

# 复几何

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本课程参考以下教材：

1. Demailly: Complex analytic and differential geometry.
2. Huybrechts: Complex geometry: an introduction.
3. Morrow, Kodaira: Complex manifolds.
4. Grauert, Remmert: Coherent analytic sheaves.
5. Hormander: An introduction to complex analysis in several variables.
6. Griffiths, Harris: Principles of algebraic geometry.

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在五道口也要红专并进、理实交融呀 ~

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# 第 1 章 多复变函数

## 1.1 多元全纯函数

首先快速回顾单复变函数的知识。我们通常用  $\Omega$  来表示  $\mathbb{C}$  的开子集,  $z = x + iy$  为  $\mathbb{C}$  的坐标。对于  $z \in \mathbb{C}$  以及实数  $R > 0$ , 我们令

$$\mathbb{D}(z, R) := \{w \in \mathbb{C} \mid |w - z| < R\}$$

为以  $z$  为圆心  $R$  为半径的开圆盘。

此外, 我们有如下常用记号:

$$\begin{cases} dz := dx + i dy \\ d\bar{z} := dx - i dy \end{cases} \quad \begin{cases} \frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \\ \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \end{cases}$$

对于函数  $f: \Omega \rightarrow \mathbb{C}$ , 称  $f$  是全纯 (holomorphic) 的, 若在  $\Omega$  中成立

$$\bar{\partial}f := \frac{\partial f}{\partial \bar{z}} d\bar{z} = 0$$

我们知道,  $f$  是全纯的当且仅当  $f$  在  $\Omega$  处处能够局部地展开为收敛幂级数。

对于  $\mathbb{C}$  中的紧致集  $K$ , 称函数  $f: K \rightarrow \mathbb{C}$  是全纯的, 如果存在  $K$  的开邻域  $\Omega \supseteq K$ , 使得  $f$  可延拓为  $\Omega$  上的全纯函数。

单复变函数论中有如下重要结果:

**定理 1.1.1.** (柯西积分公式) 设  $\mathbb{D} \subseteq \mathbb{C}$  为  $\mathbb{C}$  中的开圆盘,  $f: \mathbb{D} \rightarrow \mathbb{C}$  为  $\mathbb{D}$  上的全纯函数, 且在  $\partial\mathbb{D}$  连续, 则对于任意  $w \in \mathbb{D}$ , 成立

$$f(w) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(z)}{z - w} dz$$

此定理能推导出单变量全纯函数理论的 “almost everything”. 这里不再赘述。

我们开始考虑多变量全纯函数。

**定义 1.1.2.** 设  $\Omega \subseteq \mathbb{C}^n$  为  $\mathbb{C}^n$  的开子集, 函数  $f: \Omega \rightarrow \mathbb{C}$  称为 (多变量) 全纯函数, 如果满足以下条件:

- (1)  $f$  是连续函数;
- (2) 对任意  $1 \leq j \leq n$ , 以及任意固定的  $z_1, \dots, z_{j-1}; z_{j+1}, \dots, z_n \in \mathbb{C}$ , 关于  $z_j$  的单变量函数

$$z_j \mapsto f(z_1, \dots, z_{j-1}; z_j; z_{j+1}, \dots, z_n)$$

是 (单变量) 全纯函数。

事实上, 如果该定义中的 (2) 成立, 那么能推出 (1) 成立, 也就是说此定义中的 (1) 可以去掉。其证明比较复杂, 我们承认之。

**记号 1.1.3.** 对于  $\mathbb{C}^n$  的开子集  $\Omega$ , 我们记

$$\mathcal{O}(\Omega) := \{f: \Omega \rightarrow \mathbb{C} \mid f \text{ 是 } \Omega \text{ 上的全纯函数}\}$$

容易知道  $\mathcal{O}(\Omega)$  有显然的  $\mathbb{C}$ -代数结构。

本节将说明, 多变量全纯函数具有一些与单变量全纯函数类似的性质。

**记号 1.1.4.** 对于  $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$  以及  $R = (R_1, R_2, \dots, R_n) \in \mathbb{R}^n$ , 并且  $R_j > 0$  ( $\forall 1 \leq j \leq n$ ), 则我们记

$$\mathbb{D}(z, R) := \mathbb{D}(z_1, R_1) \times \mathbb{D}(z_2, R_2) \times \cdots \times \mathbb{D}(z_n, R_n)$$

称为以  $z$  为中心,  $R$  为半径的**多圆柱** (*polydisk*)。

对于多圆柱  $\mathbb{D}(z, R)$ , 我们记

$$\Gamma(z, R) := \partial\mathbb{D}(z_1, R_1) \times \partial\mathbb{D}(z_2, R_2) \times \cdots \times \partial\mathbb{D}(z_n, R_n)$$

称为  $\mathbb{D}(z, R)$  的**特征边界** (*distinguished boundary*)。

特别注意特征边界  $\Gamma(z, R)$  并不等于该多圆柱的边界  $\partial\mathbb{D}(z, R)$ 。

**定理 1.1.5.** (多变量全纯函数的柯西积分公式)

设  $f: \overline{\mathbb{D}(z, R)} \rightarrow \mathbb{C}$  为全纯函数, 则对任意的  $w \in \mathbb{D}(z, R)$ , 成立

$$f(w) = \frac{1}{(2\pi i)^n} \int_{\Gamma(z, R)} \frac{f(\xi) d\xi_1 d\xi_2 \cdots d\xi_n}{(\xi_1 - w_1)(\xi_2 - w_2) \cdots (\xi_n - w_n)}$$

证明. 由多变量全纯函数的定义, 反复使用单变量全纯函数的柯西积分公式即可。这是容易的。□

与单复变函数完全类似, 我们也有泰勒展开:

**推论 1.1.6.** (多元全纯函数的泰勒展开公式)

对于  $f \in \mathcal{O}(\Omega)$ , 其中  $\Omega \subseteq \mathbb{C}^n$  为开子集, 则对于任何多圆柱  $\mathbb{D}(z_0, R)$ , 如果  $\overline{\mathbb{D}(z_0, R)} \subseteq \Omega$ , 则对于任意  $w \in \mathbb{D}(z_0, R)$ , 成立

$$f(w) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (w - z_0)^\alpha$$

其中

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{\Gamma(z_0, R)} \frac{f(z)}{(z - z_0)^{\alpha+1}} dz_1 dz_2 \cdots dz_n = \frac{f^{(\alpha)}(z_0)}{\alpha!}$$

注意这里的  $\alpha$  为多重指标, 即  $\alpha = (\alpha_1, \dots, \alpha_n)$ , 其中每个  $\alpha_i$  都为非负整数。我们记

$$\begin{aligned} z^\alpha &:= z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \\ \alpha! &:= \alpha_1! \alpha_2! \cdots \alpha_n! \\ f^{(\alpha)} &:= (\partial_{z_1})^{\alpha_1} (\partial_{z_2})^{\alpha_2} \cdots (\partial_{z_n})^{\alpha_n} f \\ \alpha + 1 &:= (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_n + 1) \end{aligned}$$

其中  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ ,  $f$  为  $n$  元全纯函数。

证明. 与单复变函数的情形完全类似, 可由柯西积分公式得到。□

**定理 1.1.7.** (柯西不等式) 对于  $\mathbb{C}^n$  的开子集  $\Omega$ , 若  $f \in \mathcal{O}(\Omega)$ , 多圆柱  $\overline{\mathbb{D}(z_0, R)} \subseteq \Omega$ , 则对任意多重指标  $\alpha \in \mathbb{N}^n$ , 成立

$$|f^{(\alpha)}(z_0)| \leq \frac{\alpha!}{R^\alpha} \sup_{z \in \Gamma(z_0, R)} |f(z)|$$

证明. 与单复变函数的情形完全类似。利用多元泰勒展开（推论1.1.6）即可。  $\square$

**推论 1.1.8.** 设  $\Omega \subseteq \mathbb{C}^n$  为连通开集,  $f \in \mathcal{O}(\Omega)$  满足  $\forall 1 \leq k \leq n, \frac{\partial f}{\partial z_k}$  在  $\Omega$  上恒为 0, 则  $f$  在  $\Omega$  上为常值函数。

**推论 1.1.9.** (刘维尔定理) 设  $f \in \mathcal{O}(\mathbb{C}^n)$ , 并且满足

$$|f(z)| \leq A(1 + |z|)^B$$

其中  $A, B$  为正实数, 那么  $f$  必为次数不超过  $B$  的多项式函数。

这些性质于单变量全纯函数雷同, 证明也是类似的。

**推论 1.1.10.** (*Montel* 定理)

设  $\Omega$  为  $\mathbb{C}^n$  的开子集, 则  $\mathcal{O}(\Omega)$  中的任何局部一致有界的全纯函数列都存在一致收敛的子列。

证明. 仍类似于单复变全纯函数的情形。使用柯西积分公式, 再配合 Arzela-Ascoli 定理即可。从略。  $\square$

现在, 简单介绍一些复的微分形式。对于  $\mathbb{C}^n$ , 记其复坐标为  $(z_1, z_2, \dots, z_n)$ ; 视  $\mathbb{C}^n$  为  $2n$  维实线性空间,

$$z_k = x_k + iy_k$$

从而引入

$$dz_k = dx_k + idy_k \quad (1,0)\text{形式}$$

$$d\bar{z}_k = dx_k - idy_k \quad (0,1)\text{形式}$$

**定义 1.1.11.** ( $(p, q)$ -形式)

设  $\Omega$  为  $\mathbb{C}^n$  的非空开集, 则形如

$$u(z) = \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ}(z) dz_I \wedge d\bar{z}_J$$

的光滑张量场称为  $(p, q)$ -形式。记  $\Omega$  上的  $(p, q)$ -形式之全体为  $C_{p,q}^\infty(\Omega)$ 。

这里的  $I, J$  为多重指标。“光滑”指的是系数函数  $a_{IJ}$  为  $\Omega$  上的光滑复值函数。另外，显然  $(0,0)$ -形式即为光滑函数； $C_{p,q}^\infty(\Omega)$  具有显然的复线性空间结构，事实上还是  $C^\infty(\Omega)$ -模。

记号 1.1.12. ( $\bar{\partial}$ -算子) 定义算子

$$\bar{\partial} : C_{p,q}^\infty(\Omega) \rightarrow C_{p,q+1}^\infty(\Omega)$$

如下: 对于  $(p,q)$ -形式

$$u := \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ} dz_I \wedge d\bar{z}_J$$

则

$$\bar{\partial}u = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{k=1}^n \frac{\partial a_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J$$

类似地，也有

$$\partial : C_{p,q}^\infty(\Omega) \rightarrow C_{p+1,q}^\infty(\Omega)$$

它们与外微分算子  $d$  满足关系

$$d = \partial + \bar{\partial}$$

由  $d^2 = 0$ ，易知

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0$$

以下事实显然成立：

引理 1.1.13. 对于区域  $\Omega$  上的光滑函数  $f \in C^\infty(\Omega)$ ，则  $f$  全纯当且仅当  $\bar{\partial}f = 0$ 。

注记 1.1.14. (*Dolbeault* 上同调) 对于  $\Omega \subseteq \mathbb{C}^n$ ，注意  $\bar{\partial}^2 = 0$ ，从而对任意  $p \geq 0$ ，有上链复形  $C_{p,\bullet}^\infty(\Omega)$ ：

$$\cdots \rightarrow C_{p,q-1}^\infty(\Omega) \xrightarrow{\bar{\partial}} C_{p,q}^\infty(\Omega) \xrightarrow{\bar{\partial}} C_{p,q+1}^\infty(\Omega) \rightarrow \cdots$$

称上同调群

$$H^{p,q}(\Omega) := H^q(C_{p,\bullet}^\infty(\Omega), \bar{\partial})$$

为区域  $\Omega$  的 *Dolbeault* 上同调群。

类似于外微分  $d$  的 de-Rham 上同调群，*Dolbeault* 上同调群与  $\Omega$  的拓扑联系密切。例如，以下定理十分重要，我们先陈述，以后再证明：

**引理 1.1.15.** (*Dolbeault-Grothendieck 引理*)

设  $\mathbb{D} \subseteq \mathbb{C}^n$  为多圆柱, 则对于任意  $p, q \geq 0$ ,

$$H^{p,q}(\Omega) = 0$$

不难发现它与 de Rham 上同调的 Poincare 引理有些类似。

## 1.2 解析延拓与 Hartogs 现象

上一节介绍了多复变函数的一些“普通的”(与单变量类似)性质, 本节开始介绍多复变函数的一些独特性质。

**引理 1.2.1.** 设  $f \in C_c^\infty(\mathbb{C})$  为复平面上的紧支光滑函数, 则对任意  $z \in \mathbb{C}$ , 成立

$$\frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial f / \partial \bar{\tau}}{\tau - z} d\tau \wedge d\bar{\tau} = f(z)$$

证明. 基本的微积分练习。考虑换元  $\tau = z + re^{i\theta}$ , 则易知

$$\begin{aligned} d\tau \wedge d\bar{\tau} &= -2ir dr \wedge d\theta \\ \frac{\partial r}{\partial \bar{\tau}} &= \frac{1}{2} e^{i\theta} \\ \frac{\partial \theta}{\partial \bar{\tau}} &= -\frac{1}{2ir} e^{i\theta} \end{aligned}$$

因此有

$$\begin{aligned} \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial f / \partial \bar{\tau}}{\tau - z} d\tau \wedge d\bar{\tau} &= \frac{-1}{2\pi} \int_0^\infty dr \int_0^{2\pi} \left( -\frac{1}{ir} \frac{\partial f}{\partial \theta}(z + re^{i\theta}) \right) d\theta \\ &\quad + \frac{-1}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty \left( \frac{\partial f}{\partial r}(z + re^{i\theta}) \right) dr \\ &= 0 + \frac{-1}{2\pi} \int_0^{2\pi} -f(z) d\theta \\ &= f(z) \end{aligned}$$

证毕。 □



**引理 1.2.2.** (简单版本的  $\bar{\partial}$ -引理)

设  $n \geq 2$ ,  $\varphi \in C_{0,1}^\infty(\mathbb{C}^n)$  为具有紧支集的光滑  $(0,1)$ -形式, 且  $\bar{\partial}\varphi = 0$ , 则存在  $\mathbb{C}^n$  上的紧支光滑函数  $g$ , 使得

$$\bar{\partial}g = \varphi$$

证明. 记光滑  $(0,1)$ -形式  $\varphi$  为

$$\varphi = \sum_{k=1}^n \varphi_k(z_1, \dots, z_n) d\bar{z}_k$$

则

$$\bar{\partial}\varphi = \sum_{k,l} \frac{\partial \varphi_k}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_k = \sum_{1 \leq l < k \leq n} \left( \frac{\partial \varphi_k}{\partial \bar{z}_l} - \frac{\partial \varphi_l}{\partial \bar{z}_k} \right) d\bar{z}_l \wedge d\bar{z}_k$$

从而由  $\bar{\partial}\varphi = 0$  可得对任意  $k \neq l$ ,

$$\frac{\partial \varphi_k}{\partial \bar{z}_l} = \frac{\partial \varphi_l}{\partial \bar{z}_k}$$

考虑如下的  $\mathbb{C}^n$  上的函数  $\psi$ : 对于  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ ,

$$\psi(z) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\varphi_1(\tau; z_2, \dots, z_n)}{\tau - z_1} d\tau \wedge d\bar{\tau}$$

由  $\varphi_1$  的紧支性易知  $\psi$  为  $\mathbb{C}^n$  上的光滑函数。对于  $1 < k \leq n$ , 有

$$\begin{aligned} \frac{\partial \psi(z)}{\partial \bar{z}_k} &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_1}{\partial \bar{z}_k}(\tau; z_2, \dots, z_n)}{\tau - z_1} d\tau \wedge d\bar{\tau} \\ &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_k}{\partial \bar{\tau}}(\tau; z_2, \dots, z_n)}{\tau - z_1} d\tau \wedge d\bar{\tau} \\ &= \varphi_k(z) \end{aligned}$$

上式对  $k = 1$  显然也成立。因此  $\bar{\partial}\psi = \varphi$ .

