

复几何

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本课程参考以下教材：

1. Demailly: Complex analytic and differential geometry.
2. Huybrechts: Complex geometry: an introduction.
3. Morrow, Kodaira: Complex manifolds.
4. Grauert, Remmert: Coherent analytic sheaves.
5. Hormander: An introduction to complex analysis in several variables.
6. Griffiths, Harris: Principles of algebraic geometry.

在五道口也要红专并进、理实交融呀 ~

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第 1 章 多复变函数

1.1 多元全纯函数

首先快速回顾单复变函数的知识。我们通常用 Ω 来表示 \mathbb{C} 的开子集, $z = x + iy$ 为 \mathbb{C} 的坐标。对于 $z \in \mathbb{C}$ 以及实数 $R > 0$, 我们令

$$\mathbb{D}(z, R) := \{w \in \mathbb{C} \mid |w - z| < R\}$$

为以 z 为圆心 R 为半径的开圆盘。

此外, 我们有如下常用记号:

$$\begin{cases} dz := dx + i dy \\ d\bar{z} := dx - i dy \end{cases} \quad \begin{cases} \frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \\ \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \end{cases}$$

对于函数 $f: \Omega \rightarrow \mathbb{C}$, 称 f 是全纯 (holomorphic) 的, 若在 Ω 中成立

$$\bar{\partial}f := \frac{\partial f}{\partial \bar{z}} d\bar{z} = 0$$

我们知道, f 是全纯的当且仅当 f 在 Ω 处处能够局部地展开为收敛幂级数。

对于 \mathbb{C} 中的紧致集 K , 称函数 $f: K \rightarrow \mathbb{C}$ 是全纯的, 如果存在 K 的开邻域 $\Omega \supseteq K$, 使得 f 可延拓为 Ω 上的全纯函数。

单复变函数论中有如下重要结果:

定理 1.1.1. (柯西积分公式) 设 $\mathbb{D} \subseteq \mathbb{C}$ 为 \mathbb{C} 中的开圆盘, $f: \mathbb{D} \rightarrow \mathbb{C}$ 为 \mathbb{D} 上的全纯函数, 且在 $\partial\mathbb{D}$ 连续, 则对于任意 $w \in \mathbb{D}$, 成立

$$f(w) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(z)}{z - w} dz$$

此定理能推导出单变量全纯函数理论的 “almost everything”. 这里不再赘述。

我们开始考虑多变量全纯函数。

定义 1.1.2. 设 $\Omega \subseteq \mathbb{C}^n$ 为 \mathbb{C}^n 的开子集, 函数 $f: \Omega \rightarrow \mathbb{C}$ 称为 (多变量) 全纯函数, 如果满足以下条件:

- (1) f 是连续函数;
- (2) 对任意 $1 \leq j \leq n$, 以及任意固定的 $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n \in \mathbb{C}$, 关于 z_j 的单变量函数

$$z_j \mapsto f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n)$$

是 (单变量) 全纯函数。

事实上, 如果该定义中的 (2) 成立, 那么能推出 (1) 成立, 也就是说此定义中的 (1) 可以去掉。其证明比较复杂, 我们承认之。

记号 1.1.3. 对于 \mathbb{C}^n 的开子集 Ω , 我们记

$$\mathcal{O}(\Omega) := \{f: \Omega \rightarrow \mathbb{C} \mid f \text{ 是 } \Omega \text{ 上的全纯函数}\}$$

容易知道 $\mathcal{O}(\Omega)$ 有显然的 \mathbb{C} -代数结构。

本节将说明, 多变量全纯函数具有一些与单变量全纯函数类似的性质。

记号 1.1.4. 对于 $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ 以及 $R = (R_1, R_2, \dots, R_n) \in \mathbb{R}^n$, 并且 $R_j > 0$ ($\forall 1 \leq j \leq n$), 则我们记

$$\mathbb{D}(z, R) := \mathbb{D}(z_1, R_1) \times \mathbb{D}(z_2, R_2) \times \cdots \times \mathbb{D}(z_n, R_n)$$

称为以 z 为中心, R 为半径的**多圆柱** (*polydisk*)。

对于多圆柱 $\mathbb{D}(z, R)$, 我们记

$$\Gamma(z, R) := \partial\mathbb{D}(z_1, R_1) \times \partial\mathbb{D}(z_2, R_2) \times \cdots \times \partial\mathbb{D}(z_n, R_n)$$

称为 $\mathbb{D}(z, R)$ 的**特征边界** (*distinguished boundary*)。

特别注意特征边界 $\Gamma(z, R)$ 并不等于该多圆柱的边界 $\partial\mathbb{D}(z, R)$ 。

定理 1.1.5. (多变量全纯函数的柯西积分公式)

设 $f: \overline{\mathbb{D}(z, R)} \rightarrow \mathbb{C}$ 为全纯函数, 则对任意的 $w \in \mathbb{D}(z, R)$, 成立

$$f(w) = \frac{1}{(2\pi i)^n} \int_{\Gamma(z, R)} \frac{f(\xi) d\xi_1 d\xi_2 \cdots d\xi_n}{(\xi_1 - w_1)(\xi_2 - w_2) \cdots (\xi_n - w_n)}$$

证明. 由多变量全纯函数的定义, 反复使用单变量全纯函数的柯西积分公式即可。这是容易的。□

与单复变函数完全类似, 我们也有泰勒展开:

推论 1.1.6. (多元全纯函数的泰勒展开公式)

对于 $f \in \mathcal{O}(\Omega)$, 其中 $\Omega \subseteq \mathbb{C}^n$ 为开子集, 则对于任何多圆柱 $\mathbb{D}(z_0, R)$, 如果 $\overline{\mathbb{D}(z_0, R)} \subseteq \Omega$, 则对于任意 $w \in \mathbb{D}(z_0, R)$, 成立

$$f(w) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (w - z_0)^\alpha$$

其中

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{\Gamma(z_0, R)} \frac{f(z)}{(z - z_0)^{\alpha+1}} dz_1 dz_2 \cdots dz_n = \frac{f^{(\alpha)}(z_0)}{\alpha!}$$

注意这里的 α 为多重指标, 即 $\alpha = (\alpha_1, \dots, \alpha_n)$, 其中每个 α_i 都为非负整数。我们记

$$\begin{aligned} z^\alpha &:= z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \\ \alpha! &:= \alpha_1! \alpha_2! \cdots \alpha_n! \\ f^{(\alpha)} &:= (\partial_{z_1})^{\alpha_1} (\partial_{z_2})^{\alpha_2} \cdots (\partial_{z_n})^{\alpha_n} f \\ \alpha + 1 &:= (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_n + 1) \end{aligned}$$

其中 $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, f 为 n 元全纯函数。

证明. 与单复变函数的情形完全类似, 可由柯西积分公式得到。□

定理 1.1.7. (柯西不等式) 对于 \mathbb{C}^n 的开子集 Ω , 若 $f \in \mathcal{O}(\Omega)$, 多圆柱 $\overline{\mathbb{D}(z_0, R)} \subseteq \Omega$, 则对任意多重指标 $\alpha \in \mathbb{N}^n$, 成立

$$|f^{(\alpha)}(z_0)| \leq \frac{\alpha!}{R^\alpha} \sup_{z \in \Gamma(z_0, R)} |f(z)|$$

证明. 与单复变函数的情形完全类似. 利用多元泰勒展开 (推论1.1.6) 即可. \square

推论 1.1.8. 设 $\Omega \subseteq \mathbb{C}^n$ 为连通开集, $f \in \mathcal{O}(\Omega)$ 满足 $\forall 1 \leq k \leq n, \frac{\partial f}{\partial z_k}$ 在 Ω 上恒为 0, 则 f 在 Ω 上为常值函数。

推论 1.1.9. (刘维尔定理) 设 $f \in \mathcal{O}(\mathbb{C}^n)$, 并且满足

$$|f(z)| \leq A(1 + |z|)^B$$

其中 A, B 为正实数, 那么 f 必为次数不超过 B 的多项式函数。

这些性质于单变量全纯函数雷同, 证明也是类似的。

推论 1.1.10. (*Montel* 定理)

设 Ω 为 \mathbb{C}^n 的开子集, 则 $\mathcal{O}(\Omega)$ 中的任何局部一致有界的全纯函数列都存在一致收敛的子列。

证明. 仍类似于单复变全纯函数的情形. 使用柯西积分公式, 再配合 Arzela-Ascoli 定理即可. 从略. \square

现在, 简单介绍一些复的微分形式. 对于 \mathbb{C}^n , 记其复坐标为 (z_1, z_2, \dots, z_n) ; 视 \mathbb{C}^n 为 $2n$ 维实线性空间,

$$z_k = x_k + iy_k$$

从而引入

$$dz_k = dx_k + idy_k \quad (1,0)\text{形式}$$

$$d\bar{z}_k = dx_k - idy_k \quad (0,1)\text{形式}$$

定义 1.1.11. ((p, q) -形式)

设 Ω 为 \mathbb{C}^n 的非空开集, 则形如

$$u(z) = \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ}(z) dz_I \wedge d\bar{z}_J$$

的光滑张量场称为 (p, q) -形式. 记 Ω 上的 (p, q) -形式之全体为 $C_{p,q}^\infty(\Omega)$.

这里的 I, J 为多重指标。“光滑”指的是系数函数 a_{IJ} 为 Ω 上的光滑复值函数。另外，显然 $(0,0)$ -形式即为光滑函数； $C_{p,q}^\infty(\Omega)$ 具有显然的复线性空间结构，事实上还是 $C^\infty(\Omega)$ -模。

记号 1.1.12. ($\bar{\partial}$ -算子) 定义算子

$$\bar{\partial} : C_{p,q}^\infty(\Omega) \rightarrow C_{p,q+1}^\infty(\Omega)$$

如下: 对于 (p,q) -形式

$$u := \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ} dz_I \wedge d\bar{z}_J$$

则

$$\bar{\partial}u = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{k=1}^n \frac{\partial a_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J$$

类似地，也有

$$\partial : C_{p,q}^\infty(\Omega) \rightarrow C_{p+1,q}^\infty(\Omega)$$

它们与外微分算子 d 满足关系

$$d = \partial + \bar{\partial}$$

由 $d^2 = 0$ ，易知

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0$$

以下事实显然成立：

引理 1.1.13. 对于区域 Ω 上的光滑函数 $f \in C^\infty(\Omega)$ ，则 f 全纯当且仅当 $\bar{\partial}f = 0$ 。

注记 1.1.14. (*Dolbeault* 上同调) 对于 $\Omega \subseteq \mathbb{C}^n$ ，注意 $\bar{\partial}^2 = 0$ ，从而对任意 $p \geq 0$ ，有上链复形 $C_{p,\bullet}^\infty(\Omega)$ ：

$$\cdots \rightarrow C_{p,q-1}^\infty(\Omega) \xrightarrow{\bar{\partial}} C_{p,q}^\infty(\Omega) \xrightarrow{\bar{\partial}} C_{p,q+1}^\infty(\Omega) \rightarrow \cdots$$

称上同调群

$$H^{p,q}(\Omega) := H^q(C_{p,\bullet}^\infty(\Omega), \bar{\partial})$$

为区域 Ω 的 *Dolbeault* 上同调群。

类似于外微分 d 的 de-Rham 上同调群，*Dolbeault* 上同调群与 Ω 的拓扑联系密切。例如，以下定理十分重要，我们先陈述，以后再证明：

引理 1.1.15. (*Dolbeault-Grothendieck 引理*)

设 $\mathbb{D} \subseteq \mathbb{C}^n$ 为多圆柱, 则对于任意 $p, q \geq 0$,

$$H^{p,q}(\mathbb{D}) = 0$$

不难发现它与 de Rham 上同调的 Poincare 引理有些类似。

1.2 解析延拓与 Hartogs 现象

上一节介绍了多复变函数的一些“普通的”(与单变量类似)性质, 本节开始介绍多复变函数的一些独特性质。

引理 1.2.1. 设 $f \in C_c^\infty(\mathbb{C})$ 为复平面上的紧支光滑函数, 则对任意 $z \in \mathbb{C}$, 成立

$$\frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial f / \partial \bar{\tau}}{\tau - z} d\tau \wedge d\bar{\tau} = f(z)$$

证明. 基本的微积分练习。考虑换元 $\tau = z + re^{i\theta}$, 则易知

$$\begin{aligned} d\tau \wedge d\bar{\tau} &= -2ir dr \wedge d\theta \\ \frac{\partial r}{\partial \bar{\tau}} &= \frac{1}{2} e^{i\theta} \\ \frac{\partial \theta}{\partial \bar{\tau}} &= -\frac{1}{2ir} e^{i\theta} \end{aligned}$$

因此有

$$\begin{aligned} \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial f / \partial \bar{\tau}}{\tau - z} d\tau \wedge d\bar{\tau} &= \frac{-1}{2\pi} \int_0^\infty dr \int_0^{2\pi} \left(-\frac{1}{ir} \frac{\partial f}{\partial \theta}(z + re^{i\theta}) \right) d\theta \\ &\quad + \frac{-1}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty \left(\frac{\partial f}{\partial r}(z + re^{i\theta}) \right) dr \\ &= 0 + \frac{-1}{2\pi} \int_0^{2\pi} -f(z) d\theta \\ &= f(z) \end{aligned}$$

证毕。 □

引理 1.2.2. (简单版本的 $\bar{\partial}$ -引理)

设 $n \geq 2$, $\varphi \in C_{0,1}^\infty(\mathbb{C}^n)$ 为具有紧支集的光滑 $(0,1)$ -形式, 且 $\bar{\partial}\varphi = 0$, 则存在 \mathbb{C}^n 上的紧支光滑函数 g , 使得

$$\bar{\partial}g = \varphi$$

证明. 记光滑 $(0,1)$ -形式 φ 为

$$\varphi = \sum_{k=1}^n \varphi_k(z_1, \dots, z_n) d\bar{z}_k$$

则

$$\bar{\partial}\varphi = \sum_{k,l} \frac{\partial \varphi_k}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_k = \sum_{1 \leq l < k \leq n} \left(\frac{\partial \varphi_k}{\partial \bar{z}_l} - \frac{\partial \varphi_l}{\partial \bar{z}_k} \right) d\bar{z}_l \wedge d\bar{z}_k$$

从而由 $\bar{\partial}\varphi = 0$ 可得对任意 $k \neq l$,

$$\frac{\partial \varphi_k}{\partial \bar{z}_l} = \frac{\partial \varphi_l}{\partial \bar{z}_k}$$

考虑如下的 \mathbb{C}^n 上的函数 ψ : 对于 $z = (z_1, \dots, z_n) \in \mathbb{C}^n$,

$$\psi(z) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\varphi_1(\tau; z_2, \dots, z_n)}{\tau - z_1} d\tau \wedge d\bar{\tau}$$

由 φ_1 的紧支性易知 ψ 为 \mathbb{C}^n 上的光滑函数。对于 $1 < k \leq n$, 有

$$\begin{aligned} \frac{\partial \psi(z)}{\partial \bar{z}_k} &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_1}{\partial \bar{z}_k}(\tau; z_2, \dots, z_n)}{\tau - z_1} d\tau \wedge d\bar{\tau} \\ &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \varphi_k}{\partial \bar{\tau}}(\tau; z_2, \dots, z_n)}{\tau - z_1} d\tau \wedge d\bar{\tau} \\ &= \varphi_k(z) \end{aligned}$$

上式对 $k = 1$ 显然也成立。因此 $\bar{\partial}\psi = \varphi$.

最后还需要证明 ψ 是紧支的。由于 φ 紧支, 存在足够大的 $R > 0$, 使得

$$\text{supp } \varphi \subseteq \mathbb{D}(0, R)$$

因此任意取定 $z \in \mathbb{C}^n$, 使得 z 的分量 z_2, z_3, \dots, z_n 之中至少有一个模长大于 R , 则由 ψ 的定义式直接得到 $\psi(z) = 0$. (注意: 这一步严重依赖 $n \geq 2$!) 也就是说, 存在 $z \notin \mathbb{D}(0, R)$ 使得 $\psi = 0$ 在 z 的某邻域内都成立。另一方面, 由于 $\bar{\partial}\psi = \varphi$ 且 $\text{supp } \varphi \subseteq \mathbb{D}(0, R)$, 从而 ψ 在 $\mathbb{D}(0, R)$ 外部全纯, 因此由解析延拓唯一性, ψ 在 $\mathbb{D}(0, R)$ 外部恒为零, 因此 ψ 紧支。□

此引理在单复变 $n = 1$ 的情形不成立:

例子 1.2.3. 设 $\varphi_1 \in C_0^\infty(\mathbb{C})$ 为复平面上的紧支光滑函数, 并且

$$\iint_{\mathbb{C}} \varphi_1(z) \neq 0$$

考虑 \mathbb{C} 上的 $(0,1)$ -形式 $\varphi = \varphi_1(z)d\bar{z}$, 则 $\bar{\partial}\varphi = 0$ 是平凡的, 但不存在紧支光滑函数 ψ 使得 $\bar{\partial}\psi = \varphi$.

