

复几何

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本课程参考以下教材：

1. Demailly: Complex analytic and differential geometry.
2. Huybrechts: Complex geometry: an introduction.
3. Morrow, Kodaira: Complex manifolds.
4. Grauert, Remmert: Coherent analytic sheaves.
5. Hormander: An introduction to complex analysis in several variables.
6. Griffiths, Harris: Principles of algebraic geometry.

在五道口也要红专并进、理实交融呀 ~

目录

1	多复变函数	3
1.1	多元全纯函数	3
1.2	解析延拓与 Hartogs 现象	8
2	层与层上同调	12
2.1	层的上同调	12
2.2	Cech 上同调	15
3	Hermite 向量丛	24
3.1	联络与曲率	24
3.2	向量丛的构造	28
3.3	陈省身示性类	29
3.4	Hermite 向量丛	32
4	L^2 Hodge 理论	38
4.1	向量丛上的微分算子	38
4.2	椭圆算子的基本性质	40
4.3	紧黎曼流形的 Hodge 理论	42
4.4	Kähler 流形	49
4.5	Hodge theory on compact manifold	52

第 1 章 多复变函数

1.1 多元全纯函数

首先快速回顾单复变函数的知识。我们通常用 Ω 来表示 \mathbb{C} 的开子集, $z = x + iy$ 为 \mathbb{C} 的坐标。对于 $z \in \mathbb{C}$ 以及实数 $R > 0$, 我们令

$$\mathbb{D}(z, R) := \{w \in \mathbb{C} \mid |w - z| < R\}$$

为以 z 为圆心 R 为半径的开圆盘。

此外, 我们有如下常用记号:

$$\begin{cases} dz := dx + i dy \\ d\bar{z} := dx - i dy \end{cases} \quad \begin{cases} \frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \\ \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \end{cases}$$

对于函数 $f: \Omega \rightarrow \mathbb{C}$, 称 f 是全纯 (holomorphic) 的, 若在 Ω 中成立

$$\bar{\partial}f := \frac{\partial f}{\partial \bar{z}} d\bar{z} = 0$$

我们知道, f 是全纯的当且仅当 f 在 Ω 处处能够局部地展开为收敛幂级数。

对于 \mathbb{C} 中的紧致集 K , 称函数 $f: K \rightarrow \mathbb{C}$ 是全纯的, 如果存在 K 的开邻域 $\Omega \supseteq K$, 使得 f 可延拓为 Ω 上的全纯函数。

单复变函数论中有如下重要结果:

定理 1.1.1. (柯西积分公式) 设 $\mathbb{D} \subseteq \mathbb{C}$ 为 \mathbb{C} 中的开圆盘, $f: \mathbb{D} \rightarrow \mathbb{C}$ 为 \mathbb{D} 上的全纯函数, 且在 $\partial\mathbb{D}$ 连续, 则对于任意 $w \in \mathbb{D}$, 成立

$$f(w) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(z)}{z - w} dz$$

此定理能推导出单变量全纯函数理论的 “almost everything”. 这里不再赘述。

我们开始考虑多变量全纯函数。

定义 1.1.2. 设 $\Omega \subseteq \mathbb{C}^n$ 为 \mathbb{C}^n 的开子集, 函数 $f: \Omega \rightarrow \mathbb{C}$ 称为 (多变量) 全纯函数, 如果满足以下条件:

- (1) f 是连续函数;
- (2) 对任意 $1 \leq j \leq n$, 以及任意固定的 $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n \in \mathbb{C}$, 关于 z_j 的单变量函数

$$z_j \mapsto f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n)$$

是 (单变量) 全纯函数。

事实上, 如果该定义中的 (2) 成立, 那么能推出 (1) 成立, 也就是说此定义中的 (1) 可以去掉。其证明比较复杂, 我们承认之。

记号 1.1.3. 对于 \mathbb{C}^n 的开子集 Ω , 我们记

$$\mathcal{O}(\Omega) := \{f: \Omega \rightarrow \mathbb{C} \mid f \text{ 是 } \Omega \text{ 上的全纯函数}\}$$

容易知道 $\mathcal{O}(\Omega)$ 有显然的 \mathbb{C} -代数结构。

本节将说明, 多变量全纯函数具有一些与单变量全纯函数类似的性质。

记号 1.1.4. 对于 $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ 以及 $R = (R_1, R_2, \dots, R_n) \in \mathbb{R}^n$, 并且 $R_j > 0$ ($\forall 1 \leq j \leq n$), 则我们记

$$\mathbb{D}(z, R) := \mathbb{D}(z_1, R_1) \times \mathbb{D}(z_2, R_2) \times \cdots \times \mathbb{D}(z_n, R_n)$$

称为以 z 为中心, R 为半径的**多圆柱** (*polydisk*)。

对于多圆柱 $\mathbb{D}(z, R)$, 我们记

$$\Gamma(z, R) := \partial\mathbb{D}(z_1, R_1) \times \partial\mathbb{D}(z_2, R_2) \times \cdots \times \partial\mathbb{D}(z_n, R_n)$$

称为 $\mathbb{D}(z, R)$ 的**特征边界** (*distinguished boundary*)。

特别注意特征边界 $\Gamma(z, R)$ 并不等于该多圆柱的边界 $\partial\mathbb{D}(z, R)$ 。

定理 1.1.5. (多变量全纯函数的柯西积分公式)

设 $f: \overline{\mathbb{D}(z, R)} \rightarrow \mathbb{C}$ 为全纯函数, 则对任意的 $w \in \mathbb{D}(z, R)$, 成立

$$f(w) = \frac{1}{(2\pi i)^n} \int_{\Gamma(z, R)} \frac{f(\xi) d\xi_1 d\xi_2 \cdots d\xi_n}{(\xi_1 - w_1)(\xi_2 - w_2) \cdots (\xi_n - w_n)}$$

证明. 由多变量全纯函数的定义, 反复使用单变量全纯函数的柯西积分公式即可。这是容易的。□

与单复变函数完全类似, 我们也有泰勒展开:

推论 1.1.6. (多元全纯函数的泰勒展开公式)

对于 $f \in \mathcal{O}(\Omega)$, 其中 $\Omega \subseteq \mathbb{C}^n$ 为开子集, 则对于任何多圆柱 $\mathbb{D}(z_0, R)$, 如果 $\overline{\mathbb{D}(z_0, R)} \subseteq \Omega$, 则对于任意 $w \in \mathbb{D}(z_0, R)$, 成立

$$f(w) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (w - z_0)^\alpha$$

其中

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{\Gamma(z_0, R)} \frac{f(z)}{(z - z_0)^{\alpha+1}} dz_1 dz_2 \cdots dz_n = \frac{f^{(\alpha)}(z_0)}{\alpha!}$$

注意这里的 α 为多重指标, 即 $\alpha = (\alpha_1, \dots, \alpha_n)$, 其中每个 α_i 都为非负整数。我们记

$$\begin{aligned} z^\alpha &:= z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \\ \alpha! &:= \alpha_1! \alpha_2! \cdots \alpha_n! \\ f^{(\alpha)} &:= (\partial_{z_1})^{\alpha_1} (\partial_{z_2})^{\alpha_2} \cdots (\partial_{z_n})^{\alpha_n} f \\ \alpha + 1 &:= (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_n + 1) \end{aligned}$$

其中 $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, f 为 n 元全纯函数。

证明. 与单复变函数的情形完全类似, 可由柯西积分公式得到。□

定理 1.1.7. (柯西不等式) 对于 \mathbb{C}^n 的开子集 Ω , 若 $f \in \mathcal{O}(\Omega)$, 多圆柱 $\overline{\mathbb{D}(z_0, R)} \subseteq \Omega$, 则对任意多重指标 $\alpha \in \mathbb{N}^n$, 成立

$$|f^{(\alpha)}(z_0)| \leq \frac{\alpha!}{R^\alpha} \sup_{z \in \Gamma(z_0, R)} |f(z)|$$

证明. 与单复变函数的情形完全类似. 利用多元泰勒展开 (推论1.1.6) 即可. \square

推论 1.1.8. 设 $\Omega \subseteq \mathbb{C}^n$ 为连通开集, $f \in \mathcal{O}(\Omega)$ 满足 $\forall 1 \leq k \leq n, \frac{\partial f}{\partial z_k}$ 在 Ω 上恒为 0, 则 f 在 Ω 上为常值函数。

推论 1.1.9. (刘维尔定理) 设 $f \in \mathcal{O}(\mathbb{C}^n)$, 并且满足

$$|f(z)| \leq A(1 + |z|)^B$$

其中 A, B 为正实数, 那么 f 必为次数不超过 B 的多项式函数。

这些性质于单变量全纯函数雷同, 证明也是类似的。

推论 1.1.10. (*Montel* 定理)

设 Ω 为 \mathbb{C}^n 的开子集, 则 $\mathcal{O}(\Omega)$ 中的任何局部一致有界的全纯函数列都存在一致收敛的子列。

证明. 仍类似于单复变全纯函数的情形. 使用柯西积分公式, 再配合 Arzela-Ascoli 定理即可. 从略. \square

现在, 简单介绍一些复的微分形式. 对于 \mathbb{C}^n , 记其复坐标为 (z_1, z_2, \dots, z_n) ; 视 \mathbb{C}^n 为 $2n$ 维实线性空间,

$$z_k = x_k + iy_k$$

从而引入

$$dz_k = dx_k + idy_k \quad (1,0)\text{形式}$$

$$d\bar{z}_k = dx_k - idy_k \quad (0,1)\text{形式}$$

定义 1.1.11. ((p, q) -形式)

设 Ω 为 \mathbb{C}^n 的非空开集, 则形如

$$u(z) = \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ}(z) dz_I \wedge d\bar{z}_J$$

的光滑张量场称为 (p, q) -形式. 记 Ω 上的 (p, q) -形式之全体为 $C_{p,q}^\infty(\Omega)$.

这里的 I, J 为多重指标。“光滑”指的是系数函数 a_{IJ} 为 Ω 上的光滑复值函数。另外，显然 $(0,0)$ -形式即为光滑函数； $C_{p,q}^\infty(\Omega)$ 具有显然的复线性空间结构，事实上还是 $C^\infty(\Omega)$ -模。

记号 1.1.12. ($\bar{\partial}$ -算子) 定义算子

$$\bar{\partial} : C_{p,q}^\infty(\Omega) \rightarrow C_{p,q+1}^\infty(\Omega)$$

如下: 对于 (p,q) -形式

$$u := \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ} dz_I \wedge d\bar{z}_J$$

则

$$\bar{\partial} u = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{k=1}^n \frac{\partial a_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J$$

类似地，也有

$$\partial : C_{p,q}^\infty(\Omega) \rightarrow C_{p+1,q}^\infty(\Omega)$$

它们与外微分算子 d 满足关系

$$d = \partial + \bar{\partial}$$

由 $d^2 = 0$ ，易知

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0$$

以下事实显然成立：

引理 1.1.13. 对于区域 Ω 上的光滑函数 $f \in C^\infty(\Omega)$ ，则 f 全纯当且仅当 $\bar{\partial}f = 0$ 。

注记 1.1.14. (*Dolbeault* 上同调) 对于 $\Omega \subseteq \mathbb{C}^n$ ，注意 $\bar{\partial}^2 = 0$ ，从而对任意 $p \geq 0$ ，有上链复形 $C_{p,\bullet}^\infty(\Omega)$ ：

$$\cdots \rightarrow C_{p,q-1}^\infty(\Omega) \xrightarrow{\bar{\partial}} C_{p,q}^\infty(\Omega) \xrightarrow{\bar{\partial}} C_{p,q+1}^\infty(\Omega) \rightarrow \cdots$$

称上同调群

$$H^{p,q}(\Omega) := H^q(C_{p,\bullet}^\infty(\Omega), \bar{\partial})$$

为区域 Ω 的 *Dolbeault* 上同调群。

类似于外微分 d 的 de-Rham 上同调群，*Dolbeault* 上同调群与 Ω 的拓扑联系密切。例如，以下定理十分重要，我们先陈述，以后再证明：

引理 1.1.15. (*Dolbeault-Grothendieck 引理*)

设 $\mathbb{D} \subseteq \mathbb{C}^n$ 为多圆柱, 则对于任意 $p, q \geq 0$,

$$H^{p,q}(\Omega) = 0$$

不难发现它与 de Rham 上同调的 Poincare 引理有些类似。

1.2 解析延拓与 Hartogs 现象

上一节介绍了多复变函数的一些“普通的”(与单变量类似)性质, 本节开始介绍多复变函数的一些独特性质。

引理 1.2.1. 设 $f \in C_c^\infty(\mathbb{C})$ 为复平面上的紧支光滑函数, 则对任意 $z \in \mathbb{C}$, 成立

$$\frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial f / \partial \bar{\tau}}{\tau - z} d\tau \wedge d\bar{\tau} = f(z)$$

