意 $2\le p\le n$，令 $\mathbb{C}^n$ 的子集

$$S:=(z_1,...,z_n)\in\mathbb{C}^n|z_1=\cdots=z_p=0$$

则对任意 $f\in\mathcal{O}(\mathbb{D}(0,R)\setminus S)$，$f$ 都可（唯一地）解析延拓至 $\mathbb{D}(0,R)$.

证明. 这是 Hartogs figure 的显然推论。记 $R=(R_1,R_2,...,R_n)$，以及 $R':=(R_2,...,R_n)\in\mathbb{R}^{n-1}$. 考虑 $\mathbb{C}^{n-1}$ 的开子集

$$\begin{aligned}\omega &:= \mathbb{D}(0,R')\\ \omega' &:= \omega\setminus\{z_2=\cdots=z_p=0\}\end{aligned}$$

则易知

$$\mathbb{D}(0,R)\setminus S=\Big(\mathbb{D}(0,R_1)\setminus\{0\}\times\omega\Big)\cup\Big(\mathbb{D}(0,R_1)\times\omega'\Big)$$

为 Hartogs figure，从而完。 □

1.3 Weierstrass 预备定理与除法定理

回顾单复变函数，若 f 在 $0\in\mathbb{C}$ 附近全纯，且 $f(0)=0$，则在 0 附近 f 可以唯一地分解为 $f=z^dg(z)$，其中 g 全纯且 $g(0)\neq 0$，d 为 f 在 0 处的**零点阶数**。

现在，设 $f=f(z,w)$ 在 $0\in\mathbb{C}^n(n\geq 2)$ 附近全纯，其中 $z\in\mathbb{C}$，$w\in\mathbb{C}^{n-1}$. 固定 w，记

$$f_w(z):=f(z,w)$$

为关于 z 的单复变函数。如果 $f_0(0)=0$ 且 $f_0(z)$ 不恒为零，则 $f_0(z)=z^dg_0(z)$。我们的一个结果是，若“f_0”的下标“0”稍微“扰动”一下，则相应的多项式 z^k 也“随之扰动”。

记号 1.3.1. （*Weierstrass 多项式*）

对于 $(z_0,w_0)\in\mathbb{C}\times\mathbb{C}^{n-1}$，则 (z_0,w_0) 处的 ***Weierstrass 多项式*** *是指形如下述的定义于 (z_0,w_0) 附近的 n 元全纯函数：*

$$P(z,w)=z^k+a_1(w)z^{k-1}+\cdots+a_k(w)$$

其中 $a_i(1\leq i\leq k)$ 为定义在 $w_0\in\mathbb{C}^{n-1}$ 附近的全纯函数，且 $a_i(w_0)=0$.

关于多元全纯函数在其零点附近的行为，首先有如下：

定理 1.3.2. *(Weierstrass 预备定理)*

设 $f(z,w)$ 为定义在 $(0,0)\in\mathbb{C}\times\mathbb{C}^{n-1}$ 附近的全纯函数，$f(0,0)=0$，且 $f_w(z)$ 在 $z=0$ 附近不恒为零，则存在唯一的 $(0,0)$ 处的 Weierstrass 多项式 $P(z,w)$，使得

$$f(z,w)=P(z,w)h(z,w)$$

其中 $h(z,w)$ 在 $(0,0)$ 附近全纯，且 $h(0,0)\neq 0$.

证明. 分若干步。

Step1 设 $f_0(z)$ 在 $z=0\in\mathbb{C}$ 处的零点阶数为 $d\geq 1$, 取足够小的 $\varepsilon>0$ 使得 $f_0(z)$ 在 $|z|\leq\varepsilon$ 之中不再有 $z=0$ 之外的零点。再由 f 的连续性以及 $\{|z|=\varepsilon\}\subseteq\mathbb{C}$ 的紧性，存在足够小的 $\varepsilon'>0$，使得对任意 $|z|=\varepsilon,|w|<\varepsilon'$，$f_w(z)\neq 0$.

对于 $w\in\mathbb{C}^{n-1}$ 且 $|w|<\varepsilon'$，由辐角原理，$f_w(z)$ 在 $|z|<\varepsilon$ 内的零点个数（记重数）为

$$d(w)=\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{f_w'(\xi)}{f_w(\xi)}\mathrm{d}\xi$$

这是关于 w 的连续函数，且 $d(0)=d$. 从而不妨缩小 ε'，使得任意 $|w|<\varepsilon'$，$f_w(z)$ 在 $|z|<\varepsilon$ 内的零点个数（计重数）均为 d.

Step2 对于 $w\in\mathbb{C}^{n-1}$ 且 $|w|<\varepsilon'$，记 $f_w(z)$ 的 d 个零点为 $s_1(w),s_2(w),...,s_d(w)$，它们允许相同，则 $|s_j(w)|<\varepsilon$（注意 $s_j(w)$ 未必为关于 w 的全纯函数）。特别地 $s_1(0)=s_2(0)=\cdots=s_d(0)=0$. 考虑多项式

$$\begin{aligned}P(z,w) &:= \prod_{j=1}^{d}(z-s_j(w))\\ &= z^d+\sum_{j=1}^{d}a_j(w)z^{d-j}\end{aligned}$$

显然系数 $a_j(w)$ 满足 $a_j(0)=0$. 断言 $P(z,w)$ 为 Weierstrass 多项式。为此只需证明 $z_j(w)$ 关于 w 全纯。由代数学可知，系数 a_j 可以写为形如 $s_1^k(w)+s_2^k(w)+\cdots s_d^k(w)\,(k\geq 0)$ 的 $\mathbb{C}$-线性组合；而由留数定理易知

$$\sum_{j=1}^{d}s_j^k(w) = \frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\xi^k\frac{f_w'(\xi)}{f_w(\xi)}\mathrm{d}\xi$$

从而关于 w 全纯。这就说明了 $P(z,w)$ 的系数函数 $a_j(w)$ 关于 w 全纯。

Step3 令 $h(z,w):=\frac{f(z,w)}{P(z,w)}$，断言 h 在 $(0,0)$ 附近全纯，又因为显然 $h(0,0)\neq 0$，从而 Weierstrass 预备定理的存在性得证。由单复变易知 $h(z,w)$ 关于 z 全纯，于是只需证明 h 关于 w 全纯。

任取 $w\in\mathbb{C}^{n-1}$ 且 $|w|<\varepsilon'$，由于 $h_w(z):=h(z,w)$ 关于 z 全纯，从而

$$h(z,w)=\frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\frac{h_w(\xi)}{\xi-z}\mathrm{d}\xi$$

而被积函数 $(\xi,w)\mapsto\frac{h_w(\xi)}{\xi-z}$ 在 $\{(z,w)||z|=\varepsilon,|w|<\varepsilon'\}$ 的某个邻域全纯，从而 $h(z,w)$ 关于 w 也全纯。存在性证毕。

Step4 唯一性几乎显然，因为 f（在 $(0,0)$ 附近）的零点完全由 Weierstrass 多项式贡献：对于 w，以 $s_1(w),...,s_d(w)$ 为零点的关于 z 的首一多项式只能是 $P(z,w)$. □

定理 1.3.3. （*Weierstrass* 除法定理）

设 $f(z,w)$ 为定义在 $(0,0)\in\mathbb{C}\times\mathbb{C}^{n-1}$ 附近的全纯函数，$g(z,w)=z^d+\sum\limits_{j=1}^{d}a_j(w)z^{d-j}$ 为次数为 d 的 *Weierstrass* 多项式。那么存在唯一的 $h(z,w)$ 与 $r(z,w)$，其中 h 为定义在 $(0,0)\in\mathbb{C}\times\mathbb{C}^{n-1}$ 附近的全纯函数，r 为关于 z 的在 $(0,0)$ 处的次数 $<d$ 的多项式，使得

$$f=gh+r$$

在 $(0,0)$ 附近成立。

证明. 先看唯一性。

Step1 唯一性是容易的。如果 $f=gh_1+r_1=gh_2+r_2$，则

$$r_1-r_2=g(h_2-h_1)$$

注意 g,r_1,r_2 为 Weierstrass 多项式，从而由之前讨论，存在足够小的 $\varepsilon,\varepsilon'>0$ 使得对任意 $w\in\mathbb{C}^{n-1}$ 且 $|w|<\varepsilon'$，$g_w(z)$ 在 $\{|z|<\varepsilon\}$ 内的零点个数（计重数）恰为 g 的次数 d，并且 $(r_1-r_2)_w(z)$ 在此范围内的零点个数（计重数）恰为 (r_1-r_2) 的次数。注意 r_1,r_2 的次数均小于 d，从而若 $r_1\neq r_2$，则导致 $(r_1-r_2)_w(z)$ 的零点个数小于 $g_w(z)(h_2-h_1)_w(z)$，因此导致矛盾。这迫使 $r_1=r_2$.

Step2 再看存在性。取 $\varepsilon,\varepsilon'>0$ 使得对任意 $|z|=\varepsilon$，$|w|\le\varepsilon'$，$g_w(z)\neq 0$。对任意 $|z|<\varepsilon,|w|<\varepsilon'$，定义

$$h(z,w)=\frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\frac{f_w(\xi)}{g_w(\xi)(\xi-z)}\mathrm{d}\xi$$

则易知 $h(z,w)$ 在 $(0,0)$ 附近全纯。再令 $r:=f-gh$，只需证明 r 为关于 z 的次数小于 d 的 Weierstrass 多项式即可。事实上，

$$\begin{aligned}
r(z,w) &= f(z,w)-g(z,w)h(z,w)\\
&= \frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\frac{f_w(\xi)}{\xi-z}\mathrm{d}\xi-\frac{g_w(z)}{2\pi i}\int_{|\xi|=\varepsilon}\frac{f_w(\xi)}{g_w(\xi)(\xi-z)}\mathrm{d}\xi\\
&= \frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\frac{f_w(\xi)(g_w(\xi)-g_w(z))}{g_w(\xi)(\xi-z)}\mathrm{d}\xi\\
&= \frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\frac{f_w(\xi)}{g_w(\xi)}\frac{(\xi^d-z^d)+a_1(w)(\xi^{d-1}-z^{d-1})+\cdots}{\xi-z}\mathrm{d}\xi\\
&= \frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\frac{f_w(\xi)}{g_w(\xi)}\left(z^{d-1}+\beta_1(\xi,w)z^{d-2}+\cdots\right)\mathrm{d}\xi
\end{aligned}$$

其中函数 $\beta_j(\xi,w)$ 由 g 的系数函数 $a_k(w)$ 决定。容易看出 $r(z,w)$ 的确为关于 z 的次数 $\le d-1$ 的多项式。存在性证毕。 $\square$

注意 r 未必是 Weierstrass 多项式，因为 $r(z,w)$ 的 z^{d-1} 的系数

$$\frac{1}{2\pi i}\int_{|\zeta|=\varepsilon}\frac{f_w(\zeta)}{g_w(\zeta)}\mathrm{d}\zeta$$

不见得是 1（若此积分为 0，则 r 的首项系数甚至可以是关于 w 的函数）。

注记 1.3.4. 事实上，*Weierstrass* 除法定理对单复变 $n=1$ 的情形也成立。设 $f(z)=\sum_{k=0}^{\infty}a_kz^k$ 在 $0\in\mathbb{C}$ 附近全纯，$g(z)=z^d$ 为次数为 d 的 *Weierstrass* 多项式。则令

$$\begin{aligned} h(z) &= \sum_{k=d}^{\infty}a_kz^{k-d} \\ r(z) &= \sum_{k=0}^{d-1}a_kz^k \end{aligned}$$

则 $f=gh+r$ 满足要求。

1.4 解析函数芽环 $\mathcal{O}_{\mathbb{C}^n,z}$ 及其代数结构

本节继续研究多元解析函数的性质。首先回顾函数芽的概念。

定义 1.4.1. （解析函数芽环）

对于 $z\in\mathbb{C}^n$，记

$$\mathcal{O}_{\mathbb{C}^n,z}:=\{(U,f)|U \text{ 是 } z \text{ 在 } \mathbb{C}^n \text{ 的一个开邻域，} f \text{ 为定义在 } U \text{ 上的全纯函数}\}/\sim$$

其中模掉的关系 $\sim$ 为

$$(U,f)\sim(V,g) \quad\Longleftrightarrow\quad \text{存在 } z \text{ 的开邻域 } W\text{，使得 } W\subseteq U\cap V\text{，且 } f|_W=g|_W$$

粗俗地说，$\mathcal{O}_{\mathbb{C}^n,z}$ 就是“定义在 $z\in\mathbb{C}^n$ 附近的全纯函数之全体”。之前介绍的 Weierstrass 预备定理、Weierstrass 除法定理其实都是解析函数芽环的性质。容易验证，$\mathcal{O}_{\mathbb{C}^n,z}$ 在通常的函数加法、乘法下构成**环**。

我们记 $\mathcal{O}_n:=\mathcal{O}_{\mathbb{C}^n,0}$. 本节介绍环 $\mathcal{O}_n$ 的代数性质。假定读者熟悉基础的交换代数。本讲义中的“环”默认为含幺、交换的。

定理 1.4.2. $\mathcal{O}_n$ 是局部诺特环（$\forall n\geq 1$）。

回顾：环 A 称为**局部环**（local ring），若 A 存在唯一极大理想 $\mathfrak{m}$（等价定义：A 的全体不可逆元构成 A 的理想）；环 A 称为**诺特环**（Noetherian ring），若满足理想升链条件（等价定义：A 的每个理想都是有限生成的）。

证明. 显然 $\mathcal{O}_n$ 为局部环，其极大理想 $\mathfrak{m}$ 由定义在 0 附近、在 0 处取值为 0 的函数芽构成。我们对 n 归纳证明 $\mathcal{O}_n$ 为诺特环。

$n=1$ 时，在单复变中我们早已熟知 $\mathcal{O}_1 \cong \{$收敛半径 ≥ 0 的幂级数$\}$ 为主理想整环（PID），其理想形如 $J_k=(z^k)$。特别地，为诺特环。

一般地，对于 $n\geq 2$，若 $\mathcal{O}_{n-1}$ 为诺特环，则对 $\mathcal{O}_n$ 的任意非零理想 J，断言 J 时有限生成的。任取 $0\neq h\in J\subseteq \mathfrak{m}$，则 $h(0)=0$，不妨 $h(z,0)$ 不恒为零（其中 $z\in\mathbb{C}, 0\in\mathbb{C}^{n-1}$），则由 Weierstrass 预备定理，存在 Weierstrass 多项式 $P(z,w)\in\mathcal{O}_{n-1}[z]\subseteq\mathcal{O}_n$ 以及函数芽 $h'\in\mathcal{O}_n\setminus\mathfrak{m}$，使得 $h(z,w)=P(z,w)h'(z,w)$. 注意 $h'(0,0)$ 为 $\mathcal{O}_n$ 的可逆元，又 $h\in J$ 且 J 为 $\mathcal{O}_n$ 的理想，从而 $P(z,w)\in J$.

这说明 J 当中必存在 Weierstrass 多项式。取定

$$P(z,w)=z^d+\sum_{j=1}^{d}a_j(w)z^{d-j}\in J$$

则对任意 $f\in J$，对 f,P 使用 Weierstrass 除法定理，存在 $g(z,w)\in\mathcal{O}_n$，以及

$$r(z,w)=\sum_{k=0}^{d-1}c_k(w)z^k\in\mathcal{O}_{\mathbb{C}^{n-1}}[z]$$

为次数至多为 $(d-1)$ 的多项式，使得

$$f=gP+r$$

则 $r(z,w)\in J$，并且容易验证，这诱导了 $\mathcal{O}_{n-1}$-模同态

$$\begin{aligned}\varphi: J &\to \mathcal{O}_{n-1}^{\oplus d}\cong\{r\in\mathcal{O}_{n-1}[z]\,|\,\deg_z r<d\}\\ f &\mapsto \sum_{k=0}^{d-1}c_k(w)z^k\end{aligned}$$

由归纳假设，$\mathcal{O}_{n-1}$ 为诺特环，从而 $\mathcal{O}_{n-1}^{\oplus d}$ 作为有限生成 $\mathcal{O}_{n-1}$-模为诺特模，从而其子模 $\operatorname{Im}\varphi$ 也为有限生成的。注意 $\operatorname{Im}\varphi\subseteq J$，记 $\{\beta_1,\dots,\beta_N\}\subseteq\operatorname{Im}\varphi$ 为 $\operatorname{Im}\varphi$ 的一组 $\mathcal{O}_{n-1}$-生成元，其中

$$\beta_j(w)=\sum_{l=0}^{d-1}\beta_{j,l}(w)z^l\in\mathcal{O}_{n-1}^{\oplus d}$$

则易知

$$\{\beta_j\}_{1\leq j\leq N}\cup\{P(z,w)\}$$

为理想 J 的一组生成元，因此 J 是有限生成的。从而 $\mathcal{O}_n$ 为诺特环。 □

引理 1.4.3. *设 $P, Q \in \mathcal{O}_{n-1}[z] \subseteq \mathcal{O}_n$，其中 P 为 Weierstrass 多项式，则 P 整除 Q 在 $\mathcal{O}_n$ 成立，当且仅当 P 整除 Q 在 $\mathcal{O}_{n-1}[z]$ 中成立。*

证明. “当”是显然的，只证“仅当”。若 $P|Q$ 在 $\mathcal{O}_n$ 中成立，则令

$$Q(z,w) = f(z,w)P(z,w)$$

其中 $f \in \mathcal{O}_n$. 另一方面，考虑 $\mathcal{O}_{n-1}[z]$ 中标准的欧几里得带余除法，

$$Q(z,w) = g(z,w)P(z,w) + r(z,w)$$

其中 $g, r \in \mathcal{O}_{n-1}[z]$. 则 Weierstrass 除法定理的唯一性迫使 $f = g, r = 0$，从而得证。 □

引理 1.4.4. *设 $P(z,w) \in \mathcal{O}_{n-1}[z]$ 为 Weierstrass 多项式，则：*

（1）若在 $\mathcal{O}_{n-1}[z]$ 中有分解

$$P = P_1 P_2 \cdots P_N$$

则在相差 $\mathcal{O}_{n-1}$ 中的可逆元的意义下，每个 P_j 都为 Weierstrass 多项式；

（2）P 为 $\mathcal{O}_n$ 中的不可约元当且仅当 P 为 $\mathcal{O}_{n-1}[z]$ 中的不可约元。

证明.

（1）记 $\deg_z P = s$，以及 $\deg_z P_j = s_j$，则 $s = \sum_{j=1}^{N} s_j$. 不妨每个 $s_j > 0$. 考虑 P 的最高次项，有

$$z^s = z^s \prod_{j=1}^{N} \left(P_j \text{ 的 } z^{s_j} \text{ 系数}\right)$$

从而相差 $\mathcal{O}_{n-1}$ 中某个可逆元倍，不妨每个 P_j 的 z^{s_j} 系数都为 1. 再注意

$$z^s = P(0,z) = \prod_{j=1}^{N} P_j(0,z) = \prod_{j=1}^{N} (z^{s_j} + \cdots)$$

从而迫使 $P_j(0,z) = z^{s_j}$，因此 P_j 为 Weierstrass 多项式。

（2）“仅当”是显然的，只证“当”。仍记 $P(z,w)$ 关于 z 的次数为 s. 如果 P 在 $\mathcal{O}_n$ 中可约，令 $P = g_1 g_2$，其中 g_1, g_2 为 $\mathcal{O}_n$ 中的不可逆元，从而关于 z 的函数 $g_1(z,0), g_2(z,0)$ 在 $z = 0$ 处的零点阶数大于 0，分别记为 s_1, s_2. 由 Weierstrass 预备定理，存在分解

$$g_j(z,w) = P_j(z,w) u_j(z,w) \quad (j = 1,2)$$

使得 $P_j \in \mathcal{O}_{n-1}[z]$ 为次数为 s_j 的 Weierstrass 多项式，u_j 为 $\mathcal{O}_n$ 中的可逆元。所以在 $\mathcal{O}_n$ 中成立 $(P_1 P_2)|P$；再根据引理1.4.3，可知 $(P_1 P_2)|P$ 在 $\mathcal{O}_{n-1}[z]$ 中也成立。而 P, P_1, P_2 都为首一多项式，从而必有 $P = P_1 P_2$，因此 P 在 $\mathcal{O}_{n-1}$ 中可约。 □

定理 1.4.5. $\mathcal{O}_n$ *是唯一分解整环 (UFD).*

证明. 对 n 归纳。$n=1$ 时，$\mathcal{O}_1$ 为主理想整环，从而为唯一分解整环。对于 $n\geq 2$，如果 $\mathcal{O}_{n-1}$ 为唯一分解整环，则由代数学中的高斯引理，多项式环 $\mathcal{O}_{n-1}[z]$ 也是唯一分解整环。

现在，对于 $\mathcal{O}_n$ 中的不可逆元 f，不妨 $z\mapsto f(z,w)|_{w=0}$ 不恒为零（$w\in\mathbb{C}^{n-1}$），从而由 Weierstrass 预备定理，存在分解 $f(z,w)=u(z,w)P(z,w)$，其中 u 为 $\mathcal{O}_n$ 中的可逆元，$P\in\mathcal{O}_{n-1}[z]$ 为 Weierstrass 多项式。由归纳假设，$\mathcal{O}_{n-1}[z]$ 为唯一分解整环，从而存在 P 在 $\mathcal{O}_{n-1}[z]$ 中的分解 $P=P_1P_2\cdots P_s$，使得每个 P_j 都为 $\mathcal{O}_{n-1}[z]$ 中的不可约元。从而由引理1.4.4的（1），不妨每个 P_j 都为 Weierstrass 多项式；再对每个 P_j 使用引理1.4.4的（2），知 P_j 为 $\mathcal{O}_n$ 中的不可约元。从而 $f\in\mathcal{O}_n$ 的不可约分解的存在性证毕。

再看分解的唯一性。只需再证明 $\mathcal{O}_n$ 的不可约元都是素元。若 f 为 $\mathcal{O}_n$ 中的不可约元，以及 $g,h\in\mathcal{O}_n$ 使得 $f|gh$，断言 $f|g$ 或者 $f|h$. 由 Weierstrass 预备定理，不妨假设 $f=f(z,w)$ 为关于第一个分量 z 的 Weierstrass 多项式，从而由 $f|gh$ 知 $g(z,0),h(z,0)$ 也不恒为零，于是由 Weierstrass 预备定理也不妨 $g,h\in\mathcal{O}_{n-1}[z]$ 为 Weierstrass 多项式。因此 $f|gh$ 在 $\mathcal{O}_{n-1}[z]$ 中成立，而由归纳假设 $\mathcal{O}_{n-1}[z]$ 是唯一分解整环，且 f 在 $\mathcal{O}_{n-1}[z]$ 不可约，所以 $f|g$ 或者 $f|h$ 在 $\mathcal{O}_{n-1}[z]$ 中成立，从而在 $\mathcal{O}_n$ 中成立。证毕。 □

1.5 解析集与局部解析零点定理

多复变函数与单复变的一个显著区别是解析延拓的难易程度，Hartogs 现象表明多复变函数“更容易被解析延拓”；而单复变与多复变函数令一个区别是零点集的形态：在单复变中我们熟知全纯函数零点离散（除非函数恒为零），这在多复变中显然不对，例如 $\mathbb{C}^2$ 上的全纯函数 $f(z_1,z_2)=z_1$.

事实上，多元全纯函数的零点集十分重要，而且是代数几何学中的某些概念（代数簇）的源头。

定义 1.5.1. （解析集）

设 $n\geq 2$，$\mathbb{C}^n$ 的子集 A 称为**解析集**（*analytic set*），若对任意 $z\in A$，存在 z 在 $\mathbb{C}^n$ 中的开邻域 Ω，以及 $f_1,f_2,...,f_N\in\mathcal{O}(\Omega)$，使得

$$A\cap\Omega=\{w\in\Omega|f_1(w)=f_2(w)=\cdots=f_N(w)\}$$

也就是说，“局部上看是若干全纯函数的公共零点集”。对于一个解析集，我们首先局部地研究之——类似于解析函数芽环，我们引入如下概念：

定义 1.5.2. （解析集芽）对于 $x \in \mathbb{C}^n$，定义

$$\mathcal{A}_x := \{(A,x) | x \in A,\ A \text{ 是 } \mathbb{C}^n \text{ 中的解析集}\} / \sim$$

其中关系 $\sim$ 为：$(A_1,x) \sim (A_2,x) \iff$ 存在 x 在 $\mathbb{C}^n$ 中的开邻域 Ω，使得 $A_1 \cap \Omega = A_2 \cap \Omega$. 称 $\mathcal{A}_x$ 中的元素为 x 处的**解析集芽**。

$\mathcal{A}_x$ 中的元素可以认为是包含 x 的“无穷小解析集”。容易知道它与解析函数芽的关系：任意 $(A,x) \in \mathcal{A}_x$，(A,x) 为 $\mathcal{O}_{\mathbb{C}^n,x}$ 中某些函数的公共零点集。

定义 1.5.3. 对于 $x \in \mathbb{C}^n$，

（*1*）对与 x 处的解析集芽 $(A,x) \in \mathcal{A}_x$，定义 $\mathcal{O}_{\mathbb{C}^n,x}$ 的理想

$$J_{(A,x)} := \{f \in \mathcal{O}_{\mathbb{C}^n,x} | f(z) = 0 \forall z \in A\}$$

（*2*）对于 $\mathcal{O}_{\mathbb{C}^n,x}$ 中的理想 J，定义 x 处的解析集芽

$$(V(J),x) := \{z \in \mathbb{C}^n | g(z) \equiv 0, \forall g \in J\}\text{的等价类}$$

这里并未仔细写清楚，需要验证良定性：注意解析集芽、函数芽实际上都为等价类，我们需要验证与代表元选取无关，留给读者。

注意 $\mathcal{O}_{\mathbb{C}^n,x}$ 为诺特环，从而任何理想 J 都是有限生成的，记 $\{g_1, g_2, ..., g_N\}$ 为其一组生成元，则易知

$$V(J) = \{g_1(x) = g_2(x) = \cdots = g_N(x) = 0\}$$

在 x 附近为有限个解析函数的公共零点集，从而的确为解析集（芽）。

引理 1.5.4. 设 $x \in \mathbb{C}^n$，$(A,x) \in \mathcal{A}_x$ 为 x 处的解析集芽，$J \subseteq \mathcal{O}_{\mathbb{C}^n,x}$ 为理想，则

$$\begin{aligned} J &\subseteq J_{(V(J),x)} \\ (V(J_{(A,x)}),x) &= (A,x) \end{aligned}$$

证明. 直接按定义验证即可。第一式是容易的；至于第二式，由解析集的定义，(A,x) 必形如

$$\{g_1(x) = g_2(x) = \cdots = g_N(x) = 0\}$$

其中 $g_j \in \mathcal{O}_{\mathbb{C}^n,x}$，从而 $J_{(A,x)} = (g_1, ..., g_N)$，之后容易。 □

注记 1.5.5. 不过要注意，第一式的等号未必成立，例如对于 $0 \in \mathbb{C}^2$，$f(z_1, z_2) = z_1^2$，令 $J := (f) \subseteq \mathcal{O}_{\mathbb{C}^2,0}$ 为由 f 生成的理想，则 $V(J) = \{z_1^2 = 0\} = \{z_1 = 0\}$，于是 $J_{(V(J),0)} = (z_1)$，即为由 $\widetilde{f}(z_1, z_2) = z_1$ 生成的理想。很明显，$J \subsetneqq J_{(V(J),0)}$.

对于 $x \in \mathbb{C}^n$，则 $\mathcal{A}_x$ 中的解析集芽可以进行交、并运算：

引理 1.5.6. 对于 $x \in \mathbb{C}^n$，$\{J_\alpha | \alpha \in \mathcal{I}\}$ 为 $\mathcal{O}_{\mathbb{C}^n,x}$ 的一族理想，则对任意 $\alpha, \beta \in \mathcal{I}$，

$$(V(J_\alpha) \cup V(J_\beta), x) = (V(J_\alpha J_\beta), x)$$

$$(\bigcap_{\alpha \in \mathcal{I}} V(J_\alpha), x) = (V(\sum_{\gamma \in \mathcal{I}} J_\gamma), x)$$

自行补全解析集芽交、并的定义（无非是取代表元作交、并）

证明. 直接定义验证。 □

此引理表明，一点处的解析集芽可以“有限并，任意交”，与拓扑学中的“闭集”类似。

接下来研究解析集芽的局部结构。

定义 1.5.7. （不可约解析集芽）

对于 $x \in \mathbb{C}^n$，以及 $(A, x) \in \mathcal{A}_x$，称解析集芽 (A, x) 是**不可约** *(irreducible)* 的，若不存在 $(A_1, x), (A_2, x) \in \mathcal{A}_x$，使得 $(A, x) = (A_1 \cup A_2, x)$，且 $(A_i, x) \subsetneqq (A, x), i = 1, 2$.

由引理1.5.6，以及基本的交换代数，容易知道：解析集芽 (A, x) 不可约，当且仅当 $J_{(A,x)}$ 为 $\mathcal{O}_{\mathbb{C}^n,x}$ 的**素理想**。此外，解析函数芽环的诺特性等价于如下：

引理 1.5.8. 对于 $x \in \mathbb{C}^n$，以及 $(A_k, x) \in \mathcal{A}_x, k \geq 1$，若 $(A_k, x) \supseteq (A_{k+1}, x)$ 对任意 $k \geq 1$ 都成立（即 $\{A_k\}_{k=1}^\infty$ 为解析集芽降链），则存在 $k_0 \geq 1$，使得对任意 $l \geq k_0$，都有 $(A_k, x) = (A_l, x)$.

证明. 考察理想 $J_{(A_k,x)} \subseteq \mathcal{O}_{\mathbb{C}^n,x}$，则 $(A_k, x) \supseteq (A_{k+1}, x)$ 表明

$$J_{(A_k,x)} \subseteq J_{(A_{k+1},x)}$$

即 $\{J_{(A_k,x)}\}_{k=1}^\infty$ 为理想升链，从而由 $\mathcal{O}_{\mathbb{C}^n,x}$ 的诺特性，以及引理1.5.4，得证。 □

定理 1.5.9. （解析集芽的不可约分解）

给定 $x \in \mathbb{C}^n$，则对任意 $(A, x) \in \mathcal{A}_x$，存在 $N \geq 1$，以及对任意 $1 \leq k \leq N$ 存在 $(A_k, x) \in \mathcal{A}_x$ 为不可约解析集芽，使得这些解析集芽**互不包含**，并满足

$$(A, x) = \bigcup_{k=1}^{N} (A_k, x)$$

并且上述分解是唯一的（不计次序）。

证明. **存在性**：先断言，若 (A, x) 可约，则存在分解 $(A, x) = (A^{(1)}, x) \cup (A^{(2)}, x)$，其中 $(A^{(1)}, x)$ 与 $(A^{(2)}, x)$ 都为 (A, x) 的真子芽，并且 $(A^{(1)}, x)$ 不可约。

这是因为，由 (A, x) 可约，取真子芽 $(A_1, x), (A'_1, x)$ 使得 $(A, x) = (A_1, x) \cup (A'_1, x)$（但至此无法保证 A_1, A_2 至少有一个不可约）。如果 (A_1, x) 不可约，则继续对其分解：$(A_1, x) = (A_2, x) \cup (A'_2, x)$，然后再考察 (A_2, x) 的可约性，不断做下去，总会得到不可约的 (A_k, x)；若不然就有解析集芽降链

$$(A_1, x) \supsetneqq (A_2, x) \supsetneqq (A_3, x) \supsetneqq \cdots$$

与引理1.5.8矛盾。因此必存在 $k > 0$，使得 (A_k, x) 不可约，此时

$$(A, x) = (A_k, x) \cup \left(\bigcup_{j=1}^{k} (A'_j, x) \right)$$

为所希望的分解，断言证毕。

反复使用此断言：令 $(A, x) = (A^{(1)}, x) \cup (B_1, x)$，其中 $(A^{(1)}, x)$ 不可约，若 (B_1, x) 可约，则再对 (B_1, x) 使用此断言：$(B_1, x) = (A^{(2)}, x) \cup (B_2, x)$，其中 $(A^{(2)}, x)$ 不可约；若 (B_2, x) 可约，则再继续对 (B_2, x) 使用断言……该操作必在有限步停止，停止于某个 $(B_{\widetilde{N}}, x)$ 不可约，否则就有解析集芽降链

$$(B_1, x) \supsetneqq (B_2, x) \supsetneqq (B_3, x) \cdots$$

与引理1.5.8矛盾。从而得到不可约分解

$$(A, x) = (B_{\widetilde{N}}, x) \cup \left(\bigcup_{k=1}^{\widetilde{N}} (A_k, x) \right)$$

之后适当取 $\{A_1, A_2, ..., A_{\widetilde{N}}; B_{\widetilde{N}}\}$ 的子集使得其中元素之并仍是 (A, x) 并且其中元素互不包含。因此存在性证毕。

唯一性：假设

$$(A, x) = \bigcup_{k=1}^{N} (A_k, x) = \bigcup_{k=1}^{N'} (A'_k, x)$$

都为 (A,x) 的满足题设的不可约分解，则需要证明 $N=N'$，并且有集合相等

$$\{A_1,A_2,...,A_N\}=\{A'_1,A'_2,...,A'_{N'}\}$$

对任意 A_i，因为

$$(A_i,x)=\bigcup_{k=1}^{N'}(A_i\cap A'_k,x)$$

从而 (A_i,x) 的不可约性迫使存在某个 (A'_j,x) 使得 $(A_i,x)=(A_i\cap A'_j,x)$，即 $(A_i,x)\subseteq(A'_j,x)$. 同理，对于此 (A'_j,x)，存在某个 $(A_{i'},x)$，使得 $(A'_j,x)\subseteq(A_{i'},x)$，因此

$$(A_i,x)\subseteq(A'_j,x)\subseteq(A_{i'},x)$$

但由于 $\{(A_k,x)\}_{k=1}^{N}$ 中任何两元素互不包含，因此上式等号成立。也就是说对任意 $1\le j\le N$，存在（唯一）$1\le j'\le N'$，使得 $(A_j,x)=(A'_{j'},x)$；同理对任意 $1\le j'\le N'$ 也有类似结果。这就给出了集合一一对应

$$\{A_1,A_2,...,A_N\}\cong\{A'_1,A'_2,...,A'_{N'}\}$$

从而证毕。 □

注记 1.5.10. 此定理表明，欲研究解析集芽的局部性态，只需要研究不可约解析集芽；一般的解析集芽无非是不可约解析集芽的有限并。

现在，考虑 $\mathcal{O}_n:=\mathcal{O}_{\mathbb{C}^n,0}$ 的素理想 $\mathfrak{p}$，我们研究解析集芽 $(V(\mathfrak{p}),0)$ 的性质。

记号 1.5.11. 给定 $\mathbb{C}^n$ 的一组基 $\{e_1,e_2,...,e_n\}$，关于此基的坐标函数记作 $z_1,z_2,...,z_n$，对 $1\le k\le n$，记

$$\mathbb{C}\{z_1,...,z_k\}:=\{f\in\mathcal{O}_n|\frac{\partial f}{\partial z_l}\equiv 0\ ,\forall k+1\le l\le n\}$$

为 $\mathcal{O}_n$ 中“只显含前 k 个变量的函数芽”，则明显有

$$\mathcal{O}_k\cong\mathbb{C}\{z_1,...,z_k\}\hookrightarrow\mathcal{O}_n$$

于是对于 $\mathcal{O}_n$ 的素理想 $\mathfrak{p}$，

$$\mathfrak{p}_k:=\mathfrak{p}\cap\mathbb{C}\{z_1,...,z_k\}$$

为子环 $\mathcal{O}_k\cong\mathbb{C}\{z_1,...,z_k\}$ 的素理想。

引理 1.5.12. 对于环 $\mathcal{O}_n$ 的素理想 $\mathfrak{p}$，则存在 $\mathbb{C}_n$ 的一组基 $\{f_1, f_2, ..., f_n\}$，（记在该基下的坐标函数为 $w_1, w_2, ..., w_n$）以及存在 $0 \le d \le n$，使得

$$\mathfrak{p}_d := \mathfrak{p} \cap \mathbb{C}\{w_1, w_2, ..., w_d\} = 0$$

并且对任意 $d+1 \le k \le n$，$\mathfrak{p}_k$ 当中存在 *Weierestrass* 多项式

$$P_k(\widetilde{w}_k, w_k) = w_k^{s_k} + \sum_{j=1}^{s_k} a_{jk}(\widetilde{w}_k) w_k^{s_k - j}$$

其中 $\widetilde{w}_k := (w_1, w_2, ..., w_{k-1}) \in \mathbb{C}^{k-1}$.