最后还需要证明  $\psi$  是紧支的。由于  $\varphi$  紧支, 存在足够大的  $R > 0$ , 使得

$$\text{supp } \varphi \subseteq \mathbb{D}(0, R)$$

因此任意取定  $z \in \mathbb{C}^n$ , 使得  $z$  的分量  $z_2, z_3, \dots, z_n$  之中至少有一个模长大于  $R$ , 则由  $\psi$  的定义式直接得到  $\psi(z) = 0$ . (注意: 这一步严重依赖  $n \geq 2$ !) 也就是说, 存在  $z \notin \mathbb{D}(0, R)$  使得  $\psi = 0$  在  $z$  的某邻域内都成立。另一方面, 由于  $\bar{\partial}\psi = \varphi$  且  $\text{supp } \varphi \subseteq \mathbb{D}(0, R)$ , 从而  $\psi$  在  $\mathbb{D}(0, R)$  外部全纯, 因此由解析延拓唯一性,  $\psi$  在  $\mathbb{D}(0, R)$  外部恒为零, 因此  $\psi$  紧支。□

此引理在单复变  $n = 1$  的情形不成立:

例子 1.2.3. 设  $\varphi_1 \in C_0^\infty(\mathbb{C})$  为复平面上的紧支光滑函数, 并且

$$\iint_{\mathbb{C}} \varphi_1(z) \neq 0$$

考虑  $\mathbb{C}$  上的  $(0,1)$ -形式  $\varphi = \varphi_1(z)d\bar{z}$ , 则  $\bar{\partial}\varphi = 0$  是平凡的, 但不存在紧支光滑函数  $\psi$  使得  $\bar{\partial}\psi = \varphi$ .

证明. 若存在紧支光滑函数  $\psi$  使得  $\bar{\partial}\psi = \varphi$ , 则  $\frac{\partial\psi}{\partial\bar{z}} = \varphi_1$ . 于是

$$0 \neq \iint_{\mathbb{C}} \varphi_1(z) dz \wedge d\bar{z} = \iint_{\mathbb{C}} \frac{\partial\psi}{\partial\bar{z}} dz \wedge d\bar{z} = 0$$

产生矛盾。 □

以下是多复变函数解析延拓的令人惊讶的性质, 它与单复变函数有本质不同:

**定理 1.2.4.** (*Hartogs* 现象)

设  $\Omega \subseteq \mathbb{C}^n$  为开集 ( $n \geq 2$ ),  $K \subset\subset \Omega$  且为  $\mathbb{C}^n$  的紧子集, 则对任意的  $f \in \mathcal{O}(\Omega \setminus K)$ , 都存在解析延拓  $F \in \mathcal{O}(\Omega)$ , 使得

$$F|_{\Omega \setminus K} = f$$

证明. 取  $K$  与  $\Omega$  直接的截断函数  $\psi \in C_0^\infty(\mathbb{C}^n)$ , 使得  $0 \leq \psi \leq 1$ ,

$$K \subset\subset \text{supp } \psi \subset\subset \Omega$$

并且  $\psi|_K \equiv 1$ . 考虑

$$\tilde{f} := (1 - \psi)f$$

则  $\tilde{f}$  在整个  $\Omega$  上都有定义。注意

$$\bar{\partial}\tilde{f} = -(\bar{\partial}\psi)f + (1 - \psi)\bar{\partial}f$$

易知  $\text{supp } \bar{\partial}\tilde{f} \subseteq \text{supp } \psi$ . 于是由引理1.2.2, 存在光滑函数  $v$ , 使得  $\text{supp } v \subseteq \text{supp } \psi$ , 并且  $\bar{\partial}v = \bar{\partial}\tilde{f}$ , 从而考虑函数

$$F := (1 - \psi)f - v$$

则  $\bar{\partial}F = 0$ , 从而  $F \in \mathcal{O}(\Omega)$ . 又因为易知

$$F = f \quad (\forall z \in \Omega \setminus \text{supp } \psi)$$

从而由解析延拓唯一性, 有  $F|_{\Omega \setminus K} = f$ . □

关于解析延拓, 再介绍如下结果:

**引理 1.2.5.** (*Hartogs figure*)

对于  $n > 1$ , 正实数  $0 < r < R$ , 以及  $\mathbb{C}^{n-1}$  的开子集  $\omega' \subseteq \omega$ , 记  $\mathbb{C}^n$  的开子集

$$\Omega := ((\mathbb{D}(0, R) \setminus \mathbb{D}(0, r)) \times \omega) \cup (\mathbb{D}(0, R) \times \omega')$$

其中  $\mathbb{D}(0, r)$  与  $\mathbb{D}(0, R)$  分别为  $\mathbb{C}$  上的以原点为中心,  $r, R$  为半径的开圆盘。则任意  $f \in \mathcal{O}(\Omega)$  都可以 (唯一地) 解析延拓至

$$\tilde{\Omega} := \mathbb{D}(0, R) \times \omega$$

如此的区域  $\Omega$  称之为 “**Hartogs figure**”。 $\Omega$  的几何图像大致如下:

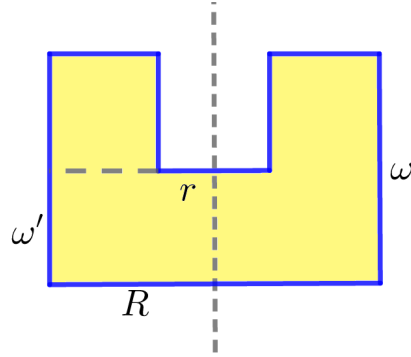


图: Hartogs figure 示意

证明. 容易知道

$$\Omega = \{(z_1, \tilde{z}) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid r < |z_1| < R, \tilde{z} \in \omega \text{ 或者 } |z_1| \leq r, \tilde{z} \in \omega'\}$$

对于  $f \in \mathcal{O}(\Omega)$ , 定义  $\tilde{\Omega}$  上的函数

$$\tilde{f}(z_1, \tilde{z}) := \frac{1}{2\pi i} \int_{|w|=\rho} \frac{f(w, \tilde{z})}{z_1 - w} dw$$

其中  $\rho$  为满足  $\max\{r, |z_1|\} < \rho < R$  的任意实数。则易知如此定义的  $\tilde{f}$  为  $f$  在  $\tilde{\Omega}$  上的解析延拓。  $\square$

## 第2章 层与层上同调

### 2.1 层的上同调

Today:

Sheaf cohomology

$X$  a topological space,  $\mathcal{F}$ - sheaf (of abelian groups).

定义 2.1.1. (*resolution*)

(1) a resolution of  $\mathcal{F}$  is an exact sequence

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{F} \xrightarrow{d^0} \mathcal{F} \xrightarrow{d^1} \rightarrow \dots$$

定义 2.1.2. A sheaf  $\mathcal{A}$  is called injective, if if for any injective morphism  $j : \mathcal{A} \rightarrow \mathcal{B}$  and for any morphism  $\varphi : \mathcal{A} \rightarrow \mathcal{S}$ , there exists an extension  $\psi : \mathcal{B} \rightarrow \mathcal{S}$ , such that

定理 2.1.3. the category of sheaves of abelian sheaves have enough injective objects, i.e. any  $\mathcal{F}$  can be embedded in some injective sheaf.

定义 2.1.4. Consider an injective resolution of  $\mathcal{F}$ , i.e. an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \mathcal{I}^2 \rightarrow \dots$$

where every  $\mathcal{I}^k (k \geq 0)$  is injective.

$\leadsto$  induces a sequence

$$0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{I}^0) \xrightarrow{d} \Gamma(X, \mathcal{I}^1) \xrightarrow{d} \Gamma(X, \mathcal{I}^2) \rightarrow \dots$$

Then

$$H^q(X, \mathcal{F}) := H^q(\Gamma(X, \mathcal{I}^\bullet))$$

then,  $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ .

**定义 2.1.5.** A sheaf  $\mathcal{S}$  is called a flabby (flasque, in France), if for any open set  $\Omega \subseteq X$ , the morphism

$$\mathcal{S}(X) \rightarrow \mathcal{S}(\Omega)$$

is surjective.

**定义 2.1.6.**

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1$$

is an exact sequence is called a flabby resolution, if any  $\mathcal{F}^k$  is flabby.

**定义 2.1.7.**

$$H^q(X, \mathcal{F}) := \dots \text{by flabby resolution} \dots$$

证明. Homological Algebra...omit. □

the two definitions of Sheaf Cohomology are isomorphic.

Godement's construction

$$God(\mathcal{F})(U) := \{f : U \rightarrow \bigcup_{x \in U} \mathcal{F}_x \mid f(y) \in \mathcal{F}_y, \forall y \in U\} := \prod_{x \in U} \mathcal{F}_x$$

$God(\mathcal{F})$  is a sheaf, and it is flabby. and there is a canonical morphism  $\mathcal{F}(U) \rightarrow God(\mathcal{F})(U)$  by  $x \mapsto (x \mapsto s_x)$  is injective.

$$\mathcal{F}^0 := God(\mathcal{F})$$

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \twoheadrightarrow \text{coker}(j) = \mathcal{F}^0 / \mathcal{F}$$

and consider

$$\mathcal{F}^1 := \text{God}(\text{coker}(j))$$

.....then construct by induction... this is a flabby resolution of  $\mathcal{F}$ .

**定义 2.1.8.** (resolution by fine sheaves)

$\mathcal{A}$  is a sheaf of ring,  $X$  is a paracompact topological space,  $\mathcal{A}$  is called a fine sheaf, if for any open covering

$$X = \bigcup_{\alpha} V_{\alpha} \quad , \mathcal{V} := \{V_{\alpha}\}$$

there exists a partition of unit subordinate to  $\mathcal{V}$ , (i.e.  $\exists f_{\alpha} \in \mathcal{A}(V_{\alpha}), \text{supp}(\alpha) := \overline{\{x \in V_{\alpha} | f_{\alpha,x} \neq 0\}} \subseteq V_{\alpha}$ , and  $\sum_{\alpha} f_{\alpha} = 1$  (the sum is locally finite) )

**例子 2.1.9.**  $X$  is a differential manifold,  $\mathcal{C}^{\infty}$  is the sheaf of smooth functions, then  $\mathcal{C}^{\infty}$  is a fine sheaf.

**定理 2.1.10.**  $\mathcal{S}$  is a sheaf of  $\mathcal{A}$ -modules,  $\mathcal{A}$  is a fine sheaf. then for any  $q \geq 1$ ,

$$H^q(X, \mathcal{S}) = 0$$

**证明.** Consider a flabby(or injective) resolution

$$0 \rightarrow \mathcal{S} \xrightarrow{j} \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \mathcal{I}^2 \dots$$

where any  $\mathcal{I}^k (k \geq 0)$  is a sheaf of  $\mathcal{A}$ -modules.

by definition,

$$H^q(X, m\mathcal{S}) := \frac{\ker d : \Gamma(\mathcal{I}^q) \rightarrow \Gamma(\mathcal{I}^{q+1})}{\Im d : \Gamma(\mathcal{I}^{q-1}) \rightarrow \Gamma(\mathcal{I}^q)}$$

Let  $\alpha \in \ker\{d : \Gamma(\mathcal{I}^q) \rightarrow \Gamma(\mathcal{I}^{q+1})\}$  by the exactness of resolution,  $\exists$  an open covering  $\mathcal{U} = (U_i)_i$ , s.t.  $\alpha|_{U_i} = d\beta_i$  where  $\beta_i \in \mathcal{T}^{q-1}(U_i)$ . Let  $(\beta_i)_i$  be the partition of unit w.r.t.  $\mathcal{U}$ . consider

$$\beta := \sum_i f_i \beta_i$$

(well defined). Then  $d\beta = \alpha$ ....

□

## 2.2 Čech 上同调

### Čech cohomology

$X$ - a topological space,  $\mathcal{F}$ - a sheaf of abelian group.

$$\mathcal{U} = (U_\alpha)_{\alpha \in I}$$

is an open covering.

notation:  $U_{\alpha_1, \dots, \alpha_q} := \bigcap_{i=1}^q U_{\alpha_i}$ .

