复几何

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本课程参考以下教材：

1. Demailly: Complex analytic and differential geometry.
2. Huybrechts: Complex geometry: an introduction.
3. Morrow, Kodaira: Complex manifolds.
4. Grauert, Remmert: Coherent analytic sheaves.
5. Hormander: An introduction to complex analysis in several variables.
6. Griffiths, Harris: Principles of algebraic geometry.

在五道口也要红专并进、理实交融呀 ～

目录

第 1 章　多复变函数

1.1　多元全纯函数

首先快速回顾单复变函数的知识。我们通常用 Ω 来表示 $\mathbb{C}$ 的开子集，$z = x + iy$ 为 $\mathbb{C}$ 的坐标。对于 $z \in \mathbb{C}$ 以及实数 $R > 0$，我们令

$$\mathbb{D}(z, R) := \{w \in \mathbb{C} \mid |w - z| < R\}$$

为以 z 为圆心 R 为半径的开圆盘。

此外，我们有如下常用记号：

$$\begin{cases} \mathrm{d}z := \mathrm{d}x + i\mathrm{d}y \\ \mathrm{d}\bar{z} := \mathrm{d}x - i\mathrm{d}y \end{cases} \qquad \begin{cases} \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{cases}$$

对于函数 $f : \Omega \to \mathbb{C}$，称 f 是**全纯**（holomorphic）的，若在 Ω 中成立

$$\bar{\partial} f := \frac{\partial f}{\partial \bar{z}} \mathrm{d}\bar{z} = 0$$

我们知道，f 是全纯的当且仅当 f 在 Ω 处处能够局部地展开为收敛幂级数。

对于 $\mathbb{C}$ 中的紧致集 K，称函数 $f : K \to \mathbb{C}$ 是全纯的，如果存在 K 的开邻域 $\Omega \supseteq K$，使得 f 可延拓为 Ω 上的全纯函数。

单复变函数论中有如下重要结果：

定理 1.1.1. （柯西积分公式）设 $\mathbb{D} \subseteq \mathbb{C}$ 为 $\mathbb{C}$ 中的开圆盘，$f : \mathbb{D} \to \mathbb{C}$ 为 $\mathbb{D}$ 上的全纯函数，且在 $\partial\mathbb{D}$ 连续，则对于任意 $w \in \mathbb{D}$，成立

$$f(w) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(z)}{z - w} \mathrm{d}z$$

此定理能推导出单变量全纯函数理论的“almost everything”. 这里不再赘述。

我们开始考虑多变量全纯函数。

定义 1.1.2. 设 $\Omega \subseteq \mathbb{C}^n$ 为 $\mathbb{C}^n$ 的开子集，函数 $f: \Omega \to \mathbb{C}$ 称为（多变量）**全纯函数**，如果满足以下条件：

（*1*）f 是连续函数；

（*2*）对任意 $1 \le j \le n$，以及任意固定的 $z_1, ..., z_{j-1}; z_{j+1}, ..., z_n \in \mathbb{C}$，关于 z_j 的单变量函数

$$z_j \mapsto f(z_1, ..., z_{j-1}; z_j; z_{j+1}, ..., z_n)$$

是（单变量）全纯函数。

事实上，如果该定义中的（2）成立，那么能推出（1）成立，也就是说此定义中的（1）可以去掉。其证明比较复杂，我们承认之。

记号 1.1.3. 对于 $\mathbb{C}^n$ 的开子集 Ω，我们记

$$\mathcal{O}(\Omega) := \{f : \Omega \to \mathbb{C} | f \text{是 } \Omega \text{ 上的全纯函数}\}$$

容易知道 $\mathcal{O}(\Omega)$ 有显然的 $\mathbb{C}$-代数结构。

本节将说明，多变量全纯函数具有一些与单变量全纯函数类似的性质。

记号 1.1.4. 对于 $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$ 以及 $R = (R_1, R_2, ..., R_n) \in \mathbb{R}^n$，并且 $R_j > 0$ $(\forall 1 \le j \le n)$，则我们记

$$\mathbb{D}(z, R) := \mathbb{D}(z_1, R_1) \times \mathbb{D}(z_2, R_2) \times \cdots \times \mathbb{D}(z_n, R_n)$$

称为以 z 为中心，R 为半径的**多圆柱**（*polydisk*）。

对于多圆柱 $\mathbb{D}(z, R)$，我们记

$$\Gamma(z, R) := \partial\mathbb{D}(z_1, R_1) \times \partial\mathbb{D}(z_2, R_2) \times \cdots \times \partial\mathbb{D}(z_n, R_n)$$

称为 $\mathbb{D}(z, R)$ 的**特征边界**（*distinguished boundary*）。

特别注意特征边界 $\Gamma(z, R)$ 并不等于该多圆柱的边界 $\partial\mathbb{D}(z, R)$.

定理 1.1.5. （多变量全纯函数的柯西积分公式）

设 $f:\overline{\mathbb{D}(z,R)}\to\mathbb{C}$ 为全纯函数，则对任意的 $w\in\mathbb{D}(z,R)$，成立

$$f(w)=\frac{1}{(2\pi i)^n}\int_{\Gamma(z,R)}\frac{f(\xi)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_n}{(\xi_1-w_1)(\xi_2-w_2)\cdots(\xi_n-w_n)}$$

证明. 由多变量全纯函数的定义，反复使用单变量全纯函数的柯西积分公式即可。这是容易的。 □

与单复变函数完全类似，我们也有泰勒展开：

推论 1.1.6. （多元全纯函数的泰勒展开公式）

对于 $f\in\mathcal{O}(\Omega)$，其中 $\Omega\subseteq\mathbb{C}^n$ 为开子集，则对于任何多圆柱 $\mathbb{D}(z_0,R)$，如果 $\overline{\mathbb{D}(z_0,R)}\subseteq\Omega$，则对于任意 $w\in\mathbb{D}(z_0,R)$，成立

$$f(w)=\sum_{\alpha\in\mathbb{N}^n}a_\alpha(w-z_0)^\alpha$$

其中

$$a_\alpha=\frac{1}{(2\pi i)^n}\int_{\Gamma(z_0,R)}\frac{f(z)}{(z-z_0)^{\alpha+1}}\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_n=\frac{f^{(\alpha)}(z_0)}{\alpha!}$$

注意这里的 α 为多重指标，即 $\alpha=(\alpha_1,...,\alpha_n)$，其中每个 α_i 都为非负整数。我们记

$$\begin{aligned}
z^\alpha &:= z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n}\\
\alpha! &:= \alpha_1!\alpha_2!\cdots\alpha_n!\\
f^{(\alpha)} &:= (\partial_{z_1})^{\alpha_1}(\partial_{z_2})^{\alpha_2}\cdots(\partial_{z_n})^{\alpha_n}f\\
\alpha+1 &:= (\alpha_1+1,\alpha_2+1,...,\alpha_n+1)
\end{aligned}$$

其中 $z=(z_1,...,z_n)\in\mathbb{C}^n$，$f$ 为 n 元全纯函数。

证明. 与单复变函数的情形完全类似，可由柯西积分公式得到。 □

定理 1.1.7. （柯西不等式）对于 $\mathbb{C}^n$ 的开子集 Ω，若 $f\in\mathcal{O}(\Omega)$，多圆柱 $\overline{\mathbb{D}(z_0,R)}\subseteq\Omega$，则对任意多重指标 $\alpha\in\mathbb{N}^n$，成立

$$\left|f^{(\alpha)}(z_0)\right|\leq\frac{\alpha!}{R^\alpha}\sup_{z\in\Gamma(z_0,R)}|f(z)|$$

证明. 与单复变函数的情形完全类似。利用多元泰勒展开（推论1.1.6）即可。 □

推论 1.1.8. 设 $\Omega \subseteq \mathbb{C}^n$ 为**连通**开集，$f \in \mathcal{O}(\Omega)$ 满足 $\forall 1 \le k \le n$，$\frac{\partial f}{\partial z_k}$ 在 Ω 上恒为 0，则 f 在 Ω 上为常值函数。

推论 1.1.9. （刘维尔定理）设 $f \in \mathcal{O}(\mathbb{C}^n)$，并且满足

$$|f(z)| \le A(1+|z|)^B$$

其中 A, B 为正实数，那么 f 必为次数不超过 B 的多项式函数。

这些性质于单变量全纯函数雷同，证明也是类似的。

推论 1.1.10. （*Montel* 定理）

设 Ω 为 $\mathbb{C}^n$ 的开子集，则 $\mathcal{O}(\Omega)$ 中的任何局部一致有界的全纯函数列都存在一致收敛的子列。

证明. 仍类似于单复变全纯函数的情形。使用柯西积分公式，再配合 Arzela-Ascoli 定理即可。从略。 □

现在，简单介绍一些复的微分形式。对于 $\mathbb{C}^n$，记其复坐标为 $(z_1, z_2, \dots, z_n)$；视 $\mathbb{C}^n$ 为 $2n$ 维实线性空间，

$$z_k = x_k + iy_k$$

从而引入

$$\begin{aligned} \mathrm{d}z_k &= \mathrm{d}x_k + i\mathrm{d}y_k \qquad (1,0)\text{形式} \\ \mathrm{d}\bar{z}_k &= \mathrm{d}x_k - i\mathrm{d}y_k \qquad (0,1)\text{形式} \end{aligned}$$

定义 1.1.11. （(p,q)-形式）

设 Ω 为 $\mathbb{C}^n$ 的非空开集，则形如

$$u(z) = \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ}(z)\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

的光滑张量场称为 (p,q)-形式。记 Ω 上的 (p,q)-形式之全体为 $C^\infty_{p,q}(\Omega)$.

这里的 I, J 为多重指标。“光滑”指的是系数函数 a_{IJ} 为 Ω 上的光滑复值函数。另外，显然 $(0,0)$-形式即为光滑函数；$C^{\infty}_{p,q}(\Omega)$ 具有显然的复线性空间结构，事实上还是 $C^{\infty}(\Omega)$-模。

记号 1.1.12. （$\bar\partial$-算子）定义算子

$$\bar\partial : C^{\infty}_{p,q}(\Omega) \to C^{\infty}_{p,q+1}(\Omega)$$

如下: 对于 (p,q)-形式

$$u := \sum_{\substack{|I|=p\\|J|=q}} a_{IJ}\mathrm{d}z_I \wedge \mathrm{d}\bar z_J$$

则

$$\bar\partial u = \sum_{\substack{|I|=p\\|J|=q}} \sum_{k=1}^{n} \frac{\partial a_{IJ}}{\partial \bar z_k} \mathrm{d}\bar z_k \wedge \mathrm{d}z_I \wedge \mathrm{d}\bar z_J$$

类似地，也有

$$\partial : C^{\infty}_{p,q}(\Omega) \to C^{\infty}_{p+1,q}(\Omega)$$

它们与外微分算子 d 满足关系

$$\mathrm{d} = \partial + \bar\partial$$

由 $\mathrm{d}^2 = 0$，易知

$$\partial^2 = 0, \quad \bar\partial^2 = 0, \quad \partial\bar\partial + \bar\partial\partial = 0$$

以下事实显然成立：

引理 1.1.13. 对于区域 Ω 上的光滑函数 $f \in C^{\infty}(\Omega)$，则 f 全纯当且仅当 $\bar\partial f = 0$.

注记 1.1.14. （*Dolbeault* 上同调）对于 $\Omega \subseteq \mathbb{C}^n$，注意 $\bar\partial^2 = 0$，从而对任意 $p \geq 0$，有上链复形 $C^{\infty}_{p,\bullet}(\Omega)$：

$$\cdots \to C^{\infty}_{p,q-1}(\Omega) \xrightarrow{\bar\partial} C^{\infty}_{p,q}(\Omega) \xrightarrow{\bar\partial} C^{\infty}_{p,q+1}(\Omega) \to \cdots$$

称上同调群

$$H^{p,q}(\Omega) := H^q(C^{\infty}_{p,\bullet}(\Omega), \bar\partial)$$

为区域 Ω 的 ***Dolbeault* 上同调群**。

类似于外微分 d 的 de-Rham 上同调群，Dolbeault 上同调群与 Ω 的拓扑联系密切。例如，以下定理十分重要，我们先陈述，以后再证明：

引理 1.1.15. *(Dolbeault-Grothendieck 引理)*

设 $\mathbb{D} \subseteq \mathbb{C}^n$ 为多圆柱，则对于任意 $p, q \geq 0$，

$$H^{p,q}(\mathbb{D}) = 0$$

不难发现它与 de Rham 上同调的 Poincare 引理有些类似。

1.2 解析延拓与 Hartogs 现象

上一节介绍了多复变函数的一些“普通的”（与单变量类似）性质，本节开始介绍多复变函数的一些独特性质。

引理 1.2.1. 设 $f \in C_c^\infty(\mathbb{C})$ 为复平面上的紧支光滑函数，则对任意 $z \in \mathbb{C}$，成立

$$\frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\partial f/\partial\overline{\tau}}{\tau - z}\mathrm{d}\tau \wedge \mathrm{d}\overline{\tau} = f(z)$$

证明. 基本的微积分练习。考虑换元 $\tau = z + re^{i\theta}$，则易知

$$\begin{aligned}
\mathrm{d}\tau \wedge \mathrm{d}\overline{\tau} &= -2ir\mathrm{d}r \wedge \mathrm{d}\theta \\
\frac{\partial r}{\partial\overline{\tau}} &= \frac{1}{2}e^{i\theta} \\
\frac{\partial\theta}{\partial\overline{\tau}} &= -\frac{1}{2ir}e^{i\theta}
\end{aligned}$$

因此有

$$\begin{aligned}
\frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\partial f/\partial\overline{\tau}}{\tau - z}\mathrm{d}\tau \wedge \mathrm{d}\overline{\tau} &= \frac{-1}{2\pi}\int_0^\infty \mathrm{d}r \int_0^{2\pi}\left(-\frac{1}{ir}\frac{\partial f}{\partial\theta}(z + re^{i\theta})\right)\mathrm{d}\theta \\
&\quad + \frac{-1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\int_0^\infty\left(\frac{\partial f}{\partial r}(z + re^{i\theta})\right)\mathrm{d}r \\
&= 0 + \frac{-1}{2\pi}\int_0^{2\pi} -f(z)\mathrm{d}\theta \\
&= f(z)
\end{aligned}$$

证毕。 □

引理 1.2.2. （简单版本的 $\bar{\partial}$-引理）

设 $n \geq 2$，$\varphi \in C_{0,1}^{\infty}(\mathbb{C}^n)$ 为具有紧支集的光滑 $(0,1)$-形式，且 $\bar{\partial}\varphi = 0$，则存在 $\mathbb{C}^n$ 上的紧支光滑函数 g，使得

$$\bar{\partial} g = \varphi$$

证明. 记光滑 $(0,1)$-形式 φ 为

$$\varphi = \sum_{k=1}^{n} \varphi_k(z_1, ..., z_n) \mathrm{d}\bar{z}_k$$

则

$$\bar{\partial}\varphi = \sum_{k,l} \frac{\partial \varphi_k}{\partial \bar{z}_l} \mathrm{d}\bar{z}_l \wedge \mathrm{d}\bar{z}_k = \sum_{1 \leq l < k \leq n} \left(\frac{\partial \varphi_k}{\partial \bar{z}_l} - \frac{\partial \varphi_l}{\partial \bar{z}_k} \right) \mathrm{d}\bar{z}_l \wedge \mathrm{d}\bar{z}_k$$

从而由 $\bar{\partial}\varphi = 0$ 可得对任意 $k \neq l$，

$$\frac{\partial \varphi_k}{\partial \bar{z}_l} = \frac{\partial \varphi_l}{\partial \bar{z}_k}$$

考虑如下的 $\mathbb{C}^n$ 上的函数 ψ：对于 $z = (z_1, ..., z_n) \in \mathbb{C}^n$，

$$\psi(z) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\varphi_1(\tau; z_2, ..., z_n)}{\tau - z_1} \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}$$

由 φ_1 的紧支性易知 ψ 为 $\mathbb{C}^n$ 上的光滑函数。对于 $1 < k \leq n$，有

$$\begin{aligned}
\frac{\partial \psi(z)}{\partial \bar{z}_k} &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_1}{\partial \bar{z}_k}(\tau; z_2, ..., z_n)}{\tau - z_1} \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau} \\
&= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_k}{\partial \bar{\tau}}(\tau; z_2, ..., z_n)}{\tau - z_1} \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau} \\
&= \varphi_k(z)
\end{aligned}$$

上式对 $k = 1$ 显然也成立。因此 $\bar{\partial}\psi = \varphi$.

最后还需要证明 ψ 是紧支的。由于 φ 紧支，存在足够大的 $R > 0$，使得

$$\operatorname{supp} \varphi \subseteq \mathbb{D}(0, R)$$

因此任意取定 $z \in \mathbb{C}^n$，使得 z 的分量 $z_2, z_3, ..., z_n$ 之中至少有一个模长大于 R，则由 ψ 的定义式直接得到 $\psi(z) = 0$. (注意：这一步严重依赖 $n \geq 2$！) 也就是说，存在 $z \notin \mathbb{D}(0, R)$ 使得 $\psi = 0$ 在 z 的某邻域内都成立。另一方面，由于 $\bar{\partial}\psi = \varphi$ 且 $\operatorname{supp} \varphi \subseteq \mathbb{D}(0, R)$，从而 ψ 在 $\mathbb{D}(0, \mathbb{R})$ 外部全纯，因此由解析延拓唯一性，ψ 在 $\mathbb{D}(0, R)$ 外部恒为零，因此 ψ 紧支。 □

此引理在单复变 $n = 1$ 的情形**不成立**：

例子 1.2.3. 设 $\varphi_1 \in C_0^\infty(\mathbb{C})$ 为复平面上的紧支光滑函数，并且

$$\iint_{\mathbb{C}} \varphi_1(z) \neq 0$$

考虑 $\mathbb{C}$ 上的 $(0,1)$-形式 $\varphi = \varphi_1(z)\mathrm{d}\bar{z}$，则 $\bar{\partial}\varphi = 0$ 是平凡的，但**不存在**紧支光滑函数 ψ 使得 $\bar{\partial}\psi = \varphi$.

证明. 若存在紧支光滑函数 ψ 使得 $\bar{\partial}\psi = \varphi$，则 $\frac{\partial\psi}{\partial\bar{z}} = \varphi_1$. 于是

$$0 \neq \iint_{\mathbb{C}} \varphi_1(z)\mathrm{d}z \wedge \mathrm{d}\bar{z} = \iint_{\mathbb{C}} \frac{\partial\psi}{\partial\bar{z}}\mathrm{d}z \wedge \mathrm{d}\bar{z} = 0$$

产生矛盾。 □

以下是多复变函数解析延拓的令人惊讶的性质，它与单复变函数有本质不同：

定理 1.2.4. （*Hartogs* 现象）

设 $\Omega \subseteq \mathbb{C}^n$ 为开集 $(n \geq 2)$，$K \subset\subset \Omega$ 且为 $\mathbb{C}^n$ 的紧子集，则对任意的 $f \in \mathcal{O}(\Omega \setminus K)$，都存在解析延拓 $F \in \mathcal{O}(\Omega)$，使得

$$F|_{\Omega\setminus K} = f$$

证明. 取 K 与 Ω 直接的截断函数 $\psi \in C_0^\infty(\mathbb{C}^n)$，使得 $0 \leq \psi \leq 1$，

$$K \subset\subset \operatorname{supp}\psi \subset\subset \Omega$$

并且 $\psi|_K \equiv 1$. 考虑

$$\widetilde{f} := (1-\psi)f$$

则 $\widetilde{f}$ 在整个 Ω 上都有定义。注意

$$\bar{\partial}\widetilde{f} = -(\bar{\partial}\psi)f + (1-\psi)\bar{\partial}f$$

易知 $\operatorname{supp}\bar{\partial}\widetilde{f} \subseteq \operatorname{supp}\psi$. 于是由引理1.2.2，存在光滑函数 v，使得 $\operatorname{supp} v \subseteq \psi$，并且 $\bar{\partial}v = \bar{\partial}\widetilde{f}$，从而考虑函数

$$F := (1-\psi)f - v$$

则 $\bar{\partial}F = 0$，从而 $F \in \mathcal{O}(\Omega)$. 又因为易知

$$F = f \quad (\forall z \in \Omega \setminus \operatorname{supp}\psi)$$

从而由解析延拓唯一性，有 $F_{\Omega\setminus K} = f$. □

关于解析延拓，再介绍如下结果：

引理 1.2.5. （*Hartogs figure*）

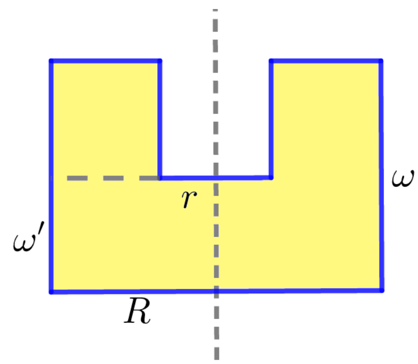
对于 $n>1$，正实数 $0\le r<R$，以及 $\mathbb{C}^{n-1}$ 的开子集 $\omega'\subseteq\omega$，其中 ω 是连通的。记 $\mathbb{C}^n$ 的开子集

$$\Omega := \left((\mathbb{D}(0,R)\setminus\mathbb{D}(0,r))\times\omega\right)\cup\left(\mathbb{D}(0,R)\times\omega'\right)$$

其中 $\mathbb{D}(0,r)$ 与 $\mathbb{D}(0,R)$ 分别为 $\mathbb{C}$ 上的以原点为中心，r,R 为半径的开圆盘。则任意 $f\in\mathcal{O}(\Omega)$ 都可以（唯一地）解析延拓至

$$\widetilde{\Omega} := \mathbb{D}(0,R)\times\omega$$

如此的区域 Ω 称之为“**Hartogs figure**”。Ω 的几何图像大致如下：



图：Hartogs figure 示意

证明. 容易知道

$$\Omega=\left\{(z_1,\widetilde{z})\in\mathbb{C}\times\mathbb{C}^{n-1}\middle| r<|z_1|<R,\widetilde{z}\in\omega\text{或者}|z_1|\le r,\widetilde{z}\in\omega'\right\}$$

对于 $f\in\mathcal{O}(\Omega)$，定义 $\widetilde{\Omega}$ 上的函数

$$\widetilde{f}(z_1,\widetilde{z}) := \frac{1}{2\pi i}\int_{|w|=\rho}\frac{f(w,\widetilde{z})}{z_1-w}\mathrm{d}w$$

其中 ρ 为满足 $\max\{r,|z_1|\}<\rho<R$ 的任意实数。则易知如此定义的 $\widetilde{f}$ 为 f 在 $\widetilde{\Omega}$ 上的解析延拓。 □

定理 1.2.6. （*Riemann* 延拓定理）

考虑 $\mathbb{C}^n$ 中的多圆柱 $\mathbb{D}(0,R)$，其中 $n\ge 2$，$R\in\mathbb{R}^n_+$。对任意 $2\le p\le n$，令 $\mathbb{C}^n$ 的子集

$$S := (z_1,...,z_n)\in\mathbb{C}^n|z_1=\cdots=z_p=0$$

则对任意 $f\in\mathcal{O}(\mathbb{D}(0,R)\setminus S)$，$f$ 都可（唯一地）解析延拓至 $\mathbb{D}(0,R)$.