证明. 对 n 归纳，$n = 1$ 时平凡。

Step1对于 $n \ge 2$，先给定 $\mathbb{C}^n$ 的一组基 $\{e_1, ..., e_n\}$ 并记坐标函数为 $z_1, z_2, ..., z_n$，如果 $\mathfrak{p} = \{0\}$，则仍取这组基，并取 $d = n$ 即可。若 $\mathfrak{p} \neq 0$，则任取 $0 \neq g_n \in \mathfrak{p}$，注意 $g_n(0) = 0$；取 $\mathbb{C}^n$ 中的非零向量 f_n，使得定义在 $0 \in \mathbb{C}$ 附近的函数

$$t \mapsto g_n(tf_n)$$

在 $t = 0$ 处的零点阶数最低，记为 s_n. 注意满足如此性质的向量 f_n 在 $\mathbb{C}^n$ 中是稠密的（只需要使得 g_n 沿 f_n 方向的 s_n 阶方向导数非零），从而不妨取 f_n 充分接近基向量 e_n，使得 $\{e_1, e_2, ..., e_{n-1}; f_n\}$ 仍是 $\mathbb{C}^n$ 的一组基。

Step2现在考虑基 $\{e_1, e_2, ..., e_{n-1}; f_n\}$，该基下的坐标记为 $z'_1, z'_2, ..., z'_n$，则由 Weierstrass 预备定理，注意 $z'_n = 0$ 是函数 $z'_n \mapsto g_n(0, z'_n)$ 的 s_n 阶零点，则由 Weierstrass 预备定理，存在 Weierstrass 多项式

$$P_n(\widetilde{z}'_n, z'_n) = (z'_n)^{s_n} + \sum_{j=1}^{s_n} a_{jn}(\widetilde{z}'_n)(z'_n)^{s_k - j}$$

以及 $h \in \mathcal{O}_n$ 使得 $h(0) \neq 0$，以及 $g_n = P_n h$. （其中 $\widetilde{z}'_n = (z'_1, ..., z'_{n-1}) \in \mathbb{C}^{n-1}$）由于 h 在 $\mathcal{O}_n$ 中可逆，所以 Weierstrass 多项式 $P_n \in \mathfrak{p} = \mathfrak{p}_n$.

Step3如果 $\mathfrak{p}_{n-1} := \mathfrak{p} \cap \mathbb{C}\{z'_1, z'_2, ..., z'_{n-1}\} = 0$，则取 $\mathbb{C}^n$ 的基 $\{e_1, ..., e_{n-1}; f_n\}$，以及 $d = n - 1$ 即可。如果 $\mathfrak{p}_{n-1} \neq 0$，则 $\mathfrak{p}_{n-1}$ 为子环 $\mathcal{O}_{n-1} \cong \mathbb{C}\{z'_1, ..., z'_{n-1}\}$ 的素理想，之后对 $\mathbb{C}^{n-1} \cong \mathrm{span}_{\mathbb{C}}\{e_1, e_2, ..., e_{n-1}\}$ 以及 $\mathfrak{p}_{n-1}$ 使用归纳假设即可。 □

注记 1.5.13. 容易知道，对事先任意给定的 $\mathbb{C}^n$ 的基 $\{e_1, e_2, ..., e_n\}$，上述引理中的基 $\{f_1, f_2, ..., f_n\}$ 可以适当选取使得与 $\{e_1, e_2, ..., e_n\}$ 任意接近。

（这个引理证明过程中，哪里利用了“素理想”？）

本节有坑待填，尚未完成。笔者打算完整证明如下：

定理 1.5.14. （局部解析零点定理）

设 I 为 $\mathcal{O}_n$ 的理想，则

$$J_{(V(J),x)} = \sqrt{J}$$

回顾 $\sqrt{J} := \left\{ f \in \mathcal{O}_n \middle| \exists N \geq 0, f^n \in J \right\}$ 为 J 的**根式理想**。交换代数当中有以下基本结果：

$$\sqrt{J} = \bigcap_{\substack{\mathfrak{p} \supseteq J \\ \mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_n)}} \mathfrak{p}$$

证明大意. $J_{(V(J),x)} \supseteq \sqrt{J}$ 是容易验证的，而另一边“$\subseteq$”，由交换代数，只需对 $J = \mathfrak{p}$ 为素理想的情形证明。

这是非常不显然的结果，需要利用引理1.5.12 等多复变函数的结果，以及较多的交换代数。从略。 □

（这里待完善）

1.6 局部参数化

本节陈述关于不可约解析集芽的如下重要定理

定理 1.6.1. （不可约解析集芽的局部参数化定理）

设 $\mathfrak{p}$ 为环 $\mathcal{O}_n$ 的素理想，任取解析集 A 为解析集芽 $(V(\mathfrak{p}),0)$ 的代表元，则：存在 $\mathbb{C}^n$ 的基 $\{e_1, e_2, \ldots, e_n\}$（该基下的坐标函数记为 $z_1, z_2, \ldots, z_n$），存在 $1 \leq d \leq n$，以及存在足够小的正实数 $r', r'' > 0$，以及常数 $C > 0$，使得：

（*1*）$\mathfrak{p} \cap \mathbb{C}\{z_1, \ldots, z_d\} = 0$，并且环同态

$$\mathbb{C}\{z_1, \ldots, z_d\} \hookrightarrow \mathcal{O}_n / \mathfrak{p}$$

为**有限整扩张**。

（*2*） 在坐标 $z' = (z_1, \ldots, z_d), z'' = (z_{d+1}, \ldots, z_n)$ 下，

$$A \cap (\triangle' \times \triangle'') \subseteq \left\{ (z', z'') \in \mathbb{C}^d \times \mathbb{C}^{n-d} \middle| |z''| \leq C|z'| \right\}$$

其中 $\triangle'$ 为 $\mathbb{C}^d$ 中以原点为中心，半径 r' 的多圆柱；$\triangle''$ 为 $\mathbb{C}^{n-d}$ 中以原点为中心，半径 r'' 的多圆柱。

（3） 记 q 为 $\mathbb{C}\{z_1,...,z_d\} \hookrightarrow \mathcal{O}_n/\mathfrak{p}$ 的扩张次数，则投影映射

$$\begin{aligned} \pi : A \cap (\triangle' \times \triangle'') &\to \triangle' \\ (z', z'') &\mapsto z' \end{aligned}$$

为次数为 q 的**分歧映射**（*ramified map*），并且存在某个 $\delta \in \mathcal{O}_d$，使得 π 的所有**分歧值**都位于集合

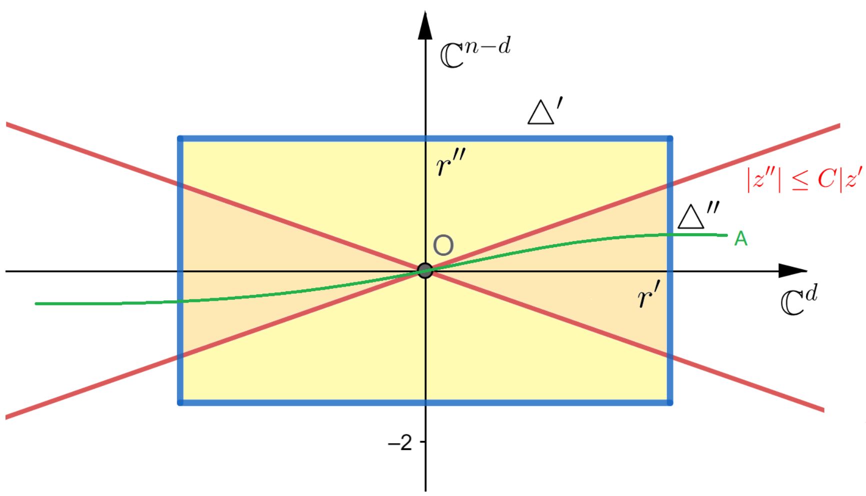
$$S := \left\{ z' \in \triangle' \middle| \delta(z') = 0 \right\}$$

之中，并且 $\triangle' \setminus S$ 为 $\triangle'$ 的连通、稠密子集。

第（3）条的"分歧映射"、"分歧值"具体指：投影

$$\begin{aligned} \pi' : A \cap \left[(\triangle' \setminus S) \times \triangle'' \right] &\to \triangle' \\ (z', z'') &\mapsto z' \end{aligned}$$

为 q 叶覆盖映射，并且对任意 $z' \in S$，$\#\pi^{-1}(z') \leq q$.



图：性质1.6.1示意

证明. 异常复杂，从略。承认之。 □

不过我们可以考虑一种简单的特殊情形——$\mathfrak{p}$ 为主理想：

例子 1.6.2. （超曲面的参数化）

设 $\mathcal{O}_n$ 的素理想 $\mathfrak{p} = (f)$ 为主理想，证明此种情形的局部参数化定理。

证明. 由 Weierstrass 预备定理，不妨取 $\mathfrak{p}$ 的生成元 f 为 weierstrass 多项式

$$f(\tilde{z}, z_n) = z_n^q + \sum_{j=1}^{q} a_j(\tilde{z}) z_n^{s-j} = \prod_{j=1}^{q} (z_n - w_j(\tilde{z}))$$

其中 $\tilde{z} = (z_1, z_2, ..., z_{n-1}) \in \mathbb{C}^{n-1}$，$w_j(\tilde{z})$ 为多项式 $z_n \mapsto f(\tilde{z}, z_n)$ 的根。取 $d = n-1$，显然

$$\mathfrak{p} \cap \mathbb{C}\{z_1, z_2, ..., z_d\} = 0$$

现在对任意 $F \in \mathcal{O}_n$，对 F 以及 Weierstrass 多项式 f 使用 Weierstrass 除法定理，有 $F = hf + R$，其中 $R \in \mathcal{O}_{n-1}[z_n]$ 并且次数 $< q$. 这表明 $\widetilde{F} \in \mathcal{O}_n/\mathfrak{p}$ 为有限生成 $\mathcal{O}_d = \mathcal{O}_{n-1}$-模，并且 $\{1, z_n, z_n^2, ..., z_n^{q-1}\}$ 为其一组 $\mathcal{O}_d$-模生成元。因此

$$\mathcal{O}_d \hookrightarrow \mathcal{O}_n/\mathfrak{p}$$

为有限整扩张。从而定理1.6.1的（1）证毕。

而（3）几乎显然，取

$$S := \left\{ \tilde{z} \in \triangle' \middle| \text{多项式 } z_n \mapsto f(\tilde{z}, z_n) \text{ 无重根} \right\}$$

即可。利用代数学中关于重根的判别式，容易知道 S 为某个 $\mathcal{O}_d$ 中的函数（芽）的零点集。从而（3）易证。

至于（2），常数 C 的存在性显然吗？如果有对 f 的根的估计

$$w_j(\tilde{z}) = O(|\tilde{z}|)$$

那么就没问题。（待补） □

1.7 正则点、奇异点，全纯隐函数定理

（待补）

第 2 章　复流形（待补）

计划详细介绍复流形、复微分形式，以及复流形的例子。

2.1　复流形与全纯向量丛（暂定）

2.2　微分形式（暂定）

2.3　例子（暂定）

第 3 章　层与层上同调

本章介绍层论、层上同调的语言。这套理论是 J-Leray 于 1945-1946 年在监狱中创立的。在正式介绍这套抽象的理论之前，先通过一个例子来大致了解引入此理论的动机。

问题：设 S 为一个黎曼曲面，$\{p_n\} \subseteq S$ 为 S 的一个离散点集，我们希望找一个 S 上的亚纯函数 f，使得 f 在 $S\setminus\{p_n\}$ 全纯，并且在每个 p_i 处具有事先给定的主部。

这样的函数 f 在局部上的存在性是显然的；而在 S 上的整体存在性并不平凡。

思路 (Cěch). 取 S 的一族开覆盖 $\mathcal{U} := \left\{U_\alpha \middle| \alpha \in \mathcal{I}\right\}$，使得每个 U_α 均为局部坐标卡，并且至多包含 $\{p_n\}$ 中的一个点，则局部地，可在每个 U_α 上找到满足要求的亚纯函数 f_α.

之后我们希望找到 $g_\alpha \in \mathcal{O}(U_\alpha)$，使得对任意 $\alpha, \beta \in \mathcal{I}$，在 $U_\alpha \cap U_\beta$ 上成立 $f_\alpha - g_\alpha = f_\beta - g_\beta$. 于是我们可定义 S 上的亚纯函数 $f = f_\alpha - g_\alpha$. 易知 f 良定，且满足要求。

令 $f_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ 为

$$f_{\alpha\beta} := f_\alpha - f_\beta$$

则显然对于任意指标 α, β, γ，在公共部分 $U_\alpha \cap U_\beta \cap U_\gamma$ 上成立

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0 \tag{$*$}$$

而如果存在上述 $g_\alpha \in U_\alpha$，则有 $f_\alpha = g_\alpha - g_\beta$. 现在，令

$$\begin{aligned}
Z^1(\mathcal{U}, \mathcal{O}) &:= \operatorname{span}\left\{f_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta) \middle| f_{\alpha\beta}\text{满足 } (*)\right\} \\
B^1(\mathcal{U}, \mathcal{O}) &:= \operatorname{span}\left\{f_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta) \middle| \exists g_\alpha \in \mathcal{O}(U_\alpha), f_{\alpha\beta} = g_\alpha - g_\beta\right\}
\end{aligned}$$

显然 $B^1(\mathcal{U}, \mathcal{O})$ 为 $Z^1(\mathcal{U}, \mathcal{O})$ 的子空间。如果这两者相等，则满足题设的解存在。 □

我们记 $H^1(\mathcal{U}, \mathcal{O}) := \frac{Z^1(\mathcal{U}, \mathcal{O})}{B^1(\mathcal{U}, \mathcal{O})}$ 为 X 上的全纯函数“层”（sheaf）关于开覆盖 $\mathcal{U}$ 的第 1 个 **Cěch 上同调**. 我们将了解到，Cěch 上同调与 S 的拓扑有密切关系。

本章需要一定的范畴论准备。由于这不是专门介绍层论的讲义，我们会省略很多论证细节，只介绍主要结果。

3.1 预层与层的概念

定义 3.1.1. （集值预层）

设 X 为拓扑空间，X 上的**预层**（*presheaf*）$\mathcal{F}$ 是指以下资料：

（*1*）对任意 X 中的开集 U，给定集合 $\mathcal{F}(U)$，称 $\mathcal{F}(U)$ 为 $\mathcal{F}$ 在 U 上的**截面空间**，其中的元素称为 $\mathcal{F}$ 在 $\mathcal{U}$ 上的一个**截面** *(section)*.

（*2*）对于 X 的任意开子集 U, V，若 $U \subseteq V$，则配以**限制映射**

$$\begin{aligned} \rho_{UV} : \mathcal{F}(V) &\to \mathcal{F}(U) \\ s &\mapsto s|_U \end{aligned}$$

并且对 X 的任意开子集 $W \subseteq U \subseteq V$ 成立：

$$\begin{aligned} \rho_{UU} &= \mathrm{id}_{\mathcal{F}(U)} \\ \rho_{WV} &= \rho_{WU} \circ \rho_{UV} \end{aligned}$$

最典型的例子是，拓扑空间 X 上的函数之全体函数构成预层 $\mathcal{C}$. 具体地，对 X 的开子集 U，$\mathcal{C}(U) := C(U)$ 为定义在 U 上的连续函数之全体；对于 $V \subseteq U$，则限制映射 ρ_{UV} 为通常的函数定义域的限制。

注记 3.1.2. 通常来说，预层 $\mathcal{F}$ 被假定具有代数结构。具体地，对于 X 的开集 U，$\mathcal{F}(U)$ 被假定具有 *Abel* 群结构、交换环结构或者 A-模结构等等，此时分别称作取值于 *Abel* 群范畴、交换环范畴、A-模范畴的预层。

当然，若 $\mathcal{F}(U)$ 具有上述代数结构，则我们也要求限制映射 ρ_{VU} 为相应范畴中的态射，并且规定 $\mathcal{F}(\varnothing) = \{0\}$ 为相应范畴中的零对象。

例子 3.1.3. （常值预层）

对于拓扑空间 X，定义 X 上的集值预层 $\mathbb{C}_X$ 如下：对于任意开子集 U，$\mathbb{C}_X(U) := \mathbb{C}$；对于 $U \subseteq V$，限制映射 $\rho_{UV} := \begin{cases} \mathrm{id}_{\mathbb{C}} & U \neq \varnothing \\ 0 & U = \varnothing \end{cases}$，则容易验证这是 X 上的预层，称为**常值预层**.

例子 3.1.4. （全纯函数预层）

设 X 为复流形，则 $\mathcal{O}_X : U \mapsto \mathcal{O}(U)$，配以通常的函数限制，构成 X 上的预层，称为**全纯函数预层**。

例子 3.1.5. （微分形式预层）

设 X 为光滑流形，对 X 的任意开子集 U，考虑 U 上的光滑 k 形式之全体 $\bigwedge^k(U)$，配以通常的限制映射，则 $\bigwedge^k$ 构成预层，称为 **光滑 k-形式预层**。

定义 3.1.6. （层）

设 $\mathcal{F}$ 为拓扑空间 X 上的预层，称 $\mathcal{F}$ 为层（*sheaf*），若以下成立：

（*1*）（粘合公理）若 U 与 $U_\alpha(\alpha \in \mathcal{I})$ 均为 X 的开子集，并且 $U = \bigcup\limits_{\alpha\in\mathcal{I}} U_\alpha$，则对于任何 $s_\alpha \in \mathcal{F}(U_\alpha)$，如果 $s_\alpha|_{U_\alpha\cap U_\beta} = s_\beta|_{U_\alpha\cap U_\beta}$ 对任意 $\alpha,\beta \in \mathcal{I}$ 成立，则存在 $s \in \mathcal{F}(U)$，使得 $s|_{U_\alpha} = s_\alpha$ 对任意 $\alpha \in \mathcal{I}$ 成立。

（*2*）（唯一性公理）条件同上，则对于任意 $s,t \in \mathcal{F}(U)$，若对任意 $\alpha \in \mathcal{I}$，$s|_{U_\alpha} = t|_{U_\alpha}$，则 $s = t$.

类似地也可以定义取值于 Abel 范畴上的层。此时，容易验证唯一性公理等价于：（$U = \bigcup\limits_{\alpha\in\mathcal{I}} U_\alpha$）对于 $s \in \mathcal{F}(U)$，若 $s|_{U_\alpha} = 0$ 对任意 $\alpha \in \mathcal{I}$ 成立，则 $s = 0$.

例子 3.1.7. 若拓扑空间 X 包含至少两个不交的开集，则常值预层（例子*3.1.3*）$\mathbb{C}_X$ **不是**层，因为不满足粘合公理。

具体地，若 U,V 为 X 的两个不交的开子集，考虑 $1 \in \mathbb{C}_X(U)$ 以及 $2 \in \mathbb{C}_X(V)$，则显然不存在 $z \in \mathbb{C}_X(U \cup V)$ 使得 $1 = z|_U$ 以及 $2 = z|_V$.

例子 3.1.8. （向量丛是层）设 $E \to X$ 为光滑流形 X 上的向量丛，则 E 自然视为 X 上的层 $\Gamma(-,E)$：对任意 $U \subseteq X$，考虑丛 E 在 U 上的截面之全体 $\Gamma(U,E)$。易验证其满足层的公理。

类似地，复流形上的全纯函数预层是层，光滑 k-形式预层也是层。

定义 3.1.9. （预层的同态）

设 $\mathcal{F}$ 与 $\mathcal{G}$ 为拓扑空间 X 上的（取值于同一个 *Abel* 范畴的）预层，预层同态 $\varphi : \mathcal{F} \to \mathcal{G}$ 是指以下资料：对任意开集 $U \subseteq X$，配以（相应 *Abel* 范畴中的）态射 $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$，并且对于 X 的任意开子集 $U \subseteq V$，以下图表交换：

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[d, "\varphi_U"'] & \mathcal{F}(V) \arrow[l, "\rho_{UV}"'] \arrow[d, "\varphi_V"'] \\
\mathcal{G}(U) & \mathcal{G}(V) \arrow[l, "\rho_{UV}"']
\end{tikzcd}$$

设 $\varphi : \mathcal{F} \to \mathcal{G}$ 为 X 上的预层同态，则我们可以定义 $\ker^p \varphi$,$\mathrm{Im}^p \varphi$,$\mathrm{coker}^p \varphi$ 为：对任意开集 $U \subseteq X$，

$$(\ker^p \varphi)(U) := \ker(\varphi_U)$$

$\mathrm{Im}^p \varphi$ 与 $\mathrm{coker}^p \varphi$ 也完全类似。容易验证它们都是预层，分别称为预层同态 φ 的**核预层**、**像预层**、**余核预层**。这里的上标“p”是指“预层”（presheaf）。

性质 3.1.10. 设 $\mathcal{F},\mathcal{G}$ 为 X 上的层，$\varphi:\mathcal{F}\to\mathcal{G}$ 为预层同态，则预层 $\ker^p\varphi$ 是层。

证明. 直接验证 $\ker^p\varphi$ 满足层的粘合公理和唯一性公理。设 $\left\{U_\alpha \middle| \alpha\in\mathcal{I}\right\}$ 为 X 的开子集 U 的一族开覆盖，注意到 $(\ker^p\varphi)(U_\alpha)\subseteq\mathcal{F}(U_\alpha)$，以及 $\mathcal{F}$ 为层（满足粘合公理），因此易知 $\ker^p\varphi$ 也满足粘合公理。$\ker^p\varphi$ 的唯一性公理也是由 $\mathcal{F}$ 的层性质直接得到的。 □

从此以后，若 $\mathcal{F}$ 与 $\mathcal{G}$ 都为层，则我们将核预层 $\ker^p\varphi$ 简记为 $\ker\varphi$.

注记 3.1.11. 好吧，刚才的命题几乎显然。但是要注意，即使 $\mathcal{F}$ 与 $\mathcal{G}$ 都是层，$\operatorname{Im}^p\varphi$ 与 $\operatorname{coker}^p\varphi$ 未必是层。它们并没有 $\ker^p\varphi$ 的良好性质。

例子 3.1.12. 考虑拓扑空间 $X=\mathbb{C}\setminus\{0\}$，令 $\mathcal{F}:=\mathcal{O}_X$ 为 X 上的全纯函数层，$\mathcal{G}:=\mathcal{O}_X^*$ 定义为：对于 X 的开集 U，

$$\mathcal{O}_X^*(U):=\left\{f\in\mathcal{O}_X(U)\middle| f(z)\neq 0,\forall z\in U\right\}$$

容易验证 $\mathcal{O}_X^*$ 为（取值于集合的）层。考虑层同态

$$\begin{aligned}\exp:\mathcal{F} &\to \mathcal{G}\\ f\in\mathcal{F}(U) &\mapsto e^f\end{aligned}$$

则 $\operatorname{Im}^p\exp$ **不是**层。

证明. 只需要考虑函数 $z\in\mathcal{O}_X^*(X)$. 对任意单连通的开子集 $U\subseteq X$，易知 $z\in\mathcal{O}_X^*(U)$ 满足 $z\in(\operatorname{Im}^p\exp)(U)$，但是 $z\in\mathcal{O}_X^*(X)$ 并不位于 $(\operatorname{Im}^p\exp)(X)$ 当中，从而 $\operatorname{Im}^p\exp$ 不满足粘合公理。 □

记号 3.1.13. （层的限制）设 $\mathcal{F}$ 是拓扑空间 X 上的层，U 为 X 的开子集，则自然有拓扑空间 U 上的层 $\mathcal{F}|_U$ 如下：对 U 中的开集 V （注意 V 也是 X 中的开集），定义

$$\mathcal{F}|_U(V):=\mathcal{F}(V)$$

相应的限制映射也自然给出。容易验证 $\mathcal{F}|_U$ 是拓扑空间 U 上的层，称为 $\mathcal{F}$ 在 U 上的**限制**。

关于层的构造，我们再介绍层的直和：

例子 3.1.14. （层的直和）

设 $\mathcal{F}$ 与 $\mathcal{G}$ 为拓扑空间 X 上的取值于（同一个）*Abel* 范畴的层，则定义 $\mathcal{F}$ 与 $\mathcal{G}$ 的**直和层**$\mathcal{F}\oplus\mathcal{G}$ 如下：对 X 中的开集 U, $(\mathcal{F}\oplus\mathcal{G})(U):=\mathcal{F}(U)\oplus\mathcal{G}U.$

容易验证 $\mathcal{F}\oplus\mathcal{G}$ 也为 X 上的层。类似也可以定义多个层的直和。特别地，对于层 $\mathcal{F}$ 以及正整数 n，记 $\mathcal{F}^{\oplus n}:=\underbrace{\mathcal{F}\oplus\mathcal{F}\oplus\cdots\oplus\mathcal{F}}_{n\text{个}}$

3.2 预层的层化

定义 3.2.1. *(预层的芽)*

设 $\mathcal{F}$ 为 X 上的预层，$x \in X$，则称

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

为 $\mathcal{F}$ 在 x 处的**茎条**（*stalk*），其中 U 取遍 x 的开邻域。$\mathcal{F}_X$ 中的元素称为 x 处的**芽**（*germ*）。

我们不再回顾范畴论中的余极限（or 归纳极限、正向极限）的概念。典型的例子是，若 $\mathcal{O}_X$ 为复流形 X 上的解析函数环层，则对于 $x \in X$，$\mathcal{O}_{X,x}$ 即为通常的在 x 处的解析函数芽环。

回顾层的粘合公理、唯一性公理，用茎条、芽的语言可以给出上述公理的等价表述：

性质 3.2.2. 设 $\mathcal{F}$ 是拓扑空间 X 上的预层，则

（1）$\mathcal{F}$ 满足粘合公理 $\iff$ 对任意开集 U，以及对任意 $s(x) \in \mathcal{F}_x(\forall x \in U)$，如果对任意 $x \in U$，存在 x 的开邻域 $V \subseteq U$，以及 $s(x)$ 的代表元 $t \in \mathcal{F}(V)$，使得对任意 $y \in V$，成立 $s(y) = t_y$，那么存在 $S \in \mathcal{F}(U)$，使得对任意 $x \in U$ 成立 $S_x = s(x)$。

（2）$\mathcal{F}$ 满足唯一性公理 $\iff$ 对任意开集 U，以及对任意 $s \in \mathcal{F}(U)$，如果对任意 $x \in U$，$s_x = 0$，那么 $s = 0$.

证明. 由有关定义出发，几乎显然。 □

性质 3.2.3. 设 $\mathcal{F}$ 与 $\mathcal{G}$ 为 X 上的预层，$\varphi : \mathcal{F} \to \mathcal{G}$ 为预层同态，则对任意 $x \in X$，φ 自然诱导茎条同态

$$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$$

证明. 由余极限 $\varinjlim$ 的函子性直接得到。 □

具体构造是，对任意 $F_x \in \mathcal{F}_x$，取 F_x 的代表元 $F \in \mathcal{F}(U)$，其中 U 为 x 的某个开邻域。之后，$\varphi_x(F_x) = (\varphi_U(F))_x$.

定义 3.2.4. （预层的层空间）

设 $\mathcal{F}$ 为拓扑空间 X 上的预层，则定义拓扑空间

$$\widetilde{\mathcal{F}} := \coprod_{x \in X} \mathcal{F}_x$$

其拓扑由拓扑基 $\left\{ \Omega_{F,U} \middle| U \subseteq X \text{为开子集}, F \in \mathcal{F}(U) \right\}$ 生成，其中 $\Omega_{F,U} = \left\{ F_x \in \mathcal{F}_x \middle| x \in U \right\}$. 称拓扑空间 $\widetilde{\mathcal{F}}$ 为预层 $\mathcal{F}$ 的**层空间**（*sheaf space*）。

具体地，若芽 $F_x \in \widetilde{\mathcal{F}}$，取 F_x 的代表元 $F \in \mathcal{F}(U)$，其中 U 为 x 的一个（充分小的）开邻域，则 $\left\{ F_y \middle| y \in U \right\}$ 为 F_X 在 $\widetilde{\mathcal{F}}$ 中的一个开邻域。我们由自然的映射

$$\begin{aligned} \Pi : \widetilde{\mathcal{F}} &\to X \\ s \in \mathcal{F}_x &\mapsto x \end{aligned}$$

则容易验证 $\Pi : \widetilde{\mathcal{F}} \to X$ 为连续映射，且对于任意 $F \in \mathcal{F}(U)$，$\Pi : \Omega_{F,U} \to U$ 为拓扑同胚。

定义 3.2.5. （预层的层化）

设 $\mathcal{F}$ 是 X 上的预层，对 X 的开子集 U，定义

$$\mathcal{F}^+(U) := \left\{ s : U \to \widetilde{\mathcal{F}} \middle| s \text{为连续映射，并且} \Pi \circ s = \mathrm{id}_U \right\}$$

称 $\mathcal{F}^+$ 为预层 $\mathcal{F}$ 的**层化**（*sheafification*）.

具体地，对于 $s : U \to \widetilde{\mathcal{F}}$，$s \in \mathcal{F}^+(U)$ 当且仅当对任意的 $x \in U$，$s(x) \in \mathcal{F}_x$，并且存在 x 的开邻域 $V \subseteq U$，以及存在 $F \in \mathcal{F}(V)$，使得 $s(y) = F_y$ 对任意 $y \in V$ 成立。

性质 3.2.6. 设 $\mathcal{F}$ 为 X 上的预层，则 $\mathcal{F}^+$ 为 X 上的层，并且有典范的预层同态 $\theta : \mathcal{F} \to \mathcal{F}^+$ 如下：对任意开集 U，

$$\begin{aligned} \theta_U : \mathcal{F}(U) &\to \mathcal{F}^+(U) \\ s &\mapsto \widetilde{s} : U \to \widetilde{\mathcal{F}} \quad (x \mapsto s_x) \end{aligned}$$

证明. $\mathcal{F}^+$ 的粘合公理与唯一性公理几乎显然成立。 □

我们更习惯于把有预层同态 $\theta : \mathcal{F} \to \mathcal{F}^+$ 称为 $\mathcal{F}$ 的层化。容易验证，对任意 $x \in X$，由茎条同构 $\mathcal{F}_X \cong \mathcal{F}_x^+$；此外也容易验证，如果 $\mathcal{F}$ 本身是层，那么 θ 为层同构，即“层的层化同构于其本身”。

性质 3.2.7. （层化的泛性质）

设 $\mathcal{F}$ 为拓扑空间 X 上的预层，则对于 X 上的任何层 $\mathcal{G}$，以及预层同态 $\varphi : \mathcal{F} \to \mathcal{G}$，存在唯一的层同态 $\psi : \mathcal{F}^+ \to \mathcal{G}$，使得以下图表交换：

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\forall\varphi"] \arrow[d, "\theta"'] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, dashed, "\exists!\,\psi"'] & \end{tikzcd}$$

证明. 对任意 $x \in X$，$\varphi : \mathcal{F} \to \mathcal{G}$ 诱导了 $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$，再注意 $\mathcal{F}_X \cong \mathcal{F}_x^+$，从而自然给出 $\psi_x : \mathcal{F}_x^+ \to \mathcal{G}_x$. 易验证 $\left\{\psi_x \middle| x \in X\right\}$ 确定了层同态 $\psi : \mathcal{F}^+ \to \mathcal{G}$，且 $\psi \circ \theta = \varphi$.

ψ 的唯一性是显然的。 □

例子 3.2.8. 回顾常值预层 $\mathbb{C}_X$ （见例子*3.1.3*），则其层化 $\mathbb{C}_X^+$ 为，对任意开集 U，

$$\mathbb{C}_X^+(U) = \left\{ f : U \to \mathbb{C} \middle| f \text{为局部常值函数} \right\}$$

称之为 X 上的**局部常值层**。

例子 3.2.9. 回顾例子*3.1.12*中的预层同态

$$\exp : \mathcal{O}_X \to \mathcal{O}_X^*$$

则像预层 $\mathrm{Im}^p(\exp)$ 的层化 $(\mathrm{Im}^p \exp)^+ \cong \mathcal{O}_X^*$.

定义 3.2.10. （像层、余核层与商层）

设 $\mathcal{F}$ 与 $\mathcal{G}$ 为拓扑空间 X 上的层，$\varphi : \mathcal{F} \to \mathcal{G}$ 为层同态。

（*1*）定义 $\mathrm{Im}\,\varphi := (\mathrm{Im}^p\,\varphi)^+$，称之为 φ 的**像层**；

（*2*）定义 $\mathrm{coker}\,\varphi := (\mathrm{coker}^p\,\varphi)^+$，称之为 φ 的**余核层**；

（*3*）若对于任意开集 U，$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ 为单同态，则称 φ 为**层单同态**，此时也称 $\mathcal{F}$ 为 $\mathcal{G}$ 的**子层**，并且定义**商层**$\mathcal{F}/\mathcal{G} := \mathrm{coker}\,\varphi$.

无非是将相应的预层加以层化。此外容易验证，层同态 $\varphi : \mathcal{F} \to \mathcal{G}$ 为单同态，当且仅当对任意 $x \in X$，$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ 为单同态。

注记 3.2.11. 设 $\varphi : \mathcal{F} \to \mathcal{G}$ 为层同态，则像层 $\operatorname{Im}\varphi$ 自然地视为 $\mathcal{G}$ 的子层：

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\widetilde{\varphi}"'] & \mathcal{G} \\
\operatorname{Im}^p \varphi \arrow[r, "\theta"] \arrow[ur, "i'"] & \operatorname{Im}\varphi \arrow[u, dashed, "i"']
\end{tikzcd}$$

层同态 $i : \operatorname{Im}\phi \to \mathcal{G}$ 由层化的泛性质给出，并且逐茎条看，显然 i 为层单同态。

定义 3.2.12. （层满同态）

设 $\varphi : \mathcal{F} \to \mathcal{G}$ 为层同态，称 φ 为**层满同态**，若 $\operatorname{Im}\varphi := (\operatorname{Im}^p \varphi)^+ \cong \mathcal{G}$.

由有关定义可以验证，层同态 $\varphi : \mathcal{F} \to \mathcal{G}$ 为层满同态，当且仅当对任意 $x \in X$，$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ 为满同态。由此可推出，φ 为层同构，当且仅当对任意 $x \in X$，φ_x 为茎条同构。

3.3 层的顺像与逆像

记号 3.3.1. 对于拓扑空间 X，定义 X 上的 *Abel* 群层范畴 Ab_X 为：

（*1*）$\mathsf{Ab}(X)$ 中的对象为 X 上的取值于 *Abel* 群的层；

（*2*）对象之间的态射为相应的层同态。

显然这是一个范畴。类似可定义“X 上的集值层范畴”Set_X，“X 上的交换环层范畴”Ring_X，以及对于交换环 A，我们可定义 X 上的 A-模层范畴 A-Mod_X 等等。

一般地，将 X 上（所有种类的）层之全体记作 Sh_X，这自然也给出一个范畴，称为 X 上的**层范畴**。类似地，X 上的所有预层也构成范畴，记为 pSh_X.

定义 3.3.2. （层的顺像）

设 $f : X \to Y$ 为拓扑空间的连续映射，$\mathcal{F}$ 是 X 上的层，则定义 $\mathcal{F}$ 的**推出** (*push-forward*)，也称为**顺像**（*direct image*）$f_*\mathcal{F}$ 为：对 Y 的开子集 U，$(f_*\mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$.

显然 $f_*\mathcal{F}$ 为 Y 上的预层。容易验证，若 $\mathcal{F}$ 是层，则预层 $f_*\mathcal{F}$ 也是层。事实上，顺像 f_* 具有函子性，具体地说，若 $\varphi : \mathcal{F} \to \mathcal{G}$ 为 X 上的层同态，则 f 诱导了 Y 上的层同态 $f_*\varphi : f_*\mathcal{F} \to f_*\mathcal{G}$，并且使得有关图表交换。换句话说，我们有函子 $f_* : \mathsf{Sh}_X \to \mathsf{Sh}_Y$.