Čech  $q$ -chain w.r.t  $\mathcal{U}$ :

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{(\alpha_1, \dots, \alpha_q) \in \mathcal{I}^{q+1}} \mathcal{F}(U_{\alpha_1, \dots, \alpha_q})$$

$$c \in C^q(\mathcal{U}, \mathcal{F})$$

means that we have a family of sections  $c_{\alpha_1, \dots, \alpha_q} \in \mathcal{F}(U_{\alpha_1, \dots, \alpha_q})$  with the relation

$$c_{\alpha_0, \dots, \alpha_j, \dots, \alpha_i, \dots} = -c_{\dots}$$

(Č)ech differential:

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{F})$$

$$\delta^q(c)_{\alpha_0, \dots, \alpha_{q+1}} := \sum_{0 \leq k \leq q+1} (-1)^k c_{\dots \hat{\alpha}_k \dots} |_{U_{\alpha_0, \dots, \alpha_{q+1}}}$$

性质 2.2.1.

$$\delta^q \circ \delta^q = 0$$

so, we have Čech cohomology

$$H^q(\mathcal{U}, \mathcal{F}) := \ker \delta^q / \operatorname{Im} \delta^{q-1}$$

example:

$$C^0(\mathcal{U}, \mathcal{F}) := \prod_{\alpha \in I} \mathcal{F}(U_\alpha)$$

$$c = (c_\alpha)_{\alpha \in I} \in C^0(\mathcal{U}, \mathcal{F})$$

$$\delta^0 c = 0 \iff (\delta^0 c)_{\alpha_0 \alpha_1} := (c_{\alpha_1} - c_{\alpha_0})|_{U_{\alpha_0 \alpha_1}} = 0$$

so,  $c_{\alpha_0} = c_{\alpha_1}$  on  $U_{\alpha_0 \alpha_1}$ .

$$\rightsquigarrow H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X).$$

**例子 2.2.2.** (1) consider  $X = \Delta \setminus \{0\}$ , where  $\Delta = \{(z_1, z_2) | |z_1| < 1, |z_2| < 1\}$ . Consider the covering

$$\mathcal{U} = U_1 \cup U_2$$

where

$$U_1 := \{(z_1, z_2) \in \Delta | z_1 \neq 0\} = \mathbb{D}^* \times \mathbb{D}$$

$$U_2 := \{(z_1, z_2) \in \Delta | z_2 \neq 0\} = \mathbb{D} \times \mathbb{D}^*$$

then

$$U_1 \cap U_2 = \mathbb{D}^* \times \mathbb{D}^*$$

consider  $H^0(X, \mathcal{O}) = \mathcal{O}(X) \cong \mathcal{O}(\Delta) = \{f : \Delta \rightarrow \mathbb{C} \text{ holomorphic}\}$ .

$$H^1(\mathcal{U}, \mathcal{O}) = \ker \delta^1 / \text{Im } \delta^0$$

$$\delta^1 : C^1(\mathcal{U}, \mathcal{O}) \rightarrow C^2(\mathcal{U}, \mathcal{O}) \subseteq \prod_{\alpha_0, \alpha_1, \alpha_2} \mathcal{O}(U_{\alpha_0, \alpha_1, \alpha_2}) = 0$$

$$\ker \delta^1 = C^1(\mathcal{U}, \mathcal{O}) = \{c = c(\alpha_0, \alpha_1) | c_{\alpha_0, \alpha_1} \in \mathcal{O}(U_{\alpha_0, \alpha_1})\} = \{c \in \mathcal{O}(U_1 \cap U_2)\} = \{c = \sum_{m, n \in \mathbb{Z}} a_{mn} z_1^m z_2^n \text{ convergent}\}$$

$$\delta^0 : C^0(\mathcal{U}, \mathcal{O}) \rightarrow C^1(\mathcal{U}, \mathcal{O})$$

$$(\delta^0 c)_{12} = (c_2 - c_1)|_{U_{12}}$$

where  $c_2 \in \mathcal{O}(U_2)$  and  $c_1 \in \mathcal{O}(U_1)$ . note that

$$\mathcal{O}(U_1) = \{c(z_1, z_2) = \sum_{m \in \mathbb{Z}, n \geq 0} a_{mn} z_1^m z_2^n \text{ convergent}\}$$

$$\mathcal{O}(U_2) = \{c(z_1, z_2) = \sum_{n \in \mathbb{Z}, m \geq 0} a_{mn} z_1^m z_2^n \text{ convergent}\}$$

$$\text{So, } H^1(\mathcal{U}, \mathcal{O}) = \{c(z_1, z_2) = \sum_{m, n < 0} a_{mn} z_1^m z_2^n\}$$

**例子 2.2.3.** (complex projective space)

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

$$(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$$

for some  $\lambda \in \mathbb{C}^*$ .

$$\mathbb{C}P^n = \{[z_0, \dots, z_n] | \text{not all } z_k = 0, z_i \in \mathbb{C}\} = \bigcup_{0 \leq p \leq n} V_p$$

where

$$V_k = \{[z_0, \dots, z_n] | z_k \neq 0\} \cong \{(\frac{z_0}{z_k}, \dots, 1, \dots, \frac{z_n}{z_k}) | z_i \in \mathbb{C}, i \neq k, z_k \neq 0\} \cong \mathbb{C}^n$$



this is a holo chart.

$$\mathbb{C}P^1 = V_0 \cup V_1, \mathcal{V} = \{V_0, V_1\}$$

HW: compute  $H^q(\mathcal{V}, \mathcal{O})$ .

Answer:

$$H^0 \cong \mathbb{C}, H^1 \cong 0$$

**Correction:**

$\mathcal{A}$ : Sheaf of rings (with unit)

$X$ : paracompact topological space,

**定义 2.2.4.**  $\mathcal{A}$  is called fine, if for any open covering  $\mathcal{U} = (V_\alpha)_{\alpha \in \mathcal{I}}$ , there exist  $s_\alpha \in \mathcal{A}(X)$  such that such that  $\text{supp}(s_\alpha) \subseteq V_\alpha$ ,

$$\sum_{\alpha} s_\alpha = 1$$

(this is a locally finite sum)

**注记 2.2.5.** we call  $\mathcal{A}$  is a **soft sheaf**, if for any closed set  $K \subseteq X$ , the morphism

$$\mathcal{A}(X) \rightarrow \mathcal{A}(K)$$

is surjective. where  $\mathcal{A}(K) := \Gamma(K, \mathcal{A}|_K)$

fact:  $\mathcal{A}$  is fine if and only if  $\mathcal{H}om(\mathcal{A}, \mathcal{A})$  is soft. (omit)

Recall:

Cech cohomology:  $X$  topological space,  $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$ ,

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 < \dots < \alpha_q} \mathcal{F}(\alpha_1, \dots, \alpha_q)$$

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{F})$$

fact:  $H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$ .

Today:

**定义 2.2.6.** Let  $\mathcal{V} = (V_\beta)_{\beta \in \mathcal{J}}$  be another open covering, then  $\mathcal{V}$  is called a refinement of  $\mathcal{U}$ , if there exists a map

$$\rho : \mathcal{J} \rightarrow \mathcal{I}$$

such that

$$V_\beta \subseteq U_{\rho(\beta)}$$

性质 2.2.7. Let  $\mathcal{V}$  be a refinement of  $\mathcal{U}$ , then  $\rho$  induces a map

$$\rho^q : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^q(\mathcal{V}, \mathcal{F})$$

$$(\rho^q C)_{\beta_0, \dots, \beta_q} \mapsto C_{\rho(\beta_0), \dots, \rho(\beta_q)}|_{V_{\beta_0, \dots, \beta_q}}$$

$\rho$  is a morphism of complexes.

so,  $\rho$  induces a map

$$H^q(\rho) : H^q(\mathcal{U}, \mathcal{F}) \rightarrow H^q(\mathcal{V}, \mathcal{F})$$

Let  $\tilde{\rho} : \mathcal{J} \rightarrow \mathcal{I}$  be another refinement of  $\mathcal{U}$

(induces  $H^q(\tilde{\rho}) : H^q(\mathcal{U}, \mathcal{F}) \rightarrow H^q(\mathcal{V}, \mathcal{F})$ ) then  $\rho, \tilde{\rho}$  are homotopic (chain homotopy  $\rightsquigarrow H^q(\rho) = H^q(\tilde{\rho})$ )

so, if  $\rho : \mathcal{J} \rightarrow \mathcal{I}$  is refinement, then

$$H^q(\rho)$$

is independent of the refinement.

定义 2.2.8.

$$\check{H}^q(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F})$$

i.e.  $a \in H^q(\mathcal{U}, \mathcal{F}) \sim \in H^q(\mathcal{V}, \mathcal{F})$  iff  $\exists$  a refinement  $\mathcal{W}$  of  $\mathcal{U}$  and  $\mathcal{V}$  such that  $a, b$  have the same image in  $H^q(\mathcal{W}, \mathcal{F})$

注记 2.2.9.

$$\check{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$$

Exercise: For  $q = 1$ , if  $\mathcal{V}$  is a refinement of  $\mathcal{U}$ , then

$$H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(\mathcal{V}, \mathcal{F})$$

is injective.

so, for any open cover  $\mathcal{U}$ ,

$$H^1(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{F})$$

is injective.

**Homological Algebra** recall: let  $(K^\bullet, d_k), (L^\bullet, d_l)$  and  $(M^\bullet, d_M)$ , if we have a short exact sequence

$$0 \rightarrow K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} M^\bullet \rightarrow 0$$

then it induces a long exact sequence :

$$\cdots \rightarrow H^q(K^\bullet) \rightarrow H^q(L^\bullet) \rightarrow H^q(M^\bullet) \rightarrow H^{q+1}(K^\bullet) \rightarrow \cdots$$

analogy of Cech cohomology:  $X$  is a topological space,  $\mathcal{U}$  is an open covering of  $X$ .  $\mathcal{A}$  and  $\mathcal{B}$  sheaves on  $X$ , Let

$$\varphi : \mathcal{A} \rightarrow \mathcal{B}$$

be a morphism, then it induces

$$\varphi^\bullet : C^\bullet(\mathcal{U}, \mathcal{A}) \rightarrow C^\bullet(\mathcal{U}, \mathcal{B})$$

Let

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

be an exact sequence of sheaves, then we have: for any open set  $\Omega$ ,

$$0 \rightarrow \mathcal{A}(\Omega) \rightarrow \mathcal{B}(\Omega) \rightarrow \mathcal{C}(\Omega)$$

left exact.

Example: consider

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \rightarrow 0$$

is exact on  $bbC^\times := \mathbb{C} \setminus \{0\}$

but we have :

$$0 \rightarrow \mathcal{A}(\Omega) \xrightarrow{\psi} \mathcal{B}(\Omega) \rightarrow \text{Im } \psi(\Omega) \rightarrow 0$$

is exact.

First we have the following exact sequence

$$C^q(\mathcal{U}, \mathcal{A}) \rightarrow C^q(\mathcal{U}, \mathcal{B}) \rightarrow C_B^q(\mathcal{U}, \mathcal{C}) \rightarrow 0$$

where  $C_B^q$  is the image of ...

then we get an exact sequence

$$0 \rightarrow (C^\bullet(\mathcal{U}, \mathcal{A}), \delta) \rightarrow (C^\bullet(\mathcal{U}, \mathcal{B}), \delta) \rightarrow (C_B^\bullet(\mathcal{U}, \mathcal{C}), \delta) \rightarrow 0$$

it induces a long exact sequence

$$\cdots \rightarrow H^q(\mathcal{U}, \mathcal{A}) \rightarrow H^q(\mathcal{U}, \mathcal{B}) \rightarrow H_B^q(\mathcal{U}, \mathcal{C}) \rightarrow H^{q+1}(\mathcal{U}, \mathcal{A}) \rightarrow \cdots$$

**定理 2.2.10.** *If  $X$  is paracompact,*

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

*is a sheaf exact sequence. Then there is a long exact sequence*

$$\cdots \rightarrow \check{H}^q(X, \mathcal{A}) \rightarrow \check{H}^q(X, \mathcal{B}) \rightarrow \check{H}^q(X, \mathcal{C}) \rightarrow \check{H}^{q+1}(X, \mathcal{A}) \rightarrow \cdots$$

证明. Key lemma: need to prove

$$\lim_{\vec{U}} H^q(\mathcal{U}, \mathcal{C}) = \lim_{\vec{U}} H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C})$$

if  $X$  is paracompact.

Omit. □

if

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

exact,

recall: (cohomology by resolutions)

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots$$

flabby resolution. then it induces

$$0 \rightarrow \Gamma(X, \mathcal{A}) \rightarrow \Gamma(X, \mathcal{F}^0) \rightarrow \Gamma(X, \mathcal{F}^1) \rightarrow \dots$$

then define the sheaf cohomology...

we have a long exact sequence

$$\dots \rightarrow H^q(X, \mathcal{A}) \rightarrow H^q(X, \mathcal{B}) \rightarrow H^q(X, \mathcal{C}) \rightarrow H^{q+1}(X, \mathcal{A}) \rightarrow \dots$$

it is homological algebra...

**定理 2.2.11.** (*Leray's acyclic theorem*) Let  $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$  be an open covering of  $X$ , ( $\mathcal{F}$  is a sheaf on  $X$ ), if satisfying

$$H^k(U_{\alpha_0, \dots, \alpha_q}) = 0$$

for any  $k \geq 1$ , then

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{H}^q(X, \mathcal{F})$$

and if  $X$  is paracompact, we also have

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{H}^q(X, \mathcal{F}) \cong H^q(X, \mathcal{F})$$

(this  $\mathcal{U}$  is called acyclic covering)

**de Rham- Weil theorem**

定义 2.2.12.  $\mathcal{F}$  is a sheaf on  $X$ ,  $\Omega$  is an open set of  $X$ , then  $\mathcal{F}$  is called **acyclic sheaf** if

$$H^q(\Omega, \mathcal{F}) = 0$$

for any  $q \geq 1$ .

定理 2.2.13. Let

$$0 \rightarrow \mathcal{F} \rightarrow (L^\bullet, d)$$

be an acyclic resolution of  $\mathcal{F}$  (i.e.  $L^q$  is acyclic on  $X$ ) then

$$H^q(X, \mathcal{F}) \cong H^q(\Gamma(X, L^\bullet), d)$$

for any  $q \geq 0$ .

(先看例子)

例子 2.2.14. Let  $X$  be a differential manifold,  $\mathcal{E}^p$ : sheaf of smooth  $p$ -forms, then we have a resolution (de Rham complex)

$$0 \rightarrow \mathbb{R} \hookrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \mathcal{E}^3 \rightarrow \dots$$

where  $d$  differential operators. (Why it is a resolution? because of Poincare lemma...locally solvable..)

Note that

$$\mathcal{E}^0 = \mathcal{C}^\infty$$

$\mathcal{E}^p$  is a sheaf of  $\mathcal{C}^\infty$ -modules..

then we have

$$H^q(X, \mathcal{E}^p) = 0$$

for all  $q \geq 1$

and then

$$H^q(X, \mathbb{R}) \cong \frac{\ker(d : \Gamma(X, \mathcal{E}^q) \rightarrow \Gamma(X, \mathcal{E}^{q+1}))}{\text{Im}(d : \Gamma(X, \mathcal{E}^{q-1}) \rightarrow \Gamma(X, \mathcal{E}^q))} = H_{DR}^q(X, \mathbb{R})$$

例子 2.2.15. Let  $X$  be a complex manifold,  $\mathcal{E}^{p,q}$  sheaf of smooth  $(p, q)$  forms,  $\Omega^p$  is the sheaf of holomorphic  $p$ -forms (i.e.  $(p, 0)$ -form  $\varphi$  with  $\bar{\partial}\varphi = 0$ ).