证明. 基本的微积分练习。考虑换元 $\tau = z + re^{i\theta}$, 则易知

$$\begin{aligned} d\tau \wedge d\bar{\tau} &= -2ir dr \wedge d\theta \\ \frac{\partial r}{\partial \bar{\tau}} &= \frac{1}{2} e^{i\theta} \\ \frac{\partial \theta}{\partial \bar{\tau}} &= -\frac{1}{2ir} e^{i\theta} \end{aligned}$$

因此有

$$\begin{aligned} \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial f / \partial \bar{\tau}}{\tau - z} d\tau \wedge d\bar{\tau} &= \frac{-1}{2\pi} \int_0^\infty dr \int_0^{2\pi} \left(-\frac{1}{ir} \frac{\partial f}{\partial \theta}(z + re^{i\theta}) \right) d\theta \\ &\quad + \frac{-1}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty \left(\frac{\partial f}{\partial r}(z + re^{i\theta}) \right) dr \\ &= 0 + \frac{-1}{2\pi} \int_0^{2\pi} -f(z) d\theta \\ &= f(z) \end{aligned}$$

证毕。 □

引理 1.2.2. (简单版本的 $\bar{\partial}$ -引理)

设 $n \geq 2$, $\varphi \in C_{0,1}^\infty(\mathbb{C}^n)$ 为具有紧支集的光滑 $(0,1)$ -形式, 且 $\bar{\partial}\varphi = 0$, 则存在 \mathbb{C}^n 上的紧支光滑函数 g , 使得

$$\bar{\partial}g = \varphi$$

证明. 记光滑 $(0,1)$ -形式 φ 为

$$\varphi = \sum_{k=1}^n \varphi_k(z_1, \dots, z_n) d\bar{z}_k$$

则

$$\bar{\partial}\varphi = \sum_{k,l} \frac{\partial \varphi_k}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_k = \sum_{1 \leq l < k \leq n} \left(\frac{\partial \varphi_k}{\partial \bar{z}_l} - \frac{\partial \varphi_l}{\partial \bar{z}_k} \right) d\bar{z}_l \wedge d\bar{z}_k$$

从而由 $\bar{\partial}\varphi = 0$ 可得对任意 $k \neq l$,

$$\frac{\partial \varphi_k}{\partial \bar{z}_l} = \frac{\partial \varphi_l}{\partial \bar{z}_k}$$

考虑如下的 \mathbb{C}^n 上的函数 ψ : 对于 $z = (z_1, \dots, z_n) \in \mathbb{C}^n$,

$$\psi(z) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\varphi_1(\tau; z_2, \dots, z_n)}{\tau - z_1} d\tau \wedge d\bar{\tau}$$

由 φ_1 的紧支性易知 ψ 为 \mathbb{C}^n 上的光滑函数。对于 $1 < k \leq n$, 有

$$\begin{aligned} \frac{\partial \psi(z)}{\partial \bar{z}_k} &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_1}{\partial \bar{z}_k}(\tau; z_2, \dots, z_n)}{\tau - z_1} d\tau \wedge d\bar{\tau} \\ &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_k}{\partial \bar{\tau}}(\tau; z_2, \dots, z_n)}{\tau - z_1} d\tau \wedge d\bar{\tau} \\ &= \varphi_k(z) \end{aligned}$$

上式对 $k = 1$ 显然也成立。因此 $\bar{\partial}\psi = \varphi$.

最后还需要证明 ψ 是紧支的。由于 φ 紧支, 存在足够大的 $R > 0$, 使得

$$\text{supp } \varphi \subseteq \mathbb{D}(0, R)$$

因此任意取定 $z \in \mathbb{C}^n$, 使得 z 的分量 z_2, z_3, \dots, z_n 之中至少有一个模长大于 R , 则由 ψ 的定义式直接得到 $\psi(z) = 0$. (注意: 这一步严重依赖 $n \geq 2$!) 也就是说, 存在 $z \notin \mathbb{D}(0, R)$ 使得 $\psi = 0$ 在 z 的某邻域内都成立。另一方面, 由于 $\bar{\partial}\psi = \varphi$ 且 $\text{supp } \varphi \subseteq \mathbb{D}(0, R)$, 从而 ψ 在 $\mathbb{D}(0, R)$ 外部全纯, 因此由解析延拓唯一性, ψ 在 $\mathbb{D}(0, R)$ 外部恒为零, 因此 ψ 紧支。□

此引理在单复变 $n = 1$ 的情形不成立:

例子 1.2.3. 设 $\varphi_1 \in C_0^\infty(\mathbb{C})$ 为复平面上的紧支光滑函数, 并且

$$\iint_{\mathbb{C}} \varphi_1(z) \neq 0$$

考虑 \mathbb{C} 上的 $(0,1)$ -形式 $\varphi = \varphi_1(z)d\bar{z}$, 则 $\bar{\partial}\varphi = 0$ 是平凡的, 但不存在紧支光滑函数 ψ 使得 $\bar{\partial}\psi = \varphi$.

证明. 若存在紧支光滑函数 ψ 使得 $\bar{\partial}\psi = \varphi$, 则 $\frac{\partial\psi}{\partial\bar{z}} = \varphi_1$. 于是

$$0 \neq \iint_{\mathbb{C}} \varphi_1(z) dz \wedge d\bar{z} = \iint_{\mathbb{C}} \frac{\partial\psi}{\partial\bar{z}} dz \wedge d\bar{z} = 0$$

产生矛盾。 □

以下是多复变函数解析延拓的令人惊讶的性质, 它与单复变函数有本质不同:

定理 1.2.4. (*Hartogs* 现象)

设 $\Omega \subseteq \mathbb{C}^n$ 为开集 ($n \geq 2$), $K \subset\subset \Omega$ 且为 \mathbb{C}^n 的紧子集, 则对任意的 $f \in \mathcal{O}(\Omega \setminus K)$, 都存在解析延拓 $F \in \mathcal{O}(\Omega)$, 使得

$$F|_{\Omega \setminus K} = f$$

证明. 取 K 与 Ω 直接的截断函数 $\psi \in C_0^\infty(\mathbb{C}^n)$, 使得 $0 \leq \psi \leq 1$,

$$K \subset\subset \text{supp } \psi \subset\subset \Omega$$

并且 $\psi|_K \equiv 1$. 考虑

$$\tilde{f} := (1 - \psi)f$$

则 \tilde{f} 在整个 Ω 上都有定义。注意

$$\bar{\partial}\tilde{f} = -(\bar{\partial}\psi)f + (1 - \psi)\bar{\partial}f$$

易知 $\text{supp } \bar{\partial}\tilde{f} \subseteq \text{supp } \psi$. 于是由引理1.2.2, 存在光滑函数 v , 使得 $\text{supp } v \subseteq \text{supp } \psi$, 并且 $\bar{\partial}v = \bar{\partial}\tilde{f}$, 从而考虑函数

$$F := (1 - \psi)f - v$$

则 $\bar{\partial}F = 0$, 从而 $F \in \mathcal{O}(\Omega)$. 又因为易知

$$F = f \quad (\forall z \in \Omega \setminus \text{supp } \psi)$$

从而由解析延拓唯一性, 有 $F|_{\Omega \setminus K} = f$. □

关于解析延拓, 再介绍如下结果:

引理 1.2.5. (*Hartogs figure*)

对于 $n > 1$, 正实数 $0 < r < R$, 以及 \mathbb{C}^{n-1} 的开子集 $\omega' \subseteq \omega$, 记 \mathbb{C}^n 的开子集

$$\Omega := ((\mathbb{D}(0, R) \setminus \mathbb{D}(0, r)) \times \omega) \cup (\mathbb{D}(0, R) \times \omega')$$

其中 $\mathbb{D}(0, r)$ 与 $\mathbb{D}(0, R)$ 分别为 \mathbb{C} 上的以原点为中心, r, R 为半径的开圆盘。则任意 $f \in \mathcal{O}(\Omega)$ 都可以 (唯一地) 解析延拓至

$$\tilde{\Omega} := \mathbb{D}(0, R) \times \omega$$

如此的区域 Ω 称之为 “**Hartogs figure**”。 Ω 的几何图像大致如下:

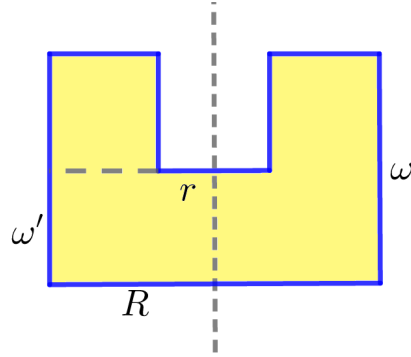


图: Hartogs figure 示意

证明. 容易知道

$$\Omega = \{(z_1, \tilde{z}) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid r < |z_1| < R, \tilde{z} \in \omega \text{ 或者 } |z_1| \leq r, \tilde{z} \in \omega'\}$$

对于 $f \in \mathcal{O}(\Omega)$, 定义 $\tilde{\Omega}$ 上的函数

$$\tilde{f}(z_1, \tilde{z}) := \frac{1}{2\pi i} \int_{|w|=\rho} \frac{f(w, \tilde{z})}{z_1 - w} dw$$

其中 ρ 为满足 $\max\{r, |z_1|\} < \rho < R$ 的任意实数。则易知如此定义的 \tilde{f} 为 f 在 $\tilde{\Omega}$ 上的解析延拓。 \square

第 2 章 层与层上同调

2.1 层的上同调

Today:

Sheaf cohomology

X a topological space, \mathcal{F} - sheaf (of abelian groups).

定义 2.1.1. (*resolution*)

(1) a resolution of \mathcal{F} is an exact sequence

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{F} \xrightarrow{d^0} \mathcal{F} \xrightarrow{d^1} \rightarrow \dots$$

定义 2.1.2. A sheaf \mathcal{A} is called injective, if if for any injective morphism $j : \mathcal{A} \rightarrow \mathcal{B}$ and for any morphism $\varphi : \mathcal{A} \rightarrow \mathcal{S}$, there exists an extension $\psi : \mathcal{B} \rightarrow \mathcal{S}$, such that

定理 2.1.3. the category of sheaves of abelian sheaves have enough injective objects, i.e. any \mathcal{F} can be embedded in some injective sheaf.

定义 2.1.4. Consider an injective resolution of \mathcal{F} , i.e. an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \mathcal{I}^2 \rightarrow \dots$$

where every $\mathcal{I}^k (k \geq 0)$ is injective.

\leadsto induces a sequence

$$0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{I}^0) \xrightarrow{d} \Gamma(X, \mathcal{I}^1) \xrightarrow{d} \Gamma(X, \mathcal{I}^2) \rightarrow \dots$$

Then

$$H^q(X, \mathcal{F}) := H^q(\Gamma(X, \mathcal{I}^\bullet))$$

then, $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

定义 2.1.5. A sheaf \mathcal{S} is called a flabby (flasque, in France), if for any open set $\Omega \subseteq X$, the morphism

$$\mathcal{S}(X) \rightarrow \mathcal{S}(\Omega)$$

is surjective.

定义 2.1.6.

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1$$

is an exact sequence is called a flabby resolution, if any \mathcal{F}^k is flabby.

定义 2.1.7.

$$H^q(X, \mathcal{F}) := \dots \text{by flabby resolution} \dots$$

证明. Homological Algebra...omit. □

the two definitions of Sheaf Cohomology are isomorphic.

Godement's construction

$$God(\mathcal{F})(U) := \{f : U \rightarrow \bigcup_{x \in U} \mathcal{F}_x \mid f(y) \in \mathcal{F}_y, \forall y \in U\} := \prod_{x \in U} \mathcal{F}_x$$

$God(\mathcal{F})$ is a sheaf, and it is flabby. and there is a canonical morphism $\mathcal{F}(U) \rightarrow God(\mathcal{F})(U)$ by $x \mapsto (x \mapsto s_x)$ is injective.

$$\mathcal{F}^0 := God(\mathcal{F})$$

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \twoheadrightarrow \text{coker}(j) = \mathcal{F}^0 / \mathcal{F}$$

and consider

$$\mathcal{F}^1 := \text{God}(\text{coker}(j))$$

.....then construct by induction... this is a flabby resolution of \mathcal{F} .