容易验证，$f_*\mathcal{F}$ 在 $y\in Y$ 处的茎条为

$$(f_*\mathcal{F})_y \cong \varinjlim_{y\in V} \mathcal{F}(f^{-1}(V))$$

定义 3.3.3. （层的逆像）

设 $f: X\to Y$ 为拓扑空间之间的连续映射，$\mathcal{G}$ 为 Y 上的层，则定义 X 上的层 $f^{-1}\mathcal{G}$ 为：对 X 的任意开集 U，

$$(f^{-1}\mathcal{G})(U) := \varinjlim_{V\in f(U)} \mathcal{G}(V)$$

其中 V 取遍 Y 中的包含 $f(U)$ 的开子集。称 $f^{-1}\mathcal{G}$ 为 $\mathcal{G}$ 关于 f 的**逆像**（*inverse image*）

显然如此定义的 $f^{-1}\mathcal{G}$ 为 X 上的预层。利用余极限的泛性质，也能验证当 $\mathcal{G}$ 为层时，$f^{-1}\mathcal{G}$ 也为层。容易验证对 Y 中的开集 V，成立

$$(f^{-1}\mathcal{G})(f^{-1}(V)) \cong \mathcal{G}(V)$$

此外对任意 $x\in X$，成立

$$(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)} \tag{$*$}$$

容易验证 $f^{-1}: \mathsf{Sh}_Y \to \mathsf{Sh}_X$ 为层范畴之间的函子。

注记 3.3.4. （逆像的层空间）

设 $f: X\to Y$ 为拓扑空间之间的连续映射，$\mathcal{G}$ 为 Y 上的层，则有层空间的拓扑同胚

$$\widetilde{f^{-1}\mathcal{G}} \cong \mathrm{X}\times_Y \widetilde{\mathcal{G}}$$

也就是说，存在下述纤维积图表：

$$\begin{tikzcd} \widetilde{f^{-1}\mathcal{G}} \arrow[r, "\alpha"] \arrow[d] & \widetilde{\mathcal{G}} \arrow[d] \\ \mathrm{X} \arrow[r, "f"] & Y \end{tikzcd}$$

其中映射 α 由 $(*)$ 式诱导。由拓扑空间纤维积的具体构造，容易验证以上。

性质 3.3.5. （伴随对）

设 $f: X\to Y$ 为拓扑空间之间的连续映射，则 f^{-1} 为 f_* 的左伴随函子。也就是说对于任意 $\mathcal{F}\in\mathsf{Sh}_X$ 以及 $\mathcal{G}\in\mathsf{Sh}_Y$，存在（关于 X,Y）自然的一一对应

$$\mathrm{Hom}_{\mathsf{Sh}_X}(f^{-1}\mathcal{G},\mathcal{F}) \xlongequal{\text{1-1}} \mathrm{Hom}_{\mathsf{Sh}_Y}(\mathcal{G},f_*\mathcal{F})$$

证明大意. 我们只给出此一一对应的构造，其余细节从略（反复使用各种泛性质）。对于任意的

$$\begin{aligned} \psi: &\quad \mathcal{G} \to f_*\mathcal{F} \\ \varphi: &\quad f^{-1}\mathcal{G} \to \mathcal{F} \end{aligned}$$

首先我们定义 $\alpha : \mathrm{Hom}_{\mathsf{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}_{\mathsf{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F})$ 如下：对 X 中开集 U，$[\alpha(\psi)]_U$ 由以下交换图表给出：

$$\begin{tikzcd}
\mathcal{G}(W) \arrow[r, "\psi_W"] \arrow[d, "\rho"] & (f_*\mathcal{F})(W) = \mathcal{F}(f^{-1}(W)) \arrow[d, "\rho"] \\
\mathcal{G}(V) \arrow[r, "\psi_V"] \arrow[d] & (f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)) \arrow[d, "\rho"] \\
\varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = (f^{-1}\mathcal{G})(U) \arrow[r, dashed, "{[\alpha(\psi)]_U}"] & \mathcal{F}(U)
\end{tikzcd}$$

其中 $W \supseteq V$ 为 Y 中的包含 $f(U)$ 的开集。

再定义 $\beta : \mathrm{Hom}_{\mathsf{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_{\mathsf{Sh}_Y}(\mathcal{G}, f_*\mathcal{F})$ 如下: 对 Y 中的开集 V，$[\beta(\varphi)]_V$ 由以下交换图表给出：

$$\begin{tikzcd}
(f^{-1}\mathcal{G})(f^{-1}(V)) \arrow[r, "\varphi_{f^{-1}(V)}"] \arrow[d, equal] & \mathcal{F}(f^{-1}(V)) \arrow[d, equal] \\
\mathcal{G}(V) \arrow[r, dashed, "{[\beta(\varphi)]_V}"] & (f_*\mathcal{F})(V)
\end{tikzcd}$$

其余细节从略。 □

3.4 局部自由模层与向量丛

定义 3.4.1. （$\mathcal{A}$-模层）

设 $\mathcal{A}$ 为拓扑空间 X 上的（含幺交换）环层，$\mathcal{M}$ 为 X 上的 *Abel* 群层，称 $\mathcal{M}$ 为 $\mathcal{A}$-模层，如果对 X 的任何开集 $V \supseteq U$，$\mathcal{M}(U)$ 具有 $\mathcal{A}(U)$-模结构 $\mathcal{A}(U) \times \mathcal{M}(U) \to \mathcal{M}(U)$，并且下述图表交换：

$$\begin{tikzcd}
\mathcal{A}(V) \times \mathcal{M}(V) \arrow[r] \arrow[d] & \mathcal{M}(V) \arrow[d] \\
\mathcal{A}(U) \times \mathcal{M}(U) \arrow[r] & \mathcal{M}(U)
\end{tikzcd}$$

例如，考虑复流形 X 上的解析函数环层 $\mathcal{O}_X$，则全纯切向量场、全纯微分形式等等，都可视为 $\mathcal{O}_X$-模层。再比如，环层 $\mathcal{A}$ 也有自然的 $\mathcal{A}$-模层结构。一般地，对于拓扑空间 X 上的环层 $\mathcal{A}$，我们有 X 上的 $\mathcal{A}$-模层范畴 $\mathcal{A}\text{-}\mathsf{Mod}_X$，自行定义此范畴中的态射“$\mathcal{A}$-模层同态”。能够验证，$\mathcal{A}\text{-}\mathsf{Mod}_X$ 为 Abel 范畴。

容易验证，对于 $\mathcal{A}$-模层 $\mathcal{M}$，则对任意 $x\in X$，茎条 $\mathcal{M}_x$ 有自然的 $\mathcal{A}_x$-模结构。

定义 3.4.2. （局部自由层）

设 $\mathcal{S}$ 为拓扑空间 X 上的 $\mathcal{A}$-模层，称 $\mathcal{S}$ 为**局部自由 $\mathcal{A}$-模层**，简称**局部自由层**（*locally free sheaf*），如果对任意 $x\in X$，存在 x 的开邻域 U，使得有层同构

$$\mathcal{S}|_U \cong (\mathcal{A}|_U)^{\oplus r}$$

其中 r 为正整数，称为局部自由层 $\mathcal{S}$ 的秩。

特别地，对任意 $x\in X$，存在 x 的开邻域 U，使得 $\mathcal{S}(U)\cong(\mathcal{A}(U))^{\oplus r}$ （但是定义中的“层限制”的语言更强）。事实上 $\mathcal{S}$ 为局部自由层当且仅当对任意 $x\in X$，存在 x 的开邻域 U，以及截面 $F_{1,x},F_{2,x},\dots,F_{r,x}\in\mathcal{S}(U)$，使得对任意 $y\in U$，环同态

$$\begin{array}{rcl} \mathcal{A}_y^{\oplus r} & \to & \mathcal{S}_y \\ (w_1,w_2,\dots,w_r) & \mapsto & \displaystyle\sum_{i=1}^{r} w_i F_{i,x} \end{array}$$

为同构。如此选取的 $\left\{F_{i,x}\in\mathcal{A}(U)\middle|1\le i\le r\right\}$ 称为 $\mathcal{S}$ 的一个**局部标架**。

记号 3.4.3. （局部自由层局部标架的转移函数）

设 $\mathcal{S}$ 为拓扑空间 X 上的秩为 r 的局部自由 $\mathcal{A}$-模层。取 X 的一族开覆盖 $X=\bigcup\limits_{\alpha\in\mathcal{I}}U_\alpha$，以及对于任意 $\alpha\in\mathcal{I}$，取 $\mathcal{S}$ 在 U_α 上的局部标架

$$F_\alpha := \left\{F_\alpha^i\in\mathcal{S}(U_\alpha)\middle|1\le i\le r\right\}$$

则 F_α 自然诱导了层同构（仍记作 F_α）

$$F_\alpha : \mathcal{A}|_{U_\alpha}^{\oplus r} \xrightarrow{\sim} \mathcal{S}|_{U_\alpha}$$

对于 $\alpha,\beta\in\mathcal{I}$，若 $U_\alpha\cap U_\beta\neq\varnothing$，则考虑如下图表：

$$\begin{tikzcd} \mathcal{A}|^{\oplus r}_{U_\alpha\cap U_\beta} \arrow[r, "F_\alpha"] & \mathcal{S}|_{U_\alpha\cap U_\beta} \arrow[d, equal] \\ \mathcal{A}|^{\oplus r}_{U_\alpha\cap U_\beta} \arrow[r, "F_\beta"] \arrow[u, dashed, "G_{\alpha\beta}"] & \mathcal{S}|_{U_\alpha\cap U_\beta} \end{tikzcd}$$

称层自同构 $G_{\alpha\beta} := F_\alpha^{-1} \circ F_\beta$ 为局部标架 F_α 与 F_β 之间的转移函数。

对于 $x \in U_\alpha \cap U_\beta$，

$$(G_{\alpha\beta})_x : \mathcal{A}_x^{\oplus r} \to \mathcal{A}_x^{\oplus r}$$

可以表达为在基 $\left\{(F_\beta^i)_x \middle| 1 \le i \le r\right\}$ 与 $\left\{(F_\alpha^i)_x \middle| 1 \le i \le r\right\}$ 下的矩阵，称此矩阵为**转移矩阵**。

对于 $\alpha, \beta, \gamma \in \mathcal{I}$，如果 $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$，则显然有 $\begin{cases} G_{\alpha\alpha} = \mathrm{id}_{\mathcal{A}|_{U_\alpha}^{\oplus r}} \\ G_{\alpha\beta} = G_{\beta\alpha}^{-1} \\ G_{\alpha\beta} \circ G_{\beta\gamma} \circ G_{\gamma\alpha} = \mathrm{id}_{\mathcal{A}|_{U_{\alpha\beta\gamma}}^{\oplus r}} \end{cases}$，其中 $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$.

上述的语言与**向量丛**十分相似，事实上局部自由层是向量丛概念的推广。

重要例子 3.4.4. （拓扑向量丛）

设 X 为拓扑空间，$\mathcal{C}_X$ 为 X 上的连续函数环层，则有自然的一一对应

$$\left\{X \text{ 上的局部自由 } \mathcal{C}_X\text{-模层}\right\} \xlongequal{\text{1-1}} \left\{X \text{ 上的（拓扑）向量丛}\right\}$$

证明. 若 $\mathcal{E}$ 为 X 上的局部自由 $\mathcal{C}_X$-模层，取 X 的一组局部标架覆盖 $X = \bigcup\limits_{\alpha\in\mathcal{I}} U_\alpha$，以及 U_α 上的局部标架 $F_\alpha = \left\{F_\alpha^i \middle| 1 \le i \le r\right\}$，则对于任意的 $\alpha, \beta \in \mathcal{I}$，若 $U_\alpha \cap U_\beta \neq \varnothing$，则对任意 $x \in U_\alpha \cap U_\beta$，转移函数 $(G_{\alpha\beta})_x$ 在相应标架上的矩阵（仍记为 $(G_{\alpha\beta})_x$）给出了映射

$$\begin{array}{rcl} U_\alpha \cap U_\beta & \to & \mathrm{GL}(r, \mathbb{C}) \\ x & \mapsto & (G_{\alpha\beta})_x \end{array}$$

易验证该映射连续，并且满足向量丛转移函数的相容条件，从而这些转移函数可以粘合成一个向量丛。反之，对于拓扑向量丛 $E \to X$，该向量丛的截面层显然为局部自由 $\mathcal{C}_X$-模层。容易验证上述给出的对应是互逆的，从而得到一一对应。 □

例子 3.4.5. （全纯向量丛）

设 X 为复流形，$\mathcal{O}_X$ 为 X 上的全纯函数环层，则类似地有一一对应

$$\left\{X \text{ 上的局部自由 } \mathcal{O}_X\text{-模层}\right\} \xlongequal{\text{1-1}} \left\{X \text{ 上的全纯向量丛}\right\}$$

光滑流形上的光滑向量丛也完全类似。

最后，需要注意局部自由层范畴不是 Abel 范畴：

重要例子 3.4.6. （摩天大厦层）

考虑拓扑空间（复流形）$X=\mathbb{C}$，X 上的局部自由 $\mathcal{O}_X$-模层 $\mathcal{S}_1=\mathcal{S}_2:=\mathcal{O}_X$. 考虑 $\mathcal{O}_X$-模层同态 $\varphi:\mathcal{S}_1\to\mathcal{S}_2$ 为：对任意开集 $U\subseteq X$，

$$\begin{aligned}\varphi_U:\mathcal{S}_1(U) &\to \mathcal{S}_2(U)\\ f(z) &\mapsto zf(z)\end{aligned}$$

则其余核层 $\operatorname{coker}\varphi$ **不是**局部自由 $\mathcal{O}_X$-模层。

容易验证，对 X 中的开集 U，成立 $\operatorname{coker}\varphi(U)\cong\begin{cases}\mathbb{C} & (0\in U)\\ 0 & (0\notin U)\end{cases}$，明显不是局部自由层。此层称为**摩天大厦层**（skyscraper sheaf）。

3.5 凝聚层及其基本性质

定义 3.5.1. （局部有限生成 $\mathcal{A}$-模层）

设 $\mathcal{M}$ 为拓扑空间 X 上的 $\mathcal{A}$-模层，称 $\mathcal{A}$ 是 **局部有限生成**的，若对任意 $x\in X$，存在 x 的邻域 U，以及正整数 r，使得有层同态短正合列

$$\mathcal{A}|_U^{\oplus r}\twoheadrightarrow\mathcal{M}|_U\to 0$$

或者等价地，存在 x 的开邻域 U，以及截面 $F_1,F_2,...,F_r\in\mathcal{M}(U)$，使得对任意 $y\in U$，$\left\{(F_i)_y\in\mathcal{M}_y\middle|1\leq i\leq r\right\}$ 是 $\mathcal{M}_y$ 的一组 $\mathcal{A}_x$-模生成元。

显然，局部自由层一定是局部有限生成的。

定义 3.5.2. （关系层）

设 $\mathcal{M}$ 是拓扑空间 X 上的 $\mathcal{A}$-模层，对于 X 的开集 U，以及 $F_1,F_2,...,F_r\in\mathcal{M}(U)$，称层同态

$$\begin{aligned}\varphi:\mathcal{A}|_U^{\oplus r} &\to \mathcal{M}|_U\\ (g_1,g_2,...,g_r) &\mapsto \sum_{i=1}^r g_iF_i\end{aligned}$$

的核层 $\mathcal{R}(F_1,F_2,...,F_r):=\ker\varphi$ 为截面 $F_1,F_2,...,F_r$ 的**关系层**。

这个定义当中并不要求 φ 为层满同态，也就是说 $\mathcal{M}$ 未必为局部有限生成的。只要给定若干局部截面，就可以定义它们的关系层。

定义 3.5.3. （凝聚层）

对于拓扑空间 X 上的 $\mathcal{A}$-模层 $\mathcal{M}$，称 $\mathcal{A}$ 为 **凝聚层**（*coherent sheaf*），*如果：*

（1）$\mathcal{A}$ 为局部有限生成的；

（2）对 X 的任意开集 U，以及任意截面 $F_1, F_2, ..., F_r \in \mathcal{M}(U)$，关系层 $\mathcal{R}(F_1, F_2, ..., F_r)$ 也是局部有限生成的。

通过适当缩小 $x \in X$ 的邻域 U，容易验证 $\mathcal{M}$ 是凝聚层一定是**局部有限呈示**的，即对任意 $x \in X$，存在 x 的开邻域 U，以及正整数 p, q，使得存在 U 上的 $\mathcal{A}|_U$-模层正合列

$$\mathcal{A}|_U^{\oplus p} \to \mathcal{A}|_U^{\oplus q} \twoheadrightarrow \mathcal{M}|_U \to 0$$

由定义容易知道，**凝聚层的局部有限生成子层也是凝聚的**。

此外，对于 X 上的交换环层 $\mathcal{A}$，称 $\mathcal{A}$ 为局部有限生成的（切转：凝聚的），如果 $\mathcal{A}$ 作为 $\mathcal{A}$-模层是局部有限生成的（切转：凝聚的）。

凝聚层的下列基本性质是纯线性代数的：

性质 3.5.4. （凝聚层的基本性质）

设 $\mathcal{A}$ 为拓扑空间 X 上的交换环层，$\mathcal{F}, \mathcal{G}$ 为凝聚 $\mathcal{A}$-模层，$\varphi : \mathcal{F} \to \mathcal{G}$ 为 $\mathcal{A}$-模层同态，则 $\ker\varphi, \operatorname{Im}\varphi, \operatorname{coker}\varphi$ 均为凝聚 $\mathcal{A}$-模层。

证明. 显然 $\operatorname{Im}\varphi$ 是局部有限生成的，从而为凝聚层 $\mathcal{G}$ 的局部有限生成子层，故也为凝聚层。再看 $\ker\varphi$ 作为凝聚层 $\mathcal{F}$ 的子层，只需要说明 $\ker\varphi$ 是局部有限生成的。对任意 $x \in X$，由于 $\mathcal{F}$ 局部有限生成，取 x 的开邻域 U，以及截面 $F_1, F_2, ..., F_q \in \mathcal{F}(U)$ 为 $\mathcal{F}|_U$ 的生成元，于是有 $\varphi(F_1), \varphi(F_2), ..., \varphi(F_q) \in \mathcal{G}(U)$. 由 $\mathcal{G}$ 的凝聚性，取关系层 $\mathcal{R}(\varphi(F_1), \varphi(F_2), ..., \varphi(F_q))$ 的一组生成元 $G_1, G_2, ..., G_r \in \mathcal{A}(U)^{\oplus q}$，其中 $G_i = (G_i^1, G_i^2, ..., G_i^q)$，即有以 $\mathcal{A}(U)$ 为系数的矩阵 (G_i^j)，其中 $1 \le i \le r, 1 \le j \le q$. 则容易验证 $\left\{ \sum_{j=1}^{q} G_i^j F_j \middle| 1 \le i \le r \right\}$ 是 $\ker\varphi|_U$ 的一组生成元，因此 $\ker\varphi$ 是局部有限生成的，进而由 $\mathcal{F}$ 的凝聚性知 $\ker\varphi$ 也是凝聚的。

再看 $\operatorname{coker}\varphi$ 的凝聚性。$\operatorname{coker}\varphi$ 作为局部有限生成层 $\mathcal{G}$ 的商层，显然也是局部有限生成的。然后对 X 的任意开集 U，以及任意截面 $G_1, G_2, ..., G_q \in \operatorname{coker}\varphi(U)$，断言关系层 $\mathcal{R}(G_1, G_2, ..., G_q) \subseteq \mathcal{A}|_U^{\oplus q}$ 是局部有限生成的。对于任意 $x \in U$，取 x 在 U 中的（足够小）邻域 U'，$G_i (1 \le i \le q)$ 在 U' 上的限制仍记为 G_i. 取截面 $G_i \in \operatorname{coker}\varphi(U')$ 在 $\mathcal{G}$ 中的代表元 $\widetilde{G}_i \in \mathcal{G}(U')$，再取 $F_1, F_2, ..., F_p \in \mathcal{F}(U')$ 为 $\mathcal{F}|_{U'}$ 的生成元，考虑关系层

$$\mathcal{R}(F_1, ..., F_p; \widetilde{G}_1, ..., \widetilde{G}_q) \subseteq \mathcal{A}|_{U'}^{\oplus(p+q)}$$

由 $\mathcal{G}$ 的凝聚性（不断缩小 U'），取其一组生成元

$$\left\{H_i=(H_i^1,H_i^2,...,H_i^{p+q})\in\mathcal{A}(U')^{\oplus(p+q)}\middle|1\le i\le r\right\}$$

则容易验证（纯线性代数，细节略）

$$\left\{\widetilde{H}_i=(\pi(H_i^{p+1}),...,\pi(H_i^{p+q}))\in\mathcal{A}(U')^{\oplus q}\middle|1\le i\le r\right\}$$

是关系层 $\mathcal{R}(G_1,G_2,...,G_q)|_{U'}$ 的生成元（其中 $\pi:\mathcal{G}\to\operatorname{coker}\varphi$ 为典范投影），从而关系层 $\mathcal{R}(G_1,G_2,...,G_q)$ 是局部有限生成的，因此 $\operatorname{coker}\varphi$ 凝聚。 □

注记 3.5.5. 对于拓扑空间 X，$\mathcal{A}$ 为 X 上的交换环层，记 $\mathcal{A}\text{-}\mathsf{Coh}_X$ 为 X 上的凝聚 $\mathcal{A}$-模层范畴，这是 $\mathcal{A}\text{-}\mathsf{Mod}_X$ 的子范畴。上述性质表明 $\mathcal{A}\text{-}\mathsf{Coh}_X$ 是 *Abel* 范畴。

性质 3.5.6. 设 $\mathcal{A}$ 为拓扑空间 X 上的交换环层，则对于 $\mathcal{A}$-模层同态短正合列

$$0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$$

此正合列中任何两个为凝聚层均可推出第三个也为凝聚层。

证明. 只需再证明 $\mathcal{F}_1,\mathcal{F}_3$ 凝聚能推出 $\mathcal{F}_2$ 凝聚。先断言 $\mathcal{F}_2$ 是局部有限生成的。对任意 $x\in X$，取 x 的（足够小的）开邻域 U，并且取 $F_1,F_2,...,F_p\in\mathcal{F}_1(U)$ 为 $\mathcal{F}_1|_U$ 的生成元，再取 $G_1,G_2,...,G_q\in\mathcal{F}_3(U)$ 为 $\mathcal{F}_3|_U$ 的生成元。将 G_j 在 $F_2(U)$ 中的代表元记为 $\widetilde{G}_j\,(1\le j\le q)$，则容易验证 $\{F_1,F_2,...,F_p;\widetilde{G}_1,\widetilde{G}_2,...,\widetilde{G}_q\}$ 为 $\mathcal{F}_2|_U$ 的一组生成元。从而 $\mathcal{F}_2$ 是局部有限生成的。

对 X 的任意开集 U，以及 $S_1,S_2,...,S_r\in\mathcal{F}_2(U)$，断言关系层 $\mathcal{R}(S_1,S_2,...,S_r)$ 是局部有限生成的。任取 $x\in U$，记截面 $S_1,...,S_r$ 在 $\mathcal{F}_3(U)$ 上的投影分别为 $\overline{S}_1,...,\overline{S}_r$. 由 $\mathcal{F}_3$ 的凝聚性，$\mathcal{R}(\overline{S}_1,...,\overline{S}_r)$ 是局部有限生成的，从而取 x 在 U 中的（足够小且不妨不断缩小的）开邻域 U'，以及 $\mathcal{R}(\overline{S}_1,...,\overline{S}_r)|_{U'}$ 的生成元矩阵

$$H:=\begin{pmatrix}H_1^1&\cdots&H_t^1\\ \vdots&&\vdots\\ H_1^r&\cdots&H_t^r\end{pmatrix}\in\mathcal{A}(U')^{r\times t}$$

即每个 $H_j^i\in\mathcal{A}(U')$，H 中的列向量 $\in\mathcal{R}(\overline{S}_1,...,\overline{S}_r)(U')$，矩阵 H 的 r 个列向量构成 $\mathcal{R}(\overline{S}_1,...,\overline{S}_r)|_{U'}$ 的生成元。再令

$$(F_1,F_2,...,F_t):=(S_1,S_2,...,S_r)H$$

则易验证 $(F_1,F_2,...,F_t)\in\mathcal{F}_1(U')^{\oplus t}$. 由 $\mathcal{F}_1$ 的凝聚性，取 $\mathcal{R}(F_1,...,F_t)|_{U'}$ 的生成元矩阵

$$K:=\begin{pmatrix}K_1^1&\cdots&K_s^1\\ \vdots&&\vdots\\ K_1^t&\cdots&K_s^t\end{pmatrix}\in\mathcal{A}(U')^{t\times s}$$

则容易验证 HK 为 $\mathcal{R}(S_1,...,S_r)|_{U'}$ 的生成元矩阵，从而 $\mathcal{R}(S_1,...,S_r)$ 是局部有限生成的。

综上所述，若 $\mathcal{F}_1$ 与 $\mathcal{F}_3$ 凝聚，则 $\mathcal{F}_2$ 也凝聚。 □

推论 3.5.7. 设 $\mathcal{F}$ 是拓扑空间 X 上的凝聚 $\mathcal{A}$-模层，则

（*1*）任意 $n \geq 1$，$\mathcal{F}^{\oplus n}$ 也是凝聚 $\mathcal{A}$-模层；

（*2*）对 X 的任意开集 U，以及任意 $F_1,F_2,...,F_p \in \mathcal{F}(U)$，则关系层 $\mathcal{R}(F_1,F_2,...,F_p)$ 也是凝聚的（$\mathcal{A}|_U$-模层）。

证明. **(1)**注意短正合列 $0 \to \mathcal{F} \to \mathcal{F} \oplus \mathcal{F}^{\oplus(n-1)} \to \mathcal{F}^{\oplus(n-1)} \to 0$，反复利用性质3.5.6作归纳即可。

(2)由（1）知 $(\mathcal{F}|_U)^{\oplus p}$ 是凝聚的，因此 $\mathcal{R}(F_1,F_2,...,F_p)$ 作为 $(\mathcal{F}|_U)^{\oplus p}$ 的局部有限生成子层，也是凝聚的。 □

推论 3.5.8. 若拓扑空间 X 上的 $\mathcal{A}$-模层 $\mathcal{M}$ 是凝聚的，并且 $\mathcal{M}$ 的子层 $\mathcal{F},\mathcal{G}$ 也是凝聚的，那么 $\mathcal{F} \cap \mathcal{G}$ 也为凝聚 $\mathcal{A}$-模层。

证明. 考虑层同态 $\varphi : \mathcal{F} \to \mathcal{M}/\mathcal{G}$ 为如下复合：

$$\mathcal{F} \hookrightarrow \mathcal{M} \twoheadrightarrow \mathcal{G}$$

注意 $\mathcal{F}$ 与 $\mathcal{M}/\mathcal{G}$ 都是凝聚的，再注意 $\mathcal{F} \cap \mathcal{G} \cong \ker\varphi$，因此 $\mathcal{F} \cap \mathcal{G}$ 也凝聚。 □

3.6 Oka 凝聚定理

本节介绍多复变函数论、复几何中的重要结果：**对于复流形 X，解析函数环层 $\mathcal{O}_X$ 是凝聚层**。这也是凝聚层的重要例子。注意凝聚性是局部性质，于是我们不妨 $X = \mathbb{C}^n$. 我们只需要证明，对 $\mathbb{C}^n$ 的任意开子集 U，以及任意 $F_1,F_2,...,F_q \in \mathcal{O}_X(U)$，关系层 $\mathcal{R}(F_1,F_2,...,F_q)$ 是局部有限生成的。

现在，对任意 $x \in X$，由于 $\mathcal{O}_{X,x}$ 为诺特环，从而 $\mathcal{R}(F_1,...,F_q)_x \subseteq \mathcal{O}_{X,x}^{\oplus q}$ 为有限生成 $\mathcal{O}_{X,x}$-模。但这与希望要证的“$\mathcal{R}(F_1,...,F_q)$ 局部有限生成”还差些东西。我们暂时只能说明存在 x 的邻域 $U' \subseteq U$，以及有限多个 $\mathcal{O}_X^{\oplus q}$ 在 U' 的截面，使得它们在 x 的芽生成 $\mathcal{R}(F_1,...,F_q)_x$；但我们希望**对 x 附近的任何点 y**，这些截面在 y 处的芽也生成 $\mathcal{R}(F_1,...,F_q)_y$——这是不显然的。

引理 3.6.1. （重要引理）

对于 $n \geq 2$，记 $\mathbb{C}^n = \left\{(z',z_n)\middle| z' = (z_1,...,z_{n-1}) \in \mathbb{C}^{n-1}, z_n \in \mathbb{C}\right\}$，设 $F_1,F_2,...,F_q$ 为定义在 $(0,0) \in \mathbb{C}^n$ 附近的解析函数，则存在 $(0,0)$ 的邻域 $\triangle := \triangle' \times \triangle_n$，其中 $\triangle'$ 与 $\triangle_n$ 分别为 $\mathbb{C}^{n-1}$

与 $\mathbb{C}$ 中的以原点为中心的多圆柱，使得对任意 $w=(w',w_n)\in\triangle$，$\left\{(K^1,K^2,...,K^q)\in\mathcal{O}_{\triangle,w}^{\oplus q}\middle|K^j\in\mathcal{K},\forall 1\leq j\leq q\right\}$ 是 $\mathcal{O}_{\triangle,w}$-模 $\mathcal{R}(F_1,F_2,...,F_q)_w$ 的一组生成元，其中

$$\mathcal{K}:=\left\{f(z',z)\in\mathcal{O}_{\triangle',w'}[z_n]\middle|\deg_{z_n}f\leq\mu\right\}$$

$$\mu:=\max\left\{\mathrm{Ord}_{z_n}(F_k)_0\middle|1\leq k\leq q\right\}$$

证明. 对 $F_1,F_2,...,F_q$ 在原点处使用 Weierstrass 预备定理，适当乘以原点附近的可逆解析函数（不会改变 $\mathcal{R}(F_1,...,F_q)$ 在原点的足够小邻域的限制），不妨设 $F_1,...,F_q\in\mathcal{O}_{\triangle',0}[z_n]$ 为定义在原点附近的关于 z_n 的 Weieretrass 多项式。此外，不妨

$$\deg_{z_n}F_q=\mu$$

Step 1 对于 $w=(w',w_n)\in\triangle$，关于 z_n 的 Weierstrass 多项式 F_q （通过平移）自然也视为关于 (z_n-w_n) 的 Weierstrass 多项式（次数仍为 μ）。对 F_q 在 w 处使用 Weierstrass 预备定理，令 $F_q=f'f''$，其中 $f'\in\mathcal{O}_{\triangle',w'}[z_n]$ 为关于 (z_n-w_n) 的 Weierstrass 多项式，$f''\in\mathcal{O}_{\triangle,w}$ 在 w 附近可逆。注意 F_q 与 f' 都为 Weierstrass 多项式，从而由引理1.4.3可知 $f''\in\mathcal{O}_{\triangle',w'}[z_n]$ 为关于 (z_n-w_n) 的多项式。分别记 μ',μ'' 为多项式 f',f'' 关于 z_n 的次数，则 $\mu=\mu'+\mu''$.

Step 2 我们习惯将 $\mathcal{R}(F_1,F_2,...,F_q)_w$ 中的元素记成列向量。对于任意的 $\begin{pmatrix}g^1\\g^2\\\vdots\\g^q\end{pmatrix}\in\mathcal{R}(F_1,F_2,...,F_q)_w$，对于 $1\leq j\leq q-1$，将 g^j 除以 Weierstrass 多项式 $F_{q,w}$，由 Weierstrass 除法定理，得

$$g^j=F_{q,w}T^j+R^j\qquad(1\leq j\leq q-1)$$

其中 $T^j\in\mathcal{O}_{\triangle,w}$ 以及 $R^j\in\mathcal{O}_{\triangle',w'}[z_n]$，且 $\deg_{z_n}R^j<\mu'$. 而对于 $j=q$，令

$$R^q:=g^q+\sum_{j=1}^{q-1}F_{j,w}T^j$$

则容易验证

$$\begin{pmatrix}g^1\\g^2\\\vdots\\g^q\end{pmatrix}=\begin{pmatrix}F_{q,w}&&&\\&F_{q,w}&&\\&&\ddots&\\&&&F_{q,w}\\-F_{1,w}&-F_{2,w}&\cdots&-F_{q-1,w}\end{pmatrix}\begin{pmatrix}T^1\\T^2\\\vdots\\T^{q-1}\end{pmatrix}+\begin{pmatrix}R^1\\R^2\\\cdots\\R^q\end{pmatrix}=\begin{pmatrix}F_{q,w}&&&R^1\\&\ddots&&\vdots\\&&F_{q,w}&R^{q-1}\\-F_{1,w}&\cdots&-F_{q-1,w}&R^q\end{pmatrix}\begin{pmatrix}T^1\\\vdots\\T^{q-1}\\1\end{pmatrix}$$

Step 3 我们得到了 q 阶方阵 $G := \begin{pmatrix} F_{q,w} & & & R^1 \\ & \ddots & & \vdots \\ & & F_{q,w} & R^{q-1} \\ -F_{1,w} & \cdots & -F_{q-1,w} & R^q \end{pmatrix}$. 容易验证 G 的每一列都位于 $\mathcal{R}(F_1, F_2, ..., F_q)_w$ 之中；并且除了第 (q,q)-分量 $G^q_q = R_q$，G 的其余矩阵元都位于 $\mathcal{K}$ 中，即为次数不超过 μ 的关于 z_n 的 $\mathcal{O}_{\triangle',w'}$-系数的多项式。最后，我们适当调整矩阵 G 的最后一列。

注意到 G 的第 q 列位于 $\mathcal{R}(F_1, F_2, ..., F_q)_w$ 之中，以及 $F_q = f'f''$，从而

$$\sum_{j=1}^{q-1} F_{j,w} R^j + f'f''R^q = 0$$

注意 $\deg_{z_n}\left(\sum_{j=1}^{q-1} F_{j,w}R^j\right) < \mu + \mu'$，因此 $f'f''R^q \in \mathcal{O}_{\triangle',w'}[z_n]$ 并且 $\deg_{z,n}(f'f''R^q) < \mu + \mu'$. 又因为 f' 是关于 z_n 的次数为 μ' 的 Weierstrass 多项式，从而由引理1.4.3可知，$f''R^q \in \mathcal{O}_{\triangle',w'}[z_n]$，并且 $\deg_{z_n}(f''R^q) < \mu$. 从而考虑

$$\begin{pmatrix} g^1 \\ g^2 \\ \vdots \\ g^q \end{pmatrix} = \begin{pmatrix} F_{q,w} & & & f''R^1 \\ & \ddots & & \vdots \\ & & F_{q,w} & f''R^{q-1} \\ -F_{1,w} & \cdots & -F_{q-1,w} & f''R^q \end{pmatrix} \begin{pmatrix} T^1 \\ \vdots \\ T^{q-1} \\ 1/f'' \end{pmatrix}$$

易知上式中的矩阵的每个矩阵元都位于 $\mathcal{K}$，并且每一列都位于 $\mathcal{R}(F_1, F_2, ..., F_q)_w$，因此 $\begin{pmatrix} g^1 \\ \vdots \\ g^n \end{pmatrix}$ 由上述矩阵（q 个列向量）$\mathcal{O}_{\triangle,w}$-生成。从而证毕。 □

定理 3.6.2. （*Oka* 凝聚定理）

对于复流形 X，X 上的解析函数环层 $\mathcal{O}_X$ 是凝聚的。

证明. 如之前所述，不妨 $X = \mathbb{C}^n$，以及对于任意开集 $U \subseteq \mathbb{C}^n$ 以及任意 $F_1, ..., F_q \in \mathcal{O}_{\mathbb{C}^n}(U)$，我们不妨 U 是以原点为中心的多圆柱区域，不妨 $F_1, ..., F_q$ 为关于 z_n 的 Weierstrass 多项式。

对 $X = \mathbb{C}^n$ 的维数 n 归纳。$n = 0$ 时平凡。对于 $n \geq 1$，如果 $\mathcal{O}_{\mathbb{C}^{n-1}}$ 是凝聚的，则对于 $(0,0) \in \mathbb{C}^{n-1} \times \mathbb{C}$ 的多圆柱邻域 $\triangle = \triangle' \times \triangle_n$，以及 $F_1, F_2, ..., F_q \in \mathcal{O}_{\triangle'}[z_n]$ 为 Weierstrass 多项式，它们关于 z_n 的最高次数记为 μ. 只需证 $\mathcal{F}(F_1, F_2, ..., F_q)$ 局部有限生成。对于任意的 $w \in \triangle$，以及 $\begin{pmatrix} g^1 \\ \vdots \\ g^q \end{pmatrix} \in \mathcal{R}(F_1, F_2, ..., F_q)_w$，由重要引理3.6.1 可知，存在 $q \times (\mu + 1)$ 矩阵 $U = (U^j_\alpha)_{\substack{1 \leq j \leq q \\ 0 \leq \alpha \leq \mu}}$，使

得

$$\begin{pmatrix} g^1 \\ \vdots \\ g^q \end{pmatrix} = \begin{pmatrix} U_0^1 & \cdots & U_\mu^1 \\ \vdots & & \vdots \\ U_0^q & \cdots & U_\mu^q \end{pmatrix} \begin{pmatrix} z_n^0 \\ \vdots \\ z_n^\mu \end{pmatrix}$$

其中 $U_\alpha^j \in \mathcal{O}_{\triangle',w'}$，视为定义在 $w' \in \triangle'$ 附近的解析函数，自然也视为定义在 $w \in \triangle$ 附近的（不显含 z_n 的）解析函数。注意 $F_k \in \mathcal{O}_{\triangle'}[z_n]$ 也为关于 z_n 的（次数不超过 μ 的）（Weierstrass）多项式，从而

$$(F_1, \cdots F_q) = (z_n^0, \cdots z_n^\mu) \begin{pmatrix} H_1^0 & \cdots & H_q^0 \\ \vdots & & \vdots \\ H_1^\mu & \cdots & H_q^\mu \end{pmatrix}$$

即得 $(\mu+1) \times q$ 的矩阵 H，H 的每个矩阵元都位于 $\mathcal{O}_{\triangle'}$ 之中，当然也是定义在 $w \in \triangle$ 附近的（不显含 z_n 的）解析函数。注意到

$$0 = (F_1, \cdots, F_q) \begin{pmatrix} g^1 \\ \vdots \\ g^q \end{pmatrix} = (z_n^0, \cdots z_n^\mu) HU \begin{pmatrix} z_n^0 \\ \vdots \\ z_n^\mu \end{pmatrix} =: \sum_{k=0}^{2\mu} L_k(U) z_n^k$$

因此比较 z_n 各次幂的系数，知 $L_k(U) = 0, \forall 0 \le k \le 2\mu$.