证明. 这是 Hartogs figure 的显然推论。记 $R=(R_1,R_2,...,R_n)$，以及 $R':=(R_2,...,R_n)\in\mathbb{R}^{n-1}$. 考虑 $\mathbb{C}^{n-1}$ 的开子集

$$\begin{aligned}\omega &:= \mathbb{D}(0,R')\\ \omega' &:= \omega\setminus\{z_2=\cdots=z_p=0\}\end{aligned}$$

则易知

$$\mathbb{D}(0,R)\setminus S=\Big(\mathbb{D}(0,R_1)\setminus\{0\}\times\omega\Big)\cup\Big(\mathbb{D}(0,R_1)\times\omega'\Big)$$

为 Hartogs figure，从而完。 □

1.3 Weierstrass 预备定理与除法定理

回顾单复变函数，若 f 在 $0\in\mathbb{C}$ 附近全纯，且 $f(0)=0$，则在 0 附近 f 可以唯一地分解为 $f=z^d g(z)$，其中 g 全纯且 $g(0)\neq 0$，d 为 f 在 0 处的**零点阶数**。

现在，设 $f=f(z,w)$ 在 $0\in\mathbb{C}^n(n\geq 2)$ 附近全纯，其中 $z\in\mathbb{C}$，$w\in\mathbb{C}^{n-1}$. 固定 w，记

$$f_w(z):=f(z,w)$$

为关于 z 的单复变函数。如果 $f_0(0)=0$ 且 $f_0(z)$ 不恒为零，则 $f_0(z)=z^d g_0(z)$。我们的一个结果是，若“f_0”的下标“0”稍微“扰动”一下，则相应的多项式 z^k 也“随之扰动”。

记号 1.3.1. （*Weierstrass 多项式*）

对于 $(z_0,w_0)\in\mathbb{C}\times\mathbb{C}^{n-1}$，则 (z_0,w_0) 处的 ***Weierstrass* 多项式** 是指形如下述的定义于 (z_0,w_0) 附近的 n 元全纯函数：

$$P(z,w)=z^k+a_1(w)z^{k-1}+\cdots+a_k(w)$$

其中 $a_i(1\leq i\leq k)$ 为定义在 $w_0\in\mathbb{C}^{n-1}$ 附近的全纯函数，且 $a_i(w_0)=0$.

关于多元全纯函数在其零点附近的行为，首先有如下：

定理 1.3.2. *(Weierstrass 预备定理)*

设 $f(z,w)$ 为定义在 $(0,0)\in\mathbb{C}\times\mathbb{C}^{n-1}$ 附近的全纯函数，$f(0,0)=0$，且 $f_w(z)$ 在 $z=0$ 附近不恒为零，则存在唯一的 $(0,0)$ 处的 Weierstrass 多项式 $P(z,w)$，使得

$$f(z,w)=P(z,w)h(z,w)$$

其中 $h(z,w)$ 在 $(0,0)$ 附近全纯，且 $h(0,0)\neq 0$.

证明. 分若干步。

Step1 设 $f_0(z)$ 在 $z=0\in\mathbb{C}$ 处的零点阶数为 $d\geq 1$, 取足够小的 $\varepsilon>0$ 使得 $f_0(z)$ 在 $|z|\leq\varepsilon$ 之中不再有 $z=0$ 之外的零点。再由 f 的连续性以及 $\{|z|=\varepsilon\}\subseteq\mathbb{C}$ 的紧性，存在足够小的 $\varepsilon'>0$，使得对任意 $|z|=\varepsilon, |w|<\varepsilon'$，$f_w(z)\neq 0$.

对于 $w\in\mathbb{C}^{n-1}$ 且 $|w|<\varepsilon'$，由辐角原理，$f_w(z)$ 在 $|z|<\varepsilon$ 内的零点个数（记重数）为

$$d(w)=\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{f'_w(\xi)}{f_w(\xi)}\mathrm{d}\xi$$

这是关于 w 的连续函数，且 $d(0)=d$. 从而不妨缩小 ε'，使得任意 $|w|<\varepsilon'$，$f_w(z)$ 在 $|z|<\varepsilon$ 内的零点个数（计重数）均为 d.

Step2 对于 $w\in\mathbb{C}^{n-1}$ 且 $|w|<\varepsilon'$，记 $f_w(z)$ 的 d 个零点为 $s_1(w),s_2(w),...,s_d(w)$，它们允许相同，则 $|s_j(w)|<\varepsilon$（注意 $s_j(w)$ 未必为关于 w 的全纯函数）。特别地 $s_1(0)=s_2(0)=\cdots=s_d(0)=0$. 考虑多项式

$$\begin{aligned}P(z,w) &:= \prod_{j=1}^{d}(z-s_j(w))\\ &= z^d+\sum_{j=1}^{d}a_j(w)z^{d-j}\end{aligned}$$

显然系数 $a_j(w)$ 满足 $a_j(0)=0$. 断言 $P(z,w)$ 为 Weierstrass 多项式。为此只需证明 $z_j(w)$ 关于 w 全纯。由代数学可知，系数 a_j 可以写为形如 $s_1^k(w)+s_2^k(w)+\cdots s_d^k(w)\ (k\geq 0)$ 的 $\mathbb{C}$-线性组合；而由留数定理易知

$$\sum_{j=1}^{d}s_j^k(w) = \frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\xi^k\frac{f'_w(\xi)}{f_w(\xi)}\mathrm{d}\xi$$

从而关于 w 全纯。这就说明了 $P(z,w)$ 的系数函数 $a_j(w)$ 关于 w 全纯。

Step3 令 $h(z,w):=\frac{f(z,w)}{P(z,w)}$，断言 h 在 $(0,0)$ 附近全纯，又因为显然 $h(0,0)\neq 0$，从而 Weierstrass 预备定理的存在性得证。由单复变易知 $h(z,w)$ 关于 z 全纯，于是只需证明 h 关于 w 全纯。

任取 $w\in\mathbb{C}^{n-1}$ 且 $|w|<\varepsilon'$，由于 $h_w(z):=h(z,w)$ 关于 z 全纯，从而

$$h(z,w)=\frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\frac{h_w(\xi)}{\xi-z}\mathrm{d}\xi$$

而被积函数 $(\xi,w)\mapsto\frac{h_w(\xi)}{\xi-z}$ 在 $\{(z,w)||z|=\varepsilon,|w|<\varepsilon'\}$ 的某个邻域全纯，从而 $h(z,w)$ 关于 w 也全纯。存在性证毕。

Step4 唯一性几乎显然，因为 f（在 $(0,0)$ 附近）的零点完全由 Weierstrass 多项式贡献：对于 w，以 $s_1(w),...,s_d(w)$ 为零点的关于 z 的首一多项式只能是 $P(z,w)$. □

定理 1.3.3. （*Weierstrass* 除法定理）

设 $f(z,w)$ 为定义在 $(0,0) \in \mathbb{C} \times \mathbb{C}^{n-1}$ 附近的全纯函数，$g(z,w) = z^d + \sum_{j=1}^{d} a_j(w) z^{d-j}$ 为次数为 d 的 *Weierstrass* 多项式。那么存在唯一的 $h(z,w)$ 与 $r(z,w)$，其中 h 为定义在 $(0,0) \in \mathbb{C} \times \mathbb{C}^{n-1}$ 附近的全纯函数，r 为关于 z 的在 $(0,0)$ 处的次数 $< d$ 的多项式，使得

$$f = gh + r$$

在 $(0,0)$ 附近成立。

证明. 先看唯一性。

Step1 唯一性是容易的。如果 $f = gh_1 + r_1 = gh_2 + r_2$，则

$$r_1 - r_2 = g(h_2 - h_1)$$

注意 g, r_1, r_2 为 Weierstrass 多项式，从而由之前讨论，存在足够小的 $\varepsilon, \varepsilon' > 0$ 使得对任意 $w \in \mathbb{C}^{n-1}$ 且 $|w| < \varepsilon'$，$g_w(z)$ 在 $\{|z| < \varepsilon\}$ 内的零点个数（计重数）恰为 g 的次数 d，并且 $(r_1 - r_2)_w(z)$ 在此范围内的零点个数（计重数）恰为 $(r_1 - r_2)$ 的次数。注意 r_1, r_2 的次数均小于 d，从而若 $r_1 \neq r_2$，则导致 $(r_1 - r_2)_w(z)$ 的零点个数小于 $g_w(z)(h_2 - h_1)_w(z)$，因此导致矛盾。这迫使 $r_1 = r_2$.

Step2 再看存在性。取 $\varepsilon, \varepsilon' > 0$ 使得对任意 $|z| = \varepsilon$，$|w| \leq \varepsilon'$，$g_w(z) \neq 0$。对任意 $|z| < \varepsilon, |w| < \varepsilon'$，定义

$$h(z,w) = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{g_w(\xi)(\xi - z)} \mathrm{d}\xi$$

则易知 $h(z,w)$ 在 $(0,0)$ 附近全纯。再令 $r := f - gh$，只需证明 r 为关于 z 的次数小于 d 的 Weierstrass 多项式即可。事实上，

$$\begin{aligned}
r(z,w) &= f(z,w) - g(z,w)h(z,w) \\
&= \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{\xi - z} \mathrm{d}\xi - \frac{g_w(z)}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{g_w(\xi)(\xi - z)} \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)(g_w(\xi) - g_w(z))}{g_w(\xi)(\xi - z)} \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{g_w(\xi)} \frac{(\xi^d - z^d) + a_1(w)(\xi^{d-1} - z^{d-1}) + \cdots}{\xi - z} \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{g_w(\xi)} \left(z^{d-1} + \beta_1(\xi, w) z^{d-2} + \cdots \right) \mathrm{d}\xi
\end{aligned}$$

其中函数 $\beta_j(\xi, w)$ 由 g 的系数函数 $a_k(w)$ 决定。容易看出 $r(z,w)$ 的确为关于 z 的次数 $\leq d-1$ 的多项式。存在性证毕。 □

注意 r 未必是 Weierstrass 多项式，因为 $r(z,w)$ 的 z^{d-1} 的系数

$$\frac{1}{2\pi i}\int_{|\xi|=\varepsilon}\frac{f_w(\xi)}{g_w(\xi)}\mathrm{d}\xi$$

不见得是 1（若此积分为 0，则 r 的首项系数甚至可以是关于 w 的函数）。

注记 1.3.4. 事实上，*Weierstrass* 除法定理对单复变 $n=1$ 的情形也成立。设 $f(z)=\sum_{k=0}^{\infty}a_kz^k$ 在 $0\in\mathbb{C}$ 附近全纯，$g(z)=z^d$ 为次数为 d 的 *Weierstrass* 多项式。则令

$$\begin{aligned} h(z) &= \sum_{k=d}^{\infty}a_kz^{k-d} \\ r(z) &= \sum_{k=0}^{d-1}a_kz^k \end{aligned}$$

则 $f=gh+r$ 满足要求。

1.4 解析函数芽环 $\mathcal{O}_{\mathbb{C}^n,z}$ 及其代数结构

本节继续研究多元解析函数的性质。首先回顾函数芽的概念。

定义 1.4.1. （解析函数芽环）

对于 $z\in\mathbb{C}^n$，记

$$\mathcal{O}_{\mathbb{C}^n,z}:=\{(U,f)|U\text{ 是 }z\text{ 在 }\mathbb{C}^n\text{ 的一个开邻域，}f\text{ 为定义在 }U\text{ 上的全纯函数}\}/\sim$$

其中模掉的关系 $\sim$ 为

$$(U,f)\sim(V,g)\quad\Longleftrightarrow\quad\text{存在 }z\text{ 的开邻域 }W\text{，使得 }W\subseteq U\cap V\text{，且 }f|_W=g|_W$$

粗俗地说，$\mathcal{O}_{\mathbb{C}^n,z}$ 就是“定义在 $z\in\mathbb{C}^n$ 附近的全纯函数之全体”。之前介绍的 Weierstrass 预备定理、Weierstrass 除法定理其实都是解析函数芽环的性质。容易验证，$\mathcal{O}_{\mathbb{C}^n,z}$ 在通常的函数加法、乘法下构成**环**。

我们记 $\mathcal{O}_n:=\mathcal{O}_{\mathbb{C}^n,0}$. 本节介绍环 $\mathcal{O}_n$ 的代数性质。假定读者熟悉基础的交换代数。本讲义中的“环”默认为含幺、交换的。

定理 1.4.2. $\mathcal{O}_n$ 是局部诺特环（$\forall n\geq 1$）。

回顾：环 A 称为**局部环**（local ring），若 A 存在唯一极大理想 $\mathfrak{m}$（等价定义：A 的全体不可逆元构成 A 的理想）；环 A 称为**诺特环**（Noetherian ring），若满足理想升链条件（等价定义：A 的每个理想都是有限生成的）。

证明. 显然 $\mathcal{O}_n$ 为局部环，其极大理想 $\mathfrak{m}$ 由定义在 0 附近、在 0 处取值为 0 的函数芽构成。我们对 n 归纳证明 $\mathcal{O}_n$ 为诺特环。

$n=1$ 时，在单复变中我们早已熟知 $\mathcal{O}_1 \cong \{$收敛半径 ≥ 0 的幂级数$\}$ 为主理想整环（PID），其理想形如 $J_k=(z^k)$。特别地，为诺特环。

一般地，对于 $n\geq 2$，若 $\mathcal{O}_{n-1}$ 为诺特环，则对 $\mathcal{O}_n$ 的任意非零理想 J，断言 J 时有限生成的。任取 $0\neq h\in J\subseteq \mathfrak{m}$，则 $h(0)=0$，不妨 $h(z,0)$ 不恒为零（其中 $z\in\mathbb{C}, 0\in\mathbb{C}^{n-1}$），则由 Weierstrass 预备定理，存在 Weierstrass 多项式 $P(z,w)\in\mathcal{O}_{n-1}[z]\subseteq\mathcal{O}_n$ 以及函数芽 $h'\in\mathcal{O}_n\setminus\mathfrak{m}$，使得 $h(z,w)=P(z,w)h'(z,w)$. 注意 $h'(0,0)$ 为 $\mathcal{O}_n$ 的可逆元，又 $h\in J$ 且 J 为 $\mathcal{O}_n$ 的理想，从而 $P(z,w)\in J$.

这说明 J 当中必存在 Weierstrass 多项式。取定

$$P(z,w)=z^d+\sum_{j=1}^{d}a_j(w)z^{d-j}\in J$$

则对任意 $f\in J$，对 f,P 使用 Weierstrass 除法定理，存在 $g(z,w)\in\mathcal{O}_n$，以及

$$r(z,w)=\sum_{k=0}^{d-1}c_k(w)z^k\in\mathcal{O}_{\mathbb{C}^{n-1}}[z]$$

为次数至多为 $(d-1)$ 的多项式，使得

$$f=gP+r$$

则 $r(z,w)\in J$，并且容易验证，这诱导了 $\mathcal{O}_{n-1}$-模同态

$$\begin{aligned}\varphi: J &\to \mathcal{O}_{n-1}^{\oplus d}\cong\{r\in\mathcal{O}_{n-1}[z]\mid \deg_z r<d\}\\ f &\mapsto \sum_{k=0}^{d-1}c_k(w)z^k\end{aligned}$$

由归纳假设，$\mathcal{O}_{n-1}$ 为诺特环，从而 $\mathcal{O}_{n-1}^{\oplus d}$ 作为有限生成 $\mathcal{O}_{n-1}$-模为诺特模，从而其子模 $\operatorname{Im}\varphi$ 也为有限生成的。注意 $\operatorname{Im}\varphi\subseteq J$，记 $\{\beta_1,...,\beta_N\}\subseteq\operatorname{Im}\varphi$ 为 $\operatorname{Im}\varphi$ 的一组 $\mathcal{O}_{n-1}$-生成元，其中

$$\beta_j(w)=\sum_{l=0}^{d-1}\beta_{j,l}(w)z^l\in\mathcal{O}_{n-1}^{\oplus d}$$

则易知

$$\{\beta_j\}_{1\leq j\leq N}\cup\{P(z,w)\}$$

为理想 J 的一组生成元，因此 J 是有限生成的。从而 $\mathcal{O}_n$ 为诺特环。 □

引理 1.4.3. *设 $P, Q \in \mathcal{O}_{n-1}[z] \subseteq \mathcal{O}_n$，其中 P 为 Weierstrass 多项式，则 P 整除 Q 在 $\mathcal{O}_n$ 成立，当且仅当 P 整除 Q 在 $\mathcal{O}_{n-1}[z]$ 中成立。*

证明. “当”是显然的，只证“仅当”。若 $P|Q$ 在 $\mathcal{O}_n$ 中成立，则令

$$Q(z,w) = f(z,w)P(z,w)$$

其中 $f \in \mathcal{O}_n$. 另一方面，考虑 $\mathcal{O}_{n-1}[z]$ 中标准的欧几里得带余除法，

$$Q(z,w) = g(z,w)P(z,w) + r(z,w)$$

其中 $g, r \in \mathcal{O}_{n-1}[z]$. 则 Weierstrass 除法定理的唯一性迫使 $f = g, r = 0$，从而得证。 □

引理 1.4.4. *设 $P(z,w) \in \mathcal{O}_{n-1}[z]$ 为 Weierstrass 多项式，则：*

（1）若在 $\mathcal{O}_{n-1}[z]$ 中有分解

$$P = P_1 P_2 \cdots P_N$$

则在相差 $\mathcal{O}_{n-1}$ 中的可逆元的意义下，每个 P_j 都为 Weierstrass 多项式；

（2）P 为 $\mathcal{O}_n$ 中的不可约元当且仅当 P 为 $\mathcal{O}_{n-1}[z]$ 中的不可约元。

证明.

（1）记 $\deg_z P = s$，以及 $\deg_z P_j = s_j$，则 $s = \sum\limits_{j=1}^{N} s_j$. 不妨每个 $s_j > 0$. 考虑 P 的最高次项，有

$$z^s = z^s \prod_{j=1}^{N} \left(P_j \text{ 的 } z^{s_j} \text{ 系数}\right)$$

从而相差 $\mathcal{O}_{n-1}$ 中某个可逆元倍，不妨每个 P_j 的 z^{s_j} 系数都为 1. 再注意

$$z^s = P(0,z) = \prod_{j=1}^{N} P_j(0,z) = \prod_{j=1}^{N} (z^{s_j} + \cdots)$$

从而迫使 $P_j(0,z) = z^{s_j}$，因此 P_j 为 Weierstrass 多项式。

（2）“仅当”是显然的，只证“当”。仍记 $P(z,w)$ 关于 z 的次数为 s. 如果 P 在 $\mathcal{O}_n$ 中可约，令 $P = g_1 g_2$，其中 g_1, g_2 为 $\mathcal{O}_n$ 中的不可逆元，从而关于 z 的函数 $g_1(z,0), g_2(z,0)$ 在 $z = 0$ 处的零点阶数大于 0，分别记为 s_1, s_2. 由 Weierstrass 预备定理，存在分解

$$g_j(z,w) = P_j(z,w) u_j(z,w) \quad (j = 1,2)$$

使得 $P_j \in \mathcal{O}_{n-1}[z]$ 为次数为 s_j 的 Weierstrass 多项式，u_j 为 $\mathcal{O}_n$ 中的可逆元。所以在 $\mathcal{O}_n$ 中成立 $(P_1P_2)|P$；再根据引理1.4.3，可知 $(P_1P_2)|P$ 在 $\mathcal{O}_{n-1}[z]$ 中也成立。而 P, P_1, P_2 都为首一多项式，从而必有 $P = P_1P_2$，因此 P 在 $\mathcal{O}_{n-1}$ 中可约。 □

定理 1.4.5. $\mathcal{O}_n$ *是唯一分解整环 (UFD).*

证明. 对 n 归纳。$n=1$ 时，$\mathcal{O}_1$ 为主理想整环，从而为唯一分解整环。对于 $n\geq 2$，如果 $\mathcal{O}_{n-1}$ 为唯一分解整环，则由代数学中的高斯引理，多项式环 $\mathcal{O}_{n-1}[z]$ 也是唯一分解整环。

现在，对于 $\mathcal{O}_n$ 中的不可逆元 f，不妨 $z\mapsto f(z,w)|_{w=0}$ 不恒为零（$w\in\mathbb{C}^{n-1}$），从而由 Weierstrass 预备定理，存在分解 $f(z,w)=u(z,w)P(z,w)$，其中 u 为 $\mathcal{O}_n$ 中的可逆元，$P\in\mathcal{O}_{n-1}[z]$ 为 Weierstrass 多项式。由归纳假设，$\mathcal{O}_{n-1}[z]$ 为唯一分解整环，从而存在 P 在 $\mathcal{O}_{n-1}[z]$ 中的分解 $P=P_1P_2\cdots P_s$，使得每个 P_j 都为 $\mathcal{O}_{n-1}[z]$ 中的不可约元。从而由引理1.4.4的（1），不妨每个 P_j 都为 Weierstrass 多项式；再对每个 P_j 使用引理1.4.4的（2），知 P_j 为 $\mathcal{O}_n$ 中的不可约元。从而 $f\in\mathcal{O}_n$ 的不可约分解的存在性证毕。

再看分解的唯一性。只需再证明 $\mathcal{O}_n$ 的不可约元都是素元。若 f 为 $\mathcal{O}_n$ 中的不可约元，以及 $g,h\in\mathcal{O}_n$ 使得 $f|gh$，断言 $f|g$ 或者 $f|h$. 由 Weierstrass 预备定理，不妨假设 $f=f(z,w)$ 为关于第一个分量 z 的 Weierstrass 多项式，从而由 $f|gh$ 知 $g(z,0),h(z,0)$ 也不恒为零，于是由 Weierstrass 预备定理也不妨 $g,h\in\mathcal{O}_{n-1}[z]$ 为 Weierstrass 多项式。因此 $f|gh$ 在 $\mathcal{O}_{n-1}[z]$ 中成立，而由归纳假设 $\mathcal{O}_{n-1}[z]$ 是唯一分解整环，且 f 在 $\mathcal{O}_{n-1}[z]$ 不可约，所以 $f|g$ 或者 $f|h$ 在 $\mathcal{O}_{n-1}[z]$ 中成立，从而在 $\mathcal{O}_n$ 中成立。证毕。 □

1.5 解析集与局部解析零点定理

多复变函数与单复变的一个显著区别是解析延拓的难易程度，Hartogs 现象表明多复变函数“更容易被解析延拓”；而单复变与多复变函数令一个区别是零点集的形态：在单复变中我们熟知全纯函数零点离散（除非函数恒为零），这在多复变中显然不对，例如 $\mathbb{C}^2$ 上的全纯函数 $f(z_1,z_2)=z_1$.