定义 2.1.8. (resolution by fine sheaves)

\mathcal{A} is a sheaf of ring, X is a paracompact topological space, \mathcal{A} is called a fine sheaf, if for any open covering

$$X = \bigcup_{\alpha} V_{\alpha} \quad , \mathcal{V} := \{V_{\alpha}\}$$

there exists a partition of unit subordinate to \mathcal{V} , (i.e. $\exists f_{\alpha} \in \mathcal{A}(V_{\alpha}), \text{supp}(\alpha) := \overline{\{x \in V_{\alpha} | f_{\alpha,x} \neq 0\}} \subseteq V_{\alpha}$, and $\sum_{\alpha} f_{\alpha} = 1$ (the sum is locally finite))

例子 2.1.9. X is a differential manifold, \mathcal{C}^{∞} is the sheaf of smooth functions, then \mathcal{C}^{∞} is a fine sheaf.

定理 2.1.10. \mathcal{S} is a sheaf of \mathcal{A} -modules, \mathcal{A} is a fine sheaf. then for any $q \geq 1$,

$$H^q(X, \mathcal{S}) = 0$$

证明. Consider a flabby(or injective) resolution

$$0 \rightarrow \mathcal{S} \xrightarrow{j} \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \mathcal{I}^2 \dots$$

where any $\mathcal{I}^k (k \geq 0)$ is a sheaf of \mathcal{A} -modules.

by definition,

$$H^q(X, m\mathcal{S}) := \frac{\ker d : \Gamma(\mathcal{I}^q) \rightarrow \Gamma(\mathcal{I}^{q+1})}{\Im d : \Gamma(\mathcal{I}^{q-1}) \rightarrow \Gamma(\mathcal{I}^q)}$$

Let $\alpha \in \ker\{d : \Gamma(\mathcal{I}^q) \rightarrow \Gamma(\mathcal{I}^{q+1})\}$ by the exactness of resolution, \exists an open covering $\mathcal{U} = (U_i)_i$, s.t. $\alpha|_{U_i} = d\beta_i$ where $\beta_i \in \mathcal{I}^{q-1}(U_i)$. Let $(\beta_i)_i$ be the partition of unit w.r.t. \mathcal{U} . consider

$$\beta := \sum_i f_i \beta_i$$

(well defined). Then $d\beta = \alpha$

□

2.2 Čech 上同调

Čech cohomology

X - a topological space, \mathcal{F} - a sheaf of abelian group.

$$\mathcal{U} = (U_\alpha)_{\alpha \in I}$$

is an open covering.

notation: $U_{\alpha_1, \dots, \alpha_q} := \bigcap_{i=1}^q U_{\alpha_i}$.

Čech q -chain w.r.t \mathcal{U} :

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{(\alpha_1, \dots, \alpha_q) \in \mathcal{I}^{q+1}} \mathcal{F}(U_{\alpha_1, \dots, \alpha_q})$$

$$c \in C^q(\mathcal{U}, \mathcal{F})$$

means that we have a family of sections $C_{\alpha_1, \dots, \alpha_q} \in \mathcal{F}(U_{\alpha_1, \dots, \alpha_q})$ with the relation

$$C_{\alpha_0, \dots, \alpha_j, \dots, \alpha_i, \dots} = -C_{\dots}$$

(Č)ech differential:

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{F})$$

$$\delta^q(c)_{\alpha_0, \dots, \alpha_{q+1}} := \sum_{0 \leq k \leq q+1} (-1)^k c_{\dots \hat{\alpha}_k \dots} |_{U_{\alpha_0, \dots, \alpha_{q+1}}}$$

性质 2.2.1.

$$\delta^q \circ \delta^q = 0$$

so, we have Čech cohomology

$$H^q(\mathcal{U}, \mathcal{F}) := \ker \delta^q / \operatorname{Im} \delta^{q-1}$$

example:

$$C^0(\mathcal{U}, \mathcal{F}) := \prod_{\alpha \in I} \mathcal{F}(U_\alpha)$$

$$c = (c_\alpha)_{\alpha \in I} \in C^0(\mathcal{U}, \mathcal{F})$$

$$\delta^0 c = 0 \iff (\delta^0 c)_{\alpha_0 \alpha_1} := (c_{\alpha_1} - c_{\alpha_0})|_{U_{\alpha_0 \alpha_1}} = 0$$

so, $c_{\alpha_0} = c_{\alpha_1}$ on $U_{\alpha_0 \alpha_1}$.

$$\rightsquigarrow H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X).$$

例子 2.2.2. (1) consider $X = \Delta \setminus \{0\}$, where $\Delta = \{(z_1, z_2) | |z_1| < 1, |z_2| < 1\}$. Consider the covering

$$\mathcal{U} = U_1 \cup U_2$$

where

$$U_1 := \{(z_1, z_2) \in \Delta | z_1 \neq 0\} = \mathbb{D}^* \times \mathbb{D}$$

$$U_2 := \{(z_1, z_2) \in \Delta | z_2 \neq 0\} = \mathbb{D} \times \mathbb{D}^*$$

then

$$U_1 \cap U_2 = \mathbb{D}^* \times \mathbb{D}^*$$

consider $H^0(X, \mathcal{O}) = \mathcal{O}(X) \cong \mathcal{O}(\Delta) = \{f : \Delta \rightarrow \mathbb{C} \text{ holomorphic}\}$.

$$H^1(\mathcal{U}, \mathcal{O}) = \ker \delta^1 / \text{Im } \delta^0$$

$$\delta^1 : C^1(\mathcal{U}, \mathcal{O}) \rightarrow C^2(\mathcal{U}, \mathcal{O}) \subseteq \prod_{\alpha_0, \alpha_1, \alpha_2} \mathcal{O}(U_{\alpha_0, \alpha_1, \alpha_2}) = 0$$

$$\ker \delta^1 = C^1(\mathcal{U}, \mathcal{O}) = \{c = c(\alpha_0, \alpha_1) | c_{\alpha_0, \alpha_1} \in \mathcal{O}(U_{\alpha_0, \alpha_1})\} = \{c \in \mathcal{O}(U_1 \cap U_2)\} = \{c = \sum_{m, n \in \mathbb{Z}} a_{mn} z_1^m z_2^n \text{ convergent}\}$$

$$\delta^0 : C^0(\mathcal{U}, \mathcal{O}) \rightarrow C^1(\mathcal{U}, \mathcal{O})$$

$$(\delta^0 c)_{12} = (c_2 - c_1)|_{U_{12}}$$

where $c_2 \in \mathcal{O}(U_2)$ and $c_1 \in \mathcal{O}(U_1)$. note that

$$\mathcal{O}(U_1) = \{c(z_1, z_2) = \sum_{m \in \mathbb{Z}, n \geq 0} a_{mn} z_1^m z_2^n \text{ convergent}\}$$

$$\mathcal{O}(U_2) = \{c(z_1, z_2) = \sum_{n \in \mathbb{Z}, m \geq 0} a_{mn} z_1^m z_2^n \text{ convergent}\}$$

$$\text{So, } H^1(\mathcal{U}, \mathcal{O}) = \{c(z_1, z_2) = \sum_{m, n < 0} a_{mn} z_1^m z_2^n\}$$

例子 2.2.3. (complex projective space)

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

$$(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$$

for some $\lambda \in \mathbb{C}^*$.

$$\mathbb{C}P^n = \{[z_0, \dots, z_n] | \text{not all } z_k = 0, z_i \in \mathbb{C}\} = \bigcup_{0 \leq p \leq n} V_p$$

where

$$V_k = \{[z_0, \dots, z_n] | z_k \neq 0\} \cong \{(\frac{z_0}{z_k}, \dots, 1, \dots, \frac{z_n}{z_k}) | z_i \in \mathbb{C}, i \neq k, z_k \neq 0\} \cong \mathbb{C}^n$$

this is a holo chart.

$$\mathbb{C}P^1 = V_0 \cup V_1, \mathcal{V} = \{V_0, V_1\}$$

HW: compute $H^q(\mathcal{V}, \mathcal{O})$.

Answer:

$$H^0 \cong \mathbb{C}, H^1 \cong 0$$

Correction:

\mathcal{A} : Sheaf of rings (with unit)

X : paracompact topological space,

定义 2.2.4. \mathcal{A} is called fine, if for any open covering $\mathcal{U} = (V_\alpha)_{\alpha \in \mathcal{I}}$, there exist $s_\alpha \in \mathcal{A}(X)$ such that $\text{supp}(s_\alpha) \subseteq V_\alpha$,

$$\sum_{\alpha} s_\alpha = 1$$

(this is a locally finite sum)

注记 2.2.5. we call \mathcal{A} is a **soft sheaf**, if for any closed set $K \subseteq X$, the morphism

$$\mathcal{A}(X) \rightarrow \mathcal{A}(K)$$

is surjective. where $\mathcal{A}(K) := \Gamma(K, \mathcal{A}|_K)$

fact: \mathcal{A} is fine if and only if $\mathcal{H}om(\mathcal{A}, \mathcal{A})$ is soft. (omit)

Recall:

Cech cohomology: X topological space, $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$,

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 < \dots < \alpha_q} \mathcal{F}(\alpha_1, \dots, \alpha_q)$$

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{F})$$

fact: $H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Today:

定义 2.2.6. Let $\mathcal{V} = (V_\beta)_{\beta \in \mathcal{J}}$ be another open covering, then \mathcal{V} is called a refinement of \mathcal{U} , if there exists a map

$$\rho : \mathcal{J} \rightarrow \mathcal{I}$$

such that

$$V_\beta \subseteq U_{\rho(\beta)}$$

性质 2.2.7. Let \mathcal{V} be a refinement of \mathcal{U} , then ρ induces a map

$$\rho^q : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^q(\mathcal{V}, \mathcal{F})$$

$$(\rho^q C)_{\beta_0, \dots, \beta_q} \mapsto C_{\rho(\beta_0), \dots, \rho(\beta_q)}|_{V_{\beta_0, \dots, \beta_q}}$$

ρ is a morphism of complexes.

so, ρ induces a map

$$H^q(\rho) : H^q(\mathcal{U}, \mathcal{F}) \rightarrow H^q(\mathcal{V}, \mathcal{F})$$

Let $\tilde{\rho} : \mathcal{J} \rightarrow \mathcal{I}$ be another refinement of \mathcal{U}

(induces $H^q(\tilde{\rho}) : H^q(\mathcal{U}, \mathcal{F}) \rightarrow H^q(\mathcal{V}, \mathcal{F})$) then $\rho, \tilde{\rho}$ are homotopic (chain homotopy $\rightsquigarrow H^q(\rho) = H^q(\tilde{\rho})$)

so, if $\rho : \mathcal{J} \rightarrow \mathcal{I}$ is refinement, then

$$H^q(\rho)$$

is independent of the refinement.

定义 2.2.8.

$$\check{H}^q(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F})$$

i.e. $a \in H^q(\mathcal{U}, \mathcal{F}) \sim \in H^q(\mathcal{V}, \mathcal{F})$ iff \exists a refinement \mathcal{W} of \mathcal{U} and \mathcal{V} such that a, b have the same image in $H^q(\mathcal{W}, \mathcal{F})$

注记 2.2.9.

$$\check{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$$

Exercise: For $q = 1$, if \mathcal{V} is a refinement of \mathcal{U} , then

$$H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(\mathcal{V}, \mathcal{F})$$

is injective.

so, for any open cover \mathcal{U} ,

$$H^1(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{F})$$

is injective.

Homological Algebra recall: let $(K^\bullet, d_K), (L^\bullet, d_L)$ and (M^\bullet, d_M) , if we have a short exact sequence

$$0 \rightarrow K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} M^\bullet \rightarrow 0$$

then it induces a long exact sequence :

$$\dots \rightarrow H^q(K^\bullet) \rightarrow H^q(L^\bullet) \rightarrow H^q(M^\bullet) \rightarrow H^{q+1}(K^\bullet) \rightarrow \dots$$

analogy of Cech cohomology: X is a topological space, \mathcal{U} is an open covering of X . \mathcal{A} and \mathcal{B} sheaves on X , Let

$$\varphi : \mathcal{A} \rightarrow \mathcal{B}$$

be a morphism, then it induces

$$\varphi^\bullet : C^\bullet(\mathcal{U}, \mathcal{A}) \rightarrow C^\bullet(\mathcal{U}, \mathcal{B})$$

Let

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

be an exact sequence of sheaves, then we have: for any open set Ω ,

$$0 \rightarrow \mathcal{A}(\Omega) \rightarrow \mathcal{B}(\Omega) \rightarrow \mathcal{C}(\Omega)$$

left exact.

Example: consider

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \rightarrow 0$$

is exact on $bbC^\times := \mathbb{C} \setminus \{0\}$

but we have :

$$0 \rightarrow \mathcal{A}(\Omega) \xrightarrow{\psi} \mathcal{B}(\Omega) \rightarrow \text{Im } \psi(\Omega) \rightarrow 0$$

is exact.