我们将矩阵 U 视为层 $\mathcal{O}_{\triangle'}^{\oplus q(\mu+1)}$ 在 w' 附近的截面，对于 $0 \le k \le 2\mu$，L_k 为层同态

$$L_k : \mathcal{O}_{\triangle'}^{\oplus q(\mu+1)}|_{\Omega'} \to \mathcal{O}_{\triangle'}|_{\Omega'}$$

并且 L_k 只与 $F_1, F_2, ..., F_q$ 有关。其中 Ω' 为 w' 在 $\triangle'$ 中的（足够小、不断缩小的）邻域。

由归纳假设，$\mathcal{O}_{\triangle'}$ 是凝聚的，因此 $\mathcal{O}_{\triangle'}^{\oplus q(\mu+1)}$ 也凝聚，因此对任意 $0 \le k \le 2\mu$，核层 $\ker L_k$ 也凝聚，从而 $\bigcap_{k=0}^{2\mu} \ker L_k$ 凝聚，故局部有限生成。因此存在截面 $U_1, U_2, ..., U_N \in \mathcal{O}_{\triangle'}^{\oplus q(\mu+1)}(\Omega')$，使得 $\{U_1, U_2, ..., U_N\}$ 为 $\mathcal{O}_{\triangle'}^{\oplus q(\mu+1)}|_{\Omega'}$ 的子层 $\bigcap_{k=0}^{2\mu} \ker L_k$ 的生成元。其中对于 $1 \le l \le N$，U_l 为 $q \times (\mu+1)$ 矩阵，其矩阵元取值于 $\mathcal{O}_{\triangle'}(\Omega')$.

容易验证，以下 N 个 q 维列向量

$$\left\{ U_l \begin{pmatrix} z_n^0 \\ \vdots \\ z_n^\mu \end{pmatrix} \middle| 1 \le l \le N \right\}$$

构成关系层 $\mathcal{R}(F_1, ..., F_q)|_{\Omega'}$ 的一组生成元。这就证明了 $\mathcal{O}_\triangle$ 的凝聚性，证毕。 □

3.7 层的上同调

Today:

Sheaf cohomology

X a topological space, $\mathcal{F}$- sheaf (of abelian groups).

定义 3.7.1. *(resolution)*

(1)a resolution of $\mathcal{F}$ is an exact sequence

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F} \xrightarrow{d^0} \mathcal{F} \xrightarrow{d^1} \to \cdots$$

定义 3.7.2. *A sheaf $\mathcal{A}$ is called injective, if if for any injective morphism $j : \mathcal{A} \to \mathcal{B}$ and for any morphism $\varphi : \mathcal{A} \to \mathcal{S}$, there exists an extension $\psi : \mathcal{B} \to \mathcal{S}$,such that*

定理 3.7.3. *the category of sheaves of abelian sheaves have enough injective objects, i.e. any $\mathcal{F}$ can be embedded in some injective sheaf.*

定义 3.7.4. *Consider an injective resolution of $\mathcal{F}$, i.e. an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \mathcal{I}^2 \to \cdots$$

where every $\mathcal{I}^k (k \geq 0)$ is injective.

$\rightsquigarrow$induces a sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}^0) \xrightarrow{\mathrm{d}} \Gamma(X, \mathcal{I}^1) \xrightarrow{\mathrm{d}} \Gamma(X, \mathcal{I}^2) \to \cdots$$

Then

$$H^q(X, \mathcal{F}) := H^q(\Gamma(X, \mathcal{I}^\bullet))$$

then, $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

定义 3.7.5. *A sheaf $\mathcal{S}$ is called a flabby (flasque ,in France) ,if for any open set $\Omega \subseteq X$, the morphism*

$$\mathcal{S}(X) \to \mathcal{S}(\Omega)$$

is surjective.

定义 3.7.6.

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \xrightarrow{d^0} \to \mathcal{F}^1$$

is an exact sequence is called a flabby resolution, if any $\mathcal{F}^k$ *is flabby.*

定义 3.7.7.

$$H^q(X, \mathcal{F}) := \text{...by flabby resolution...}$$

证明. Homological Algebra...omit. □

the two definitions of Sheaf Cohomology are isomorphic.

Godement's construction

$$God(\mathcal{F})(U) := \{f : U \to \bigcup_{x \in U} \mathcal{F}_x | f(y) \in \mathcal{F}_y, \forall y \in U\} := \prod_{x \in U} \mathcal{F}_x$$

$God(\mathcal{F})$ is a sheaf, and it is flabby. and there is a canonical morphism $\mathcal{F}(U) \to God(F)(U)$ by $x \mapsto (x \mapsto s_x)$ is injective.

$$\mathcal{F}^0 := God(\mathcal{F})$$

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \twoheadrightarrow \text{coker}(j) = \mathcal{F}^0 / \mathcal{F}$$

and consider

$$\mathcal{F}^1 := God(\text{coker}(j))$$

......then construct by induction... this is a flabby resolution of $\mathcal{F}$.

定义 3.7.8. *(resolution by fine sheaves)*

$\mathcal{A}$ *is a sheaf of ring,* X *is a paracompact topological space,* $\mathcal{A}$ *is called a fine sheaf, if for any open covering*

$$X = \bigcup_{\alpha} V_\alpha \quad , \mathcal{V} := \{V_\alpha\}$$

there exists a partition of unit subordinate to $\mathcal{V}$*, (i.e.* $\exists f_\alpha \in \mathcal{A}(V_\alpha), supp(\alpha) := \overline{\{x \in V_\alpha | f_{\alpha,x} \neq 0\}} \subseteq V_\alpha$*, and* $\sum_\alpha f_\alpha = 1$*(the sum is locally finite))*

例子 **3.7.9.** *X is a differential manifold, $\mathcal{C}^\infty$ is the sheaf of smooth functions, then $\mathcal{C}^\infty$ is a fine sheaf.*

定理 **3.7.10.** *$\mathcal{S}$ is a sheaf of $\mathcal{A}$-modules, $\mathcal{A}$ is a fine sheaf. then for any $q \geq 1$,*

$$H^q(X, \mathcal{S}) = 0$$

证明. Consider a flabby(or injective) resolution

$$0 \to \mathcal{S} \xrightarrow{j} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \mathcal{I}^2 \cdots$$

where any $\mathcal{I}^k (k \geq 0)$ is a sheaf of $\mathcal{A}$-modules.

by definition,

$$H^q(X, m\mathcal{S}) := \frac{\ker \mathrm{d} : \Gamma(\mathcal{I}^q) \to \Gamma(\mathcal{I}^{q+1})}{\Im \mathrm{d} : \Gamma(\mathcal{I}^{q-1}) \to \Gamma(\mathcal{I}^q)}$$

Let $\alpha \in \ker\{\mathrm{d} : \Gamma(\mathcal{I}^q) \to \Gamma(\mathcal{I}^{q+1})\}$ by the exactness of resolution, $\exists$ an open covering $\mathcal{U} = (U_i)_i$, s.t. $\alpha|_{U_i} = \mathrm{d}\beta_i$ where $\beta_i \in \mathcal{T}^{q-1}(U_i)$. Let $(\beta_i)_i$ be the partition of unit w.r.t. $\mathcal{U}$. consider

$$\beta := \sum_i f_i \beta_i$$

(well defined). Then $\mathrm{d}\beta = \alpha$.... □

3.8 Cech 上同调

Cech cohomology

X- a topological space, $\mathcal{F}$- a sheaf of abelian group.

$$\mathcal{U} = (U_\alpha)_{\alpha \in I}$$

is an open covering.

notation:$U_{\alpha_1,\ldots,\alpha_q} := \bigcap_{i=1}^q U_{\alpha_i}$.

Cech q-chain w.r.t $\mathcal{U}$:

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{(\alpha_1,\ldots,\alpha_q) \in \mathcal{I}^{q+1}} \mathcal{F}(U_{\alpha_1,\ldots,\alpha_q})$$

$$c \in C^q(\mathcal{U}, \mathcal{F})$$

means that we have a family of sections $C_{\alpha_1,\ldots,\alpha_q} \in \mathcal{F}(U_{\alpha_1,\ldots,\alpha_q})$ with the relation

$$C_{\alpha_0,\ldots,\alpha_j,\ldots,\alpha_i,\ldots} = -C_{\ldots}$$

(C)ech differential:

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$$

$$\delta^q(c)_{\alpha_0,\ldots,\alpha_{q+1}} := \sum_{0\le k\le q+1} (-1)^k c_{\ldots\hat{\alpha_k}\ldots}|_{U_{\alpha_0,\ldots,\alpha_{q+1}}}$$

性质 3.8.1.

$$\delta^q \circ \delta^q = 0$$

so, we have Čech cohomology

$$H^q(\mathcal{U}, \mathcal{F}) := \ker \delta^q / \operatorname{Im} \delta^{q-1}$$

example:

$$C^0(\mathcal{U}, \mathcal{F}) := \prod_{\alpha\in I} \mathcal{F}(U_\alpha)$$

$$c = (c_\alpha)_{\alpha\in I} \in C^0(\mathcal{U}, \mathcal{F})$$

$$\delta^0 c = 0 \iff (\delta^0 c)_{\alpha_0\alpha_1} := (c_{\alpha_1} - c_{\alpha_0})|_{U_{\alpha_0\alpha_1}} = 0$$

so, $c_{\alpha_0} = c_{\alpha_1}$ on $U_{\alpha_0\alpha_1}$.

$\rightsquigarrow H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

例子 3.8.2. *(1) consider* $X = \triangle \setminus \{0\}$, *where* $\triangle = \{(z_1, z_2)||z_1| < 1, |z_2| < 1\}$. *Consider the covering*

$$\mathcal{U} = U_1 \cup U_2$$

where

$$U_1 := \{(z_1, z_2) \in \triangle | z_1 \neq 0\} = \mathbb{D}^* \times \mathbb{D}$$

$$U_2 := \{(z_1, z_2) \in \triangle | z_2 \neq 0\} = \mathbb{D} \times \mathbb{D}^*$$

then

$$U_1 \cap U_2 = \mathbb{D}^* \times \mathbb{D}^*$$

consider $H^0(X, \mathcal{O}) = \mathcal{O}(X) \cong \mathcal{O}(\triangle) = \{f : \triangle \to \mathbb{C} holomorphic\}$.

$$H^1(\mathcal{U}, \mathcal{O}) = \ker \delta^1 / \operatorname{Im} \delta^0$$

$$\delta^1 : C^1(\mathcal{U}, \mathcal{O}) \to C^2(\mathcal{U}, \mathcal{O}) \subseteq \prod_{\alpha_0,\alpha_1,\alpha_2} \mathcal{O}(U_{\alpha_0,\alpha_1,\alpha_2}) = 0$$

$$\ker \delta^1 = C^1(\mathcal{U}, \mathcal{O}) = \{c = c(\alpha_0, \alpha_1) | c_{\alpha_0,\alpha_1} \in \mathcal{O}(U_{\alpha_0\alpha_1})\} = \{c \in \mathcal{O}(U_1 \cap U_2)\} = \{c = \sum_{m,n\in\mathbb{Z}} a_{mn} z_1^m z_2^n convergent\}$$

$$\delta^0 : C^0(\mathcal{U}, \mathcal{O}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{O})$$

$$(\delta^0 c)_{12} = (c_2 - c_1)|_{U_{12}}$$

where $c_2 \in \mathcal{O}(U_2)$ *and* $c_1 \in \mathcal{O}(U_1)$. *note that*

$$\mathcal{O}(U_1) = \{c(z_1, z_2) = \sum_{m\in\mathbb{Z}, n\geq 0} a_{mn} z_1^m z_2^n \, convergent\}$$

$$\mathcal{O}(U_2) = \{c(z_1, z_2) = \sum_{n\in\mathbb{Z}, m\geq 0} a_{mn} z_1^m z_2^n \, convergent\}$$

So, $H^1(\mathcal{U}, \mathcal{O}) = \{c(z_1, z_2) = \sum_{m,n<0} a_{mn} z_1^m z_2^n\}$

例子 3.8.3. *(complex projective space)*

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

$$(z_0, ..., z_n) \sim \lambda(z_0, ..., z_n)$$

for some $\lambda \in \mathbb{C}^*$.

$$\mathbb{C}P^n = \{[z_0, ..., z_n] | not \ all \ z_k = 0, z_i \in \mathbb{C}\} = \bigcup_{0\leq p\leq n} V_k$$

where

$$V_k = \{[z_0, ..., z_n] | z_k \neq 0\} \cong \{(\frac{z_0}{z_k}, ..., 1, ..., \frac{z_n}{z_k}) | z_i \in \mathbb{C}, i \neq k, z_k \neq 0\} \cong \mathbb{C}^n$$

this is a holo chart.

$$\mathbb{C}P^1 = V_0 \cup V_1, \mathcal{V} = \{V_0, \mathcal{V}_1\}$$

HW: compute $H^q(\mathcal{V}, \mathcal{O})$.

Answer:

$$H^0 \cong \mathbb{C}, H^1 \cong 0$$

Correction:

$\mathcal{A}$: Sheaf of rings (with unit)

X: paracompact topological space,

定义 3.8.4. $\mathcal{A}$ *is called fine, if for any open covering* $\mathcal{U} = (V_\alpha)_{\alpha\in\mathcal{I}}$,*there exist* $s_\alpha \in \mathcal{A}(X)$ *such that such that* $supp(s_\alpha) \subseteq V_\alpha$,

$$\sum_\alpha s_\alpha = 1$$

(this is a locally finite sum)

注记 3.8.5. *we call $\mathcal{A}$ is a* ***soft sheaf****, if for any closed set $K \subseteq X$, the morphism*

$$\mathcal{A}(X) \to \mathcal{A}(K)$$

is surjective. where $\mathcal{A}(K) := \Gamma(K, \mathcal{A}|_K)$

fact: $\mathcal{A}$ is fine if and only if $\mathcal{H}om(\mathcal{A}, \mathcal{A})$ is soft. (omit)

Recall:

Cech cohomology: X topological space, $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$,

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 < \ldots < \alpha_q} \mathcal{F}(\alpha_1, \ldots, \alpha_q)$$

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$$

fact: $H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Today:

定义 3.8.6. *Let $\mathcal{V} = (V_\beta)_{\beta \in J}$ be another open covering, then $\mathcal{V}$ is called a refinement of $\mathcal{U}$, if there exists a map*

$$\rho : \mathcal{J} \to \mathcal{I}$$

such that

$$V_\beta \subseteq U_{\rho(\beta)}$$

性质 3.8.7. *Let $\mathcal{V}$ be a refinement of $\mathcal{U}$, then ρ induces a map*

$$\rho^q : C^q(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{V}, \mathcal{F})$$

$$(\rho^q C)_{\beta_0, \ldots, \beta_q} \mapsto C_{\rho(\beta_0), \ldots, \rho(\beta_q)}|_{V_{\beta_0, \ldots, \beta_q}}$$

ρ is a morphism of complexes.

so, ρ induces a map

$$H^q(\rho) : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{V}, \mathcal{F})$$

Let $\tilde{\rho} : \mathcal{J} \to \mathcal{I}$ be another refinement of $\mathcal{U}$

(induces $H^q(\tilde{\rho}) : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{V}, \mathcal{F})$) then $\rho, \tilde{\rho}$ are homotopic (chain homotopy$\rightsquigarrow H^q(\rho) = H^q(\tilde{(\rho)}))$

so, if $\rho : \mathcal{J} \to \mathcal{I}$ is refinement, then

$$H^q(\rho)$$

is independent of the refinement.

定义 3.8.8.

$$\check{H}^q(X,\mathcal{F}) := \lim_{\overrightarrow{\mathcal{U}}} H^q(\mathcal{U},\mathcal{F})$$

i.e. $a \in H^q(\mathcal{U},\mathcal{F}) \sim \in H^q(\mathcal{V},\mathcal{F})$ *iff* $\exists$ *a refinement* $\mathcal{W}$ *of* $\mathcal{U}$ *and* $\mathcal{V}$ *such that* a, b *have the same image in* $H^q(\mathcal{W},\mathcal{F})$

注记 3.8.9.

$$\check{H}^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$$

Exercise: For $q = 1$, *if* $\mathcal{V}$ *is a refinement of* $\mathcal{U}$, *then*

$$H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F})$$

is injective.

so ,for any open cover $\mathcal{U}$,

$$H^1(\mathcal{U},\mathcal{F}) \to \check{H}^1(X,\mathcal{F})$$

is injective.

Homological Algebra recall: let $(K^\bullet, \mathrm{d}_k), (L^\bullet, \mathrm{d}_l)$ and $(M^\bullet, \mathrm{d}_M)$, if we have a short exact sequence

$$0 \to K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} M^\bullet \to 0$$

then it induces a long exact sequence :

$$\cdots \to H^q(K^\bullet) \to H^q(L^\bullet) \to H^q(M^\bullet) \to H^{q+1}(K^\bullet) \to \cdots$$

analogy of Cech cohomology: X is a topological space, $\mathcal{U}$ is an open covering of X. $\mathcal{A}$ and $\mathcal{B}$ sheaves on X, Let

$$\varphi : \mathcal{A} \to \mathcal{B}$$

be a morphism, then it induces

$$\varphi^\bullet : C^\bullet(\mathcal{U},\mathcal{A}) \to C^\bullet(\mathcal{U},\mathcal{B})$$

Let

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

be an exact sequence of sheaves, then we have: for any open set Ω,

$$0 \to \mathcal{A}(\Omega) \to \mathcal{B}(\Omega) \to \mathcal{C}(\Omega)$$

left exact.

Example: consider

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{exp} \to 0$$

is exact on $bbC^{\times} := \mathbb{C} \setminus \{0\}$

but we have :

$$0 \to \mathcal{A}(\Omega) \xrightarrow{\psi} \mathcal{B}(\Omega) \to \operatorname{Im} \psi(\Omega) \to 0$$

is exact.

First we have the following exact sequence

$$C^q(\mathcal{U}, \mathcal{A}) \to C^q(\mathcal{U}, \mathcal{B}) \to C^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \to 0$$

where $C^q_{\mathcal{B}}$ is the image of ...

then we get an exact sequence

$$0 \to (C^\bullet(\mathcal{U}, \mathcal{A}), \delta) \to (C^\bullet(\mathcal{U}, \mathcal{B}), \delta) \to (C^\bullet_{\mathcal{B}}(\mathcal{U}, \mathcal{C}), \delta) \to 0$$

it induces a long exact sequence

$$\cdots \to H^q(\mathcal{U}, \mathcal{A}) \to H^q(\mathcal{U}, \mathcal{B}) \to H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \to H^{q+1}(\mathcal{U}, \mathcal{A}) \to \cdots$$

定理 3.8.10. *If X is paracompact,*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a sheaf exact sequence. Then there is a long exact sequence

$$\cdots \to \check{H}^q(X, \mathcal{A}) \to \check{H}^q(X, \mathcal{B}) \to \check{H}^q(X, \mathcal{C}) \to \check{H}^{q+1}(X, \mathcal{Z}) \to \cdots$$

证明. Key lemma: need to prove

$$\lim_{\overrightarrow{\mathcal{U}}} H^q(\mathcal{U}, \mathcal{C}) = \lim_{\overrightarrow{\mathcal{U}}} H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C})$$

if X is paracompact.

Omit. □

if

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

exact,

recall:(cohomology by resolutions)

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$

flabby resolution. then it induces

$$0 \to \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \cdots$$

then define the sheaf cohomology...

we have a long exact sequence

$$\cdots \to H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B}) \to H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A}) \to \cdots$$

it is homological algebra...

定理 3.8.11. *(Leray's acyclic theorem) Let $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$ be an open covering of X, ($\mathcal{F}$ is a sheaf on X), if satisfying*

$$H^k(U_{\alpha_0,\ldots,\alpha_q}) = 0$$

for any $k \geq 1$,then

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{(H)}^q(X, \mathcal{F})$$

and if X is paracompact ,we also have

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{(H)}^q(X, \mathcal{F}) \cong H^q(X, \mathcal{F})$$

(this $\mathcal{U}$ is called acyclic covering)

de Rham- Weil theorem

定义 3.8.12. *$\mathcal{F}$ is a sheaf on X, Ω is an open set of X, then $\mathcal{F}$ is called* ***acyclic sheaf*** *if*

$$H^q(\Omega, \mathcal{F}) = 0$$

for any $q \geq 1$.

定理 3.8.13. *Let*

$$0 \to \mathcal{F} \to (L^\bullet, \mathrm{d})$$

be an acyclic resolution of $\mathcal{F}$ (i.e. $L^q$is acyclic on X) then

$$H^q(X, \mathcal{F}) \cong H^q(\Gamma(X, L^\bullet), \mathrm{d})$$

for any $q \geq 0$.

(先看例子)

例子 3.8.14. *Let X be a differential manifold, $\mathcal{E}^p$:sheaf of smooth p-forms, then we have a resolution (de Rham complex)*

$$0 \to \mathbb{R} \hookrightarrow \mathcal{E}^0 \xrightarrow{\mathrm{d}} \mathcal{E}^1 \xrightarrow{\mathrm{d}} \mathcal{E}^2 \xrightarrow{\mathrm{d}} \mathcal{E}^3 \to \cdots$$

where d differential operators. (Why it is a resolution? because of Poincare lemma...locally solvable..)

Note that

$$\mathcal{E}^0 = \mathcal{C}^\infty$$

$\mathcal{E}^p$ is a sheaf of C^∞-modules..

then we have

$$H^q(X, \mathcal{E}^p) = 0$$

for all $q \geq 1$

and then

$$H^q(X, \mathbb{R}) \cong \frac{\ker(\mathrm{d} : \Gamma(X, \mathcal{E}^q) \to \Gamma(X, \mathcal{E}^{q+1}))}{\operatorname{Im}(\mathrm{d} : \Gamma(X, \mathcal{E}^{q-1}) \to \Gamma(X, \mathcal{E}^q))} = H^q_{DR}(X, \mathcal{R})$$

例子 3.8.15. *Let X be a complex manifold, $\mathcal{E}^{p,q}$ sheaf of smooth (p,q) forms, Ω^p is the sheaf of holomorphic p-forms (i.e. $(p,0)$-form φ with $\bar{\partial}\varphi = 0$).*

Then we have resolution

$$0 \to \Omega^p \xrightarrow{j} \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \to \cdots$$

(Why it is a resolution? because of the Dolbeault lemma),remain to Exercise...

$$H^q(X, \Omega^p) \cong H^{p,q}_{\bar{\partial}}(X, \mathbb{C})$$

Today：de Rham-Weil Isomorphism Thm

定理 3.8.16. *Let X be a topological space, $\mathcal{F}$ be a sheaf of abelian groups on X,*

$$0 \to \mathcal{F} \to (\mathcal{L}^\bullet, \mathrm{d})$$

be an acyclic resolution, i.e.

$$H^k(X, \mathcal{L}^q) = 0$$

for all $k \geq 1$ and $q \geq 0$. Then,

$$H^q(X, \mathcal{F}) \cong H^q((\Gamma(\mathcal{L}^\bullet), \mathrm{d}))$$

证明. Since

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{\mathrm{d}^0} \mathcal{L}^1 \xrightarrow{\mathrm{d}^1} \mathcal{L}^2 \to \cdots$$

be an exact sequence, denote

$$\mathcal{Z}^q := \ker \mathrm{d}^q$$

then we have short exact sequences

$$0 \to \mathcal{Z}^q \to \mathcal{L}^q \to \mathcal{Z}^{q+1} \to 0$$

for any q. They induce long exact sequence of cohomology groups:

$$\cdots \to H^k(X, \mathcal{Z}^q) \to H^k(X, \mathcal{L}^q) \to H^k(X, \mathcal{Z}^{q+1}) \xrightarrow{\partial} H^{k+1}(X, \mathcal{L}^q) \to H^{q+1}(X, \mathcal{L}^q) \to \cdots$$

For any $k \geq 1$, since $\mathcal{L}^q$ are acyclic on X,

$$H^k(X, \mathcal{Z}^{q+1}) \cong H^{k+1}(X, \mathcal{Z}^q)$$

and for $k = 0$, we have

$$0 \to H^0(X, \mathcal{Z}^q) \to H^0(X, \mathcal{L}^q) \to H^0(X, \mathcal{Z}^{q+1}) \to H^1(X, \mathcal{Z}^q) \to H^1(X, \mathcal{L}^q) = 0 \to \cdots$$

so,

$$H^1(X, \mathcal{Z}^q) \cong H^0(X, \mathcal{Z}^{q+1}) / \operatorname{Im} \mathrm{d}^q \cong H^{q+1}((\Gamma(\mathcal{L}^\bullet), \mathrm{d}))$$

$$H^{q+1}(\Gamma(\mathcal{L}^\bullet)) \cong H^1(X, \mathcal{Z}^q) \cong H^2(X, \mathcal{Z}^{q-1}) \cong \cdots H^{q+1}(X, \mathcal{Z}^0) = H^{q+1}(X, \mathcal{F})$$

□

$$0 \to \mathbb{R} \to \mathcal{E}^0 \xrightarrow{\mathrm{d}} \mathcal{E}^1 \xrightarrow{\mathrm{d}} \mathcal{E}^2 \to \cdots$$

(de Rham resolution) then we have

$$H^k(X,\mathcal{R}) \cong H^k_{DR}(X;\mathcal{R})$$

(if X is compact ,then by Hodge theory, it also isomorphic to $\ker(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})$)

Another example:X is a complex manifold, then

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \to \cdots$$

then

$$H^q(X,\Omega^p) \cong H^{p,q}_{\bar{\partial}}(X,\mathbb{C})$$

(RHS= Dolbeault cohomology)

X be a smooth manifold, we define

$$C_q(X,\mathbb{Z}) := \text{the free abelian group generated by continuous map}$$

$$\phi : \triangle_q := \{(t_1,...,t_{q+1}) \in [0,1]^{q+1} | \sum_{i=1}^{n} t_i = 1\}$$

and we define (for $\phi \in C_q(X,\mathbb{Z})$)

$$\partial\phi := \sum_{i=1}^{q+1} (-1)^q \phi|_{\triangle_{q,i}}$$

$$\triangle_{q,i} := \{t \in \triangle_q | t_i = 0\}$$

we define

$$(C^\bullet_{sing}, \partial)$$

be the dual complex of $(C^{sing}_\bullet), \partial$.

(These are all Basic Algebraic Topology)

For any open $U \subseteq X$, we have

$$U \to C^q_{sing}(U,\mathbb{Z})$$

we get a sheaf

$$\mathcal{C}^q_{sing}$$

FACT: $(\mathcal{C}^\bullet_{sing}, \partial)$ is a flabby resolution of $\mathbb{Z}$. (check!)So,

$$H^q_{sing}(X,\mathbb{Z}) = H^q(\Gamma(\mathcal{C}^\bullet_{sing}), \partial) \cong H^q(X,\mathbb{Z})$$

第 4 章　Hermite 向量丛

4.1　联络与曲率

Recall: X is a smooth manifold, E is a vector bundle of rank r, if

(1)$\pi : E \to X$ is smooth map,

(2)for any $x \in X$, $E_x := \pi^{-1}(x)$ is a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of dimension r.

(3)there an open covering $\mathcal{U} = (\mathcal{U}_\alpha)_{\alpha \in I}$ and trivializations

$$\theta_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

and for any intersection $U_\alpha \cap U_\beta$, we have

注记 4.1.1.

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$$

(cocycle condition)

Special Case: line bundle rank E=1.

then $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, \mathbb{K}^*) = \mathcal{E}^*(U_{\alpha\beta})$ invertible smooth function on $U_{\alpha\beta}$. then, Cech cohomology,

$$(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = 1$$

so,

$$(g_{\alpha,\beta}) \in \mathcal{Z}^1(\mathcal{U}, \mathcal{E}^*) \twoheadrightarrow H^1(\mathcal{U}, \mathcal{E}^*) \hookrightarrow \check{H}^1(X, \mathcal{E}^*)$$

we get a map

$$\{\text{line bundles}\} \to \check{H}^1(X, \mathcal{E}^*)$$

actually, we have

$$\{\text{isomorphic classes of line bundles}\} \longleftrightarrow H^1(X, \mathcal{E}^*)$$

1-1 correspondence.

Now, X be a complex manifold, a complex vector bundle E is called homomorphic, if ... the transition matrix $g_{\alpha\beta}$ is holomorphic...

Holomorphic line bundles :

$$g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$$

$\mathcal{O}^*$:sheaf of invertible holomorphic functions...

FACT: there is a map

$$\{\text{holomorphic line bundle}\} \to \check{H}^1(X, \mathcal{O}^*)$$

例子 4.1.2. *trivial vector bundle* $X \times \mathbb{K}^r$

例子 4.1.3. *Tangent bundle* TX. *(transition matrix* $g_{\alpha\beta}$ *are given by Jacobi matrix..)*

定义 4.1.4. *(Local frame of vector bundles)*

$$\theta_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{K}^r$$

be a trivialization, we define

$$e_\lambda(x) := \theta_\alpha^{-1}(x, \begin{pmatrix} 0 \\ \dots \\ 1(\leftarrow ith) \\ \dots \\ 0 \end{pmatrix})$$

then, $\{e_1, ..., e_r\}$ *be a local smooth section* $s \in \Gamma(U_\alpha, E)$ *can be written as*

$$s(x) = \sum \sigma_\lambda(x)$$

where $\sigma_\lambda \in C^\infty(U_\alpha, \mathbb{K})$.

(Connection)

记号 4.1.5. *For* X *be a smooth manifold,* E *is a vector bundle(real or complex), denote*

$$C_p^k(\Omega, E) := C^k(\Omega, \bigwedge^p T^*M \otimes E)$$

is the space of k*-differential* p*-forms with values in* E.

Locally, consider a trivialization of E,

$$\theta_\alpha E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

($\rightsquigarrow$ *frame* $(e_1, ...e_r)$*)*

$$s \in \sum \varphi_\lambda(x) \otimes e_\lambda(x)$$

where φ_λ *is a* p*-form.*

定义 4.1.6. *a (linear) connection on E is a linear differential operator of order 1 acting on $C^\infty_\bullet(X,E)$:*

$$D : C^\infty_p(X,E) \to C^\infty_{p+1}(X,E)$$

$$D(f\wedge x) := \mathrm{d}f\wedge s + (-1)^p f\wedge Ds$$

where $f\in C^\infty(X,\bigwedge^p T^*M)$, $s\in C^\infty(X,E)$.

Locally, consider a local trivialization

$$\theta : E|_\Omega \xrightarrow{\sim} \Omega\times\mathbb{K}^r$$

with a frame $\{e_1,...,e_r\}$. any section $t\in C^\infty_p(\Omega,E)$ can be written as

$$t=\sum_{1\le\lambda\le r}\sigma_\lambda\otimes e_\lambda$$

$$Ds=\sum_{\lambda=1}^r \mathrm{d}\sigma_\lambda\wedge e_\lambda + (-1)^p\sigma_\lambda\wedge De_\lambda$$

where

$$De_\lambda\in C^\infty_1(\Omega,E)$$

can be written as

$$De_\lambda=\sum_{\mu=1}^r a_{\mu\lambda}\otimes e_\mu$$

where "$a_{\mu\lambda}$" is called the coefficients of D with respect to frame $\{e_1,...,e_r\}$.

so,

$$D(t)=\sum_{\lambda,\mu}\mathrm{d}\sigma_\lambda\wedge e_\lambda+(-1)^p\sigma_\lambda\wedge a_{\mu\lambda}\wedge e_\mu=\sum_\mu\sum_\lambda\left(\mathrm{d}\sigma_\mu+a_{\mu\lambda}\wedge\sigma_\lambda\right)$$

$$Dt=\mathrm{d}\sigma+A\wedge\sigma$$

where $A=(a_{\mu\lambda})$.

RMK: connection always exists!

Recall: for any (connected) smooth manifold, $E\to X$ is a smooth vector bundle,

Connection:

$$D : C^\infty_p(X,E)\to C^\infty_{p+1}(X,E)$$

where $C^\infty_p(X,E) := C^\infty(X,\wedge^p T^*M\otimes E)$

$$D(f\wedge s)=\mathrm{d}f\wedge s+(-1)^{\deg f}f\wedge Ds$$

Essentially,

$$D : C^\infty(X, E) \to C_1^\infty(X, E)$$

Locally, consider a trivialization $\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$, and a local frame $(e_1, ..., e_r)$ where $e_k(x) = \theta^{-1}(x, \begin{pmatrix} 0 \\ \vdots \\ 1(k^{th}) \\ \vdots \\ 0 \end{pmatrix})$.

Let $s \in C^\infty(\Omega, E)$, i.e.

$$s = \sum_{i=1}^{r} \sigma_i e_i$$

where σ_i are smooth functions.

$$Ds = \mathrm{d}\sigma + A \wedge \sigma$$

where

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \end{pmatrix} \quad A = a_{ij}$$

consider another trivialization

$$\tilde{\theta} : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

$\rightsquigarrow$ a local frame $(\tilde{e_1}, ..., \tilde{e_r})$. Then there exists a invertible linear transform s.t.

$$\tilde{e_k} = g_k^m e_m$$

assume

$$De_k = a_k^l e_l \qquad D\tilde{e_k} = \tilde{a}_k^l \tilde{e}_l$$

we have

$$\begin{aligned} \mathrm{d}g_k^n e_n + g_k^m a_m^n e_n &= \tilde{a}_k^l g_l^n e_n \\ \rightsquigarrow \quad \tilde{a}_k^l g_l^n (g^{-1})_n^p &= \mathrm{d}g_k^n (g^{-1})_n^p + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \quad \tilde{a}_l^p &= \mathrm{d}g_k^n (g^{-1})_n^p) + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \quad \tilde{A} &= \mathrm{d}g \cdot g^{-1} + g \cdot A \cdot g^{-1} \end{aligned}$$

Curvature

$$H_D := D^2$$

locally,

$$D^2 s = D(\mathrm{d}\sigma + A \wedge \sigma) = \mathrm{d}(\mathrm{d}\sigma + A \wedge \sigma) + A \wedge (\mathrm{d}\sigma + A \wedge \sigma)$$

$$= \mathrm{d}A \wedge \sigma - A \wedge \mathrm{d}\sigma + A \wedge \mathrm{d}\sigma + A \wedge A \wedge \sigma = (\mathrm{d}A + A \wedge A) \wedge \sigma$$

so we have

$$H = \mathrm{d}A + A \wedge A$$

Similarly to $\tilde{A}, A$ we have

Exercise:

$$\tilde{H} = gHg^{-1}$$

曲率在不同平凡化下的表达式。where

$$\tilde{e} = ge$$

$\rightsquigarrow$ H can be considered as a section of $C_2^\infty(X, \mathrm{Hom}(E, E))$. because

$$\tilde{H}\tilde{e} = gHg^{-1}\tilde{e} = gHe$$

independent of the choice of local frames.

4.2 向量丛的构造

定义 4.2.1. *(dual of vector bundles) $E \to X$, and $g_{\alpha\beta}$:transition matrix of E, the dual is given by $(g_{\alpha\beta})^{-1}$.* （用转移函数来定义向量丛）

定义 4.2.2. *direct sum of two vector bundles $(E, F) \to E \oplus F$. locally,*

$$(g_{\alpha,\beta}) \oplus (h_{\alpha\beta})$$

direct sum of transition matrices.