证明. 若存在紧支光滑函数 ψ 使得 $\bar{\partial}\psi = \varphi$, 则 $\frac{\partial\psi}{\partial\bar{z}} = \varphi_1$. 于是

$$0 \neq \iint_{\mathbb{C}} \varphi_1(z) dz \wedge d\bar{z} = \iint_{\mathbb{C}} \frac{\partial\psi}{\partial\bar{z}} dz \wedge d\bar{z} = 0$$

产生矛盾。 □

以下是多复变函数解析延拓的令人惊讶的性质, 它与单复变函数有本质不同:

定理 1.2.4. (*Hartogs 现象*)

设 $\Omega \subseteq \mathbb{C}^n$ 为开集 ($n \geq 2$), $K \subset\subset \Omega$ 且为 \mathbb{C}^n 的紧子集, 则对任意的 $f \in \mathcal{O}(\Omega \setminus K)$, 都存在解析延拓 $F \in \mathcal{O}(\Omega)$, 使得

$$F|_{\Omega \setminus K} = f$$

证明. 取 K 与 Ω 直接的截断函数 $\psi \in C_0^\infty(\mathbb{C}^n)$, 使得 $0 \leq \psi \leq 1$,

$$K \subset\subset \text{supp } \psi \subset\subset \Omega$$

并且 $\psi|_K \equiv 1$. 考虑

$$\tilde{f} := (1 - \psi)f$$

则 \tilde{f} 在整个 Ω 上都有定义。注意

$$\bar{\partial}\tilde{f} = -(\bar{\partial}\psi)f + (1 - \psi)\bar{\partial}f$$

易知 $\text{supp } \bar{\partial}\tilde{f} \subseteq \text{supp } \psi$. 于是由引理1.2.2, 存在光滑函数 v , 使得 $\text{supp } v \subseteq \text{supp } \psi$, 并且 $\bar{\partial}v = \bar{\partial}\tilde{f}$, 从而考虑函数

$$F := (1 - \psi)f - v$$

则 $\bar{\partial}F = 0$, 从而 $F \in \mathcal{O}(\Omega)$. 又因为易知

$$F = f \quad (\forall z \in \Omega \setminus \text{supp } \psi)$$

从而由解析延拓唯一性, 有 $F|_{\Omega \setminus K} = f$. □

关于解析延拓, 再介绍如下结果:

引理 1.2.5. (*Hartogs figure*)

对于 $n > 1$, 正实数 $0 \leq r < R$, 以及 \mathbb{C}^{n-1} 的开子集 $\omega' \subseteq \omega$, 其中 ω 是连通的。记 \mathbb{C}^n 的开子集

$$\Omega := ((\mathbb{D}(0, R) \setminus \mathbb{D}(0, r)) \times \omega) \cup (\mathbb{D}(0, R) \times \omega')$$

其中 $\mathbb{D}(0, r)$ 与 $\mathbb{D}(0, R)$ 分别为 \mathbb{C} 上的以原点为中心, r, R 为半径的开圆盘。则任意 $f \in \mathcal{O}(\Omega)$ 都可以 (唯一地) 解析延拓至

$$\tilde{\Omega} := \mathbb{D}(0, R) \times \omega$$

如此的区域 Ω 称之为 “**Hartogs figure**”。 Ω 的几何图像大致如下:

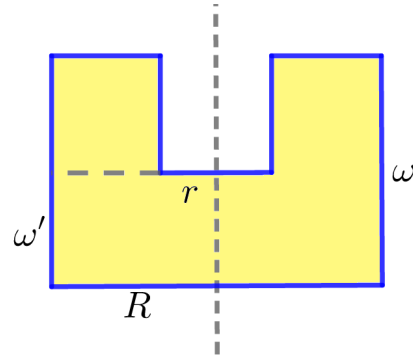


图: Hartogs figure 示意

证明. 容易知道

$$\Omega = \{(z_1, \tilde{z}) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid r < |z_1| < R, \tilde{z} \in \omega \text{ 或者 } |z_1| \leq r, \tilde{z} \in \omega'\}$$

对于 $f \in \mathcal{O}(\Omega)$, 定义 $\tilde{\Omega}$ 上的函数

$$\tilde{f}(z_1, \tilde{z}) := \frac{1}{2\pi i} \int_{|w|=\rho} \frac{f(w, \tilde{z})}{z_1 - w} dw$$

其中 ρ 为满足 $\max\{r, |z_1|\} < \rho < R$ 的任意实数。则易知如此定义的 \tilde{f} 为 f 在 $\tilde{\Omega}$ 上的解析延拓。 □

定理 1.2.6. (*Riemann 延拓定理*)

考虑 \mathbb{C}^n 中的多圆柱 $\mathbb{D}(0, R)$, 其中 $n \geq 2$, $R \in \mathbb{R}_+^n$ 。对任意 $2 \leq p \leq n$, 令 \mathbb{C}^n 的子集

$$S := (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1 = \dots = z_p = 0$$

则对任意 $f \in \mathcal{O}(\mathbb{D}(0, R) \setminus S)$, f 都可 (唯一地) 解析延拓至 $\mathbb{D}(0, R)$ 。

证明. 这是 Hartogs figure 的显然推论. 记 $R = (R_1, R_2, \dots, R_n)$, 以及 $R' := (R_2, \dots, R_n) \in \mathbb{R}^{n-1}$. 考虑 \mathbb{C}^{n-1} 的开子集

$$\begin{aligned}\omega &:= \mathbb{D}(0, R') \\ \omega' &:= \omega \setminus \{z_2 = \dots = z_p = 0\}\end{aligned}$$

则易知

$$\mathbb{D}(0, R) \setminus S = \left(\mathbb{D}(0, R_1) \setminus \{0\} \times \omega \right) \cup \left(\mathbb{D}(0, R_1) \times \omega' \right)$$

为 Hartogs figure, 从而完。 □

1.3 Weierstrass 预备定理与除法定理

回顾单复变函数, 若 f 在 $0 \in \mathbb{C}$ 附近全纯, 且 $f(0) = 0$, 则在 0 附近 f 可以唯一地分解为 $f = z^d g(z)$, 其中 g 全纯且 $g(0) \neq 0$, d 为 f 在 0 处的零点阶数。

现在, 设 $f = f(z, w)$ 在 $0 \in \mathbb{C}^n (n \geq 2)$ 附近全纯, 其中 $z \in \mathbb{C}$, $w \in \mathbb{C}^{n-1}$. 固定 w , 记

$$f_w(z) := f(z, w)$$

为关于 z 的单复变函数。如果 $f_0(0) = 0$ 且 $f_0(z)$ 不恒为零, 则 $f_0(z) = z^d g_0(z)$ 。我们的一个结果是, 若 “ f_0 ” 的下标 “0” 稍微 “扰动” 一下, 则相应的多项式 z^k 也 “随之扰动”。

记号 1.3.1. (*Weierstrass* 多项式)

对于 $(z_0, w_0) \in \mathbb{C} \times \mathbb{C}^{n-1}$, 则 (z_0, w_0) 处的 **Weierstrass 多项式** 是指形如下述的定义于 (z_0, w_0) 附近的 n 元全纯函数:

$$P(z, w) = z^k + a_1(w)z^{k-1} + \dots + a_k(w)$$

其中 $a_i (1 \leq i \leq k)$ 为定义在 $w_0 \in \mathbb{C}^{n-1}$ 附近的全纯函数, 且 $a_i(w_0) = 0$.

关于多元全纯函数在其零点附近的行为, 首先有如下:

定理 1.3.2. (*Weierstrass* 预备定理)

设 $f(z, w)$ 为定义在 $(0, 0) \in \mathbb{C} \times \mathbb{C}^{n-1}$ 附近的全纯函数, $f(0, 0) = 0$, 且 $f_w(z)$ 在 $z = 0$ 附近不恒为零, 则存在唯一的 $(0, 0)$ 处的 *Weierstrass* 多项式 $P(z, w)$, 使得

$$f(z, w) = P(z, w)h(z, w)$$

其中 $h(z, w)$ 在 $(0, 0)$ 附近全纯, 且 $h(0, 0) \neq 0$.

证明. 分若干步。

Step1 设 $f_0(z)$ 在 $z = 0 \in \mathbb{C}$ 处的零点阶数为 $d \geq 1$, 取足够小的 $\varepsilon > 0$ 使得 $f_0(z)$ 在 $|z| \leq \varepsilon$ 之中不再有 $z = 0$ 之外的零点。再由 f 的连续性以及 $\{|z| = \varepsilon\} \subseteq \mathbb{C}$ 的紧性, 存在足够小的 $\varepsilon' > 0$, 使得对任意 $|z| = \varepsilon, |w| < \varepsilon'$, $f_w(z) \neq 0$.

对于 $w \in \mathbb{C}^{n-1}$ 且 $|w| < \varepsilon'$, 由辐角原理, $f_w(z)$ 在 $|z| < \varepsilon$ 内的零点个数 (记重数) 为

$$d(w) = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{f'_w(\zeta)}{f_w(\zeta)} d\zeta$$

这是关于 w 的连续函数, 且 $d(0) = d$. 从而不妨缩小 ε' , 使得任意 $|w| < \varepsilon'$, $f_w(z)$ 在 $|z| < \varepsilon$ 内的零点个数 (计重数) 均为 d .

Step2 对于 $w \in \mathbb{C}^{n-1}$ 且 $|w| < \varepsilon'$, 记 $f_w(z)$ 的 d 个零点为 $s_1(w), s_2(w), \dots, s_d(w)$, 它们允许相同, 则 $|s_j(w)| < \varepsilon$ (注意 $s_j(w)$ 未必为关于 w 的全纯函数)。特别地 $s_1(0) = s_2(0) = \dots = s_d(0) = 0$. 考虑多项式

$$\begin{aligned} P(z, w) &:= \prod_{j=1}^d (z - s_j(w)) \\ &= z^d + \sum_{j=1}^d a_j(w) z^{d-j} \end{aligned}$$

显然系数 $a_j(w)$ 满足 $a_j(0) = 0$. 断言 $P(z, w)$ 为 Weierstrass 多项式。为此只需证明 $s_j(w)$ 关于 w 全纯。由代数学可知, 系数 a_j 可以写为形如 $s_1^k(w) + s_2^k(w) + \dots + s_d^k(w)$ ($k \geq 0$) 的 \mathbb{C} -线性组合; 而由留数定理易知

$$\sum_{j=1}^d s_j^k(w) = \frac{1}{2\pi i} \int_{|\zeta|=\varepsilon} \zeta^k \frac{f'_w(\zeta)}{f_w(\zeta)} d\zeta$$

从而关于 w 全纯。这就说明了 $P(z, w)$ 的系数函数 $a_j(w)$ 关于 w 全纯。

Step3 令 $h(z, w) := \frac{f(z, w)}{P(z, w)}$, 断言 h 在 $(0, 0)$ 附近全纯, 又因为显然 $h(0, 0) \neq 0$, 从而 Weierstrass 预备定理的存在性得证。由单复变易知 $h(z, w)$ 关于 z 全纯, 于是只需证明 h 关于 w 全纯。

任取 $w \in \mathbb{C}^{n-1}$ 且 $|w| < \varepsilon'$, 由于 $h_w(z) := h(z, w)$ 关于 z 全纯, 从而

$$h(z, w) = \frac{1}{2\pi i} \int_{|\zeta|=\varepsilon} \frac{h_w(\zeta)}{\zeta - z} d\zeta$$

而被积函数 $(\zeta, w) \mapsto \frac{h_w(\zeta)}{\zeta - z}$ 在 $\{(z, w) \mid |z| = \varepsilon, |w| < \varepsilon'\}$ 的某个邻域全纯, 从而 $h(z, w)$ 关于 w 也全纯。存在性证毕。

Step4 唯一性几乎显然, 因为 f (在 $(0, 0)$ 附近) 的零点完全由 Weierstrass 多项式贡献: 对于 w , 以 $s_1(w), \dots, s_d(w)$ 为零点的关于 z 的首一多项式只能是 $P(z, w)$. \square

定理 1.3.3. (Weierstrass 除法定理)

设 $f(z, w)$ 为定义在 $(0, 0) \in \mathbb{C} \times \mathbb{C}^{n-1}$ 附近的全纯函数, $g(z, w) = z^d + \sum_{j=1}^d a_j(w)z^{d-j}$ 为次数为 d 的 Weierstrass 多项式。那么存在唯一的 $h(z, w)$ 与 $r(z, w)$, 其中 h 为定义在 $(0, 0) \in \mathbb{C} \times \mathbb{C}^{n-1}$ 附近的全纯函数, r 为关于 z 的在 $(0, 0)$ 处的次数 $< d$ 的多项式, 使得

$$f = gh + r$$

在 $(0, 0)$ 附近成立。

证明. 先看唯一性。

Step1 唯一性是容易的。如果 $f = gh_1 + r_1 = gh_2 + r_2$, 则

$$r_1 - r_2 = g(h_2 - h_1)$$

注意 g, r_1, r_2 为 Weierstrass 多项式, 从而由之前讨论, 存在足够小的 $\varepsilon, \varepsilon' > 0$ 使得对任意 $w \in \mathbb{C}^{n-1}$ 且 $|w| < \varepsilon'$, $g_w(z)$ 在 $\{|z| < \varepsilon\}$ 内的零点个数 (计重数) 恰为 g 的次数 d , 并且 $(r_1 - r_2)_w(z)$ 在此范围内的零点个数 (计重数) 恰为 $(r_1 - r_2)$ 的次数。注意 r_1, r_2 的次数均小于 d , 从而若 $r_1 \neq r_2$, 则导致 $(r_1 - r_2)_w(z)$ 的零点个数小于 $g_w(z)(h_2 - h_1)_w(z)$, 因此导致矛盾。这迫使 $r_1 = r_2$ 。

Step2 再看存在性。取 $\varepsilon, \varepsilon' > 0$ 使得对任意 $|z| = \varepsilon$, $|w| \leq \varepsilon'$, $g_w(z) \neq 0$ 。对任意 $|z| < \varepsilon, |w| < \varepsilon'$, 定义

$$h(z, w) = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{g_w(\xi)(\xi - z)} d\xi$$

则易知 $h(z, w)$ 在 $(0, 0)$ 附近全纯。再令 $r := f - gh$, 只需证明 r 为关于 z 的次数小于 d 的 Weierstrass 多项式即可。事实上,

$$\begin{aligned} r(z, w) &= f(z, w) - g(z, w)h(z, w) \\ &= \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{\xi - z} d\xi - \frac{g_w(z)}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{g_w(\xi)(\xi - z)} d\xi \\ &= \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)(g_w(\xi) - g_w(z))}{g_w(\xi)(\xi - z)} d\xi \\ &= \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{g_w(\xi)} \frac{(\xi^d - z^d) + a_1(w)(\xi^{d-1} - z^{d-1}) + \dots}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \frac{f_w(\xi)}{g_w(\xi)} (z^{d-1} + \beta_1(\xi, w)z^{d-2} + \dots) d\xi \end{aligned}$$

其中函数 $\beta_j(\xi, w)$ 由 g 的系数函数 $a_k(w)$ 决定。容易看出 $r(z, w)$ 的确为关于 z 的次数 $\leq d-1$ 的多项式。存在性证毕。 \square

注意 r 未必是 Weierstrass 多项式, 因为 $r(z, w)$ 的 z^{d-1} 的系数

$$\frac{1}{2\pi i} \int_{|\xi|=\epsilon} \frac{f_w(\xi)}{g_w(\xi)} d\xi$$

不见得是 1 (若此积分为 0, 则 r 的首项系数甚至可以是关于 w 的函数)。

注记 1.3.4. 事实上, Weierstrass 除法定理对单复变 $n = 1$ 的情形也成立。设 $f(z) = \sum_{k=0}^{\infty} a_k z^k$ 在 $0 \in \mathbb{C}$ 附近全纯, $g(z) = z^d$ 为次数为 d 的 Weierstrass 多项式。则令

$$\begin{aligned} h(z) &= \sum_{k=d}^{\infty} a_k z^{k-d} \\ r(z) &= \sum_{k=0}^{d-1} a_k z^k \end{aligned}$$

则 $f = gh + r$ 满足要求。

1.4 解析函数芽环 $\mathcal{O}_{\mathbb{C}^n, z}$ 及其代数结构

本节继续研究多元解析函数的性质。首先回顾函数芽的概念。

定义 1.4.1. (解析函数芽环)