Then we have resolution

$$0 \rightarrow \Omega^p \xrightarrow{j} \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \rightarrow \dots$$

(Why it is a resolution? because of the Dolbeault lemma), remain to Exercise...

$$H^q(X, \Omega^p) \cong H^{p,q}_\partial(X, \mathbb{C})$$

Today: de Rham-Weil Isomorphism Thm

**定理 2.2.16.** *Let  $X$  be a topological space,  $\mathcal{F}$  be a sheaf of abelian groups on  $X$ ,*

$$0 \rightarrow \mathcal{F} \rightarrow (\mathcal{L}^\bullet, d)$$

*be an acyclic resolution, i.e.*

$$H^k(X, \mathcal{L}^q) = 0$$

*for all  $k \geq 1$  and  $q \geq 0$ . Then,*

$$H^q(X, \mathcal{F}) \cong H^q((\Gamma(\mathcal{L}^\bullet), d))$$

证明. Since

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{d^0} \mathcal{L}^1 \xrightarrow{d^1} \mathcal{L}^2 \rightarrow \dots$$

be an exact sequence, denote

$$\mathcal{Z}^q := \ker d^q$$

then we have short exact sequences

$$0 \rightarrow \mathcal{Z}^q \rightarrow \mathcal{L}^q \rightarrow \mathcal{Z}^{q+1} \rightarrow 0$$

for any  $q$ . They induce long exact sequence of cohomology groups:

$$\dots \rightarrow H^k(X, \mathcal{Z}^q) \rightarrow H^k(X, \mathcal{L}^q) \rightarrow H^k(X, \mathcal{Z}^{q+1}) \xrightarrow{\partial} H^{k+1}(X, \mathcal{L}^q) \rightarrow H^{q+1}(X, \mathcal{L}^q) \rightarrow \dots$$

For any  $k \geq 1$ , since  $\mathcal{L}^q$  are acyclic on  $X$ ,

$$H^k(X, \mathcal{Z}^{q+1}) \cong H^{k+1}(X, \mathcal{Z}^q)$$

and for  $k = 0$ , we have

$$0 \rightarrow H^0(X, \mathcal{Z}^q) \rightarrow H^0(X, \mathcal{L}^q) \rightarrow H^0(X, \mathcal{Z}^{q+1}) \rightarrow H^1(X, \mathcal{Z}^q) \rightarrow H^1(X, \mathcal{L}^q) = 0 \rightarrow \dots$$

so,

$$H^1(X, \mathcal{Z}^q) \cong H^0(X, \mathcal{Z}^{q+1}) / \text{Im } d^q \cong H^{q+1}((\Gamma(\mathcal{L}^\bullet), d))$$

$$H^{q+1}(\Gamma(\mathcal{L}^\bullet)) \cong H^1(X, \mathcal{Z}^q) \cong H^2(X, \mathcal{Z}^{q-1}) \cong \dots \cong H^{q+1}(X, \mathcal{Z}^0) = H^{q+1}(X, \mathcal{F})$$

□

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \rightarrow \dots$$

(de Rham resolution) then we have

$$H^k(X, \mathcal{R}) \cong H_{DR}^k(X; \mathcal{R})$$

(if  $X$  is compact, then by Hodge theory, it also isomorphic to  $\ker(\bar{\partial}\partial + \partial\bar{\partial})$ )

Another example:  $X$  is a complex manifold, then

$$0 \rightarrow \Omega^p \rightarrow \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \rightarrow \dots$$

then

$$H^q(X, \Omega^p) \cong H_{\bar{\partial}}^{p,q}(X, \mathbb{C})$$

(RHS= Dolbeault cohomology)

$X$  be a smooth manifold, we define

$C_q(X, \mathbb{Z}) :=$  the free abelian group generated by continuous map

$$\phi : \Delta_q := \{(t_1, \dots, t_{q+1}) \in [0, 1]^{q+1} \mid \sum_{i=1}^n t_i = 1\}$$

and we define (for  $\phi \in C_q(X, \mathbb{Z})$ )

$$\partial\phi := \sum_{i=1}^{q+1} (-1)^i \phi|_{\Delta_{q,i}}$$

$$\Delta_{q,i} := \{t \in \Delta_q \mid t_i = 0\}$$

we define

$$(C_{sing}^\bullet, \partial)$$

be the dual complex of  $(C_{sing}^\bullet, \partial)$ .

(These are all Basic Algebraic Topology)

For any open  $U \subseteq X$ , we have

$$U \rightarrow C_{sing}^q(U, \mathbb{Z})$$

we get a sheaf

$$\mathcal{C}_{sing}^q$$

FACT:  $(C_{sing}^\bullet, \partial)$  is a flabby resolution of  $\mathbb{Z}$ . (check!) So,

$$H_{sing}^q(X, \mathbb{Z}) = H^q(\Gamma(C_{sing}^\bullet), \partial) \cong H^q(X, \mathbb{Z})$$

## 第3章 Hermite 向量丛

### 3.1 联络与曲率

Recall:  $X$  is a smooth manifold,  $E$  is a vector bundle of rank  $r$ , if

- (1)  $\pi : E \rightarrow X$  is smooth map,
- (2) for any  $x \in X$ ,  $E_x := \pi^{-1}(x)$  is a vector space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) of dimension  $r$ .
- (3) there an open covering  $\mathcal{U} = (U_\alpha)_{\alpha \in I}$  and trivializations

$$\theta_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

and for any intersection  $U_\alpha \cap U_\beta$ , we have

注记 3.1.1.

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$$

(cocycle condition)

**Special Case: line bundle** rank  $E=1$ .

then  $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, \mathbb{K}^*) = \mathcal{E}^*(U_{\alpha\beta})$  invertible smooth function on  $U_{\alpha\beta}$ . then, Cech cohomology,

$$(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1$$

so,

$$(g_{\alpha,\beta}) \in \mathcal{Z}^1(\mathcal{U}, \mathcal{E}^*) \rightarrow H^1(\mathcal{U}, \mathcal{E}^*) \hookrightarrow \check{H}^1(X, \mathcal{E}^*)$$

we get a map

$$\{\text{line bundles}\} \rightarrow \check{H}^1(X, \mathcal{E}^*)$$

actually, we have

$$\{\text{isomorphic classes of line bundles}\} \longleftrightarrow H^1(X, \mathcal{E}^*)$$

1-1 correspondence.

Now,  $X$  be a complex manifold, a complex vector bundle  $E$  is called holomorphic, if ... the transition matrix  $g_{\alpha\beta}$  is holomorphic...



Holomorphic line bundles :

$$g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$$

$\mathcal{O}^*$ :sheaf of invertible holomorphic functions...

FACT: there is a map

$$\{\text{holomorphic line bundle}\} \rightarrow \check{H}^1(X, \mathcal{O}^*)$$

例子 3.1.2. *trivial vector bundle*  $X \times \mathbb{K}^r$

例子 3.1.3. *Tangent bundle*  $TX$ . (transition matrix  $g_{\alpha\beta}$  are given by Jacobi matrix..)

定义 3.1.4. (*Local frame of vector bundles*)

$$\theta_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{K}^r$$

be a trivialization, we define

$$e_\lambda(x) := \theta_\alpha^{-1}(x, \begin{pmatrix} 0 \\ \vdots \\ 1(\leftarrow \text{ith}) \\ \vdots \\ 0 \end{pmatrix})$$

then,  $\{e_1, \dots, e_r\}$  be a local smooth section  $s \in \Gamma(U_\alpha, E)$  can be written as

$$s(x) = \sum \sigma_\lambda(x)$$

where  $\sigma_\lambda \in C^\infty(U_\alpha, \mathbb{K})$ .

(Connection)

记号 3.1.5. For  $X$  be a smooth manifold,  $E$  is a vector bundle(real or complex), denote

$$C_p^k(\Omega, E) := C^k(\Omega, \bigwedge^p T^*M \otimes E)$$

is the space of  $k$ -differential  $p$ -forms with values in  $E$ .

Locally, consider a trivialization of  $E$ ,

$$\theta_\alpha E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

( $\rightsquigarrow$  frame  $(e_1, \dots, e_r)$ )

$$s \in \sum \varphi_\lambda(x) \otimes e_\lambda(x)$$

where  $\varphi_\lambda$  is a  $p$ -form.

定义 3.1.6. a (linear) connection on  $E$  is a linear differential operator of order 1 acting on  $C^\bullet_\bullet(X, E)$ :

$$D : C^\infty_p(X, E) \rightarrow C^\infty_{p+1}(X, E)$$

$$D(f \wedge s) := df \wedge s + (-1)^p f \wedge Ds$$

where  $f \in C^\infty(X, \wedge^p T^*M)$ ,  $s \in C^\infty(X, E)$ .

Locally, consider a local trivialization

$$\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

with a frame  $\{e_1, \dots, e_r\}$ . any section  $t \in C^\infty_p(\Omega, E)$  can be written as

$$t = \sum_{1 \leq \lambda \leq r} \sigma_\lambda \otimes e_\lambda$$

$$Ds = \sum_{\lambda=1}^r d\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge De_\lambda$$

where

$$De_\lambda \in C^\infty_1(\Omega, E)$$

can be written as

$$De_\lambda = \sum_{\mu=1}^r a_{\mu\lambda} \otimes e_\mu$$

where " $a_{\mu\lambda}$ " is called the coefficients of  $D$  with respect to frame  $\{e_1, \dots, e_r\}$ .

so,

$$D(t) = \sum_{\lambda, \mu} d\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge a_{\mu\lambda} \wedge e_\mu = \sum_{\mu} \sum_{\lambda} (d\sigma_\mu + a_{\mu\lambda} \wedge \sigma_\lambda)$$

$$Dt = d\sigma + A \wedge \sigma$$

where  $A = (a_{\mu\lambda})$ .

RMK: connection always exists!

Recall: for any (connected) smooth manifold,  $E \rightarrow X$  is a smooth vector bundle,

Connection:

$$D : C^\infty_p(X, E) \rightarrow C^\infty_{p+1}(X, E)$$

where  $C^\infty_p(X, E) := C^\infty(X, \wedge^p T^*M \otimes E)$

$$D(f \wedge s) = df \wedge s + (-1)^{\deg f} f \wedge Ds$$

Essentially,

$$D : C^\infty(X, E) \rightarrow C_1^\infty(X, E)$$

Locally, consider a trivialization  $\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$ , and a local frame  $(e_1, \dots, e_r)$  where  $e_k(x) =$

$$\theta^{-1}\left(x, \begin{pmatrix} 0 \\ \vdots \\ 1(k^{th}) \\ \vdots \\ 0 \end{pmatrix}\right).$$

Let  $s \in C^\infty(\Omega, E)$ , i.e.

$$s = \sum_{i=1}^r \sigma_i e_i$$

where  $\sigma_i$  are smooth functions.

$$Ds = d\sigma + A \wedge \sigma$$

where

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \end{pmatrix} \quad A = a_{ij}$$

consider another trivialization

$$\tilde{\theta} : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

$\rightsquigarrow$  a local frame  $(\tilde{e}_1, \dots, \tilde{e}_r)$ . Then there exists a invertible linear transform s.t.

$$\tilde{e}_k = g_k^m e_m$$

assume

$$De_k = a_k^l e_l \quad D\tilde{e}_k = \tilde{a}_k^l \tilde{e}_l$$

we have

$$\begin{aligned} dg_k^n e_n + g_k^m a_m^n e_n &= \tilde{a}_k^l g_l^n e_n \\ \rightsquigarrow \tilde{a}_k^l g_l^n (g^{-1})_n^p &= dg_k^n (g^{-1})_n^p + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \tilde{a}_l^p &= dg_k^n (g^{-1})_n^p + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \tilde{A} &= dg \cdot g^{-1} + g \cdot A \cdot g^{-1} \end{aligned}$$

## Curvature

$$H_D := D^2$$

locally,

$$D^2 s = D(d\sigma + A \wedge \sigma) = d(d\sigma + A \wedge \sigma) + A \wedge (d\sigma + A \wedge \sigma)$$

$$= dA \wedge \sigma - A \wedge d\sigma + A \wedge d\sigma + A \wedge A \wedge \sigma = (dA + A \wedge A) \wedge \sigma$$

so we have

$$H = dA + A \wedge A$$

Similarly to  $\tilde{A}, A$  we have

Exercise:

$$\tilde{H} = gHg^{-1}$$

曲率在不同平凡化下的表达式。where

$$\tilde{e} = ge$$

$\rightsquigarrow H$  can be considered as a section of  $C_2^\infty(X, \text{Hom}(E, E))$ . because

$$\tilde{H}\tilde{e} = gHg^{-1}\tilde{e} = gHe$$

independent of the choice of local frames.

## 3.2 向量丛的构造

**定义 3.2.1.** (dual of vector bundles)  $E \rightarrow X$ , and  $g_{\alpha\beta}$  :transition matrix of  $E$ , the dual is given by  $(g_{\alpha\beta})^{-1}$ . (用转移函数来定义向量丛)

**定义 3.2.2.** direct sum of two vector bundles  $(E, F) \rightarrow E \oplus F$ . locally,

$$(g_{\alpha,\beta}) \oplus (h_{\alpha\beta})$$

direct sum of transition matrices.

**定义 3.2.3.** tensor product of two vector bundles.

locally, tensor product of two transition matrices.

fact: let  $D_E$  be a connection on  $E$ , then it induces a connection  $D_{E^*}$ . Let  $u$  be a local section of  $E^*$ ,  $s$  local section of  $E$ , then we define

$$d\langle u, s \rangle = \langle D_{E^*}u, s \rangle + \langle u, D_E s \rangle$$

Exercise:

$$H(D_{E^*}) = -H(D_E)^T$$

and for two vector bundles  $E, F$ , connections  $D_E, D_F$ , then

$$D_{E \oplus F} := D_E \oplus D_F$$

$$H(E \oplus F) = H_E \oplus H_F$$

as for tensor product, we define  $D_{E \otimes F}$  as follows:

$$D_{E \otimes F}(s \otimes t) = D_E s \otimes t + s \otimes D_F t$$

check the curvature

$$H_{E \otimes F} = H_E \otimes id_F + id_E \otimes H_F$$

**注记 3.2.4.** we can also consider wedge product of vector bundles. Consider vector bundles  $E_1, \dots, E_k$ , with connections  $D_{E_1}, \dots, D_{E_k}$ , let  $s_i \in C_{p_i}^\infty(X, E^i)$  then

$$D_{E_1 \wedge \dots \wedge E_k}(s_1 \wedge \dots \wedge s_k) = \sum_{i=1}^k (-1)^{p_1 + \dots + p_{i-1}} s_1 \wedge \dots \wedge D_{E_i} s_i \wedge \dots \wedge s_k$$

Let  $E$  be a vector bundle of rank  $r$ , then  $\bigwedge^r E$  is a line bundle, with transition matrix by  $\det(g_{\alpha\beta})$ . this bundle is denoted by  $\det E$ . (Det-bundle)

Let  $s_1, \dots, s_r$  be local sections of  $E$ , then we have

$$D_{\det E}(s_1 \wedge \dots \wedge s_r) = \text{tr}(H_E) s_1 \wedge \dots \wedge s_r$$

### 3.3 陈省身示性类

chern classes (defined by curvature).

Let  $E \rightarrow X$  be a smooth complex vector bundle of rank  $r$ , where  $X$  be a complex manifold.