事实上，多元全纯函数的零点集十分重要，而且是代数几何学中的某些概念（代数簇）的源头。

定义 1.5.1. （解析集）

设 $n\geq 2$，$\mathbb{C}^n$ 的子集 A 称为**解析集**（*analytic set*），若对任意 $z\in A$，存在 z 在 $\mathbb{C}^n$ 中的开邻域 Ω，以及 $f_1,f_2,...,f_N\in\mathcal{O}(\Omega)$，使得

$$A\cap\Omega=\{w\in\Omega|f_1(w)=f_2(w)=\cdots=f_N(w)\}$$

也就是说，“局部上看是若干全纯函数的公共零点集”。对于一个解析集，我们首先局部地研究之——类似于解析函数芽环，我们引入如下概念：

定义 1.5.2. (解析集芽)对于 $x \in \mathbb{C}^n$，定义

$$\mathcal{A}_x := \{(A,x) | x \in A,\ A \text{ 是 } \mathbb{C}^n \text{ 中的解析集}\}/\sim$$

其中关系 $\sim$ 为：$(A_1,x) \sim (A_2,x) \iff$ 存在 x 在 $\mathbb{C}^n$ 中的开邻域 Ω，使得 $A_1 \cap \Omega = A_2 \cap \Omega$. 称 $\mathcal{A}_x$ 中的元素为 x 处的**解析集芽**。

$\mathcal{A}_x$ 中的元素可以认为是包含 x 的"无穷小解析集"。容易知道它与解析函数芽的关系：任意 $(A,x) \in \mathcal{A}_x$，(A,x) 为 $\mathcal{O}_{\mathbb{C}^n,x}$ 中某些函数的公共零点集。

定义 1.5.3. 对于 $x \in \mathbb{C}^n$，

(1) 对与 x 处的解析集芽 $(A,x) \in \mathcal{A}_x$，定义 $\mathcal{O}_{\mathbb{C}^n,x}$ 的理想

$$J_{(A,x)} := \{f \in \mathcal{O}_{\mathbb{C}^n,x} | f(z) = 0 \forall z \in A\}$$

(2) 对于 $\mathcal{O}_{\mathbb{C}^n,x}$ 中的理想 J，定义 x 处的解析集芽

$$(V(J),x) := \{z \in \mathbb{C}^n | g(z) \equiv 0, \forall g \in J\}\text{的等价类}$$

这里并未仔细写清楚，需要验证良定性：注意解析集芽、函数芽实际上都为等价类，我们需要验证与代表元选取无关，留给读者。

注意 $\mathcal{O}_{\mathbb{C}^n,x}$ 为诺特环，从而任何理想 J 都是有限生成的，记 $\{g_1, g_2, ..., g_N\}$ 为其一组生成元，则易知

$$V(J) = \{g_1(x) = g_2(x) = \cdots = g_N(x) = 0\}$$

在 x 附近为有限个解析函数的公共零点集，从而的确为解析集（芽）。

引理 1.5.4. 设 $x \in \mathbb{C}^n$，$(A,x) \in \mathcal{A}_x$ 为 x 处的解析集芽，$J \subseteq \mathcal{O}_{\mathbb{C}^n,x}$ 为理想，则

$$\begin{aligned} J &\subseteq J_{(V(J),x)} \\ (V(J_{(A,x)}),x) &= (A,x) \end{aligned}$$

证明. 直接按定义验证即可。第一式是容易的；至于第二式，由解析集的定义，(A,x) 必形如

$$\{g_1(x) = g_2(x) = \cdots = g_N(x) = 0\}$$

其中 $g_j \in \mathcal{O}_{\mathbb{C}^n,x}$，从而 $J_{(A,x)} = (g_1, ..., g_N)$，之后容易。 □

注记 1.5.5. 不过要注意，第一式的等号未必成立，例如对于 $0 \in \mathbb{C}^2$，$f(z_1, z_2) = z_1^2$，令 $J := (f) \subseteq \mathcal{O}_{\mathbb{C}^2,0}$ 为由 f 生成的理想，则 $V(J) = \{z_1^2 = 0\} = \{z_1 = 0\}$，于是 $J_{(V(J),0)} = (z_1)$，即为由 $\widetilde{f}(z_1, z_2) = z_1$ 生成的理想。很明显，$J \subsetneqq J_{(V(J),0)}$.

对于 $x \in \mathbb{C}^n$，则 $\mathcal{A}_x$ 中的解析集芽可以进行交、并运算：

引理 1.5.6. 对于 $x \in \mathbb{C}^n$，$\{J_\alpha | \alpha \in \mathcal{I}\}$ 为 $\mathcal{O}_{\mathbb{C}^n,x}$ 的一族理想，则对任意 $\alpha, \beta \in \mathcal{I}$，

$$(V(J_\alpha) \cup V(J_\beta), x) = (V(J_\alpha J_\beta), x)$$

$$(\bigcap_{\alpha \in \mathcal{I}} V(J_\alpha), x) = (V(\sum_{\gamma \in \mathcal{I}} J_\gamma), x)$$

自行补全解析集芽交、并的定义（无非是取代表元作交、并）

证明. 直接定义验证。 □

此引理表明，一点处的解析集芽可以“有限并，任意交”，与拓扑学中的“闭集”类似。

接下来研究解析集芽的局部结构。

定义 1.5.7. （不可约解析集芽）

对于 $x \in \mathbb{C}^n$，以及 $(A, x) \in \mathcal{A}_x$，称解析集芽 (A, x) 是**不可约** *(irreducible)* 的，若不存在 $(A_1, x), (A_2, x) \in \mathcal{A}_x$，使得 $(A, x) = (A_1 \cup A_2, x)$，且 $(A_i, x) \subsetneqq (A, x), i = 1, 2$.

由引理1.5.6，以及基本的交换代数，容易知道：解析集芽 (A, x) 不可约，当且仅当 $J_{(A,x)}$ 为 $\mathcal{O}_{\mathbb{C}^n,x}$ 的**素理想**。此外，解析函数芽环的诸特性等价于如下：

引理 1.5.8. 对于 $x \in \mathbb{C}^n$，以及 $(A_k, x) \in \mathcal{A}_x, k \geq 1$，若 $(A_k, x) \supseteq (A_{k+1}, x)$ 对任意 $k \geq 1$ 都成立（即 $\{A_k\}_{k=1}^\infty$ 为解析集芽降链），则存在 $k_0 \geq 1$，使得对任意 $l \geq k_0$，都有 $(A_k, x) = (A_l, x)$.

证明. 考察理想 $J_{(A_k,x)} \subseteq \mathcal{O}_{\mathbb{C}^n,x}$，则 $(A_k, x) \supseteq (A_{k+1}, x)$ 表明

$$J_{(A_k,x)} \subseteq J_{(A_{k+1},x)}$$

即 $\{J_{(A_k,x)}\}_{k=1}^\infty$ 为理想升链，从而由 $\mathcal{O}_{\mathbb{C}^n,x}$ 的诸特性，以及引理1.5.4，得证。 □

定理 1.5.9. （解析集芽的不可约分解）

给定 $x \in \mathbb{C}^n$，则对任意 $(A,x) \in \mathcal{A}_x$，存在 $N \geq 1$，以及对任意 $1 \leq k \leq N$ 存在 $(A_k,x) \in \mathcal{A}_x$ 为不可约解析集芽，使得这些解析集芽**互不包含**，并满足

$$(A,x) = \bigcup_{k=1}^{N} (A_k,x)$$

并且上述分解是唯一的（不计次序）。

证明. **存在性**：先断言，若 (A,x) 可约，则存在分解 $(A,x) = (A^{(1)},x) \cup (A^{(2)},x)$，其中 $(A^{(1)},x)$ 与 $(A^{(2)},x)$ 都为 (A,x) 的真子芽，并且 $(A^{(1)},x)$ 不可约。

这是因为，由 (A,x) 可约，取真子芽 $(A_1,x),(A_1',x)$ 使得 $(A,x) = (A_1,x) \cup (A_1',x)$（但至此无法保证 A_1, A_2 至少有一个不可约）。如果 (A_1,x) 不可约，则继续对其分解：$(A_1,x) = (A_2,x) \cup (A_2',x)$，然后再考察 (A_2,x) 的可约性，不断做下去，总会得到不可约的 (A_k,x)；若不然就有解析集芽降链

$$(A_1,x) \supsetneqq (A_2,x) \supsetneqq (A_3,x) \supsetneqq \cdots$$

与引理1.5.8矛盾。因此必存在 $k>0$，使得 (A_k,x) 不可约，此时

$$(A,x) = (A_k,x) \cup \left(\bigcup_{j=1}^{k} (A_j',x) \right)$$

为所希望的分解，断言证毕。

反复使用此断言：令 $(A,x) = (A^{(1)},x) \cup (B_1,x)$，其中 $(A^{(1)},x)$ 不可约，若 (B_1,x) 可约，则再对 (B_1,x) 使用此断言：$(B_1,x) = (A^{(2)},x) \cup (B_2,x)$，其中 $(A^{(2)},x)$ 不可约；若 (B_2,x) 可约，则再继续对 (B_2,x) 使用断言……该操作必在有限步停止，停止于某个 $(B_{\widetilde{N}},x)$ 不可约，否则就有解析集芽降链

$$(B_1,x) \supsetneqq (B_2,x) \supsetneqq (B_3,x) \cdots$$

与引理1.5.8矛盾。从而得到不可约分解

$$(A,x) = (B_{\widetilde{N}},x) \cup \left(\bigcup_{k=1}^{\widetilde{N}} (A_k,x) \right)$$

之后适当取 $\{A_1, A_2, ..., A_{\widetilde{N}}; B_{\widetilde{N}}\}$ 的子集使得其中元素之并仍是 (A,x) 并且其中元素互不包含。因此存在性证毕。

唯一性：假设

$$(A,x) = \bigcup_{k=1}^{N} (A_k,x) = \bigcup_{k=1}^{N'} (A_k',x)$$

都为 (A,x) 的满足题设的不可约分解，则需要证明 $N=N'$，并且有集合相等

$$\{A_1,A_2,...,A_N\}=\{A'_1,A'_2,...,A'_{N'}\}$$

对任意 A_i，因为

$$(A_i,x)=\bigcup_{k=1}^{N'}(A_i\cap A'_k,x)$$

从而 (A_i,x) 的不可约性迫使存在某个 (A'_j,x) 使得 $(A_i,x)=(A_i\cap A'_j,x)$，即 $(A_i,x)\subseteq(A'_j,x)$. 同理，对于此 (A'_j,x)，存在某个 $(A_{i'},x)$，使得 $(A'_j,x)\subseteq(A_{i'},x)$，因此

$$(A_i,x)\subseteq(A'_j,x)\subseteq(A_{i'},x)$$

但由于 $\{(A_k,x)\}_{k=1}^{N}$ 中任何两元素互不包含，因此上式等号成立。也就是说对任意 $1\leq j\leq N$，存在（唯一）$1\leq j'\leq N'$，使得 $(A_j,x)=(A'_{j'},x)$；同理对任意 $1\leq j'\leq N'$ 也有类似结果。这就给出了集合一一对应

$$\{A_1,A_2,...,A_N\}\cong\{A'_1,A'_2,...,A'_{N'}\}$$

从而证毕。 □

注记 1.5.10. 此定理表明，欲研究解析集芽的局部性态，只需要研究不可约解析集芽；一般的解析集芽无非是不可约解析集芽的有限并。

现在，考虑 $\mathcal{O}_n:=\mathcal{O}_{\mathbb{C}^n,0}$ 的素理想 $\mathfrak{p}$，我们研究解析集芽 $(V(\mathfrak{p}),0)$ 的性质。

记号 1.5.11. 给定 $\mathbb{C}^n$ 的一组基 $\{e_1,e_2,...,e_n\}$，关于此基的坐标函数记作 $z_1,z_2,...,z_n$，对 $1\leq k\leq n$，记

$$\mathbb{C}\{z_1,...,z_k\}:=\{f\in\mathcal{O}_n|\frac{\partial f}{\partial z_l}\equiv 0\ ,\forall k+1\leq l\leq n\}$$

为 $\mathcal{O}_n$ 中“只显含前 k 个变量的函数芽”，则明显有

$$\mathcal{O}_k\cong\mathbb{C}\{z_1,...,z_k\}\hookrightarrow\mathcal{O}_n$$

于是对于 $\mathcal{O}_n$ 的素理想 $\mathfrak{p}$，

$$\mathfrak{p}_k:=\mathfrak{p}\cap\mathbb{C}\{z_1,...,z_k\}$$

为子环 $\mathcal{O}_k\cong\mathbb{C}\{z_1,...,z_k\}$ 的素理想。

引理 1.5.12. 对于环 $\mathcal{O}_n$ 的素理想 $\mathfrak{p}$，则存在 $\mathbb{C}_n$ 的一组基 $\{f_1, f_2, ..., f_n\}$，（记在该基下的坐标函数为 $w_1, w_2, ..., w_n$）以及存在 $0 \le d \le n$，使得

$$\mathfrak{p}_d := \mathfrak{p} \cap \mathbb{C}\{w_1, w_2, ..., w_d\} = 0$$

并且对任意 $d + 1 \le k \le n$，$\mathfrak{p}_k$ 当中存在 *Weierestrass* 多项式

$$P_k(\widetilde{w}_k, w_k) = w_k^{s_k} + \sum_{j=1}^{s_k} a_{jk}(\widetilde{w}_k) w_k^{s_k - j}$$

其中 $\widetilde{w}_k := (w_1, w_2, ..., w_{k-1}) \in \mathbb{C}^{k-1}$.

证明. 对 n 归纳，$n = 1$ 时平凡。

Step1对于 $n \ge 2$，先给定 $\mathbb{C}^n$ 的一组基 $\{e_1, ..., e_n\}$ 并记坐标函数为 $z_1, z_2, ..., z_n$，如果 $\mathfrak{p} = \{0\}$，则仍取这组基，并取 $d = n$ 即可。若 $\mathfrak{p} \ne 0$，则任取 $0 \ne g_n \in \mathfrak{p}$，注意 $g_n(0) = 0$；取 $\mathbb{C}^n$ 中的非零向量 f_n，使得定义在 $0 \in \mathbb{C}$ 附近的函数

$$t \mapsto g_n(t f_n)$$

在 $t = 0$ 处的零点阶数最低，记为 s_n. 注意满足如此性质的向量 f_n 在 $\mathbb{C}^n$ 中是稠密的（只需要使得 g_n 沿 f_n 方向的 s_n 阶方向导数非零），从而不妨取 f_n 充分接近基向量 e_n，使得 $\{e_1, e_2, ..., e_{n-1}; f_n\}$ 仍是 $\mathbb{C}^n$ 的一组基。

Step2现在考虑基 $\{e_1, e_2, ..., e_{n-1}; f_n\}$，该基下的坐标记为 $z'_1, z'_2, ..., z'_n$，则由 Weierstrass 预备定理，注意 $z'_n = 0$ 是函数 $z'_n \mapsto g_n(0, z'_n)$ 的 s_n 阶零点，则由 Weierstrass 预备定理，存在 Weierstrass 多项式

$$P_n(\widetilde{z}'_n, z'_n) = (z'_n)^{s_n} + \sum_{j=1}^{s_n} a_{jn}(\widetilde{z}'_n)(z'_n)^{s_k - j}$$

以及 $h \in \mathcal{O}_n$ 使得 $h(0) \ne 0$，以及 $g_n = P_n h$. （其中 $\widetilde{z}'_n = (z'_1, ..., z'_{n-1}) \in \mathbb{C}^{n-1}$）由于 h 在 $\mathcal{O}_n$ 中可逆，所以 Weierstrass 多项式 $P_n \in \mathfrak{p} = \mathfrak{p}_n$.

Step3如果 $\mathfrak{p}_{n-1} := \mathfrak{p} \cap \mathbb{C}\{z'_1, z'_2, ..., z'_{n-1}\} = 0$，则取 $\mathbb{C}^n$ 的基 $\{e_1, ..., e_{n-1}; f_n\}$，以及 $d = n - 1$ 即可。如果 $\mathfrak{p}_{n-1} \ne 0$，则 $\mathfrak{p}_{n-1}$ 为子环 $\mathcal{O}_{n-1} \cong \mathbb{C}\{z'_1, ..., z'_{n-1}\}$ 的素理想，之后对 $\mathbb{C}^{n-1} \cong \mathrm{span}_{\mathbb{C}}\{e_1, e_2, ..., e_{n-1}\}$ 以及 $\mathfrak{p}_{n-1}$ 使用归纳假设即可。 □

注记 1.5.13. 容易知道，对事先任意给定的 $\mathbb{C}^n$ 的基 $\{e_1, e_2, ..., e_n\}$，上述引理中的基 $\{f_1, f_2, ..., f_n\}$ 可以适当选取使得与 $\{e_1, e_2, ..., e_n\}$ 任意接近。

（这个引理证明过程中，哪里利用了“素理想”？）

本节有坑待填，尚未完成。笔者打算完整证明如下：

定理 1.5.14. （局部解析零点定理）

设 I 为 $\mathcal{O}_n$ 的理想，则

$$J_{(V(J),x)} = \sqrt{J}$$

回顾 $\sqrt{J} := \left\{ f \in \mathcal{O}_n \middle| \exists N \geq 0, f^n \in J \right\}$ 为 J 的**根式理想**。交换代数当中有以下基本结果：

$$\sqrt{J} = \bigcap_{\substack{\mathfrak{p} \supseteq J \\ \mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_n)}} \mathfrak{p}$$

证明大意. $J_{(V(J),x)} \supseteq \sqrt{J}$ 是容易验证的，而另一边“$\subseteq$”，由交换代数，只需对 $J = \mathfrak{p}$ 为素理想的情形证明。

这是非常不显然的结果，需要利用引理1.5.12 等多复变函数的结果，以及较多的交换代数。从略。 □

（这里待完善）

1.6 局部参数化

本节陈述关于不可约解析集芽的如下重要定理

定理 1.6.1. （不可约解析集芽的局部参数化定理）

设 $\mathfrak{p}$ 为环 $\mathcal{O}_n$ 的素理想，任取解析集 A 为解析集芽 $(V(\mathfrak{p}),0)$ 的代表元，则：存在 $\mathbb{C}^n$ 的基 $\{e_1, e_2, ..., e_n\}$（该基下的坐标函数记为 $z_1, z_2, ..., z_n$），存在 $1 \leq d \leq n$，以及存在足够小的正实数 $r', r'' > 0$，以及常数 $C > 0$，使得：

（*1*）$\mathfrak{p} \cap \mathbb{C}\{z_1, ..., z_d\} = 0$，并且环同态

$$\mathbb{C}\{z_1, ..., z_d\} \hookrightarrow \mathcal{O}_n/\mathfrak{p}$$

为**有限整扩张**。

（*2*） 在坐标 $z' = (z_1, ..., z_d), z'' = (z_{d+1}, ..., z_n)$ 下，

$$A \cap (\triangle' \times \triangle'') \subseteq \left\{ (z', z'') \in \mathbb{C}^d \times \mathbb{C}^{n-d} \middle| |z''| \leq C|z'| \right\}$$

其中 $\triangle'$ 为 $\mathbb{C}^d$ 中以原点为中心，半径 r' 的多圆柱；$\triangle''$ 为 $\mathbb{C}^{n-d}$ 中以原点为中心，半径 r'' 的多圆柱。

（3） 记 q 为 $\mathbb{C}\{z_1,...,z_d\} \hookrightarrow \mathcal{O}_n/\mathfrak{p}$ 的扩张次数，则投影映射

$$\begin{aligned} \pi : A \cap (\triangle' \times \triangle'') &\to \triangle' \\ (z', z'') &\mapsto z' \end{aligned}$$

为次数为 q 的**分歧映射**（*ramified map*），并且存在某个 $\delta \in \mathcal{O}_d$，使得 π 的所有**分歧值**都位于集合

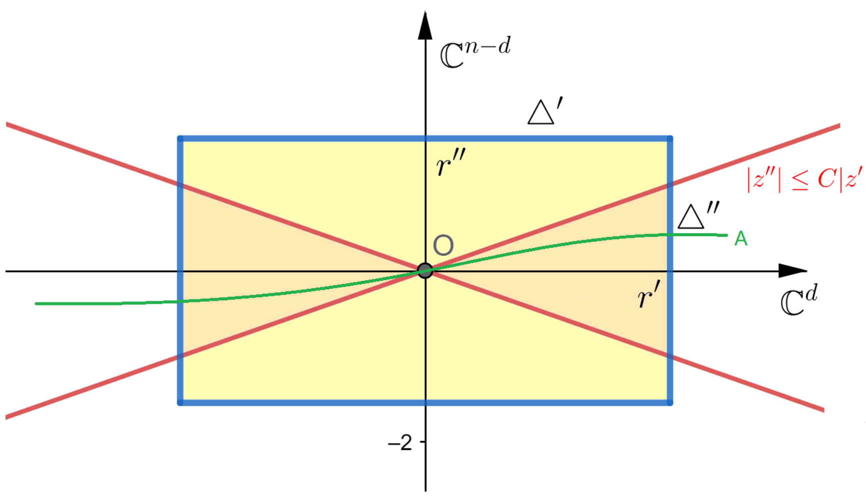
$$S := \left\{ z' \in \triangle' \middle| \delta(z') = 0 \right\}$$

之中，并且 $\triangle' \setminus S$ 为 $\triangle'$ 的连通、稠密子集。

第（3）条的“分歧映射”、“分歧值”具体指：投影

$$\begin{aligned} \pi' : A \cap \left[(\triangle' \setminus S) \times \triangle'' \right] &\to \triangle' \\ (z', z'') &\mapsto z' \end{aligned}$$

为 q 叶覆盖映射，并且对任意 $z' \in S$，$\#\pi^{-1}(z') \leq q$.