对于 $z \in \mathbb{C}^n$, 记

$$\mathcal{O}_{\mathbb{C}^n, z} := \{(U, f) | U \text{ 是 } z \text{ 在 } \mathbb{C}^n \text{ 的一个开邻域, } f \text{ 为定义在 } U \text{ 上的全纯函数}\} / \sim$$

其中模掉的关系 \sim 为

$$(U, f) \sim (V, g) \iff \text{存在 } z \text{ 的开邻域 } W, \text{ 使得 } W \subseteq U \cap V, \text{ 且 } f|_W = g|_W$$

粗俗地说, $\mathcal{O}_{\mathbb{C}^n, z}$ 就是“定义在 $z \in \mathbb{C}^n$ 附近的全纯函数之全体”。之前介绍的 Weierstrass 预备定理、Weierstrass 除法定理其实都是解析函数芽环的性质。容易验证, $\mathcal{O}_{\mathbb{C}^n, z}$ 在通常的函数加法、乘法下构成环。

我们记 $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n, 0}$. 本节介绍环 \mathcal{O}_n 的代数性质。假定读者熟悉基础的交换代数。本讲义中的“环”默认为含么、交换的。

定理 1.4.2. \mathcal{O}_n 是局部诺特环 ($\forall n \geq 1$)。

回顾：环 A 称为**局部环** (local ring)，若 A 存在唯一极大理想 \mathfrak{m} （等价定义： A 的全体不可逆元构成 A 的理想）；环 A 称为**诺特环** (Noetherian ring)，若满足理想升链条件（等价定义： A 的每个理想都是有限生成的）。

证明. 显然 \mathcal{O}_n 为局部环，其极大理想 \mathfrak{m} 由定义在 0 附近、在 0 处取值为 0 的函数芽构成。我们对 n 归纳证明 \mathcal{O}_n 为诺特环。

$n = 1$ 时，在单复变中我们早已熟知 $\mathcal{O}_1 \cong \{\text{收敛半径} \geq 0 \text{ 的幂级数}\}$ 为主理想整环 (PID)，其理想形如 $J_k = (z^k)$ 。特别地，为诺特环。

一般地，对于 $n \geq 2$ ，若 \mathcal{O}_{n-1} 为诺特环，则对 \mathcal{O}_n 的任意非零理想 J ，断言 J 时有限生成的。任取 $0 \neq h \in J \subseteq \mathfrak{m}$ ，则 $h(0) = 0$ ，不妨 $h(z, 0)$ 不恒为零（其中 $z \in \mathbb{C}, 0 \in \mathbb{C}^{n-1}$ ），则由 Weierstrass 预备定理，存在 Weierstrass 多项式 $P(z, w) \in \mathcal{O}_{n-1}[z] \subseteq \mathcal{O}_n$ 以及函数芽 $h' \in \mathcal{O}_n \setminus \mathfrak{m}$ ，使得 $h(z, w) = P(z, w)h'(z, w)$ 。注意 $h'(0, 0)$ 为 \mathcal{O}_n 的可逆元，又 $h \in J$ 且 J 为 \mathcal{O}_n 的理想，从而 $P(z, w) \in J$ 。

这说明 J 当中必存在 Weierstrass 多项式。取定

$$P(z, w) = z^d + \sum_{j=1}^d a_j(w)z^{d-j} \in J$$

则对任意 $f \in J$ ，对 f, P 使用 Weierstrass 除法定理，存在 $g(z, w) \in \mathcal{O}_n$ ，以及

$$r(z, w) = \sum_{k=0}^{d-1} c_k(w)z^k \in \mathcal{O}_{\mathbb{C}^{n-1}}[z]$$

为次数至多为 $(d-1)$ 的多项式，使得

$$f = gP + r$$

则 $r(z, w) \in J$ ，并且容易验证，这诱导了 \mathcal{O}_{n-1} -模同态

$$\begin{aligned} \varphi: J &\rightarrow \mathcal{O}_{n-1}^{\oplus d} \cong \{r \in \mathcal{O}_{n-1}[z] \mid \deg_z r < d\} \\ f &\mapsto \sum_{k=0}^{d-1} c_k(w)z^k \end{aligned}$$

由归纳假设， \mathcal{O}_{n-1} 为诺特环，从而 $\mathcal{O}_{n-1}^{\oplus d}$ 作为有限生成 \mathcal{O}_{n-1} -模为诺特模，从而其子模 $\text{Im } \varphi$ 也为有限生成的。注意 $\text{Im } \varphi \subseteq J$ ，记 $\{\beta_1, \dots, \beta_N\} \subseteq \text{Im } \varphi$ 为 $\text{Im } \varphi$ 的一组 \mathcal{O}_{n-1} -生成元，其中

$$\beta_j(w) = \sum_{l=0}^{d-1} \beta_{j,l}(w)z^l \in \mathcal{O}_{n-1}^{\oplus d}$$

则易知

$$\{\beta_j\}_{1 \leq j \leq N} \cup \{P(z, w)\}$$

为理想 J 的一组生成元，因此 J 是有限生成的。从而 \mathcal{O}_n 为诺特环。 \square

引理 1.4.3. 设 $P, Q \in \mathcal{O}_{n-1}[z] \subseteq \mathcal{O}_n$, 其中 P 为 Weierstrass 多项式, 则 P 整除 Q 在 \mathcal{O}_n 成立, 当且仅当 P 整除 Q 在 $\mathcal{O}_{n-1}[z]$ 中成立。

证明. “当”是显然的, 只证“仅当”。若 $P|Q$ 在 \mathcal{O}_n 中成立, 则令

$$Q(z, w) = f(z, w)P(z, w)$$

其中 $f \in \mathcal{O}_n$. 另一方面, 考虑 $\mathcal{O}_{n-1}[z]$ 中标准的欧几里得带余除法,

$$Q(z, w) = g(z, w)P(z, w) + r(z, w)$$

其中 $g, r \in \mathcal{O}_{n-1}[z]$. 则 Weierstrass 除法定理的唯一性迫使 $f = g, r = 0$, 从而得证。□

引理 1.4.4. 设 $P(z, w) \in \mathcal{O}_{n-1}[z]$ 为 Weierstrass 多项式, 则:

(1) 若在 $\mathcal{O}_{n-1}[z]$ 中有分解

$$P = P_1 P_2 \cdots P_N$$

则在相差 \mathcal{O}_{n-1} 中的可逆元的意义下, 每个 P_j 都为 Weierstrass 多项式;

(2) P 为 \mathcal{O}_n 中的不可约元当且仅当 P 为 $\mathcal{O}_{n-1}[z]$ 中的不可约元。

证明.

(1) 记 $\deg_z P = s$, 以及 $\deg_z P_j = s_j$, 则 $s = \sum_{j=1}^N s_j$. 不妨每个 $s_j > 0$. 考虑 P 的最高次项, 有

$$z^s = z^s \prod_{j=1}^N (P_j \text{ 的 } z^{s_j} \text{ 系数})$$

从而相差 \mathcal{O}_{n-1} 中某个可逆元倍, 不妨每个 P_j 的 z^{s_j} 系数都为 1. 再注意

$$z^s = P(0, z) = \prod_{j=1}^N P_j(0, z) = \prod_{j=1}^N (z^{s_j} + \cdots)$$

从而迫使 $P_j(0, z) = z^{s_j}$, 因此 P_j 为 Weierstrass 多项式。

(2) “仅当”是显然的, 只证“当”。仍记 $P(z, w)$ 关于 z 的次数为 s . 如果 P 在 \mathcal{O}_n 中可约, 令 $P = g_1 g_2$, 其中 g_1, g_2 为 \mathcal{O}_n 中的不可逆元, 从而关于 z 的函数 $g_1(z, 0), g_2(z, 0)$ 在 $z = 0$ 处的零点阶数大于 0, 分别记为 s_1, s_2 . 由 Weierstrass 预备定理, 存在分解

$$g_j(z, w) = P_j(z, w)u_j(z, w) \quad (j = 1, 2)$$

使得 $P_j \in \mathcal{O}_{n-1}[z]$ 为次数为 s_j 的 Weierstrass 多项式, u_j 为 \mathcal{O}_n 中的可逆元. 所以在 \mathcal{O}_n 中成立 $(P_1 P_2)|P$; 再根据引理 1.4.3, 可知 $(P_1 P_2)|P$ 在 $\mathcal{O}_{n-1}[z]$ 中也成立. 而 P, P_1, P_2 都为首一多项式, 从而必有 $P = P_1 P_2$, 因此 P 在 \mathcal{O}_{n-1} 中可约。□

定理 1.4.5. \mathcal{O}_n 是唯一分解整环 (UFD).

证明. 对 n 归纳. $n = 1$ 时, \mathcal{O}_1 为主理想整环, 从而为唯一分解整环. 对于 $n \geq 2$, 如果 \mathcal{O}_{n-1} 为唯一分解整环, 则由代数学中的高斯引理, 多项式环 $\mathcal{O}_{n-1}[z]$ 也是唯一分解整环.

现在, 对于 \mathcal{O}_n 中的不可逆元 f , 不妨 $z \mapsto f(z, w)|_{w=0}$ 不恒为零 ($w \in \mathbb{C}^{n-1}$), 从而由 Weierstrass 预备定理, 存在分解 $f(z, w) = u(z, w)P(z, w)$, 其中 u 为 \mathcal{O}_n 中的可逆元, $P \in \mathcal{O}_{n-1}[z]$ 为 Weierstrass 多项式. 由归纳假设, $\mathcal{O}_{n-1}[z]$ 为唯一分解整环, 从而存在 P 在 $\mathcal{O}_{n-1}[z]$ 中的分解 $P = P_1 P_2 \cdots P_s$, 使得每个 P_j 都为 $\mathcal{O}_{n-1}[z]$ 中的不可约元. 从而由引理 1.4.4 的 (1), 不妨每个 P_j 都为 Weierstrass 多项式; 再对每个 P_j 使用引理 1.4.4 的 (2), 知 P_j 为 \mathcal{O}_n 中的不可约元. 从而 $f \in \mathcal{O}_n$ 的不可约分解的存在性证毕.

再看分解的唯一性. 只需再证明 \mathcal{O}_n 的不可约元都是素元. 若 f 为 \mathcal{O}_n 中的不可约元, 以及 $g, h \in \mathcal{O}_n$ 使得 $f|gh$, 断言 $f|g$ 或者 $f|h$. 由 Weierstrass 预备定理, 不妨假设 $f = f(z, w)$ 为关于第一个分量 z 的 Weierstrass 多项式, 从而由 $f|gh$ 知 $g(z, 0), h(z, 0)$ 也不恒为零, 于是由 Weierstrass 预备定理也不妨 $g, h \in \mathcal{O}_{n-1}[z]$ 为 Weierstrass 多项式. 因此 $f|gh$ 在 $\mathcal{O}_{n-1}[z]$ 中成立, 而由归纳假设 $\mathcal{O}_{n-1}[z]$ 是唯一分解整环, 且 f 在 $\mathcal{O}_{n-1}[z]$ 不可约, 所以 $f|g$ 或者 $f|h$ 在 $\mathcal{O}_{n-1}[z]$ 中成立, 从而在 \mathcal{O}_n 中成立. 证毕. \square

1.5 解析集与局部解析零点定理

多复变函数与单复变的一个显著区别是解析延拓的难易程度, Hartogs 现象表明多复变函数“更容易被解析延拓”; 而单复变与多复变函数另一个区别是零点集的形态: 在单复变中我们熟知全纯函数零点离散 (除非函数恒为零), 这在多复变中显然不对, 例如 \mathbb{C}^2 上的全纯函数 $f(z_1, z_2) = z_1$.

事实上, 多元全纯函数的零点集十分重要, 而且是代数几何学中的某些概念 (代数簇) 的源头.

定义 1.5.1. (解析集)

设 $n \geq 2$, \mathbb{C}^n 的子集 A 称为**解析集** (analytic set), 若对任意 $z \in A$, 存在 z 在 \mathbb{C}^n 中的开邻域 Ω , 以及 $f_1, f_2, \dots, f_N \in \mathcal{O}(\Omega)$, 使得

$$A \cap \Omega = \{w \in \Omega | f_1(w) = f_2(w) = \cdots = f_N(w)\}$$

也就是说, “局部上看是若干全纯函数的公共零点集”. 对于一个解析集, 我们首先局部地研究之——类似于解析函数芽环, 我们引入如下概念:

定义 1.5.2. (解析集芽) 对于 $x \in \mathbb{C}^n$, 定义

$$\mathcal{A}_x := \{(A, x) | x \in A, A \text{ 是 } \mathbb{C}^n \text{ 中的解析集}\} / \sim$$

其中关系 \sim 为: $(A_1, x) \sim (A_2, x) \iff$ 存在 x 在 \mathbb{C}^n 中的开邻域 Ω , 使得 $A_1 \cap \Omega = A_2 \cap \Omega$. 称 \mathcal{A}_x 中的元素为 x 处的解析集芽。

\mathcal{A}_x 中的元素可以认为是包含 x 的“无穷小解析集”。容易知道它与解析函数芽的关系: 任意 $(A, x) \in \mathcal{A}_x$, (A, x) 为 $\mathcal{O}_{\mathbb{C}^n, x}$ 中某些函数的公共零点集。

定义 1.5.3. 对于 $x \in \mathbb{C}^n$,

(1) 对与 x 处的解析集芽 $(A, x) \in \mathcal{A}_x$, 定义 $\mathcal{O}_{\mathbb{C}^n, x}$ 的理想

$$J_{(A, x)} := \{f \in \mathcal{O}_{\mathbb{C}^n, x} | f(z) = 0 \forall z \in A\}$$

(2) 对于 $\mathcal{O}_{\mathbb{C}^n, x}$ 中的理想 J , 定义 x 处的解析集芽

$$(V(J), x) := \{z \in \mathbb{C}^n | g(z) \equiv 0, \forall g \in J\} \text{ 的等价类}$$

这里并未仔细写清楚, 需要验证良定性: 注意解析集芽、函数芽实际上都为等价类, 我们需要验证与代表元选取无关, 留给读者。

注意 $\mathcal{O}_{\mathbb{C}^n, x}$ 为诺特环, 从而任何理想 J 都是有限生成的, 记 $\{g_1, g_2, \dots, g_N\}$ 为其一组生成元, 则易知

$$V(J) = \{g_1(x) = g_2(x) = \dots = g_N(x) = 0\}$$

在 x 附近为有限个解析函数的公共零点集, 从而的确为解析集 (芽)。

引理 1.5.4. 设 $x \in \mathbb{C}^n$, $(A, x) \in \mathcal{A}_x$ 为 x 处的解析集芽, $J \subseteq \mathcal{O}_{\mathbb{C}^n, x}$ 为理想, 则

$$\begin{aligned} J &\subseteq J_{(V(J), x)} \\ (V(J_{(A, x)}), x) &= (A, x) \end{aligned}$$

证明. 直接按定义验证即可。第一式是容易的; 至于第二式, 由解析集的定义, (A, x) 必形如

$$\{g_1(x) = g_2(x) = \dots = g_N(x) = 0\}$$

其中 $g_j \in \mathcal{O}_{\mathbb{C}^n, x}$, 从而 $J_{(A, x)} = (g_1, \dots, g_N)$, 之后容易。 □

注记 1.5.5. 不过要注意, 第一式的等号未必成立, 例如对于 $0 \in \mathbb{C}^2$, $f(z_1, z_2) = z_1^2$, 令 $J := (f) \subseteq \mathcal{O}_{\mathbb{C}^2, 0}$ 为由 f 生成的理想, 则 $V(J) = \{z_1^2 = 0\} = \{z_1 = 0\}$, 于是 $J_{(V(J), 0)} = (z_1)$, 即为由 $\tilde{f}(z_1, z_2) = z_1$ 生成的理想。很明显, $J \subsetneq J_{(V(J), 0)}$.

对于 $x \in \mathbb{C}^n$, 则 \mathcal{A}_x 中的解析集芽可以进行交、并运算:

引理 1.5.6. 对于 $x \in \mathbb{C}^n$, $\{J_\alpha | \alpha \in \mathcal{I}\}$ 为 $\mathcal{O}_{\mathbb{C}^n, x}$ 的一族理想, 则对任意 $\alpha, \beta \in \mathcal{I}$,

$$(V(J_\alpha) \cup V(J_\beta), x) = (V(J_\alpha J_\beta), x)$$

$$\left(\bigcap_{\alpha \in \mathcal{I}} V(J_\alpha), x\right) = \left(V\left(\sum_{\gamma \in \mathcal{I}} J_\gamma\right), x\right)$$

自行补全解析集芽交、并的定义（无非是取代表元作交、并）

证明. 直接定义验证。 □

此引理表明, 一点处的解析集芽可以“有限并, 任意交”, 与拓扑学中的“闭集”类似。
接下来研究解析集芽的局部结构。

定义 1.5.7. (不可约解析集芽)

对于 $x \in \mathbb{C}^n$, 以及 $(A, x) \in \mathcal{A}_x$, 称解析集芽 (A, x) 是不可约 (*irreducible*) 的, 若不存在 $(A_1, x), (A_2, x) \in \mathcal{A}_x$, 使得 $(A, x) = (A_1 \cup A_2, x)$, 且 $(A_i, x) \subsetneq (A, x), i = 1, 2$.