(Chern-Weil theory)

$V$  be a complex vector space,  $f : \underbrace{V \times \dots \times V}_k \rightarrow \mathbb{C}$  be a symmetric multi-linear form of degree  $k$ .

$\rightsquigarrow f(v) := f(v, v, \dots, v)$  is a homogeneous polynomial of degree  $k$ .

**定义 3.3.1.** assume  $G$  is a group (left) acting on  $V$ , s.t.

$$f(g(v_1), \dots, g(v_k)) = f(v_1, \dots, v_k)$$

for any  $g \in G, v_i \in V$ , then we say  $f$  is  $G$ -invariant.

Special case:  $G = GL(r, \mathbb{C})$  and  $V = \text{Lie}G = \mathfrak{gl}(r, \mathbb{C})$  be the Lie algebra of  $G$ . the action is

$$(g, M) \mapsto gMg^{-1}$$

Consider

$$\det(I + \frac{i}{2\pi}tm) = I + tf_1(M) + t^2f_2(M) + \cdots t^rf_r(M)$$

$\rightsquigarrow \forall 1 \leq k \leq r, f_k$  is  $G$ -invariant.

Let  $E \rightarrow X$  complex vector bundle on a complex manifold, let  $D_E$  be a connection, curvature  $H_E \in C_2^\infty(X, \text{Hom}(E, E))$ . Let  $f \in GL(r, \mathbb{C})$ - invariant "k-form", then

(1) Let  $H_\alpha, H_\beta$  be the curvature forms of  $E$  in different trivialization, then  $f(H_\alpha) = f(H_\beta)$ , so we get a globally defined  $2k$ -form.

assume  $H_\alpha = gH_\beta g^{-1}$ , then

$$f(H_\alpha) = f(gH_\beta g^{-1}) = f(H_\beta)$$

(2) we also have

$$df(H) = 0$$

locally,  $H = H_\alpha = da_\alpha + A_\alpha \wedge A_\alpha$ , then

$$\begin{aligned} df(H) &= df(H_\alpha, H_\alpha, \dots, H_\alpha) = \sum_{i=1}^k f(H_\alpha, \dots, \underbrace{dH_\alpha}_{i}, \dots, \alpha) \\ &= \sum_{i=1}^k f(H_\alpha, \dots, dA_\alpha \wedge A_\alpha - A_\alpha \wedge dA_\alpha, \dots, H_\alpha) \end{aligned}$$

Fact: (in Riemannian geometry) For any  $x \in X$ , we always can find a local frame s.t.  $A_\alpha(x) = 0$ . so, choose this frame,

$$df(H) = 0$$

So,  $[f(H)] \in H^{2k}(X, \mathbb{C})$

(3) Claim : the class  $[f(H)]$  is independent of the choice of the connections  $D_E$ .

Let  $D_0, D_1$  be two connections, consider

$$D_t = (1-t)D_0 + tD_1$$

$t \in [0, 1]$ , curvature  $H_t$

Fact:  $\alpha := A_1 - A_0$  is globally defined, and in  $C_1^\infty(X, \text{Hom}(E, E))$ .

Fact:

$$\frac{d}{dt}f(H_t) = kdf(\alpha, H_t, H_t, \dots, H_t)$$

So,

$$f(H_1) - f(H_0) = \int_0^1 \frac{d}{dt}f(H_t)dt = d \int_0^1 f(\alpha, H_t, H_t, \dots, H_t)dt$$

So,

$$[f(H_1)] - [f(H_0)]$$

**定义 3.3.2.** *the  $k$ -th Chern class of  $E$*

$$c_k(E) := [f_k(\Theta_E)] \in H^{2k}(X, \mathbb{C})$$

Recall: Chern Class

$X$  complex manifold,  $E \rightarrow X$  is a smooth complex vector bundle of rank  $r$ .  $D$  is a connection, curvature  $\Theta(D) \in C_2^\infty(X, \text{Hom}(E, E))$ .

linear algebra:

$$\det(I + \frac{i}{2\pi} tM) = I + t f_1(M) + t^2 f_2(M) + \cdots + t^r f_r(M)$$

Chern class  $\{f_k(\Theta)\} \in H_{DR}^{2k}(X, \mathbb{C})$  is independent of choice of connection.

Today:

Special case:  $E$  is a complex line bundle. Let  $D_0$  be a connection on  $E$ , locally  $D_0 e = A_0 e$ ,  $A_0$  is 1-form. curvature

$$\Theta(D_0) = D_0^2 = dA_0 + A_0 \wedge A_0 = dA_0$$

so, curvature is  $d$ -exact, so  $d\Theta(D_0) = 0$ .

$$\det(I + \frac{i}{2\pi} tM) = I + \frac{i}{2\pi} tM$$

so, the first Chern class of line bundle is

$$c_1(E) = \{\frac{i}{2\pi} \Theta(D_0)\}$$

Let  $D_1$  be another connection, locally  $D_1 e = A_1 e$ , so  $\Theta(D_1) = dA_1$ .so,

$$\Theta(D_1) - \Theta(D_0) = d(A_1 - A_0)$$

where

$$A_1 - A_0 \in C_1^\infty(X, \text{Hom}(E, E))$$

(when  $E$  is line bundle,  $\text{Hom}(E, E) \cong E^* \otimes E$  is trivial bundle)

so,  $A_1 - A_0$  is a globally defined smooth function on  $X$ . So,

$$\{\Theta(D_1)\} = \{\Theta(D_0)\} \in H^2(X, \mathbb{C})$$

independent of the choice of connection.

### 3.4 Hermite 向量丛

**定义 3.4.1.** a complex vector bundle  $E \rightarrow X$  of rank  $r$  is called a Hermitian vector bundle, if we have an inner product on  $E$ , i.e. locally, consider a local frame  $\{e_1, \dots, e_r\}$ , we have

$$\{e_i(x), e_j(x)\} = h_{ij}(x)$$

s.t.  $(h_{ij}(x))$  is a positive definite Hermitian matrix depending smoothly on  $x$ .

**注记 3.4.2.** For any complex vector bundle, Hermitian structure always exists.

证明与黎曼几何类似。(黎曼度量的存在性)

**定义 3.4.3.** (Hermitian connection)

A connection  $D$  on  $E$  is called Hermitian, if

$$d\{e_i, e_j\} = \{De_i, e_j\} + \{e_i, De_j\}$$

More generally, let  $t \in C_p^\infty(X, E)$ ,  $s \in C_q^\infty(X, Y)$ ,

$$d\{s, t\} = \{dt, s\} + (-1)^p \{t, Ds\}$$

**性质 3.4.4.**  $D$  is a Hermitian connection, then the curvature

$$\Theta(D)^* = -\Theta(D)$$

(where  $(-)^*$  is conjugate transpose of matrix)

it means that,  $i\Theta(D) \in C_2^\infty(X, \text{Herm}(E, E))$

证明.

$$\begin{aligned} 0 &= d^2\{e_i, e_j\} = d\{De_i, e_j\} + d\{e_i, De_j\} \\ &= \{D^2e_i, e_j\} - \{De_i, De_j\} + \{De_i, De_j\} + \{e_i, D^2e_j\} = \{(\Theta + \Theta^*)e_i, e_j\} \end{aligned}$$

□



**注记 3.4.5.**  $E$  is a Hermitian line bundle,  $D$  is a Hermitian connection, then  $i\Theta(D)$  is a real 2-form,  $c_1(E) \in H^2(X, \mathbb{R})$ .

(Chern connection)

**定义 3.4.6.** Let  $X$  be a complex manifold.  $D'$  is called a connection of type  $(1,0)$  on  $E$ , if for any section  $s \in C_{p,q}^\infty(X, E)$ , we have  $D's \in C_{p+1,q}^\infty(X, E)$ .

A connection  $D''$  is called a connection of type  $(0,1)$ , if ...  $D''s \in C_{p,q+1}^\infty(X, E)$ .

**注记 3.4.7.** Let  $E \rightarrow X$  be a vector bundle. Let  $D$  be a connection on  $E$ , locally

$$Ds \xrightarrow{\sim} d\sigma + A \wedge \sigma$$

$$d\sigma = \partial\sigma + \bar{\partial}\sigma$$

so, let  $A'$  be the  $(1,0)$ -part of  $A$ , ...,

$$Ds = \partial\sigma + A' \wedge \sigma + (\bar{\partial}\sigma + A'' \wedge \sigma) =: D's + D''s$$

**性质 3.4.8.**  $E$ : Hermitian vector bundle,  $D$  is a Hermitian connection, locally, take a  $C^\infty$ -frame  $e_1, \dots, e_r$  which is orthonormal (i.e.  $\{e_i(x), e_j(x)\} = \delta_{ij}$ ), then the connection coefficient  $A = A' + A''$  satisfies

$$(A')^* = -A''$$

$$(\iff \bar{i}A = iA)$$

证明. because

$$0 = de_i, e_j = \{De_i, e_j\} + \{e_i, De_j\} = \{a_i^k e_k, e_j\} + \{e_i, a_j^l e_l\} = a_i^j + \bar{a}_j^i$$

so,  $A^* = -A$ . □

**推论 3.4.9.**  $E \rightarrow X$  is a Hermitian vector bundle,  $D_0''$  is a connection of type  $(0,1)$  on  $E$ . Then exists a unique Hermitian connection  $D$  such that  $D'' = D_0''$ .

证明. Let  $A'' = A_0''$  and  $A' = -(A_0'')^* \rightsquigarrow A = A' + A''$ , and  $D$  is given by  $A$ . □

Let  $E \rightarrow X$  is a holomorphic Hermitian vector bundle, observe that  $\bar{\partial}$  defines a connection of type  $(0,1)$  on  $E$ (check!)

assume  $E$  is a holomorphic line bundle, take a section  $s \in C_p^\infty(X, E)$ , i.e. we have a family of  $p$ -forms  $(s_\alpha)$  such that  $s_\alpha = g_{\alpha\beta} s_\beta$  where  $g_{\alpha,\beta}$  is the holomorphic transition matrix.

$$\bar{\partial}s \xrightarrow{\sim} \bar{\partial}s_\beta$$

then

$$\bar{\partial}s_\alpha = g_{\alpha,\beta} \bar{\partial}s_\beta$$

(so,  $\bar{\partial}$  is a connection of  $(0,1)$ )

this connection is called the canonical connection of type  $(0,1)$ .

**定义 3.4.10.** Let  $E \rightarrow X$  holomorphic Hermitian vector bundle, the connection  $D$  on  $E$  is called Chern connection if

$$D'' = \bar{\partial}$$

### Curvature of Chern connection

$E \rightarrow X$  is holomorphic Hermite vector bundle ,  $D$  is the Chern connection, Locally let  $\{e_1, \dots, e_r\}$  be a holomorphic frame, and two local sections

$$s, t \in C^\infty(\Omega, E)$$

where

$$s = \sum_{i=1}^r \sigma_i e_i$$

$$t = \sum_{i=1}^r t_i e_i$$

Since  $D$  is Hermitian ,

$$d\{s, t\} = d((\sigma_1, \dots, \sigma_r) H \begin{pmatrix} t_1 \\ \vdots \\ t_r \end{pmatrix}) = (d\sigma)^T H t + \sigma^T (dH) t + \sigma^T H d(t)$$

so, we have

$$\{Ds, t\} + \{s, Dt\} = (d\sigma + \bar{H}^{-1} \bar{\partial} \bar{H} \wedge \sigma)^T \wedge H \bar{t} + \sigma^T \wedge \overline{H(dt + \bar{H}^{-1} \bar{\partial} \bar{H} \wedge t)}$$

so ,

$$Ds = d\sigma + \bar{H}^{-1} \bar{\partial} \bar{H} \wedge \sigma$$

$$D's = \partial\sigma + \bar{H}^{-1}\partial\bar{H} \wedge \sigma = \bar{H}^{-1}\partial(\bar{H}\sigma)$$

$$D''s = \bar{\partial}\sigma$$

so,

$$(D')^2s = \bar{H}^{-1}\partial(\bar{H}(\bar{H}^{-1}\partial(\bar{H}\sigma))) = \dots = 0$$

$$(D'')^2s = \dots = 0$$

So we have

$$\Theta(D) = (D' + D'')^2 = D'D'' + D''D'$$

Locally ,

$$\begin{aligned}\Theta s &= D'D''s + D''D's = \bar{H}^{-1}\partial(\bar{H}\bar{\partial}\sigma) + \bar{\partial}(\bar{H}^{-1}\partial(\bar{H}\sigma)) = \dots = \bar{H}^{-1}\partial\bar{H} \wedge \bar{\partial}\sigma + \bar{\partial}(\bar{H}^{-1})\sigma \\ &= \bar{\partial}(\bar{H}^{-1}\partial\bar{H})\sigma\end{aligned}$$

So, Chern curvature

$$\Theta_D = \bar{\partial}(\bar{H}^{-1}\partial\bar{H})$$

Last time:  $E \rightarrow X$  is a holomorphic vector bundle with a Hermitian metric  $H$ . Then there is a unique connection  $D_E$  s.t. ... called Chern connection.

Curvature of Chern Connection:

$$\Theta(D_E) = \bar{\partial}(\bar{H}^{-1}\partial\bar{H})$$

so,

$$i\Theta(D_E) \in C_{1,1}^\infty(X, \text{Hom}(E, E))$$

**例子 3.4.11.** (Special case:  $E$  is a holomorphic line bundle)

locally, let  $e$  be a holomorphic frame,  $\langle e, e \rangle = h$  is the metric. then,

$$\Theta = \bar{\partial}(h^{-1}\partial h) = \bar{\partial}\partial \log h$$

so,

$$i\Theta(E) = -i\bar{\partial}\partial \log h$$

if  $h = e^{-2\varphi}$  where  $\varphi$  is a smooth function, then

$$i\Theta(E) = 2i\bar{\partial}\partial\varphi = 2\sqrt{-1} \sum_{k,l} \frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_l} dz_k \wedge d\bar{z}_l$$

**Question:** let  $s$  be a local holomorphic section of  $E$ ,

$$-i\bar{\partial}\partial \log |s|_h^2 = ?$$

(Hint:  $\frac{i}{\pi} \bar{\partial}\partial \log z = ?$  单复变, 按分布意义下求导. 等于狄拉克测度 2333333) 可能是期末题目?