图：性质1.6.1示意

证明. 异常复杂，从略。承认之。 □

不过我们可以考虑一种简单的特殊情形——$\mathfrak{p}$ 为主理想：

例子 1.6.2. （超曲面的参数化）

设 $\mathcal{O}_n$ 的素理想 $\mathfrak{p} = (f)$ 为主理想，证明此种情形的局部参数化定理。

证明. 由 Weierstrass 预备定理，不妨取 $\mathfrak{p}$ 的生成元 f 为 weierstrass 多项式

$$f(\widetilde{z},z_n)=z_n^q+\sum_{j=1}^{q}a_j(\widetilde{z})z_n^{s-j}=\prod_{j=1}^{q}(z_n-w_j(\widetilde{z}))$$

其中 $\widetilde{z}=(z_1,z_2,...,z_{n-1})\in\mathbb{C}^{n-1}$，$w_j(\widetilde{z})$ 为多项式 $z_n\mapsto f(\widetilde{z},z_n)$ 的根。取 $d=n-1$，显然

$$\mathfrak{p}\cap\mathbb{C}\{z_1,z_2,...,z_d\}=0$$

现在对任意 $F\in\mathcal{O}_n$，对 F 以及 Weierstrass 多项式 f 使用 Weierstrass 除法定理，有 $F=hf+R$，其中 $R\in\mathcal{O}_{n-1}[z_n]$ 并且次数 $<q$. 这表明 $\widetilde{F}\in\mathcal{O}_n/\mathfrak{p}$ 为有限生成 $\mathcal{O}_d=\mathcal{O}_{n-1}$-模，并且 $\{1,z_n,z_n^2,...,z_n^{q-1}\}$ 为其一组 $\mathcal{O}_d$-模生成元。因此

$$\mathcal{O}_d\hookrightarrow\mathcal{O}_n/\mathfrak{p}$$

为有限整扩张。从而定理1.6.1的（1）证毕。

而（3）几乎显然，取

$$S:=\left\{\widetilde{z}\in\triangle'\middle|\text{多项式 } z_n\mapsto f(\widetilde{z},z_n)\text{ 无重根}\right\}$$

即可。利用代数学中关于重根的判别式，容易知道 S 为某个 $\mathcal{O}_d$ 中的函数（芽）的零点集。从而（3）易证。

至于（2），常数 C 的存在性显然吗？如果有对 f 的根的估计

$$w_j(\widetilde{z})=O(|\widetilde{z}|)$$

那么就没问题。（待补） □

1.7 正则点、奇异点，全纯隐函数定理

（待补）

第 2 章 层与层上同调

2.1 层的上同调

Today:

Sheaf cohomology

X a topological space, $\mathcal{F}$- sheaf (of abelian groups).

定义 2.1.1. *(resolution)*

(1)a resolution of $\mathcal{F}$ is an exact sequence

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F} \xrightarrow{d^0} \mathcal{F} \xrightarrow{d^1} \to \cdots$$

定义 2.1.2. *A sheaf $\mathcal{A}$ is called injective, if if for any injective morphism $j : \mathcal{A} \to \mathcal{B}$ and for any morphism $\varphi : \mathcal{A} \to \mathcal{S}$, there exists an extension $\psi : \mathcal{B} \to \mathcal{S}$,such that*

定理 2.1.3. *the category of sheaves of abelian sheaves have enough injective objects, i.e. any $\mathcal{F}$ can be embedded in some injective sheaf.*

定义 2.1.4. *Consider an injective resolution of $\mathcal{F}$, i.e. an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \mathcal{I}^2 \to \cdots$$

where every $\mathcal{I}^k (k \geq 0)$ is injective.

$\rightsquigarrow$*induces a sequence*

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}^0) \xrightarrow{\mathrm{d}} \Gamma(X, \mathcal{I}^1) \xrightarrow{\mathrm{d}} \Gamma(X, \mathcal{I}^2) \to \cdots$$

Then

$$H^q(X, \mathcal{F}) := H^q(\Gamma(X, \mathcal{I}^\bullet))$$

then, $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

定义 2.1.5. *A sheaf $\mathcal{S}$ is called a flabby (flasque ,in France) ,if for any open set $\Omega \subseteq X$, the morphism*

$$\mathcal{S}(X) \to \mathcal{S}(\Omega)$$

is surjective.

定义 2.1.6.

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \xrightarrow{d^0} \to \mathcal{F}^1$$

is an exact sequence is called a flabby resolution, if any $\mathcal{F}^k$ is flabby.

定义 2.1.7.

$$H^q(X, \mathcal{F}) := \text{...by flabby resolution...}$$

证明. Homological Algebra...omit. □

the two definitions of Sheaf Cohomology are isomorphic.

Godement's construction

$$God(\mathcal{F})(U) := \{f : U \to \bigcup_{x \in U} \mathcal{F}_x | f(y) \in \mathcal{F}_y, \forall y \in U\} := \prod_{x \in U} \mathcal{F}_x$$

$God(\mathcal{F})$ is a sheaf, and it is flabby. and there is a canonical morphism $\mathcal{F}(U) \to God(F)(U)$ by $x \mapsto (x \mapsto s_x)$ is injective.

$$\mathcal{F}^0 := God(\mathcal{F})$$

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \twoheadrightarrow \mathrm{coker}(j) = \mathcal{F}^0 / \mathcal{F}$$

and consider

$$\mathcal{F}^1 := God(\mathrm{coker}(j))$$

......then construct by induction... this is a flabby resolution of $\mathcal{F}$.

定义 2.1.8. *(resolution by fine sheaves)*

$\mathcal{A}$ is a sheaf of ring, X is a paracompact topological space, $\mathcal{A}$ is called a fine sheaf, if for any open covering

$$X = \bigcup_\alpha V_\alpha \quad , \mathcal{V} := \{V_\alpha\}$$

there exists a partition of unit subordinate to $\mathcal{V}$, (i.e. $\exists f_\alpha \in \mathcal{A}(V_\alpha), supp(\alpha) := \overline{\{x \in V_\alpha | f_{\alpha,x} \neq 0\}} \subseteq V_\alpha$, and $\sum_\alpha f_\alpha = 1$(the sum is locally finite))

例子 2.1.9. *X is a differential manifold, $\mathcal{C}^\infty$ is the sheaf of smooth functions, then $\mathcal{C}^\infty$ is a fine sheaf.*

定理 2.1.10. *$\mathcal{S}$ is a sheaf of $\mathcal{A}$-modules, $\mathcal{A}$ is a fine sheaf. then for any $q \geq 1$,*

$$H^q(X, \mathcal{S}) = 0$$

证明. Consider a flabby(or injective) resolution

$$0 \to \mathcal{S} \xrightarrow{j} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \mathcal{I}^2 \cdots$$

where any $\mathcal{I}^k (k \geq 0)$ is a sheaf of $\mathcal{A}$-modules.

by definition,

$$H^q(X, m\mathcal{S}) := \frac{\ker \mathrm{d} : \Gamma(\mathcal{I}^q) \to \Gamma(\mathcal{I}^{q+1})}{\Im \mathrm{d} : \Gamma(\mathcal{I}^{q-1}) \to \Gamma(\mathcal{I}^q)}$$

Let $\alpha \in \ker\{\mathrm{d} : \Gamma(\mathcal{I}^q) \to \Gamma(\mathcal{I}^{q+1})\}$ by the exactness of resolution, $\exists$ an open covering $\mathcal{U} = (U_i)_i$, s.t. $\alpha|_{U_i} = \mathrm{d}\beta_i$ where $\beta_i \in \mathcal{T}^{q-1}(U_i)$. Let $(\beta_i)_i$ be the partition of unit w.r.t. $\mathcal{U}$. consider

$$\beta := \sum_i f_i \beta_i$$

(well defined). Then $\mathrm{d}\beta = \alpha$.... □

2.2 Čech 上同调

Čech cohomology

X- a topological space, $\mathcal{F}$- a sheaf of abelian group.

$$\mathcal{U} = (U_\alpha)_{\alpha \in I}$$

is an open covering.

notation:$U_{\alpha_1,\dots,\alpha_q} := \bigcap_{i=1}^q U_{\alpha_i}$.

Čech q-chain w.r.t $\mathcal{U}$:

$$C^q(\mathcal{U},\mathcal{F}) := \prod_{(\alpha_1,\dots,\alpha_q)\in \mathcal{I}^{q+1}} \mathcal{F}(U_{\alpha_1,\dots,\alpha_q})$$

$$c \in C^q(\mathcal{U},\mathcal{F})$$

means that we have a family of sections $C_{\alpha_1,\dots,\alpha_q} \in \mathcal{F}(U_{\alpha_1,\dots,\alpha_q})$ with the relation

$$C_{\alpha_0,\dots,\alpha_j,\dots,\alpha_i,\dots} = -C_{\dots}$$

(C)ech differential:

$$\delta^q : C^q(\mathcal{U},\mathcal{F}) \to C^{q+1}(\mathcal{U},\mathcal{F})$$

$$\delta^q(c)_{\alpha_0,\dots,\alpha_{q+1}} := \sum_{0\le k\le q+1} (-1)^k c_{\dots\hat{\alpha_k}\dots}|_{U_{\alpha_0,\dots,\alpha_{q+1}}}$$

性质 2.2.1.

$$\delta^q \circ \delta^q = 0$$

so, we have Čech cohomology

$$H^q(\mathcal{U},\mathcal{F}) := \ker\delta^q / \operatorname{Im}\delta^{q-1}$$

example:

$$C^0(\mathcal{U},\mathcal{F}) := \prod_{\alpha\in I} \mathcal{F}(U_\alpha)$$

$$c = (c_\alpha)_{\alpha\in I} \in C^0(\mathcal{U},\mathcal{F})$$

$$\delta^0 c = 0 \iff (\delta^0 c)_{\alpha_0\alpha_1} := (c_{\alpha_1} - c_{\alpha_0})|_{U_{\alpha_0\alpha_1}} = 0$$

so, $c_{\alpha_0} = c_{\alpha_1}$ on $U_{\alpha_0\alpha_1}$.

$\rightsquigarrow H^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(X)$.

例子 **2.2.2.** *(1) consider* $X = \triangle \setminus \{0\}$*, where* $\triangle = \{(z_1, z_2) | |z_1| < 1, |z_2| < 1\}$*. Consider the covering*

$$\mathcal{U} = U_1 \cup U_2$$

where

$$U_1 := \{(z_1, z_2) \in \triangle | z_1 \neq 0\} = \mathbb{D}^* \times \mathbb{D}$$

$$U_2 := \{(z_1, z_2) \in \triangle | z_2 \neq 0\} = \mathbb{D} \times \mathbb{D}^*$$

then

$$U_1 \cap U_2 = \mathbb{D}^* \times \mathbb{D}^*$$

consider $H^0(X, \mathcal{O}) = \mathcal{O}(X) \cong \mathcal{O}(\triangle) = \{f : \triangle \to \mathbb{C} holomorphic\}$*.*

$$H^1(\mathcal{U}, \mathcal{O}) = \ker \delta^1 / \operatorname{Im} \delta^0$$

$$\delta^1 : C^1(\mathcal{U}, \mathcal{O}) \to C^2(\mathcal{U}, \mathcal{O}) \subseteq \prod_{\alpha_0, \alpha_1, \alpha_2} \mathcal{O}(U_{\alpha_0, \alpha_1, \alpha_2}) = 0$$

$$\ker \delta^1 = C^1(\mathcal{U}, \mathcal{O}) = \{c = c(\alpha_0, \alpha_1) | c_{\alpha_0, \alpha_1} \in \mathcal{O}(U_{\alpha_0 \alpha_1})\} = \{c \in \mathcal{O}(U_1 \cap U_2)\} = \{c = \sum_{m,n \in \mathbb{Z}} a_{mn} z_1^m z_2^n \, convergent\}$$

$$\delta^0 : C^0(\mathcal{U}, \mathcal{O}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{O})$$

$$(\delta^0 c)_{12} = (c_2 - c_1)|_{U_{12}}$$

where $c_2 \in \mathcal{O}(U_2)$ *and* $c_1 \in \mathcal{O}(U_1)$*. note that*

$$\mathcal{O}(U_1) = \{c(z_1, z_2) = \sum_{m \in \mathbb{Z}, n \geq 0} a_{mn} z_1^m z_2^n \, convergent\}$$

$$\mathcal{O}(U_2) = \{c(z_1, z_2) = \sum_{n \in \mathbb{Z}, m \geq 0} a_{mn} z_1^m z_2^n \, convergent\}$$

So, $H^1(\mathcal{U}, \mathcal{O}) = \{c(z_1, z_2) = \sum_{m,n<0} a_{mn} z_1^m z_2^n\}$

例子 **2.2.3.** *(complex projective space)*

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

$$(z_0, ..., z_n) \sim \lambda(z_0, ..., z_n)$$

for some $\lambda \in \mathbb{C}^*$*.*

$$\mathbb{C}P^n = \{[z_0, ..., z_n] | not \ all \ z_k = 0, z_i \in \mathbb{C}\} = \bigcup_{0 \leq p \leq n} V_k$$

where

$$V_k = \{[z_0, ..., z_n] | z_k \neq 0\} \cong \{(\frac{z_0}{z_k}, ..., 1, ..., \frac{z_n}{z_k}) | z_i \in \mathbb{C}, i \neq k, z_k \neq 0\} \cong \mathbb{C}^n$$

this is a holo chart.

$$\mathbb{C}P^1 = V_0 \cup V_1, \mathcal{V} = \{V_0, \mathcal{V}_1\}$$

HW: compute $H^q(\mathcal{V}, \mathcal{O})$.

Answer:

$$H^0 \cong \mathbb{C}, H^1 \cong 0$$

Correction:

$\mathcal{A}$: Sheaf of rings (with unit)

X: paracompact topological space,

定义 2.2.4. *$\mathcal{A}$ is called fine, if for any open covering $\mathcal{U} = (V_\alpha)_{\alpha \in \mathcal{I}}$,there exist $s_\alpha \in \mathcal{A}(X)$ such that such that $supp(s_\alpha) \subseteq V_\alpha$,*

$$\sum_\alpha s_\alpha = 1$$

(this is a locally finite sum)

注记 2.2.5. *we call $\mathcal{A}$ is a **soft sheaf**, if for any closed set $K \subseteq X$, the morphism*

$$\mathcal{A}(X) \to \mathcal{A}(K)$$

is surjective. where $\mathcal{A}(K) := \Gamma(K, \mathcal{A}|_K)$

fact: $\mathcal{A}$ is fine if and only if $\mathcal{H}om(\mathcal{A}, \mathcal{A})$ is soft. (omit)

Recall:

Cech cohomology: X topological space, $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$,

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 < \ldots < \alpha_q} \mathcal{F}(\alpha_1, \ldots, \alpha_q)$$

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$$

fact: $H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Today:

定义 2.2.6. *Let $\mathcal{V} = (V_\beta)_{\beta \in J}$ be another open covering, then $\mathcal{V}$ is called a refinement of $\mathcal{U}$, if there exists a map*

$$\rho : \mathcal{J} \to \mathcal{I}$$

such that

$$V_\beta \subseteq U_{\rho(\beta)}$$

性质 2.2.7. *Let $\mathcal{V}$ be a refinement of $\mathcal{U}$, then ρ induces a map*

$$\rho^q : C^q(\mathcal{U},\mathcal{F}) \to C^q(\mathcal{V},\mathcal{F})$$

$$(\rho^q C)_{\beta_0,\ldots,\beta_q} \mapsto C_{\rho(\beta_0),\ldots,\rho(\beta_q)}|_{V_{\beta_0,\ldots,\beta_q}}$$

ρ is a morphism of complexes.

so, ρ induces a map

$$H^q(\rho) : H^q(\mathcal{U},\mathcal{F}) \to H^q(\mathcal{V},\mathcal{F})$$

Let $\tilde{\rho} : \mathcal{J} \to \mathcal{I}$ be another refinement of $\mathcal{U}$

(induces $H^q(\tilde{\rho}) : H^q(\mathcal{U},\mathcal{F}) \to H^q(\mathcal{V},\mathcal{F})$) then $\rho, \tilde{\rho}$ are homotopic (chain homotopy$\rightsquigarrow H^q(\rho) = H^q(\tilde{(\rho)})$)

so, if $\rho : \mathcal{J} \to \mathcal{I}$ is refinement, then

$$H^q(\rho)$$

is independent of the refinement.

定义 2.2.8.

$$\check{H}^q(X,\mathcal{F}) := \varinjlim_{\mathcal{U}} H^q(\mathcal{U},\mathcal{F})$$

i.e. $a \in H^q(\mathcal{U},\mathcal{F}) \sim \in H^q(\mathcal{V},\mathcal{F})$ iff $\exists$ a refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$ such that a, b have the same image in $H^q(\mathcal{W},\mathcal{F})$

注记 2.2.9.

$$\check{H}^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$$

Exercise: For $q = 1$, if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then

$$H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F})$$

is injective.

so ,for any open cover $\mathcal{U}$,

$$H^1(\mathcal{U},\mathcal{F}) \to \check{H}^1(X,\mathcal{F})$$

is injective.

Homological Algebra recall: let $(K^\bullet, \mathrm{d}_k), (L^\bullet, \mathrm{d}_l)$ and $(M^\bullet, \mathrm{d}_M)$, if we have a short exact sequence

$$0 \to K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} M^\bullet \to 0$$

then it induces a long exact sequence :

$$\cdots \to H^q(K^\bullet) \to H^q(L^\bullet) \to H^q(M^\bullet) \to H^{q+1}(K^\bullet) \to \cdots$$

analogy of Cech cohomology: X is a topological space, $\mathcal{U}$ is an open covering of X. $\mathcal{A}$ and $\mathcal{B}$ sheaves on X, Let

$$\varphi : \mathcal{A} \to \mathcal{B}$$

be a morphism, then it induces

$$\varphi^\bullet : C^\bullet(\mathcal{U}, \mathcal{A}) \to C^\bullet(\mathcal{U}, \mathcal{B})$$

Let

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

be an exact sequence of sheaves, then we have: for any open set Ω,

$$0 \to \mathcal{A}(\Omega) \to \mathcal{B}(\Omega) \to \mathcal{C}(\Omega)$$

left exact.

Example: consider

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{exp} \to 0$$

is exact on $bbC^\times := \mathbb{C} \setminus \{0\}$

but we have :

$$0 \to \mathcal{A}(\Omega) \xrightarrow{\psi} \mathcal{B}(\Omega) \to \operatorname{Im} \psi(\Omega) \to 0$$

is exact.

First we have the following exact sequence

$$C^q(\mathcal{U}, \mathcal{A}) \to C^q(\mathcal{U}, \mathcal{B}) \to C^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \to 0$$

where $C^q_{\mathcal{B}}$ is the image of ...

then we get an exact sequence

$$0 \to (C^\bullet(\mathcal{U}, \mathcal{A}), \delta) \to (C^\bullet(\mathcal{U}, \mathcal{B}), \delta) \to (C^\bullet_{\mathcal{B}}(\mathcal{U}, \mathcal{C}), \delta) \to 0$$

it induces a long exact sequence

$$\cdots \to H^q(\mathcal{U}, \mathcal{A}) \to H^q(\mathcal{U}, \mathcal{B}) \to H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \to H^{q+1}(\mathcal{U}, \mathcal{A}) \to \cdots$$

定理 2.2.10. *If X is paracompact,*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a sheaf exact sequence. Then there is a long exact sequence

$$\cdots \to \check{H}^q(X, \mathcal{A}) \to \check{H}^q(X, \mathcal{B}) \to \check{H}^q(X, \mathcal{C}) \to \check{H}^{q+1}(X, \mathcal{Z}) \to \cdots$$

证明. Key lemma: need to prove

$$\lim_{\overrightarrow{\mathcal{U}}} H^q(\mathcal{U}, \mathcal{C}) = \lim_{\overrightarrow{\mathcal{U}}} H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C})$$

if X is paracompact.

Omit. □

if

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

exact,

recall:(cohomology by resolutions)

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$

flabby resolution. then it induces

$$0 \to \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \cdots$$

then define the sheaf cohomology...

we have a long exact sequence

$$\cdots \to H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B}) \to H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A}) \to \cdots$$

it is homological algebra...

定理 2.2.11. *(Leray's acyclic theorem) Let $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$ be an open covering of X, ($\mathcal{F}$ is a sheaf on X), if satisfying*

$$H^k(U_{\alpha_0, \ldots, \alpha_q}) = 0$$

for any $k \geq 1$,then

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{(H)}^q(X, \mathcal{F})$$

and if X is paracompact ,we also have

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{(H)}^q(X, \mathcal{F}) \cong H^q(X, \mathcal{F})$$

(this $\mathcal{U}$ is called acyclic covering)

de Rham- Weil theorem

定义 2.2.12. *$\mathcal{F}$ is a sheaf on X, Ω is an open set of X, then $\mathcal{F}$ is called* ***acyclic sheaf*** *if*

$$H^q(\Omega, \mathcal{F}) = 0$$

for any $q \geq 1$.

定理 2.2.13. *Let*

$$0 \to \mathcal{F} \to (L^\bullet, \mathrm{d})$$

be an acyclic resolution of $\mathcal{F}$ (i.e. L^q is acyclic on X) then

$$H^q(X, \mathcal{F}) \cong H^q(\Gamma(X, L^\bullet), \mathrm{d})$$

for any $q \geq 0$.

(先看例子)

例子 2.2.14. *Let X be a differential manifold, $\mathcal{E}^p$:sheaf of smooth p-forms, then we have a resolution (de Rham complex)*

$$0 \to \mathbb{R} \hookrightarrow \mathcal{E}^0 \xrightarrow{\mathrm{d}} \mathcal{E}^1 \xrightarrow{\mathrm{d}} \mathcal{E}^2 \xrightarrow{\mathrm{d}} \mathcal{E}^3 \to \cdots$$

where d differential operators. (Why it is a resolution? because of Poincare lemma...locally solvable..)