定义 4.2.3. *tensor product of two vector bundles.*

locally, tensor product of two transition matrices.

fact: let D_E be a connection on E,then it induces a connection D_{E^*}. Let u be a local section of E^*, s local section of E, then we define

$$\mathrm{d}\langle u, s\rangle = \langle D_{E^*}u, s\rangle + \langle u, D_E s\rangle$$

Exercise:

$$H(D_{E^*}) = -H(D_E)^T$$

and for two vector bundles E, F, connections D_E, D_F, then

$$D_{E\oplus F} := D_E \oplus D_F$$

$$H(E \oplus F) = H_E \oplus H_F$$

as for tensor product, we define $D_{E\otimes F}$ as follows:

$$D_{E\otimes F}(s \otimes t) = D_E s \otimes t + s \otimes D_F t$$

check the curvature

$$H_{E\otimes F} = H_E \otimes id_F + id_E \otimes H_F$$

注记 4.2.4. *we can also consider wedge product of vector bundles. Consider vector bundles $E_1, ..., E_k$, with connections $D_{E_1}, ..., D_{E_k}$, let $s_i \in C^\infty_{p_i}(X, E^i)$ then*

$$D_{E_1\wedge,...,\wedge E_k}(s_1 \wedge ... \wedge s_k) = \sum_{i=1}^{k} (-1)^{p_1+...+p_{i-1}} s_1 \wedge ... \wedge D_{E_i} s_i \wedge ... \wedge s_k$$

Let E be a vector bundle of rank r, then $\bigwedge^r E$ is a line bundle, with transition matrix by $\det(g_{\alpha\beta})$. this bundle is denoted by $\det E$.(Det-bundle)

Let $s_1, ..., s_r$ be local sections of E, then we have

$$D_{\det E}(s_1 \wedge \cdots \wedge s_r) = tr(H_E) s_1 \wedge \cdots \wedge s_r$$

4.3 陈省身示性类

chern classes (defined by curvature).

Let $E \to X$ be a smooth complex vector bundle of rank r, where X be a complex manifold.

(Chern-Weil theory)

V be a complex vector space, $f : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{C}$ be a symmetric multi-linear form of degree k.

$\rightsquigarrow f(v) := f(v, v, ..., v)$ is a homogeneous polynomial of degree k.

定义 4.3.1. *assume G is a group (left) acting on V, s.t.*

$$f(g(v_1), ..., g(v_k)) = f(v_1, ..., v_k)$$

for any $g \in G, v_i \in V$, then we say f is G-invariant.

Special case: $G = GL(r, \mathbb{C})$ and $V = LieG = \mathfrak{gl}r, \mathbb{C}$ be the Lie algebra of G. the action is

$$(g, M) \mapsto gMg^{-1}$$

Consider

$$\det(I + \frac{i}{2\pi}tm) = I + tf_1(M) + t^2 f_2(M) + \cdots t^r f_r(M)$$

$\rightsquigarrow \forall 1 \leq k \leq r$, f_k is G-invariant.

Let $E \to X$ complex vector bundle on a complex manifold, let D_E be a connection, curvature $H_E \in C_2^\infty(X, \mathrm{Hom}(E, E))$. Let $f \in GL(r, \mathbb{C})$- invariant "k-form",then

(1)Let H_α, H_β be the curvature forms of E in different trivialization, then $f(H_\alpha) = f(H_\beta)$, so we get a globally defined $2k$-form.

assume $H_\alpha = gH_\beta g^{-1}$, then

$$f(H_\alpha) = f(gH_\beta g^{-1}) = f(H_\beta)$$

(2) we also have

$$\mathrm{d}f(H) = 0$$

locally , $H = H_\alpha = \mathrm{d}a_\alpha + A_\alpha \wedge A_\alpha$,then

$$\mathrm{d}f(H) = \mathrm{d}f(H_\alpha, H_\alpha, ..., H_\alpha) = \sum_{i=1}^{k} f(H_\alpha, ..., \underbrace{\mathrm{d}H_\alpha,}_{i} ..., \alpha)$$

$$= \sum_{i=1}^{k} f(H_\alpha, ..., \mathrm{d}A_\alpha \wedge A_\alpha - A_\alpha \wedge \mathrm{d}A_\alpha, ..., H_\alpha)$$

Fact:(in Riemannian geometry) For any $x \in X$, we always can find a local frame s.t. $A_\alpha(x) = 0$. so, choose this frame,

$$\mathrm{d}f(H) = 0$$

So, $[f(H)] \in H^{2k}(X, \mathbb{C})$

(3) Claim : the class $[f(H)]$ is independent of the choice of the connections D_E.

Let D_0, D_1 be two connections, consider

$$D_t = (1-t)D_0 + tD_1$$

$t \in [0, 1]$, curvature H_t

Fact: $\alpha := A_1 - A_0$ is globally defined, and in $C_1^\infty(X, \mathrm{Hom}(E, E))$.

Fact:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(H_t) = k\mathrm{d}f(\alpha, H_t, H_t, ..., H_t)$$

So,

$$f(H_1) - f(H_0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(H_t)\mathrm{d}t = \mathrm{d}\int_0^1 f(\alpha, H_t, H_t, ..., H_t)\mathrm{d}t$$

So,

$$[f(H_1)] - [f(H_0)]$$

定义 4.3.2. *the k-th Chern class of E*

$$c_k(E) := [f_k(\Theta_E)] \in H^{2k}(X, \mathbb{C})$$

Recall: Chern Class

X complex manifold, $E \to X$ is a smooth complex vector bundle of rank r. D is a connection, curvature $\Theta(D) \in C_2^\infty(X, \mathrm{Hom}(E, E))$.

linear algebra:

$$\det(I + \frac{i}{2\pi} tM) = I + tf_1(M) + t^2 f_2(M) + \cdots + t^r f_r(M)$$

Chern class $\{f_k(\Theta)\} \in H_{DR}^{2k}(X, \mathbb{C})$ is independent of choice of connection.

Today:

Special case: E is a complex line bundle. Let D_0 be a connection on E, locally $D_0 e = A_0 e$, A_0 is 1-form. curvature

$$\Theta(D_0) = D_0^2 = \mathrm{d}A_0 + A_0 \wedge A_0 = \mathrm{d}A_0$$

so, curvature is d-exact, so $\mathrm{d}\Theta(D_0) = 0$.

$$\det(I + \frac{i}{2\pi} tM) = I + \frac{i}{2\pi} tM$$

so, the first Chern class of line bundle is

$$c_1(E) = \{\frac{i}{2\pi}\Theta(D_0)\}$$

Let D_1 be another connection, locally $D_1 e = A_1 e$, so $\Theta(D_1) = \mathrm{d}A_1$.so,

$$\Theta(D_1) - \Theta(D_0) = \mathrm{d}(A_1 - A_0)$$

where

$$A_1 - A_0 \in C_1^\infty(X, \mathrm{Hom}(E, E))$$

(when E is line bundle ,$\mathrm{Hom}(E, E) \cong E^* \otimes E$ is trivial bundle)

so, $A_1 - A_0$ is a globally defined smooth function on X. So,

$$\{\Theta(D_1)\} = \{\Theta(D_0)\} \in H^2(X, \mathbb{C})$$

independent of the choice of connection.

4.4 Hermite 向量丛

定义 4.4.1. *a complex vector bundle $E \to X$ of rank r is called a Hermitian vector bundle, if we have an inner product on E, i.e. locally, consider a local frame $\{e_1, ..., e_r\}$, we have*

$$\{e_i(x), e_j(x)\} = h_{ij}(x)$$

s.t. $(h_{ij}(x))$ is a positive definite Hermitian matrix depending smoothly on x.

注记 4.4.2. *For any complex vector bundle, Hermitian structure always exists.*

证明与黎曼几何类似。（黎曼度量的存在性）

定义 4.4.3. *(Hermitian connection)*

A connection D on E is called Hermitian, if

$$\mathrm{d}\{e_i, e_j\} = \{De_i, e_j\} + \{e_i, De_j\}$$

More generally, let $t \in C_p^\infty(X, E)$, $s \in C_q^\infty(X, Y)$,

$$\mathrm{d}\{s, t\} = \{\mathrm{d}t, s\} + (-1)^p\{t, Ds\}$$

性质 4.4.4. *D is a Hermitian connection ,then the curvature*

$$\Theta(D)^* = -\Theta(D)$$

(where $(-)^*$ is conjugate transpose of matrix)

it means that, $i\Theta(D) \in C_2^\infty(X, \mathrm{Herm}(E, E))$

证明.

$$0 = \mathrm{d}^2\{e_i, e_j\} = \mathrm{d}\{De_i, e_j\} + \mathrm{d}\{e_i, De_j\}$$
$$= \{D^2e_i, e_j\} - \{De_i, De_j\} + \{De_i, De_j\} + \{e_i, D^2e_j\} = \{(\Theta + \Theta^*)e_i, e_j\}$$

□

注记 4.4.5. *E is a Hermitian line bundle, D is a Hermitian connection, then $i\Theta(D)$ is a real 2-form , $c_1(E) \in H^2(X, \mathbb{R})$.*

(Chern connection)

定义 4.4.6. *Let X be a complex manifold. D' is called a connection of type $(1,0)$ on E, if for any section $s \in C^\infty_{p,q}(X, E)$, we have $D's \in C^\infty_{p+1,q}(X, E)$.*

A connection D'' is called a connection of type $(0,1)$, if ... $D''s \in C^\infty_{p,q+1}(X, E)$.

注记 4.4.7. *Let $E \to X$ be a vector bundle. Let D be a connection on E, locally*

$$Ds \xrightarrow{\sim} \mathrm{d}\sigma + A \wedge \sigma$$

$$\mathrm{d}\sigma = \partial\sigma + \bar{\partial}\sigma$$

so,let A' be the $(1,0)$-part of A,...,

$$Ds = \partial\sigma + A' \wedge \sigma + (\bar{\partial}\sigma + A'' \wedge \sigma) =: D's + D''s$$

性质 4.4.8. *E:Hermitian vector bundle, D is a Hermitian connection, locally, take a C^∞-frame $e_1, ..., e_r$ which is orthonomal (i.e. $\{e_i(x), e_j(x)\} = \delta_{ij}$), then the connection coefficient $A = A' + A''$ satisfies*

$$(A')^* = -A''$$

($\iff$ $\bar{(}iA) = iA$)

证明. because

$$0 = \mathrm{d}e_i, e_j = \{De_i, e_j\} + \{e_i, De_j\} = \{a_i^k e_k, e_j\} + \{e_i, a_j^l e_l\} = a_i^j + \overline{a_j^i}$$

so, $A^* = -A$. □

推论 4.4.9. *$E \to X$ is a Hermitian vector bundle, D_0'' is a connection of type $(0,1)$ on E. Then exists a unique Hermitian connection D such that $D'' = D_0''$.*

证明. Let $A'' = A_0''$ and $A' = -(A_0'')^* \rightsquigarrow A = A' + A''$, and D is given by A. □

Let $E \to X$ is a holomorphic Hermitian vector bundle, observe that $\overline{\partial}$ defines a connection of type $(0,1)$ on E(check!)

assume E is a holomorphic line bundle, take a section $s \in C_p^\infty(X, E)$, i.e. we have a family of p-forms (s_α) such that $s_\alpha = g_{\alpha\beta} s_\beta$ where $g_{\alpha,\beta}$ is the holomorphic transition matrix.

$$\overline{\partial} s \xrightarrow{\sim} \overline{\partial} s_\beta$$

then

$$\overline{\partial} s_\alpha = g_{\alpha,\beta} \overline{\partial} s_\beta$$

(so, $\overline{\partial}$ is a connection of $(0,1)$)

this connection is called the canonical connection of type $(0,1)$.

定义 4.4.10. *Let $E \to X$ holomorphic Hermitian vector bundle, the connection D on E is called Chern connection if*

$$D'' = \overline{\partial}$$

Curvature of Chern connection

$E \to X$ is holomorphic Hermite vector bundle , D is the Chern connection, Locally let $\{e_1, ..., e_r\}$ be a holomorphic frame, and two local sections

$$s, t \in C^\infty(\Omega, E)$$

where

$$s = \sum_{i=1}^r \sigma_i e_i$$

$$t = \sum_{i=1}^r t_i e_i$$

Since D is Hermitian ,

$$\mathrm{d}\{s,t\} = \mathrm{d}((\sigma_1, ..., \sigma_r) H \begin{pmatrix} t_1 \\ \vdots \\ t_r \end{pmatrix}) = (\mathrm{d}\sigma)^T H t + \sigma^T (\mathrm{d}H) t + \sigma^T H \mathrm{d}(t)$$

so, we have

$$\{Ds, t\} + \{s, Dt\} = (\mathrm{d}\sigma + \overline{H}^{-1} \partial \overline{H} \wedge \sigma)^T \wedge H\bar{t} + \sigma^T \wedge H \overline{(\mathrm{d}t + \overline{H}^{-1} \partial \overline{H} \wedge t)}$$

so ,

$$Ds = \mathrm{d}\sigma + \overline{H}^{-1} \partial \overline{H} \wedge \sigma$$

$$D's = \partial\sigma + \overline{H}^{-1}\partial\overline{H}\wedge\sigma = \overline{H}^{-1}\partial(\overline{H}\sigma)$$
$$D''s = \overline{\partial}\sigma$$

so,

$$(D')^2 s = \overline{H}^{-1}\partial(\overline{H}(\overline{H}^{-1}\partial(\overline{H}\sigma))) = \cdots = 0$$

$$(D'')^2 s = \cdots = 0$$

So we have

$$\Theta(D) = (D' + D'')^2 = D'D'' + D''D'$$

Locally ,

$$\Theta s = D'D''s + D''D's = \overline{H}^{-1}\partial(\overline{H}\overline{\partial}\sigma) + \overline{\partial}(\overline{H}^{-1}\overline{\partial}(\overline{H}\sigma)) = \cdots = \overline{H}^{-1}\partial\overline{H}\wedge\overline{\partial}\sigma + \overline{\partial}(\overline{H}^{-1})\sigma$$
$$= \overline{\partial}(\overline{H}^{-1}\partial\overline{H})\sigma$$

So, Chern curvature

$$\Theta_D = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})$$

Last time: $E \to X$ is a holomorphic vector bundle with a Hermitian metric H. Then there is a unique connection D_Es.t. ... called Chern connection.

Curvature of Chern Connection:

$$\Theta(D_E) = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})$$

so,

$$i\Theta(D_E) \in C^{\infty}_{1,1}(X, \mathrm{Hom}(E,E))$$

例子 4.4.11. *(Special case: E is a holomorphic line bundle)*

locally, let e be ha holomorphic frame, $\langle e, e\rangle = h$ is the metric. then,

$$\Theta = \overline{\partial}(h^{-1}\partial h) = \overline{\partial}\partial\log h$$

so,

$$i\Theta(E) = -i\partial\overline{\partial}\log h$$

if $h = e^{-2\varphi}$ where φ is a smooth function, then

$$i\Theta(E) = 2i\partial\overline{\partial}\varphi = 2\sqrt{-1}\sum_{k,l}\frac{\partial^2\varphi}{\partial z_k\partial\overline{z_l}}\mathrm{d}z_k\wedge\mathrm{d}\overline{z_l}$$

Question: let s be a local holomorphic section of E,

$$-i\partial\overline{\partial}\log|s|_h^2 = ?$$

(Hint:$\frac{i}{\pi}\partial\overline{\partial}\log z = ?$ 单复变，按分布意义下求导. 等于狄拉克测度 2333333) 可能是期末题目？

例子 4.4.12. *$\mathcal{O}(-1)$ on $\mathbb{C}P^n$, tautological line bundle. (Recall: $\mathbb{C}P^n$ is a compact complex manifold with holomorphic charts*

$$\Omega_j := \{[z_0; z_1; ...; z_n] | z_j \neq 0\} \to \left(\frac{z_0}{z_j}, \cdots, \hat{1}, \cdots \frac{z_n}{z_j}\right) \in \mathbb{C}^n$$

)

Let V be a complex vector space, $\dim_{\mathbb{C}} V = n+1$. Denote the projective space by

$$\mathbb{P}(V) = (V \setminus \{0\})/\mathbb{C}^*$$

Let $\underline{V} := \mathbb{P}(V) \times V$ be the trivial vector bundle, define

$$\mathcal{O}(-1) := \{([x], \xi) | \xi \in \mathbb{C} \cdot x\}$$

性质 4.4.13. *$\mathcal{O}(-1)$ is a holomorphic line bundle on $\mathbb{P}(V)$.*

证明. $\mathcal{O}(-1)|_{\Omega_j}$ has a non-vanishing holomorphic section $\mathcal{E}_j$ defined by

$$\mathcal{E}_j([x]) = \frac{x}{x_j}$$

for $0 \leq j \leq n$. □

Assume V has a Hermitian inner product, then $\mathcal{O}(-1)$ has an Hermitian structure induced from V.

Let $e_0, ..., e_n$ be an orthonormal basis of V, then $\mathcal{O}(-1)|_{\Omega_0}$ has a non-vanishing holomorphic section:

$$\mathcal{E}_0(z_1, ..., z_n) = e_0 + z_1 e_1 + ... + z_n e_n$$

where

$$\Omega_0 = \{[1; z_1; ...; z_n] | z_j \in \mathbb{C}\} \cong \mathbb{C}^n$$

then,

$$|\mathcal{E}_0|_h^2 = 1 + |z_1|^2 + ... + |z_n|^2$$

so the Chern curvature of $\mathcal{O}(-1)$ on Ω_0 is given by

$$\Theta = \bar{\partial}\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2)$$

Denote $\mathcal{O}(1) := \mathcal{O}(-1)^*$, then

$$\Theta(\mathcal{O}(1)) = -\bar{\partial}\partial \log(1 + |z_1|^2 + ... + |z_n|^2)$$

on Ω_0.

$$i\Theta(\mathcal{O}(1)) = i\partial\overline{\partial}\log(1+|z_0|^2+...+|z_n|^2) = \sqrt{-1}\sum_{1\le k,l\le n} c_{k,l}\mathrm{d}z_k\wedge \mathrm{d}\overline{z_l}$$

Exercise: (c_{kl}) is a positive definite Hermitian matrix.

"Fubini-Study metric" on $\mathbb{P}(V).\mathcal{O}(1)$ is "hyperplane line bundle of $\mathbb{P}(V)$".

Exercise: calculate

$$\int_{\mathbb{P}(V)}\left(\frac{i}{2\pi}\Theta(\mathcal{O}(1))\right)^{\wedge n} = ?$$

(Hint: $\mathbb{P}(V)\setminus\Omega_0$ is a zero-measure set)

$E\to X$: holomorphic line bundle, D_E is a Chern connection.

$$c_1(E) = \{\frac{i}{2\pi}\Theta(D_E)\} \in H^2_{DR}(X,\mathbb{R})$$

Exercise: 60% 的概率出现于期末试题

Consider the sequence

$$0\to\mathbb{Z}\to\mathcal{O}\xrightarrow{e^{2\pi i *}}\mathcal{O}^*\to 0$$

it induces a long exact sequence

$$\cdots\to H^1(X,\mathcal{O})\to H^1(X,\mathcal{O}^*)\xrightarrow{\delta} H^2(X,\mathbb{Z})\to H^2(X,\mathcal{O})\to\cdots$$

prove: Consider E as an element of $H^1(X,\mathcal{O}^*)$, then the image of $\delta(E)$ in $H^2(X,\mathbb{R})\cong H^2_{DR}(X,\mathbb{R})$ is $c_1(E)$.

Exercise: E is a holomorphic line bundle, denote $\theta := \frac{i}{2\pi}\Theta(D_E)$ real $(1,1)-form$, where D_E is Chern connection with a metric h. Prove: for any smooth function $f\in C^\infty(X,\mathbb{R})$, there exists a Hermitian metric h_f s.t.

$$\frac{i}{2\pi}\Theta_{E,h_f} = \theta + i\partial\overline{\partial}f$$

第 5 章　L^2 Hodge 理论

5.1　向量丛上的微分算子

Differential operators on vector bundles.

Let X is a (connected) smooth manifold of ($\mathbb{R}$-)dimension n. E, F : $\mathbb{K}$-vector bundle of rank r, r' respectively.

定义 5.1.1. *a linear differential operator of degree k from E to F is a $\mathbb{K}$-linear map*

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

$$u \mapsto Pu$$

locally given by

$$Pu(x) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x)$$

where $a_\alpha(x) = (a_{afa,\lambda\mu}(x))$ be a $r' \times r$ matrix.

$$u(x) = (u_1(x), ..., u_r(x))^T$$

Let $t \in \mathbb{K}$, $f \in C^\infty(M, \mathbb{K})$, $u \in C^\infty(M, E)$, then

$$e^{-tf(x)} P(e^{tf(x)} u(x)) = t^k \sigma_P(x, \mathrm{d}f(x)) u(x) + \text{terms } c_j(x)^{t_j} \quad (j < k)$$

定义 5.1.2.

$$\sigma_P : T^*M \to \mathrm{Hom}(E, F)$$

is called the principal symbol of P, which is a polynomial on T^*M.

locally,

$$\sigma_P(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

$(\xi^\alpha := \xi_1^{\alpha_1}...\xi_n^{\alpha_n})$

例子 5.1.3. *Consider* $\mathrm{d} : C^\infty(M,\mathbb{K}) \to C^\infty(M,T^*M)$. *then*

$$\mathrm{d}u = \sum_{j=1}^{n} \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \frac{\partial u}{\partial x^i}$$

i.e.

$$\sigma_d(x,\xi) = \sum_{j=1}^{n} \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \xi_j$$

定义 5.1.4. *P is called elliptic, if* $\forall x \in M, \xi \in T_x^*M \setminus \{0\}$,

$$\sigma_P(x,\xi) \in \mathrm{Hom}(E_x, E_x)$$

is injective.

For example, d is elliptic.

L^2-inner product

Let M be an oriented C^∞-manifold with a smooth volume form, locally

$$\mathrm{d}V(x) = \gamma(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

$\gamma(x) > 0$. Assume E has a Euclidean(or Hermitian) structure...

Let $u, v \in C^\infty(M,E)$,define

$$\langle\langle u,v\rangle\rangle := \int_M \langle u,v\rangle \mathrm{d}V(x)$$

define $L^2(M,E) :=$ space of sections with measurable coefficients with are L^2 w.r.t $\langle\langle,\rangle\rangle$.

定义 5.1.5. *Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be a differential operator, E,F have Euclidean (or Hermitian) structure, then there exists unique differential operator*

$$P^* : C^\infty(M,F) \to C^\infty(M,E)$$

s.t.

$$\langle\langle Pu, v\rangle\rangle = \langle\langle u, P^*v\rangle\rangle$$

for all u,v s.t. $Supp u \cap Supp v \subset\subset M$(relative compact...)

P^* is called the formal adjoint of P.

证明. Existence: Assume that $SuppU, Suppv \subset\subset$ some coordinate chart Ω with coordinates $(x_1,...,x_n)$, then

$$\ll Pv, u \gg = \int_\Omega \sum_{\alpha,\lambda,\mu} a_{\alpha,\lambda\mu}(x) D^\alpha u_\mu(x) \overline{v_\lambda(x)} \gamma(x) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

integration by parts, it

$$= \int_\Omega \sum_{\alpha,\lambda,\mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha(\gamma(x)\overline{a_{\alpha,\lambda\mu}} v_\lambda(x))} \mathrm{d}x_1..\mathrm{d}x_n$$

Locally,

$$P^*v = \sum_{|\alpha|\le k} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha (\gamma(x) \overline{a_\alpha(x)}^T v(x))$$

Uniqueness: use the density of C^∞-section with compact support in $L^2(M,-)$. □

推论 5.1.6. *If $\sigma_P(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$,then*

$$\sigma_{P^*} = (-1)^k \overline{\sigma_P(x,\xi)}^T$$

推论 5.1.7. *If rank E = rank$F$, P is differential operator, then P^* is elliptic $\iff$ P^* is elliptic.*

5.2 椭圆算子的基本性质

Fundamental results of elliptic operators

M is a compact (oriented) C^∞-manifold, $\dim_{\mathbb{R}} M = n$, with a smooth volume form $\mathrm{d}V$.

E is an Hermite vector bundle, $rank_{\mathbb{C}} E = r$.

Sobolev space:$W^k(M,E) :=$ the space of section $s : M \to E$ whose derivations up to order $= k$, := the completion of space of smooth sections w.r.t W^k-norm.

$(\Omega_j)_{j\in I}$: a finite open covering of M, $E|_{\Omega_j}$ trivial, Let $(\rho_j)_{j\in I}$ be a partition of unity w.r.t. $(\Omega_j)_{j\in I}$, s.t. $\sum_j \rho_j^2 = 1$. locally, choose an orhtonormal frame $(e_{j,\lambda})_{1\le\lambda\le r}$ on Ω_j, then $u = \sum_{\lambda=1}^r u_{j,\lambda} e_{j,\lambda}$ on Ω_j. Define

$$||u||_k^2 := \sum_{j,\lambda} ||e_j u_{j,\lambda}||_k^2$$

where

$$||e_j u_{j,\lambda}||_k^2 := \int_{\Omega_j} \sum_{|\alpha|\le k} |D^\alpha(e_j u_{j,\lambda})|^2 \mathrm{d}V(x)$$

注记 5.2.1. *On a compact manifold, the equivalence of class of* $||\cdot||_k$ *is independent of the choice of : partition of unity, local trivialization, holomorphic covering...*

引理 5.2.2. *(Sobolev lemma)*

For $k > l + \frac{n}{2}$*, then we have*

$$W^k(M,E) \subseteq C^l(M,E)$$

引理 5.2.3. *(Rellich lemma)*

For any $k \in \mathbb{Z}_{\ge 0}$*, the inclusion*

$$W^{k+1}(M,E) \hookrightarrow W^k(M,E)$$

is a compact operator.

引理 5.2.4. *(Garding inequality)*

If

$$P : C^\infty(M,E) \to C^\infty(M,F)$$

is elliptic, and rankE =rank F, $\tilde{P}$ *:the extension of* P *to sections with distribution coefficients, then : for all* $u \in W^0(M,E)$*, if* $\tilde{P}u \in W^k(M,F)$*, then* $u \in W^{k+d}(M,E)$*, where* $d = \deg P$*, and*

$$||u||_{k+d} \le C_k \left(||\tilde{P}u||_k + ||u||_0\right)$$

where C_k *depending on* k, M*.*

证明. Reference: `Kodaira: deformation of complex structures (Appendix)` □

推论 5.2.5. *If $u \in \ker \tilde{P} \cap W^0(M, E)$, then $u \in C^\infty(M, E)$.*

引理 5.2.6. *(Finiteness theorem)*

Setting M be a compact manifold, rank$E$ =rank$F$,

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

elliptic,then:

(1) $\ker P$ is of finite dimension

(2) $P(C^\infty(M, E))$ is closed and of finite codimension in $C^\infty(M, F)$. If P^* is the formal adjoint of P, then $\exists$ decomposition

$$C^\infty(M, F) = P(C^\infty(M, E)) \oplus \ker P^*$$

which is orthogonal in $W^0(M, F) = L^2(M, F)$

证明. 椭圆算子的一般结果，分析的东西 233333333. 可以参考小平邦彦复流形与复结构形变的附录。 □

5.3 紧黎曼流形的 Hodge 理论

Hodge theory in compact Riemannian manifold

Hodge star operator.

M compact Riemannian manifold, $\dim_{\mathbb{R}} = n$, E is a Hermitian vector bundle. Assume $(\xi_1, ..., \xi_n), (e_1, ..., e_n)$ be orthonormal frame of TM, E on some local chart Ω, denote $(\xi_1^*, ..., \xi_n^*), (e_1^*, ..., e_n^*)$ be the co-frame of T^*M, T^*E.

$\wedge^\bullet T^*M$ is endowed with an inner product frame from TM. locally,

$$\langle u_1 \wedge \cdots \wedge u_p, u_1 \wedge \cdots \wedge u_p \rangle := \det(\langle u_i, v_j \rangle)$$

for $u_i, v_j \in T^*M$. Then , get an inner product on $\wedge^p T^*M$.

Assume

$$U = \sum_{\substack{|I|=p \\ i_1 \leq \ldots \leq i_p}} u_I \xi_I^*$$

$$V = \sum_{\substack{|I|=p \\ i_1 \leq \ldots \leq i_p}} v_I \xi_I^*$$

be p-forms, then

$$\langle u, v \rangle = \sum_{|I|=p} u_I v_I$$

i.e. $\{\xi_T^*\}$ is an orthonormal basis of $\wedge^p T^* M$.

$\wedge^* T^* M \otimes E$ has an inner product induced from $\wedge^* T^* M, E$,

定义 5.3.1. *the Hodge star operator*

$$^* : \wedge^p T^* M \to \wedge^{n-p} T^* M$$

is defined by

$$u \wedge *v = \langle u, v \rangle \mathrm{d}V$$

Locally, let

$$U = \sum_{|I|=p} u_I \xi_I^*, \ V = \sum_{|I|=p} v_I \xi_I^*$$

assume

$$*V = \sum_{|J|=n-p} a_J \xi_J^*$$

then

$$U \wedge * \sum u_I a_{I^c} \xi_I^* \wedge \xi_{I^c}^* = \sum u_I a_{I^c} \varepsilon(I, I^c) \xi_1^* \wedge \cdots \wedge \xi_n^*$$

$$\langle u, v \rangle \mathrm{d}V = \sum_{|I|=p} u_I v_I \xi_1^* \wedge \cdots \wedge \xi_n^*$$

so, we have

$$*V = \sum_{|I|=p} \varepsilon(I, I^c) V_I \xi_{I^c}^* \in \bigwedge^{n-p} T^* M$$

定义 5.3.2.

$$* : \bigwedge^p T^* M \otimes E \to \bigwedge^{n-p} T^* M \otimes E$$

is defined by

$$\{s, *t\} := \langle s, t \rangle \mathrm{d}V$$

Locally, assume

$$t = \sum_{\substack{|I|=p \\ 1\le\lambda\le r}} t_{I,\lambda}\xi_I^* \otimes e_\lambda$$

then

$$*t = \sum_{\substack{|I|=p \\ 1\le\lambda\le r}} \varepsilon(I,I^c)t_{I,\lambda}\xi_{I^c}^* \otimes e_\lambda$$

定义 5.3.3.

$$\# : \bigwedge^p T^*M \otimes E \to \bigwedge^{n-p} T^*M \otimes E^*$$

is defined by: for any $s,t \in \bigwedge^p T^*M \otimes E$,*such that*

$$s \wedge \#t := \langle s,t\rangle \mathrm{d}V$$

wedge product+ pairing of $E^* \times E \to \mathbb{C}$.

Locally: assume

$$t = \sum_{\substack{|I|=p \\ 1\le\lambda r}} t_{I,\lambda}\xi_T^* \otimes e_\lambda$$

then,

$$\#t = \sum_{|I|=p,\lambda} \varepsilon(I,I^c)t_{I,\lambda}\xi_c^* I \otimes e_\lambda^*$$

性质 5.3.4.

$$*^2 = (-1)^{p(n-1)} \quad on \ \bigwedge^p T^*M \otimes E$$

$$\#^2 = (-1)^{p(n-1)} \quad on \ \bigwedge^p T^*M \otimes E$$

（正负号对吗？）

Recall: For all $s,t \in C^\infty(M, \bigwedge^p T^*M \otimes E)$, we have an inner product

$$\langle\langle s,t\rangle\rangle := \int_M \langle s,t\rangle \mathrm{d}V$$

定理 5.3.5. *Let* D_E *be an Hermite connection on* E, *acting on* $\bigwedge^p T^*M \otimes E$, *then*

$$D_E^* := (-1)^{np+1} * D_E *$$

where D_E^* *is the formal adjoint of* D_E.

证明. Let $s \in C^\infty(M, \bigwedge^p T^*M \otimes E)$ and $t \in C^\infty(M, \bigwedge^{p+1} T^*M \otimes E)$. then

$$\langle\langle D_E s, t\rangle\rangle = \int_M \langle D_E s, t\rangle \mathrm{d}V = \int_M \{D_E s, *t\}$$

Since D_E is Hermitian ,by defination ,

$$\mathrm{d}\{s, *t\} = \{D_E s, t\} + (-1)^p \{s, D_E(*t)\}$$

so,

$$\langle\langle D_E s, t\rangle\rangle = \int_M \mathrm{d}\{s, *t\} + (-1)^{p+1}\{s, D_E * t\} = (-1)^{p+1}(-1)^{p(n_1)} \int_M \{s, *(*D_E * t)\} = \langle\langle s, D_E^* t\rangle\rangle$$

so,

$$D_E^* t = (-1)^{np+1} * D_E *$$

□

定义 5.3.6.

$$\triangle_E = D_E D_E^* + D_E^* D_E : C^\infty(M, \bigwedge^p T^*M \otimes E) \to C^\infty(M, \bigwedge^p T^*M \otimes E)$$

例子 5.3.7. *Let* $M = \mathbb{R}^n$, $g = \sum_{i=1}^n \mathrm{d}x_i^2$, $E = M \times \mathbb{C}$ *trivial line bundle with* $D_E = \mathrm{d}$. *then*

$$\triangle_E u = (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})u = -\sum_{i=1}^n \left(\sum_{|I|=p} \frac{\partial^2 u_I}{\partial x_I^{2^2}} \mathrm{d}x_I \right)$$

where

$$u = \sum_{|I|=p} u_I \mathrm{d}x_I$$

性质 5.3.8. $\triangle_E$ *is a self-adjoint elliptic operator. (i.e.* $\triangle_E^* = \triangle_E$*)*

证明. $\triangle_E^* = \triangle_E$ be definition.

note that

$$e^{-tf} D_E(e^{tf} s) = t\mathrm{d}f \wedge s + D_E s$$

so,

$$\sigma_{D_E}(x, \xi)s = \xi \wedge s$$

$$\sum_{D_E^*} = -\overline{\sigma_{D_E}}^T$$

$$\sigma_{D_E^*}(x,\xi)s = -\tilde{\xi} \lrcorner s$$

where $\tilde{\xi}$ be the vector field dual to ξ. □

定义 5.3.9.

$$\triangle_E = D_E D_E^* + D_E D_E^* : C^\infty(M, \bigwedge^p T^*M \otimes E) \to C^\infty(M, \bigwedge^p T^*M \otimes E)$$

so,

$$\sigma_{\triangle_E}(x,\xi)s = \left(\sigma_{D_E}\sigma_{D_E^*}(x,\xi) + \sigma_{D_E^*}\sigma_{D_E}(x,\xi)\right)s$$

so, $\sigma_{\triangle_E}$ is injective if $\xi \neq 0$, so $\triangle_E$ is elliptic.

Harmonic forms and Hodge isomorphism.

定义 5.3.10. *u is called harmonic if $\triangle_{\mathrm{d}} u = 0$.*

定理 5.3.11. *M is a compact Riemannian manifold, then de Rham cohomology*

$$H^p_{DR}(M,\mathbb{R}) \cong \ker(\triangle_d : C^\infty(M, \bigwedge^p T^*M))$$

证明. $\triangle_d$ self-adjoint elliptic, so by general result for elliptic operator,

$$C^\infty(M, \bigwedge^p T^*M) = \operatorname{Im}\triangle_d \oplus \ker\triangle_d^* = \operatorname{Im}\triangle_d \oplus \ker\triangle_d$$

Claim:

$$\operatorname{Im}\triangle_{\mathrm{d}} = \in \mathrm{d} \oplus \operatorname{Im}\mathrm{d}^*$$

Recall $\triangle_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, so

$$\operatorname{Im}\triangle_{\mathrm{d}} \subseteq \operatorname{Im}\mathrm{d} \oplus \in \mathrm{d}^*$$

on the other hand,

$$\operatorname{Im}\mathrm{d} \oplus \operatorname{Im}\mathrm{d}^* \subseteq (\ker\triangle_{\mathrm{d}})^\perp = \operatorname{Im}\triangle_{\mathrm{d}}$$

so,

$$\operatorname{Im}\triangle_{\mathrm{d}} = \operatorname{Im}\mathrm{d} \oplus \operatorname{Im}\mathrm{d}^*$$

so,

$$C^\infty(M, \bigwedge^p T^*M) = \operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^* \oplus \ker \triangle_{\mathrm{d}}$$

so,

$$H^p_{DR}(M, \mathbb{R}) = \frac{\operatorname{Im} \mathrm{d} \oplus \ker \triangle_{\mathrm{d}}}{\operatorname{Im} \mathrm{d}} = \ker \triangle_{\mathrm{d}}$$

□

推论 5.3.12.

$$\dim H^p_{DR}(M, \mathbb{R}) = \dim \ker \triangle_{\mathrm{d}} < +\infty$$

注记 5.3.13. *Consider*

$$u \mapsto \int_M (\langle u, u \rangle + \langle \mathrm{d}u, \mathrm{d}u \rangle + \langle \mathrm{d}^* u, \mathrm{d}^* u \rangle) \mathrm{d}V$$

这个泛函的变分是什么鬼?