由引理1.5.6, 以及基本的交换代数, 容易知道: 解析集芽 (A, x) 不可约, 当且仅当 $J_{(A, x)}$ 为 $\mathcal{O}_{\mathbb{C}^n, x}$ 的素理想。此外, 解析函数芽环的诺特性等价于如下:

引理 1.5.8. 对于 $x \in \mathbb{C}^n$, 以及 $(A_k, x) \in \mathcal{A}_x, k \geq 1$, 若 $(A_k, x) \supseteq (A_{k+1}, x)$ 对任意 $k \geq 1$ 都成立 (即 $\{A_k\}_{k=1}^\infty$ 为解析集芽降链), 则存在 $k_0 \geq 1$, 使得对任意 $l \geq k_0$, 都有 $(A_k, x) = (A_l, x)$.

证明. 考察理想 $J_{(A_k, x)} \subseteq \mathcal{O}_{\mathbb{C}^n, x}$, 则 $(A_k, x) \supseteq (A_{k+1}, x)$ 表明

$$J_{(A_k, x)} \subseteq J_{(A_{k+1}, x)}$$

即 $\{J_{(A_k, x)}\}_{k=1}^\infty$ 为理想升链, 从而由 $\mathcal{O}_{\mathbb{C}^n, x}$ 的诺特性, 以及引理1.5.4, 得证。 □

定理 1.5.9. (解析集芽的不可约分解)

给定 $x \in \mathbb{C}^n$, 则对任意 $(A, x) \in \mathcal{A}_x$, 存在 $N \geq 1$, 以及对任意 $1 \leq k \leq N$ 存在 $(A_k, x) \in \mathcal{A}_x$ 为不可约解析集芽, 使得这些解析集芽互不包含, 并满足

$$(A, x) = \bigcup_{k=1}^N (A_k, x)$$

并且上述分解是唯一的 (不计次序)。

证明. 存在性: 先断言, 若 (A, x) 可约, 则存在分解 $(A, x) = (A^{(1)}, x) \cup (A^{(2)}, x)$, 其中 $(A^{(1)}, x)$ 与 $(A^{(2)}, x)$ 都为 (A, x) 的真子芽, 并且 $(A^{(1)}, x)$ 不可约。

这是因为, 由 (A, x) 可约, 取真子芽 $(A_1, x), (A'_1, x)$ 使得 $(A, x) = (A_1, x) \cup (A'_1, x)$ (但至此无法保证 A_1, A_2 至少有一个不可约)。如果 (A_1, x) 不可约, 则继续对其分解: $(A_1, x) = (A_2, x) \cup (A'_2, x)$, 然后再考察 (A_2, x) 的可约性, 不断做下去, 总会得到不可约的 (A_k, x) ; 若不然就有解析集芽降链

$$(A_1, x) \supsetneq (A_2, x) \supsetneq (A_3, x) \supsetneq \cdots$$

与引理1.5.8矛盾。因此必存在 $k > 0$, 使得 (A_k, x) 不可约, 此时

$$(A, x) = (A_k, x) \cup \left(\bigcup_{j=1}^k (A'_j, x) \right)$$

为所希望的分解, 断言证毕。

反复使用此断言: 令 $(A, x) = (A^{(1)}, x) \cup (B_1, x)$, 其中 $(A^{(1)}, x)$ 不可约, 若 (B_1, x) 可约, 则再对 (B_1, x) 使用此断言: $(B_1, x) = (A^{(2)}, x) \cup (B_2, x)$, 其中 $(A^{(2)}, x)$ 不可约; 若 (B_2, x) 可约, 则再继续对 (B_2, x) 使用断言……该操作必在有限步停止, 停止于某个 $(B_{\tilde{N}}, x)$ 不可约, 否则就有解析集芽降链

$$(B_1, x) \supsetneq (B_2, x) \supsetneq (B_3, x) \cdots$$

与引理1.5.8矛盾。从而得到不可约分解

$$(A, x) = (B_{\tilde{N}}, x) \cup \left(\bigcup_{k=1}^{\tilde{N}} (A_k, x) \right)$$

之后适当取 $\{A_1, A_2, \dots, A_{\tilde{N}}; B_{\tilde{N}}\}$ 的子集使得其中元素之并仍是 (A, x) 并且其中元素互不包含。因此存在性证毕。

唯一性: 假设

$$(A, x) = \bigcup_{k=1}^N (A_k, x) = \bigcup_{k=1}^{N'} (A'_k, x)$$

都为 (A, x) 的满足题设的不可约分解, 则需要证明 $N = N'$, 并且有集合相等

$$\{A_1, A_2, \dots, A_N\} = \{A'_1, A'_2, \dots, A'_{N'}\}$$

对任意 A_i , 因为

$$(A_i, x) = \bigcup_{k=1}^{N'} (A_i \cap A'_k, x)$$

从而 (A_i, x) 的不可约性迫使存在某个 (A'_j, x) 使得 $(A_i, x) = (A_i \cap A'_j, x)$, 即 $(A_i, x) \subseteq (A'_j, x)$. 同理, 对于此 (A'_j, x) , 存在某个 $(A'_{i'}, x)$, 使得 $(A'_j, x) \subseteq (A'_{i'}, x)$, 因此

$$(A_i, x) \subseteq (A'_j, x) \subseteq (A'_{i'}, x)$$

但由于 $\{(A_k, x)\}_{k=1}^N$ 中任何两元素互不包含, 因此上式等号成立。也就是说对任意 $1 \leq j \leq N$, 存在 (唯一) $1 \leq j' \leq N'$, 使得 $(A_j, x) = (A'_{j'}, x)$; 同理对任意 $1 \leq j' \leq N'$ 也有类似结果。这就给出了集合一一对应

$$\{A_1, A_2, \dots, A_N\} \cong \{A'_1, A'_2, \dots, A'_{N'}\}$$

从而证毕。 \square

注记 1.5.10. 此定理表明, 欲研究解析集芽的局部性态, 只需要研究不可约解析集芽; 一般的解析集芽无非是不可约解析集芽的有限并。

现在, 考虑 $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n, 0}$ 的素理想 \mathfrak{p} , 我们研究解析集芽 $(V(\mathfrak{p}), 0)$ 的性质。

记号 1.5.11. 给定 \mathbb{C}^n 的一组基 $\{e_1, e_2, \dots, e_n\}$, 关于此基的坐标函数记作 z_1, z_2, \dots, z_n , 对 $1 \leq k \leq n$, 记

$$\mathbb{C}\{z_1, \dots, z_k\} := \{f \in \mathcal{O}_n \mid \frac{\partial f}{\partial z_l} \equiv 0, \forall k+1 \leq l \leq n\}$$

为 \mathcal{O}_n 中 “只显含前 k 个变量的函数芽”, 则明显有

$$\mathcal{O}_k \cong \mathbb{C}\{z_1, \dots, z_k\} \hookrightarrow \mathcal{O}_n$$

于是对于 \mathcal{O}_n 的素理想 \mathfrak{p} ,

$$\mathfrak{p}_k := \mathfrak{p} \cap \mathbb{C}\{z_1, \dots, z_k\}$$

为子环 $\mathcal{O}_k \cong \mathbb{C}\{z_1, \dots, z_k\}$ 的素理想。

引理 1.5.12. 对于环 \mathcal{O}_n 的素理想 \mathfrak{p} , 则存在 \mathbb{C}^n 的一组基 $\{f_1, f_2, \dots, f_n\}$, (记在该基下的坐标函数为 w_1, w_2, \dots, w_n) 以及存在 $0 \leq d \leq n$, 使得

$$\mathfrak{p}_d := \mathfrak{p} \cap \mathbb{C}\{w_1, w_2, \dots, w_d\} = 0$$

并且对任意 $d+1 \leq k \leq n$, \mathfrak{p}_k 当中存在 Weierstrass 多项式

$$P_k(\tilde{w}_k, w_k) = w_k^{s_k} + \sum_{j=1}^{s_k} a_{jk}(\tilde{w}_k) w_k^{s_k-j}$$

其中 $\tilde{w}_k := (w_1, w_2, \dots, w_{k-1}) \in \mathbb{C}^{k-1}$.

证明. 对 n 归纳, $n=1$ 时平凡.

Step1 对于 $n \geq 2$, 先给定 \mathbb{C}^n 的一组基 $\{e_1, \dots, e_n\}$ 并记坐标函数为 z_1, z_2, \dots, z_n , 如果 $\mathfrak{p} = \{0\}$, 则仍取这组基, 并取 $d=n$ 即可. 若 $\mathfrak{p} \neq 0$, 则任取 $0 \neq g_n \in \mathfrak{p}$, 注意 $g_n(0) = 0$; 取 \mathbb{C}^n 中的非零向量 f_n , 使得定义在 $0 \in \mathbb{C}$ 附近的函数

$$t \mapsto g_n(tf_n)$$

在 $t=0$ 处的零点阶数最低, 记为 s_n . 注意满足如此性质的向量 f_n 在 \mathbb{C}^n 中是稠密的 (只需要使得 g_n 沿 f_n 方向的 s_n 阶方向导数非零), 从而不妨取 f_n 充分接近基向量 e_n , 使得 $\{e_1, e_2, \dots, e_{n-1}; f_n\}$ 仍是 \mathbb{C}^n 的一组基.

Step2 现在考虑基 $\{e_1, e_2, \dots, e_{n-1}; f_n\}$, 该基下的坐标记为 z'_1, z'_2, \dots, z'_n , 则由 Weierstrass 预备定理, 注意 $z'_n = 0$ 是函数 $z'_n \mapsto g_n(0, z'_n)$ 的 s_n 阶零点, 则由 Weierstrass 预备定理, 存在 Weierstrass 多项式

$$P_n(\tilde{z}'_n, z'_n) = (z'_n)^{s_n} + \sum_{j=1}^{s_n} a_{jn}(\tilde{z}'_n) (z'_n)^{s_n-j}$$

以及 $h \in \mathcal{O}_n$ 使得 $h(0) \neq 0$, 以及 $g_n = P_n h$. (其中 $\tilde{z}'_n = (z'_1, \dots, z'_{n-1}) \in \mathbb{C}^{n-1}$) 由于 h 在 \mathcal{O}_n 中可逆, 所以 Weierstrass 多项式 $P_n \in \mathfrak{p} = \mathfrak{p}_n$.

Step3 如果 $\mathfrak{p}_{n-1} := \mathfrak{p} \cap \mathbb{C}\{z'_1, z'_2, \dots, z'_{n-1}\} = 0$, 则取 \mathbb{C}^n 的基 $\{e_1, \dots, e_{n-1}; f_n\}$, 以及 $d = n-1$ 即可. 如果 $\mathfrak{p}_{n-1} \neq 0$, 则 \mathfrak{p}_{n-1} 为子环 $\mathcal{O}_{n-1} \cong \mathbb{C}\{z'_1, \dots, z'_{n-1}\}$ 的素理想, 之后对 $\mathbb{C}^{n-1} \cong \text{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_{n-1}\}$ 以及 \mathfrak{p}_{n-1} 使用归纳假设即可. \square

注记 1.5.13. 容易知道, 对事先任意给定的 \mathbb{C}^n 的基 $\{e_1, e_2, \dots, e_n\}$, 上述引理中的基 $\{f_1, f_2, \dots, f_n\}$ 可以适当选取使得与 $\{e_1, e_2, \dots, e_n\}$ 任意接近.

(这个引理证明过程中, 哪里利用了“素理想”?)

本节有坑待填, 尚未完成. 笔者打算完整证明如下:

定理 1.5.14. (局部解析零点定理)

设 I 为 \mathcal{O}_n 的理想, 则

$$I_{(V(I),x)} = \sqrt{I}$$

回顾 $\sqrt{I} := \{f \in \mathcal{O}_n \mid \exists N \geq 0, f^N \in I\}$ 为 I 的**根式理想**。交换代数当中有以下基本结果:

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \supseteq I \\ \mathfrak{p} \in \text{Spec}(\mathcal{O}_n)}} \mathfrak{p}$$

证明大意. $I_{(V(I),x)} \supseteq \sqrt{I}$ 是容易验证的, 而另一边 “ \subseteq ”, 由交换代数, 只需对 $I = \mathfrak{p}$ 为素理想的情形证明。

这是非常不显然的结果, 需要利用引理1.5.12 等多复变函数的结果, 以及较多的交换代数。从略。 \square

([这里待完善](#))

1.6 局部参数化

本节陈述关于不可约解析集芽的如下重要定理

定理 1.6.1. (不可约解析集芽的局部参数化定理)

设 \mathfrak{p} 为环 \mathcal{O}_n 的素理想, 任取解析集 A 为解析集芽 $(V(\mathfrak{p}), 0)$ 的代表元, 则: 存在 \mathbb{C}^n 的基 $\{e_1, e_2, \dots, e_n\}$ (该基下的坐标函数记为 z_1, z_2, \dots, z_n), 存在 $1 \leq d \leq n$, 以及存在足够小的正实数 $r', r'' > 0$, 以及常数 $C > 0$, 使得:

(1) $\mathfrak{p} \cap \mathbb{C}\{z_1, \dots, z_d\} = 0$, 并且环同态

$$\mathbb{C}\{z_1, \dots, z_d\} \hookrightarrow \mathcal{O}_n / \mathfrak{p}$$

为有限整扩张。

(2) 在坐标 $z' = (z_1, \dots, z_d), z'' = (z_{d+1}, \dots, z_n)$ 下,

$$A \cap (\Delta' \times \Delta'') \subseteq \{(z', z'') \in \mathbb{C}^d \times \mathbb{C}^{n-d} \mid |z''| \leq C|z'|\}$$

其中 Δ' 为 \mathbb{C}^d 中以原点为中心, 半径 r' 的多圆柱; Δ'' 为 \mathbb{C}^{n-d} 中以原点为中心, 半径 r'' 的多圆柱。

(3) 记 q 为 $\mathbb{C}\{z_1, \dots, z_d\} \hookrightarrow \mathcal{O}_n/\mathfrak{p}$ 的扩张次数, 则投影映射

$$\begin{aligned}\pi: A \cap (\Delta' \times \Delta'') &\rightarrow \Delta' \\ (z', z'') &\mapsto z'\end{aligned}$$

为次数为 q 的分歧映射 (ramified map), 并且存在某个 $\delta \in \mathcal{O}_d$, 使得 π 的所有分歧值都位于集合

$$S := \{z' \in \Delta' \mid \delta(z') = 0\}$$

之中, 并且 $\Delta' \setminus S$ 为 Δ' 的连通、稠密子集。

第 (3) 条的“分歧映射”、“分歧值”具体指: 投影

$$\begin{aligned}\pi': A \cap [(\Delta' \setminus S) \times \Delta''] &\rightarrow \Delta' \\ (z', z'') &\mapsto z'\end{aligned}$$

为 q 叶覆盖映射, 并且对任意 $z' \in S$, $\#\pi^{-1}(z') \leq q$.