Note that

$$\mathcal{E}^0 = \mathcal{C}^\infty$$

$\mathcal{E}^p$ is a sheaf of C^∞-modules..

then we have

$$H^q(X, \mathcal{E}^p) = 0$$

for all $q \geq 1$

and then

$$H^q(X, \mathbb{R}) \cong \frac{\ker(\mathrm{d}: \Gamma(X, \mathcal{E}^q) \to \Gamma(X, \mathcal{E}^{q+1}))}{\mathrm{Im}(\mathrm{d}: \Gamma(X, \mathcal{E}^{q-1}) \to \Gamma(X, \mathcal{E}^q))} = H^q_{DR}(X, \mathcal{R})$$

例子 2.2.15. *Let X be a complex manifold, $\mathcal{E}^{p,q}$ sheaf of smooth (p,q) forms, Ω^p is the sheaf of holomorphic p-forms (i.e. $(p,0)$-form φ with $\bar{\partial}\varphi = 0$).*

Then we have resolution

$$0 \to \Omega^p \xrightarrow{j} \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \to \cdots$$

(Why it is a resolution? because of the Dolbeault lemma),remain to Exercise...

$$H^q(X,\Omega^p) \cong H^{p,q}_{\bar\partial}(X,\mathbb{C})$$

Today：de Rham-Weil Isomorphism Thm

定理 2.2.16. *Let X be a topological space, $\mathcal{F}$ be a sheaf of abelian groups on X,*

$$0 \to \mathcal{F} \to (\mathcal{L}^\bullet, \mathrm{d})$$

be an acyclic resolution, i.e.

$$H^k(X,\mathcal{L}^q) = 0$$

for all $k \geq 1$ and $q \geq 0$. Then,

$$H^q(X,\mathcal{F}) \cong H^q((\Gamma(\mathcal{L}^\bullet),\mathrm{d}))$$

证明. Since

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{\mathrm{d}^0} \mathcal{L}^1 \xrightarrow{\mathrm{d}^1} \mathcal{L}^2 \to \cdots$$

be an exact sequence, denote

$$\mathcal{Z}^q := \ker \mathrm{d}^q$$

then we have short exact sequences

$$0 \to \mathcal{Z}^q \to \mathcal{L}^q \to \mathcal{Z}^{q+1} \to 0$$

for any q. They induce long exact sequence of cohomology groups:

$$\cdots \to H^k(X,\mathcal{Z}^q) \to H^k(X,\mathcal{L}^q) \to H^k(X,\mathcal{Z}^{q+1}) \xrightarrow{\partial} H^{k+1}(X,\mathcal{L}^q) \to H^{q+1}(X,\mathcal{L}^q) \to \cdots$$

For any $k \geq 1$, since $\mathcal{L}^q$ are acyclic on X,

$$H^k(X,\mathcal{Z}^{q+1}) \cong H^{k+1}(X,\mathcal{Z}^q)$$

and for $k=0$, we have

$$0 \to H^0(X,\mathcal{Z}^q) \to H^0(X,\mathcal{L}^q) \to H^0(X,\mathcal{Z}^{q+1}) \to H^1(X,\mathcal{Z}^q) \to H^1(X,\mathcal{L}^q) = 0 \to \cdots$$

so,

$$H^1(X,\mathcal{Z}^q) \cong H^0(X,\mathcal{Z}^{q+1}) \big/ \operatorname{Im} \mathrm{d}^q \cong H^{q+1}((\Gamma(\mathcal{L}^\bullet),\mathrm{d}))$$

$$H^{q+1}(\Gamma(\mathcal{L}^\bullet)) \cong H^1(X,\mathcal{Z}^q) \cong H^2(X,\mathcal{Z}^{q-1}) \cong \cdots H^{q+1}(X,\mathcal{Z}^0) = H^{q+1}(X,\mathcal{F})$$

□

$$0 \to \mathbb{R} \to \mathcal{E}^0 \xrightarrow{\mathrm{d}} \mathcal{E}^1 \xrightarrow{\mathrm{d}} \mathcal{E}^2 \to \cdots$$

(de Rham resolution) then we have

$$H^k(X, \mathcal{R}) \cong H^k_{DR}(X; \mathcal{R})$$

(if X is compact ,then by Hodge theory, it also isomorphic to $\ker(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})$)

Another example:X is a complex manifold, then

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \to \cdots$$

then

$$H^q(X, \Omega^p) \cong H^{p,q}_{\bar{\partial}}(X, \mathbb{C})$$

(RHS= Dolbeault cohomology)

X be a smooth manifold, we define

$$C_q(X, \mathbb{Z}) := \text{the free abelian group generated by continuous map}$$

$$\phi : \triangle_q := \{(t_1, ..., t_{q+1}) \in [0,1]^{q+1} | \sum_{i=1}^{n} t_i = 1\}$$

and we define (for $\phi \in C_q(X, \mathbb{Z})$)

$$\partial\phi := \sum_{i=1}^{q+1} (-1)^q \phi|_{\triangle_{q,i}}$$

$$\triangle_{q,i} := \{t \in \triangle_q | t_i = 0\}$$

we define

$$(C^\bullet_{sing}, \partial)$$

be the dual complex of $(C^{sing}_\bullet), \partial$.

(These are all Basic Algebraic Topology)

For any open $U \subseteq X$, we have

$$U \to C^q_{sing}(U, \mathbb{Z})$$

we get a sheaf

$$\mathcal{C}^q_{sing}$$

FACT: $(\mathcal{C}^\bullet_{sing}, \partial)$ is a flabby resolution of $\mathbb{Z}$. (check!)So,

$$H^q_{sing}(X, \mathbb{Z}) = H^q(\Gamma(\mathcal{C}^\bullet_{sing}), \partial) \cong H^q(X, \mathbb{Z})$$

第 3 章 Hermite 向量丛

3.1 联络与曲率

Recall: X is a smooth manifold, E is a vector bundle of rank r, if

(1)$\pi : E \to X$ is smooth map,

(2)for any $x \in X$, $E_x := \pi^{-1}(x)$ is a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of dimension r.

(3)there an open covering $\mathcal{U} = (\mathcal{U}_\alpha)_{\alpha \in I}$ and trivializations

$$\theta_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

and for any intersection $U_\alpha \cap U_\beta$, we have

注记 3.1.1.

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$$

(cocycle condition)

Special Case: line bundle rank E=1.

then $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, \mathbb{K}^*) = \mathcal{E}^*(U_{\alpha\beta})$ invertible smooth function on $U_{\alpha\beta}$. then, Cech cohomology,

$$(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = 1$$

so,

$$(g_{\alpha,\beta}) \in \mathcal{Z}^1(\mathcal{U}, \mathcal{E}^*) \twoheadrightarrow H^1(\mathcal{U}, \mathcal{E}^*) \hookrightarrow \check{H}^1(X, \mathcal{E}^*)$$

we get a map

$$\{\text{line bundles}\} \to \check{H}^1(X, \mathcal{E}^*)$$

actually, we have

$$\{\text{isomorphic classes of line bundles}\} \longleftrightarrow H^1(X, \mathcal{E}^*)$$

1-1 correspondence.

Now, X be a complex manifold, a complex vector bundle E is called homomorphic, if ... the transition matrix $g_{\alpha\beta}$ is holomorphic...

Holomorphic line bundles :

$$g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$$

$\mathcal{O}^*$:sheaf of invertible holomorphic functions...

FACT: there is a map

$$\{\text{holomorphic line bundle}\} \to \check{H}^1(X, \mathcal{O}^*)$$

例子 3.1.2. *trivial vector bundle* $X \times \mathbb{K}^r$

例子 3.1.3. *Tangent bundle* TX*. (transition matrix* $g_{\alpha\beta}$ *are given by Jacobi matrix..)*

定义 3.1.4. *(Local frame of vector bundles)*

$$\theta_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{K}^r$$

be a trivialization, we define

$$e_\lambda(x) := \theta_\alpha^{-1}(x, \begin{pmatrix} 0 \\ \dots \\ 1(\leftarrow ith) \\ \dots \\ 0 \end{pmatrix})$$

then, $\{e_1, ..., e_r\}$ *be a local smooth section* $s \in \Gamma(U_\alpha, E)$ *can be written as*

$$s(x) = \sum \sigma_\lambda(x)$$

where $\sigma_\lambda \in C^\infty(U_\alpha, \mathbb{K})$*.*

(Connection)

记号 3.1.5. *For* X *be a smooth manifold,* E *is a vector bundle(real or complex), denote*

$$C_p^k(\Omega, E) := C^k(\Omega, \bigwedge^p T^*M \otimes E)$$

is the space of k*-differential* p*-forms with values in* E*.*

Locally, consider a trivialization of E*,*

$$\theta_\alpha E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

($\rightsquigarrow$ *frame* $(e_1, ...e_r)$*)*

$$s \in \sum \varphi_\lambda(x) \otimes e_\lambda(x)$$

where φ_λ *is a* p*-form.*

定义 3.1.6. *a (linear) connection on E is a linear differential operator of order 1 acting on $C^\infty_\bullet(X,E)$:*

$$D: C^\infty_p(X,E) \to C^\infty_{p+1}(X,E)$$

$$D(f \wedge x) := \mathrm{d}f \wedge s + (-1)^p f \wedge Ds$$

where $f \in C^\infty(X, \bigwedge^p T^*M)$, $s \in C^\infty(X,E)$.

Locally, consider a local trivialization

$$\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

with a frame $\{e_1, ..., e_r\}$. any section $t \in C^\infty_p(\Omega, E)$ can be written as

$$t = \sum_{1 \le \lambda \le r} \sigma_\lambda \otimes e_\lambda$$

$$Ds = \sum_{\lambda=1}^{r} \mathrm{d}\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge De_\lambda$$

where

$$De_\lambda \in C^\infty_1(\Omega, E)$$

can be written as

$$De_\lambda = \sum_{\mu=1}^{r} a_{\mu\lambda} \otimes e_\mu$$

where "$a_{\mu\lambda}$" is called the coefficients of D with respect to frame $\{e_1, ..., e_r\}$.

so,

$$D(t) = \sum_{\lambda,\mu} \mathrm{d}\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge a_{\mu\lambda} \wedge e_\mu = \sum_\mu \sum_\lambda \left(\mathrm{d}\sigma_\mu + a_{\mu\lambda} \wedge \sigma_\lambda \right)$$

$$Dt = \mathrm{d}\sigma + A \wedge \sigma$$

where $A = (a_{\mu\lambda})$.

RMK: connection always exists!

Recall: for any (connected) smooth manifold, $E \to X$ is a smooth vector bundle,

Connection:

$$D: C^\infty_p(X,E) \to C^\infty_{p+1}(X,E)$$

where $C^\infty_p(X,E) := C^\infty(X, \wedge^p T^*M \otimes E)$

$$D(f \wedge s) = \mathrm{d}f \wedge s + (-1)^{\deg f} f \wedge Ds$$

Essentially,

$$D : C^\infty(X,E) \to C_1^\infty(X,E)$$

Locally, consider a trivialization $\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$, and a local frame $(e_1,...,e_r)$ where $e_k(x) = \theta^{-1}(x, \begin{pmatrix} 0 \\ \vdots \\ 1(k^{th}) \\ \vdots \\ 0 \end{pmatrix})$.

Let $s \in C^\infty(\Omega, E)$, i.e.

$$s = \sum_{i=1}^{r} \sigma_i e_i$$

where σ_i are smooth functions.

$$Ds = \mathrm{d}\sigma + A \wedge \sigma$$

where

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \end{pmatrix} \qquad A = a_{ij}$$

consider another trivialization

$$\tilde{\theta} : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

$\rightsquigarrow$ a local frame $(\tilde{e_1},...,\tilde{e_r})$. Then there exists a invertible linear transform s.t.

$$\tilde{e_k} = g_k^m e_m$$

assume

$$De_k = a_k^l e_l \qquad D\tilde{e_k} = \tilde{a}_k^l \tilde{e}_l$$

we have

$$\begin{aligned} \mathrm{d}g_k^n e_n + g_k^m a_m^n e_n &= \tilde{a}_k^l g_l^n e_n \\ \rightsquigarrow \quad \tilde{a}_k^l g_l^n (g^{-1})_n^p &= \mathrm{d}g_k^n (g^{-1})_n^p + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \quad \tilde{a}_l^p &= \mathrm{d}g_k^n (g^{-1})_n^p) + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \quad \tilde{A} &= \mathrm{d}g \cdot g^{-1} + g \cdot A \cdot g^{-1} \end{aligned}$$

Curvature

$$H_D := D^2$$

locally,

$$D^2 s = D(\mathrm{d}\sigma + A \wedge \sigma) = \mathrm{d}(\mathrm{d}\sigma + A \wedge \sigma) + A \wedge (\mathrm{d}\sigma + A \wedge \sigma)$$

$$= \mathrm{d}A \wedge \sigma - A \wedge \mathrm{d}\sigma + A \wedge \mathrm{d}\sigma + A \wedge A \wedge \sigma = (\mathrm{d}A + A \wedge A) \wedge \sigma$$

so we have

$$H = \mathrm{d}A + A \wedge A$$

Similarly to $\tilde{A}, A$ we have

Exercise:

$$\tilde{H} = gHg^{-1}$$

曲率在不同平凡化下的表达式。where

$$\tilde{e} = ge$$

$\rightsquigarrow$ H can be considered as a section of $C_2^\infty(X, \mathrm{Hom}(E, E))$. because

$$\tilde{H}\tilde{e} = gHg^{-1}\tilde{e} = gHe$$

independent of the choice of local frames.

3.2 向量丛的构造

定义 3.2.1. *(dual of vector bundles) $E \to X$, and $g_{\alpha\beta}$:transition matrix of E, the dual is given by $(g_{\alpha\beta})^{-1}$.* （用转移函数来定义向量丛）

定义 3.2.2. *direct sum of two vector bundles $(E, F) \to E \oplus F$. locally,*

$$(g_{\alpha,\beta}) \oplus (h_{\alpha\beta})$$

direct sum of transition matrices.

定义 3.2.3. *tensor product of two vector bundles.*

locally, tensor product of two transition matrices.

fact: let D_E be a connection on E,then it induces a connection D_{E^*}. Let u be a local section of E^*, s local section of E, then we define

$$\mathrm{d}\langle u, s\rangle = \langle D_{E^*}u, s\rangle + \langle u, D_E s\rangle$$

Exercise:

$$H(D_{E^*}) = -H(D_E)^T$$

and for two vector bundles E, F, connections D_E, D_F, then

$$D_{E\oplus F} := D_E \oplus D_F$$

$$H(E \oplus F) = H_E \oplus H_F$$

as for tensor product, we define $D_{E\otimes F}$ as follows:

$$D_{E\otimes F}(s \otimes t) = D_E s \otimes t + s \otimes D_F t$$

check the curvature

$$H_{E\otimes F} = H_E \otimes id_F + id_E \otimes H_F$$

注记 3.2.4. *we can also consider wedge product of vector bundles. Consider vector bundles $E_1, ..., E_k$, with connections $D_{E_1}, ..., D_{E_k}$, let $s_i \in C^\infty_{p_i}(X, E^i)$ then*

$$D_{E_1\wedge,...,\wedge E_k}(s_1 \wedge ... \wedge s_k) = \sum_{i=1}^{k} (-1)^{p_1+...+p_{i-1}} s_1 \wedge ... \wedge D_{E_i} s_i \wedge ... \wedge s_k$$

Let E be a vector bundle of rank r, then $\bigwedge^r E$ is a line bundle, with transition matrix by $\det(g_{\alpha\beta})$. this bundle is denoted by $\det E$.(Det-bundle)

Let $s_1, ..., s_r$ be local sections of E, then we have

$$D_{\det E}(s_1 \wedge \cdots \wedge s_r) = tr(H_E) s_1 \wedge \cdots \wedge s_r$$

3.3 陈省身示性类

chern classes (defined by curvature).

Let $E \to X$ be a smooth complex vector bundle of rank r, where X be a complex manifold.

(Chern-Weil theory)

V be a complex vector space, $f : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{C}$ be a symmetric multi-linear form of degree k.

$\rightsquigarrow f(v) := f(v, v, ..., v)$ is a homogeneous polynomial of degree k.

定义 3.3.1. *assume G is a group (left) acting on V, s.t.*

$$f(g(v_1), ..., g(v_k)) = f(v_1, ..., v_k)$$

for any $g \in G, v_i \in V$, then we say f is G-invariant.

Special case: $G = GL(r, \mathbb{C})$ and $V = LieG = \mathfrak{gl}r, \mathbb{C}$ be the Lie algebra of G. the action is

$$(g, M) \mapsto gMg^{-1}$$

Consider

$$\det(I + \frac{i}{2\pi}tm) = I + tf_1(M) + t^2 f_2(M) + \cdots t^r f_r(M)$$

$\rightsquigarrow \forall 1 \le k \le r$, f_k is G-invariant.

Let $E \to X$ complex vector bundle on a complex manifold, let D_E be a connection, curvature $H_E \in C_2^\infty(X, \mathrm{Hom}(E, E))$. Let $f \in GL(r, \mathbb{C})$- invariant "k-form",then

(1)Let H_α, H_β be the curvature forms of E in different trivialization, then $f(H_\alpha) = f(H_\beta)$, so we get a globally defined $2k$-form.

assume $H_\alpha = gH_\beta g^{-1}$, then

$$f(H_\alpha) = f(gH_\beta g^{-1}) = f(H_\beta)$$

(2) we also have

$$\mathrm{d}f(H) = 0$$

locally , $H = H_\alpha = \mathrm{d}a_\alpha + A_\alpha \wedge A_\alpha$,then

$$\mathrm{d}f(H) = \mathrm{d}f(H_\alpha, H_\alpha, ..., H_\alpha) = \sum_{i=1}^{k} f(H_\alpha, ..., \underbrace{\mathrm{d}H_\alpha,}_{i} ..., \alpha)$$

$$= \sum_{i=1}^{k} f(H_\alpha, ..., \mathrm{d}A_\alpha \wedge A_\alpha - A_\alpha \wedge \mathrm{d}A_\alpha, ..., H_\alpha)$$

Fact:(in Riemannian geometry) For any $x \in X$, we always can find a local frame s.t. $A_\alpha(x) = 0$.

so, choose this frame,

$$\mathrm{d}f(H) = 0$$

So, $[f(H)] \in H^{2k}(X, \mathbb{C})$

(3) Claim : the class $[f(H)]$ is independent of the choice of the connections D_E.

Let D_0, D_1 be two connections, consider

$$D_t = (1-t)D_0 + tD_1$$

$t \in [0, 1]$, curvature H_t

Fact: $\alpha := A_1 - A_0$ is globally defined, and in $C_1^\infty(X, \mathrm{Hom}(E, E))$.

Fact:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(H_t) = k\mathrm{d}f(\alpha, H_t, H_t, ..., H_t)$$

So,

$$f(H_1) - f(H_0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(H_t)\mathrm{d}t = \mathrm{d}\int_0^1 f(\alpha, H_t, H_t, ..., H_t)\mathrm{d}t$$

So,

$$[f(H_1)] - [f(H_0)]$$

定义 3.3.2. *the k-th Chern class of E*

$$c_k(E) := [f_k(\Theta_E)] \in H^{2k}(X, \mathbb{C})$$

Recall: Chern Class

X complex manifold, $E \to X$ is a smooth complex vector bundle of rank r. D is a connection, curvature $\Theta(D) \in C_2^\infty(X, \mathrm{Hom}(E, E))$.

linear algebra:

$$\det(I + \frac{i}{2\pi} tM) = I + tf_1(M) + t^2 f_2(M) + \cdots + t^r f_r(M)$$

Chern class $\{f_k(\Theta)\} \in H_{DR}^{2k}(X, \mathbb{C})$ is independent of choice of connection.

Today:

Special case: E is a complex line bundle. Let D_0 be a connection on E, locally $D_0 e = A_0 e$, A_0 is 1-form. curvature

$$\Theta(D_0) = D_0^2 = \mathrm{d}A_0 + A_0 \wedge A_0 = \mathrm{d}A_0$$

so, curvature is d-exact, so $\mathrm{d}\Theta(D_0) = 0$.

$$\det(I + \frac{i}{2\pi} tM) = I + \frac{i}{2\pi} tM$$

so, the first Chern class of line bundle is

$$c_1(E) = \{\frac{i}{2\pi} \Theta(D_0)\}$$

Let D_1 be another connection, locally $D_1 e = A_1 e$, so $\Theta(D_1) = \mathrm{d}A_1$.so,

$$\Theta(D_1) - \Theta(D_0) = \mathrm{d}(A_1 - A_0)$$

where

$$A_1 - A_0 \in C_1^\infty(X, \mathrm{Hom}(E, E))$$

(when E is line bundle ,$\mathrm{Hom}(E, E) \cong E^* \otimes E$ is trivial bundle)

so, $A_1 - A_0$ is a globally defined smooth function on X. So,

$$\{\Theta(D_1)\} = \{\Theta(D_0)\} \in H^2(X, \mathbb{C})$$

independent of the choice of connection.

3.4 Hermite 向量丛

定义 3.4.1. *a complex vector bundle $E \to X$ of rank r is called a Hermitian vector bundle, if we have an inner product on E, i.e. locally, consider a local frame $\{e_1, ..., e_r\}$, we have*

$$\{e_i(x), e_j(x)\} = h_{ij}(x)$$

s.t. $(h_{ij}(x))$ is a positive definite Hermitian matrix depending smoothly on x.

注记 3.4.2. *For any complex vector bundle, Hermitian structure always exists.*

证明与黎曼几何类似。（黎曼度量的存在性）

定义 3.4.3. *(Hermitian connection)*

A connection D on E is called Hermitian, if

$$\mathrm{d}\{e_i, e_j\} = \{De_i, e_j\} + \{e_i, De_j\}$$

More generally, let $t \in C_p^\infty(X, E)$, $s \in C_q^\infty(X, Y)$,

$$\mathrm{d}\{s, t\} = \{\mathrm{d}t, s\} + (-1)^p\{t, Ds\}$$

性质 3.4.4. *D is a Hermitian connection ,then the curvature*

$$\Theta(D)^* = -\Theta(D)$$

(where $(-)^*$ is conjugate transpose of matrix)

it means that, $i\Theta(D) \in C_2^\infty(X, \mathrm{Herm}(E, E))$

证明.

$$0 = \mathrm{d}^2\{e_i, e_j\} = \mathrm{d}\{De_i, e_j\} + \mathrm{d}\{e_i, De_j\}$$
$$= \{D^2e_i, e_j\} - \{De_i, De_j\} + \{De_i, De_j\} + \{e_i, D^2e_j\} = \{(\Theta + \Theta^*)e_i, e_j\}$$

□

注记 3.4.5. *E is a Hermitian line bundle, D is a Hermitian connection, then* $i\Theta(D)$ *is a real* 2*-form , $c_1(E) \in H^2(X, \mathbb{R})$.*

(Chern connection)

定义 3.4.6. *Let X be a complex manifold. D' is called a connection of type $(1,0)$ on E, if for any section $s \in C^\infty_{p,q}(X, E)$, we have $D's \in C^\infty_{p+1,q}(X, E)$.*

A connection D'' is called a connection of type $(0,1)$, if ... $D''s \in C^\infty_{p,q+1}(X, E)$.

注记 3.4.7. *Let $E \to X$ be a vector bundle. Let D be a connection on E, locally*

$$Ds \xrightarrow{\sim} \mathrm{d}\sigma + A \wedge \sigma$$

$$\mathrm{d}\sigma = \partial\sigma + \overline{\partial}\sigma$$

so,let A' be the $(1,0)$-part of A,...,

$$Ds = \partial\sigma + A' \wedge \sigma + (\overline{\partial}\sigma + A'' \wedge \sigma) =: D's + D''s$$

性质 3.4.8. *E:Hermitian vector bundle, D is a Hermitian connection, locally, take a C^∞-frame $e_1, ..., e_r$ which is orthonomal (i.e. $\{e_i(x), e_j(x)\} = \delta_{ij}$), then the connection coefficient $A = A' + A''$ satisfies*

$$(A')^* = -A''$$

($\Longleftrightarrow$ $\overline{(}iA) = iA$)

证明. because

$$0 = \mathrm{d}e_i, e_j = \{De_i, e_j\} + \{e_i, De_j\} = \{a_i^k e_k, e_j\} + \{e_i, a_j^l e_l\} = a_i^j + \overline{a_j^i}$$

so, $A^* = -A$. □

推论 3.4.9. *$E \to X$ is a Hermitian vector bundle, D_0'' is a connection of type $(0,1)$ on E. Then exists a unique Hermitian connection D such that $D'' = D_0''$.*

证明. Let $A'' = A_0''$ and $A' = -(A_0'')^* \rightsquigarrow A = A' + A''$, and D is given by A. □

Let $E \to X$ is a holomorphic Hermitian vector bundle, observe that $\overline{\partial}$ defines a connection of type $(0,1)$ on E(check!)

assume E is a holomorphic line bundle, take a section $s \in C_p^\infty(X,E)$, i.e. we have a family of p-forms (s_α) such that $s_\alpha = g_{\alpha\beta}s_\beta$ where $g_{\alpha,\beta}$ is the holomorphic transition matrix.

$$\overline{\partial}s \xrightarrow{\sim} \overline{\partial}s_\beta$$

then

$$\overline{\partial}s_\alpha = g_{\alpha,\beta}\overline{\partial}s_\beta$$

(so, $\overline{\partial}$ is a connection of $(0,1)$)

this connection is called the canonical connection of type $(0,1)$.