First we have the following exact sequence

$$C^q(\mathcal{U}, \mathcal{A}) \rightarrow C^q(\mathcal{U}, \mathcal{B}) \rightarrow C_B^q(\mathcal{U}, \mathcal{C}) \rightarrow 0$$

where C_B^q is the image of ...

then we get an exact sequence

$$0 \rightarrow (C^\bullet(\mathcal{U}, \mathcal{A}), \delta) \rightarrow (C^\bullet(\mathcal{U}, \mathcal{B}), \delta) \rightarrow (C_B^\bullet(\mathcal{U}, \mathcal{C}), \delta) \rightarrow 0$$

it induces a long exact sequence

$$\dots \rightarrow H^q(\mathcal{U}, \mathcal{A}) \rightarrow H^q(\mathcal{U}, \mathcal{B}) \rightarrow H_B^q(\mathcal{U}, \mathcal{C}) \rightarrow H^{q+1}(\mathcal{U}, \mathcal{A}) \rightarrow \dots$$

定理 2.2.10. *If X is paracompact,*

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

is a sheaf exact sequence. Then there is a long exact sequence

$$\dots \rightarrow \check{H}^q(X, \mathcal{A}) \rightarrow \check{H}^q(X, \mathcal{B}) \rightarrow \check{H}^q(X, \mathcal{C}) \rightarrow \check{H}^{q+1}(X, \mathcal{A}) \rightarrow \dots$$

证明. Key lemma: need to prove

$$\lim_{\vec{U}} H^q(\mathcal{U}, \mathcal{C}) = \lim_{\vec{U}} H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C})$$

if X is paracompact.

Omit. □

if

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

exact,

recall: (cohomology by resolutions)

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots$$

flabby resolution. then it induces

$$0 \rightarrow \Gamma(X, \mathcal{A}) \rightarrow \Gamma(X, \mathcal{F}^0) \rightarrow \Gamma(X, \mathcal{F}^1) \rightarrow \dots$$

then define the sheaf cohomology...

we have a long exact sequence

$$\dots \rightarrow H^q(X, \mathcal{A}) \rightarrow H^q(X, \mathcal{B}) \rightarrow H^q(X, \mathcal{C}) \rightarrow H^{q+1}(X, \mathcal{A}) \rightarrow \dots$$

it is homological algebra...

定理 2.2.11. (*Leray's acyclic theorem*) Let $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$ be an open covering of X , (\mathcal{F} is a sheaf on X), if satisfying

$$H^k(U_{\alpha_0, \dots, \alpha_q}) = 0$$

for any $k \geq 1$, then

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{H}^q(X, \mathcal{F})$$

and if X is paracompact, we also have

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{H}^q(X, \mathcal{F}) \cong H^q(X, \mathcal{F})$$

(this \mathcal{U} is called acyclic covering)

de Rham- Weil theorem

定义 2.2.12. \mathcal{F} is a sheaf on X , Ω is an open set of X , then \mathcal{F} is called **acyclic sheaf** if

$$H^q(\Omega, \mathcal{F}) = 0$$

for any $q \geq 1$.

定理 2.2.13. Let

$$0 \rightarrow \mathcal{F} \rightarrow (L^\bullet, d)$$

be an acyclic resolution of \mathcal{F} (i.e. L^q is acyclic on X) then

$$H^q(X, \mathcal{F}) \cong H^q(\Gamma(X, L^\bullet), d)$$

for any $q \geq 0$.

(先看例子)

例子 2.2.14. Let X be a differential manifold, \mathcal{E}^p : sheaf of smooth p -forms, then we have a resolution (de Rham complex)

$$0 \rightarrow \mathbb{R} \hookrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \mathcal{E}^3 \rightarrow \dots$$

where d differential operators. (Why it is a resolution? because of Poincare lemma...locally solvable..)

Note that

$$\mathcal{E}^0 = \mathcal{C}^\infty$$

\mathcal{E}^p is a sheaf of \mathcal{C}^∞ -modules..

then we have

$$H^q(X, \mathcal{E}^p) = 0$$

for all $q \geq 1$

and then

$$H^q(X, \mathbb{R}) \cong \frac{\ker(d : \Gamma(X, \mathcal{E}^q) \rightarrow \Gamma(X, \mathcal{E}^{q+1}))}{\operatorname{Im}(d : \Gamma(X, \mathcal{E}^{q-1}) \rightarrow \Gamma(X, \mathcal{E}^q))} = H_{DR}^q(X, \mathbb{R})$$

例子 2.2.15. Let X be a complex manifold, $\mathcal{E}^{p,q}$ sheaf of smooth (p, q) forms, Ω^p is the sheaf of holomorphic p -forms (i.e. $(p, 0)$ -form φ with $\bar{\partial}\varphi = 0$).

Then we have resolution

$$0 \rightarrow \Omega^p \xrightarrow{j} \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \rightarrow \dots$$

(Why it is a resolution? because of the Dolbeault lemma), remain to Exercise...

$$H^q(X, \Omega^p) \cong H^{p,q}_\partial(X, \mathbb{C})$$

Today: de Rham-Weil Isomorphism Thm

定理 2.2.16. *Let X be a topological space, \mathcal{F} be a sheaf of abelian groups on X ,*

$$0 \rightarrow \mathcal{F} \rightarrow (\mathcal{L}^\bullet, d)$$

be an acyclic resolution, i.e.

$$H^k(X, \mathcal{L}^q) = 0$$

for all $k \geq 1$ and $q \geq 0$. Then,

$$H^q(X, \mathcal{F}) \cong H^q((\Gamma(\mathcal{L}^\bullet), d))$$

证明. Since

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{d^0} \mathcal{L}^1 \xrightarrow{d^1} \mathcal{L}^2 \rightarrow \dots$$

be an exact sequence, denote

$$\mathcal{Z}^q := \ker d^q$$

then we have short exact sequences

$$0 \rightarrow \mathcal{Z}^q \rightarrow \mathcal{L}^q \rightarrow \mathcal{Z}^{q+1} \rightarrow 0$$

for any q . They induce long exact sequence of cohomology groups:

$$\dots \rightarrow H^k(X, \mathcal{Z}^q) \rightarrow H^k(X, \mathcal{L}^q) \rightarrow H^k(X, \mathcal{Z}^{q+1}) \xrightarrow{\partial} H^{k+1}(X, \mathcal{L}^q) \rightarrow H^{q+1}(X, \mathcal{L}^q) \rightarrow \dots$$

For any $k \geq 1$, since \mathcal{L}^q are acyclic on X ,

$$H^k(X, \mathcal{Z}^{q+1}) \cong H^{k+1}(X, \mathcal{Z}^q)$$

and for $k = 0$, we have

$$0 \rightarrow H^0(X, \mathcal{Z}^q) \rightarrow H^0(X, \mathcal{L}^q) \rightarrow H^0(X, \mathcal{Z}^{q+1}) \rightarrow H^1(X, \mathcal{Z}^q) \rightarrow H^1(X, \mathcal{L}^q) = 0 \rightarrow \dots$$

so,

$$H^1(X, \mathcal{Z}^q) \cong H^0(X, \mathcal{Z}^{q+1}) / \text{Im } d^q \cong H^{q+1}((\Gamma(\mathcal{L}^\bullet), d))$$

$$H^{q+1}(\Gamma(\mathcal{L}^\bullet)) \cong H^1(X, \mathcal{Z}^q) \cong H^2(X, \mathcal{Z}^{q-1}) \cong \dots \cong H^{q+1}(X, \mathcal{Z}^0) = H^{q+1}(X, \mathcal{F})$$

□

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \rightarrow \dots$$

(de Rham resolution) then we have

$$H^k(X, \mathcal{R}) \cong H_{DR}^k(X; \mathcal{R})$$

(if X is compact, then by Hodge theory, it also isomorphic to $\ker(\bar{\partial}\partial + \partial\bar{\partial})$)

Another example: X is a complex manifold, then

$$0 \rightarrow \Omega^p \rightarrow \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \rightarrow \dots$$

then

$$H^q(X, \Omega^p) \cong H_{\bar{\partial}}^{p,q}(X, \mathbb{C})$$

(RHS= Dolbeault cohomology)

X be a smooth manifold, we define

$C_q(X, \mathbb{Z}) :=$ the free abelian group generated by continuous map

$$\phi : \Delta_q := \{(t_1, \dots, t_{q+1}) \in [0, 1]^{q+1} \mid \sum_{i=1}^n t_i = 1\}$$

and we define (for $\phi \in C_q(X, \mathbb{Z})$)

$$\partial\phi := \sum_{i=1}^{q+1} (-1)^i \phi|_{\Delta_{q,i}}$$

$$\Delta_{q,i} := \{t \in \Delta_q \mid t_i = 0\}$$

we define

$$(C_{sing}^\bullet, \partial)$$

be the dual complex of $(C_{sing}^\bullet, \partial)$.

(These are all Basic Algebraic Topology)

For any open $U \subseteq X$, we have

$$U \rightarrow C_{sing}^q(U, \mathbb{Z})$$

we get a sheaf

$$\mathcal{C}_{sing}^q$$

FACT: $(C_{sing}^\bullet, \partial)$ is a flabby resolution of \mathbb{Z} . (check!) So,

$$H_{sing}^q(X, \mathbb{Z}) = H^q(\Gamma(C_{sing}^\bullet), \partial) \cong H^q(X, \mathbb{Z})$$

第3章 Hermite 向量丛

3.1 联络与曲率

Recall: X is a smooth manifold, E is a vector bundle of rank r , if

- (1) $\pi : E \rightarrow X$ is smooth map,
- (2) for any $x \in X$, $E_x := \pi^{-1}(x)$ is a vector space over \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) of dimension r .
- (3) there an open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ and trivializations

$$\theta_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

and for any intersection $U_\alpha \cap U_\beta$, we have

注记 3.1.1.

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$$

(cocycle condition)

Special Case: line bundle rank $E=1$.

then $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, \mathbb{K}^*) = \mathcal{E}^*(U_{\alpha\beta})$ invertible smooth function on $U_{\alpha\beta}$. then, Cech cohomology,

$$(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1$$

so,

$$(g_{\alpha,\beta}) \in \mathcal{Z}^1(\mathcal{U}, \mathcal{E}^*) \rightarrow H^1(\mathcal{U}, \mathcal{E}^*) \hookrightarrow \check{H}^1(X, \mathcal{E}^*)$$

we get a map

$$\{\text{line bundles}\} \rightarrow \check{H}^1(X, \mathcal{E}^*)$$

actually, we have

$$\{\text{isomorphic classes of line bundles}\} \longleftrightarrow H^1(X, \mathcal{E}^*)$$

1-1 correspondence.

Now, X be a complex manifold, a complex vector bundle E is called holomorphic, if ... the transition matrix $g_{\alpha\beta}$ is holomorphic...

Holomorphic line bundles :

$$g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$$

\mathcal{O}^* :sheaf of invertible holomorphic functions...

FACT: there is a map

$$\{\text{holomorphic line bundle}\} \rightarrow \check{H}^1(X, \mathcal{O}^*)$$

例子 3.1.2. *trivial vector bundle* $X \times \mathbb{K}^r$

例子 3.1.3. *Tangent bundle* TX . (transition matrix $g_{\alpha\beta}$ are given by Jacobi matrix..)

定义 3.1.4. (*Local frame of vector bundles*)

$$\theta_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{K}^r$$

be a trivialization, we define

$$e_\lambda(x) := \theta_\alpha^{-1}\left(x, \begin{pmatrix} 0 \\ \vdots \\ 1(\leftarrow \text{ith}) \\ \vdots \\ 0 \end{pmatrix}\right)$$

then, $\{e_1, \dots, e_r\}$ be a local smooth section $s \in \Gamma(U_\alpha, E)$ can be written as

$$s(x) = \sum \sigma_\lambda(x)$$

where $\sigma_\lambda \in C^\infty(U_\alpha, \mathbb{K})$.

(Connection)

记号 3.1.5. For X be a smooth manifold, E is a vector bundle(real or complex), denote

$$C_p^k(\Omega, E) := C^k(\Omega, \bigwedge^p T^*M \otimes E)$$

is the space of k -differential p -forms with values in E .

Locally, consider a trivialization of E ,

$$\theta_\alpha E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

(\rightsquigarrow frame (e_1, \dots, e_r))

$$s \in \sum \varphi_\lambda(x) \otimes e_\lambda(x)$$

where φ_λ is a p -form.

定义 3.1.6. a (linear) connection on E is a linear differential operator of order 1 acting on $C^\bullet_\bullet(X, E)$:

$$D : C^\infty_p(X, E) \rightarrow C^\infty_{p+1}(X, E)$$

$$D(f \wedge s) := df \wedge s + (-1)^p f \wedge Ds$$

where $f \in C^\infty(X, \wedge^p T^*M)$, $s \in C^\infty(X, E)$.