**例子 3.4.12.**  $\mathcal{O}(-1)$  on  $\mathbb{C}P^n$ , tautological line bundle. (Recall:  $\mathbb{C}P^n$  is a compact complex manifold with holomorphic charts

$$\Omega_j := \{[z_0; z_1; \dots; z_n] | z_j \neq 0\} \rightarrow \left( \frac{z_0}{z_j}, \dots, \hat{1}, \dots, \frac{z_n}{z_j} \right) \in \mathbb{C}^n$$

)

Let  $V$  be a complex vector space,  $\dim_{\mathbb{C}} V = n + 1$ . Denote the projective space by

$$\mathbb{P}(V) = (V \setminus \{0\}) / \mathbb{C}^*$$

Let  $\underline{V} := \mathbb{P}(V) \times V$  be the trivial vector bundle, define

$$\mathcal{O}(-1) := \{([x], \xi) | \xi \in \mathbb{C} \cdot x\}$$

**性质 3.4.13.**  $\mathcal{O}(-1)$  is a holomorphic line bundle on  $\mathbb{P}(V)$ .

证明.  $\mathcal{O}(-1)|_{\Omega_j}$  has a non-vanishing holomorphic section  $\mathcal{E}_j$  defined by

$$\mathcal{E}_j([x]) = \frac{x}{x_j}$$

for  $0 \leq j \leq n$ . □

Assume  $V$  has a Hermitian inner product, then  $\mathcal{O}(-1)$  has an Hermitian structure induced from  $V$ .

Let  $e_0, \dots, e_n$  be an orthonormal basis of  $V$ , then  $\mathcal{O}(-1)|_{\Omega_0}$  has a non-vanishing holomorphic section:

$$\mathcal{E}_0(z_1, \dots, z_n) = e_0 + z_1 e_1 + \dots + z_n e_n$$

where

$$\Omega_0 = \{[1; z_1; \dots; z_n] | z_j \in \mathbb{C}\} \cong \mathbb{C}^n$$

then,

$$|\mathcal{E}_0|_h^2 = 1 + |z_1|^2 + \dots + |z_n|^2$$

so the Chern curvature of  $\mathcal{O}(-1)$  on  $\Omega_0$  is given by

$$\Theta = \bar{\partial} \partial \log(1 + |z_1|^2 + \dots + |z_n|^2)$$

Denote  $\mathcal{O}(1) := \mathcal{O}(-1)^*$ , then

$$\Theta(\mathcal{O}(1)) = -\bar{\partial} \partial \log(1 + |z_1|^2 + \dots + |z_n|^2)$$

on  $\Omega_0$ .

$$i\Theta(\mathcal{O}(1)) = i\partial\bar{\partial}\log(1 + |z_0|^2 + \dots + |z_n|^2) = \sqrt{-1} \sum_{1 \leq k, l \leq n} c_{k,l} dz_k \wedge d\bar{z}_l$$

Exercise:  $(c_{kl})$  is a positive definite Hermitian matrix.

"Fubini-Study metric" on  $\mathbb{P}(V)$ .  $\mathcal{O}(1)$  is "hyperplane line bundle of  $\mathbb{P}(V)$ ".

Exercise: calculate

$$\int_{\mathbb{P}(V)} \left( \frac{i}{2\pi} \Theta(\mathcal{O}(1)) \right)^{\wedge n} = ?$$

(Hint:  $\mathbb{P}(V) \setminus \Omega_0$  is a zero-measure set)

$E \rightarrow X$  : holomorphic line bundle,  $D_E$  is a Chern connection.

$$c_1(E) = \left\{ \frac{i}{2\pi} \Theta(D_E) \right\} \in H_{DR}^2(X, \mathbb{R})$$

Exercise: 60% 的概率出现于期末试题

Consider the sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{e^{2\pi i *}} \mathcal{O}^* \rightarrow 0$$

it induces a long exact sequence

$$\dots \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}) \rightarrow \dots$$

prove: Consider  $E$  as an element of  $H^1(X, \mathcal{O}^*)$ , then the image of  $\delta(E)$  in  $H^2(X, \mathbb{R}) \cong H_{DR}^2(X, \mathbb{R})$  is  $c_1(E)$ .

Exercise:  $E$  is a holomorphic line bundle, denote  $\theta := \frac{i}{2\pi} \Theta(D_E)$  real  $(1,1)$ -form, where  $D_E$  is Chern connection with a metric  $h$ . Prove: for any smooth function  $f \in C^\infty(X, \mathbb{R})$ , there exists a Hermitian metric  $h_f$  s.t.

$$\frac{i}{2\pi} \Theta_{E, h_f} = \theta + i\partial\bar{\partial}f$$

## 第 4 章 $L^2$ Hodge 理论

### 4.1 向量丛上的微分算子

Differential operators on vector bundles.

Let  $X$  is a (connected) smooth manifold of  $(\mathbb{R})$ -dimension  $n$ .  $E, F : \mathbb{K}$ -vector bundle of rank  $r, r'$  respectively.

**定义 4.1.1.** a linear differential operator of degree  $k$  from  $E$  to  $F$  is a  $\mathbb{K}$ -linear map

$$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

$$u \mapsto Pu$$

locally given by

$$Pu(x) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x)$$

where  $a_\alpha(x) = (a_{af_a, \lambda \mu}(x))$  be a  $r' \times r$  matrix.

$$u(x) = (u_1(x), \dots, u_r(x))^T$$

Let  $t \in \mathbb{K}$ ,  $f \in C^\infty(M, \mathbb{K})$ ,  $u \in C^\infty(M, E)$ , then

$$e^{-tf(x)} P(e^{tf(x)} u(x)) = t^k \sigma_P(x, df(x)) u(x) + \text{terms } c_j(x) t^j \quad (j < k)$$

**定义 4.1.2.**

$$\sigma_P : T^*M \rightarrow \text{Hom}(E, F)$$

is called the principal symbol of  $P$ , which is a polynomial on  $T^*M$ .

locally,

$$\sigma_P(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$$

$$(\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n})$$

例子 4.1.3. Consider  $d : C^\infty(M, \mathbb{K}) \rightarrow C^\infty(M, T^*M)$ . then

$$du = \sum_{j=1}^n \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \frac{\partial u}{\partial x^j}$$

i.e.

$$\sigma_d(x, \xi) = \sum_{j=1}^n \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \xi_j$$

定义 4.1.4.  $P$  is called elliptic, if  $\forall x \in M, \xi \in T_x^*M \setminus \{0\}$ ,

$$\sigma_P(x, \xi) \in \text{Hom}(E_x, E_x)$$

is injective.

For example,  $d$  is elliptic.

**$L^2$ -inner product**

Let  $M$  be an oriented  $C^\infty$ -manifold with a smooth volume form, locally

$$dV(x) = \gamma(x) dx_1 \wedge \dots \wedge dx_n$$

$\gamma(x) > 0$ . Assume  $E$  has a Euclidean(or Hermitian) structure...

Let  $u, v \in C^\infty(M, E)$ , define

$$\langle\langle u, v \rangle\rangle := \int_M \langle u, v \rangle dV(x)$$

define  $L^2(M, E) :=$  space of sections with measurable coefficients with are  $L^2$  w.r.t  $\langle\langle \cdot, \cdot \rangle\rangle$ .

**定义 4.1.5.** Let  $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$  be a differential operator,  $E, F$  have Euclidean (or Hermitian) structure, then there exists unique differential operator

$$P^* : C^\infty(M, F) \rightarrow C^\infty(M, E)$$

s.t.

$$\langle \langle Pu, v \rangle \rangle = \langle \langle u, P^*v \rangle \rangle$$

for all  $u, v$  s.t.  $\text{Supp}u \cap \text{Supp}v \subset\subset M$  (relative compact...)

$P^*$  is called the formal adjoint of  $P$ .

证明. Existence: Assume that  $\text{Supp}u, \text{Supp}v \subset\subset$  some coordinate chart  $\Omega$  with coordinates  $(x_1, \dots, x_n)$ , then

$$\langle \langle Pv, u \rangle \rangle = \int_{\Omega} \sum_{\alpha, \lambda, \mu} a_{\alpha, \lambda, \mu}(x) D^\alpha u_\mu(x) \overline{v_\lambda(x)} \gamma(x) dx_1 \cdots dx_n$$

integration by parts, it

$$= \int_{\Omega} \sum_{\alpha, \lambda, \mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha (\gamma(x) \overline{a_{\alpha, \lambda, \mu}} v_\lambda(x))} dx_1 \cdots dx_n$$

Locally,

$$P^*v = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha (\gamma(x) \overline{a_\alpha(x)})^T v(x)$$

Uniqueness: use the density of  $C^\infty$ -section with compact support in  $L^2(M, -)$ .  $\square$

**推论 4.1.6.** If  $\sigma_P(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$ , then

$$\sigma_{P^*} = (-1)^k \overline{\sigma_P(x, \xi)}^T$$

**推论 4.1.7.** If  $\text{rank } E = \text{rank } F$ ,  $P$  is differential operator, then  $P^*$  is elliptic  $\iff P$  is elliptic.

## 4.2 椭圆算子的基本性质

### Fundamental results of elliptic operators



$M$  is a compact (oriented)  $C^\infty$ -manifold,  $\dim_{\mathbb{R}} M = n$ , with a smooth volume form  $dV$ .

$E$  is an Hermite vector bundle,  $\text{rank}_{\mathbb{C}} E = r$ .

Sobolev space:  $W^k(M, E) :=$  the space of section  $s : M \rightarrow E$  whose derivations up to order  $= k$ ,  
 $:=$  the completion of space of smooth sections w.r.t  $W^k$ -norm.

$(\Omega_j)_{j \in I}$ : a finite open covering of  $M$ ,  $E|_{\Omega_j}$  trivial, Let  $(\rho_j)_{j \in I}$  be a partition of unity w.r.t.  
 $(\Omega_j)_{j \in I}$ , s.t.  $\sum_j \rho_j^2 = 1$ . locally, choose an orthonormal frame  $(e_{j,\lambda})_{1 \leq \lambda \leq r}$  on  $\Omega_j$ , then  $u = \sum_{\lambda=1}^r u_{j,\lambda} e_{j,\lambda}$  on  $\Omega_j$ . Define

$$\|u\|_k^2 := \sum_{j,\lambda} \|e_j u_{j,\lambda}\|_k^2$$

where

$$\|e_j u_{j,\lambda}\|_k^2 := \int_{\Omega_j} \sum_{|\alpha| \leq k} |D^\alpha(e_j u_{j,\lambda})|^2 dV(x)$$

**注记 4.2.1.** *On a compact manifold, the equivalence of class of  $\|\cdot\|_k$  is independent of the choice of : partition of unity, local trivialization, holomorphic covering...*

**引理 4.2.2.** *(Sobolev lemma)*

*For  $k > l + \frac{n}{2}$ , then we have*

$$W^k(M, E) \subseteq C^l(M, E)$$

**引理 4.2.3.** *(Rellich lemma)*

*For any  $k \in \mathbb{Z}_{\geq 0}$ , the inclusion*

$$W^{k+1}(M, E) \hookrightarrow W^k(M, E)$$

*is a compact operator.*

**引理 4.2.4.** *(Garding inequality)*

*If*

$$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

*is elliptic, and  $\text{rank} E = \text{rank} F$ ,  $\tilde{P}$ : the extension of  $P$  to sections with distribution coefficients, then  
: for all  $u \in W^0(M, E)$ , if  $\tilde{P}u \in W^k(M, F)$ , then  $u \in W^{k+d}(M, E)$ , where  $d = \deg P$ , and*

$$\|u\|_{k+d} \leq C_k (\|\tilde{P}u\|_k + \|u\|_0)$$

*where  $C_k$  depending on  $k, M$ .*

证明. Reference: Kodaira: deformation of complex structures (Appendix) □

**推论 4.2.5.** *If  $u \in \ker \tilde{P} \cap W^0(M, E)$ , then  $u \in C^\infty(M, E)$ .*

**引理 4.2.6.** *(Finiteness theorem)*

*Setting  $M$  be a compact manifold,  $\text{rank} E = \text{rank} F$ ,*

$$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

*elliptic, then:*

*(1)  $\ker P$  is of finite dimension*

*(2)  $P(C^\infty(M, E))$  is closed and of finite codimension in  $C^\infty(M, F)$ . If  $P^*$  is the formal adjoint of  $P$ , then  $\exists$  decomposition*

$$C^\infty(M, F) = P(C^\infty(M, E)) \oplus \ker P^*$$

*which is orthogonal in  $W^0(M, F) = L^2(M, F)$*

证明. 椭圆算子的一般结果, 分析的东西 233333333. 可以参考小平邦彦复流形与复结构形变的附录. □

## 4.3 紧黎曼流形的 Hodge 理论

### Hodge theory in compact Riemannian manifold

Hodge star operator.

$M$  compact Riemannian manifold,  $\dim_{\mathbb{R}} = n$ ,  $E$  is a Hermitian vector bundle. Assume  $(\xi_1, \dots, \xi_n), (e_1, \dots, e_n)$  be orthonormal frame of  $TM, E$  on some local chart  $\Omega$ , denote  $(\tilde{\xi}_1^*, \dots, \tilde{\xi}_n^*), (e_1^*, \dots, e_n^*)$  be the co-frame of  $T^*M, T^*E$ .

$\wedge^\bullet T^*M$  is endowed with an inner product frame from  $TM$ . locally,

$$\langle u_1 \wedge \dots \wedge u_p, u_1 \wedge \dots \wedge u_p \rangle := \det(\langle u_i, v_j \rangle)$$

for  $u_i, v_j \in T^*M$ . Then , get an inner product on  $\wedge^p T^*M$ .

Assume

$$U = \sum_{\substack{|I|=p \\ i_1 \leq \dots \leq i_p}} u_I \tilde{\xi}_I^*$$

$$V = \sum_{\substack{|I|=p \\ i_1 \leq \dots \leq i_p}} v_I \tilde{\zeta}_I^*$$

be  $p$ -forms, then

$$\langle u, v \rangle = \sum_{|I|=p} u_I v_I$$

i.e.  $\{\tilde{\zeta}_I^*\}$  is an orthonormal basis of  $\wedge^p T^*M$ .

$\wedge^* T^*M \otimes E$  has an inner product induced from  $\wedge^* T^*M, E$ ,

**定义 4.3.1.** *the Hodge star operator*

$$* : \wedge^p T^*M \rightarrow \wedge^{n-p} T^*M$$

is defined by

$$u \wedge *v = \langle u, v \rangle dV$$

Locally, let

$$U = \sum_{|I|=p} u_I \tilde{\zeta}_I^*, \quad V = \sum_{|I|=p} v_I \tilde{\zeta}_I^*$$

assume

$$*V = \sum_{|J|=n-p} a_J \tilde{\zeta}_J^*$$

then

$$\begin{aligned} U \wedge * \sum u_I a_{I^c} \tilde{\zeta}_I^* \wedge \tilde{\zeta}_{I^c}^* &= \sum u_I a_{I^c} \varepsilon(I, I^c) \tilde{\zeta}_1^* \wedge \dots \wedge \tilde{\zeta}_n^* \\ \langle u, v \rangle dV &= \sum_{|I|=p} u_I v_I \tilde{\zeta}_1^* \wedge \dots \wedge \tilde{\zeta}_n^* \end{aligned}$$

so, we have

$$*V = \sum_{|I|=p} \varepsilon(I, I^c) V_I \tilde{\zeta}_{I^c}^* \in \wedge^{n-p} T^*M$$

**定义 4.3.2.**

$$* : \wedge^p T^*M \otimes E \rightarrow \wedge^{n-p} T^*M \otimes E$$

is defined by

$$\{s, *t\} := \langle s, t \rangle dV$$

Locally, assume

$$t = \sum_{\substack{|I|=p \\ 1 \leq \lambda \leq r}} t_{I,\lambda} \tilde{\zeta}_I^* \otimes e_\lambda$$

then

$$*t = \sum_{\substack{|I|=p \\ 1 \leq \lambda \leq r}} \varepsilon(I, I^c) t_{I,\lambda} \tilde{\zeta}_{I^c}^* \otimes e_\lambda$$

**定义 4.3.3.**

$$\# : \bigwedge^p T^*M \otimes E \rightarrow \bigwedge^{n-p} T^*M \otimes E^*$$

is defined by: for any  $s, t \in \bigwedge^p T^*M \otimes E$ , such that

$$s \wedge \#t := \langle s, t \rangle dV$$

wedge product + pairing of  $E^* \times E \rightarrow \mathbb{C}$ .