定义 3.4.10. *Let $E \to X$ holomorphic Hermitian vector bundle, the connection D on E is called Chern connection if*

$$D'' = \overline{\partial}$$

Curvature of Chern connection

$E \to X$ is holomorphic Hermite vector bundle , D is the Chern connection, Locally let $\{e_1, ..., e_r\}$ be a holomorphic frame, and two local sections

$$s, t \in C^\infty(\Omega, E)$$

where

$$s = \sum_{i=1}^r \sigma_i e_i$$

$$t = \sum_{i=1}^r t_i e_i$$

Since D is Hermitian ,

$$\mathrm{d}\{s,t\} = \mathrm{d}((\sigma_1, ..., \sigma_r)H \begin{pmatrix} t_1 \\ \vdots \\ t_r \end{pmatrix}) = (\mathrm{d}\sigma)^T H t + \sigma^T(\mathrm{d}H)t + \sigma^T H \mathrm{d}(t)$$

so, we have

$$\{Ds,t\} + \{s,Dt\} = (\mathrm{d}\sigma + \overline{H}^{-1}\partial\overline{H} \wedge \sigma)^T \wedge H\bar{t} + \sigma^T \wedge H\overline{(\mathrm{d}t + \overline{H}^{-1}\partial\overline{H} \wedge t)}$$

so ,

$$Ds = \mathrm{d}\sigma + \overline{H}^{-1}\partial\overline{H} \wedge \sigma$$

$$D's = \partial\sigma + \overline{H}^{-1}\partial\overline{H}\wedge\sigma = \overline{H}^{-1}\partial(\overline{H}\sigma)$$
$$D''s = \overline{\partial}\sigma$$

so,

$$(D')^2 s = \overline{H}^{-1}\partial(\overline{H}(\overline{H}^{-1}\partial(\overline{H}\sigma))) = \cdots = 0$$

$$(D'')^2 s = \cdots = 0$$

So we have

$$\Theta(D) = (D' + D'')^2 = D'D'' + D''D'$$

Locally ,

$$\Theta s = D'D''s + D''D's = \overline{H}^{-1}\partial(\overline{H}\overline{\partial}\sigma) + \overline{\partial}(\overline{H}^{-1}\overline{\partial}(\overline{H}\sigma)) = \cdots = \overline{H}^{-1}\partial\overline{H}\wedge\overline{\partial}\sigma + \overline{\partial}(\overline{H}^{-1})\sigma$$
$$= \overline{\partial}(\overline{H}^{-1}\partial\overline{H})\sigma$$

So, Chern curvature

$$\Theta_D = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})$$

Last time: $E \to X$ is a holomorphic vector bundle with a Hermitian metric H. Then there is a unique connection D_Es.t. ... called Chern connection.

Curvature of Chern Connection:

$$\Theta(D_E) = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})$$

so,

$$i\Theta(D_E) \in C^\infty_{1,1}(X, \mathrm{Hom}(E, E))$$

例子 3.4.11. *(Special case: E is a holomorphic line bundle)*

locally, let e be ha holomorphic frame, $\langle e, e\rangle = h$ is the metric. then,

$$\Theta = \overline{\partial}(h^{-1}\partial h) = \overline{\partial}\partial\log h$$

so,

$$i\Theta(E) = -i\partial\overline{\partial}\log h$$

if $h = e^{-2\varphi}$ where φ is a smooth function, then

$$i\Theta(E) = 2i\partial\overline{\partial}\varphi = 2\sqrt{-1}\sum_{k,l}\frac{\partial^2\varphi}{\partial z_k\partial\overline{z_l}}\mathrm{d}z_k\wedge\mathrm{d}\overline{z_l}$$

Question: let s be a local holomorphic section of E,

$$-i\partial\overline{\partial}\log|s|^2_h = ?$$

(Hint:$\frac{i}{\pi}\partial\overline{\partial}\log z =$? 单复变，按分布意义下求导. 等于狄拉克测度 2333333) 可能是期末题目？

例子 3.4.12. *$\mathcal{O}(-1)$ on $\mathbb{C}P^n$, tautological line bundle. (Recall: $\mathbb{C}P^n$ is a compact complex manifold with holomorphic charts*

$$\Omega_j := \{[z_0; z_1; ...; z_n] | z_j \neq 0\} \to \left(\frac{z_0}{z_j}, \cdots, \hat{1}, \cdots \frac{z_n}{z_j}\right) \in \mathbb{C}^n$$

)

Let V be a complex vector space, $\dim_{\mathbb{C}} V = n + 1$. Denote the projective space by

$$\mathbb{P}(V) = (V \setminus \{0\})/\mathbb{C}^*$$

Let $\underline{V} := \mathbb{P}(V) \times V$ be the trivial vector bundle, define

$$\mathcal{O}(-1) := \{([x], \xi) | \xi \in \mathbb{C} \cdot x\}$$

性质 3.4.13. *$\mathcal{O}(-1)$ is a holomorphic line bundle on $\mathbb{P}(V)$.*

证明. $\mathcal{O}(-1)|_{\Omega_j}$ has a non-vanishing holomorphic section $\mathcal{E}_j$ defined by

$$\mathcal{E}_j([x]) = \frac{x}{x_j}$$

for $0 \leq j \leq n$. □

Assume V has a Hermitian inner product, then $\mathcal{O}(-1)$ has an Hermitian structure induced from V.

Let $e_0, ..., e_n$ be an orthonormal basis of V, then $\mathcal{O}(-1)|_{\Omega_0}$ has a non-vanishing holomorphic section:

$$\mathcal{E}_0(z_1, ..., z_n) = e_0 + z_1 e_1 + ... + z_n e_n$$

where

$$\Omega_0 = \{[1; z_1; ...; z_n] | z_j \in \mathbb{C}\} \cong \mathbb{C}^n$$

then,

$$|\mathcal{E}_0|_h^2 = 1 + |z_1|^2 + ... + |z_n|^2$$

so the Chern curvature of $\mathcal{O}(-1)$ on Ω_0 is given by

$$\Theta = \bar{\partial}\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2)$$

Denote $\mathcal{O}(1) := \mathcal{O}(-1)^*$, then

$$\Theta(\mathcal{O}(1)) = -\bar{\partial}\partial \log(1 + |z_1|^2 + ... + |z_n|^2)$$

on Ω_0.

$$i\Theta(\mathcal{O}(1)) = i\partial\bar{\partial}\log(1+|z_0|^2+...+|z_n|^2) = \sqrt{-1}\sum_{1\le k,l\le n} c_{k,l}\mathrm{d}z_k\wedge\mathrm{d}\overline{z_l}$$

Exercise: (c_{kl}) is a positive definite Hermitian matrix.

"Fubini-Study metric" on $\mathbb{P}(V)$.$\mathcal{O}(1)$ is "hyperplane line bundle of $\mathbb{P}(V)$".

Exercise: calculate

$$\int_{\mathbb{P}(V)}\left(\frac{i}{2\pi}\Theta(\mathcal{O}(1))\right)^{\wedge n} =?$$

(Hint: $\mathbb{P}(V)\setminus\Omega_0$ is a zero-measure set)

$E\to X$: holomorphic line bundle, D_E is a Chern connection.

$$c_1(E) = \{\frac{i}{2\pi}\Theta(D_E)\}\in H^2_{DR}(X,\mathbb{R})$$

Exercise: 60% 的概率出现于期末试题

Consider the sequence

$$0\to\mathbb{Z}\to\mathcal{O}\xrightarrow{e^{2\pi i *}}\mathcal{O}^*\to 0$$

it induces a long exact sequence

$$\cdots\to H^1(X,\mathcal{O})\to H^1(X,\mathcal{O}^*)\xrightarrow{\delta} H^2(X,\mathbb{Z})\to H^2(X,\mathcal{O})\to\cdots$$

prove: Consider E as an element of $H^1(X,\mathcal{O}^*)$, then the image of $\delta(E)$ in $H^2(X,\mathbb{R})\cong H^2_{DR}(X,\mathbb{R})$ is $c_1(E)$.

Exercise: E is a holomorphic line bundle, denote $\theta := \frac{i}{2\pi}\Theta(D_E)$ real $(1,1)-form$, where D_E is Chern connection with a metric h. Prove: for any smooth function $f\in C^\infty(X,\mathbb{R})$, there exists a Hermitian metric h_f s.t.

$$\frac{i}{2\pi}\Theta_{E,h_f} = \theta + i\partial\bar{\partial}f$$

第 4 章　L^2 Hodge 理论

4.1　向量丛上的微分算子

Differential operators on vector bundles.

Let X is a (connected) smooth manifold of ($\mathbb{R}$-)dimension n. E, F : $\mathbb{K}$-vector bundle of rank r, r' respectively.

定义 4.1.1. *a linear differential operator of degree k from E to F is a $\mathbb{K}$-linear map*

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

$$u \mapsto Pu$$

locally given by

$$Pu(x) = \sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u(x)$$

where $a_\alpha(x) = (a_{afa,\lambda\mu}(x))$ be a $r' \times r$ matrix.

$$u(x) = (u_1(x), ..., u_r(x))^T$$

Let $t \in \mathbb{K}$, $f \in C^\infty(M, \mathbb{K})$, $u \in C^\infty(M, E)$, then

$$e^{-tf(x)} P(e^{tf(x)} u(x)) = t^k \sigma_P(x, \mathrm{d}f(x)) u(x) + \text{terms } c_j(x)^{t_j} \quad (j < k)$$

定义 4.1.2.

$$\sigma_P : T^*M \to \mathrm{Hom}(E, F)$$

is called the principal symbol of P, which is a polynomial on T^*M.

locally,

$$\sigma_P(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

$(\xi^\alpha := \xi_1^{\alpha_1}...\xi_n^{\alpha_n})$

例子 **4.1.3.** *Consider* $\mathrm{d} : C^\infty(M,\mathbb{K}) \to C^\infty(M,T^*M)$*. then*

$$\mathrm{d}u = \sum_{j=1}^{n} \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \frac{\partial u}{\partial x^i}$$

i.e.

$$\sigma_d(x,\xi) = \sum_{j=1}^{n} \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \xi_j$$

定义 **4.1.4.** *P is called elliptic, if* $\forall x \in M, \xi \in T_x^*M \setminus \{0\}$,

$$\sigma_P(x,\xi) \in \mathrm{Hom}(E_x,E_x)$$

is injective.

For example, d is elliptic.

L^2-inner product

Let M be an oriented C^∞-manifold with a smooth volume form, locally

$$\mathrm{d}V(x) = \gamma(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

$\gamma(x) > 0$. Assume E has a Euclidean(or Hermitian) structure...

Let $u,v \in C^\infty(M,E)$,define

$$\langle\langle u,v\rangle\rangle := \int_M \langle u,v\rangle \mathrm{d}V(x)$$

define $L^2(M,E) :=$ space of sections with measurable coefficients with are L^2 w.r.t $\langle\langle,\rangle\rangle$.

定义 4.1.5. *Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be a differential operator, E, F have Euclidean (or Hermitian) structure, then there exists unique differential operator*

$$P^* : C^\infty(M,F) \to C^\infty(M,E)$$

s.t.

$$\langle\langle Pu, v\rangle\rangle = \langle\langle u, P^*v\rangle\rangle$$

for all u, v s.t. $Supp u \cap Supp v \subset\subset M$(relative compact...)

P^* is called the formal adjoint of P.

证明. Existence: Assume that $SuppU, Suppv \subset\subset$ some coordinate chart Ω with coordinates $(x_1, ..., x_n)$, then

$$\ll Pv, u \gg = \int_\Omega \sum_{\alpha,\lambda,\mu} a_{\alpha,\lambda\mu}(x) D^\alpha u_\mu(x) \overline{v_\lambda(x)} \gamma(x) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

integration by parts, it

$$= \int_\Omega \sum_{\alpha,\lambda,\mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha(\gamma(x)\overline{a_{\alpha,\lambda\mu}} v_\lambda(x))} \mathrm{d}x_1..\mathrm{d}x_n$$

Locally,

$$P^*v = \sum_{|\alpha| \le k} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha(\gamma(x) \overline{a_\alpha(x)}^T v(x))$$

Uniqueness: use the density of C^∞-section with compact support in $L^2(M, -)$. □

推论 4.1.6. *If $\sigma_P(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$,then*

$$\sigma_{P^*} = (-1)^k \overline{\sigma_P(x,\xi)}^T$$

推论 4.1.7. *If rank E = rank$F$, P is differential operator, then P^* is elliptic $\iff$ P^* is elliptic.*

4.2 椭圆算子的基本性质

Fundamental results of elliptic operators

M is a compact (oriented) C^∞-manifold, $\dim_{\mathbb{R}} M = n$, with a smooth volume form $\mathrm{d}V$.

E is an Hermite vector bundle, $rank_{\mathbb{C}} E = r$.

Sobolev space:$W^k(M, E) :=$ the space of section $s : M \to E$ whose derivations up to order $= k$, := the completion of space of smooth sections w.r.t W^k-norm.

$(\Omega_j)_{j\in I}$: a finite open covering of M, $E|_{\Omega_j}$ trivial, Let $(\rho_j)_{j\in I}$ be a partition of unity w.r.t. $(\Omega_j)_{j\in I}$, s.t. $\sum_j \rho_j^2 = 1$. locally, choose an orhtonormal frame $(e_{j,\lambda})_{1\le\lambda\le r}$ on Ω_j, then $u = \sum_{\lambda=1}^r u_{j,\lambda} e_{j,\lambda}$ on Ω_j. Define

$$||u||_k^2 := \sum_{j,\lambda} ||e_j u_{j,\lambda}||_k^2$$

where

$$||e_j u_{j,\lambda}||_k^2 := \int_{\Omega_j} \sum_{|\alpha|\le k} |D^\alpha(e_j u_{j,\lambda})|^2 \mathrm{d}V(x)$$

注记 4.2.1. *On a compact manifold, the equivalence of class of* $||\cdot||_k$ *is independent of the choice of : partition of unity, local trivialization, holomorphic covering...*

引理 4.2.2. *(Sobolev lemma)*

For $k > l + \frac{n}{2}$*, then we have*

$$W^k(M, E) \subseteq C^l(M, E)$$

引理 4.2.3. *(Rellich lemma)*

For any $k \in \mathbb{Z}_{\ge 0}$*, the inclusion*

$$W^{k+1}(M, E) \hookrightarrow W^k(M, E)$$

is a compact operator.

引理 4.2.4. *(Garding inequality)*

If

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

is elliptic, and rankE =rank F, $\tilde{P}$ *:the extension of* P *to sections with distribution coefficients, then : for all* $u \in W^0(M, E)$*, if* $\tilde{P}u \in W^k(M, F)$*, then* $u \in W^{k+d}(M, E)$*, where* $d = \deg P$*, and*

$$||u||_{k+d} \le C_k \left(||\tilde{P}u||_k + ||u||_0\right)$$

where C_k *depending on* k, M*.*

证明. Reference: `Kodaira: deformation of complex structures (Appendix)` □

推论 4.2.5. *If $u \in \ker \tilde{P} \cap W^0(M,E)$, then $u \in C^\infty(M,E)$.*

引理 4.2.6. *(Finiteness theorem)*

Setting M be a compact manifold, rank$E$ =rank$F$,

$$P : C^\infty(M,E) \to C^\infty(M,F)$$

elliptic,then:

(1) $\ker P$ is of finite dimension

(2) $P(C^\infty(M,E))$ is closed and of finite codimension in $C^\infty(M,F)$. If P^* is the formal adjoint of P, then $\exists$ decomposition

$$C^\infty(M,F) = P(C^\infty(M,E)) \oplus \ker P^*$$

which is orthogonal in $W^0(M,F) = L^2(M,F)$

证明. 椭圆算子的一般结果，分析的东西 233333333. 可以参考小平邦彦复流形与复结构形变的附录。 □

4.3 紧黎曼流形的 Hodge 理论

Hodge theory in compact Riemannian manifold

Hodge star operator.

M compact Riemannian manifold, $\dim_{\mathbb{R}} = n$, E is a Hermitian vector bundle. Assume $(\xi_1, ..., \xi_n), (e_1, ..., e_n)$ be orthonormal frame of TM, E on some local chart Ω, denote $(\xi_1^*, ..., \xi_n^*), (e_1^*, ..., e_n^*)$ be the co-frame of T^*M, T^*E.

$\wedge^\bullet T^*M$ is endowed with an inner product frame from TM. locally,

$$\langle u_1 \wedge \cdots \wedge u_p, u_1 \wedge \cdots \wedge u_p \rangle := \det(\langle u_i, v_j \rangle)$$

for $u_i, v_j \in T^*M$. Then , get an inner product on $\wedge^p T^*M$.

Assume

$$U = \sum_{\substack{|I|=p \\ i_1 \le \dots \le i_p}} u_I \xi_I^*$$

$$V = \sum_{\substack{|I|=p \\ i_1 \leq \dots \leq i_p}} v_I \xi_I^*$$

be p-forms, then

$$\langle u, v \rangle = \sum_{|I|=p} u_I v_I$$

i.e. $\{\xi_T^*\}$ is an orthonormal basis of $\wedge^p T^* M$.

$\wedge^* T^* M \otimes E$ has an inner product induced from $\wedge^* T^* M, E$,

定义 4.3.1. *the Hodge star operator*

$$^* : \wedge^p T^* M \to \wedge^{n-p} T^* M$$

is defined by

$$u \wedge *v = \langle u, v \rangle \mathrm{d}V$$

Locally, let

$$U = \sum_{|I|=p} u_I \xi_I^*, \; V = \sum_{|I|=p} v_I \xi_I^*$$

assume

$$*V = \sum_{|J|=n-p} a_J \xi_J^*$$

then

$$U \wedge * \sum u_I a_{I^c} \xi_I^* \wedge \xi_{I^c}^* = \sum u_I a_{I^c} \varepsilon(I, I^c) \xi_1^* \wedge \cdots \wedge \xi_n^*$$
$$\langle u, v \rangle \mathrm{d}V = \sum_{|I|=p} u_I v_I \xi_1^* \wedge \cdots \wedge \xi_n^*$$

so, we have

$$*V = \sum_{|I|=p} \varepsilon(I, I^c) V_I \xi_{I^c}^* \in \bigwedge^{n-p} T^* M$$

定义 4.3.2.

$$* : \bigwedge^p T^* M \otimes E \to \bigwedge^{n-p} T^* M \otimes E$$

is defined by

$$\{s, *t\} := \langle s, t \rangle \mathrm{d}V$$

Locally, assume

$$t = \sum_{\substack{|I|=p \\ 1\le\lambda\le r}} t_{I,\lambda}\xi_I^* \otimes e_\lambda$$

then

$$*t = \sum_{\substack{|I|=p \\ 1\le\lambda\le r}} \varepsilon(I, I^c) t_{I,\lambda}\xi_{I^c}^* \otimes e_\lambda$$

定义 4.3.3.

$$\# : \bigwedge^p T^*M \otimes E \to \bigwedge^{n-p} T^*M \otimes E^*$$

is defined by: for any $s, t \in \bigwedge^p T^*M \otimes E$,*such that*

$$s \wedge \# t := \langle s, t\rangle \mathrm{d}V$$

wedge product+ pairing of $E^* \times E \to \mathbb{C}$.

Locally: assume

$$t = \sum_{\substack{|I|=p \\ 1\le\lambda r}} t_{I,\lambda}\xi_T^* \otimes e_\lambda$$

then,

$$\# t = \sum_{|I|=p,\lambda} \varepsilon(I, I^c) t_{I,\lambda}\xi_c^* I \otimes e_\lambda^*$$

性质 4.3.4.

$$*^2 = (-1)^{p(n-1)} \quad on \ \bigwedge^p T^*M \otimes E$$
$$\#^2 = (-1)^{p(n-1)} \quad on \ \bigwedge^p T^*M \otimes E$$

（正负号对吗？）

Recall: For all $s, t \in C^\infty(M, \bigwedge^p T^*M \otimes E)$, we have an inner product

$$\langle\langle s, t\rangle\rangle := \int_M \langle s, t\rangle \mathrm{d}V$$

定理 4.3.5. *Let* D_E *be an Hermite connection on* E*, acting on* $\bigwedge^p T^*M \otimes E$*, then*

$$D_E^* := (-1)^{np+1} * D_E *$$

where D_E^* *is the formal adjoint of* D_E.

证明. Let $s \in C^\infty(M, \bigwedge^p T^*M \otimes E)$ and $t \in C^\infty(M, \bigwedge^{p+1} T^*M \otimes E)$. then

$$\langle\langle D_E s, t\rangle\rangle = \int_M \langle D_E s, t\rangle \mathrm{d}V = \int_M \{D_E s, *t\}$$

Since D_E is Hermitian ,by defination ,

$$\mathrm{d}\{s, *t\} = \{D_E s, t\} + (-1)^p\{s, D_E(*t)\}$$

so,

$$\langle\langle D_E s, t\rangle\rangle = \int_M \mathrm{d}\{s, *t\} + (-1)^{p+1}\{s, D_E * t\} = (-1)^{p+1}(-1)^{p(n_1)}\int_M \{s, *(*D_E * t)\} = \langle\langle s, D_E^* t\rangle\rangle$$

so,

$$D_E^* t = (-1)^{np+1} * D_E *$$

□

定义 4.3.6.

$$\triangle_E = D_E D_E^* + D_E^* D_E : C^\infty(M, \bigwedge^p T^*M \otimes E) \to C^\infty(M, \bigwedge^p T^*M \otimes E)$$

例子 4.3.7. *Let $M = \mathbb{R}^n$, $g = \sum\limits_{i=1}^n \mathrm{d}x_i^2$, $E = M \times \mathbb{C}$ trivial line bundle with $D_E = \mathrm{d}$. then*

$$\triangle_E u = (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})u = -\sum_{i=1}^n \left(\sum_{|I|=p} \frac{\partial^2 u_I}{\partial x_I^{2\,2}} \mathrm{d}x_I\right)$$

where

$$u = \sum_{|I|=p} u_I \mathrm{d}x_I$$

性质 4.3.8. *$\triangle_E$ is a self-adjoint elliptic operator. (i.e. $\triangle_E^* = \triangle_E$)*

证明. $\triangle_E^* = \triangle_E$ be definition.

note that

$$e^{-tf} D_E(e^{tf} s) = t\mathrm{d}f \wedge s + D_E s$$

so,

$$\sigma_{D_E}(x, \xi)s = \xi \wedge s$$

$$\sum_{D_E^*} = -\overline{\sigma_{D_E}}^T$$

$$\sigma_{D_E^*}(x,\xi)s = -\tilde{\xi} \lrcorner s$$

where $\tilde{\xi}$ be the vector field dual to ξ. □

定义 4.3.9.

$$\triangle_E = D_E D_E^* + D_E D_E^* : C^\infty(M, \bigwedge^p T^*M \otimes E) \to C^\infty(M, \bigwedge^p T^*M \otimes E)$$

so,

$$\sigma_{\triangle_E}(x,\xi)s = \left(\sigma_{D_E}\sigma_{D_E^*}(x,\xi) + \sigma_{D_E^*}\sigma_{D_E}(x,\xi)\right)s$$

so, $\sigma_{\triangle_E}$ is injective if $\xi \neq 0$, so $\triangle_E$ is elliptic.