Harmonic forms and Hodge isomorphism

Recall: M is a compact Riemann manifold,

$$\mathrm{d} : C^\infty(M, \bigwedge^* T^*M) \to C^\infty(M, \bigwedge^{*+1} T^*M)$$

adjoint d^*,

$$\triangle_d = dd^* + d^*d$$

is a self-adjoint elliptic operator.

Hodge decomposition:

$$C^\infty(M, \bigwedge^p T^*M) = \ker \triangle_d \oplus \operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^*$$

$$\mathcal{H}^p(M, \mathbb{R}) := \ker \triangle_d \quad \text{finite dimension}$$

$$\mathcal{H}^p(M, \mathbb{R}) \cong H^p_{DR} \cong H^p(M, \mathbb{R})$$

(Hodge isomorphism, and, de Rham-Weil)

Poincare duality

定理 5.3.14. *The pairing*

$$H^p_{DR}(M, \mathbb{R}) \times H^{n-p}_{DR}(M, \mathbb{R}) \to \mathbb{R}$$

$$(s, t) \mapsto \int_M s \wedge t$$

(is well defined) is non-degenerated. In particular, $H^p_{DR}(M, \mathbb{R})^* \cong H^{n-p}_{DR}(M, \mathbb{R})$

证明. the pairing factors through the pairing on

$$\mathcal{H}^p(M,\mathbb{R}) \times \mathcal{H}^{n-p}(M,\mathbb{R}) \to \mathbb{R}$$

$$(s,t) \mapsto \int_M s \wedge t$$

need to verify:(1) it is independent of the choice of representations.(Easy, check) (2) Pairing $\mathcal{H}... \times \mathcal{H}...$ is non-degenerated..

claim(Exercise): Hodge star $*$ s.t. $*\triangle_{\mathrm{d}} = \triangle_{\mathrm{d}}*$.

so, s is a harmonic p-form $\iff$ $*s$ is a harmonic $(n-p)$-form.

note that

$$s \wedge *s = \langle s,s\rangle \mathrm{d}V = \int_M s \wedge *s = \int_M \langle s,s\rangle \mathrm{d}V = ||s||^2$$

□

推论 5.3.15.

$$\dim \mathcal{H}^p(M,\mathbb{R}) = \dim \mathcal{H}^{n-p}(M,\mathbb{R})$$

Generalization to flat bundle. M is a compact Riemannian manifold, $\dim_{\mathbb{R}} M = n$, $E \to M$ is a complex Hermitian vector bundle.

定义 5.3.16. *$E \to X$ is called flat, if it admit a connection D_E s.t.*

$$D_E^2 = 0$$

注记 5.3.17. *E is flat $\iff$ E is given by a representation*

$$\pi_1(M) \to GL(r,\mathbb{C})$$

(我们不证)

Consider the complex :

$$(C^\infty(M, \bigwedge^* T^*M \otimes E), D_E)$$

$$\rightsquigarrow H^p_{DR}(M,E) := \frac{\ker D_E}{\operatorname{Im} D_E}$$

Exercise: we have decomposition

$$C^\infty(M, \bigwedge^p T^*M \otimes E) = \ker \triangle_{D_E} \oplus \operatorname{Im} D_E \oplus \operatorname{Im} D_E^*$$

$$H^p_{DR}(M,E) \cong \ker \triangle_{D_E}$$

and the pairing

$$H^p_{DR}(M,E)\times H^{n-p}_{DR}(M,E^*)\to\mathbb{C}$$
$$(s,t)\mapsto\int_M s\wedge t$$

is non-degenerate..

以上是实的 Hodge 理论。

5.4 Kähler 流形

定义 5.4.1. *Let X be a complex manifold,* $\dim_{\mathbb{C}} X=n$*, X is called a Hermitian manifold, if X has a Hermitian metric, i.e. locally* $h(z):=\sum_{1\le j,k\le n}h_{jk}(z)\mathrm{d}z_j\otimes\mathrm{d}\bar{z}_k$*, where* (h_{jk}) *is positive definition Hermitian matrix.*

Check:the positivity of h is independent of the choice of holomorphic local coordinate

Rmk: Any complex manifold has a Hermitian metric...(Exercise)

Fundamental $(1,1)$-form associated to $h(z)$ is defined by

$$\omega:=-\operatorname{Im}h=\frac{\sqrt{-1}}{2}\sum_{j,k}h_{jk}\mathrm{d}z_j\mathrm{d}\bar{z}_k$$

we also call ω is the Hermitian metric on X

Fact: ω is real (i.e. $\overline{\omega}=\omega$).

注记 5.4.2. *h is a Hermite structure on TX(holomorphic tangent bundle of X). locally,*

$$\langle\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\rangle(z)=h_{ij}(z)$$

定义 5.4.3. (X,ω) *is an Hermitian manifold, X is Kähler if* $\mathrm{d}\omega=0$.

性质 5.4.4. *Locally,* $\omega=\frac{\sqrt{-1}}{2}\sum_{jk}h_{jk}\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_k$ *is Kaehler,* $\iff$ $\partial\omega=0$ *and* $\bar{\partial}\omega=0$*, i.e.*

$$\frac{\partial h_{jk}}{\partial z_l}=\frac{\partial h_{lk}}{\partial z_j}$$

If (X,ω) *is a compact Kaehler manifold, then*

$$H^{2k}(X,\mathbb{R})\neq 0$$

证明. $d\omega = 0$, so $\omega \in H^2(M, \mathbb{R})$. Claim:

$$0 \neq \omega^k \in H^{2k}(M, \mathbb{R})$$

proof of the claim:

$$[\omega^k][\omega^{n-k}] = \int_X \omega^k \wedge \omega^{n-k} = \int_X \omega^n$$

Since ω is positive, locally

$$\omega^n = n! \det(h_{jk}) \bigwedge_{l=1}^{n} \left(\frac{\sqrt{-1}}{2} dzl \wedge d\bar{z}_l \right) > 0$$

is a volume form. So,

$$[\omega^k][\omega^{n-k}] = \int_X \omega^n > 0$$

(Using Poincare dual) □

例子 5.4.5. *(Exists a complex manifold NOT Kaehler) (Hopf Surface)*

$$X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

where discrete group $\Gamma := \{\lambda^n | n \in \mathbb{Z}\}$, $0 < \lambda < 1$ *fixed.*

Exercise: $X \cong S^1 \times S^3$ C^∞ homeomorphism.. and X is compact complex manifold.
and $H^2(X, \mathbb{R}) = H^2(S^1 \times S^3, \mathbb{R}) = 0$ by Künneth Formula...
So, X is non-Kahler...

例子 5.4.6. *Examples of Kaehler manifold)*

(1)Riemann surface must be Kaehler...(trivial)

(2)(complex torus) $X = C^n/\Gamma$, Γ *is a lattice. (this manifold may not compact...)*

$$\omega = \sqrt{-1} \sum_{j,k} h_{jk} dz_j \wedge d\bar{z}_k$$

is a Kahler metric on X *if* $(H_{jk}) > 0$, h_{jk} *are constant.*

(3) Projective space $\mathbb{C}P^n$.

$$\omega := \sqrt{-1}\Theta_h(\mathcal{O}(1))$$

locally,

$$\omega = \sqrt{-1}\partial\bar{\partial} \log(1 + |z_1|^2 + ... + |z_n|^2)$$

on Ω. *This* ω *is a Kahler metric,*

例子 5.4.7. *Let (X,ω) is a Kahler manifold, then any complex submanifold $Y \subseteq X$ is also Kahler.*

$$i : Y \hookrightarrow X$$

with the Kahler metric $i^*\omega$.

Exercise: Let $f : Y \to X$ be a holomorphic immersion, and assume X is Kahler, then Y is Kahler.

推论 5.4.8. *Any projective manifold (i.e. $X \hookrightarrow \mathbb{C}P^N$) is Kähler.*

(Algebraic Geometry.....)

性质 5.4.9. *(Equivalent definition of Kaehler metrics) a Hermitian metric ω is Kahler, iff for all $x_0 \in X$, there exists a holomorphic chart $(z_1, ..., z_n)$ centered at x_0, s.t.*

$$\omega(z) = \sqrt{-1}\sigma_{jk}\delta_{jk}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k + O(|z|^2)$$

($\Leftarrow$ is trivial...) (left to HW)

定理 5.4.10. *(Exercise)*

If (X,ω) is Kahler, then for all $x_0 \in X$, $\exists$ holomorphic chart $z_1, ..., z_n$ centered at x_0, s.t. assume

$$\omega = \sqrt{-1}h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$$

then

$$h_{lm}(z) = \delta_{lm} - \sum_{j,k} c_{jk,lm} z_j \bar{z}_k + O(|z|^3)$$

where $c_{jk,lm}$ is the coefficients of the Chern curvature tensor,

$$\Theta(TX)_x := \sum c_{jk,lm}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes (\frac{\partial}{\partial z_l})^* \otimes \frac{\partial}{\partial z_m}$$

（查书）

5.5 紧复流形上的 Hodge 理论

(X, ω) is a compact Hermitian manifold, $E \to X$ is a homomorphic Hermitian vector bundle.

$$D_E := D'_E + D''_E$$

Chern connection, $D''_E = \overline{\partial}$.

定义 5.5.1.

$$\triangle_E := D_E D_E^* + D_E^* D_E$$

$$(D'_E)^* = - * D''_E *$$
$$(D''_E)^* = - * D'_E *$$
$$\triangle'_E = D'_E (D'_E)^* + ...$$
$$\triangle''_E = ...$$

Note that $(D''_E)^2 = 0$, consider the complex

$$C^\infty(X, \bigwedge^{p,q} \otimes E) \xrightarrow{D''_E} C^\infty(X, \bigwedge^{p,q+1} \otimes E)$$
$$\rightsquigarrow H^{p,q}_{D''_E}(X, E)$$

Dolbeaut cohomology... it isom to $\ker \triangle''_E$

Hodge theory in compact complex manifold.

Let (X, ω) be a compact complex manifold of dimension n. $E \to X$ holomorphic Hermitian vector bundle, with Chern connection D_E, $D_E = D'_E + D''_E$ where $D''_E = \overline{\partial}$.

Recall: L^2 inner product: $u \in C^\infty(X \bigwedge^{p,q} \otimes E)$,

$$\langle\langle u, v \rangle\rangle := \int_X \langle u, v \rangle \mathrm{dvol}$$

Hodge star operator $*$: $u, v \in C^\infty(X, \bigwedge^{p,q} \otimes E)$,

定义 5.5.2.

$$* : \bigwedge^{p,q} \otimes E \to \bigwedge^{n-q,n-p} \otimes E$$

s.t.

$$u \wedge *v = \langle u, v \rangle \mathrm{d}vol$$

(wedge product from $\bigwedge^{p,q}$, with inner product from E)

Exercise: Take a holomorphic chart $(z_1, ..., z_n)$ s.t.

$$\omega = \sqrt{-1}\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$$

at some point p. An orthonormal frame $\{e_1, ..., e_r\}$, Let

$$u = \sum_{\substack{|I|=p\\|J|=q}} \sum_{\lambda=1}^{r} u_{IJ}\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_j \otimes e_\lambda \in \bigwedge^{p,q} \otimes E$$

WHAT IS $*u$?

Formal adjoint of D_E, D'_E, D''_E?

性质 5.5.3.

$$D_E^* = -*D_E*$$

$$(D'_E)^* = -*D''_E*$$

$$(D''_E)^* = -*D'_E*$$

定义 5.5.4.

$$\triangle_E := D_E D_E^* + D_E^* D_E$$

$$\triangle'_E := D'_E D'^*_E + D'^*_E D'_E$$

$$\triangle''_E := \cdots$$

Check: $\triangle_E, \triangle'_E, \triangle''_E$ are self adjoint, elliptic operators.

Hodge theory w.r.t. $\triangle''_E$.

定理 5.5.5. *We have a decomposition*

$$C^\infty(X, \bigwedge^{p,q} \otimes E) = \ker \triangle''_E \oplus \operatorname{Im} D''_E \oplus \operatorname{Im} D''^*_E$$

As a consequence, Dolbeault cohomology

$$H^{p,q}_{D''_E}(X, \mathbb{C}) \cong \ker \triangle''_E$$

推论 5.5.6.

$$\dim_{\mathbb{C}} H^{p,q}_{D''_E}(X,\mathbb{C}) < +\infty$$

Cohomology group

$$H^{p,q}_{D''_E}(X,\mathbb{C})$$

Ω^p:sheaf of holomorphic p-forms on X (i.e. a $(p,0)$-form φ is holomorphic if $\overline{\partial}\varphi = 0$).

$\mathcal{E}^{p,q}$:Sheaf of smooth (p,q)-forms on X.

Similarly, we have $\Omega^p(E)$ the sheaf of holomorphic p-forms with values in E,and $\mathcal{E}^{p,q}(E)$ the sheaf...smooth (p,q)-forms ...

we have an acyclic resolutions

$$0 \to \Omega^p(E) \xrightarrow{D''_E} \mathcal{E}^{p,1}(E) \xrightarrow{D''_E} \mathcal{E}^{p,2}(E) \xrightarrow{D''_E} \cdots$$

(check, it is a resolution)

By de Rham-Weil theorem,

$$H^q(X,\Omega^p(E)) \cong D^{p,q}_{D''_E}(X,\mathbb{C}) \cong \mathcal{H}^{p,q}_{D''_E}(X,\mathbb{C}) := \ker \triangle''_E$$

定理 5.5.7. *(Serre duality)*

The pairing

$$H^{p,q}_{D''_E}(X,E) \times H^{n-p,n-q}_{D''_E}(X,E^*) \to \mathbb{C}$$

$$(s,t) \mapsto \int_X s \wedge t$$

is non-degenerate

证明. Define

$$\# : \bigwedge^{p,q} \otimes E \to \bigwedge^{n-p,n-q} \otimes E^*$$

by: for $u,v \in \bigwedge^{p,q} \otimes E$,

$$u \wedge \# v := \langle u,v \rangle \mathrm{dvol}$$

Fact:

$$\triangle''_{E^*}\# = \#\triangle''_E$$

□

Remark: take $E = X \times \mathbb{C}, D_E = \mathrm{d} = \mathrm{d}' + \mathrm{d}''$,$(\mathrm{d}' = \partial, \mathrm{d}'' = \overline{\partial})$ then we have

$$\triangle' = \mathrm{d}'\mathrm{d}'^{*} + \mathrm{d}'^{*}\mathrm{d}'$$

$$\triangle'' = \cdots$$

then

$$H^{p,q}_{\mathrm{d}''}(X, \mathbb{C}) \cong \ker \triangle'' \curvearrowright C^{\infty}(X, \bigwedge\nolimits^{p,q})$$

the pairing

$$H^{p,q}(X, \mathbb{C}) \times H^{n-p,n-q}(X, \mathbb{C}) \to \mathbb{C}$$

is non-degenerate.

第 6 章　Lefschitz 分解

6.1　线性代数版本的 Lefschitz 算子

Three goals:

Kahler package

Lefschetz decomposition

Hodge-Riemann bilinear relations

Linear algebra(baby representation theory)(local case)

$\mathbb{C}^n$,

$$\omega = \sqrt{-1}\sum_{i,j} h_{ij} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$$

Kahler metric with constant coefficients.(i.e. h_{ij} is constant, (h_{ij}) is positive Hermite matrix)

W.L.O.G, by taking a linear transformation, we can assume

$$\omega = \sqrt{-1}\sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$$

记号 6.1.1. *An operator is of pure degree r if it transform a form of $\deg = k$ to as form of degree $k+r$.*

An operator ..of bi-degree (p,q) if ...$(s,t) \to (s+p, t+q)$ (in this case, degree $= p+q$)

if A, B with degree $\deg A, \deg B$,define

$$[A,B] := AB - (-1)^{\deg A \deg B} BA$$

定义 6.1.2.

$$L : \bigwedge^{p,q} \to \bigwedge^{p+1,q+1}$$
$$u \mapsto \omega \wedge u$$

is called Lefschetz operator.

Denote Λ to be the adjoint of L, adjointed by : Let $v \in \bigwedge^{p-1,q-1}$ and $u \in \bigwedge^{p,q}$

$$\langle Lv, u\rangle := \langle u, \Lambda u\rangle$$

The operator Λ is of bi-degree $(-1,-1)$.

性质 6.1.3. *If*

$$u = \sum_{\substack{|I|=p\\|J|=q}} u_{IJ}\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_j$$

then

$$Lu = \sqrt{-1}\sum_{\substack{|I|=p\\|J|=q}}\sum_{m=1}^{n} u_{IJ}\mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m \wedge \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

$$\Lambda u = \sqrt{-1}(-1)^p \sum_{\substack{|I|=p\\|J|=q}}\sum_{m=1}^{n} u_{IJ}\left(\frac{\partial}{\partial z_m}\lrcorner\,\mathrm{d}z_I\right) \wedge \left(\frac{\partial}{\partial \bar{z}_m}\lrcorner\,\mathrm{d}\bar{z}_J\right)$$

where "$\lrcorner$" is contraction.

推论 6.1.4. *(Exercise) Let*

$$\alpha = \sqrt{-1}\sum_{j=1}^{n}\alpha_j \mathrm{d}z_j \wedge \bar{z}_j$$

then,(α is a operator of bi-degree $(1,1)$)

$$[\alpha, \Lambda]u = \sum_{\substack{|I|=p\\|J|=q}}\left(\sum_{i\in I}\alpha_i + \sum_{j\in J}\alpha_j - \sum_{k=1}^{n}\alpha_k\right)u_{IJ}\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

where

$$u = \sum_{\substack{|I|=p\\|J|=q}} u_{IJ}\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

推论 6.1.5. *if $u \in \bigwedge^{p,q}$,then*

$$[L, \Lambda]u = (p+q-n)u$$

推论 6.1.6. *Denote* $B := [L, \lambda]$, *then*

$$[B, L] = 2L$$

$$[B, \Lambda] = -2\Lambda$$

证明. Take $u \in \bigwedge^{p,q}$, then

$$[B, L] = BLu - LBu = (p+q-n+2)Lu - (p+q-n)Lu = 2Lu$$

the second is similar.. □

$\mathfrak{sl}(2, \mathbb{C})$-representation

$$\mathfrak{sl}(2, \mathbb{C}) = \operatorname{span}_{\mathbb{C}} l, \lambda, b$$

where

$$l = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have

$$[l, \lambda] = b \quad [b, l] = 2l \quad [b, \lambda] = -2\lambda$$

性质 6.1.7. *There exists a natural action*

$$\rho : \mathfrak{sl}(2, \mathbb{C}) \to \operatorname{End}(\bigoplus_{p,q} \bigwedge^{p,q})$$

with

$$\rho(l) = L$$

$$\rho(\lambda) = \Lambda$$

$$\rho(b) = B$$

定理 6.1.8. *(HL)*

$$L^{n-k} : \bigwedge^k \to \bigwedge^{2n-k}$$

$$u \to \omega^{n-k} \wedge u$$

is an isomorphism.

$$L^{n-k}:\bigwedge^{p,q}\to\bigwedge^{n-k+p,n-k+q}$$

is also an isomorphism.

证明. Lemma:

$$[L^r,\Lambda]u=r(k-n+r-1)L^{r-1}u$$

(induction, omit)

Assume $\alpha\in\bigwedge^k_{\mathbb{C}}$, $L^{n-k}\alpha=0$, need to verify $\alpha=0$.

Claim:

$$L^r:\bigwedge^k\to\bigwedge^{k+2r}$$

is injective whenever $r\leq n-k$.

proof of the claim:

claim is true when $k=0$ or $k=1$.(check)

Let $\alpha\in\bigwedge^k$ s.t. $L^r\alpha=0(r\leq n-k)$. By the lemma,

$$L^r\Lambda\alpha-\lambda L^r\alpha=r(k-n+r-1)L^{r-1}\alpha$$

so,

$$L^{r-1}(L\Lambda\alpha-r(k-n+r-1)\alpha)=0$$

by the induction on r,

$$L\Lambda\alpha=r(k-n+r-1)\alpha$$

since $r(k-n+r-1)\neq 0$, $\alpha=L\beta$ for some $\beta\in\bigwedge^{k-2}$. so,$L^r\alpha=L^{r+1}\beta=0$, by induction on k, we have $\beta=0$,so $\alpha=0$.

The claim is proved. □

定义 6.1.9. *(Primitive form)*

$\alpha\in\bigwedge^k(k\leq n)$ *is called primitive form, if*

$$L^{n-k+1}\alpha=0$$

推论 6.1.10. *(Lefischtz Decomposition)(LD)*

For any $\alpha\in\bigwedge^k,(1\leq k\leq 2n)$, *we have a unique decomposition:*

$$\alpha=\sum_{\gamma\geq(k-n)_+}L^{\gamma}\alpha_r$$

($(k-n)_+ := \max\{k-n, 0\}$) with $\alpha_r \in \bigwedge^{k-2r}$ is primitive

证明. Existence: assume $k \leq n$, consider

$$L^{n-k+1}\alpha \in \bigwedge^{2n-k+2}$$

by HL, $\exists!\, \beta \in \bigwedge^{k-2}$ s.t. $L^{n-k+2}\beta = L^{n-k+1}\alpha$, so $L^{n-k+1}(\alpha - L\beta) = 0$, i.e. $\alpha_0 = \alpha - L\beta$ is primitive. $\alpha = \alpha_0 + L\beta$, then induction on degrees, we get the decomposition for α.

If $k > n$, we apply HL to reduce it to case 1.

Uniqueness: Next time.. □

Today: Continuous to Hard Lef decomposition, Hodge-Riemann bilinear relations.

Hard-Lefschitz: HL

Lefschitz decomposition :LD

Hodge-Riemann bilinear relations :HRR

Recall: $\mathbb{C}^n, \bigwedge^k = \bigoplus_{p+q=k} \bigwedge^{p,q}$, ω: a Kahler metric on $\mathbb{C}^n$ with constant coefficient $\in \bigwedge^{1,1}_{\mathbb{R}}$.

Lefschitz operator : $Lu = \omega \wedge u$.

定理 6.1.11. *(HL)*

Assume $k \leq n, p+q \leq n$, then

$$L^{n-k} : \bigwedge^k \to \bigwedge^{2n-k}$$

is a linear isomorphism.

$$L^{n-k} : \bigwedge^{p,q} \to \bigwedge^{p+n-k, q+n-k}$$

is also a linear isomorphism.

Linear algebra..

定理 6.1.12. *(LD) for any $u \in \bigwedge^k$,we have a unique decomposition*

$$u = \sum_{r \geq (k-n)_+} L^r u_r$$

where $u_r \in \bigwedge^{k-2r}_{prim}$ is a primitive form.

Recall: a k-form $u \in \bigwedge^k (k \leq n)$ is called primitive, if $L^{n-k+1}(u) = 0$. When $k > n$, u is called primitive, $\Lambda(u) = 0$, where Λ is the adjoint of L.

证明. Existence: application of HL.

Uniqueness: Omit. □

性质 6.1.13. *Assume $\alpha \in \bigwedge^{p,q}_{prim}$, and $p + q \leq n$. (i.e. $L^{n-p-q+1}\alpha = 0$),then*

$$*\alpha = (-1)^{\frac{(p+q)(p+q-1)}{2}} (\sqrt{-1})^{p-q} \frac{1}{(n-p-q)!} L^{n-p-q}\alpha$$

证明. See [Humphreys, Prop 1.2.31] □

定理 6.1.14. *(HRR) Define the bilinear form Q on $\bigwedge^k$ $(k \leq n)$ as follows:*

$$Q(\alpha, \beta) := L^{n-k} \wedge \alpha \wedge \overline{\beta}$$

Then

$$(\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} Q(u, u) \geq 0$$

for any $u \in \bigwedge^{p,q}_{prim}, p + q = k \leq n$, and equal holds

$$\iff u = 0$$

(i.e. $Q|_{\bigwedge^{p,q}_{prim}}$ is positive definite up to a factor)

证明. Take $u \in \bigwedge^{p,q}_{prim}$,

$$Q(u, u) = L^{n-k} \wedge u \wedge \overline{u} = *u \wedge \overline{u} = \langle \overline{u}, \overline{u} \rangle \mathrm{d}Vol = |u|^2 \mathrm{d}Vol \geq 0$$

(up to a factor!)

(We apply the following result: $\overline{*\varphi} = *\overline{\varphi}$, i.e. $*$ is a real operator) □

Summary: $\bigwedge^{\bullet} = \bigoplus\limits_{1 \leq k \leq n} \bigwedge^k_{\mathbb{C}}$, where$\bigwedge^k_{\mathbb{C}} = \bigoplus\limits_{p+q=k} \bigwedge^{p,q}_{\mathbb{C}}$.

Lefschitz operator $L \rightsquigarrow$ HL,LD,HRR.

6.2 紧 Kahler 流形的上同调群

The analogue of compact Kahler manifolds,

$$H^k_{DR}(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{Dol}(X,\mathbb{C})$$

ω: A Kahler metric $\in H^{1,1}_{Dol}(X,\mathbb{R})$.
Denote $L \curvearrowright H^k_{DR}(X,\mathbb{C})$,

$$L(u) = [\omega, u] = [\omega] \wedge u$$

Commutative relations on Kahler manifolds

$$(\mathbb{C}^n, \omega = \sqrt{-1}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j)$$

$u \in C^\infty(\mathbb{C}^n, \bigwedge^{p,q})$, locally

$$u = \sum_{|I|=p,|J|=q} u_{I,J} dz_I \wedge dz_j, \quad v = \sum_{|I|=p,|J|=q} v_{I,J} dz_I \wedge dz_j$$

$$\langle\langle u, v\rangle\rangle = \int_{\mathbb{C}^n} \sum_{|I|=p,|J|=q} u_{I,J}\overline{V_{I,J}} dVol$$

$d = d' + d''$, $d' = \partial, d'' = \bar{\partial}$.

$$d'u = \sum_{I,J}\sum_k \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge dz_J$$

$$d''u = \cdots$$

定理 6.2.1.

$$(d'')^* u = -\sum_{I,J}\sum_k \frac{\partial u_{I,J}}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_k} \lrcorner (dz_I \wedge d\bar{z}_J)$$

$$(d')^* u = -\sum_{I,J}\sum_k \frac{\partial u_{I,J}}{\partial \bar{z}_k} \frac{\partial}{\partial z_k} \lrcorner (dz_I \wedge d\bar{z}_J)$$

性质 6.2.2.

$$[(d'')^*, L] = \sqrt{-1} d'$$

证明. Exercise. □

定理 6.2.3. *Let X be a Kahler manifold (may not compact), with Kahler metric ω, then we have*

$$[(\mathrm{d}'')^*, L] = \sqrt{-1}\mathrm{d}'$$

证明. Only need to verify $u \in C_c^\infty(X, \bigwedge^{p,q})$ with compact support in a holomorphic chart at x.

Assume the holomorphic chart near x is choosen s.t.

$$\omega(z) = \sqrt{-1} \sum_{1 \le j \le n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j + O(|z|^2)$$

$$u \in \sum_{I,J} u_{I,J} \mathrm{d}z_I \wedge \bar{z}_J$$

is a (p,q)-form, v is also...

$$\langle u, q \rangle = u_{I,J} \overline{v_{M,N}} \langle \mathrm{d}z_I, \mathrm{d}z_M \rangle \langle \mathrm{d}\bar{z}_J, \mathrm{d}\bar{z}_N \rangle = u_{IJ} \overline{V_{ij}} + a_{IJMN}(z) u_{IJ} \overline{V_{MN}}$$

where $a_{IJMN} = O(|z|^2)$.

So,

$$(\mathrm{d}'')^* u = -\sum_{IJk} \frac{\partial u_{IJ}}{\partial z_k} \frac{\partial}{\partial \bar{z}_k} \lrcorner (\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J) + \sum_{IJMN} b_{IJMN} u_{IJ} \mathrm{d}z_M \wedge \mathrm{d}\bar{z}_N$$

where $b_{IJMN}(z) = O(|z|)$. So,

$$[(\mathrm{d}'')^*, L] u(x) = \sqrt{-1}\mathrm{d}' u(x)$$

$$\implies [(\mathrm{d}'')^*, L] = \sqrt{-1}\mathrm{d}'$$

□

性质 6.2.4. *In Kahler manifold,*

$$[(\mathrm{d}')^*, L] = -\sqrt{-1}\mathrm{d}''$$

$$[\Lambda, \mathrm{d}''] = -\sqrt{-1}(\mathrm{d}')^*$$

$$[\Lambda, \mathrm{d}'] = \sqrt{-1}(\mathrm{d}'')^*$$

推论 6.2.5. *(X,ω) is a Kahler manifold, then*

$$\triangle_{\mathrm{d}} = 2\triangle_{\mathrm{d}'} = 2\triangle_{\mathrm{d}''}$$

证明. For example, $\triangle_{\mathrm{d}} = 2\triangle_{\mathrm{d}''}$,

$\triangle_{\mathrm{d}} = (\mathrm{d}'+\mathrm{d}'')(\mathrm{d}'+\mathrm{d}'')^* + (\mathrm{d}'+\mathrm{d}'')^*(\mathrm{d}'+\mathrm{d}'') = (\mathrm{d}'+\mathrm{d}'')(\mathrm{d}'^* - \sqrt{-1}[\Lambda,\mathrm{d}']) + (\mathrm{d}'^* - \sqrt{-1}[\Lambda,\mathrm{d}'])(\mathrm{d}'+\mathrm{d}'')$

然后暴力展开，12 项??? $\cdots$

从略。 □

推论 6.2.6. *If (X,ω) is a Kahler manifold, then*

$$\triangle_{\mathrm{d}} : C^\infty(C, \bigwedge^{p,q}) \to C^\infty(C, \bigwedge^{p,q})$$

证明. Since $\triangle_{\mathrm{d}} = 2\triangle_{\mathrm{d}'}$, $\triangle_{td'}$ preserves the bi-degree. □

推论 6.2.7. *If (X,ω) is a compact Kahler manifold, u is a $\triangle_{\mathrm{d}}$-harmonic k-form. Assume*

$$u = \sum_{p+q=k} u^{p,q}$$

$$u^{p,q} \in C^\infty(X, \bigwedge^{p,q})$$

then each $u^{p,q}$ is also harmonic.

定理 6.2.8. *(Hodge decomposition)*

X is a compact Kahler manifold, then we have a decomposition

$$H^k_{\mathrm{d}}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\mathrm{d}''}(X,\mathbb{C})$$

Equivalently, (sheaf cohomology)

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X,\Omega^p)$$

证明. take a Kahler metric ω, we can define $\triangle_d, \triangle_{td'}, \triangle_{\mathrm{d}''}$, then

$$\ker \triangle_{\mathrm{d}} := \mathcal{H}^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\mathrm{d}''}(X,\mathbb{C})$$

then $\Longrightarrow$ the decomposition for $H^k_{\mathrm{d}}(X,\mathbb{C})$

the decomposition for $H^k_{\mathrm{d}}(X,\mathbb{C})$ is independent of the choice of ω (Next time) □

Recall: Hodge decomposition,

X compact Kahler manifold, $\dim_{\mathbb{C}} X = n$,

Thm:(Hodge decomposition)

$$H^k_{DR}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}_{\mathrm{d}''}(X,\mathbb{C})$$

where

$$H^{p,q}(X,\mathbb{C}) = \{[\alpha] \in H^k_{DR}(X,\mathbb{C}) | \alpha \text{is a d-closed s.m. } (p,q)\text{-form}\}$$

Proof: take a Kahler metric ω,

$$H^k_{DR}(X,\mathbb{C}) \cong \mathcal{H}^k_{\mathrm{d}}(X,\mathbb{C}) = \bigoplus \mathcal{H}^{p,q}_{\mathrm{d}}(X,\mathbb{C}) = \bigoplus \mathcal{H}^{p,q}_{\mathrm{d}''}(X,\mathbb{C})$$

性质 6.2.9. *There is a canonical isomorphism*

$$H^{p,q}_{\mathrm{d}}(X,\mathbb{C}) \xrightarrow{\sim} H^{p,q}_{\mathrm{d}''}(X,\mathbb{C})$$

$$[\alpha]_{\mathrm{d}} \mapsto [\alpha]_{\mathrm{d}''}$$

where $\mathrm{d}\alpha = 0$,α *is a* (p,q)*-form.* $\Rightarrow \mathrm{d}''\alpha = 0$

证明. Check: this map is well defined. Need to verify: if $\alpha = \mathrm{d}\beta$ is a (p,q)-form, then $[\alpha]_{\mathrm{d}''} = 0$, i.e. α is also d''-exact.

α is a (p,q)-form,

$$\Rightarrow \alpha = \mathrm{d}'\beta^{p-1,q} + \mathrm{d}''\beta^{p,q-1}$$

we have $\mathrm{d}''\mathrm{d}'\beta^{p-1,q} = 0$, $\mathrm{d}'\mathrm{d}''\beta^{p,q-1} = 0$

We need a very important lemma: □

引理 6.2.10. *($\partial\bar{\partial}$-lemma)*

Let X is a Kahler manifold, α is a smooth form which is d' and d'' closed. Then, if α is d or d' or d''-exact, then $\alpha = \mathrm{d}'\mathrm{d}''\gamma$ for some γ.

Using $\partial\bar{\partial}$-lemma, this map is well-defined.

Now, notice that the two space has the same dimension. So, we need to show the map is injective(or, surjective). Claim : this map is injective. If α is a d-closed with $[\alpha]_{\mathrm{d}''} = 0$, i.e. $\alpha = \mathrm{d}''\beta^{p,q-1}$. α is d-closed$\Rightarrow$ $\mathrm{d}'\mathrm{d}''\beta^{p,q-1} = 0$, $\partial\bar{\partial}$-lemma applying to $\mathrm{d}''\beta^{p,q-1}$, we have

$$\mathrm{d}''\beta^{p,q-1} = \mathrm{d}'\mathrm{d}''\gamma = \mathrm{d}(\mathrm{d}''\gamma)$$

for some γ.

Proof of $\partial\bar{\partial}$-lemma:

证明. Assume α is d'' exact, 1.e. $\alpha = \mathrm{d}''\beta$, write

$$\beta = H(\beta) + \triangle_{\mathrm{d}}\gamma$$

where $H(\beta)$ is $\triangle_{\mathrm{d}}$-harmonic, so

$$\alpha = \mathrm{d}''H(\beta) + \mathrm{d}''\triangle_{\mathrm{d}}\gamma - 2\mathrm{d}''\triangle_{\mathrm{d}'}\gamma$$

(Since $\triangle_{\mathrm{d}} = 2\triangle_{\mathrm{d}''}$)

$$\Rightarrow \alpha = 2\mathrm{d}''(\mathrm{d}'\mathrm{d}'^{*} + \mathrm{d}'^{*}\mathrm{d}') = 2\mathrm{d}''\mathrm{d};\mathrm{d}'^{*}\gamma - 2\mathrm{d}'^{*}\mathrm{d}''\mathrm{d}'\gamma$$

By the assumption, $\mathrm{d}'\alpha = 0$, so $\mathrm{d}'^{*}\mathrm{d}''\mathrm{d}'\gamma = 0$

$$\alpha = -2\mathrm{d}'\mathrm{d}''\mathrm{d}'^{*}\gamma$$

□

注记 6.2.11. *(Deligne-Griffiths-Morrora)*

If $\widehat{X}$ is bimeromapic to X, where X is acompact Kahler, then$\hat{X}$ is also satisfys the $\partial\bar{\partial}$-lemma.