Locally, consider a local trivialization

$$\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

with a frame $\{e_1, \dots, e_r\}$. any section $t \in C^\infty_p(\Omega, E)$ can be written as

$$t = \sum_{1 \leq \lambda \leq r} \sigma_\lambda \otimes e_\lambda$$

$$Ds = \sum_{\lambda=1}^r d\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge De_\lambda$$

where

$$De_\lambda \in C^\infty_1(\Omega, E)$$

can be written as

$$De_\lambda = \sum_{\mu=1}^r a_{\mu\lambda} \otimes e_\mu$$

where " $a_{\mu\lambda}$ " is called the coefficients of D with respect to frame $\{e_1, \dots, e_r\}$.

so,

$$D(t) = \sum_{\lambda, \mu} d\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge a_{\mu\lambda} \wedge e_\mu = \sum_{\mu} \sum_{\lambda} (d\sigma_\mu + a_{\mu\lambda} \wedge \sigma_\lambda)$$

$$Dt = d\sigma + A \wedge \sigma$$

where $A = (a_{\mu\lambda})$.

RMK: connection always exists!

Recall: for any (connected) smooth manifold, $E \rightarrow X$ is a smooth vector bundle,

Connection:

$$D : C^\infty_p(X, E) \rightarrow C^\infty_{p+1}(X, E)$$

where $C^\infty_p(X, E) := C^\infty(X, \wedge^p T^*M \otimes E)$

$$D(f \wedge s) = df \wedge s + (-1)^{\deg f} f \wedge Ds$$

Essentially,

$$D : C^\infty(X, E) \rightarrow C_1^\infty(X, E)$$

Locally, consider a trivialization $\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$, and a local frame (e_1, \dots, e_r) where $e_k(x) =$

$$\theta^{-1}\left(x, \begin{pmatrix} 0 \\ \vdots \\ 1(k^{th}) \\ \vdots \\ 0 \end{pmatrix}\right).$$

Let $s \in C^\infty(\Omega, E)$, i.e.

$$s = \sum_{i=1}^r \sigma_i e_i$$

where σ_i are smooth functions.

$$Ds = d\sigma + A \wedge \sigma$$

where

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \end{pmatrix} \quad A = a_{ij}$$

consider another trivialization

$$\tilde{\theta} : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

\rightsquigarrow a local frame $(\tilde{e}_1, \dots, \tilde{e}_r)$. Then there exists a invertible linear transform s.t.

$$\tilde{e}_k = g_k^m e_m$$

assume

$$De_k = a_k^l e_l \quad D\tilde{e}_k = \tilde{a}_k^l \tilde{e}_l$$

we have

$$\begin{aligned} dg_k^n e_n + g_k^m a_m^n e_n &= \tilde{a}_k^l g_l^n e_n \\ \rightsquigarrow \tilde{a}_k^l g_l^n (g^{-1})_n^p &= dg_k^n (g^{-1})_n^p + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \tilde{a}_l^p &= dg_k^n (g^{-1})_n^p + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \tilde{A} &= dg \cdot g^{-1} + g \cdot A \cdot g^{-1} \end{aligned}$$

Curvature

$$H_D := D^2$$

locally,

$$D^2 s = D(d\sigma + A \wedge \sigma) = d(d\sigma + A \wedge \sigma) + A \wedge (d\sigma + A \wedge \sigma)$$

$$= dA \wedge \sigma - A \wedge d\sigma + A \wedge d\sigma + A \wedge A \wedge \sigma = (dA + A \wedge A) \wedge \sigma$$

so we have

$$H = dA + A \wedge A$$

Similarly to \tilde{A}, A we have

Exercise:

$$\tilde{H} = gHg^{-1}$$

曲率在不同平凡化下的表达式。where

$$\tilde{e} = ge$$

$\rightsquigarrow H$ can be considered as a section of $C_2^\infty(X, \text{Hom}(E, E))$. because

$$\tilde{H}\tilde{e} = gHg^{-1}\tilde{e} = gHe$$

independent of the choice of local frames.

3.2 向量丛的构造

定义 3.2.1. (dual of vector bundles) $E \rightarrow X$, and $g_{\alpha\beta}$:transition matrix of E , the dual is given by $(g_{\alpha\beta})^{-1}$. (用转移函数来定义向量丛)

定义 3.2.2. direct sum of two vector bundles $(E, F) \rightarrow E \oplus F$. locally,

$$(g_{\alpha,\beta}) \oplus (h_{\alpha\beta})$$

direct sum of transition matrices.

定义 3.2.3. tensor product of two vector bundles.

locally, tensor product of two transition matrices.

fact: let D_E be a connection on E , then it induces a connection D_{E^*} . Let u be a local section of E^* , s local section of E , then we define

$$d\langle u, s \rangle = \langle D_{E^*}u, s \rangle + \langle u, D_E s \rangle$$

Exercise:

$$H(D_{E^*}) = -H(D_E)^T$$

and for two vector bundles E, F , connections D_E, D_F , then

$$D_{E \oplus F} := D_E \oplus D_F$$

$$H(E \oplus F) = H_E \oplus H_F$$

as for tensor product, we define $D_{E \otimes F}$ as follows:

$$D_{E \otimes F}(s \otimes t) = D_E s \otimes t + s \otimes D_F t$$

check the curvature

$$H_{E \otimes F} = H_E \otimes id_F + id_E \otimes H_F$$

注记 3.2.4. we can also consider wedge product of vector bundles. Consider vector bundles E_1, \dots, E_k , with connections D_{E_1}, \dots, D_{E_k} , let $s_i \in C_{p_i}^\infty(X, E^i)$ then

$$D_{E_1 \wedge \dots \wedge E_k}(s_1 \wedge \dots \wedge s_k) = \sum_{i=1}^k (-1)^{p_1 + \dots + p_{i-1}} s_1 \wedge \dots \wedge D_{E_i} s_i \wedge \dots \wedge s_k$$

Let E be a vector bundle of rank r , then $\bigwedge^r E$ is a line bundle, with transition matrix by $\det(g_{\alpha\beta})$. this bundle is denoted by $\det E$. (Det-bundle)

Let s_1, \dots, s_r be local sections of E , then we have

$$D_{\det E}(s_1 \wedge \dots \wedge s_r) = \text{tr}(H_E) s_1 \wedge \dots \wedge s_r$$

3.3 陈省身示性类

chern classes (defined by curvature).

Let $E \rightarrow X$ be a smooth complex vector bundle of rank r , where X be a complex manifold.

(Chern-Weil theory)

V be a complex vector space, $f : \underbrace{V \times \dots \times V}_k \rightarrow \mathbb{C}$ be a symmetric multi-linear form of degree k .

$\rightsquigarrow f(v) := f(v, v, \dots, v)$ is a homogeneous polynomial of degree k .

定义 3.3.1. assume G is a group (left) acting on V , s.t.

$$f(g(v_1), \dots, g(v_k)) = f(v_1, \dots, v_k)$$

for any $g \in G, v_i \in V$, then we say f is G -invariant.

Special case: $G = GL(r, \mathbb{C})$ and $V = \text{Lie}G = \mathfrak{gl}(r, \mathbb{C})$ be the Lie algebra of G . the action is

$$(g, M) \mapsto gMg^{-1}$$

Consider

$$\det(I + \frac{i}{2\pi}tm) = I + tf_1(M) + t^2f_2(M) + \cdots t^rf_r(M)$$

$\rightsquigarrow \forall 1 \leq k \leq r, f_k$ is G -invariant.

Let $E \rightarrow X$ complex vector bundle on a complex manifold, let D_E be a connection, curvature $H_E \in C_2^\infty(X, \text{Hom}(E, E))$. Let $f \in GL(r, \mathbb{C})$ - invariant "k-form", then

(1) Let H_α, H_β be the curvature forms of E in different trivialization, then $f(H_\alpha) = f(H_\beta)$, so we get a globally defined $2k$ -form.

assume $H_\alpha = gH_\beta g^{-1}$, then

$$f(H_\alpha) = f(gH_\beta g^{-1}) = f(H_\beta)$$

(2) we also have

$$df(H) = 0$$

locally, $H = H_\alpha = da_\alpha + A_\alpha \wedge A_\alpha$, then

$$\begin{aligned} df(H) &= df(H_\alpha, H_\alpha, \dots, H_\alpha) = \sum_{i=1}^k f(H_\alpha, \dots, \underbrace{dH_\alpha}_{i}, \dots, \alpha) \\ &= \sum_{i=1}^k f(H_\alpha, \dots, dA_\alpha \wedge A_\alpha - A_\alpha \wedge dA_\alpha, \dots, H_\alpha) \end{aligned}$$

Fact: (in Riemannian geometry) For any $x \in X$, we always can find a local frame s.t. $A_\alpha(x) = 0$. so, choose this frame,

$$df(H) = 0$$

So, $[f(H)] \in H^{2k}(X, \mathbb{C})$

(3) Claim : the class $[f(H)]$ is independent of the choice of the connections D_E .

Let D_0, D_1 be two connections, consider

$$D_t = (1-t)D_0 + tD_1$$

$t \in [0, 1]$, curvature H_t

Fact: $\alpha := A_1 - A_0$ is globally defined, and in $C_1^\infty(X, \text{Hom}(E, E))$.

Fact:

$$\frac{d}{dt}f(H_t) = kdf(\alpha, H_t, H_t, \dots, H_t)$$

So,

$$f(H_1) - f(H_0) = \int_0^1 \frac{d}{dt}f(H_t)dt = d \int_0^1 f(\alpha, H_t, H_t, \dots, H_t)dt$$

So,

$$[f(H_1)] - [f(H_0)]$$

定义 3.3.2. *the k -th Chern class of E*

$$c_k(E) := [f_k(\Theta_E)] \in H^{2k}(X, \mathbb{C})$$

Recall: Chern Class

X complex manifold, $E \rightarrow X$ is a smooth complex vector bundle of rank r . D is a connection, curvature $\Theta(D) \in C_2^\infty(X, \text{Hom}(E, E))$.

linear algebra:

$$\det(I + \frac{i}{2\pi} tM) = I + t f_1(M) + t^2 f_2(M) + \cdots + t^r f_r(M)$$

Chern class $\{f_k(\Theta)\} \in H_{DR}^{2k}(X, \mathbb{C})$ is independent of choice of connection.

Today:

Special case: E is a complex line bundle. Let D_0 be a connection on E , locally $D_0 e = A_0 e$, A_0 is 1-form. curvature

$$\Theta(D_0) = D_0^2 = dA_0 + A_0 \wedge A_0 = dA_0$$

so, curvature is d -exact, so $d\Theta(D_0) = 0$.

$$\det(I + \frac{i}{2\pi} tM) = I + \frac{i}{2\pi} tM$$

so, the first Chern class of line bundle is

$$c_1(E) = \{\frac{i}{2\pi} \Theta(D_0)\}$$

Let D_1 be another connection, locally $D_1 e = A_1 e$, so $\Theta(D_1) = dA_1$.so,

$$\Theta(D_1) - \Theta(D_0) = d(A_1 - A_0)$$

where

$$A_1 - A_0 \in C_1^\infty(X, \text{Hom}(E, E))$$

(when E is line bundle, $\text{Hom}(E, E) \cong E^* \otimes E$ is trivial bundle)

so, $A_1 - A_0$ is a globally defined smooth function on X . So,

$$\{\Theta(D_1)\} = \{\Theta(D_0)\} \in H^2(X, \mathbb{C})$$

independent of the choice of connection.

3.4 Hermite 向量丛

定义 3.4.1. a complex vector bundle $E \rightarrow X$ of rank r is called a Hermitian vector bundle, if we have an inner product on E , i.e. locally, consider a local frame $\{e_1, \dots, e_r\}$, we have

$$\{e_i(x), e_j(x)\} = h_{ij}(x)$$

s.t. $(h_{ij}(x))$ is a positive definite Hermitian matrix depending smoothly on x .

注记 3.4.2. For any complex vector bundle, Hermitian structure always exists.

证明与黎曼几何类似。(黎曼度量的存在性)

定义 3.4.3. (Hermitian connection)

A connection D on E is called Hermitian, if

$$d\{e_i, e_j\} = \{De_i, e_j\} + \{e_i, De_j\}$$

More generally, let $t \in C_p^\infty(X, E)$, $s \in C_q^\infty(X, Y)$,

$$d\{s, t\} = \{dt, s\} + (-1)^p \{t, Ds\}$$

性质 3.4.4. D is a Hermitian connection, then the curvature

$$\Theta(D)^* = -\Theta(D)$$

(where $(-)^*$ is conjugate transpose of matrix)

it means that, $i\Theta(D) \in C_2^\infty(X, \text{Herm}(E, E))$

证明.

$$\begin{aligned} 0 &= d^2\{e_i, e_j\} = d\{De_i, e_j\} + d\{e_i, De_j\} \\ &= \{D^2e_i, e_j\} - \{De_i, De_j\} + \{De_i, De_j\} + \{e_i, D^2e_j\} = \{(\Theta + \Theta^*)e_i, e_j\} \end{aligned}$$

□

注记 3.4.5. E is a Hermitian line bundle, D is a Hermitian connection, then $i\Theta(D)$ is a real 2-form, $c_1(E) \in H^2(X, \mathbb{R})$.