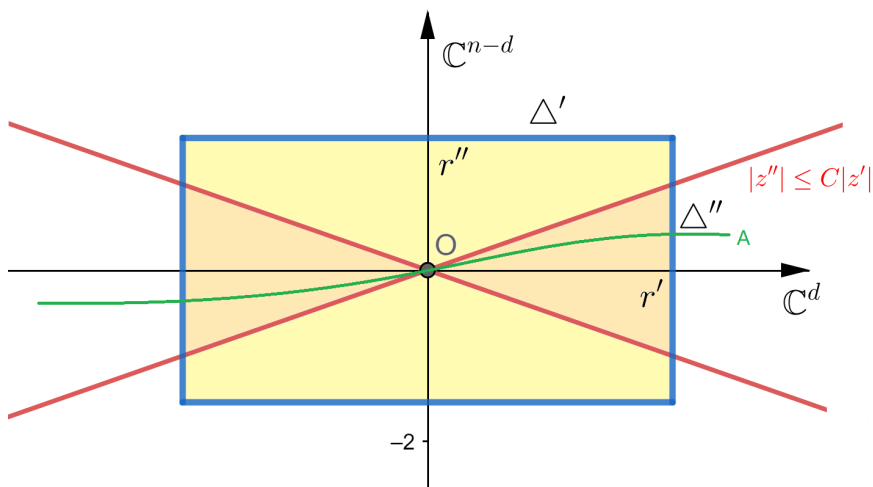


图: 性质1.6.1示意

证明. 异常复杂, 从略. 承认之。

□

不过我们可以考虑一种简单的特殊情形—— \mathfrak{p} 为主理想:

例子 1.6.2. (超曲面的参数化)

设 \mathcal{O}_n 的素理想 $\mathfrak{p} = (f)$ 为主理想, 证明此种情形的局部参数化定理。

证明. 由 Weierstrass 预备定理, 不妨取 \mathfrak{p} 的生成元 f 为 weierstrass 多项式

$$f(\tilde{z}, z_n) = z_n^q + \sum_{j=1}^q a_j(\tilde{z}) z_n^{s-j} = \prod_{j=1}^q (z_n - w_j(\tilde{z}))$$

其中 $\tilde{z} = (z_1, z_2, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$, $w_j(\tilde{z})$ 为多项式 $z_n \mapsto f(\tilde{z}, z_n)$ 的根. 取 $d = n - 1$, 显然

$$\mathfrak{p} \cap \mathbb{C}\{z_1, z_2, \dots, z_d\} = 0$$

现在对任意 $F \in \mathcal{O}_n$, 对 F 以及 Weierstrass 多项式 f 使用 Weierstrass 除法定理, 有 $F = hf + R$, 其中 $R \in \mathcal{O}_{n-1}[z_n]$ 并且次数 $< q$. 这表明 $\tilde{F} \in \mathcal{O}_n/\mathfrak{p}$ 为有限生成 $\mathcal{O}_d = \mathcal{O}_{n-1}$ -模, 并且 $\{1, z_n, z_n^2, \dots, z_n^{q-1}\}$ 为其一组 \mathcal{O}_d -模生成元. 因此

$$\mathcal{O}_d \hookrightarrow \mathcal{O}_n/\mathfrak{p}$$

为有限整扩张. 从而定理1.6.1的 (1) 证毕.

而 (3) 几乎显然, 取

$$S := \left\{ \tilde{z} \in \Delta' \mid \text{多项式 } z_n \mapsto f(\tilde{z}, z_n) \text{ 无重根} \right\}$$

即可. 利用代数学中关于重根的判别式, 容易知道 S 为某个 \mathcal{O}_d 中的函数 (芽) 的零点集. 从而 (3) 易证.

至于 (2), 常数 C 的存在性显然吗? 如果有对 f 的根的估计

$$w_j(\tilde{z}) = O(|\tilde{z}|)$$

那么就没问题. (待补)

□

1.7 正则点、奇异点, 全纯隐函数定理

(待补)

第2章 复流形（待补）

计划详细介绍复流形、复微分形式，以及复流形的例子。

2.1 复流形与全纯向量丛（暂定）

2.2 微分形式（暂定）

2.3 例子（暂定）

第3章 层与层上同调

3.1 层的上同调

Today:

Sheaf cohomology

X a topological space, \mathcal{F} - sheaf (of abelian groups).

定义 3.1.1. (*resolution*)

(1) a resolution of \mathcal{F} is an exact sequence

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{F} \xrightarrow{d^0} \mathcal{F} \xrightarrow{d^1} \rightarrow \dots$$

定义 3.1.2. A sheaf \mathcal{A} is called injective, if if for any injective morphism $j : \mathcal{A} \rightarrow \mathcal{B}$ and for any morphism $\varphi : \mathcal{A} \rightarrow \mathcal{S}$, there exists an extension $\psi : \mathcal{B} \rightarrow \mathcal{S}$, such that

定理 3.1.3. the category of sheaves of abelian sheaves have enough injective objects, i.e. any \mathcal{F} can be embedded in some injective sheaf.

定义 3.1.4. Consider an injective resolution of \mathcal{F} , i.e. an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \mathcal{I}^2 \rightarrow \dots$$

where every $\mathcal{I}^k (k \geq 0)$ is injective.

\leadsto induces a sequence

$$0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{I}^0) \xrightarrow{d} \Gamma(X, \mathcal{I}^1) \xrightarrow{d} \Gamma(X, \mathcal{I}^2) \rightarrow \dots$$

Then

$$H^q(X, \mathcal{F}) := H^q(\Gamma(X, \mathcal{I}^\bullet))$$

then, $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

定义 3.1.5. A sheaf \mathcal{S} is called a flabby (flasque, in France), if for any open set $\Omega \subseteq X$, the morphism

$$\mathcal{S}(X) \rightarrow \mathcal{S}(\Omega)$$

is surjective.

定义 3.1.6.

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1$$

is an exact sequence is called a flabby resolution, if any \mathcal{F}^k is flabby.

定义 3.1.7.

$$H^q(X, \mathcal{F}) := \dots \text{by flabby resolution} \dots$$

证明. Homological Algebra...omit. □

the two definitions of Sheaf Cohomology are isomorphic.

Godement's construction

$$God(\mathcal{F})(U) := \{f : U \rightarrow \bigcup_{x \in U} \mathcal{F}_x \mid f(y) \in \mathcal{F}_y, \forall y \in U\} := \prod_{x \in U} \mathcal{F}_x$$

$God(\mathcal{F})$ is a sheaf, and it is flabby. and there is a canonical morphism $\mathcal{F}(U) \rightarrow God(\mathcal{F})(U)$ by $x \mapsto (x \mapsto s_x)$ is injective.

$$\mathcal{F}^0 := God(\mathcal{F})$$

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \twoheadrightarrow \text{coker}(j) = \mathcal{F}^0 / \mathcal{F}$$

and consider

$$\mathcal{F}^1 := \text{God}(\text{coker}(j))$$

.....then construct by induction... this is a flabby resolution of \mathcal{F} .

定义 3.1.8. (resolution by fine sheaves)

\mathcal{A} is a sheaf of ring, X is a paracompact topological space, \mathcal{A} is called a fine sheaf, if for any open covering

$$X = \bigcup_{\alpha} V_{\alpha} \quad , \mathcal{V} := \{V_{\alpha}\}$$

there exists a partition of unit subordinate to \mathcal{V} , (i.e. $\exists f_{\alpha} \in \mathcal{A}(V_{\alpha}), \text{supp}(\alpha) := \overline{\{x \in V_{\alpha} | f_{\alpha,x} \neq 0\}} \subseteq V_{\alpha}$, and $\sum_{\alpha} f_{\alpha} = 1$ (the sum is locally finite))

例子 3.1.9. X is a differential manifold, \mathcal{C}^{∞} is the sheaf of smooth functions, then \mathcal{C}^{∞} is a fine sheaf.

定理 3.1.10. \mathcal{S} is a sheaf of \mathcal{A} -modules, \mathcal{A} is a fine sheaf. then for any $q \geq 1$,

$$H^q(X, \mathcal{S}) = 0$$

证明. Consider a flabby(or injective) resolution

$$0 \rightarrow \mathcal{S} \xrightarrow{j} \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \mathcal{I}^2 \dots$$

where any $\mathcal{I}^k (k \geq 0)$ is a sheaf of \mathcal{A} -modules.

by definition,

$$H^q(X, m\mathcal{S}) := \frac{\ker d : \Gamma(\mathcal{I}^q) \rightarrow \Gamma(\mathcal{I}^{q+1})}{\Im d : \Gamma(\mathcal{I}^{q-1}) \rightarrow \Gamma(\mathcal{I}^q)}$$

Let $\alpha \in \ker\{d : \Gamma(\mathcal{I}^q) \rightarrow \Gamma(\mathcal{I}^{q+1})\}$ by the exactness of resolution, \exists an open covering $\mathcal{U} = (U_i)_i$, s.t. $\alpha|_{U_i} = d\beta_i$ where $\beta_i \in \mathcal{I}^{q-1}(U_i)$. Let $(\beta_i)_i$ be the partition of unit w.r.t. \mathcal{U} . consider

$$\beta := \sum_i f_i \beta_i$$

(well defined). Then $d\beta = \alpha$

□

3.2 Čech 上同调

Čech cohomology

X - a topological space, \mathcal{F} - a sheaf of abelian group.

$$\mathcal{U} = (U_\alpha)_{\alpha \in I}$$

is an open covering.

notation: $U_{\alpha_1, \dots, \alpha_q} := \bigcap_{i=1}^q U_{\alpha_i}$.

Čech q -chain w.r.t \mathcal{U} :

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{(\alpha_1, \dots, \alpha_q) \in \mathcal{I}^{q+1}} \mathcal{F}(U_{\alpha_1, \dots, \alpha_q})$$

$$c \in C^q(\mathcal{U}, \mathcal{F})$$

means that we have a family of sections $c_{\alpha_1, \dots, \alpha_q} \in \mathcal{F}(U_{\alpha_1, \dots, \alpha_q})$ with the relation

$$c_{\alpha_0, \dots, \alpha_j, \dots, \alpha_i, \dots} = -c_{\dots}$$

(Č)ech differential:

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{F})$$

$$\delta^q(c)_{\alpha_0, \dots, \alpha_{q+1}} := \sum_{0 \leq k \leq q+1} (-1)^k c_{\dots \hat{\alpha}_k \dots} |_{U_{\alpha_0, \dots, \alpha_{q+1}}}$$

性质 3.2.1.

$$\delta^q \circ \delta^q = 0$$

so, we have Čech cohomology

$$H^q(\mathcal{U}, \mathcal{F}) := \ker \delta^q / \operatorname{Im} \delta^{q-1}$$

example:

$$C^0(\mathcal{U}, \mathcal{F}) := \prod_{\alpha \in I} \mathcal{F}(U_\alpha)$$

$$c = (c_\alpha)_{\alpha \in I} \in C^0(\mathcal{U}, \mathcal{F})$$

$$\delta^0 c = 0 \iff (\delta^0 c)_{\alpha_0 \alpha_1} := (c_{\alpha_1} - c_{\alpha_0})|_{U_{\alpha_0 \alpha_1}} = 0$$

so, $c_{\alpha_0} = c_{\alpha_1}$ on $U_{\alpha_0 \alpha_1}$.

$$\rightsquigarrow H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X).$$

例子 3.2.2. (1) consider $X = \Delta \setminus \{0\}$, where $\Delta = \{(z_1, z_2) | |z_1| < 1, |z_2| < 1\}$. Consider the covering

$$\mathcal{U} = U_1 \cup U_2$$

where

$$U_1 := \{(z_1, z_2) \in \Delta | z_1 \neq 0\} = \mathbb{D}^* \times \mathbb{D}$$

$$U_2 := \{(z_1, z_2) \in \Delta | z_2 \neq 0\} = \mathbb{D} \times \mathbb{D}^*$$

then

$$U_1 \cap U_2 = \mathbb{D}^* \times \mathbb{D}^*$$

consider $H^0(X, \mathcal{O}) = \mathcal{O}(X) \cong \mathcal{O}(\Delta) = \{f : \Delta \rightarrow \mathbb{C} \text{ holomorphic}\}$.

$$H^1(\mathcal{U}, \mathcal{O}) = \ker \delta^1 / \text{Im } \delta^0$$

$$\delta^1 : C^1(\mathcal{U}, \mathcal{O}) \rightarrow C^2(\mathcal{U}, \mathcal{O}) \subseteq \prod_{\alpha_0, \alpha_1, \alpha_2} \mathcal{O}(U_{\alpha_0, \alpha_1, \alpha_2}) = 0$$

$$\ker \delta^1 = C^1(\mathcal{U}, \mathcal{O}) = \{c = c(\alpha_0, \alpha_1) | c_{\alpha_0, \alpha_1} \in \mathcal{O}(U_{\alpha_0, \alpha_1})\} = \{c \in \mathcal{O}(U_1 \cap U_2)\} = \{c = \sum_{m, n \in \mathbb{Z}} a_{mn} z_1^m z_2^n \text{ convergent}\}$$

$$\delta^0 : C^0(\mathcal{U}, \mathcal{O}) \rightarrow C^1(\mathcal{U}, \mathcal{O})$$

$$(\delta^0 c)_{12} = (c_2 - c_1)|_{U_{12}}$$

where $c_2 \in \mathcal{O}(U_2)$ and $c_1 \in \mathcal{O}(U_1)$. note that

$$\mathcal{O}(U_1) = \{c(z_1, z_2) = \sum_{m \in \mathbb{Z}, n \geq 0} a_{mn} z_1^m z_2^n \text{ convergent}\}$$

$$\mathcal{O}(U_2) = \{c(z_1, z_2) = \sum_{n \in \mathbb{Z}, m \geq 0} a_{mn} z_1^m z_2^n \text{ convergent}\}$$

$$\text{So, } H^1(\mathcal{U}, \mathcal{O}) = \{c(z_1, z_2) = \sum_{m, n < 0} a_{mn} z_1^m z_2^n\}$$

例子 3.2.3. (complex projective space)

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

$$(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$$

for some $\lambda \in \mathbb{C}^*$.

$$\mathbb{C}P^n = \{[z_0, \dots, z_n] | \text{not all } z_k = 0, z_i \in \mathbb{C}\} = \bigcup_{0 \leq p \leq n} V_k$$

where

$$V_k = \{[z_0, \dots, z_n] | z_k \neq 0\} \cong \{(\frac{z_0}{z_k}, \dots, 1, \dots, \frac{z_n}{z_k}) | z_i \in \mathbb{C}, i \neq k, z_k \neq 0\} \cong \mathbb{C}^n$$

this is a holomorphic chart.

$$\mathbb{C}P^1 = V_0 \cup V_1, \mathcal{V} = \{V_0, V_1\}$$

HW: compute $H^q(\mathcal{V}, \mathcal{O})$.

Answer:

$$H^0 \cong \mathbb{C}, H^1 \cong 0$$

Correction:

\mathcal{A} : Sheaf of rings (with unit)

X : paracompact topological space,

定义 3.2.4. \mathcal{A} is called *fine*, if for any open covering $\mathcal{U} = (V_\alpha)_{\alpha \in \mathcal{I}}$, there exist $s_\alpha \in \mathcal{A}(X)$ such that $\text{supp}(s_\alpha) \subseteq V_\alpha$,

$$\sum_{\alpha} s_\alpha = 1$$

(this is a locally finite sum)

注记 3.2.5. we call \mathcal{A} is a **soft sheaf**, if for any closed set $K \subseteq X$, the morphism

$$\mathcal{A}(X) \rightarrow \mathcal{A}(K)$$

is surjective. where $\mathcal{A}(K) := \Gamma(K, \mathcal{A}|_K)$

fact: \mathcal{A} is fine if and only if $\mathcal{H}om(\mathcal{A}, \mathcal{A})$ is soft. (omit)

Recall:

Cech cohomology: X topological space, $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$,

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 < \dots < \alpha_q} \mathcal{F}(\alpha_0, \dots, \alpha_q)$$

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{F})$$

fact: $H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Today:

定义 3.2.6. Let $\mathcal{V} = (V_\beta)_{\beta \in \mathcal{J}}$ be another open covering, then \mathcal{V} is called a *refinement* of \mathcal{U} , if there exists a map

$$\rho : \mathcal{J} \rightarrow \mathcal{I}$$

such that

$$V_\beta \subseteq U_{\rho(\beta)}$$

性质 3.2.7. Let \mathcal{V} be a refinement of \mathcal{U} , then ρ induces a map

$$\rho^q : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^q(\mathcal{V}, \mathcal{F})$$

$$(\rho^q C)_{\beta_0, \dots, \beta_q} \mapsto C_{\rho(\beta_0), \dots, \rho(\beta_q)}|_{V_{\beta_0, \dots, \beta_q}}$$

ρ is a morphism of complexes.

so, ρ induces a map

$$H^q(\rho) : H^q(\mathcal{U}, \mathcal{F}) \rightarrow H^q(\mathcal{V}, \mathcal{F})$$

Let $\tilde{\rho} : \mathcal{J} \rightarrow \mathcal{I}$ be another refinement of \mathcal{U}

(induces $H^q(\tilde{\rho}) : H^q(\mathcal{U}, \mathcal{F}) \rightarrow H^q(\mathcal{V}, \mathcal{F})$) then $\rho, \tilde{\rho}$ are homotopic (chain homotopy $\rightsquigarrow H^q(\rho) = H^q(\tilde{\rho})$)

so, if $\rho : \mathcal{J} \rightarrow \mathcal{I}$ is refinement, then

$$H^q(\rho)$$

is independent of the refinement.