X is a kahler manifold, then

$$H^{p,q}_{\mathrm{d}}(X,\mathbb{C}) \cong H^{p,q}_{\mathrm{d}''}(C,\mathbb{C}) \cong H^{p,q}X,\mathbb{C}$$

X us a compact complex manifold,define

$$H^{p,q}_{BC} := \frac{\text{d-closed } (p,q))}{\mathrm{d}'\mathrm{d};\text{; exact}}$$

Bott-Chern cohomology

Exercise" If X is Kahler , then $H^{p,q}_{BC} = H^{p,q}_{\mathrm{d}}$

$$H^{p,q}_{A}(X,\mathbb{C}) := \frac{\mathrm{d}'\mathrm{d}''\text{closed}}{(\mathrm{d}')\text{-exact} + \{\mathrm{d}''\text{exact}\}}$$

(Appeli cohomology)

denote

$$h^k_{BC} := \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}$$

$$h^k_{A} := \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{A}$$

定理 6.2.12. *X satisfies* $\partial\bar{\partial}$*-lemma* $\iff$

$$h^k_B + h^k_A = 2b_k$$

where

$$b_k = \dim_{\mathbb{C}} H^k_{DR}(X, \mathbb{C})$$

定理 6.2.13. *(Hard Lef)*

X is a compact Kahler, $\dim_{\mathbb{C}} X = n$*, denote* $L = \{\omega\} \curvearrowright H^k_{DR}(X, \mathbb{C})$*,* ω *is a Kahler metric, Then we have:*

$$L^{n-k} : H^k_{DR}(X, \mathbb{C}) \cong H^{2n-k}_{DR}(X, \mathbb{C})$$

$$H^{p,q}(X, \mathbb{C}) \cong H^{p+n-k, q+n-k}(X, \mathbb{C})$$

where $k \leq n$*,* $p + q \leq n$*.*

证明. Fox a Kahler metric ω,

$$L^{n-k} : H^k_{DR} \to H^{2n-k}_{DR}$$

($\cong \mathcal{H}^k_{\mathrm{d}}$, $\cong \mathcal{H}^{2n-k}_{\mathrm{d}}$ respectively) (there is a commutative diagram...)

need to proof: For any $\varphi \in \mathcal{H}^k_{\mathrm{d}}$, then

$$L^{n-k}(\varphi) = \omega^{n-k} \wedge \varphi$$

is also harmonic. □

引理 6.2.14.

$$[\triangle_{\mathrm{d}}, L] = 0$$

证明.

$$[\triangle_{\mathrm{d}}, L] = 2[\triangle_{\mathrm{d}'}, L] = 2\left([\mathrm{d}'\mathrm{d}'^*, L] + [\mathrm{d}'^*\mathrm{d}', L]\right) = 2\left(\mathrm{d}'[\mathrm{d}'*, L] + [\mathrm{d}'^*, L]\mathrm{d}'\right)$$

(check: $[L, \mathrm{d}'] = 0$) So,

$$= -2\sqrt{-1}(\mathrm{d}'\mathrm{d}'' + \mathrm{d}''\mathrm{d}') = 0$$

□

Exercise: Complex tori

$$\mathbb{T}^n := \mathbb{C}^n/\Gamma$$

where $\Gamma = \mathbb{Z}^n$. $\mathbb{T}^n$ is a compact Kahler manifold. Then

$$H^{1,1}(\mathbb{T}^n, \mathbb{C}) \cong \bigwedge\nolimits_{\mathbb{C}}^{1,1}$$

the space of $(1,1)$-forms on $\mathbb{C}^n$ with constant coefficient, in particular,

$$\dim_{\mathbb{C}} H^{1,1}(\mathbb{T}^n, \mathbb{C}) = n^2$$

Exercise: the set of all the Kahler class on $\mathbb{T}^n \subseteq H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$ is equal to the set of $n \times n$ positive definite Hermitian metrics.

(Hint: using Hodge theory)

定理 6.2.15. *(Lefschitz decomposition)*

Define a class $\alpha \in H^k_{DR}(X, \mathbb{C})$ to be positive if

$$L^{n-k+1}(\alpha) = 0$$

if $k \leq n$.

(When $\alpha \in H^k_{DR}(X, \mathbb{C})$, $k > n$, we call α positive)

Then $\forall \varphi \in H^k_{DR}(X, \mathbb{C})$, exist unique decomposition

$$\varphi = \sum_{\gamma \geq (k-n)_+} L^{\gamma}\varphi_{\gamma}$$

where $\varphi_{\gamma} \in H^{k-2\gamma}_{prim}(X, \mathbb{C})$.

Similarly,

$$H^{p,q}(X, \mathbb{C}) = \bigoplus_{r \geq (p+q-n)_+} H^{p-r,q-r}_{prim}(X, \mathbb{C})$$

证明. Exercise. □

定理 6.2.16. *(HRR)*

X compact Kahler, $\dim_{\mathbb{C}} X = n$, ω *is Kahler metric, define*

$$Q(\alpha, \beta) = L^{n-k}\alpha \wedge \overline{\beta}$$

where $\alpha, \beta \in H^{p,q}(X, \mathbb{C})$, *and* $p + q = k$.

Then $Q|_{H^{p,q}_{prim}}$ *is positive defined (up to a factor).*

证明. Exercise. □

Exercise:Consider X-compact Kahler,$\dim_{\mathbb{C}} X = n$, ω-Kahler metric, Then $\forall \alpha, \beta \in H^{1,1}(X, \mathbb{R}) = H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$, Then

$$\left(\{\omega^{n-2}\} \cdot \alpha \cdot \beta\right)^2 \geq \left(\{\omega^{n-2}\} \cdot \alpha^2\right)\left(\{\omega^{n-2}\} \cdot \beta^2\right)$$

with equality if and only if $\alpha = \lambda\beta$ for some $\lambda \in \mathbb{R}$

Eg: $\mathbb{C}^2$, α, β real $(1,1)$-forms,

$$(\alpha, \beta)^2 \geq \alpha^2\beta^2$$

Hint: Using HRR, and Lefschitz decomposition... "Alg-Geom-inequality over Kahler manifold".

性质 6.2.17. *X is a compact Kahler, then*

$$\overline{H^{p,q}(X, \mathbb{C})} = H^{q,p}(X, \mathbb{C})$$

证明. Use harmonic form.. and $\triangle_{\mathrm{d}}$ is a real operator... □

Summary X-compact Kahler with a Kahler metric ω, then define Lefschitz operator $L = [\omega]\wedge$, then:

Hodge decomposition:

$$H^k = \bigoplus_{p+q=k} H^{p,q}$$

$$\overline{H^{p,q}} = H^{q,p}$$

Hard Lefschitz:

$$L^{n-k} : H^{p,q} \cong H^{p+n-k, q+n-k}$$

where $p + q = k$

Lefschitz decomposition:

$$H^{p,q} = \bigoplus_{r \geq (p+q-1)_+} L^r H_{prim}^{p-r,q-r}$$

HRR:...

References Kahler pairing in other settings..

`Adiprusito-Huh-Katz: Hodge theory in combinatorial geometries`

`McMullen: On simple polytopes`

`Deligne: Weil II`

`Beillinson-Bernstein-Deligne-Gabber: Faisceaux Pervers`

`Adiprasito: Combinatorial Lefschetz theorem beyond positivity, 2018`

Recall: Kahler pairing: X-compact Kahler manifold of complex dimension n, ω-Kahler metric. Lefischitz operator

$$L = \{\omega\} \curvearrowright H^\bullet$$

Hodge decomposition

$$H^k = \bigoplus_{p+q=k} H^{p,q}, \qquad \overline{H^{p,q}} = H^{q,p}$$

(Corollary: if k is odd, then $b_k := \dim_{\mathbb{C}} H^k(X, \mathbb{C})$ is even.)

Rmk: if X is compact complex surface($\dim_{\mathbb{C}} = 2$),X is Kahler $\iff$ b_1 is even. (The proof of "$\Leftarrow$" we not given...Ref: Kodaira&Siu,Lamari 1999)

Hard Lef. ($p + q = k$)

$$L^{n-k} : H^{p,q} \xrightarrow{\sim} H^{p+n-k,q+n-k}$$

Lef. decomposition:

$$H^{p,q} = \bigoplus_{r \geq (k-n)_+} L^r H_{prim}^{p-r,q-r}$$

Denote $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$,"Hodge number". Cor:

$$h^{p,q} = \begin{cases} h_{prim}^{p,q} + h_{prim}^{p-1,q-1} + \cdots & p+q \leq n \\ h_{prim}^{n-q,n-p} + h_{prim}^{n-q-1,n-p-1} + \cdots & p+q \geq n \end{cases}$$

(Using the property of L^r)

If $p + q \leq n$, $h^{p,q} \geq h^{p-1,q-1} \Rightarrow b_k \geq b_{k-2}$ if $k \leq n$.

If $p + q \geq n$, $h^{p,q} \leq h^{p-1,q-1} \Rightarrow b_k \leq b_{k-2}$ if $k \geq n$.

(Hodge-Frolicher spectral sequence)

X-compact Kahler, then Hodge decomposition

$$\Rightarrow b_k = \sum_{p+q=k} h^{p,q}$$

Question: X compact complex manifold, relation between b_k and $\sum\limits_{p+q=k} h^{p,q}$?

定理 6.2.18. *(Hodge-Frolicher inequality) X compact complex manifold, then*

$$b_k \leq \sum_{p+q=k} h^{p,q}$$

Spectral sequence: $(K^{p,q}, \mathrm{d} = \mathrm{d}' + \mathrm{d}'')$ a double complex of modules.

$$K^{p,q} \xrightarrow{\mathrm{d}'} K^{p+1,q} \quad K^{p,q} \xrightarrow{\mathrm{d}''} K^{p,q+1}$$

with $\mathrm{d}'^2 = 0, \mathrm{d}''^2 = 0, \mathrm{d}^2 = 0$.

Assume $K^{p,q} = 0$ if $p \leq 0$ or $q \leq 0$.

$\rightsquigarrow$ total complex $(K^\bullet, \mathrm{d})$where

$$K^l := \bigoplus_{p+q=l} K^{p,q}$$

$\exists$ a natural filtration

$$F_p K^l := \bigoplus_{l \geq i \geq p} K^{i,l-i}$$

F induces a filtration on $H^\bullet(K^\bullet)$.

$$F_p H^l(K^\bullet) = \mathrm{Im}(H^l(F_p K^\bullet) \to H^l(K^\bullet)) = \frac{F_p Z^l}{F_p B^l}$$

where $Z^l = \ker \mathrm{d} \curvearrowright K^l$ and $B^l = \mathrm{Im}\, \mathrm{d} \curvearrowright K^{l-1}$

Denote $G_p H^l(K^\bullet) = F_p H^l / F_{p+1} H^l$.

定理 6.2.19. *There exists a sequence*

$$\{E_r, \mathrm{d}_r\}_{r \geq 0}$$

satisfying:

(1) $E_r = \bigoplus_{p,q \geq 0} E_r^{p,q}$

(2)$\mathrm{d}_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$, $\mathrm{d}_r^2 = 0$.

(3) $E_{r+1} = H^\bullet((E_r, \mathrm{d}_r))$.

$$E_0^{p,q} = \frac{F_p K^{p+q}}{F_{p+1} K^{p+q}} = K^{p,q}$$

d_0 induced by d.

$$E_1^{p,q} = H^q((K^{p,\bullet}, \mathrm{d}''))$$

d_1 induced by d.

查任何一本同调代数的书。

定义 6.2.20. *We call the sequence E_r converges at E_{r_0}, if $E_{r+1} = E_r$ for any $r \geq r_0$,（$\iff \mathrm{d}_r = 0$ for any $r \geq r_0$）then we denote $E_\infty = E_{r_0}$*

In our setting, $E_\infty^{p,q} = G_p H^{p+q}(K^\bullet)$

Application: X compact complex manifold,

$$K^{p,q} = C^\infty(X, \bigwedge^{p,q}) \quad \mathrm{d} = \mathrm{d}' + \mathrm{d}''$$

$\rightsquigarrow E_0^{p,q} = K^{p,q}$, $E_1^{p,q} = H^{p,q}(X, \mathbb{C})$.

推论 6.2.21.

$$E_\infty^{p,q} = G_p H^{p+q}(X, \mathbb{C})$$

定理 6.2.22. *X is a compact complex manifold of complex dimension n,then*

$$b_l = \dim_{\mathbb{C}} H^l(X, \mathbb{C}) = \sum_{p+q=l} \dim_{\mathbb{C}} E_\infty^{p,q} \leq \sum_{p+q=l} \dim_{\mathbb{C}} E_1^{p,q} = \sum_{p+q=l} h^{p,q}$$

with equality holds if and only if $\mathrm{d}_1 = 0$ (i.e $\{E_r\}$ converges at E_1.)

定理 6.2.23. *X compact Kahler $\Rightarrow \{E_r\}$ converges at E_1 ($\iff b_l = \sum_{p+q=l} h^{p,q}$)*

Remark: algebraic proof by Deligne-Illusive 1987.

Relèvement module p^2 et décomposition du complexe de de Rham

remark: Assume X is bimeromorphic to a compact Kahler manifold, then we still have the convergence of $\{E_r\}$ ($\iff$ Hodge decomposition)

(Deligne-Griffiths-Morgan)

Picard group $H^1(X, \mathcal{O}^*)$.

Recall:

$$\{\text{isomorphic class of holomorphic line bundle}\} \xrightarrow{1-1} H^1(X, \mathcal{O}^*)$$

Consider the sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e^{2\pi\sqrt{-1}}} \mathcal{O}^* \to 0$$

$$\rightsquigarrow 0 \to H^0(X,\mathbb{Z}) \to H^0(X,\mathcal{O}) \to H^0(X,\mathcal{O}^*) \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \to \cdots$$

Assume X is a compact complex manifold, then

$$H^0(X,\mathcal{O}) = \mathbb{C}$$

$$H^0(X,\mathcal{O}^*) = \mathbb{C}^*$$

$\Rightarrow H^0(X,\mathcal{O}) \to H^0(X,\mathcal{O}^*)$ is surjective,

$\Rightarrow H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O})$ is injective.

So we have an exact sequence

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z})$$

so we have an isomorphism

$$\ker\{c_1 : H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z})\} \cong H^1(X,\mathcal{O})/H^1(X,\mathbb{Z})$$

定义 6.2.24. *(Irregularity of X)*

$$q(X) = \dim_{\mathbb{C}} H^1(X,\mathcal{O}) = h^{0,1}$$

if X is also complex Kahler, then $h^{0,1} = h^{1,0}$.

Assume X is compact Kahler:

引理 6.2.25. $H^1(X,\mathbb{Z})$ *is also a lattice in* $H^1(X,\mathcal{O})$*of*

$$rank_{\mathbb{Z}} H^1(X,\mathbb{Z}) = 2q$$

$\Rightarrow H^1(X,\mathcal{O})/H^1(X,\mathbb{Z})$ *is a compact torus of* $\dim_{\mathbb{C}} = q$.

$$H^1(C,\mathcal{O})/H^1(X,\mathbb{Z}) := \ker\{c_1 : H^1(X,\mathcal{O}^*) to H^2(X,\mathbb{Z})\}$$

is called **Jacobian variety**($Jac(X)$) or **Picard variety** ($Pic^\circ(X)$)

Denote $NS(X)_{\mathbb{Z}} = \mathrm{Im}(c_1 : H^1(X,\mathcal{O}^*) to H^2(X,\mathbb{Z}))$ the Neron-Severi group of X,

$$\rightsquigarrow \quad 0 \to Pic^\circ(X) \to H^1(X,\mathcal{O}^*) \xrightarrow{c_1} NS(X,\mathbb{Z}) \to 0$$

proof of the lemma. $\mathbb{Z} \to \mathcal{O}$ can be decomposed : $\mathbb{Z} \to \mathbb{R} \to \mathbb{C} \to \mathcal{O}$. It induces a sequence

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{R}) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O})$$

$H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O})$ is an isomorphism.
Consider the diagram
then $H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O})$ corresponds to

$$H^1_{DR}(X, \mathbb{R}) \hookrightarrow H^1_{DR}(X, \mathbb{C}) \twoheadrightarrow H^{0,1}(X, \mathbb{C})$$

$H^1(X, \mathbb{Z})$ is a lattice in $H^1(X, \mathbb{R})$ of $rank_{\mathbb{Z}} = 2q$ □

Albanese map, Albanese torus
X-compact Kahler $\Rightarrow$ any holomorphic p-forms are d-closed.
(Exercise!!)
Special case: holo 1-forms is d-closed.

$$Alb(X) := H^0(X, \Omega^1)^* / \operatorname{Im}(H_1(X, \mathbb{Z}))$$

where $H^1(X, \mathbb{Z})$ is mapped to $H^0(X, \Omega^1)^*$ in the following way:

$$[\gamma] \mapsto (\alpha \in H^0(X, \Omega^1) \mapsto \int_\gamma \alpha)$$

(Fact: $\int_\gamma \alpha$ depends only on the class on $[\gamma]$)
Then $Alb(X)$ is compact complex of $\dim_{\mathbb{C}} = q(X)$. More precisely, we have a map:

$$alb : X \to Alb(X)$$

Fix a base point $x_0 \in X$, then

$$alb(x) = \left(u \mapsto \int_{x_0}^{x} u\right) \mod \Lambda$$

where

$$\Lambda := \left\{ (\int_\gamma u_1, ..., \int_\gamma u_q) \middle| [\gamma] \in H_1(X, \mathbb{Z}) \right\}$$

$\{u_1, ..., u_q\}$ is a basis of $H^0(X, \Omega^1)$. Then Λ is a lattice of $rank_{\mathbb{Z}} = 2q$.
The map

$$alb : X \to Alb(X)$$

is holomorphic.

第 7 章　正性与消灭定理

positivity and vanishing theorem

X-Kahler manifold, i.e. $\exists$ Hermitian metric ω s.t. $\mathrm{d}\omega = 0$, $\mathrm{d} = \mathrm{d}' + \mathrm{d}''$, $\mathrm{d}' = \partial, \mathrm{d}'' = \bar{\partial}$.

$$\triangle_{\mathrm{d}} = [\mathrm{d}, \mathrm{d}^*] = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$$

$$\triangle_{\mathrm{d}'} = [\mathrm{d}', \mathrm{d}'^*]$$

$$\triangle_{\mathrm{d}''} = [\mathrm{d}'', \mathrm{d}''^*]$$

$\mathrm{d} \curvearrowright C^\infty(X, \bigwedge^{p,q})$.

Fact: ω is Kahler $\iff$ $\triangle_{\mathrm{d}'} = \triangle_{\mathrm{d}''} = \frac{1}{2}\triangle_{\mathrm{d}}$.

Let $\underline{\mathbb{C}} := X \times \mathbb{C}$ be the trivial line bundle, d can be regraded as the Chern connection on $\underline{\mathbb{C}}$.

(E, h)-Hermitian holomorphic vector bundle over (X, ω), with Chern connection $D_E = D'_E + D''_E$. ($D''_E = \bar{\partial}$).

$$C^\infty(X, \bigwedge^{p,q} \otimes E)$$

has an inner product induced by ω, h. $\rightsquigarrow$ adjoint operators $D_E^* = D_E'^* + D_E''^*$.

$\rightsquigarrow \triangle_E = [D_E, D_E^*] = D_E D_E^* + D_E^* D_E$, and $\triangle'_E$, $\triangle''_E$. (self adjoint, elliptic operators)

Question: relation between $\triangle'_E$ and $\triangle''_E$?

定理 7.0.26. *(Bochner-Kodaira-Nakaino identity)*

$$\triangle''_E - \triangle'_E = \left[\sqrt{-1}\Theta_E, \Lambda\right]$$

where Θ_E is the Chern curvature of D_E.

Recall: $\Theta_E = D_E^2$, when D_E is Chern connectoin, we have

$$D_E'^2 = 0 \qquad D_E''^2 = 0$$

i.e. $\Theta_E = [D'_E, D''_E]$.

Remark: E is flat(i.e. $D_E^2 = 0$) $\iff$ $\triangle'_E = \triangle''_E$.

证明. based on following identities:

$$[D_E''^*, L] = \sqrt{-1} D_E'$$

$$[D_E'^*, L] = -\sqrt{-1} D_E''$$

$$[\Lambda, D_E'] = -\sqrt{-1} D_E'^*$$

$$[\Lambda, D_E''] = \sqrt{-1} D_E''^*$$

then (by super Jacobi identity):

$$\begin{aligned} \triangle_E'' = [D_E'', D_E''^*] &= -\sqrt{-1}\,[D_E'', [\Lambda, D_E']] = -\sqrt{-1}\,([\Lambda, [D_E', D_E'']] + [D_E', [D_E'', \Lambda]]) \\ &= -\sqrt{-1}\left([\Lambda, \Theta_E] + [D_E', \sqrt{-1} D_E'^*]\right) \end{aligned}$$

so,

$$\triangle_E'' - \triangle_E' = [\sqrt{-1}\Theta_E, \Lambda]$$

□

引理 7.0.27. *(normal frame)*

Let X be a complex manifold, then for any $x_0 \in X$, and any holomorphic chart $(z_1, ..., z_n)$ centered at x_0, there exists a holomorphic frame $\{e_\lambda\}_{\lambda=1}^{r:=rankE}$ of E near x_0 such that

$$\langle e_\lambda(z), e_\mu(z) \rangle = \delta_{\lambda,\mu} - \sum_{1\le j,k\le n} C_{jk\lambda\mu} z_j \bar{z}_k + O(|z|^3)$$

where $(C_{jk\lambda\mu})$ are the coefficients of the Chern curvature

$$\Theta_E(x_0) = \sum_{\substack{1\le j,k\le n \\ 1\le\lambda,\mu\le r}} C_{jk\lambda\mu} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

need to verify: $\forall s \in C^\infty(X, \bigwedge^{p,q} \otimes E), x_0 \in X$,

$$[D_E''^*, L]s(x_0) = \sqrt{-1} D_E' s(x_0)$$

w.r.t the normal frame $(e_\lambda)_{\lambda=1}^r$ near x_0, assume

$$s = \sum_{\lambda=1}^n \sigma_\lambda \otimes e_\lambda$$

then

$$D_E s(z) = \sum_{\lambda=1}^n \mathrm{d}\sigma_\lambda \otimes e_\lambda + O(|z|)$$

$$D_E^* s(z) = \sum_{\lambda=1}^{n} \mathrm{d}^* \sigma_\lambda \otimes e_\lambda + O(|z|)$$

$$D_E''^* = \sum_{\lambda=1}^{r} \mathrm{d}''^* \sigma_\lambda \otimes e_\lambda + O(|z|)$$

$$\Rightarrow [D_E''^*, L]s = D_E''^*(\sum \omega \wedge \sigma_\lambda \otimes e_\lambda) - \omega \wedge \left(\sum_{\lambda=1}^{r} \mathrm{d}''^* \sigma_\lambda \otimes e_\lambda + O(|z|) \right) = \sum_{\lambda=1}^{r} [\mathrm{d}''^*, L] \sigma_\lambda \otimes e_\lambda + O(|z|)$$

Similarly,

$$D_E' s = \sum_{\lambda=1}^{r} \mathrm{d}' \sigma_\lambda \otimes e_\lambda + O(|z|)$$

we have:

$$[d''^*, L] = \sqrt{-1}\mathrm{d}'$$

(because ω is Kahler)

...

(E, h) hermitian holomorphic vector bundle over Kahler manifold (X, ω). we have BKN identity

$$\triangle_E'' - \triangle_E' = [\sqrt{-1}\Theta_E, \Lambda]$$

Recall: L^2-Hodge theory. X compact manifold,then

$$H^{p,q}(X, E) := \frac{\ker D_E''}{\operatorname{Im} D_E''} \cong \ker \triangle_E''$$

(harmonic form)

Take $u \in C^\infty(X, \bigwedge^(p, q) \otimes E)$,applying BKN identity to u,

$$\triangle_E'' u - \triangle_E' u = [\sqrt{-1}\Theta_E, \Lambda]u$$

note that

$$\langle\!\langle \triangle_E' u, u \rangle\!\rangle = \|D_E' u\|^2 + \|D_E'^* u\|^2 \geq 0$$

$$\Rightarrow \|D_E'' u\|^2 + \|D_E''^* u\|^2 \geq \langle\!\langle [\sqrt{-1}\Theta_E, \Lambda], u \rangle\!\rangle$$

i.e.

$$\|D_E'' u\|^2 + \|D_E''^* u\|^2 \geq \int_X \langle [\sqrt{-1}\Theta_E, \Lambda], u \rangle \mathrm{d}Vol$$

Observation: if $u \in \ker \triangle_E''$, and $[\sqrt{-1}\Theta_E, \Lambda]$ has ”positivity”, then $LHS = 0$. So, $H^{p,q}(X, E) = 0$.

定义 7.0.28. *(Positivity)*

We call $[\sqrt{-1}\Theta_E, \Lambda]$ *is positive at* $x_0 \in X$*, if for any* $0 \neq v \in (\bigwedge^{p,q} \otimes E)_{x_0}$*, we have*

$$\langle[\sqrt{-1}\Theta_E, \Lambda]v, v\rangle > 0$$

....positive on X*, if ... at each point*

定理 7.0.29. *If* $[\sqrt{-1}\Theta_E, \Lambda]$ *is positive on* X*, then*

$$H^{p,q}(X, E) = 0$$

Special case: E is a holomorphic line bundle, with Hermitian metric h,

$$\Theta_E = -\mathrm{d}'\mathrm{d}'' \log h$$

$\Rightarrow \sqrt{-1}\Theta_E$ is a real d-closed $(1,1)$-form on X.
locally,

$$\alpha = \sqrt{-1} \sum_{1 \leq i,j \leq n} a_{ij} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$$

α is real $\iff$ $\alpha = \bar{\alpha}$, (i.e. locally (a_{ij}) is an hermitian matrix)

定义 7.0.30. *a real* $(1,1)$*-form* α *is called positive, if* $(a_{ij})_{ij}$ *is positive definite.*

引理 7.0.31. *If* $\sqrt{-1}\Theta_E$ *is positive, then* $\omega := \sqrt{-1}\Theta_E$ *gives a Kahler metric on* X*.*

引理 7.0.32. *If* $\omega = \sqrt{-1}\Theta_E > 0$*, and* Λ *is the adjoint of* $L = \omega\wedge$*, then*

$$[\sqrt{-1}\Theta_E, \Lambda]$$

is positive on $\bigwedge^{p,q} \otimes E$ *whenever* $p + q \geq n + 1$*.*

引理 7.0.33. *Let α be a real $(1,1)$-form, ω a Kahler metric, assume the eigenvalue of α at x_0 is $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, then (in the coordinate chart $(z_1, z_2, ..., z_n)$, and $u = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$)*

$$[\alpha, L] = \sum_{I,J} \left(\sum_{i \in I} \alpha_i + \sum_{j \in J} \alpha_j - \sum_{k=1}^{n} \alpha_k \right) u_{IJ} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

推论 7.0.34. *$\alpha = \omega$, then*

$$[\omega, \Lambda]u = (p + q - n)u$$

推论 7.0.35. *Take an orthonormal frame e of E, then for any $u = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J \otimes e$, we have*

$$\langle [\sqrt{-1}\Theta_E, \Lambda]u, u \rangle = (p + q - n)|u|^2$$

定理 7.0.36. *If $[\sqrt{-1}\Theta_E, \Lambda]$ is positive on X, then*

$$H^{p,q}(X, E) = 0$$

定理 7.0.37. *If E is a holomorphic line bundle with a smooth hermitian metric h s.t. $\sqrt{-1}\Theta_{(E,h)} \geq 0$, then$H^{p,q}(X, E) = 0$ whenever $p + q \geq n + 1$.*

de Rham-Weil...$\cong H^q(X, \Omega^p \otimes E)$.

定义 7.0.38. *(canonical bundle)*

$$K_X = \det T^* X$$

determinate bundle of cotangent bundle, is called canonical bundle. ($\mathcal{O}(K_X) = \Omega_X^n$)

定义 7.0.39. *X is called Fano, if $K_X^* = \det(TX)$ has a matric with positive curvature.*

X is called Calabi-Yau,if K_X has a metric with vanishing curvature.

X is of general type, if K_X has a metric with positive curvature.

推论 7.0.40. *(Kodaira vanishing theorem) E is a positive line bundle, then*

$$H^q(X, K_X \otimes E) = 0$$

for any $q \geq 1$.

So, if X is Fano, ($\iff$ K_X^*) positive, $K_X \otimes K_X^* = \underline{\mathbb{C}}$, $\Rightarrow H^1(X, \mathcal{O}) = 0, \Rightarrow H^1(X, \mathbb{R}) = 0$,

Recall: BKN-inequality.

holomorphic Hermitian vector bundle $(E, h) \to (X, \omega)$, ω is Kahler. For any $u \in C^\infty(X, \bigwedge^{p,q} \otimes E)$, we have

$$\|D''u\|^2 + \|D''^*u\|^2 \geq \int_X \langle [\sqrt{-1}\Theta_E, \Lambda_\omega]u, u\rangle \mathrm{d}Vol$$

Recall: If $[\sqrt{-1}\Theta_E, \Lambda_\omega]$ is positive on $C^\infty(X, \bigwedge^{p,q} \otimes E)$, then $H^{p,q}(X, E) = 0$.

定理 7.0.41. *(Kodaira-Nakano vanishing theorem)*

If E is a holomorphic line bundle with a smooth metric h s.t. $\sqrt{-1}\Theta_{(E,h)} > 0$,then $[\sqrt{-1}\Theta_E, \Lambda_\omega]$ is positive on $C^\infty(X, \bigwedge^{p,q} \otimes E)$ whenever $p + q \geq n + 1$.

$\Rightarrow H^{p,q}(X, E) = 0$ when $p + q \geq n + 1$.

(Last time)

Today:

定理 7.0.42. *(Girbau vanishing theorem, 1976)*

E is a holomorphic line bundle over compact Kahler manifold, with smooth metric h s.t. $\sqrt{-1}\Theta_{(E,h)} \geq 0$, and has at least $n - s + 1$ positive eigenvalues at every points of X, then

$$H^{p,q}(X, E) = 0$$

if $p + q \geq n + s$.

α: a **real** $(1,1)$-form on X, locally $\alpha = \sqrt{-1}\sum \alpha_{ij} dz_i \wedge d\bar{z}_j$. then we have a matrix $M(\alpha) = (\alpha_{ij})_{n\times n}$, ($\alpha$ is real $\Rightarrow$)a hermite matrix.

we call α has at least k positive eigenvalues at x, if $M(\alpha)(x)$ has k positive eigenvalues. (Remark: It is well defined)

证明. Claim: there exists some Kahler metric ω s.t. $[\sqrt{-1}\Theta, \Lambda]$ is positive.

Fix a Kahler metric ω, for $p \in X$, choose a holomorphic chart $(z_1, ..., z_n)$, s.t. $\omega(p) = \sqrt{-1}\sum dz_j \wedge d\bar{z}_j$ and $\sqrt{-1}\Theta_E(p) = \sqrt{-1}\sum\limits_{j=1}^{n} \gamma_j dz_j \wedge d\bar{z}_j$. WLOG, $0 \le \gamma_1 \le \gamma_2 \le \cdots \le \gamma_n$, and for any $j \ge s$, $\gamma_j > 0$.

Consider

$$\omega_\varepsilon := \varepsilon\omega + \sqrt{-1}\Theta_E$$

for $\varepsilon > 0$, then ω_ε is a Kahler metric. $\omega_\varepsilon(p) = \sqrt{-1}\sum\limits_j (\varepsilon + \gamma_j) dz_j \wedge d\bar{z}_j$.

$\Rightarrow$ the eigenvalue of $\sqrt{-1}\Theta$ with respective to $\omega_\varepsilon(p)$ is given by

$$\gamma_{j,\varepsilon} = \frac{\gamma_j}{\varepsilon + \gamma_j} = \frac{1}{1 + \frac{\varepsilon}{\gamma_j}}$$

Claim: $[\sqrt{-1}\Theta, \Lambda_{\omega_\varepsilon}]$ is positive on $\bigwedge^{p,q} \otimes E$ when $p + q \ge n + s$, $0 < \varepsilon << 1$.

Take $u = \sum u_{IJ} dw_T \wedge d\overline{w}_J \otimes e$,then

$$\langle [\sqrt{-1}\Theta_E, \Lambda_{\omega_\varepsilon}], u\rangle = \sum_{\substack{|I|=p\\|J|=q}} \left(\sum_{i\in I} \gamma_{i,\varepsilon} + \sum_{j\in J} \gamma_{j,\varepsilon} + \sum_{k=1}^{n} \gamma_{k,\varepsilon} \right) |u_{IJ}|^2 \ge (\gamma_{1,\varepsilon} + ... + \gamma_{p,\varepsilon} - \gamma_{q+1,\varepsilon} - ... - \gamma_{n,\varepsilon})|u|^2$$

note that $\gamma_{j,\varepsilon} \ge 1 - \frac{\varepsilon}{\gamma_s}$ if $j \ge s$, $\gamma_{j,\varepsilon} \in [0, 1)$ for all j. it

$$\ge \left((q + s - 1)(1 - \frac{\varepsilon}{\gamma_s}) - (n - p) \right) |u|^2 > 0$$

if $p + q \ge n + s$ and $0 < \varepsilon << 1$. □

注记 7.0.43. *(Kawamata-Viewheg vanishing theorem)*

$E \to (X, \omega)$ is a holomorphic line bundle over a compact Kahler manifold.

Definition: E is called positive, ...(positive="ample" in AG). numerically effective(nef) if for any $\varepsilon > 0$, there is a smooth metric h_ε s.t. $\sqrt{-1}\Theta_{h_\varepsilon} \ge -\varepsilon\omega$.

Theorem: If E is nef, and $\int_X c_1(E)^n > 0$, then $H^q(X, K_X \otimes E) = 0$ for $q \ge 1$.

Positivity concept of vector bundles(rank > 1)

$(E, h) \to (X, \omega)$ Hermitian vector bundle of rank r, over a complex manifold(may not Kahler).

Denote $(e_1, ..., e_r)$ a local orthonormal frame of E, $(z_1, ..., z_n)$ local holomorphic chart, Chern curvature of (E, h):

$$\Theta_{(E,h)} = \sum_{\substack{1\le j,k\le n\\ 1\lambda,\mu\le r}} c_{ik\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

Fact: $\sqrt{-1}\Theta_E$ induces a Hermitian operator θ_E on $TX \otimes E$.

Let u, v be local sections of $TX \otimes E$,

$$u = \sum_{\substack{1 \le j \le n \\ 1 \le \lambda \le r}} u_{k\mu} \frac{\partial}{\partial z_k} \otimes e_\mu$$

$$\theta_E(u, v) := \sum_{\substack{1 \le j,k \le n \\ 1 \le \lambda,\mu \le r}} c_{jk\lambda\mu} u_{j\lambda} \overline{v_{k\mu}}$$

定义 7.0.44. *We call E Nakano positive, if θ_E is positive. (i.e for any non-zero local section $u \in TX \otimes E$, $\theta_E(u, u) > 0$)*

We call E Griffith positive, if for any $0 \neq \xi \in T_x X$, $s \in E_x, s \neq 0$,

$$\theta_E(\xi \otimes s, \xi \otimes s) > 0$$

注记 7.0.45. *By definition, Nakano positivity $\Rightarrow$ Griffith positivity.*

If E is line bundle, Nakano positivity $\iff$ Griffith positivity. (and $\iff$ positivity of lines bundles)

定理 7.0.46. *(Demailly-Skota, 1979)*

E is Griffith positive $\Rightarrow$ $E \otimes \det E$ is Nakano positive.

证明. Omit. Non-trivial. □

Notation: $E >_{Nak} 0$ (E is Nakano positive). Similarly, $E >_{Giff} 0$...

性质 7.0.47. *(1)E is Griffith positive if and only if E^* is Griffith negative.*

(2) Consider an exact sequence of holomorphic vector bundles:

$$0 \to S \to E \to Q \to 0$$

then if E is Griffith positive, then Q is Griffith positive. If E is Griffith negative, then S is Griffith negative. If E is Nakano negative, then S is Nakano negative.

证明. Omit. Compute curvature... □

Remark: In general, E is Nakano positive, $\not\Rightarrow Q$ is Nakano positive.

定理 7.0.48. *(Nakano vanishing theorem)*

(X,ω) is compact Kahler of dimension n, (E,h) is a Nakano positive holomorphic Hermitian vector bundle, then

$$H^{n,q}(X,E)=0 \qquad \forall q\geq 1$$

证明. E is Nakano positive, check:

$$[\sqrt{-1}\Theta_E,\Lambda_\omega]$$

is positive on $\bigwedge^{n,q}\otimes E$ for $(q\geq 1)$ □

Ampleness

$E\to X$, E: holomorphic line bundle of rank r, X:complex manifold.

定义 7.0.49. *(Jet vector bundle)*

$$J^kE=\bigcup_{x\in X}(J^kE)_x$$

where

$$(J^kE)_x=\mathcal{O}_x(E)\Big/\mathfrak{m}_x^{k+1}\mathcal{O}_x(E)$$

$\mathfrak{m}_x\subseteq\mathcal{O}_x$ be the maximal ideal of $\mathcal{O}_x$.