Locally: assume

$$t = \sum_{\substack{|I|=p \\ 1 \leq \lambda \leq r}} t_{I,\lambda} \tilde{\zeta}_T^* \otimes e_\lambda$$

then,

$$\#t = \sum_{|I|=p, \lambda} \varepsilon(I, I^c) t_{I,\lambda} \tilde{\zeta}_c^* I \otimes e_\lambda^*$$

**性质 4.3.4.**

$$*^2 = (-1)^{p(n-1)} \quad \text{on } \bigwedge^p T^*M \otimes E$$

$$\#^2 = (-1)^{p(n-1)} \quad \text{on } \bigwedge^p T^*M \otimes E$$

(正负号对吗?)

Recall: For all  $s, t \in C^\infty(M, \bigwedge^p T^*M \otimes E)$ , we have an inner product

$$\langle \langle s, t \rangle \rangle := \int_M \langle s, t \rangle dV$$

**定理 4.3.5.** Let  $D_E$  be an Hermite connection on  $E$ , acting on  $\bigwedge^p T^*M \otimes E$ , then

$$D_E^* := (-1)^{np+1} * D_E *$$

where  $D_E^*$  is the formal adjoint of  $D_E$ .

证明. Let  $s \in C^\infty(M, \bigwedge^p T^*M \otimes E)$  and  $t \in C^\infty(M, \bigwedge^{p+1} T^*M \otimes E)$ . then

$$\langle \langle D_E s, t \rangle \rangle = \int_M \langle D_E s, t \rangle dV = \int_M \{D_E s, *t\}$$

Since  $D_E$  is Hermitian ,by definetion ,

$$d\{s, *t\} = \{D_E s, t\} + (-1)^p \{s, D_E(*t)\}$$

so,

$$\langle \langle D_E s, t \rangle \rangle = \int_M d\{s, *t\} + (-1)^{p+1} \{s, D_E *t\} = (-1)^{p+1} (-1)^{p(n_1)} \int_M \{s, *(D_E *t)\} = \langle \langle s, D_E^* t \rangle \rangle$$

so,

$$D_E^* t = (-1)^{np+1} * D_E *$$

□

**定义 4.3.6.**

$$\Delta_E = D_E D_E^* + D_E^* D_E : C^\infty(M, \bigwedge^p T^*M \otimes E) \rightarrow C^\infty(M, \bigwedge^p T^*M \otimes E)$$

**例子 4.3.7.** Let  $M = \mathbb{R}^n$ ,  $g = \sum_{i=1}^n dx_i^2$ ,  $E = M \times \mathbb{C}$  trivial line bundle with  $D_E = d$ . then

$$\Delta_E u = (dd^* + d^*d)u = - \sum_{i=1}^n \left( \sum_{|I|=p} \frac{\partial^2 u_I}{\partial x_I^2} dx_I \right)$$

where

$$u = \sum_{|I|=p} u_I dx_I$$

**性质 4.3.8.**  $\Delta_E$  is a self-adjoint elliptic operator. (i.e.  $\Delta_E^* = \Delta_E$ )

证明.  $\Delta_E^* = \Delta_E$  be definition.

note that

$$e^{-tf} D_E (e^{tf} s) = t df \wedge s + D_E s$$

so,

$$\sigma_{D_E}(x, \xi) s = \xi \wedge s$$

$$\sum_{D_E^*} = -\overline{\sigma_{D_E}}^T$$

$$\sigma_{D_E^*}(x, \xi)s = -\tilde{\xi} \lrcorner s$$

where  $\tilde{\xi}$  be the vector field dual to  $\xi$ . □

**定义 4.3.9.**

$$\Delta_E = D_E D_E^* + D_E^* D_E : C^\infty(M, \bigwedge^p T^* M \otimes E) \rightarrow C^\infty(M, \bigwedge^p T^* M \otimes E)$$

so,

$$\sigma_{\Delta_E}(x, \xi)s = \left( \sigma_{D_E} \sigma_{D_E^*}(x, \xi) + \sigma_{D_E^*} \sigma_{D_E}(x, \xi) \right) s$$

so,  $\sigma_{\Delta_E}$  is injective if  $\xi \neq 0$ , so  $\Delta_E$  is elliptic.

Harmonic forms and Hodge isomorphism.

**定义 4.3.10.**  $u$  is called harmonic if  $\Delta_d u = 0$ .

**定理 4.3.11.**  $M$  is a compact Riemannian manifold, then de Rham cohomology

$$H_{DR}^p(M, \mathbb{R}) \cong \ker(\Delta_d : C^\infty(M, \bigwedge^p T^* M))$$

证明.  $\Delta_d$  self-adjoint elliptic, so by general result for elliptic operator,

$$C^\infty(M, \bigwedge^p T^* M) = \text{Im } \Delta_d \oplus \ker \Delta_d^* = \text{Im } \Delta_d \oplus \ker \Delta_d$$

Claim:

$$\text{Im } \Delta_d = \text{Im } d \oplus \text{Im } d^*$$

Recall  $\Delta_d = dd^* + d^*d$ , so

$$\text{Im } \Delta_d \subseteq \text{Im } d \oplus \text{Im } d^*$$

on the other hand,

$$\text{Im } d \oplus \text{Im } d^* \subseteq (\ker \Delta_d)^\perp = \text{Im } \Delta_d$$

so,

$$\text{Im } \Delta_d = \text{Im } d \oplus \text{Im } d^*$$

so,

$$C^\infty(M, \bigwedge^p T^*M) = \text{Im } d \oplus \text{Im } d^* \oplus \ker \Delta_d$$

so,

$$H_{DR}^p(M, \mathbb{R}) = \frac{\text{Im } d \oplus \ker \Delta_d}{\text{Im } d} = \ker \Delta_d$$

□

**推论 4.3.12.**

$$\dim H_{DR}^p(M, \mathbb{R}) = \dim \ker \Delta_d < +\infty$$

**注记 4.3.13.** *Consider*

$$u \mapsto \int_M (\langle u, u \rangle + \langle du, du \rangle + \langle d^*u, d^*u \rangle) dV$$

这个泛函的变分是什么鬼?

Harmonic forms and Hodge isomorphism

Recall:  $M$  is a compact Riemann manifold,

$$d : C^\infty(M, \bigwedge^* T^*M) \rightarrow C^\infty(M, \bigwedge^{*+1} T^*M)$$

adjoint  $d^*$ ,

$$\Delta_d = dd^* + d^*d$$

is a self-adjoint elliptic operator.

Hodge decomposition:

$$C^\infty(M, \bigwedge^p T^*M) = \ker \Delta_d \oplus \text{Im } d \oplus \text{Im } d^*$$

$$\mathcal{H}^p(M, \mathbb{R}) := \ker \Delta_d \quad \text{finite dimension}$$

$$\mathcal{H}^p(M, \mathbb{R}) \cong H_{DR}^p \cong H^p(M, \mathbb{R})$$

(Hodge isomorphism, and, de Rham-Weil)

**Poincare duality**

**定理 4.3.14.** *The pairing*

$$H_{DR}^p(M, \mathbb{R}) \times H_{DR}^{n-p}(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$(s, t) \mapsto \int_M s \wedge t$$

(is well defined) is non-degenerated. In particular,  $H_{DR}^p(M, \mathbb{R})^* \cong H_{DR}^{n-p}(M, \mathbb{R})$

证明. the pairing factors through the pairing on

$$\mathcal{H}^p(M, \mathbb{R}) \times \mathcal{H}^{n-p}(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$(s, t) \mapsto \int_M s \wedge t$$

need to verify: (1) it is independent of the choice of representations. (Easy, check) (2) Pairing  $\mathcal{H}^p \times \mathcal{H}^{n-p}$  is non-degenerated..

claim(Exercise): Hodge star  $*$  s.t.  $*\Delta_d = \Delta_d*$ .

so,  $s$  is a harmonic  $p$ -form  $\iff *s$  is a harmonic  $(n-p)$ -form.

note that

$$s \wedge *s = \langle s, s \rangle dV = \int_M s \wedge *s = \int_M \langle s, s \rangle dV = \|s\|^2$$

□

**推论 4.3.15.**

$$\dim \mathcal{H}^p(M, \mathbb{R}) = \dim \mathcal{H}^{n-p}(M, \mathbb{R})$$

Generalization to flat bundle.  $M$  is a compact Riemannian manifold,  $\dim_{\mathbb{R}} M = n$ ,  $E \rightarrow M$  is a complex Hermitian vector bundle.

**定义 4.3.16.**  $E \rightarrow X$  is called flat, if it admit a connection  $D_E$  s.t.

$$D_E^2 = 0$$

**注记 4.3.17.**  $E$  is flat  $\iff E$  is given by a representation

$$\pi_1(M) \rightarrow GL(r, \mathbb{C})$$

(我们不证)

Consider the complex :

$$(C^\infty(M, \bigwedge^* T^*M \otimes E), D_E) \\ \rightsquigarrow H_{DR}^p(M, E) := \frac{\ker D_E}{\text{Im } D_E}$$

Exercise: we have decomposition

$$C^\infty(M, \bigwedge^p T^*M \otimes E) = \ker \Delta_{D_E} \oplus \text{Im } D_E \oplus \text{Im } D_E^*$$

$$H_{DR}^p(M, E) \cong \ker \Delta_{D_E}$$



and the pairing

$$H_{DR}^p(M, E) \times H_{DR}^{n-p}(M, E^*) \rightarrow \mathbb{C}$$

$$(s, t) \mapsto \int_M s \wedge t$$

is non-degenerate..

以上是实的 Hodge 理论。

## 4.4 Kähler 流形

**定义 4.4.1.** Let  $X$  be a complex manifold,  $\dim_{\mathbb{C}} X = n$ ,  $X$  is called a Hermitian manifold, if  $X$  has a Hermitian metric, i.e. locally  $h(z) := \sum_{1 \leq j, k \leq n} h_{jk}(z) dz_j \otimes d\bar{z}_k$ , where  $(h_{jk})$  is positive definition Hermitian matrix.

Check: the positivity of  $h$  is independent of the choice of holomorphic local coordinate

Rmk: Any complex manifold has a Hermitian metric... (Exercise)

Fundamental  $(1, 1)$ -form associated to  $h(z)$  is defined by

$$\omega := -\operatorname{Im} h = \frac{\sqrt{-1}}{2} \sum_{j, k} h_{jk} dz_j d\bar{z}_k$$

we also call  $\omega$  is the Hermitian metric on  $X$

Fact:  $\omega$  is real (i.e.  $\bar{\omega} = \omega$ ).

**注记 4.4.2.**  $h$  is a Hermite structure on  $TX$  (holomorphic tangent bundle of  $X$ ). locally,

$$\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle(z) = h_{ij}(z)$$

**定义 4.4.3.**  $(X, \omega)$  is an Hermitian manifold,  $X$  is Kähler if  $d\omega = 0$ .

**性质 4.4.4.** Locally,  $\omega = \frac{\sqrt{-1}}{2} \sum_{jk} h_{jk} dz_j \wedge d\bar{z}_k$  is Kaehler,  $\iff \partial\omega = 0$  and  $\bar{\partial}\omega = 0$ , i.e.

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}$$

If  $(X, \omega)$  is a compact Kaehler manifold, then

$$H^{2k}(X, \mathbb{R}) \neq 0$$

证明.  $d\omega = 0$ , so  $\omega \in H^2(M, \mathbb{R})$ . Claim:

$$0 \neq \omega^k \in H^{2k}(M, \mathbb{R})$$

proof of the claim:

$$[\omega^k][\omega^{n-k}] = \int_X \omega^k \wedge \omega^{n-k} = \int_X \omega^n$$

Since  $\omega$  is positive, locally

$$\omega^n = n! \det(h_{j\bar{k}}) \bigwedge_{l=1}^n \left( \frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l \right) > 0$$

is a volume form. So,

$$[\omega^k][\omega^{n-k}] = \int_X \omega^n > 0$$

(Using Poincare dual)

□

例子 4.4.5. (*Exists a complex manifold NOT Kaehler*) (*Hopf Surface*)

$$X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

where discrete group  $\Gamma := \{\lambda^n | n \in \mathbb{Z}\}$ ,  $0 < \lambda < 1$  fixed.

Exercise:  $X \cong S^1 \times S^3$   $C^\infty$  homeomorphism.. and  $X$  is compact complex manifold.  
and  $H^2(X, \mathbb{R}) = H^2(S^1 \times S^3, \mathbb{R}) = 0$  by Künneth Formula...  
So,  $X$  is non-Kahler...

例子 4.4.6. (*Examples of Kaehler manifold*)

(1) *Riemann surface must be Kaehler...(trivial)*

(2) (*complex torus*)  $X = \mathbb{C}^n/\Gamma$ ,  $\Gamma$  is a lattice. (*this manifold may not compact...*)

$$\omega = \sqrt{-1} \sum_{j,k} h_{j\bar{k}} dz_j \wedge d\bar{z}_k$$

is a Kahler metric on  $X$  if  $(H_{j\bar{k}}) > 0$ ,  $h_{j\bar{k}}$  are constant.

(3) *Projective space  $\mathbb{CP}^n$ .*

$$\omega := \sqrt{-1} \Theta_h(\mathcal{O}(1))$$

locally,

$$\omega = \sqrt{-1} \partial \bar{\partial} \log(1 + |z_1|^2 + \dots + |z_n|^2)$$

on  $\Omega$ . This  $\omega$  is a Kahler metric,

例子 4.4.7. Let  $(X, \omega)$  is a Kahler manifold, then any complex submanifold  $Y \subseteq X$  is also Kahler.

$$i : Y \hookrightarrow X$$

with the Kahler metric  $i^*\omega$ .