定义 3.2.8.

$$\check{H}^q(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F})$$

i.e. $a \in H^q(\mathcal{U}, \mathcal{F}) \sim \in H^q(\mathcal{V}, \mathcal{F})$ iff \exists a refinement \mathcal{W} of \mathcal{U} and \mathcal{V} such that a, b have the same image in $H^q(\mathcal{W}, \mathcal{F})$

注记 3.2.9.

$$\check{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$$

Exercise: For $q = 1$, if \mathcal{V} is a refinement of \mathcal{U} , then

$$H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(\mathcal{V}, \mathcal{F})$$

is injective.

so, for any open cover \mathcal{U} ,

$$H^1(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{F})$$

is injective.

Homological Algebra recall: let $(K^\bullet, d_k), (L^\bullet, d_l)$ and (M^\bullet, d_M) , if we have a short exact sequence

$$0 \rightarrow K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} M^\bullet \rightarrow 0$$

then it induces a long exact sequence :

$$\cdots \rightarrow H^q(K^\bullet) \rightarrow H^q(L^\bullet) \rightarrow H^q(M^\bullet) \rightarrow H^{q+1}(K^\bullet) \rightarrow \cdots$$

analogy of Cech cohomology: X is a topological space, \mathcal{U} is an open covering of X . \mathcal{A} and \mathcal{B} sheaves on X , Let

$$\varphi : \mathcal{A} \rightarrow \mathcal{B}$$

be a morphism, then it induces

$$\varphi^\bullet : C^\bullet(\mathcal{U}, \mathcal{A}) \rightarrow C^\bullet(\mathcal{U}, \mathcal{B})$$

Let

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

be an exact sequence of sheaves, then we have: for any open set Ω ,

$$0 \rightarrow \mathcal{A}(\Omega) \rightarrow \mathcal{B}(\Omega) \rightarrow \mathcal{C}(\Omega)$$

left exact.

Example: consider

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \rightarrow 0$$

is exact on $bbC^\times := \mathbb{C} \setminus \{0\}$

but we have :

$$0 \rightarrow \mathcal{A}(\Omega) \xrightarrow{\psi} \mathcal{B}(\Omega) \rightarrow \text{Im } \psi(\Omega) \rightarrow 0$$

is exact.

First we have the following exact sequence

$$C^q(\mathcal{U}, \mathcal{A}) \rightarrow C^q(\mathcal{U}, \mathcal{B}) \rightarrow C_B^q(\mathcal{U}, \mathcal{C}) \rightarrow 0$$

where C_B^q is the image of ...

then we get an exact sequence

$$0 \rightarrow (C^\bullet(\mathcal{U}, \mathcal{A}), \delta) \rightarrow (C^\bullet(\mathcal{U}, \mathcal{B}), \delta) \rightarrow (C_B^\bullet(\mathcal{U}, \mathcal{C}), \delta) \rightarrow 0$$

it induces a long exact sequence

$$\cdots \rightarrow H^q(\mathcal{U}, \mathcal{A}) \rightarrow H^q(\mathcal{U}, \mathcal{B}) \rightarrow H_B^q(\mathcal{U}, \mathcal{C}) \rightarrow H^{q+1}(\mathcal{U}, \mathcal{A}) \rightarrow \cdots$$

定理 3.2.10. *If X is paracompact,*

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

is a sheaf exact sequence. Then there is a long exact sequence

$$\cdots \rightarrow \check{H}^q(X, \mathcal{A}) \rightarrow \check{H}^q(X, \mathcal{B}) \rightarrow \check{H}^q(X, \mathcal{C}) \rightarrow \check{H}^{q+1}(X, \mathcal{A}) \rightarrow \cdots$$

证明. Key lemma: need to prove

$$\lim_{\vec{U}} H^q(\mathcal{U}, \mathcal{C}) = \lim_{\vec{U}} H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C})$$

if X is paracompact.

Omit. □

if

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

exact,

recall: (cohomology by resolutions)

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots$$

flabby resolution. then it induces

$$0 \rightarrow \Gamma(X, \mathcal{A}) \rightarrow \Gamma(X, \mathcal{F}^0) \rightarrow \Gamma(X, \mathcal{F}^1) \rightarrow \dots$$

then define the sheaf cohomology...

we have a long exact sequence

$$\dots \rightarrow H^q(X, \mathcal{A}) \rightarrow H^q(X, \mathcal{B}) \rightarrow H^q(X, \mathcal{C}) \rightarrow H^{q+1}(X, \mathcal{A}) \rightarrow \dots$$

it is homological algebra...

定理 3.2.11. (*Leray's acyclic theorem*) Let $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$ be an open covering of X , (\mathcal{F} is a sheaf on X), if satisfying

$$H^k(U_{\alpha_0, \dots, \alpha_q}) = 0$$

for any $k \geq 1$, then

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{H}^q(X, \mathcal{F})$$

and if X is paracompact, we also have

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{H}^q(X, \mathcal{F}) \cong H^q(X, \mathcal{F})$$

(this \mathcal{U} is called acyclic covering)

de Rham- Weil theorem

定义 3.2.12. \mathcal{F} is a sheaf on X , Ω is an open set of X , then \mathcal{F} is called **acyclic sheaf** if

$$H^q(\Omega, \mathcal{F}) = 0$$

for any $q \geq 1$.

定理 3.2.13. Let

$$0 \rightarrow \mathcal{F} \rightarrow (L^\bullet, d)$$

be an acyclic resolution of \mathcal{F} (i.e. L^q is acyclic on X) then

$$H^q(X, \mathcal{F}) \cong H^q(\Gamma(X, L^\bullet), d)$$

for any $q \geq 0$.

(先看例子)

例子 3.2.14. Let X be a differential manifold, \mathcal{E}^p : sheaf of smooth p -forms, then we have a resolution (de Rham complex)

$$0 \rightarrow \mathbb{R} \hookrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \mathcal{E}^3 \rightarrow \dots$$

where d differential operators. (Why it is a resolution? because of Poincare lemma...locally solvable..)

Note that

$$\mathcal{E}^0 = \mathcal{C}^\infty$$

\mathcal{E}^p is a sheaf of \mathcal{C}^∞ -modules..

then we have

$$H^q(X, \mathcal{E}^p) = 0$$

for all $q \geq 1$

and then

$$H^q(X, \mathbb{R}) \cong \frac{\ker(d : \Gamma(X, \mathcal{E}^q) \rightarrow \Gamma(X, \mathcal{E}^{q+1}))}{\text{Im}(d : \Gamma(X, \mathcal{E}^{q-1}) \rightarrow \Gamma(X, \mathcal{E}^q))} = H_{DR}^q(X, \mathbb{R})$$

例子 3.2.15. Let X be a complex manifold, $\mathcal{E}^{p,q}$ sheaf of smooth (p, q) forms, Ω^p is the sheaf of holomorphic p -forms (i.e. $(p, 0)$ -form φ with $\bar{\partial}\varphi = 0$).

Then we have resolution

$$0 \rightarrow \Omega^p \xrightarrow{j} \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \rightarrow \dots$$

(Why it is a resolution? because of the Dolbeault lemma), remain to Exercise...

$$H^q(X, \Omega^p) \cong H_{\bar{\partial}}^{p,q}(X, \mathbb{C})$$

Today: de Rham-Weil Isomorphism Thm

定理 3.2.16. *Let X be a topological space, \mathcal{F} be a sheaf of abelian groups on X ,*

$$0 \rightarrow \mathcal{F} \rightarrow (\mathcal{L}^\bullet, d)$$

be an acyclic resolution, i.e.

$$H^k(X, \mathcal{L}^q) = 0$$

for all $k \geq 1$ and $q \geq 0$. Then,

$$H^q(X, \mathcal{F}) \cong H^q((\Gamma(\mathcal{L}^\bullet), d))$$

证明. Since

$$0 \rightarrow \mathcal{F} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{d^0} \mathcal{L}^1 \xrightarrow{d^1} \mathcal{L}^2 \rightarrow \dots$$

be an exact sequence, denote

$$\mathcal{Z}^q := \ker d^q$$

then we have short exact sequences

$$0 \rightarrow \mathcal{Z}^q \rightarrow \mathcal{L}^q \rightarrow \mathcal{Z}^{q+1} \rightarrow 0$$

for any q . They induce long exact sequence of cohomology groups:

$$\dots \rightarrow H^k(X, \mathcal{Z}^q) \rightarrow H^k(X, \mathcal{L}^q) \rightarrow H^k(X, \mathcal{Z}^{q+1}) \xrightarrow{\partial} H^{k+1}(X, \mathcal{L}^q) \rightarrow H^{q+1}(X, \mathcal{L}^q) \rightarrow \dots$$

For any $k \geq 1$, since \mathcal{L}^q are acyclic on X ,

$$H^k(X, \mathcal{Z}^{q+1}) \cong H^{k+1}(X, \mathcal{Z}^q)$$

and for $k = 0$, we have

$$0 \rightarrow H^0(X, \mathcal{Z}^q) \rightarrow H^0(X, \mathcal{L}^q) \rightarrow H^0(X, \mathcal{Z}^{q+1}) \rightarrow H^1(X, \mathcal{Z}^q) \rightarrow H^1(X, \mathcal{L}^q) = 0 \rightarrow \dots$$

so,

$$H^1(X, \mathcal{Z}^q) \cong H^0(X, \mathcal{Z}^{q+1}) / \text{Im } d^q \cong H^{q+1}((\Gamma(\mathcal{L}^\bullet), d))$$

$$H^{q+1}(\Gamma(\mathcal{L}^\bullet)) \cong H^1(X, \mathcal{Z}^q) \cong H^2(X, \mathcal{Z}^{q-1}) \cong \dots \cong H^{q+1}(X, \mathcal{Z}^0) = H^{q+1}(X, \mathcal{F})$$

□

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \rightarrow \dots$$

(de Rham resolution) then we have

$$H^k(X, \mathcal{R}) \cong H_{DR}^k(X; \mathcal{R})$$

(if X is compact, then by Hodge theory, it also isomorphic to $\ker(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})$)

Another example: X is a complex manifold, then

$$0 \rightarrow \Omega^p \rightarrow \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \rightarrow \dots$$

then

$$H^q(X, \Omega^p) \cong H_{\bar{\partial}}^{p,q}(X, \mathbb{C})$$

(RHS= Dolbeault cohomology)

X be a smooth manifold, we define

$C_q(X, \mathbb{Z}) :=$ the free abelian group generated by continuous map

$$\phi : \Delta_q := \{(t_1, \dots, t_{q+1}) \in [0, 1]^{q+1} \mid \sum_{i=1}^n t_i = 1\}$$

and we define (for $\phi \in C_q(X, \mathbb{Z})$)

$$\partial\phi := \sum_{i=1}^{q+1} (-1)^i \phi|_{\Delta_{q,i}}$$

$$\Delta_{q,i} := \{t \in \Delta_q \mid t_i = 0\}$$

we define

$$(C_{sing}^\bullet, \partial)$$

be the dual complex of $(C_{sing}^\bullet, \partial)$.

(These are all Basic Algebraic Topology)

For any open $U \subseteq X$, we have

$$U \rightarrow C_{sing}^q(U, \mathbb{Z})$$

we get a sheaf

$$\mathcal{C}_{sing}^q$$

FACT: $(C_{sing}^\bullet, \partial)$ is a flabby resolution of \mathbb{Z} . (check!) So,

$$H_{sing}^q(X, \mathbb{Z}) = H^q(\Gamma(C_{sing}^\bullet), \partial) \cong H^q(X, \mathbb{Z})$$

第4章 Hermite 向量丛

4.1 联络与曲率

Recall: X is a smooth manifold, E is a vector bundle of rank r , if

- (1) $\pi : E \rightarrow X$ is smooth map,
- (2) for any $x \in X$, $E_x := \pi^{-1}(x)$ is a vector space over \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) of dimension r .
- (3) there an open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ and trivializations

$$\theta_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

and for any intersection $U_\alpha \cap U_\beta$, we have

注记 4.1.1.

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$$

(cocycle condition)

Special Case: line bundle rank $E=1$.

then $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, \mathbb{K}^*) = \mathcal{E}^*(U_{\alpha\beta})$ invertible smooth function on $U_{\alpha\beta}$. then, Cech cohomology,

$$(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = 1$$

so,

$$(g_{\alpha,\beta}) \in \mathcal{Z}^1(\mathcal{U}, \mathcal{E}^*) \rightarrow H^1(\mathcal{U}, \mathcal{E}^*) \hookrightarrow \check{H}^1(X, \mathcal{E}^*)$$

we get a map

$$\{\text{line bundles}\} \rightarrow \check{H}^1(X, \mathcal{E}^*)$$

actually, we have

$$\{\text{isomorphic classes of line bundles}\} \longleftrightarrow H^1(X, \mathcal{E}^*)$$

1-1 correspondence.

Now, X be a complex manifold, a complex vector bundle E is called holomorphic, if ... the transition matrix $g_{\alpha\beta}$ is holomorphic...

Holomorphic line bundles :

$$g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$$

\mathcal{O}^* :sheaf of invertible holomorphic functions...

FACT: there is a map

$$\{\text{holomorphic line bundle}\} \rightarrow \check{H}^1(X, \mathcal{O}^*)$$

例子 4.1.2. *trivial vector bundle* $X \times \mathbb{K}^r$

例子 4.1.3. *Tangent bundle* TX . (transition matrix $g_{\alpha\beta}$ are given by Jacobi matrix..)

定义 4.1.4. (Local frame of vector bundles)

$$\theta_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{K}^r$$

be a trivialization, we define

$$e_\lambda(x) := \theta_\alpha^{-1}\left(x, \begin{pmatrix} 0 \\ \vdots \\ 1(\leftarrow \text{ith}) \\ \vdots \\ 0 \end{pmatrix}\right)$$

then, $\{e_1, \dots, e_r\}$ be a local smooth section $s \in \Gamma(U_\alpha, E)$ can be written as

$$s(x) = \sum \sigma_\lambda(x)$$

where $\sigma_\lambda \in C^\infty(U_\alpha, \mathbb{K})$.

(Connection)

记号 4.1.5. For X be a smooth manifold, E is a vector bundle(real or complex), denote

$$C_p^k(\Omega, E) := C^k(\Omega, \bigwedge^p T^*M \otimes E)$$

is the space of k -differential p -forms with values in E .

Locally, consider a trivialization of E ,

$$\theta_\alpha E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

(\rightsquigarrow frame (e_1, \dots, e_r))

$$s \in \sum \varphi_\lambda(x) \otimes e_\lambda(x)$$

where φ_λ is a p -form.

定义 4.1.6. a (linear) connection on E is a linear differential operator of order 1 acting on $C^\bullet_\bullet(X, E)$:

$$D : C^\infty_p(X, E) \rightarrow C^\infty_{p+1}(X, E)$$

$$D(f \wedge s) := df \wedge s + (-1)^p f \wedge Ds$$

where $f \in C^\infty(X, \wedge^p T^*M)$, $s \in C^\infty(X, E)$.

Locally, consider a local trivialization

$$\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

with a frame $\{e_1, \dots, e_r\}$. any section $t \in C^\infty_p(\Omega, E)$ can be written as

$$t = \sum_{1 \leq \lambda \leq r} \sigma_\lambda \otimes e_\lambda$$

$$Ds = \sum_{\lambda=1}^r d\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge De_\lambda$$

where

$$De_\lambda \in C^\infty_1(\Omega, E)$$

can be written as

$$De_\lambda = \sum_{\mu=1}^r a_{\mu\lambda} \otimes e_\mu$$

where " $a_{\mu\lambda}$ " is called the coefficients of D with respect to frame $\{e_1, \dots, e_r\}$.

so,

$$D(t) = \sum_{\lambda, \mu} d\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge a_{\mu\lambda} \wedge e_\mu = \sum_\mu \sum_\lambda (d\sigma_\mu + a_{\mu\lambda} \wedge \sigma_\lambda)$$

$$Dt = d\sigma + A \wedge \sigma$$

where $A = (a_{\mu\lambda})$.

RMK: connection always exists!