Harmonic forms and Hodge isomorphism.

定义 4.3.10. *u is called harmonic if $\triangle_{\mathrm{d}} u = 0$.*

定理 4.3.11. *M is a compact Riemannian manifold, then de Rham cohomology*

$$H^p_{DR}(M,\mathbb{R}) \cong \ker(\triangle_d : C^\infty(M, \bigwedge^p T^*M))$$

证明. $\triangle_d$ self-adjoint elliptic, so by general result for elliptic operator,

$$C^\infty(M, \bigwedge^p T^*M) = \operatorname{Im}\triangle_d \oplus \ker\triangle_d^* = \operatorname{Im}\triangle_d \oplus \ker\triangle_d$$

Claim:

$$\operatorname{Im}\triangle_{\mathrm{d}} = \in \mathrm{d} \oplus \operatorname{Im}\mathrm{d}^*$$

Recall $\triangle_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, so

$$\operatorname{Im}\triangle_{\mathrm{d}} \subseteq \operatorname{Im}\mathrm{d} \oplus \in \mathrm{d}^*$$

on the other hand,

$$\operatorname{Im}\mathrm{d} \oplus \operatorname{Im}\mathrm{d}^* \subseteq (\ker\triangle_{\mathrm{d}})^\perp = \operatorname{Im}\triangle_{\mathrm{d}}$$

so,

$$\operatorname{Im}\triangle_{\mathrm{d}} = \operatorname{Im}\mathrm{d} \oplus \operatorname{Im}\mathrm{d}^*$$

so,

$$C^\infty(M, \bigwedge^p T^*M) = \operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^* \oplus \ker \triangle_{\mathrm{d}}$$

so,

$$H^p_{DR}(M, \mathbb{R}) = \frac{\operatorname{Im} \mathrm{d} \oplus \ker \triangle_{\mathrm{d}}}{\operatorname{Im} \mathrm{d}} = \ker \triangle_{\mathrm{d}}$$

□

推论 4.3.12.

$$\dim H^p_{DR}(M, \mathbb{R}) = \dim \ker \triangle_{\mathrm{d}} < +\infty$$

注记 4.3.13. *Consider*

$$u \mapsto \int_M (\langle u, u \rangle + \langle \mathrm{d}u, \mathrm{d}u \rangle + \langle \mathrm{d}^* u, \mathrm{d}^* u \rangle) \mathrm{d}V$$

这个泛函的变分是什么鬼?

Harmonic forms and Hodge isomorphism

Recall: M is a compact Riemann manifold,

$$\mathrm{d} : C^\infty(M, \bigwedge^* T^*M) \to C^\infty(M, \bigwedge^{*+1} T^*M)$$

adjoint d^*,

$$\triangle_d = dd^* + d^*d$$

is a self-adjoint elliptic operator.

Hodge decomposition:

$$C^\infty(M, \bigwedge^p T^*M) = \ker \triangle_d \oplus \operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^*$$

$$\mathcal{H}^p(M, \mathbb{R}) := \ker \triangle_d \quad \text{finite dimension}$$

$$\mathcal{H}^p(M, \mathbb{R}) \cong H^p_{DR} \cong H^p(M, \mathbb{R})$$

(Hodge isomorphism, and, de Rham-Weil)

Poincare duality

定理 4.3.14. *The pairing*

$$H^p_{DR}(M, \mathbb{R}) \times H^{n-p}_{DR}(M, \mathbb{R}) \to \mathbb{R}$$

$$(s, t) \mapsto \int_M s \wedge t$$

(is well defined) is non-degenerated. In particular, $H^p_{DR}(M, \mathbb{R})^* \cong H^{n-p}_{DR}(M, \mathbb{R})$

证明. the pairing factors through the pairing on

$$\mathcal{H}^p(M,\mathbb{R}) \times \mathcal{H}^{n-p}(M,\mathbb{R}) \to \mathbb{R}$$

$$(s,t) \mapsto \int_M s \wedge t$$

need to verify:(1) it is independent of the choice of representations.(Easy, check) (2) Pairing $\mathcal{H}... \times \mathcal{H}...$ is non-degenerated..

claim(Exercise): Hodge star $*$ s.t. $*\triangle_{\mathrm{d}} = \triangle_{\mathrm{d}}*$.

so, s is a harmonic p-form $\iff$ $*s$ is a harmonic $(n-p)$-form.

note that

$$s \wedge *s = \langle s,s\rangle \mathrm{d}V = \int_M s \wedge *s = \int_M \langle s,s\rangle \mathrm{d}V = ||s||^2$$

□

推论 4.3.15.

$$\dim \mathcal{H}^p(M,\mathbb{R}) = \dim \mathcal{H}^{n-p}(M,\mathbb{R})$$

Generalization to flat bundle. M is a compact Riemannian manifold, $\dim_{\mathbb{R}} M = n$, $E \to M$ is a complex Hermitian vector bundle.

定义 4.3.16. *$E \to X$ is called flat, if it admit a connection D_E s.t.*

$$D_E^2 = 0$$

注记 4.3.17. *E is flat $\iff$ E is given by a representation*

$$\pi_1(M) \to GL(r,\mathbb{C})$$

(我们不证)

Consider the complex :

$$(C^\infty(M,\bigwedge^* T^*M \otimes E), D_E)$$

$$\rightsquigarrow H^p_{DR}(M,E) := \frac{\ker D_E}{\operatorname{Im} D_E}$$

Exercise: we have decomposition

$$C^\infty(M,\bigwedge^p T^*M \otimes E) = \ker \triangle_{D_E} \oplus \operatorname{Im} D_E \oplus \operatorname{Im} D_E^*$$

$$H^p_{DR}(M,E) \cong \ker \triangle_{D_E}$$

and the pairing

$$H^p_{DR}(M,E) \times H^{n-p}_{DR}(M,E^*) \to \mathbb{C}$$
$$(s,t) \mapsto \int_M s \wedge t$$

is non-degenerate..

以上是实的 Hodge 理论。

4.4 Kähler 流形

定义 4.4.1. *Let X be a complex manifold,* $\dim_{\mathbb{C}} X = n$*, X is called a Hermitian manifold, if X has a Hermitian metric, i.e. locally* $h(z) := \sum_{1\le j,k\le n} h_{jk}(z)\mathrm{d}z_j \otimes \mathrm{d}\bar{z}_k$*, where* (h_{jk}) *is positive definition Hermitian matrix.*

Check:the positivity of h is independent of the choice of holomorphic local coordinate

Rmk: Any complex manifold has a Hermitian metric...(Exercise)

Fundamental $(1,1)$-form associated to $h(z)$ is defined by

$$\omega := -\operatorname{Im} h = \frac{\sqrt{-1}}{2}\sum_{j,k} h_{jk}\mathrm{d}z_j \mathrm{d}\bar{z}_k$$

we also call ω is the Hermitian metric on X

Fact: ω is real (i.e. $\overline{\omega} = \omega$).

注记 4.4.2. *h is a Hermite structure on* TX*(holomorphic tangent bundle of X). locally,*

$$\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\rangle(z) = h_{ij}(z)$$

定义 4.4.3. (X,ω) *is an Hermitian manifold, X is Kähler if* $\mathrm{d}\omega = 0$.

性质 4.4.4. *Locally,* $\omega = \frac{\sqrt{-1}}{2}\sum_{jk} h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$ *is Kaehler,* $\iff$ $\partial\omega = 0$ *and* $\bar{\partial}\omega = 0$*, i.e.*

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}$$

If (X,ω) *is a compact Kaehler manifold, then*

$$H^{2k}(X,\mathbb{R}) \neq 0$$

证明. $\mathrm{d}\omega = 0$, so $\omega \in H^2(M, \mathbb{R})$. Claim:

$$0 \neq \omega^k \in H^{2k}(M, \mathbb{R})$$

proof of the claim:

$$[\omega^k][\omega^{n-k}] = \int_X \omega^k \wedge \omega^{n-k} = \int_X \omega^n$$

Since ω is positive, locally

$$\omega^n = n! \det(h_{jk}) \bigwedge_{l=1}^{n} \left(\frac{\sqrt{-1}}{2} \mathrm{d}zl \wedge \mathrm{d}\bar{z}_l \right) > 0$$

is a volume form. So,

$$[\omega^k][\omega^{n-k}] = \int_X \omega^n > 0$$

(Using Poincare dual) □

例子 4.4.5. *(Exists a complex manifold NOT Kaehler) (Hopf Surface)*

$$\mathrm{X} = (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

where discrete group $\Gamma := \{\lambda^n | n \in \mathbb{Z}\}$, $0 < \lambda < 1$ *fixed.*

Exercise: $X \cong S^1 \times S^3$ C^∞ homeomorphism.. and X is compact complex manifold.
and $H^2(X, \mathbb{R}) = H^2(S^1 \times S^3, \mathbb{R}) = 0$ by Künneth Formula...
So, X is non-Kahler...

例子 4.4.6. *Examples of Kaehler manifold)*

(1)Riemann surface must be Kaehler...(trivial)

(2)(complex torus) $\mathrm{X} = \mathrm{C}^n/\Gamma$, Γ *is a lattice. (this manifold may not compact...)*

$$\omega = \sqrt{-1} \sum_{j,k} h_{jk} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$$

is a Kahler metric on X if $(H_{jk}) > 0$, h_{jk} *are constant.*

(3) Projective space $\mathbb{C}P^n$.

$$\omega := \sqrt{-1}\Theta_h(\mathcal{O}(1))$$

locally,

$$\omega = \sqrt{-1}\partial\bar{\partial} \log(1 + |z_1|^2 + ... + |z_n|^2)$$

on Ω. *This* ω *is a Kahler metric,*

例子 4.4.7. *Let (X, ω) is a Kahler manifold, then any complex submanifold $Y \subseteq X$ is also Kahler.*

$$i : Y \hookrightarrow X$$

with the Kahler metric $i^*\omega$.

Exercise: Let $f : Y \to X$ be a holomorphic immersion, and assume X is Kahler, then Y is Kahler.

推论 4.4.8. *Any projective manifold (i.e. $X \hookrightarrow \mathbb{C}P^N$) is Kähler.*

(Algebraic Geometry.....)

性质 4.4.9. *(Equivalent definition of Kaehler metrics) a Hermitian metric ω is Kahler, iff for all $x_0 \in X$, there exists a holomorphic chart $(z_1, ..., z_n)$ centered at x_0, s.t.*

$$\omega(z) = \sqrt{-1}\sigma_{jk}\delta_{jk}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k + O(|z|^2)$$

($\Leftarrow$ is trivial...) (left to HW)

定理 4.4.10. *(Exercise)*

If (X, ω) is Kahler, then for all $x_0 \in X$, $\exists$ holomorphic chart $z_1, ..., z_n$ centered at x_0, s.t. assume

$$\omega = \sqrt{-1}h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$$

then

$$h_{lm}(z) = \delta_{lm} - \sum_{j,k} c_{jk,lm} z_j \bar{z}_k + O(|z|^3)$$

where $c_{jk,lm}$ is the coefficients of the Chern curvature tensor,

$$\Theta(TX)_x := \sum c_{jk,lm}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes (\frac{\partial}{\partial z_l})^* \otimes \frac{\partial}{\partial z_m}$$

（查书）

4.5 紧复流形上的 Hodge 理论

(X,ω) is a compact Hermitian manifold, $E \to X$ is a homomorphic Hermitian vector bundle.

$$D_E := D'_E + D''_E$$

Chern connection, $D''_E = \overline{\partial}$.

定义 4.5.1.

$$\triangle_E := D_E D_E^* + D_E^* D_E$$

$$(D'_E)^* = - * D''_E *$$
$$(D''_E)^* = - * D'_E *$$
$$\triangle'_E = D'_E (D'_E)^* + ...$$
$$\triangle''_E = ...$$

Note that $(D''_E)^2 = 0$, consider the complex

$$C^\infty(X, \bigwedge^{p,q} \otimes E) \xrightarrow{D''_E} C^\infty(X, \bigwedge^{p,q+1} \otimes E)$$
$$\rightsquigarrow H^{p,q}_{D''_E}(X,E)$$

Dolbeaut cohomology... it isom to $\ker \triangle''_E$

Hodge theory in compact complex manifold.

Let (X,ω) be a compact complex manifold of dimension n. $E \to X$ holomorphic Hermitian vector bundle, with Chern connection D_E, $D_E = D'_E + D''_E$ where $D''_E = \overline{\partial}$.

Recall: L^2 inner product: $u \in C^\infty(X \bigwedge^{p,q} \otimes E)$,

$$\langle\langle u, v \rangle\rangle := \int_X \langle u, v \rangle \mathrm{dvol}$$

Hodge star operator $*$: $u, v \in C^\infty(X, \bigwedge^{p,q} \otimes E)$,

定义 4.5.2.

$$* : \bigwedge^{p,q} \otimes E \to \bigwedge^{n-q,n-p} \otimes E$$

s.t.

$$u \wedge *v = \langle u, v \rangle \mathrm{d}vol$$

(wedge product from $\bigwedge^{p,q}$, with inner product from E)

Exercise: Take a holomorphic chart $(z_1, ..., z_n)$ s.t.

$$\omega = \sqrt{-1}\sum_j dz_j \wedge d\bar{z}_j$$

at some point p. An orthonormal frame $\{e_1, ..., e_r\}$, Let

$$u = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{\lambda=1}^{r} u_{IJ} dz_I \wedge d\bar{z}_J \otimes e_\lambda \in \bigwedge^{p,q} \otimes E$$

WHAT IS $*u$?

Formal adjoint of D_E, D'_E, D''_E?

性质 4.5.3.

$$D_E^* = - * D_E *$$

$$(D'_E)^* = - * D''_E *$$

$$(D''_E)^* = - * D'_E *$$

定义 4.5.4.

$$\triangle_E := D_E D_E^* + D_E^* D_E$$

$$\triangle'_E := D'_E D'^*_E + D'^*_E D'_E$$

$$\triangle''_E := \cdots$$

Check: $\triangle_E, \triangle'_E, \triangle''_E$ are self adjoint, elliptic operators.

Hodge theory w.r.t. $\triangle''_E$.

定理 4.5.5. *We have a decomposition*

$$C^\infty(X, \bigwedge^{p,q} \otimes E) = \ker \triangle''_E \oplus \operatorname{Im} D''_E \oplus \operatorname{Im} D''^*_E$$

As a consequence, Dolbeault cohomology

$$H^{p,q}_{D''_E}(X, \mathbb{C}) \cong \ker \triangle''_E$$

推论 4.5.6.

$$\dim_{\mathbb{C}} H^{p,q}_{D''_E}(X,\mathbb{C}) < +\infty$$

Cohomology group

$$H^{p,q}_{D''_E}(X,\mathbb{C})$$

Ω^p:sheaf of holomorphic p-forms on X (i.e. a $(p,0)$-form φ is holomorphic if $\bar{\partial}\varphi = 0$).

$\mathcal{E}^{p,q}$:Sheaf of smooth (p,q)-forms on X.

Similarly, we have $\Omega^p(E)$ the sheaf of holomorphic p-forms with values in E,and $\mathcal{E}^{p,q}(E)$ the sheaf...smooth (p,q)-forms ...

we have an acyclic resolutions

$$0 \to \Omega^p(E) \xrightarrow{D''_E} \mathcal{E}^{p,1}(E) \xrightarrow{D''_E} \mathcal{E}^{p,2}(E) \xrightarrow{D''_E} \cdots$$

(check, it is a resolution)

By de Rham-Weil theorem,

$$H^q(X,\Omega^p(E)) \cong D^{p,q}_{D''_E}(X,\mathbb{C}) \cong \mathcal{H}^{p,q}_{D''_E}(X,\mathbb{C}) := \ker \triangle''_E$$

定理 4.5.7. *(Serre duality)*

The pairing

$$H^{p,q}_{D''_E}(X,E) \times H^{n-p,n-q}_{D''_E}(X,E^*) \to \mathbb{C}$$

$$(s,t) \mapsto \int_X s \wedge t$$

is non-degenerate

证明. Define

$$\# : \bigwedge^{p,q} \otimes E \to \bigwedge^{n-p,n-q} \otimes E^*$$

by: for $u,v \in \bigwedge^{p,q} \otimes E$,

$$u \wedge \# v := \langle u,v \rangle \mathrm{dvol}$$

Fact:

$$\triangle''_{E^*}\# = \#\triangle''_E$$

□

Remark: take $E = X \times \mathbb{C}, D_E = \mathrm{d} = \mathrm{d}' + \mathrm{d}''$,$(\mathrm{d}' = \partial, \mathrm{d}'' = \overline{\partial})$ then we have

$$\triangle' = \mathrm{d}'\mathrm{d}'^* + \mathrm{d}'^*\mathrm{d}'$$

$$\triangle'' = \cdots$$

then

$$H^{p,q}_{\mathrm{d}''}(X, \mathbb{C}) \cong \ker \triangle'' \curvearrowright C^\infty(X, \bigwedge^{p,q})$$

the pairing

$$H^{p,q}(X, \mathbb{C}) \times H^{n-p,n-q}(X, \mathbb{C}) \to \mathbb{C}$$

is non-degenerate.

第 5 章 Lefschitz 分解

5.1 线性代数版本的 Lefschitz 算子

Three goals:

Kahler package

Lefschetz decomposition

Hodge-Riemann bilinear relations

Linear algebra(baby representation theory)(local case)

$\mathbb{C}^n$,

$$\omega = \sqrt{-1}\sum_{i,j} h_{ij} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$$

Kahler metric with constant coefficients.(i.e. h_{ij} is constant, (h_{ij}) is positive Hermite matrix)

W.L.O.G, by taking a linear transformation, we can assume

$$\omega = \sqrt{-1}\sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$$

记号 5.1.1. *An operator is of pure degree r if it transform a form of $\deg = k$ to as form of degree $k+r$.*

An operator ..of bi-degree (p,q) if ...$(s,t) \to (s+p, t+q)$ (in this case, degree $= p+q$)

if A, B with degree $\deg A, \deg B$,define

$$[A,B] := AB - (-1)^{\deg A \deg B} BA$$

定义 5.1.2.

$$L : \bigwedge^{p,q} \to \bigwedge^{p+1,q+1}$$

$$u \mapsto \omega \wedge u$$

is called Lefschetz operator.

Denote Λ to be the adjoint of L, adjointed by : Let $v \in \bigwedge^{p-1,q-1}$ and $u \in \bigwedge^{p,q}$

$$\langle Lv, u\rangle := \langle u, \Lambda u\rangle$$

The operator Λ is of bi-degree $(-1,-1)$.

性质 5.1.3. *If*

$$u = \sum_{\substack{|I|=p\\|J|=q}} u_{IJ} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_j$$

then

$$Lu = \sqrt{-1} \sum_{\substack{|I|=p\\|J|=q}} \sum_{m=1}^{n} u_{IJ} \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m \wedge \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

$$\Lambda u = \sqrt{-1}(-1)^p \sum_{\substack{|I|=p\\|J|=q}} \sum_{m=1}^{n} u_{IJ} \left(\frac{\partial}{\partial z_m} \lrcorner \mathrm{d}z_I \right) \wedge \left(\frac{\partial}{\partial \bar{z}_m} \lrcorner \mathrm{d}\bar{z}_J \right)$$

where "$\lrcorner$" is contraction.

推论 5.1.4. *(Exercise) Let*

$$\alpha = \sqrt{-1} \sum_{j=1}^{n} \alpha_j \mathrm{d}z_j \wedge \bar{z}_j$$

then,(α is a operator of bi-degree $(1,1)$)

$$[\alpha, \Lambda]u = \sum_{\substack{|I|=p\\|J|=q}} \left(\sum_{i\in I} \alpha_i + \sum_{j\in J} \alpha_j - \sum_{k=1}^{n} \alpha_k \right) u_{IJ} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

where

$$u = \sum_{\substack{|I|=p\\|J|=q}} u_{IJ} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

推论 5.1.5. *if $u \in \bigwedge^{p,q}$,then*

$$[L, \Lambda]u = (p+q-n)u$$

推论 5.1.6. *Denote* $B := [L, \lambda]$, *then*

$$[B, L] = 2L$$

$$[B, \Lambda] = -2\Lambda$$

证明. Take $u \in \bigwedge^{p,q}$, then

$$[B, L] = BLu - LBu = (p+q-n+2)Lu - (p+q-n)Lu = 2Lu$$

the second is similar.. □

$\mathfrak{sl}(2, \mathbb{C})$-representation

$$\mathfrak{sl}(2, \mathbb{C}) = \operatorname{span}_{\mathbb{C}} l, \lambda, b$$

where

$$l = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have

$$[l, \lambda] = b \quad [b, l] = 2l \quad [b, \lambda] = -2\lambda$$

性质 5.1.7. *There exists a natural action*

$$\rho : \mathfrak{sl}(2, \mathbb{C}) \to \operatorname{End}(\bigoplus_{p,q} \bigwedge^{p,q})$$

with

$$\rho(l) = L$$

$$\rho(\lambda) = \Lambda$$

$$\rho(b) = B$$

定理 5.1.8. *(HL)*

$$L^{n-k} : \bigwedge^{k} \to \bigwedge^{2n-k}$$

$$u \to \omega^{n-k} \wedge u$$

is an isomorphism.

$$L^{n-k}:\bigwedge^{p,q}\to\bigwedge^{n-k+p,n-k+q}$$

is also an isomorphism.

证明. Lemma:

$$[L^r,\Lambda]u=r(k-n+r-1)L^{r-1}u$$

(induction, omit)

Assume $\alpha\in\bigwedge_{\mathbb{C}}^k$, $L^{n-k}\alpha=0$, need to verify $\alpha=0$.

Claim:

$$L^r:\bigwedge^k\to\bigwedge^{k+2r}$$

is injective whenever $r\le n-k$.

proof of the claim:

claim is true when $k=0$ or $k=1$.(check)

Let $\alpha\in\bigwedge^k$ s.t. $L^r\alpha=0(r\le n-k)$. By the lemma,

$$L^r\Lambda\alpha-\lambda L^r\alpha=r(k-n+r-1)L^{r-1}\alpha$$

so,

$$L^{r-1}(L\Lambda\alpha-r(k-n+r-1)\alpha)=0$$

by the induction on r,

$$L\Lambda\alpha=r(k-n+r-1)\alpha$$

since $r(k-n+r-1)\neq 0$, $\alpha=L\beta$ for some $\beta\in\bigwedge^{k-2}$. so,$L^r\alpha=L^{r+1}\beta=0$, by induction on k, we have $\beta=0$,so $\alpha=0$.