(Chern connection)

定义 3.4.6. Let X be a complex manifold. D' is called a connection of type $(1,0)$ on E , if for any section $s \in C_{p,q}^\infty(X, E)$, we have $D's \in C_{p+1,q}^\infty(X, E)$.

A connection D'' is called a connection of type $(0,1)$, if ... $D''s \in C_{p,q+1}^\infty(X, E)$.

注记 3.4.7. Let $E \rightarrow X$ be a vector bundle. Let D be a connection on E , locally

$$Ds \xrightarrow{\sim} d\sigma + A \wedge \sigma$$

$$d\sigma = \partial\sigma + \bar{\partial}\sigma$$

so, let A' be the $(1,0)$ -part of A , ...,

$$Ds = \partial\sigma + A' \wedge \sigma + (\bar{\partial}\sigma + A'' \wedge \sigma) =: D's + D''s$$

性质 3.4.8. E : Hermitian vector bundle, D is a Hermitian connection, locally, take a C^∞ -frame e_1, \dots, e_r which is orthonormal (i.e. $\{e_i(x), e_j(x)\} = \delta_{ij}$), then the connection coefficient $A = A' + A''$ satisfies

$$(A')^* = -A''$$

$$(\iff \bar{i}A = iA)$$

证明. because

$$0 = de_i, e_j = \{De_i, e_j\} + \{e_i, De_j\} = \{a_i^k e_k, e_j\} + \{e_i, a_j^l e_l\} = a_i^j + \bar{a}_j^i$$

so, $A^* = -A$. □

推论 3.4.9. $E \rightarrow X$ is a Hermitian vector bundle, D_0'' is a connection of type $(0,1)$ on E . Then exists a unique Hermitian connection D such that $D'' = D_0''$.

证明. Let $A'' = A_0''$ and $A' = -(A_0'')^* \rightsquigarrow A = A' + A''$, and D is given by A . □

Let $E \rightarrow X$ is a holomorphic Hermitian vector bundle, observe that $\bar{\partial}$ defines a connection of type $(0,1)$ on E (check!)

assume E is a holomorphic line bundle, take a section $s \in C_p^\infty(X, E)$, i.e. we have a family of p -forms (s_α) such that $s_\alpha = g_{\alpha\beta} s_\beta$ where $g_{\alpha,\beta}$ is the holomorphic transition matrix.

$$\bar{\partial}s \xrightarrow{\sim} \bar{\partial}s_\beta$$

then

$$\bar{\partial}s_\alpha = g_{\alpha,\beta} \bar{\partial}s_\beta$$

(so, $\bar{\partial}$ is a connection of $(0,1)$)

this connection is called the canonical connection of type $(0,1)$.

定义 3.4.10. Let $E \rightarrow X$ holomorphic Hermitian vector bundle, the connection D on E is called Chern connection if

$$D'' = \bar{\partial}$$

Curvature of Chern connection

$E \rightarrow X$ is holomorphic Hermite vector bundle , D is the Chern connection, Locally let $\{e_1, \dots, e_r\}$ be a holomorphic frame, and two local sections

$$s, t \in C^\infty(\Omega, E)$$

where

$$s = \sum_{i=1}^r \sigma_i e_i$$

$$t = \sum_{i=1}^r t_i e_i$$

Since D is Hermitian ,

$$d\{s, t\} = d((\sigma_1, \dots, \sigma_r) H \begin{pmatrix} t_1 \\ \vdots \\ t_r \end{pmatrix}) = (d\sigma)^T H t + \sigma^T (dH) t + \sigma^T H d(t)$$

so, we have

$$\{Ds, t\} + \{s, Dt\} = (d\sigma + \bar{H}^{-1} \bar{\partial} \bar{H} \wedge \sigma)^T \wedge H \bar{t} + \sigma^T \wedge \overline{H(dt + \bar{H}^{-1} \bar{\partial} \bar{H} \wedge t)}$$

so ,

$$Ds = d\sigma + \bar{H}^{-1} \bar{\partial} \bar{H} \wedge \sigma$$

$$D's = \partial\sigma + \bar{H}^{-1}\partial\bar{H} \wedge \sigma = \bar{H}^{-1}\partial(\bar{H}\sigma)$$

$$D''s = \bar{\partial}\sigma$$

so,

$$(D')^2s = \bar{H}^{-1}\partial(\bar{H}(\bar{H}^{-1}\partial(\bar{H}\sigma))) = \cdots = 0$$

$$(D'')^2s = \cdots = 0$$

So we have

$$\Theta(D) = (D' + D'')^2 = D'D'' + D''D'$$

Locally ,

$$\begin{aligned}\Theta s &= D'D''s + D''D's = \bar{H}^{-1}\partial(\bar{H}\bar{\partial}\sigma) + \bar{\partial}(\bar{H}^{-1}\partial(\bar{H}\sigma)) = \cdots = \bar{H}^{-1}\partial\bar{H} \wedge \bar{\partial}\sigma + \bar{\partial}(\bar{H}^{-1})\sigma \\ &= \bar{\partial}(\bar{H}^{-1}\partial\bar{H})\sigma\end{aligned}$$

So, Chern curvature

$$\Theta_D = \bar{\partial}(\bar{H}^{-1}\partial\bar{H})$$

Last time: $E \rightarrow X$ is a holomorphic vector bundle with a Hermitian metric H . Then there is a unique connection D_E s.t. ... called Chern connection.

Curvature of Chern Connection:

$$\Theta(D_E) = \bar{\partial}(\bar{H}^{-1}\partial\bar{H})$$

so,

$$i\Theta(D_E) \in C_{1,1}^\infty(X, \text{Hom}(E, E))$$

例子 3.4.11. (Special case: E is a holomorphic line bundle)

locally, let e be a holomorphic frame, $\langle e, e \rangle = h$ is the metric. then,

$$\Theta = \bar{\partial}(h^{-1}\partial h) = \bar{\partial}\partial \log h$$

so,

$$i\Theta(E) = -i\bar{\partial}\partial \log h$$

if $h = e^{-2\varphi}$ where φ is a smooth function, then

$$i\Theta(E) = 2i\bar{\partial}\partial\varphi = 2\sqrt{-1} \sum_{k,l} \frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_l} dz_k \wedge d\bar{z}_l$$

Question: let s be a local holomorphic section of E ,

$$-i\bar{\partial}\partial \log |s|_h^2 = ?$$

(Hint: $\frac{i}{\pi} \bar{\partial}\partial \log z = ?$ 单复变, 按分布意义下求导. 等于狄拉克测度 2333333) 可能是期末题目?

例子 3.4.12. $\mathcal{O}(-1)$ on $\mathbb{C}P^n$, tautological line bundle. (Recall: $\mathbb{C}P^n$ is a compact complex manifold with holomorphic charts

$$\Omega_j := \{[z_0; z_1; \dots; z_n] | z_j \neq 0\} \rightarrow \left(\frac{z_0}{z_j}, \dots, \hat{1}, \dots, \frac{z_n}{z_j} \right) \in \mathbb{C}^n$$

)

Let V be a complex vector space, $\dim_{\mathbb{C}} V = n + 1$. Denote the projective space by

$$\mathbb{P}(V) = (V \setminus \{0\}) / \mathbb{C}^*$$

Let $\underline{V} := \mathbb{P}(V) \times V$ be the trivial vector bundle, define

$$\mathcal{O}(-1) := \{([x], \xi) | \xi \in \mathbb{C} \cdot x\}$$

性质 3.4.13. $\mathcal{O}(-1)$ is a holomorphic line bundle on $\mathbb{P}(V)$.

证明. $\mathcal{O}(-1)|_{\Omega_j}$ has a non-vanishing holomorphic section \mathcal{E}_j defined by

$$\mathcal{E}_j([x]) = \frac{x}{x_j}$$

for $0 \leq j \leq n$. □

Assume V has a Hermitian inner product, then $\mathcal{O}(-1)$ has an Hermitian structure induced from V .

Let e_0, \dots, e_n be an orthonormal basis of V , then $\mathcal{O}(-1)|_{\Omega_0}$ has a non-vanishing holomorphic section:

$$\mathcal{E}_0(z_1, \dots, z_n) = e_0 + z_1 e_1 + \dots + z_n e_n$$

where

$$\Omega_0 = \{[1; z_1; \dots; z_n] | z_j \in \mathbb{C}\} \cong \mathbb{C}^n$$

then,

$$|\mathcal{E}_0|_h^2 = 1 + |z_1|^2 + \dots + |z_n|^2$$

so the Chern curvature of $\mathcal{O}(-1)$ on Ω_0 is given by

$$\Theta = \bar{\partial} \partial \log(1 + |z_1|^2 + \dots + |z_n|^2)$$

Denote $\mathcal{O}(1) := \mathcal{O}(-1)^*$, then

$$\Theta(\mathcal{O}(1)) = -\bar{\partial} \partial \log(1 + |z_1|^2 + \dots + |z_n|^2)$$

on Ω_0 .

$$i\Theta(\mathcal{O}(1)) = i\partial\bar{\partial}\log(1 + |z_0|^2 + \dots + |z_n|^2) = \sqrt{-1} \sum_{1 \leq k, l \leq n} c_{k,l} dz_k \wedge d\bar{z}_l$$

Exercise: (c_{kl}) is a positive definite Hermitian matrix.

"Fubini-Study metric" on $\mathbb{P}(V)$. $\mathcal{O}(1)$ is "hyperplane line bundle of $\mathbb{P}(V)$ ".

Exercise: calculate

$$\int_{\mathbb{P}(V)} \left(\frac{i}{2\pi} \Theta(\mathcal{O}(1)) \right)^{\wedge n} = ?$$

(Hint: $\mathbb{P}(V) \setminus \Omega_0$ is a zero-measure set)

$E \rightarrow X$: holomorphic line bundle, D_E is a Chern connection.

$$c_1(E) = \left\{ \frac{i}{2\pi} \Theta(D_E) \right\} \in H_{DR}^2(X, \mathbb{R})$$

Exercise: 60% 的概率出现于期末试题

Consider the sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{e^{2\pi i *}} \mathcal{O}^* \rightarrow 0$$

it induces a long exact sequence

$$\dots \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}) \rightarrow \dots$$

prove: Consider E as an element of $H^1(X, \mathcal{O}^*)$, then the image of $\delta(E)$ in $H^2(X, \mathbb{R}) \cong H_{DR}^2(X, \mathbb{R})$ is $c_1(E)$.

Exercise: E is a holomorphic line bundle, denote $\theta := \frac{i}{2\pi} \Theta(D_E)$ real $(1,1)$ -form, where D_E is Chern connection with a metric h . Prove: for any smooth function $f \in C^\infty(X, \mathbb{R})$, there exists a Hermitian metric h_f s.t.

$$\frac{i}{2\pi} \Theta_{E, h_f} = \theta + i\partial\bar{\partial}f$$

第 4 章 L^2 Hodge 理论

4.1 向量丛上的微分算子

Differential operators on vector bundles.

Let X is a (connected) smooth manifold of (\mathbb{R}) -dimension n . $E, F : \mathbb{K}$ -vector bundle of rank r, r' respectively.

定义 4.1.1. a linear differential operator of degree k from E to F is a \mathbb{K} -linear map

$$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

$$u \mapsto Pu$$

locally given by

$$Pu(x) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x)$$

where $a_\alpha(x) = (a_{af_a, \lambda_\mu}(x))$ be a $r' \times r$ matrix.

$$u(x) = (u_1(x), \dots, u_r(x))^T$$

Let $t \in \mathbb{K}$, $f \in C^\infty(M, \mathbb{K})$, $u \in C^\infty(M, E)$, then

$$e^{-tf(x)} P(e^{tf(x)} u(x)) = t^k \sigma_P(x, df(x)) u(x) + \text{terms } c_j(x) t^j \quad (j < k)$$

定义 4.1.2.

$$\sigma_P : T^*M \rightarrow \text{Hom}(E, F)$$

is called the principal symbol of P , which is a polynomial on T^*M .

locally,

$$\sigma_P(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$$

$$(\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n})$$

例子 4.1.3. Consider $d : C^\infty(M, \mathbb{K}) \rightarrow C^\infty(M, T^*M)$. then

$$du = \sum_{j=1}^n \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \frac{\partial u}{\partial x^j}$$

i.e.

$$\sigma_d(x, \xi) = \sum_{j=1}^n \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \xi_j$$

定义 4.1.4. P is called elliptic, if $\forall x \in M, \xi \in T_x^*M \setminus \{0\}$,

$$\sigma_P(x, \xi) \in \text{Hom}(E_x, E_x)$$

is injective.