Exercise: Let  $f : Y \rightarrow X$  be a holomorphic immersion, and assume  $X$  is Kahler, then  $Y$  is Kahler.

推论 4.4.8. Any projective manifold (i.e.  $X \hookrightarrow \mathbb{CP}^N$ ) is Kähler.

(Algebraic Geometry.....)

性质 4.4.9. (Equivalent definition of Kaehler metrics) a Hermitian metric  $\omega$  is Kahler, iff for all  $x_0 \in X$ , there exists a holomorphic chart  $(z_1, \dots, z_n)$  centered at  $x_0$ , s.t.

$$\omega(z) = \sqrt{-1} \sigma_{jk} \delta_{jk} dz_j \wedge d\bar{z}_k + O(|z|^2)$$

( $\Leftarrow$  is trivial...) (left to HW)

定理 4.4.10. (Exercise)

If  $(X, \omega)$  is Kahler, then for all  $x_0 \in X$ ,  $\exists$  holomorphic chart  $z_1, \dots, z_n$  centered at  $x_0$ , s.t. assume

$$\omega = \sqrt{-1} h_{jk} dz_j \wedge d\bar{z}_k$$

then

$$h_{lm}(z) = \delta_{lm} - \sum_{j,k} c_{jk,lm} z_j \bar{z}_k + O(|z|^3)$$

where  $c_{jk,lm}$  is the coefficients of the Chern curvature tensor,

$$\Theta(TX)_x := \sum c_{jk,lm} dz_j \wedge d\bar{z}_k \otimes \left(\frac{\partial}{\partial z_l}\right)^* \otimes \frac{\partial}{\partial z_m}$$

(查书)

## 4.5 紧复流形上的 Hodge 理论

$(X, \omega)$  is a compact Hermitian manifold,  $E \rightarrow X$  is a homomorphic Hermitian vector bundle.

$$D_E := D'_E + D''_E$$

Chern connection,  $D''_E = \bar{\partial}$ .

定义 4.5.1.

$$\Delta_E := D_E D_E^* + D_E^* D_E$$

$$(D'_E)^* = - * D''_E *$$

$$(D''_E)^* = - * D'_E *$$

$$\Delta'_E = D'_E (D'_E)^* + \dots$$

$$\Delta''_E = \dots$$

Note that  $(D''_E)^2 = 0$ , consider the complex

$$\begin{aligned} C^\infty(X, \bigwedge^{p,q} \otimes E) &\xrightarrow{D''_E} C^\infty(X, \bigwedge^{p,q+1} \otimes E) \\ &\rightsquigarrow H_{D''_E}^{p,q}(X, E) \end{aligned}$$

Dolbeaut cohomology... it isom to  $\ker \Delta''_E$

Hodge theory in compact complex manifold.

Let  $(X, \omega)$  be a compact complex manifold of dimension  $n$ .  $E \rightarrow X$  holomorphic Hermitian vector bundle, with Chern connection  $D_E$ ,  $D_E = D'_E + D''_E$  where  $D''_E = \bar{\partial}$ .

Recall:  $L^2$  inner product:  $u \in C^\infty(X, \bigwedge^{p,q} \otimes E)$ ,

$$\langle \langle u, v \rangle \rangle := \int_X \langle u, v \rangle \mathrm{dvol}$$

Hodge star operator  $*$ :  $u, v \in C^\infty(X, \bigwedge^{p,q} \otimes E)$ ,

定义 4.5.2.

$$*: \bigwedge^{p,q} \otimes E \rightarrow \bigwedge^{n-q, n-p} \otimes E$$

s.t.

$$u \wedge *v = \langle u, v \rangle \mathrm{dvol}$$

(wedge product from  $\bigwedge^{p,q}$ , with inner product from  $E$ )

Exercise: Take a holomorphic chart  $(z_1, \dots, z_n)$  s.t.

$$\omega = \sqrt{-1} \sum_j dz_j \wedge d\bar{z}_j$$

at some point  $p$ . An orthonormal frame  $\{e_1, \dots, e_r\}$ , Let

$$u = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{\lambda=1}^r u_{IJ} dz_I \wedge d\bar{z}_J \otimes e_\lambda \in \bigwedge^{p,q} \otimes E$$

WHAT IS  $*u$ ?

Formal adjoint of  $D_E, D'_E, D''_E$ ?

性质 4.5.3.

$$D_E^* = - * D_E *$$

$$(D'_E)^* = - * D''_E *$$

$$(D''_E)^* = - * D'_E *$$

定义 4.5.4.

$$\Delta_E := D_E D_E^* + D_E^* D_E$$

$$\Delta'_E := D'_E D_E'^* + D_E'^* D'_E$$

$$\Delta''_E := \dots$$

Check:  $\Delta_E, \Delta'_E, \Delta''_E$  are self adjoint, elliptic operators.

Hodge theory w.r.t.  $\Delta''_E$ .

定理 4.5.5. We have a decomposition

$$C^\infty(X, \bigwedge^{p,q} \otimes E) = \ker \Delta''_E \oplus \text{Im } D''_E \oplus \text{Im } D''_E^*$$

As a consequence, Dolbeault cohomology

$$H_{D''_E}^{p,q}(X, \mathbb{C}) \cong \ker \Delta''_E$$

推论 4.5.6.

$$\dim_{\mathbb{C}} H_{D_E''}^{p,q}(X, \mathbb{C}) < +\infty$$

Cohomology group

$$H_{D_E''}^{p,q}(X, \mathbb{C})$$

$\Omega^p$ : sheaf of holomorphic  $p$ -forms on  $X$  (i.e. a  $(p, 0)$ -form  $\varphi$  is holomorphic if  $\bar{\partial}\varphi = 0$ ).

$\mathcal{E}^{p,q}$ : Sheaf of smooth  $(p, q)$ -forms on  $X$ .

Similarly, we have  $\Omega^p(E)$  the sheaf of holomorphic  $p$ -forms with values in  $E$ , and  $\mathcal{E}^{p,q}(E)$  the sheaf...smooth  $(p, q)$ -forms ...

we have an acyclic resolutions

$$0 \rightarrow \Omega^p(E) \xrightarrow{D_E''} \mathcal{E}^{p,1}(E) \xrightarrow{D_E''} \mathcal{E}^{p,2}(E) \xrightarrow{D_E''} \dots$$

(check, it is a resolution)

By de Rham-Weil theorem,

$$H^q(X, \Omega^p(E)) \cong D_{D_E''}^{p,q}(X, \mathbb{C}) \cong \mathcal{H}_{D_E''}^{p,q}(X, \mathbb{C}) := \ker \Delta_E''$$

定理 4.5.7. (*Serre duality*)

*The pairing*

$$H_{D_E''}^{p,q}(X, E) \times H_{D_E''}^{n-p, n-q}(X, E^*) \rightarrow \mathbb{C}$$

$$(s, t) \mapsto \int_X s \wedge t$$

*is non-degenerate*

证明. Define

$$\# : \bigwedge^{p,q} \otimes E \rightarrow \bigwedge^{n-p, n-q} \otimes E^*$$

by: for  $u, v \in \bigwedge^{p,q} \otimes E$ ,

$$u \wedge \#v := \langle u, v \rangle \text{dvol}$$

Fact:

$$\Delta_{E^*}'' \# = \# \Delta_E''$$

□

Remark: take  $E = X \times \mathbb{C}, D_E = d = d' + d'', (d' = \partial, d'' = \bar{\partial})$  then we have

$$\Delta' = d'd'^* + d'^*d'$$

$$\Delta'' = \dots$$

then

$$H_{d''}^{p,q}(X, \mathbb{C}) \cong \ker \Delta'' \hookrightarrow C^\infty(X, \bigwedge^{p,q})$$

the pairing

$$H^{p,q}(X, \mathbb{C}) \times H^{n-p, n-q}(X, \mathbb{C}) \rightarrow \mathbb{C}$$

is non-degenerate.

## 4.6 紧 Kahler 流形的上同调群

Three goals:

**Kahler package**

**Lefschetz decomposition**

**Hodge-Riemann bilinear relations**

Linear algebra(baby representation theory)(local case)

$\mathbb{C}^n$ ,

$$\omega = \sqrt{-1} \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j$$

Kahler metric with constant coefficients.(i.e.  $h_{ij}$  is constant,  $(h_{ij})$  is positive Hermite matrix)

W.L.O.G, by taking a linear transformation, we can assume

$$\omega = \sqrt{-1} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$$

**记号 4.6.1.** An operator is of pure degree  $r$  if it transform a form of  $\deg = k$  to as form of degree  $k + r$ .

An operator ..of bi-degree  $(p, q)$  if  $...(s, t) \rightarrow (s + p, t + q)$  (in this case, degree =  $p + q$ )  
if  $A, B$  with degree  $\deg A, \deg B$ , define

$$[A, B] := AB - (-1)^{\deg A \deg B} BA$$

定义 4.6.2.

$$L : \bigwedge^{p,q} \rightarrow \bigwedge^{p+1,q+1}$$

$$u \mapsto \omega \wedge u$$

is called Lefschetz operator.

Denote  $\Lambda$  to be the adjoint of  $L$ , adjointed by : Let  $v \in \bigwedge^{p-1,q-1}$  and  $u \in \bigwedge^{p,q}$

$$\langle Lv, u \rangle := \langle u, \Lambda u \rangle$$

The operator  $\Lambda$  is of bi-degree  $(-1, -1)$ .

性质 4.6.3. If

$$u = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ} dz_I \wedge d\bar{z}_J$$

then

$$Lu = \sqrt{-1} \sum_{\substack{|I|=p \\ |J|=q}} \sum_{m=1}^n u_{IJ} dz_m \wedge d\bar{z}_m \wedge dz_I \wedge d\bar{z}_J$$

$$\Lambda u = \sqrt{-1}(-1)^p \sum_{\substack{|I|=p \\ |J|=q}} \sum_{m=1}^n u_{IJ} \left( \frac{\partial}{\partial z_m} \lrcorner dz_I \right) \wedge \left( \frac{\partial}{\partial \bar{z}_m} \lrcorner d\bar{z}_J \right)$$

where " $\lrcorner$ " is contraction.

推论 4.6.4. (Exercise) Let

$$\alpha = \sqrt{-1} \sum_{j=1}^n \alpha_j dz_j \wedge \bar{z}_j$$

then, ( $\alpha$  is a operator of bi-degree  $(1, 1)$ )

$$[\alpha, \Lambda]u = \sum_{\substack{|I|=p \\ |J|=q}} \left( \sum_{i \in I} \alpha_i + \sum_{j \in J} \alpha_j - \sum_{k=1}^n \alpha_k \right) u_{IJ} dz_I \wedge d\bar{z}_J$$

where

$$u = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ} dz_I \wedge d\bar{z}_J$$



**推论 4.6.5.** *if  $u \in \bigwedge^{p,q}$ , then*

$$[L, \Lambda]u = (p + q - n)u$$

**推论 4.6.6.** *Denote  $B := [L, \lambda]$ , then*

$$[B, L] = 2L$$

$$[B, \Lambda] = -2\Lambda$$

**证明.** Take  $u \in \bigwedge^{p,q}$ , then

$$[B, L]u = BLu - LBu = (p + q - n + 2)Lu - (p + q - n)Lu = 2Lu$$

the second is similar.. □

**$\mathfrak{sl}(2, \mathbb{C})$ -representation**

$$\mathfrak{sl}(2, \mathbb{C}) = \text{span}_{\mathbb{C}} l, \lambda, b$$

where

$$l = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have

$$[l, \lambda] = b \quad [b, l] = 2l \quad [b, \lambda] = -2\lambda$$

**性质 4.6.7.** *There exists a natural action*

$$\rho : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \text{End}\left(\bigoplus_{p,q} \bigwedge^{p,q}\right)$$

*with*

$$\rho(l) = L$$

$$\rho(\lambda) = \Lambda$$

$$\rho(b) = B$$

定理 4.6.8. (HL)

$$L^{n-k} : \bigwedge^k \rightarrow \bigwedge^{2n-k}$$

$$u \rightarrow \omega^{n-k} \wedge u$$

is an isomorphism.

$$L^{n-k} : \bigwedge^{p,q} \rightarrow \bigwedge^{n-k+p, n-k+q}$$

is also an isomorphism.

证明. Lemma:

$$[L^r, \Lambda]u = r(k - n + r - 1)L^{r-1}u$$

(induction, omit)

Assume  $\alpha \in \bigwedge_{\mathbb{C}}^k$ ,  $L^{n-k}\alpha = 0$ , need to verify  $\alpha = 0$ .

Claim:

$$L^r : \bigwedge^k \rightarrow \bigwedge^{k+2r}$$

is injective whenever  $r \leq n - k$ .

proof of the claim:

claim is true when  $k = 0$  or  $k = 1$ . (check)

Let  $\alpha \in \bigwedge^k$  s.t.  $L^r\alpha = 0$  ( $r \leq n - k$ ). By the lemma,

$$L^r\Lambda\alpha - \lambda L^r\alpha = r(k - n + r - 1)L^{r-1}\alpha$$

so,

$$L^{r-1}(L\Lambda\alpha - r(k - n + r - 1)\alpha) = 0$$

by the induction on  $r$ ,

$$L\Lambda\alpha = r(k - n + r - 1)\alpha$$

since  $r(k - n + r - 1) \neq 0$ ,  $\alpha = L\beta$  for some  $\beta \in \bigwedge^{k-2}$ . so,  $L^r\alpha = L^{r+1}\beta = 0$ , by induction on  $k$ , we have  $\beta = 0$ , so  $\alpha = 0$ .

The claim is proved. □

定义 4.6.9. (Primitive form)

$\alpha \in \bigwedge^k$  ( $k \leq n$ ) is called primitive form, if

$$L^{n-k+1}\alpha = 0$$

**推论 4.6.10.** (*Lefschitz Decomposition*)(LD)

For any  $\alpha \in \bigwedge^k$ , ( $1 \leq k \leq 2n$ ), we have a unique decomposition:

$$\alpha = \sum_{\gamma \geq (k-n)_+} L^\gamma \alpha_\gamma$$

( $(k-n)_+ := \max\{k-n, 0\}$ ) with  $\alpha_\gamma \in \bigwedge^{k-2r}$  is primitive

证明. Existence: assume  $k \leq n$ , consider

$$L^{n-k+1}\alpha \in \bigwedge^{2n-k+2}$$

by HL,  $\exists! \beta \in \bigwedge^{k-2}$  s.t.  $L^{n-k+2}\beta = L^{n-k+1}\alpha$ , so  $L^{n-k+1}(\alpha - L\beta) = 0$ , i.e.  $\alpha_0 = \alpha - L\beta$  is primitive.  $\alpha = \alpha_0 + L\beta$ , then induction on degrees, we get the decomposition for  $\alpha$ .

If  $k > n$ , we apply HL to reduce it to case 1.

Uniqueness: Next time..

□