The claim is proved. □

定义 5.1.9. *(Primitive form)*

$\alpha\in\bigwedge^k(k\le n)$ *is called primitive form, if*

$$L^{n-k+1}\alpha=0$$

推论 5.1.10. *(Lefischtz Decomposition)(LD)*

For any $\alpha\in\bigwedge^k,(1\le k\le 2n)$*, we have a unique decomposition:*

$$\alpha=\sum_{\gamma\ge(k-n)_+}L^\gamma\alpha_r$$

($(k-n)_+ := \max\{k-n, 0\}$) with $\alpha_r \in \bigwedge^{k-2r}$ is primitive

证明. Existence: assume $k \leq n$, consider

$$L^{n-k+1}\alpha \in \bigwedge^{2n-k+2}$$

by HL, $\exists! \, \beta \in \bigwedge^{k-2}$ s.t. $L^{n-k+2}\beta = L^{n-k+1}\alpha$, so $L^{n-k+1}(\alpha - L\beta) = 0$, i.e. $\alpha_0 = \alpha - L\beta$ is primitive. $\alpha = \alpha_0 + L\beta$, then induction on degrees, we get the decomposition for α.

If $k > n$, we apply HL to reduce it to case 1.

Uniqueness: Next time.. □

Today: Continuous to Hard Lef decomposition, Hodge-Riemann bilinear relations.

Hard-Lefschitz: HL

Lefschitz decomposition :LD

Hodge-Riemann bilinear relations :HRR

Recall: $\mathbb{C}^n, \bigwedge^k = \bigoplus\limits_{p+q=k} \bigwedge^{p,q}$, ω: a Kahler metric on $\mathbb{C}^n$ with constant coefficient $\in \bigwedge^{1,1}_{\mathbb{R}}$.

Lefschitz operator : $Lu = \omega \wedge u$.

定理 5.1.11. *(HL)*

Assume $k \leq n, p+q \leq n$, then

$$L^{n-k} : \bigwedge^k \to \bigwedge^{2n-k}$$

is a linear isomorphism.

$$L^{n-k} : \bigwedge^{p,q} \to \bigwedge^{p+n-k,q+n-k}$$

is also a linear isomorphism.

Linear algebra..

定理 5.1.12. *(LD) for any $u \in \bigwedge^k$,we have a unique decomposition*

$$u = \sum_{r \geq (k-n)_+} L^r u_r$$

where $u_r \in \bigwedge^{k-2r}_{prim}$ is a primitive form.

Recall: a k-form $u \in \bigwedge^k (k \leq n)$ is called primitive, if $L^{n-k+1}(u) = 0$. When $k > n$, u is called primitive, $\Lambda(u) = 0$, where Λ is the adjoint of L.

证明. Existence: application of HL.

Uniqueness: Omit. □

性质 5.1.13. *Assume $\alpha \in \bigwedge^{p,q}_{prim}$, and $p+q \leq n$. (i.e. $L^{n-p-q+1}\alpha = 0$),then*

$$*\alpha = (-1)^{\frac{(p+q)(p+q-1)}{2}} (\sqrt{-1})^{p-q} \frac{1}{(n-p-q)!} L^{n-p-q}\alpha$$

证明. See [Humphreys, Prop 1.2.31] □

定理 5.1.14. *(HRR) Define the bilinear form Q on $\bigwedge^k$ $(k \leq n)$ as follows:*

$$Q(\alpha, \beta) := L^{n-k} \wedge \alpha \wedge \overline{\beta}$$

Then

$$(\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} Q(u,u) \geq 0$$

for any $u \in \bigwedge^{p,q}_{prim}, p+q = k \leq n$, and equal holds

$$\iff u = 0$$

(i.e. $Q|_{\bigwedge^{p,q}_{prim}}$ is positive definite up to a factor)

证明. Take $u \in \bigwedge^{p,q}_{prim}$,

$$Q(u,u) = L^{n-k} \wedge u \wedge \overline{u} = *u \wedge \overline{u} = \langle \overline{u}, \overline{u} \rangle \mathrm{d}Vol = |u|^2 \mathrm{d}Vol \geq 0$$

(up to a factor!)

(We apply the following result: $\overline{*\varphi} = *\overline{\varphi}$, i.e. $*$ is a real operator) □

Summary: $\bigwedge^\bullet = \bigoplus_{1 \leq k \leq n} \bigwedge^k_{\mathbb{C}}$, where$\bigwedge^k_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q}_{\mathbb{C}}$.

Lefschitz operator $L \rightsquigarrow$ HL,LD,HRR.

5.2 紧 Kahler 流形的上同调群

The analogue of compact Kahler manifolds,

$$H^k_{DR}(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{Dol}(X, \mathbb{C})$$

ω: A Kahler metric $\in H^{1,1}_{Dol}(X, \mathbb{R})$.
Denote $L \curvearrowright H^k_{DR}(X, \mathbb{C})$,

$$L(u) = [\omega, u] = [\omega] \wedge u$$

Commutative relations on Kahler manifolds

$$(\mathbb{C}^n, \omega = \sqrt{-1} \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j)$$

$u \in C^\infty(\mathbb{C}^n, \bigwedge^{p,q})$, locally

$$u = \sum_{|I|=p,|J|=q} u_{I,J} \mathrm{d}z_I \wedge \mathrm{d}z_j, \quad v = \sum_{|I|=p,|J|=q} v_{I,J} \mathrm{d}z_I \wedge \mathrm{d}z_j$$

$$\langle\langle u, v \rangle\rangle = \int_{\mathbb{C}^n} \sum_{|I|=p,|J|=q} u_{I,J} \overline{V_{I,J}} \mathrm{d}Vol$$

$\mathrm{d} = \mathrm{d}' + \mathrm{d}''$, $\mathrm{d}' = \partial, \mathrm{d}'' = \bar{\partial}$.

$$\mathrm{d}'u = \sum_{I,J} \sum_k \frac{\partial u_{I,J}}{\partial z_k} \mathrm{d}z_k \wedge \mathrm{d}z_I \wedge \mathrm{d}z_J$$

$$\mathrm{d}''u = \cdots$$

定理 5.2.1.

$$(\mathrm{d}'')^* u = -\sum_{I,J} \sum_k \frac{\partial u_{I,J}}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_k} \lrcorner (\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J)$$

$$(\mathrm{d}')^* u = -\sum_{I,J} \sum_k \frac{\partial u_{I,J}}{\partial \bar{z}_k} \frac{\partial}{\partial z_k} \lrcorner (\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J)$$

性质 5.2.2.

$$[(\mathrm{d}'')^*, L] = \sqrt{-1}\mathrm{d}'$$

证明. Exercise. □

定理 5.2.3. *Let X be a Kahler manifold (may not compact), with Kahler metric ω, then we have*

$$[(\mathrm{d}'')^*, L] = \sqrt{-1}\mathrm{d}'$$

证明. Only need to verify $u \in C_c^\infty(X, \bigwedge^{p,q})$ with compact support in a holomorphic chart at x.

Assume the holomorphic chart near x is choosen s.t.

$$\omega(z) = \sqrt{-1} \sum_{1\le j\le n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j + O(|z|^2)$$

$$u \in \sum_{I,J} u_{I,J} \mathrm{d}z_I \wedge \bar{z}_J$$

is a (p,q)-form, v is also...

$$\langle u, q\rangle = u_{I,J}\overline{v_{M,N}}\langle \mathrm{d}z_I, \mathrm{d}z_M\rangle\langle \mathrm{d}\bar{z}_J, \mathrm{d}\bar{z}_N\rangle = u_{IJ}\overline{V_{ij}} + a_{IJMN}(z)u_{IJ}\overline{V_{MN}}$$

where $a_{IJMN} = O(|z|^2)$.

So,

$$(\mathrm{d}'')^* u = -\sum_{IJk} \frac{\partial u_{IJ}}{\partial z_k} \frac{\partial}{\partial \bar{z}_k} \lrcorner (\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J) + \sum_{IJMN} b_{IJMN} u_{IJ} \mathrm{d}z_M \wedge \mathrm{d}\bar{z}_N$$

where $b_{IJMN}(z) = O(|z|)$. So,

$$[(\mathrm{d}'')^*, L]u(x) = \sqrt{-1}\mathrm{d}'u(x)$$

$$\Longrightarrow [(\mathrm{d}'')^*, L] = \sqrt{-1}\mathrm{d}'$$

□

性质 5.2.4. *In Kahler manifold,*

$$[(\mathrm{d}')^*, L] = -\sqrt{-1}\mathrm{d}''$$
$$[\Lambda, \mathrm{d}''] = -\sqrt{-1}(\mathrm{d}')^*$$
$$[\Lambda, \mathrm{d}'] = \sqrt{-1}(\mathrm{d}'')^*$$

推论 5.2.5. *(X,ω) is a Kahler manifold, then*

$$\triangle_{\mathrm{d}} = 2\triangle_{\mathrm{d}'} = 2\triangle_{\mathrm{d}''}$$

证明. For example, $\triangle_{\mathrm{d}} = 2\triangle_{\mathrm{d}''}$,

$$\triangle_{\mathrm{d}} = (\mathrm{d}'+\mathrm{d}'')(\mathrm{d}'+\mathrm{d}'')^* + (\mathrm{d}'+\mathrm{d}'')^*(\mathrm{d}'+\mathrm{d}'') = (\mathrm{d}'+\mathrm{d}'')(\mathrm{d}'^* - \sqrt{-1}[\Lambda,\mathrm{d}']) + (\mathrm{d}'^* - \sqrt{-1}[\Lambda,\mathrm{d}'])(\mathrm{d}'+\mathrm{d}'')$$

然后暴力展开，12 项??? $\cdots$

从略。 □

推论 5.2.6. *If (X,ω) is a Kahler manifold, then*

$$\triangle_{\mathrm{d}} : C^\infty(C,\bigwedge\nolimits^{p,q}) \to C^\infty(C,\bigwedge\nolimits^{p,q})$$

证明. Since $\triangle_{\mathrm{d}} = 2\triangle_{\mathrm{d}'}$, $\triangle_{td'}$ preserves the bi-degree. □

推论 5.2.7. *If (X,ω) is a compact Kahler manifold, u is a $\triangle_{\mathrm{d}}$-harmonic k-form. Assume*

$$u = \sum_{p+q=k} u^{p,q}$$

$$u^{p,q} \in C^\infty(X,\bigwedge\nolimits^{p,q})$$

then each $u^{p,q}$ is also harmonic.

定理 5.2.8. *(Hodge decomposition)*

X is a compact Kahler manifold, then we have a decomposition

$$H^k_{\mathrm{d}}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\mathrm{d}''}(X,\mathbb{C})$$

Equivalently, (sheaf cohomology)

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X,\Omega^p)$$

证明. take a Kahler metric ω, we can define $\triangle_d, \triangle_{td'}, \triangle_{\mathrm{d}''}$, then

$$\ker \triangle_{\mathrm{d}} := \mathcal{H}^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\mathrm{d}''}(X, \mathbb{C})$$

then $\Longrightarrow$ the decomposition for $H^k_{\mathrm{d}}(X, \mathbb{C})$

the decomposition for $H^k_{\mathrm{d}}(X, \mathbb{C})$ is independent of the choice of ω (Next time) □

Recall: Hodge decomposition,

X compact Kahler manifold, $\dim_{\mathbb{C}} X = n$,

Thm:(Hodge decomposition)

$$H^k_{DR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}_{\mathrm{d}''}(X, \mathbb{C})$$

where

$$H^{p,q}(X, \mathbb{C}) = \{[\alpha] \in H^k_{DR}(X, \mathbb{C}) | \alpha \text{is a d-closed s.m. } (p,q)\text{-form}\}$$

Proof: take a Kahler metric ω,

$$H^k_{DR}(X, \mathbb{C}) \cong \mathcal{H}^k_{\mathrm{d}}(X, \mathbb{C}) = \bigoplus \mathcal{H}^{p,q}_{\mathrm{d}}(X, \mathbb{C}) = \bigoplus \mathcal{H}^{p,q}_{\mathrm{d}''}(X, \mathbb{C})$$

性质 5.2.9. *There is a canonical isomorphism*

$$H^{p,q}_{\mathrm{d}}(X, \mathbb{C}) \xrightarrow{\sim} H^{p,q}_{\mathrm{d}''}(X, \mathbb{C})$$

$$[\alpha]_{\mathrm{d}} \mapsto [\alpha]_{\mathrm{d}''}$$

where $\mathrm{d}\alpha = 0$,α *is a* (p,q)*-form.* $\Rightarrow \mathrm{d}''\alpha = 0$

证明. Check: this map is well defined. Need to verify: if $\alpha = \mathrm{d}\beta$ is a (p,q)-form, then $[\alpha]_{\mathrm{d}''} = 0$, i.e. α is also d''-exact.

α is a (p,q)-form,

$$\Rightarrow \alpha = \mathrm{d}'\beta^{p-1,q} + \mathrm{d}''\beta^{p,q-1}$$

we have $\mathrm{d}''\mathrm{d}'\beta^{p-1,q} = 0$, $\mathrm{d}'\mathrm{d}''\beta^{p,q-1} = 0$

We need a very important lemma: □

引理 5.2.10. *($\partial\bar{\partial}$-lemma)*

Let X *is a Kahler manifold,* α *is a smooth form which is* d' *and* d'' *closed. Then, if* α *is* d *or* d' *or* d''*-exact, then* $\alpha = \mathrm{d}'\mathrm{d}''\gamma$ *for some* γ.

Using $\partial\bar{\partial}$-lemma, this map is well-defined.

Now, notice that the two space has the same dimension. So, we need to show the map is injective(or, surjective). Claim : this map is injective. If α is a d-closed with $[\alpha]_{\mathrm{d}''} = 0$, i.e. $\alpha = \mathrm{d}''\beta^{p,q-1}$. α is d-closed$\Rightarrow$ $\mathrm{d}'\mathrm{d}''\beta^{p,q-1} = 0$, $\partial\bar{\partial}$-lemma applying to $\mathrm{d}''\beta^{p,q-1}$, we have

$$\mathrm{d}''\beta^{p,q-1} = \mathrm{d}'\mathrm{d}''\gamma = \mathrm{d}(\mathrm{d}''\gamma)$$

for some γ.

Proof of $\partial\bar{\partial}$-lemma:

证明. Assume α is d'' exact, 1.e. $\alpha = \mathrm{d}''\beta$, write

$$\beta = H(\beta) + \triangle_{\mathrm{d}}\gamma$$

where $H(\beta)$ is $\triangle_{\mathrm{d}}$-harmonic, so

$$\alpha = \mathrm{d}''H(\beta) + \mathrm{d}''\triangle_{\mathrm{d}}\gamma - 2\mathrm{d}''\triangle_{\mathrm{d}'}\gamma$$

(Since $\triangle_{\mathrm{d}} = 2\triangle_{\mathrm{d}''}$)

$$\Rightarrow \alpha = 2\mathrm{d}''(\mathrm{d}'\mathrm{d}'^{*} + \mathrm{d}'^{*}\mathrm{d}') = 2\mathrm{d}''\mathrm{d};\mathrm{d}'^{*}\gamma - 2\mathrm{d}'^{*}\mathrm{d}''\mathrm{d}'\gamma$$

By the assumption, $\mathrm{d}'\alpha = 0$, so $\mathrm{d}'^{*}\mathrm{d}''\mathrm{d}'\gamma = 0$

$$\alpha = -2\mathrm{d}'\mathrm{d}''\mathrm{d}'^{*}\gamma$$

$\square$

注记 5.2.11. *(Deligne-Griffiths-Morrora)*

If $\widehat{X}$ is bimeromapic to X, where X is acompact Kahler, then$\hat{X}$ is also satisfys the $\partial\bar{\partial}$-lemma.

X is a kahler manifold, then

$$H^{p,q}_{\mathrm{d}}(X,\mathbb{C}) \cong H^{p,q}_{\mathrm{d}''}(C,\mathbb{C}) \cong H^{p,q}X,\mathbb{C}$$

X us a compact complex manifold,define

$$H^{p,q}_{BC} := \frac{\text{d-closed } (p,q))}{\mathrm{d}'\mathrm{d};;\text{exact}}$$

Bott-Chern cohomology

Exercise" If X is Kahler , then $H^{p,q}_{BC} = H^{p,q}_{\mathrm{d}}$

$$H^{p,q}_{A}(X,\mathbb{C}) := \frac{\mathrm{d}'\mathrm{d}''\text{closed}}{(\mathrm{d}')\text{-exact} + \{\mathrm{d}''\text{exact}\}}$$

(Appeli cohomology)

denote

$$h^k_{BC} := \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}$$

$$h^k_A := \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_A$$

定理 5.2.12. *X satisfies $\partial\bar{\partial}$-lemma* $\iff$

$$h^k_B + h^k_A = 2b_k$$

where

$$b_k = \dim_{\mathbb{C}} H^k_{DR}(X, \mathbb{C})$$

定理 5.2.13. *(Hard Lef)*

X is a compact Kahler, $\dim_{\mathbb{C}} X = n$, denote $L = \{\omega\} \curvearrowright H^k_{DR}(X, \mathbb{C})$, ω is a Kahler metric, Then we have:

$$L^{n-k} : H^k_{DR}(X, \mathbb{C}) \cong H^{2n-k}_{DR}(X, \mathbb{C})$$

$$H^{p,q}(X, \mathbb{C}) \cong H^{p+n-k, q+n-k}(X, \mathbb{C})$$

where $k \leq n$, $p + q \leq n$.

证明. Fox a Kahler metric ω,

$$L^{n-k} : H^k_{DR} \to H^{2n-k}_{DR}$$

($\cong \mathcal{H}^k_{\mathrm{d}}$, $\cong \mathcal{H}^{2n-k}_{\mathrm{d}}$ respectively) (there is a commutative diagram...)

need to proof: For any $\varphi \in \mathcal{H}^k_{\mathrm{d}}$, then

$$L^{n-k}(\varphi) = \omega^{n-k} \wedge \varphi$$

is also harmonic. □

引理 5.2.14.

$$[\triangle_{\mathrm{d}}, L] = 0$$

证明.

$$[\triangle_{\mathrm{d}}, L] = 2[\triangle_{\mathrm{d}'}, L] = 2\left([\mathrm{d}'\mathrm{d}'^*, L] + [\mathrm{d}'^*\mathrm{d}', L]\right) = 2\left(\mathrm{d}'[\mathrm{d}'*, L] + [\mathrm{d}'^*, L]\mathrm{d}'\right)$$

(check: $[L, \mathrm{d}'] = 0$) So,

$$= -2\sqrt{-1}(\mathrm{d}'\mathrm{d}'' + \mathrm{d}''\mathrm{d}') = 0$$

□

Exercise: Complex tori

$$\mathbb{T}^n := \mathbb{C}^n/\Gamma$$

where $\Gamma = \mathbb{Z}^n$. $\mathbb{T}^n$ is a compact Kahler manifold. Then

$$H^{1,1}(\mathbb{T}^n, \mathbb{C}) \cong \bigwedge\nolimits_{\mathbb{C}}^{1,1}$$

the space of $(1,1)$-forms on $\mathbb{C}^n$ with constant coefficient, in particular,

$$\dim_{\mathbb{C}} H^{1,1}(\mathbb{T}^n, \mathbb{C}) = n^2$$

Exercise: the set of all the Kahler class on $\mathbb{T}^n \subseteq H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$ is equal to the set of $n \times n$ positive definite Hermitian metrics.

(Hint: using Hodge theory)

定理 5.2.15. *(Lefschitz decomposition)*

Define a class $\alpha \in H^k_{DR}(X, \mathbb{C})$ to be positive if

$$L^{n-k+1}(\alpha) = 0$$

if $k \leq n$.

(When $\alpha \in H^k_{DR}(X, \mathbb{C})$, $k > n$, we call α positive)

Then $\forall \varphi \in H^k_{DR}(X, \mathbb{C})$, exist unique decomposition

$$\varphi = \sum_{\gamma \geq (k-n)_+} L^\gamma \varphi_\gamma$$

where $\varphi_\gamma \in H^{k-2\gamma}_{prim}(X, \mathbb{C})$.

Similarly,

$$H^{p,q}(X, \mathbb{C}) = \bigoplus_{r \geq (p+q-n)_+} H^{p-r,q-r}_{prim}(X, \mathbb{C})$$

证明. Exercise. □

定理 5.2.16. *(HRR)*

X compact Kahler, $\dim_{\mathbb{C}} X = n$, ω is Kahler metric, define

$$Q(\alpha, \beta) = L^{n-k}\alpha \wedge \overline{\beta}$$

where $\alpha, \beta \in H^{p,q}(X, \mathbb{C})$, and $p + q = k$.

Then $Q|_{H^{p,q}_{prim}}$ is positive defined (up to a factor).

证明. Exercise. □

Exercise:Consider X-compact Kahler,$\dim_{\mathbb{C}} X = n$, ω-Kahler metric, Then $\forall \alpha, \beta \in H^{1,1}(X, \mathbb{R}) = H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$, Then

$$\left(\{\omega^{n-2}\} \cdot \alpha \cdot \beta\right)^2 \geq \left(\{\omega^{n-2}\} \cdot \alpha^2\right)\left(\{\omega^{n-2}\} \cdot \beta^2\right)$$

with equality if and only if $\alpha = \lambda\beta$ for some $\lambda \in \mathbb{R}$

Eg: $\mathbb{C}^2$, α, β real $(1, 1)$-forms,

$$(\alpha, \beta)^2 \geq \alpha^2\beta^2$$

Hint: Using HRR, and Lefschitz decomposition... "Alg-Geom-inequality over Kahler manifold".

性质 5.2.17. *X is a compact Kahler, then*

$$\overline{H^{p,q}(X, \mathbb{C})} = H^{q,p}(X, \mathbb{C})$$

证明. Use harmonic form.. and $\triangle_{\mathrm{d}}$ is a real operator... □

Summary X-compact Kahler with a Kahler metric ω, then define Lefschitz operator $L = [\omega]\wedge$, then:

Hodge decomposition:

$$H^k = \bigoplus_{p+q=k} H^{p,q}$$

$$\overline{H^{p,q}} = H^{q,p}$$

Hard Lefschitz:

$$L^{n-k} : H^{p,q} \cong H^{p+n-k, q+n-k}$$

where $p + q = k$

Lefschitz decomposition:

$$H^{p,q} = \bigoplus_{r \geq (p+q-1)_+} L^r H^{p-r,q-r}_{prim}$$

HRR:...

References Kahler pairing in other settings..

`Adiprusito-Huh-Katz: Hodge theory in combinatorial geometries`

`McMullen: On simple polytopes`

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`Beillinson-Bernstein-Deligne-Gabber: Faisceaux Pervers`

`Adiprasito: Combinatorial Lefschetz theorem beyond positivity, 2018`

术语索引