For example, d is elliptic.

L^2 -inner product

Let M be an oriented C^∞ -manifold with a smooth volume form, locally

$$dV(x) = \gamma(x) dx_1 \wedge \dots \wedge dx_n$$

$\gamma(x) > 0$. Assume E has a Euclidean(or Hermitian) structure...

Let $u, v \in C^\infty(M, E)$, define

$$\langle\langle u, v \rangle\rangle := \int_M \langle u, v \rangle dV(x)$$

define $L^2(M, E) :=$ space of sections with measurable coefficients with are L^2 w.r.t $\langle\langle, \rangle\rangle$.

定义 4.1.5. Let $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ be a differential operator, E, F have Euclidean (or Hermitian) structure, then there exists unique differential operator

$$P^* : C^\infty(M, F) \rightarrow C^\infty(M, E)$$

s.t.

$$\langle \langle Pu, v \rangle \rangle = \langle \langle u, P^*v \rangle \rangle$$

for all u, v s.t. $\text{Supp}u \cap \text{Supp}v \subset\subset M$ (relative compact...)

P^* is called the formal adjoint of P .

证明. Existence: Assume that $\text{Supp}u, \text{Supp}v \subset\subset$ some coordinate chart Ω with coordinates (x_1, \dots, x_n) , then

$$\langle \langle Pv, u \rangle \rangle = \int_{\Omega} \sum_{\alpha, \lambda, \mu} a_{\alpha, \lambda, \mu}(x) D^\alpha u_\mu(x) \overline{v_\lambda(x)} \gamma(x) dx_1 \cdots dx_n$$

integration by parts, it

$$= \int_{\Omega} \sum_{\alpha, \lambda, \mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha (\gamma(x) \overline{a_{\alpha, \lambda, \mu}} v_\lambda(x))} dx_1 \cdots dx_n$$

Locally,

$$P^*v = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha (\gamma(x) \overline{a_\alpha(x)})^T v(x)$$

Uniqueness: use the density of C^∞ -section with compact support in $L^2(M, -)$. \square

推论 4.1.6. If $\sigma_P(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$, then

$$\sigma_{P^*} = (-1)^k \overline{\sigma_P(x, \xi)}^T$$

推论 4.1.7. If $\text{rank } E = \text{rank } F$, P is differential operator, then P^* is elliptic $\iff P$ is elliptic.

4.2 椭圆算子的基本性质

Fundamental results of elliptic operators

M is a compact (oriented) C^∞ -manifold, $\dim_{\mathbb{R}} M = n$, with a smooth volume form dV .

E is an Hermite vector bundle, $\text{rank}_{\mathbb{C}} E = r$.

Sobolev space: $W^k(M, E) :=$ the space of section $s : M \rightarrow E$ whose derivations up to order $= k$,
 $:=$ the completion of space of smooth sections w.r.t W^k -norm.

$(\Omega_j)_{j \in I}$: a finite open covering of M , $E|_{\Omega_j}$ trivial, Let $(\rho_j)_{j \in I}$ be a partition of unity w.r.t.
 $(\Omega_j)_{j \in I}$, s.t. $\sum_j \rho_j^2 = 1$. locally, choose an orthonormal frame $(e_{j,\lambda})_{1 \leq \lambda \leq r}$ on Ω_j , then $u = \sum_{\lambda=1}^r u_{j,\lambda} e_{j,\lambda}$ on Ω_j . Define

$$\|u\|_k^2 := \sum_{j,\lambda} \|e_j u_{j,\lambda}\|_k^2$$

where

$$\|e_j u_{j,\lambda}\|_k^2 := \int_{\Omega_j} \sum_{|\alpha| \leq k} |D^\alpha(e_j u_{j,\lambda})|^2 dV(x)$$

注记 4.2.1. *On a compact manifold, the equivalence of class of $\|\cdot\|_k$ is independent of the choice of : partition of unity, local trivialization, holomorphic covering...*

引理 4.2.2. *(Sobolev lemma)*

For $k > l + \frac{n}{2}$, then we have

$$W^k(M, E) \subseteq C^l(M, E)$$

引理 4.2.3. *(Rellich lemma)*

For any $k \in \mathbb{Z}_{\geq 0}$, the inclusion

$$W^{k+1}(M, E) \hookrightarrow W^k(M, E)$$

is a compact operator.

引理 4.2.4. *(Garding inequality)*

If

$$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

*is elliptic, and $\text{rank} E = \text{rank} F$, \tilde{P} : the extension of P to sections with distribution coefficients, then
: for all $u \in W^0(M, E)$, if $\tilde{P}u \in W^k(M, F)$, then $u \in W^{k+d}(M, E)$, where $d = \deg P$, and*

$$\|u\|_{k+d} \leq C_k (\|\tilde{P}u\|_k + \|u\|_0)$$

where C_k depending on k, M .

证明. Reference: Kodaira: deformation of complex structures (Appendix) □

推论 4.2.5. *If $u \in \ker \tilde{P} \cap W^0(M, E)$, then $u \in C^\infty(M, E)$.*

引理 4.2.6. *(Finiteness theorem)*

Setting M be a compact manifold, $\text{rank} E = \text{rank} F$,

$$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

elliptic, then:

(1) $\ker P$ is of finite dimension

(2) $P(C^\infty(M, E))$ is closed and of finite codimension in $C^\infty(M, F)$. If P^ is the formal adjoint of P , then \exists decomposition*

$$C^\infty(M, F) = P(C^\infty(M, E)) \oplus \ker P^*$$

which is orthogonal in $W^0(M, F) = L^2(M, F)$

证明. 椭圆算子的一般结果, 分析的东西 233333333. 可以参考小平邦彦复流形与复结构形变的附录. □

4.3 紧黎曼流形的 Hodge 理论

Hodge theory in compact Riemannian manifold

Hodge star operator.

M compact Riemannian manifold, $\dim_{\mathbb{R}} = n$, E is a Hermitian vector bundle. Assume $(\xi_1, \dots, \xi_n), (e_1, \dots, e_n)$ be orthonormal frame of TM, E on some local chart Ω , denote $(\tilde{\xi}_1^*, \dots, \tilde{\xi}_n^*), (e_1^*, \dots, e_n^*)$ be the co-frame of T^*M, T^*E .

$\wedge^\bullet T^*M$ is endowed with an inner product frame from TM . locally,

$$\langle u_1 \wedge \dots \wedge u_p, u_1 \wedge \dots \wedge u_p \rangle := \det(\langle u_i, v_j \rangle)$$

for $u_i, v_j \in T^*M$. Then , get an inner product on $\wedge^p T^*M$.

Assume

$$U = \sum_{\substack{|I|=p \\ i_1 \leq \dots \leq i_p}} u_I \tilde{\xi}_I^*$$

$$V = \sum_{\substack{|I|=p \\ i_1 \leq \dots \leq i_p}} v_I \tilde{\zeta}_I^*$$

be p -forms, then

$$\langle u, v \rangle = \sum_{|I|=p} u_I v_I$$

i.e. $\{\tilde{\zeta}_I^*\}$ is an orthonormal basis of $\wedge^p T^*M$.

$\wedge^* T^*M \otimes E$ has an inner product induced from $\wedge^* T^*M, E$,

定义 4.3.1. *the Hodge star operator*

$$* : \wedge^p T^*M \rightarrow \wedge^{n-p} T^*M$$

is defined by

$$u \wedge *v = \langle u, v \rangle dV$$

Locally, let

$$U = \sum_{|I|=p} u_I \tilde{\zeta}_I^*, \quad V = \sum_{|I|=p} v_I \tilde{\zeta}_I^*$$

assume

$$*V = \sum_{|J|=n-p} a_J \tilde{\zeta}_J^*$$

then

$$\begin{aligned} U \wedge * \sum u_I a_{I^c} \tilde{\zeta}_I^* \wedge \tilde{\zeta}_{I^c}^* &= \sum u_I a_{I^c} \varepsilon(I, I^c) \tilde{\zeta}_1^* \wedge \dots \wedge \tilde{\zeta}_n^* \\ \langle u, v \rangle dV &= \sum_{|I|=p} u_I v_I \tilde{\zeta}_1^* \wedge \dots \wedge \tilde{\zeta}_n^* \end{aligned}$$

so, we have

$$*V = \sum_{|I|=p} \varepsilon(I, I^c) V_I \tilde{\zeta}_{I^c}^* \in \wedge^{n-p} T^*M$$

定义 4.3.2.

$$* : \wedge^p T^*M \otimes E \rightarrow \wedge^{n-p} T^*M \otimes E$$

is defined by

$$\{s, *t\} := \langle s, t \rangle dV$$

Locally, assume

$$t = \sum_{\substack{|I|=p \\ 1 \leq \lambda \leq r}} t_{I,\lambda} \tilde{\zeta}_I^* \otimes e_\lambda$$

then

$$*t = \sum_{\substack{|I|=p \\ 1 \leq \lambda \leq r}} \varepsilon(I, I^c) t_{I,\lambda} \tilde{\zeta}_{I^c}^* \otimes e_\lambda$$

定义 4.3.3.

$$\# : \bigwedge^p T^*M \otimes E \rightarrow \bigwedge^{n-p} T^*M \otimes E^*$$

is defined by: for any $s, t \in \bigwedge^p T^*M \otimes E$, such that

$$s \wedge \#t := \langle s, t \rangle dV$$

wedge product + pairing of $E^* \times E \rightarrow \mathbb{C}$.

Locally: assume

$$t = \sum_{\substack{|I|=p \\ 1 \leq \lambda \leq r}} t_{I,\lambda} \tilde{\zeta}_T^* \otimes e_\lambda$$

then,

$$\#t = \sum_{|I|=p, \lambda} \varepsilon(I, I^c) t_{I,\lambda} \tilde{\zeta}_c^* I \otimes e_\lambda^*$$

性质 4.3.4.

$$*^2 = (-1)^{p(n-1)} \quad \text{on } \bigwedge^p T^*M \otimes E$$

$$\#^2 = (-1)^{p(n-1)} \quad \text{on } \bigwedge^p T^*M \otimes E$$

(正负号对吗?)

Recall: For all $s, t \in C^\infty(M, \bigwedge^p T^*M \otimes E)$, we have an inner product

$$\langle \langle s, t \rangle \rangle := \int_M \langle s, t \rangle dV$$

定理 4.3.5. Let D_E be an Hermite connection on E , acting on $\bigwedge^p T^*M \otimes E$, then

$$D_E^* := (-1)^{np+1} * D_E *$$

where D_E^* is the formal adjoint of D_E .

证明. Let $s \in C^\infty(M, \bigwedge^p T^*M \otimes E)$ and $t \in C^\infty(M, \bigwedge^{p+1} T^*M \otimes E)$. then

$$\langle \langle D_E s, t \rangle \rangle = \int_M \langle D_E s, t \rangle dV = \int_M \{D_E s, *t\}$$

Since D_E is Hermitian ,by definetion ,

$$d\{s, *t\} = \{D_E s, t\} + (-1)^p \{s, D_E(*t)\}$$

so,

$$\langle \langle D_E s, t \rangle \rangle = \int_M d\{s, *t\} + (-1)^{p+1} \{s, D_E *t\} = (-1)^{p+1} (-1)^{p(n_1)} \int_M \{s, *(D_E *t)\} = \langle \langle s, D_E^* t \rangle \rangle$$

so,

$$D_E^* t = (-1)^{np+1} * D_E *$$

□

定义 4.3.6.

$$\Delta_E = D_E D_E^* + D_E^* D_E : C^\infty(M, \bigwedge^p T^*M \otimes E) \rightarrow C^\infty(M, \bigwedge^p T^*M \otimes E)$$

例子 4.3.7. Let $M = \mathbb{R}^n$, $g = \sum_{i=1}^n dx_i^2$, $E = M \times \mathbb{C}$ trivial line bundle with $D_E = d$. then

$$\Delta_E u = (dd^* + d^*d)u = - \sum_{i=1}^n \left(\sum_{|I|=p} \frac{\partial^2 u_I}{\partial x_I^2} dx_I \right)$$

where

$$u = \sum_{|I|=p} u_I dx_I$$

性质 4.3.8. Δ_E is a self-adjoint elliptic operator. (i.e. $\Delta_E^* = \Delta_E$)

证明. $\Delta_E^* = \Delta_E$ be definition.

note that

$$e^{-tf} D_E (e^{tf} s) = t df \wedge s + D_E s$$

so,

$$\sigma_{D_E}(x, \xi) s = \xi \wedge s$$

$$\sum_{D_E^*} = -\overline{\sigma_{D_E}}^T$$

$$\sigma_{D_E^*}(x, \xi)s = -\tilde{\xi} \lrcorner s$$

where $\tilde{\xi}$ be the vector field dual to ξ . □

定义 4.3.9.

$$\Delta_E = D_E D_E^* + D_E^* D_E : C^\infty(M, \bigwedge^p T^* M \otimes E) \rightarrow C^\infty(M, \bigwedge^p T^* M \otimes E)$$

so,

$$\sigma_{\Delta_E}(x, \xi)s = \left(\sigma_{D_E} \sigma_{D_E^*}(x, \xi) + \sigma_{D_E^*} \sigma_{D_E}(x, \xi) \right) s$$

so, σ_{Δ_E} is injective if $\xi \neq 0$, so Δ_E is elliptic.