Recall: for any (connected) smooth manifold, $E \rightarrow X$ is a smooth vector bundle,

Connection:

$$D : C^\infty_p(X, E) \rightarrow C^\infty_{p+1}(X, E)$$

where $C^\infty_p(X, E) := C^\infty(X, \wedge^p T^*M \otimes E)$

$$D(f \wedge s) = df \wedge s + (-1)^{\deg f} f \wedge Ds$$

Essentially,

$$D : C^\infty(X, E) \rightarrow C_1^\infty(X, E)$$

Locally, consider a trivialization $\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$, and a local frame (e_1, \dots, e_r) where $e_k(x) =$

$$\theta^{-1}\left(x, \begin{pmatrix} 0 \\ \vdots \\ 1(k^{th}) \\ \vdots \\ 0 \end{pmatrix}\right).$$

Let $s \in C^\infty(\Omega, E)$, i.e.

$$s = \sum_{i=1}^r \sigma_i e_i$$

where σ_i are smooth functions.

$$Ds = d\sigma + A \wedge \sigma$$

where

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \end{pmatrix} \quad A = a_{ij}$$

consider another trivialization

$$\tilde{\theta} : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

\rightsquigarrow a local frame $(\tilde{e}_1, \dots, \tilde{e}_r)$. Then there exists a invertible linear transform s.t.

$$\tilde{e}_k = g_k^m e_m$$

assume

$$De_k = a_k^l e_l \quad D\tilde{e}_k = \tilde{a}_k^l \tilde{e}_l$$

we have

$$\begin{aligned} dg_k^n e_n + g_k^m a_m^n e_n &= \tilde{a}_k^l g_l^n e_n \\ \rightsquigarrow \tilde{a}_k^l g_l^n (g^{-1})_n^p &= dg_k^n (g^{-1})_n^p + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \tilde{a}_l^p &= dg_k^n (g^{-1})_n^p + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \tilde{A} &= dg \cdot g^{-1} + g \cdot A \cdot g^{-1} \end{aligned}$$

Curvature

$$H_D := D^2$$

locally,

$$D^2 s = D(d\sigma + A \wedge \sigma) = d(d\sigma + A \wedge \sigma) + A \wedge (d\sigma + A \wedge \sigma)$$

$$= dA \wedge \sigma - A \wedge d\sigma + A \wedge d\sigma + A \wedge A \wedge \sigma = (dA + A \wedge A) \wedge \sigma$$

so we have

$$H = dA + A \wedge A$$

Similarly to \tilde{A}, A we have

Exercise:

$$\tilde{H} = gHg^{-1}$$

曲率在不同平凡化下的表达式。where

$$\tilde{e} = ge$$

$\rightsquigarrow H$ can be considered as a section of $C_2^\infty(X, \text{Hom}(E, E))$. because

$$\tilde{H}\tilde{e} = gHg^{-1}\tilde{e} = gHe$$

independent of the choice of local frames.

4.2 向量丛的构造

定义 4.2.1. (dual of vector bundles) $E \rightarrow X$, and $g_{\alpha\beta}$:transition matrix of E , the dual is given by $(g_{\alpha\beta})^{-1}$. (用转移函数来定义向量丛)

定义 4.2.2. direct sum of two vector bundles $(E, F) \rightarrow E \oplus F$. locally,

$$(g_{\alpha,\beta}) \oplus (h_{\alpha\beta})$$

direct sum of transition matrices.

定义 4.2.3. tensor product of two vector bundles.

locally, tensor product of two transition matrices.

fact: let D_E be a connection on E , then it induces a connection D_{E^*} . Let u be a local section of E^* , s local section of E , then we define

$$d\langle u, s \rangle = \langle D_{E^*}u, s \rangle + \langle u, D_E s \rangle$$

Exercise:

$$H(D_{E^*}) = -H(D_E)^T$$

and for two vector bundles E, F , connections D_E, D_F , then

$$D_{E \oplus F} := D_E \oplus D_F$$

$$H(E \oplus F) = H_E \oplus H_F$$

as for tensor product, we define $D_{E \otimes F}$ as follows:

$$D_{E \otimes F}(s \otimes t) = D_E s \otimes t + s \otimes D_F t$$

check the curvature

$$H_{E \otimes F} = H_E \otimes id_F + id_E \otimes H_F$$

注记 4.2.4. we can also consider wedge product of vector bundles. Consider vector bundles E_1, \dots, E_k , with connections D_{E_1}, \dots, D_{E_k} , let $s_i \in C_{p_i}^\infty(X, E^i)$ then

$$D_{E_1 \wedge \dots \wedge E_k}(s_1 \wedge \dots \wedge s_k) = \sum_{i=1}^k (-1)^{p_1 + \dots + p_{i-1}} s_1 \wedge \dots \wedge D_{E_i} s_i \wedge \dots \wedge s_k$$

Let E be a vector bundle of rank r , then $\bigwedge^r E$ is a line bundle, with transition matrix by $\det(g_{\alpha\beta})$. this bundle is denoted by $\det E$. (Det-bundle)

Let s_1, \dots, s_r be local sections of E , then we have

$$D_{\det E}(s_1 \wedge \dots \wedge s_r) = \text{tr}(H_E) s_1 \wedge \dots \wedge s_r$$

4.3 陈省身示性类

chern classes (defined by curvature).

Let $E \rightarrow X$ be a smooth complex vector bundle of rank r , where X be a complex manifold.

(Chern-Weil theory)

V be a complex vector space, $f : \underbrace{V \times \dots \times V}_k \rightarrow \mathbb{C}$ be a symmetric multi-linear form of degree k .

$\rightsquigarrow f(v) := f(v, v, \dots, v)$ is a homogeneous polynomial of degree k .

定义 4.3.1. assume G is a group (left) acting on V , s.t.

$$f(g(v_1), \dots, g(v_k)) = f(v_1, \dots, v_k)$$

for any $g \in G, v_i \in V$, then we say f is G -invariant.

Special case: $G = GL(r, \mathbb{C})$ and $V = \text{Lie}G = \mathfrak{gl}(r, \mathbb{C})$ be the Lie algebra of G . the action is

$$(g, M) \mapsto gMg^{-1}$$

Consider

$$\det(I + \frac{i}{2\pi}tm) = I + tf_1(M) + t^2f_2(M) + \cdots t^rf_r(M)$$

$\rightsquigarrow \forall 1 \leq k \leq r, f_k$ is G -invariant.

Let $E \rightarrow X$ complex vector bundle on a complex manifold, let D_E be a connection, curvature $H_E \in C_2^\infty(X, \text{Hom}(E, E))$. Let $f \in GL(r, \mathbb{C})$ - invariant "k-form", then

(1) Let H_α, H_β be the curvature forms of E in different trivialization, then $f(H_\alpha) = f(H_\beta)$, so we get a globally defined $2k$ -form.

assume $H_\alpha = gH_\beta g^{-1}$, then

$$f(H_\alpha) = f(gH_\beta g^{-1}) = f(H_\beta)$$

(2) we also have

$$df(H) = 0$$

locally, $H = H_\alpha = da_\alpha + A_\alpha \wedge A_\alpha$, then

$$\begin{aligned} df(H) &= df(H_\alpha, H_\alpha, \dots, H_\alpha) = \sum_{i=1}^k f(H_\alpha, \dots, \underbrace{dH_\alpha}_{i}, \dots, \alpha) \\ &= \sum_{i=1}^k f(H_\alpha, \dots, dA_\alpha \wedge A_\alpha - A_\alpha \wedge dA_\alpha, \dots, H_\alpha) \end{aligned}$$

Fact: (in Riemannian geometry) For any $x \in X$, we always can find a local frame s.t. $A_\alpha(x) = 0$. so, choose this frame,

$$df(H) = 0$$

So, $[f(H)] \in H^{2k}(X, \mathbb{C})$

(3) Claim : the class $[f(H)]$ is independent of the choice of the connections D_E .

Let D_0, D_1 be two connections, consider

$$D_t = (1-t)D_0 + tD_1$$

$t \in [0, 1]$, curvature H_t

Fact: $\alpha := A_1 - A_0$ is globally defined, and in $C_1^\infty(X, \text{Hom}(E, E))$.

Fact:

$$\frac{d}{dt}f(H_t) = kdf(\alpha, H_t, H_t, \dots, H_t)$$

So,

$$f(H_1) - f(H_0) = \int_0^1 \frac{d}{dt}f(H_t)dt = d \int_0^1 f(\alpha, H_t, H_t, \dots, H_t)dt$$

So,

$$[f(H_1)] - [f(H_0)]$$

定义 4.3.2. *the k -th Chern class of E*

$$c_k(E) := [f_k(\Theta_E)] \in H^{2k}(X, \mathbb{C})$$

Recall: Chern Class

X complex manifold, $E \rightarrow X$ is a smooth complex vector bundle of rank r . D is a connection, curvature $\Theta(D) \in C_2^\infty(X, \text{Hom}(E, E))$.

linear algebra:

$$\det(I + \frac{i}{2\pi} tM) = I + t f_1(M) + t^2 f_2(M) + \cdots + t^r f_r(M)$$

Chern class $\{f_k(\Theta)\} \in H_{DR}^{2k}(X, \mathbb{C})$ is independent of choice of connection.

Today:

Special case: E is a complex line bundle. Let D_0 be a connection on E , locally $D_0 e = A_0 e$, A_0 is 1-form. curvature

$$\Theta(D_0) = D_0^2 = dA_0 + A_0 \wedge A_0 = dA_0$$

so, curvature is d-exact, so $d\Theta(D_0) = 0$.

$$\det(I + \frac{i}{2\pi} tM) = I + \frac{i}{2\pi} tM$$

so, the first Chern class of line bundle is

$$c_1(E) = \{\frac{i}{2\pi} \Theta(D_0)\}$$

Let D_1 be another connection, locally $D_1 e = A_1 e$, so $\Theta(D_1) = dA_1$.so,

$$\Theta(D_1) - \Theta(D_0) = d(A_1 - A_0)$$

where

$$A_1 - A_0 \in C_1^\infty(X, \text{Hom}(E, E))$$

(when E is line bundle, $\text{Hom}(E, E) \cong E^* \otimes E$ is trivial bundle)

so, $A_1 - A_0$ is a globally defined smooth function on X . So,

$$\{\Theta(D_1)\} = \{\Theta(D_0)\} \in H^2(X, \mathbb{C})$$

independent of the choice of connection.

4.4 Hermite 向量丛

定义 4.4.1. a complex vector bundle $E \rightarrow X$ of rank r is called a Hermitian vector bundle, if we have an inner product on E , i.e. locally, consider a local frame $\{e_1, \dots, e_r\}$, we have

$$\{e_i(x), e_j(x)\} = h_{ij}(x)$$

s.t. $(h_{ij}(x))$ is a positive definite Hermitian matrix depending smoothly on x .

注记 4.4.2. For any complex vector bundle, Hermitian structure always exists.

证明与黎曼几何类似。(黎曼度量的存在性)

定义 4.4.3. (Hermitian connection)

A connection D on E is called Hermitian, if

$$d\{e_i, e_j\} = \{De_i, e_j\} + \{e_i, De_j\}$$

More generally, let $t \in C_p^\infty(X, E)$, $s \in C_q^\infty(X, Y)$,

$$d\{s, t\} = \{dt, s\} + (-1)^p \{t, Ds\}$$

性质 4.4.4. D is a Hermitian connection, then the curvature

$$\Theta(D)^* = -\Theta(D)$$

(where $(-)^*$ is conjugate transpose of matrix)

it means that, $i\Theta(D) \in C_2^\infty(X, \text{Herm}(E, E))$

证明.

$$\begin{aligned} 0 &= d^2\{e_i, e_j\} = d\{De_i, e_j\} + d\{e_i, De_j\} \\ &= \{D^2e_i, e_j\} - \{De_i, De_j\} + \{De_i, De_j\} + \{e_i, D^2e_j\} = \{(\Theta + \Theta^*)e_i, e_j\} \end{aligned}$$

□

注记 4.4.5. E is a Hermitian line bundle, D is a Hermitian connection, then $i\Theta(D)$ is a real 2-form, $c_1(E) \in H^2(X, \mathbb{R})$.

(Chern connection)

定义 4.4.6. Let X be a complex manifold. D' is called a connection of type $(1,0)$ on E , if for any section $s \in C_{p,q}^\infty(X, E)$, we have $D's \in C_{p+1,q}^\infty(X, E)$.

A connection D'' is called a connection of type $(0,1)$, if ... $D''s \in C_{p,q+1}^\infty(X, E)$.

注记 4.4.7. Let $E \rightarrow X$ be a vector bundle. Let D be a connection on E , locally

$$Ds \xrightarrow{\sim} d\sigma + A \wedge \sigma$$

$$d\sigma = \partial\sigma + \bar{\partial}\sigma$$

so, let A' be the $(1,0)$ -part of A , ...,

$$Ds = \partial\sigma + A' \wedge \sigma + (\bar{\partial}\sigma + A'' \wedge \sigma) =: D's + D''s$$

性质 4.4.8. E : Hermitian vector bundle, D is a Hermitian connection, locally, take a C^∞ -frame e_1, \dots, e_r which is orthonormal (i.e. $\{e_i(x), e_j(x)\} = \delta_{ij}$), then the connection coefficient $A = A' + A''$ satisfies

$$(A')^* = -A''$$

$$(\iff \bar{i}A = iA)$$

证明. because

$$0 = d\langle e_i, e_j \rangle = \{De_i, e_j\} + \{e_i, De_j\} = \{a_i^k e_k, e_j\} + \{e_i, a_j^l e_l\} = a_i^j + \bar{a}_j^i$$

so, $A^* = -A$. □

推论 4.4.9. $E \rightarrow X$ is a Hermitian vector bundle, D_0'' is a connection of type $(0,1)$ on E . Then exists a unique Hermitian connection D such that $D'' = D_0''$.

证明. Let $A'' = D_0''$ and $A' = -(A_0'')^* \rightsquigarrow A = A' + A''$, and D is given by A . □

Let $E \rightarrow X$ is a holomorphic Hermitian vector bundle, observe that $\bar{\partial}$ defines a connection of type $(0,1)$ on E (check!)

assume E is a holomorphic line bundle, take a section $s \in C_p^\infty(X, E)$, i.e. we have a family of p -forms (s_α) such that $s_\alpha = g_{\alpha\beta} s_\beta$ where $g_{\alpha,\beta}$ is the holomorphic transition matrix.

$$\bar{\partial}s \xrightarrow{\sim} \bar{\partial}s_\beta$$

then

$$\bar{\partial}s_\alpha = g_{\alpha,\beta} \bar{\partial}s_\beta$$

(so, $\bar{\partial}$ is a connection of $(0,1)$)

this connection is called the canonical connection of type $(0,1)$.

定义 4.4.10. Let $E \rightarrow X$ holomorphic Hermitian vector bundle, the connection D on E is called Chern connection if

$$D'' = \bar{\partial}$$

Curvature of Chern connection

$E \rightarrow X$ is holomorphic Hermite vector bundle , D is the Chern connection, Locally let $\{e_1, \dots, e_r\}$ be a holomorphic frame, and two local sections

$$s, t \in C^\infty(\Omega, E)$$

where

$$s = \sum_{i=1}^r \sigma_i e_i$$

$$t = \sum_{i=1}^r t_i e_i$$

Since D is Hermitian ,

$$d\{s, t\} = d((\sigma_1, \dots, \sigma_r) H \begin{pmatrix} t_1 \\ \vdots \\ t_r \end{pmatrix}) = (d\sigma)^T H t + \sigma^T (dH) t + \sigma^T H d(t)$$

so, we have

$$\{Ds, t\} + \{s, Dt\} = (d\sigma + \bar{H}^{-1} \bar{\partial} \bar{H} \wedge \sigma)^T \wedge H \bar{t} + \sigma^T \wedge \overline{H(dt + \bar{H}^{-1} \bar{\partial} \bar{H} \wedge t)}$$

so ,

$$Ds = d\sigma + \bar{H}^{-1} \bar{\partial} \bar{H} \wedge \sigma$$

$$D's = \partial\sigma + \bar{H}^{-1}\partial\bar{H} \wedge \sigma = \bar{H}^{-1}\partial(\bar{H}\sigma)$$

$$D''s = \bar{\partial}\sigma$$

so,

$$(D')^2s = \bar{H}^{-1}\partial(\bar{H}(\bar{H}^{-1}\partial(\bar{H}\sigma))) = \dots = 0$$

$$(D'')^2s = \dots = 0$$

So we have

$$\Theta(D) = (D' + D'')^2 = D'D'' + D''D'$$

Locally ,

$$\begin{aligned}\Theta s &= D'D''s + D''D's = \bar{H}^{-1}\partial(\bar{H}\bar{\partial}\sigma) + \bar{\partial}(\bar{H}^{-1}\partial(\bar{H}\sigma)) = \dots = \bar{H}^{-1}\partial\bar{H} \wedge \bar{\partial}\sigma + \bar{\partial}(\bar{H}^{-1})\sigma \\ &= \bar{\partial}(\bar{H}^{-1}\partial\bar{H})\sigma\end{aligned}$$

So, Chern curvature

$$\Theta_D = \bar{\partial}(\bar{H}^{-1}\partial\bar{H})$$

Last time: $E \rightarrow X$ is a holomorphic vector bundle with a Hermitian metric H . Then there is a unique connection D_E s.t. ... called Chern connection.