In local coordinate,

$$(J^kE)_x=\Big\{\sum_{\substack{1\leq\lambda\leq r\\|\alpha|\leq k}}C_{\lambda\alpha}(z-x)^\alpha e_\lambda(z)\Big\}$$

性质 7.0.50. *J^kE is a holomorphic vector bundle of rank $=r\binom{n+k}{n}$.*

证明. Exercise. □

定义 7.0.51. *E is called very ample, if the following maps:*

$$H^0(X,E)\to(J^1E)_x$$

$$H^0(X,E) \to E_x \oplus E_y$$

are surjective, for all $x, y \in X$, $x \neq y$.

E is called ample, if $S^m E := \mathrm{Sym}^m E$ is very ample for some $m \in \mathbb{N}$.

(ample: ” 足够多的全纯截面”)

定理 7.0.52. *(Kodaira)*

L-holomorphic line bundle, X is a compact complex manifold. Then L is positive if and only if L is ample.

We will prove:

定理 7.0.53. *$L \to X$ holomorphic line bundle over a compact complex manifold, then L is positive $\iff$ L is ample.*

We need:

(1)Kodiara vanishing theorem.

(2)Blow-up of complex manifold

(3)Relation between divisor and line bundles.

analytic cycles, divisors and meromorphic functions

定义 7.0.54. *X be a analytic set in some complex manifold, then the set X_{reg} is a dense subset of X. Denote the connected component of X_{reg} by X_α, $\overline{X_\alpha}$ is the closure of X_α in X, then $\overline{X_\alpha}$ is called a global irreducible component of X.*

In particular, X is the union of global irreducible components.

例子 7.0.55. *(Global irreducibility is different from local irreducibility)*

$V = \left\{(x,y) \in \mathbb{C}^2 \middle| y^2 = x^2(1+x)\right\}$ is an analytic set in $\mathbb{C}^2$, $V_{reg} = V \setminus \{0\}$ is connected. So, $V = \overline{V_{reg}}$ is globally irreducible.

On the other hand, $(V, 0)$ is a reducible as an analytic germ.

定义 7.0.56. *(analytic cycles)*

X is a complex manifold, a q-cycle (with integer coefficient) is a formal linear combination $\sum \lambda_j V_j$, $\lambda_j \in \mathbb{Z}$, and V_j is a global analytic sets of X of dimension q.

So, we get a group $C^q_{cyl}(X)$.

an element of $Cycl^{n-1}(X)$ is called a divisor. (Weil divisor) ($Div(X)$)

If D is an irreducible analytic set of dimension $n-1$ then the divisor given by D is called a prime divisor.

注记 7.0.57. *For any open set $U \subseteq X$, $U \to Cycl^q(U)$ induces a sheaf $Cycl^q$ of X with the germ $Cycl^q_x$ given by q-dimension analytic germs at X.*

定理 7.0.58. *X is a connected complex manifold, $f \in \mathcal{O}(X)$, then we have $f^{-1}(0)$ is emply of $\dim_{\mathbb{C}}$ isempty of $n-1$.*

定义 7.0.59. （*Cartier-dividiot*）

A divisor $D = \sum \lambda_j D_j$ locally giveb by a $\mathbb{X}$ linear combination of $div(f)$. f is locally holomorphic functions.

定义 7.0.60. *X is a compact , $\beta \in \mathcal{O}(X)$, D_j is a global irreponent of $f^{(-1)0}$,*

$$m_j := Ord_z(f)$$

for all $z \in D_j reg \setminus \bigcup_{k \neq j} D_k$ m_j be the vanishing order along D_j.

定理 7.0.61. *(A, x) an analytic germ of $\dim_{\mathbb{C}}$=n-1. $(A, x) = (g)$for sone $g \in \mathcal{O}X$,and g is a product of $(J_{A_j, x}) = (g_j)$.*

(2) Let $f \in \theta_x$ with $(f^{-1}(0), x) \subseteq (A, x)$,then $f = u \coprod_j g_j m^{m_j}$, where $m_j = ord_z(f)$

性质 7.0.62. *If X is a complex manifold, then any Weil divisor is also a Cantier divisor.*

Remark: NOT true for singular points.

Meromorphic function: X complex manifold, $\mathcal{O}_X$ sheaf of functions on X.

$$\mathfrak{m}_x := \left\{ \frac{g_x}{h_x} \middle| g_x, h_x \in \mathcal{O}_x \text{and } h_x \text{ is not zeor in } \mathcal{O}_x \right\}$$

$$\mathcal{M} := \bigcup_{x \in X} \mathfrak{m}_x$$

with the topology given by the basis

$$\left\{ \frac{G_x}{H_x} \middle| x \in V, G, H \in \mathcal{O}(V) \right\}$$

例子 7.0.63. $f(z_1, z_2) = \frac{z_1}{z_2}$

定义 7.0.64. *Let $F \in \mathfrak{m}(X)$,denote $P(X) :=\notin \left\{ x \in X \middle| f_x \notin \mathcal{O}_x \right\}$. Pole set pf f, and $Z(f) := P(\frac{1}{z})$ zero set of f.*

定理 7.0.65. *$f \in \mathfrak{m}(X)$, if $P(d)$ (or$Z(f)$) is not empty, then $P(f)$ is analytic set of* $\dim = \mathbb{H}$.

定义 7.0.66. *$P(f) \cup Z(f)$ is called the indeterminiary of set of f, (in particular, codimension $P(M) \cap Z(f) \geq 2$)*

性质 7.0.67. *Given $f \in \mathcal{M}(X)$, we get a divisor:*

$$div(f) = \sum a_j A_j - \sum b_j B_j$$

where a_j = the vanishing order of f along A_j, A_j a globally irreducible component of $Z(f)$, b_j =...along of $\frac{1}{f}$ along B_j, B_j:...component of $P(f)$.

例子 **7.0.68.** *$f = \frac{z_1}{z_2} \in \mathcal{M}(\mathbb{C}^2)$, then $P(f) = \{z_2 = 0\}$ and $Z(f) = \{z_1 = 0\}$, and*

$$div(f) = [z_1 = 0] - [z_2 = 0]$$

Consider: X - complex manifold, $\mathcal{O}^*$: sheaf of invertible holomorphic functions,
$\mathcal{M}^*$: Sheaf of non-zero meromorphic functions
$\mathcal{D}iv$: Sheaf of $(n-1)$-cycles.

性质 **7.0.69.** *We have an exact sequences:*

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D}iv \to 0$$

In particular, $\mathcal{D}iv = \mathcal{M}^*/\mathcal{O}^*$.
long exact sequence:

$$0 \to H^0(X, \mathcal{O}^*) \to H^0(X, \mathcal{M}^*) \to H^0(X, \mathcal{D}iv) \to H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{M}^*) \to \cdots$$

where, note that :

$$H^0(X, \mathcal{D}iv) = Div(X) \qquad H^1(X, \mathcal{O}^*) = Pic(X)$$

Consider $Div(X) = H^0(X, \mathcal{M}^*/\mathcal{O}^*) \to Pic(X)$, $f \in H^0(X, \mathcal{M}^*/\mathcal{O}^*) \iff$ we have an open covering $X = \bigcup_i U_i$ and $f_i \in \mathcal{M}^*(U_i)$ with $\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$.

$f \in H^0(X, \mathcal{M}^*/\mathcal{O}^*) \xrightarrow{\varphi} (U_i \cap U_j, g_{ij} \in \mathcal{O}^*(U_i \cap U_j)) \in \check{H}^1(\mathcal{U}, \mathcal{O}^*) \hookrightarrow H^1(X, \mathcal{O}^*)$.

定义 **7.0.70.** *A divisor D is called principal divisor, if $D = div(h)$ for some $h \in \mathcal{M}^*(X)$.*

性质 **7.0.71.** $\ker \varphi = \{$*principal divisors*$\}$, *i.e. $\mathcal{O}(D)$ is trivial $\iff$ $D = div(f)$ for some global meromorphic functions.*

性质 **7.0.72.**

$$\mathcal{O}(D_1 + D_2) = \mathcal{O}(D_1) \otimes \mathcal{O}(D_2)$$

$$\mathcal{O}(-D) = \mathcal{O}(D)^*$$

定义 7.0.73. *$D_1, D_2 \in Div(X)$ is called linear equivalent, if $D_1 - D_2$ is principal, denoted by $D_1 \sim D_2$. We have an injection:*

$$Div(X)/\sim \hookrightarrow Pic(X)$$

Remark: in general, $D \to \mathcal{O}(D)$ is not surjective.
If $X \hookrightarrow \mathbb{P}^n$, then $Div(X)/\sim \cong Pic(X)$.

性质 7.0.74. *$L \to X$ holomorphic line bundle over a complex manifold, we have a canonical map:*

$$H^0(X, L) \setminus \{0\} \to Div(X)$$

$$s \to Z(s)$$

证明. $s \in H^0(X, L) \iff$ the data $(U_i, f_i \in \mathcal{O}(U_i))$, L is determined by $g_{ij} \in \mathcal{O}^*(\mathcal{U}_i \cap \mathcal{U}_j)$.
$Z(s)$ locally given by $div(f_i)$. $(div(f_i) = div(f_j)$ on $U_i \cap U_j)$ □

性质 7.0.75. *$s_i \in H^0(X, L_i) \setminus \{0\}, i = 1, 2$,we have $Z(s_1 \otimes s_2) = Z(s_1) + Z(s_2)$.*

性质 7.0.76. *Let $s \in H^0(X, L) \setminus \{0\}$, then $\mathcal{O}(Z(s)) \cong L$.*

证明. Assume $X = \bigcup U_i$ with L determined by $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$, $s \in H^0(X, L)$ determined by $(U_i, f_i \in \mathcal{O}(U_i))$.
so,$\mathcal{O}(Z(s))$ is the line bundle given by $\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$.
note that $f_i = g_{ij} f_j$. □

推论 7.0.77. *Let $s_i \in H^0(X, L_i) \setminus \{0\}, i = 1, 2$, then*

$$Z(s_1) \sim Z(s_2) \iff L_1 \cong L_2$$

use the fact: $\mathcal{O}(Z(s_i)) = L_i$ and $\mathcal{O}(\text{principal divisor}) \cong \mathcal{O}_X$ trivial line bundle.

性质 7.0.78. *Consider the map*

$$Div(X) \to Pic(X)$$

$$D \to \mathcal{O}(D)$$

then the image is generated by line bundles with non-zero holomorphic sections.

7.1 Blow-up

Local picture: $U \subseteq \mathbb{C}^n$ open subset, $Y \subseteq U$ linear subspace, $codim_U Y = k$, e.g. assume $Y = \left\{ z \in U \middle| z_1 = ... = z_k = 0 \right\}$.

Consider the space

$$U_Y := \left\{ ([w], z) \in \mathbb{P}^{k-1} \times U \middle| w_i z_j = w_j z_i, 1 \leq i, j \leq k \right\} \subseteq \mathbb{P}^{k-1} \times U \xrightarrow{\pi_2} U$$

定义 7.1.1. U_Y *is called the blow-up of* U *along* Y.

性质 7.1.2. U_Y *is a smooth complex submanifold of* $\mathbb{P}^{k-1} \times U$*, and* $\dim_{\mathbb{C}} U_Y = \dim_{\mathbb{C}} U = n$*. And* $\tau : U_Y \to U$ *is a holomorphic map with*

$$\tau|_{U_Y \setminus \tau^{-1}(Y)} : U_Y \setminus \tau^{-1}(Y) \cong U \setminus Y$$

And for any $y \in Y$, $\tau^{-1}(y) = \mathbb{P}^{k-1} \times \{y\}$ *is complex projective space.*

Locally, on then chart $w_1 \neq 0$, denote $\hat{w}_i = \frac{w_i}{w_1}$ for all $2 \leq i \leq k$. Then $z_i = \widehat{w}_i z_1$.Then $(z_1, \widehat{w}_2, ..., \widehat{w}_k, z_{k+1}, ..., z_n)$ gives a holomorphic chart of U_Y.

Denote $(z_1, ..., z_n) = (z_1, \widehat{w}_2, ..., \widehat{w}_k, z_{k+1}, ..., z_n)$, then $z_1 = \xi_1$, $z_2 = \xi_1\xi_2$,...,$z_k = \xi_1\xi_k$, and $z_{k+l} = \xi_{k+l}$for $k \geq l$.

In this coordinate system, $\tau^{-1}(Y) = \left\{ \xi \in U_Y \middle| \xi_1 = 0 \right\}$.

$\Rightarrow$ $\tau^{-1}(Y)$ is a (smooth) hypersurface in U_Y.And, $\tau^{-1}(Y) \cong \mathbb{P}(N_{Y/U})$, where $N_{Y/U}$ is the normal bundle of Y in U.

$$(0 \to T_Y \to T_U|_Y \to N_{Y/U} \to 0)$$

If $codim_U Y = 1$ hypersurface, then $U_Y \cong U$.

Global construction

Y is a complex submanifold of X, $\dim_{\mathbb{C}} = n$,$\dim_{\mathbb{C}} Y = k \leq n$.

引理 7.1.3. *If $f_1, ..., f_k$ and $g_1, ..., g_k$ are two (local) definition of Y, defining equations of Y, $Y = \left\{ f_z(z) = ... = f_k(z) = 0 \middle| , \right\}$, then $\mathrm{d}f_1, ..., \mathrm{d}f_k$ are linely independent along Y. And $\exists$ a matrix (m_{ij}) of holomorphic functions, s.t. $g_i = \sum_{j=1}^{k} M_{n,j} f_j$ for any $1 \leq i \leq k$.*

The matrix (M_i^j) is invertible along Y,and determined uniquely by $(f_1, ..., f_k)$ and $g_1, ..., g_k$.

证明. Assume $f_i = z_i$ for $1 \leq i \leq k$ is a local coordinate system $\equiv 0$.For ever g_i, $g_i|_{z_1,...,z_k=0}$

Consider the Taylor expansion of
g_i, we set

$$g_i = \sum_{j=1}^{k} M_i^j(z) z_j$$

$\mathrm{d}g_i = \sum_{j=1}^{k} \mathrm{d}M_i^j z_j + \sum_{j=1}^{k} M_i^j \mathrm{d}z_j$.

$(\mathrm{d}g_1, ..., \mathrm{d}g_k)|_Y$ and $(\mathrm{d}z_1, ..., \mathrm{d}z_k)|_Y$ are $L.I$, so $M_i^j|_Y$ is invertible.

Assume $Y \cap U = \{f_1^U = ... = f_k^U = 0\}$, $Y \cap V = \{f_1^V = f_2^V = ... = f_k^V = 0\}$ and $(M_{i,UV}^j)_{1 \leq i,j \leq k}$ is the □

$0 \to T_Y \to T_X|_Y \to N_{Y/D}$, the dual

$$N_{Y/X}^* \to T_X^*|_Y \to T_Y^*$$

(M_i^j, UV) gives the translation matrix middle of $N_{Y/X}^*$

引理 7.1.4. *$\exists$ isomorphism $\phi_{UV} : \tau_U^{-1}(U \cap V) \cong \tau_V^{-1}(U \cap V)$.*

证明. Assume $f_i^U = \sum_{j=1}^{k} = \sum_{j=1}^{k} M_{i,UV}^j f_j^V$. Define $\phi_{UV}([w], z) = ([M^{-t} w], z)$, then ϕ_{UV} satisfies the two properties. □

定义 7.1.5. *(The blow-up of X along Y)(Global blow up)*

$\mathrm{Bl}_Y X$*:the blow-up of X along Y is defined as the complex manifold by gluing the U_Y and $\Omega := X \setminus S_Y$, where S_Y is some neighborhood of Y.*

we have a holomorphic map: $\tau : \mathrm{Bl}_Y X \to X$.

性质 7.1.6. $\tau : \mathrm{Bl}_Y X \to X$ *satisfies :*

(1)$\tau^{-1}(Y)$ *is a smooth complex submanifold of* $\mathrm{Bl}_Y X$*, with* $\dim_{\mathbb{C}} = n-1$*,(It is called the excepted divisor of* τ*)*

(2)$\tau : \mathrm{Bl}_Y X \setminus \tau^{-1}(Y) \to X \setminus Y$ *is an isomorphism.*

(2)τ *is a proper map(any pre-image of compact set is compact).*

证明. Check. □

projective bundle $E \to X$ is a holomorphic vector bundle(of rank r) over a complex manifold(of complex dimension n), then we can define projective bundle $\mathbb{P}(E)$,

$$\mathbb{P}(E) := \left\{ (x, [\xi]) \middle| x \in X, \xi \in E_x \setminus \{0\} \right\}$$

$\mathbb{P}(E)$ is a complex manifold of dimension $n+r-1$ (if $X = \{pt\}$, then $\mathbb{P}(E)$ is just the projective space)

We have a tautological line bundle on $\mathbb{P}(E)$:

$$\mathcal{O}_E(-1)_{(x,[\xi])} = \mathbb{C}\xi$$

$\mathcal{O}_E(-1)$ is a holomorphic line bundle on $\mathbb{P}(E)$.

Exercise: Assume (E, h) is an hermitian vector bundle with metric h,then h induces a metric on $\widetilde{h}$ on $\mathcal{O}_E(-1)$, then the Chern curvature Θ of $\widetilde{h}$ satisfies: for any $x \in X$, $\sqrt{-1}\Theta|_{\mathbb{P}(E_x)} < 0$.

定理 7.1.7. $\tau : \mathrm{Bl}_Y X \to X$ *blow-up along* Y*,* $E := \tau^{-1}(Y)$ *exceptional divisor,* $\mathcal{O}(E)$*: the holomorphic line bundle associated to* E*, then*

(1) $\tau : E \to Y$ *is just the map* $\mathbb{P}(N_{Y/X}) \to Y$

(2) $\mathcal{O}(E)|_E \cong \mathcal{O}_{P(N_{Y/X})}(-1) \cong N_{E/\mathrm{Bl}_Y X}$ *the normal bundle of* E *in* $\mathrm{Bl}_Y X$*.*

证明. Exercise. □

推论 7.1.8. *If* X *is a (compact) Kahler manifold,* Y *is a compact submnifold of* X*, then the blow-up* $\mathrm{Bl}_Y X$ *is also a (compact) Kahler manifold.*

证明. $\tau : \mathrm{Bl}_Y X \to X$, let ω be a Kahler matric on X, then $\tau^*\omega$ is a semi-positive $(1,1)$-form on $\mathrm{Bl}_Y X$, positive on $\mathrm{Bl}_Y X \setminus E$, and the kernel of $\tau^{-1}\omega$ along E is given by the tangent space of the fiber $E \to Y$.

Define the metric h on $\mathcal{O}(E)$ as follows: on E,h is induced by the metric on $N_{Y/X}$ induced by the metric on $N_{Y/X}$, and we extend h to a neighborhood of E; outside a neighborhood of E,($\mathcal{O}(E)|_{\mathrm{Bl}_Y X\setminus E}$ is trivial), h is given by the trivial metric.

Then ,we glue these two metrics to get a matric on $\mathcal{O}(E)$. Denote the curvature $\theta := \sqrt{-1}\Theta(\mathcal{O}(-E), h)/$

Claim: $C\tau^*\omega + \theta > 0$ for $C \gg 1$ □

7.2 Kodaira Embedding Theorem

Recall: $L \to X$ holomorphic line bundle with a smooth metric h over compact complex manifold.

L is called positive if the curvature $\sqrt{-1}\Theta_{(L,h)}$ is a positive $(1,1)$-form.

L is called ample, if $L^{\otimes m} := mL$ is very ample for $m \gg 1$.

Recall: a holomorphic vector bundle E is called very ample, if the following maps

$$H^0(X, E) \to E_x \oplus E_y \qquad \forall x \neq y \in X$$

$$H^0(X, E) \to (J^1 E)_x \qquad \forall x \in X$$

are surjective.

性质 7.2.1. *X is a complex manifold of dimension n, $Y \subseteq X$ is a complex submanifold of codimension k. $\tau : \widehat{X} \to X$ blow-up along Y. $E := \tau^{-1}(Y)$ exceptional divisor. Then*

$$K_{\widehat{X}} = \tau^* K_X \otimes \mathcal{O}((k-1)E)$$

(Recall: $K_X = \det T^*X = \bigwedge^n T^*X$, locally free sheaf of holomorphic n-terms Ω_X^n).

证明. locally, τ can be written as

$$\tau : (w_1, ..., w_n) \to (z_1, ..., z_n)$$

$$z_1 = w_1,\ z_2 = w_2, ..., z_k = w_k w_1, ..., z_{k+l} = w_{k+l}$$

$$\Rightarrow \tau^*(\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \cdots \wedge \mathrm{d}z_n) = w_1^{k-1} \mathrm{d}w_1 \wedge \mathrm{d}w_2 \wedge \cdots \wedge \mathrm{d}w_n$$

(local holomorphic frame of K_X and $K_{\widehat{X}}$... w_1^{k-1}-local section of $\mathcal{O}(E)$)

Recall: L-line bundle, $\{g_{ij}\}$ transition function, a local section is the following data $f_i = g_{ij} f_j$. If e_i the local frame on U_i, then $f_i e_i = f_j e_j$ on $U_i \cap U_j$.

之后 check 两个线丛的转移函数相同. □

引理 7.2.2. *Let $\widehat{X}$ be the blow up of X along $\{x_1, ..., x_N\} \subseteq X$, ($N$ distinct points), denote E the exceptional divisor, then*

$$H^1(\widehat{X}, \mathcal{O}(-mE) \otimes \tau^*(kL)) = 0$$

for $m \geq 1$, $k \geq Cm$ for $C \gg 1$

证明.

$$H^1(\widehat{X}, \mathcal{O}(-mE) \otimes \tau^*(kL)) = H^1(\widehat{X}, K_{\widehat{X}} \otimes K_{\widehat{X}}^{-1} \otimes \mathcal{O}(-mE) \otimes \tau^*(kL)) = H^{n,1}(\widehat{X}, F)$$

where $F := K_{\widehat{X}}^{-1} \otimes \mathcal{O}(-mE) \otimes \tau^*(kL)$.

By Kodaira-Nakano vanishing, if F is positive, then $H^{n,1}(\widehat{X}, F) = 0$.

Note that

$$\begin{aligned} F &= \mathcal{O}(-mE) \otimes \tau^* K_X^{-1} \otimes \mathcal{O}((1-n)E) \otimes \tau^*(kL) \\ &= \tau^* K_X^{-1} \otimes \mathcal{O}(-(m+n-1)E) \otimes \tau^*(kL) \end{aligned}$$

We know, $\exists C_0 \gg 1$ s.t. $C_0 L \otimes K_X^{-1}$ is positive, and $\exists C \gg 1$,s.t. $C\tau^* L \otimes \mathcal{O}(-E)$ is positive.

So, For $k \geq Cm$ ($C \gg 1$), F is positive.

Let $v_j \in H^0(\Omega_j, kL)$ be a local section of kL, s.t. v_j generates the m-jet at x_j. Let $\psi_j \in C^\infty(X, \mathbb{R})$ s.t. $supp\psi_j \subset\subset \Omega_j$, $0 \leq \psi_j \leq 1$, $\psi_j \equiv 1$ around x_j. Denote

$$v := \sum_{j=1}^{n} \psi_j v_j$$

a smooth section of kL.

$$\mathrm{d}'' v = \sum_j \mathrm{d}'' \psi_j v_j \in C^\infty_{(0,1)}(X, kL)$$

satisfies $\mathrm{d}'' v = 0$ near x_j for $1 \leq j \leq N$.

Lemma:(Exercise)

$$H^0(X, M) \to H^0(\widehat{X}, \tau^* M)$$

$$s \mapsto \tau^* s$$

is an isomorphism for any line bundle M.

Lemma:(Exercise) a section of $\tau^* M$ with vanishing order$=$ k along E is the pull-back of a section of M with vanishing order $= k$ at x_j.

Denote $S_E \in H^0(\widehat{X}, \mathcal{O}(E))$ the canonical section of E,

$$w = S_E^{-(m+1)} \otimes \tau^*(\mathrm{d}'' v) \in C^\infty_{(0,1)}(\widehat{X}, \mathcal{O}(-(m+1)E) \otimes \tau^*(kL))$$

and $\mathrm{d}''w = 0$. Vanishing of $H^0(\widehat{X}, \mathcal{O}(-(m+1)) \otimes \tau^*(kL))$ implies $w = \mathrm{d}''u$ for some $u \in C^\infty(\widehat{X}, \mathcal{O}(-(m+1)E) \otimes \tau^{-1}kL)$.

$$S_E^{-(m+1)}\tau^*(\mathrm{d}''v) = \mathrm{d}''u$$

$$\Rightarrow \mathrm{d}''(\tau^*v - S_E^{(m+1)}u) = 0$$

so, $\tau^*v - s_E^{(m+1)}u$ is a holomorphic section of $\tau^*(kL)$. Using $s_E^{(m+1)}u = \tau^*f$ for some $f \in H^0(X, kL)$ with vanishing order $= m+1$ along x_j.

Claim: denote $g := v - f$ is the holomorphic sections generating the m-jets at x_j. $\mathrm{d}''(\tau^*g) = 0 \Rightarrow \tau^*g$ is holomorphic, $\mathrm{Ord}_{x_j}(f) = m+1$. So, $J^m(g)_{x_j} = J^m(v)_{x_j}$.

□

定理 7.2.3. *$L \to X$ positive line bundle, $x_1, ..., x_N \in X$ are N distinct points on X, then there exists $C > 0$, s.t.*

$$H^0(X, kL) \twoheadrightarrow \bigoplus_{j=1}^N (J^m(kL))_{x_j}$$

is surjective for all $m \geq 0$ and $k \geq Cm$

证明. □

定理 7.2.4. *(Kodaira)*

Line bundle L is positive $\iff$ it is ample.

（微分几何的正性与代数几何的正性是等价的）

证明. (有一边是显然的，留作习题)

proof of "L ample $\Rightarrow$ L positive".

Exercise: If A is a very ample line bundle on X, $H^0(X, A)$ has a basis $\{s_0, ..., s_N\}$, then the map

$$\Phi : X \to \mathbb{P}(H^0(X, A))$$

$$s \mapsto [s_0(x); s_1(x); ...; s_N()]$$

(Kodaira map) is a holomorphic embedding.

(Hint: $H^0(X, A) \twoheadrightarrow A_x \oplus A_y$ means that Φ is injective; $H^0(X, A) \twoheadrightarrow (J^1(A))_x$ means that Φ_* is injective.)

Exercise: denote the tautological line bundle on $\mathbb{P}(H^0(X, A))$ by $\mathcal{O}(1)$, then $A = \Phi^*\mathcal{O}(1)$.

Cor: A is very ample $\Rightarrow$ A is positive.

Given any inner product on $H^0(X,A)$, we get a metric h on $\mathcal{O}(1)$, the curvature $\Theta(\mathcal{O}(n))$ of h is positive.

$$\Rightarrow \Theta(A) = \Phi^*\Theta(\mathcal{O}(1))$$

Φ is embedding $\Rightarrow \Theta(A)$ is positive. $\square$

L positive $\Rightarrow L$ ample, i.e. mL is very ample,

$$\Rightarrow \Phi_{H^0(X,mL)} : X \hookrightarrow \mathbb{P}(H^0(X,mL))$$

holomorphic embedding($\Rightarrow X$ is an analytic submanifold of $\mathbb{P}(H^0(X,mL))$)

$\xrightarrow{\text{Chow theorem}}$ X is an algebraic set of $\mathbb{P}(H^0(X,mL))$ (i.e. $X = \bigcup\limits_{j=1}^{t}\{P_j = 0\}$, P_j-homogenous polynomial)

a compact complex manifold X admitting a positive line bundle L if and only if X is an algebraic manifold.

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e^{2\pi\sqrt{-1}}} \mathcal{O}^* \to 0$$

$$\rightsquigarrow H^1(X,\mathcal{O}^*) \xrightarrow{C_1} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}) \to ...$$

and $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}) \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, and $H^2(X,\mathcal{O}) \cong H^{0,2}(X,\mathbb{C})$.

$\Rightarrow \forall \alpha \in H^2(X,\mathbb{Z}) \cup H^{1,1}(X,\mathbb{C})$, we have a holomorphic line bundle L s.t. $\alpha = c_1(L)$.

L admitting a positive line bundle $\iff$ X admitting a class $\alpha \in H^2(X,\mathbb{Z}) \cup H^{1,1}$ with a positive representative.

Recall:

定理 7.2.5. *$L \to X$ positive line bundle over compact complex manifold, then $\forall x_1, ..., x_N \in X$,$\exists C > 0$(depends on X), s.t.*

$$H^0(X, L^k) \twoheadrightarrow \bigoplus_{i=1}^{N} (J^m L^k)_{x_i} \qquad (*)$$

whenever $m \geq 0$ and $k \geq C(m+1)$

For fixed $(x_1, ..., x_N)$, we proved $\exists C(x_1, ..., x_N) > 0$, s.t. $(*)$ holds.

Observation:$(*)$ is an open condition with respect to $(x_1, ..., x_N)$.

$\Rightarrow \exists$ open set $U(x_i)$ s.t. $\forall (y_1, ..., y_N) \in \prod_{i=1}^{N} U(x_i)$, $(*)$ holds for $C = C(x_1, ..., x_N)$.

$m = 0, N = 1$,$H^0(X, L^k) \twoheadrightarrow (L^k)_x \iff \exists$ section s s.t. $s(x) \neq 0$ (for y near x, $s(y) \neq 0$)

$\pi : Y \to X$ blow-up along $x_1, ..., x_N$, with exception divisor E,

FACT:$\exists C \gg 1$, s.t. $C\pi^*L + \mathcal{O}(-E)$ is positive.

（这些已证明）

(more generally, if ω is a Kahler metric on X,denote $\{\omega\} \in H^{1,1}(X,\mathbb{R})$ the Kahler associated to ω, then $\exists C \gg 1$ s.t. $C\pi^*\omega + c_1(-E)$ is a Kahler class)

性质 7.2.6. *Define the Seshadri constant*

$$\mathcal{E}(x_1,...,x_N;\omega) := \sup\left\{t \geq 0 \middle| \pi^*\omega + t \cdot c_1(-E) \text{ is a Kahler class}\right\}$$

Then $\mathcal{E}(x_1,...,x_N;\omega)$ is a lower-semi-continuous function w.r.t $x_1,...,x_N$.

So,

$$\inf\left\{\mathcal{E}(x_1,...,x_N;\omega)\middle|(x_1,...,x_N) \in \underbrace{X \times \cdots \times X}_{N}\right\} > 0$$

证明. Too difficult. omit. □

注记 7.2.7. （如果感兴趣）

Nagata conjecture

Biran-Nagata conjecture

Symplectic packing/embedding of bundles

定理 7.2.8. *L is a positive line bundle, for $k \gg 1$,*

$$\Phi_{H^0(X,L^k)} : X \hookrightarrow \mathbb{P}(H^0(X,L^k))$$

$$x \mapsto [s_0(x) : ... : s_N(x)]$$

is a holomorphic embedding.(Where $\{s_j\}_{j=0}^N$ is a basis of $H^0(X,L^k)$)

So, (Chow theorem), X is an algebraic manifold.

Chow theorem 1949:

定理 7.2.9. （*Chow theorem ,1949*）

Let A be an analytic set of $\mathbb{P}^n$, then A is an algebraic set, i.e.

$$A = \bigcap_{j=1}^{N}\{P_j(z_0,...,z_n) = 0\}$$

where P_j is a homogeneous polynomial.

Using the Remmert-Stein theorem:

X- a complex manifold, $A \subseteq X$ an analytic set, $Z \subseteq X \setminus A$ is an analytic subset (of $X \setminus A$). If $\dim(Z, x) > \dim A$ for all $x \in Z$, then the closure $\overline{Z}$ in X is also an analytic set of X.

Consider the natural map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, then $Z := \pi^{-1}(A)$ is an analytic set of $\mathbb{C}^{n+1} \setminus \{0\}$. we have $\dim Z \geq 1 > \dim\{0\}$, Using Remmert-Stein, $\overline{Z}$ is an analytic set of $\mathbb{C}^{n+1}$. So, for a small disk $\triangle$ around $0 \in \mathbb{C}^{n+1}$,

$$\overline{Z} \cap \triangle = \bigcap_{j=1}^{N} \{f_j(z_1, ..., z_n) = 0\}$$

where $f_j \in \mathcal{O}(\triangle)$.

Let $f_j = \sum_{k=0}^{\infty} P_{j,k}$ be the Taylor expansion of f_j, where $P_{j,k}$ is a homogenous polynomials of degree k.

Claim: $\overline{Z} \cap \triangle = \left(\bigcap_{j,k}\{P_{j,k} = 0\}\right) \cap \triangle$. Denote $W := \bigcap_{j,k}\{P_{j,k} = 0\}$,

$W \cap \triangle \subseteq \overline{Z} \cap$ is obvious.

By the definition of π, Z is invariant by homotheties, so, for any $z \in \overline{Z} \cap \triangle$, $|t| \ll 1$, we have $f_j(t, z) = 0$.Write

$$f_j(tz) = \sum_{k=0}^{\infty} P_{j,k}(z) t^k = 0 \quad \Rightarrow \quad P_{j,k}(z) = 0$$

so, $\overline{Z} \cap \triangle \subseteq W \cap \triangle$.

$\Rightarrow \overline{Z} = W$ by the $\mathbb{C}^*$-invariance of $\overline{Z}$ and W. By the noetherian property of $\mathbb{C}[z_0, ..., z_n]$, $\exists$ finite polynomials $P_j, \quad 1 \leq j \leq k$, s.t.

$$W = \bigcap_{j=1}^{k} \{P_j = 0\}$$

推论 7.2.10. *Any analytic subset of an algebraic variety is also algebraic.*

Lefschetz's $(1-1)$-theorem

Exercise: X is a compact complex manifold, L, A be two holomorphic line bundles over X, A is positive($\iff$ ample). Then for $k \gg 1$, $H^0(X, L \otimes A^k) \neq \{0\}$. （与之前证明几乎完全一样）

Recall:$0 \to \mathcal{O} \to \mathcal{M}^* \to Div \to 0$ induces

$$Div(X) := H^0(X, Div) \to H^1(X, \mathcal{O}^*) =: Pic(X)$$

定理 7.2.11. *If X is an algebraic manifold, then for all $L \in Pic(X)$, $\exists$ divisor D s.t. $L = \mathcal{O}(D)$.*

证明. Take non-zero sections $S \in H^0(X, L \otimes A^k)$, $t \in H^0(X, A^k)$, then $\frac{s}{t}$ is a meromorphic section of L. Let D be the divisor associated to $\frac{s}{t}$, then

$$L \cong \mathcal{O}(D)$$

□

定理 7.2.12. *(Lelong-Poincare equation)*

Let $s \in H^0(X, L) \setminus \{0\}$, then

$$\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|s|_h = [s^{-1}(0)] - \frac{\sqrt{-1}}{2\pi}\Theta_{(L,h)} \tag{$*$}$$

where $[s^{-1}(0)]$ is defined as follows:

$$\langle [s^{-1}(0)], \psi \rangle = \int_{s^{-1}(0)} \psi$$

where ψ is an $(n-1, n-1)$-form on X. （假设.. 有度量；在分布意义下求导）

(Current of integration)

证明. （以后再证） □

$(*) \Rightarrow$

$$c_1(L) = \{\frac{\sqrt{-1}}{2\pi}\Theta_{(L,h)}\} = \{[s^{-1}(0)]\}$$

注记 7.2.13. $(*)$ *also holds for moromorphic sections.*

推论 7.2.14. *X be an algebraic manifold, then $\forall \alpha \in H^{1,1}(X, \mathbb{Q})$, we have a divisor D with $\mathbb{Q}$-coefficients s.t.*

$$[\alpha] = \{[D]\}$$

(Hodge conjecture for $(1,1)$-classes)

Fact: X is a compact complex manifold, $V \subseteq X$ is an analytic set of pure $\dim_{\mathbb{C}} V = p$. Then the current $[V]$ associated to V_{reg}:

$$\langle [V], \psi \rangle := \int_{V_{\text{reg}}} \psi|_{V_{\text{reg}}}$$

where $\psi \in C^\infty(X, \bigwedge^{p,p})$, defines a class $\{[V]\} \in H^{n-p,n-p}(X, \mathbb{Z})$.

Hodge conjecture: X is a complex algebraic manifold, then for all $\alpha \in H^{n-p,n-p}(X,\mathbb{Q})$, $\exists$ analytic sets V_k of pure dimension p and rational numbers r_k, s.t.

$$\alpha \in \{\sum_{k=1}^{N} r_k[V_k]\}$$

（这个猜想作为练习，说不定就做出来了………………）

Known case: $p = n-1$, it is Lef. $(1,1)$-theorem.

Exercise: also true for $p = 1$ (Using Hard Lef) And, $p = 0, p = n$...

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Pluripotential theroy/Positive currents

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