Harmonic forms and Hodge isomorphism.

定义 4.3.10. u is called harmonic if $\Delta_d u = 0$.

定理 4.3.11. M is a compact Riemannian manifold, then de Rham cohomology

$$H_{DR}^p(M, \mathbb{R}) \cong \ker(\Delta_d : C^\infty(M, \bigwedge^p T^* M))$$

证明. Δ_d self-adjoint elliptic, so by general result for elliptic operator,

$$C^\infty(M, \bigwedge^p T^* M) = \text{Im } \Delta_d \oplus \ker \Delta_d^* = \text{Im } \Delta_d \oplus \ker \Delta_d$$

Claim:

$$\text{Im } \Delta_d = \text{Im } d \oplus \text{Im } d^*$$

Recall $\Delta_d = dd^* + d^*d$, so

$$\text{Im } \Delta_d \subseteq \text{Im } d \oplus \text{Im } d^*$$

on the other hand,

$$\text{Im } d \oplus \text{Im } d^* \subseteq (\ker \Delta_d)^\perp = \text{Im } \Delta_d$$

so,

$$\text{Im } \Delta_d = \text{Im } d \oplus \text{Im } d^*$$

so,

$$C^\infty(M, \bigwedge^p T^*M) = \text{Im } d \oplus \text{Im } d^* \oplus \ker \Delta_d$$

so,

$$H_{DR}^p(M, \mathbb{R}) = \frac{\text{Im } d \oplus \ker \Delta_d}{\text{Im } d} = \ker \Delta_d$$

□

推论 4.3.12.

$$\dim H_{DR}^p(M, \mathbb{R}) = \dim \ker \Delta_d < +\infty$$

注记 4.3.13. Consider

$$u \mapsto \int_M (\langle u, u \rangle + \langle du, du \rangle + \langle d^*u, d^*u \rangle) dV$$

这个泛函的变分是什么鬼?

Harmonic forms and Hodge isomorphism

Recall: M is a compact Riemann manifold,

$$d : C^\infty(M, \bigwedge^* T^*M) \rightarrow C^\infty(M, \bigwedge^{*+1} T^*M)$$

adjoint d^* ,

$$\Delta_d = dd^* + d^*d$$

is a self-adjoint elliptic operator.

Hodge decomposition:

$$C^\infty(M, \bigwedge^p T^*M) = \ker \Delta_d \oplus \text{Im } d \oplus \text{Im } d^*$$

$$\mathcal{H}^p(M, \mathbb{R}) := \ker \Delta_d \quad \text{finite dimension}$$

$$\mathcal{H}^p(M, \mathbb{R}) \cong H_{DR}^p \cong H^p(M, \mathbb{R})$$

(Hodge isomorphism, and, de Rham-Weil)

Poincare duality

定理 4.3.14. The pairing

$$H_{DR}^p(M, \mathbb{R}) \times H_{DR}^{n-p}(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$(s, t) \mapsto \int_M s \wedge t$$

(is well defined) is non-degenerated. In particular, $H_{DR}^p(M, \mathbb{R})^* \cong H_{DR}^{n-p}(M, \mathbb{R})$

证明. the pairing factors through the pairing on

$$\mathcal{H}^p(M, \mathbb{R}) \times \mathcal{H}^{n-p}(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$(s, t) \mapsto \int_M s \wedge t$$

need to verify: (1) it is independent of the choice of representations. (Easy, check) (2) Pairing $\mathcal{H}^p \times \mathcal{H}^{n-p}$ is non-degenerated..

claim(Exercise): Hodge star $*$ s.t. $*\Delta_d = \Delta_d*$.

so, s is a harmonic p -form $\iff *s$ is a harmonic $(n-p)$ -form.

note that

$$s \wedge *s = \langle s, s \rangle dV = \int_M s \wedge *s = \int_M \langle s, s \rangle dV = \|s\|^2$$

□

推论 4.3.15.

$$\dim \mathcal{H}^p(M, \mathbb{R}) = \dim \mathcal{H}^{n-p}(M, \mathbb{R})$$

Generalization to flat bundle. M is a compact Riemannian manifold, $\dim_{\mathbb{R}} M = n$, $E \rightarrow M$ is a complex Hermitian vector bundle.

定义 4.3.16. $E \rightarrow X$ is called flat, if it admit a connection D_E s.t.

$$D_E^2 = 0$$

注记 4.3.17. E is flat $\iff E$ is given by a representation

$$\pi_1(M) \rightarrow GL(r, \mathbb{C})$$

(我们不证)

Consider the complex :

$$(C^\infty(M, \bigwedge^* T^*M \otimes E), D_E) \\ \rightsquigarrow H_{DR}^p(M, E) := \frac{\ker D_E}{\text{Im } D_E}$$

Exercise: we have decomposition

$$C^\infty(M, \bigwedge^p T^*M \otimes E) = \ker \Delta_{D_E} \oplus \text{Im } D_E \oplus \text{Im } D_E^*$$

$$H_{DR}^p(M, E) \cong \ker \Delta_{D_E}$$

and the pairing

$$H_{DR}^p(M, E) \times H_{DR}^{n-p}(M, E^*) \rightarrow \mathbb{C}$$

$$(s, t) \mapsto \int_M s \wedge t$$

is non-degenerate..

以上是实的 Hodge 理论。

4.4 Kähler 流形

定义 4.4.1. Let X be a complex manifold, $\dim_{\mathbb{C}} X = n$, X is called a Hermitian manifold, if X has a Hermitian metric, i.e. locally $h(z) := \sum_{1 \leq j, k \leq n} h_{jk}(z) dz_j \otimes d\bar{z}_k$, where (h_{jk}) is positive definition Hermitian matrix.

Check: the positivity of h is independent of the choice of holomorphic local coordinate

Rmk: Any complex manifold has a Hermitian metric... (Exercise)

Fundamental $(1, 1)$ -form associated to $h(z)$ is defined by

$$\omega := -\operatorname{Im} h = \frac{\sqrt{-1}}{2} \sum_{j, k} h_{jk} dz_j d\bar{z}_k$$

we also call ω is the Hermitian metric on X

Fact: ω is real (i.e. $\bar{\omega} = \omega$).

注记 4.4.2. h is a Hermite structure on TX (holomorphic tangent bundle of X). locally,

$$\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle(z) = h_{ij}(z)$$

定义 4.4.3. (X, ω) is an Hermitian manifold, X is Kähler if $d\omega = 0$.

性质 4.4.4. Locally, $\omega = \frac{\sqrt{-1}}{2} \sum_{jk} h_{jk} dz_j \wedge d\bar{z}_k$ is Kaehler, $\iff \partial\omega = 0$ and $\bar{\partial}\omega = 0$, i.e.

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}$$

If (X, ω) is a compact Kaehler manifold, then

$$H^{2k}(X, \mathbb{R}) \neq 0$$

证明. $d\omega = 0$, so $\omega \in H^2(M, \mathbb{R})$. Claim:

$$0 \neq \omega^k \in H^{2k}(M, \mathbb{R})$$

proof of the claim:

$$[\omega^k][\omega^{n-k}] = \int_X \omega^k \wedge \omega^{n-k} = \int_X \omega^n$$

Since ω is positive, locally

$$\omega^n = n! \det(h_{jk}) \bigwedge_{l=1}^n \left(\frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l \right) > 0$$

is a volume form. So,

$$[\omega^k][\omega^{n-k}] = \int_X \omega^n > 0$$

(Using Poincare dual)

□

例子 4.4.5. (Exists a complex manifold NOT Kaehler) (Hopf Surface)

$$X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

where discrete group $\Gamma := \{\lambda^n | n \in \mathbb{Z}\}$, $0 < \lambda < 1$ fixed.

Exercise: $X \cong S^1 \times S^3$ C^∞ homeomorphism.. and X is compact complex manifold.

and $H^2(X, \mathbb{R}) = H^2(S^1 \times S^3, \mathbb{R}) = 0$ by Künneth Formula...

So, X is non-Kahler...

例子 4.4.6. Examples of Kaehler manifold)

(1) Riemann surface must be Kaehler...(trivial)

(2) (complex torus) $X = \mathbb{C}^n/\Gamma$, Γ is a lattice. (this manifold may not compact...)

$$\omega = \sqrt{-1} \sum_{j,k} h_{jk} dz_j \wedge d\bar{z}_k$$

is a Kahler metric on X if $(H_{jk}) > 0$, h_{jk} are constant.

(3) Projective space \mathbb{CP}^n .

$$\omega := \sqrt{-1} \Theta_h(\mathcal{O}(1))$$

locally,

$$\omega = \sqrt{-1} \partial \bar{\partial} \log(1 + |z_1|^2 + \dots + |z_n|^2)$$

on Ω . This ω is a Kahler metric,

例子 4.4.7. Let (X, ω) is a Kahler manifold, then any complex submanifold $Y \subseteq X$ is also Kahler.

$$i : Y \hookrightarrow X$$

with the Kahler metric $i^*\omega$.

Exercise: Let $f : Y \rightarrow X$ be a holomorphic immersion, and assume X is Kahler, then Y is Kahler.

推论 4.4.8. Any projective manifold (i.e. $X \hookrightarrow \mathbb{CP}^N$) is Kähler.

(Algebraic Geometry.....)

性质 4.4.9. (Equivalent definition of Kaehler metrics) a Hermitian metric ω is Kahler, iff for all $x_0 \in X$, there exists a holomorphic chart (z_1, \dots, z_n) centered at x_0 , s.t.

$$\omega(z) = \sqrt{-1} \sigma_{jk} \delta_{jk} dz_j \wedge d\bar{z}_k + O(|z|^2)$$

(\Leftarrow is trivial...) (left to HW)

定理 4.4.10. (Exercise)

If (X, ω) is Kahler, then for all $x_0 \in X$, \exists holomorphic chart z_1, \dots, z_n centered at x_0 , s.t. assume

$$\omega = \sqrt{-1} h_{jk} dz_j \wedge d\bar{z}_k$$

then

$$h_{lm}(z) = \delta_{lm} - \sum_{j,k} c_{jk,lm} z_j \bar{z}_k + O(|z|^3)$$

where $c_{jk,lm}$ is the coefficients of the Chern curvature tensor,

$$\Theta(TX)_x := \sum c_{jk,lm} dz_j \wedge d\bar{z}_k \otimes \left(\frac{\partial}{\partial z_l}\right)^* \otimes \frac{\partial}{\partial z_m}$$

(查书)

4.5 Hodge theory on compact compact manifold

(X, ω) is a compact Hermitian manifold, $E \rightarrow X$ is a homomorphic Hermitian vector bundle.

$$D_E := D'_E + D''_E$$

Chern connection, $D''_E = \bar{\partial}$.

定义 4.5.1.

$$\Delta_E := D_E D_E^* + D_E^* D_E$$

$$(D'_E)^* = - * D''_E *$$

$$(D''_E)^* = - * D'_E *$$

$$\Delta'_E = D'_E (D'_E)^* + \dots$$

$$\Delta''_E = \dots$$

Note that $(D''_E)^2 = 0$, consider the complex

$$\begin{aligned} C^\infty(X, \bigwedge^{p,q} E) &\xrightarrow{D''_E} C^\infty(X, \bigwedge^{p,q+1} E) \\ &\rightsquigarrow H_{D''_E}^{p,q}(X, E) \end{aligned}$$

Dolbeaut cohomology... it isom to $\ker \Delta''_E$

术语索引

distinguished boundary 特征边界, 4

Dolbeault cohomology, 7

Hartogs figure, 11

holomorphic function 全纯函数, 3

polydisk 多圆柱, 4