Curvature of Chern Connection:

$$\Theta(D_E) = \bar{\partial}(\bar{H}^{-1}\partial\bar{H})$$

so,

$$i\Theta(D_E) \in C_{1,1}^\infty(X, \text{Hom}(E, E))$$

例子 4.4.11. (Special case: E is a holomorphic line bundle)

locally, let e be a holomorphic frame, $\langle e, e \rangle = h$ is the metric. then,

$$\Theta = \bar{\partial}(h^{-1}\partial h) = \bar{\partial}\partial \log h$$

so,

$$i\Theta(E) = -i\bar{\partial}\partial \log h$$

if $h = e^{-2\varphi}$ where φ is a smooth function, then

$$i\Theta(E) = 2i\bar{\partial}\partial\varphi = 2\sqrt{-1} \sum_{k,l} \frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_l} dz_k \wedge d\bar{z}_l$$

Question: let s be a local holomorphic section of E ,

$$-i\bar{\partial}\partial \log |s|_h^2 = ?$$

(Hint: $\frac{i}{\pi} \bar{\partial}\partial \log z = ?$ 单复变, 按分布意义下求导. 等于狄拉克测度 2333333) 可能是期末题目?

例子 4.4.12. $\mathcal{O}(-1)$ on $\mathbb{C}P^n$, tautological line bundle. (Recall: $\mathbb{C}P^n$ is a compact complex manifold with holomorphic charts

$$\Omega_j := \{[z_0; z_1; \dots; z_n] | z_j \neq 0\} \rightarrow \left(\frac{z_0}{z_j}, \dots, \hat{1}, \dots, \frac{z_n}{z_j} \right) \in \mathbb{C}^n$$

)

Let V be a complex vector space, $\dim_{\mathbb{C}} V = n + 1$. Denote the projective space by

$$\mathbb{P}(V) = (V \setminus \{0\}) / \mathbb{C}^*$$

Let $\underline{V} := \mathbb{P}(V) \times V$ be the trivial vector bundle, define

$$\mathcal{O}(-1) := \{([x], \xi) | \xi \in \mathbb{C} \cdot x\}$$

性质 4.4.13. $\mathcal{O}(-1)$ is a holomorphic line bundle on $\mathbb{P}(V)$.

证明. $\mathcal{O}(-1)|_{\Omega_j}$ has a non-vanishing holomorphic section \mathcal{E}_j defined by

$$\mathcal{E}_j([x]) = \frac{x}{x_j}$$

for $0 \leq j \leq n$. □

Assume V has a Hermitian inner product, then $\mathcal{O}(-1)$ has an Hermitian structure induced from V .

Let e_0, \dots, e_n be an orthonormal basis of V , then $\mathcal{O}(-1)|_{\Omega_0}$ has a non-vanishing holomorphic section:

$$\mathcal{E}_0(z_1, \dots, z_n) = e_0 + z_1 e_1 + \dots + z_n e_n$$

where

$$\Omega_0 = \{[1; z_1; \dots; z_n] | z_j \in \mathbb{C}\} \cong \mathbb{C}^n$$

then,

$$|\mathcal{E}_0|_h^2 = 1 + |z_1|^2 + \dots + |z_n|^2$$

so the Chern curvature of $\mathcal{O}(-1)$ on Ω_0 is given by

$$\Theta = \bar{\partial} \partial \log(1 + |z_1|^2 + \dots + |z_n|^2)$$

Denote $\mathcal{O}(1) := \mathcal{O}(-1)^*$, then

$$\Theta(\mathcal{O}(1)) = -\bar{\partial} \partial \log(1 + |z_1|^2 + \dots + |z_n|^2)$$

on Ω_0 .

$$i\Theta(\mathcal{O}(1)) = i\partial\bar{\partial}\log(1 + |z_0|^2 + \dots + |z_n|^2) = \sqrt{-1} \sum_{1 \leq k, l \leq n} c_{k,l} dz_k \wedge d\bar{z}_l$$

Exercise: (c_{kl}) is a positive definite Hermitian matrix.

"Fubini-Study metric" on $\mathbb{P}(V)$. $\mathcal{O}(1)$ is "hyperplane line bundle of $\mathbb{P}(V)$ ".

Exercise: calculate

$$\int_{\mathbb{P}(V)} \left(\frac{i}{2\pi} \Theta(\mathcal{O}(1)) \right)^{\wedge n} = ?$$

(Hint: $\mathbb{P}(V) \setminus \Omega_0$ is a zero-measure set)

$E \rightarrow X$: holomorphic line bundle, D_E is a Chern connection.

$$c_1(E) = \left\{ \frac{i}{2\pi} \Theta(D_E) \right\} \in H_{DR}^2(X, \mathbb{R})$$

Exercise: 60% 的概率出现于期末试题

Consider the sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{e^{2\pi i *}} \mathcal{O}^* \rightarrow 0$$

it induces a long exact sequence

$$\dots \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}) \rightarrow \dots$$

prove: Consider E as an element of $H^1(X, \mathcal{O}^*)$, then the image of $\delta(E)$ in $H^2(X, \mathbb{R}) \cong H_{DR}^2(X, \mathbb{R})$ is $c_1(E)$.

Exercise: E is a holomorphic line bundle, denote $\theta := \frac{i}{2\pi} \Theta(D_E)$ real $(1,1)$ -form, where D_E is Chern connection with a metric h . Prove: for any smooth function $f \in C^\infty(X, \mathbb{R})$, there exists a Hermitian metric h_f s.t.

$$\frac{i}{2\pi} \Theta_{E, h_f} = \theta + i\partial\bar{\partial}f$$

第 5 章 L^2 Hodge 理论

5.1 向量丛上的微分算子

Differential operators on vector bundles.

Let X is a (connected) smooth manifold of (\mathbb{R}) -dimension n . $E, F : \mathbb{K}$ -vector bundle of rank r, r' respectively.

定义 5.1.1. a linear differential operator of degree k from E to F is a \mathbb{K} -linear map

$$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

$$u \mapsto Pu$$

locally given by

$$Pu(x) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x)$$

where $a_\alpha(x) = (a_{af_a, \lambda_\mu}(x))$ be a $r' \times r$ matrix.

$$u(x) = (u_1(x), \dots, u_r(x))^T$$

Let $t \in \mathbb{K}$, $f \in C^\infty(M, \mathbb{K})$, $u \in C^\infty(M, E)$, then

$$e^{-tf(x)} P(e^{tf(x)} u(x)) = t^k \sigma_P(x, df(x)) u(x) + \text{terms } c_j(x) t^j \quad (j < k)$$

定义 5.1.2.

$$\sigma_P : T^*M \rightarrow \text{Hom}(E, F)$$

is called the principal symbol of P , which is a polynomial on T^*M .

locally,

$$\sigma_P(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$$

$$(\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n})$$

例子 5.1.3. Consider $d : C^\infty(M, \mathbb{K}) \rightarrow C^\infty(M, T^*M)$. then

$$du = \sum_{j=1}^n \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \frac{\partial u}{\partial x^j}$$

i.e.

$$\sigma_d(x, \xi) = \sum_{j=1}^n \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \xi_j$$

定义 5.1.4. P is called elliptic, if $\forall x \in M, \xi \in T_x^*M \setminus \{0\}$,

$$\sigma_P(x, \xi) \in \text{Hom}(E_x, E_x)$$

is injective.

For example, d is elliptic.

L^2 -inner product

Let M be an oriented C^∞ -manifold with a smooth volume form, locally

$$dV(x) = \gamma(x) dx_1 \wedge \dots \wedge dx_n$$

$\gamma(x) > 0$. Assume E has a Euclidean(or Hermitian) structure...

Let $u, v \in C^\infty(M, E)$, define

$$\langle\langle u, v \rangle\rangle := \int_M \langle u, v \rangle dV(x)$$

define $L^2(M, E) :=$ space of sections with measurable coefficients with are L^2 w.r.t $\langle\langle \cdot, \cdot \rangle\rangle$.

定义 5.1.5. Let $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ be a differential operator, E, F have Euclidean (or Hermitian) structure, then there exists unique differential operator

$$P^* : C^\infty(M, F) \rightarrow C^\infty(M, E)$$

s.t.

$$\langle \langle Pu, v \rangle \rangle = \langle \langle u, P^*v \rangle \rangle$$

for all u, v s.t. $\text{Supp}u \cap \text{Supp}v \subset\subset M$ (relative compact...)

P^* is called the formal adjoint of P .

证明. Existence: Assume that $\text{Supp}u, \text{Supp}v \subset\subset$ some coordinate chart Ω with coordinates (x_1, \dots, x_n) , then

$$\langle \langle Pv, u \rangle \rangle = \int_{\Omega} \sum_{\alpha, \lambda, \mu} a_{\alpha, \lambda, \mu}(x) D^\alpha u_\mu(x) \overline{v_\lambda(x)} \gamma(x) dx_1 \cdots dx_n$$

integration by parts, it

$$= \int_{\Omega} \sum_{\alpha, \lambda, \mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha (\gamma(x) \overline{a_{\alpha, \lambda, \mu}} v_\lambda(x))} dx_1 \cdots dx_n$$

Locally,

$$P^*v = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha (\gamma(x) \overline{a_\alpha(x)})^T v(x)$$

Uniqueness: use the density of C^∞ -section with compact support in $L^2(M, -)$. \square

推论 5.1.6. If $\sigma_P(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$, then

$$\sigma_{P^*} = (-1)^k \overline{\sigma_P(x, \xi)}^T$$

推论 5.1.7. If $\text{rank } E = \text{rank } F$, P is differential operator, then P^* is elliptic $\iff P$ is elliptic.

5.2 椭圆算子的基本性质

Fundamental results of elliptic operators

M is a compact (oriented) C^∞ -manifold, $\dim_{\mathbb{R}} M = n$, with a smooth volume form dV .

E is an Hermite vector bundle, $\text{rank}_{\mathbb{C}} E = r$.

Sobolev space: $W^k(M, E) :=$ the space of section $s : M \rightarrow E$ whose derivations up to order $= k$,
 $:=$ the completion of space of smooth sections w.r.t W^k -norm.

$(\Omega_j)_{j \in I}$: a finite open covering of M , $E|_{\Omega_j}$ trivial, Let $(\rho_j)_{j \in I}$ be a partition of unity w.r.t.
 $(\Omega_j)_{j \in I}$, s.t. $\sum_j \rho_j^2 = 1$. locally, choose an orthonormal frame $(e_{j,\lambda})_{1 \leq \lambda \leq r}$ on Ω_j , then $u = \sum_{\lambda=1}^r u_{j,\lambda} e_{j,\lambda}$ on Ω_j . Define

$$\|u\|_k^2 := \sum_{j,\lambda} \|e_j u_{j,\lambda}\|_k^2$$

where

$$\|e_j u_{j,\lambda}\|_k^2 := \int_{\Omega_j} \sum_{|\alpha| \leq k} |D^\alpha(e_j u_{j,\lambda})|^2 dV(x)$$

注记 5.2.1. *On a compact manifold, the equivalence of class of $\|\cdot\|_k$ is independent of the choice of : partition of unity, local trivialization, holomorphic covering...*

引理 5.2.2. *(Sobolev lemma)*

For $k > l + \frac{n}{2}$, then we have

$$W^k(M, E) \subseteq C^l(M, E)$$

引理 5.2.3. *(Rellich lemma)*

For any $k \in \mathbb{Z}_{\geq 0}$, the inclusion

$$W^{k+1}(M, E) \hookrightarrow W^k(M, E)$$

is a compact operator.

引理 5.2.4. *(Garding inequality)*

If

$$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

*is elliptic, and $\text{rank} E = \text{rank} F$, \tilde{P} : the extension of P to sections with distribution coefficients, then
: for all $u \in W^0(M, E)$, if $\tilde{P}u \in W^k(M, F)$, then $u \in W^{k+d}(M, E)$, where $d = \deg P$, and*

$$\|u\|_{k+d} \leq C_k (\|\tilde{P}u\|_k + \|u\|_0)$$

where C_k depending on k, M .

证明. Reference: Kodaira: deformation of complex structures (Appendix) □

推论 5.2.5. *If $u \in \ker \tilde{P} \cap W^0(M, E)$, then $u \in C^\infty(M, E)$.*

引理 5.2.6. *(Finiteness theorem)*

Setting M be a compact manifold, $\text{rank} E = \text{rank} F$,

$$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

elliptic, then:

(1) $\ker P$ is of finite dimension

(2) $P(C^\infty(M, E))$ is closed and of finite codimension in $C^\infty(M, F)$. If P^ is the formal adjoint of P , then \exists decomposition*

$$C^\infty(M, F) = P(C^\infty(M, E)) \oplus \ker P^*$$

which is orthogonal in $W^0(M, F) = L^2(M, F)$

证明. 椭圆算子的一般结果, 分析的东西 233333333. 可以参考小平邦彦复流形与复结构形变的附录. □

5.3 紧黎曼流形的 Hodge 理论

Hodge theory in compact Riemannian manifold

Hodge star operator.

M compact Riemannian manifold, $\dim_{\mathbb{R}} = n$, E is a Hermitian vector bundle. Assume $(\xi_1, \dots, \xi_n), (e_1, \dots, e_n)$ be orthonormal frame of TM, E on some local chart Ω , denote $(\tilde{\xi}_1^*, \dots, \tilde{\xi}_n^*), (e_1^*, \dots, e_n^*)$ be the co-frame of T^*M, T^*E .

$\wedge^\bullet T^*M$ is endowed with an inner product frame from TM . locally,

$$\langle u_1 \wedge \dots \wedge u_p, u_1 \wedge \dots \wedge u_p \rangle := \det(\langle u_i, v_j \rangle)$$

for $u_i, v_j \in T^*M$. Then , get an inner product on $\wedge^p T^*M$.

Assume

$$U = \sum_{\substack{|I|=p \\ i_1 \leq \dots \leq i_p}} u_I \tilde{\xi}_I^*$$

$$V = \sum_{\substack{|I|=p \\ i_1 \leq \dots \leq i_p}} v_I \tilde{\zeta}_I^*$$

be p -forms, then

$$\langle u, v \rangle = \sum_{|I|=p} u_I v_I$$

i.e. $\{\tilde{\zeta}_I^*\}$ is an orthonormal basis of $\wedge^p T^*M$.

$\wedge^* T^*M \otimes E$ has an inner product induced from $\wedge^* T^*M, E$,

定义 5.3.1. *the Hodge star operator*

$$* : \wedge^p T^*M \rightarrow \wedge^{n-p} T^*M$$

is defined by

$$u \wedge *v = \langle u, v \rangle dV$$

Locally, let

$$U = \sum_{|I|=p} u_I \tilde{\zeta}_I^*, \quad V = \sum_{|I|=p} v_I \tilde{\zeta}_I^*$$

assume

$$*V = \sum_{|J|=n-p} a_J \tilde{\zeta}_J^*$$

then

$$\begin{aligned} U \wedge * \sum u_I a_{I^c} \tilde{\zeta}_I^* \wedge \tilde{\zeta}_{I^c}^* &= \sum u_I a_{I^c} \varepsilon(I, I^c) \tilde{\zeta}_1^* \wedge \dots \wedge \tilde{\zeta}_n^* \\ \langle u, v \rangle dV &= \sum_{|I|=p} u_I v_I \tilde{\zeta}_1^* \wedge \dots \wedge \tilde{\zeta}_n^* \end{aligned}$$

so, we have

$$*V = \sum_{|I|=p} \varepsilon(I, I^c) V_I \tilde{\zeta}_{I^c}^* \in \wedge^{n-p} T^*M$$

定义 5.3.2.

$$* : \wedge^p T^*M \otimes E \rightarrow \wedge^{n-p} T^*M \otimes E$$

is defined by

$$\{s, *t\} := \langle s, t \rangle dV$$

Locally, assume

$$t = \sum_{\substack{|I|=p \\ 1 \leq \lambda \leq r}} t_{I,\lambda} \tilde{\xi}_I^* \otimes e_\lambda$$

then

$$*t = \sum_{\substack{|I|=p \\ 1 \leq \lambda \leq r}} \varepsilon(I, I^c) t_{I,\lambda} \tilde{\xi}_{I^c}^* \otimes e_\lambda$$

定义 5.3.3.

$$\# : \bigwedge^p T^*M \otimes E \rightarrow \bigwedge^{n-p} T^*M \otimes E^*$$

is defined by: for any $s, t \in \bigwedge^p T^*M \otimes E$, such that

$$s \wedge \#t := \langle s, t \rangle dV$$

wedge product + pairing of $E^* \times E \rightarrow \mathbb{C}$.

Locally: assume

$$t = \sum_{\substack{|I|=p \\ 1 \leq \lambda \leq r}} t_{I,\lambda} \tilde{\xi}_T^* \otimes e_\lambda$$

then,

$$\#t = \sum_{|I|=p, \lambda} \varepsilon(I, I^c) t_{I,\lambda} \tilde{\xi}_c^* I \otimes e_\lambda^*$$

性质 5.3.4.

$$*^2 = (-1)^{p(n-1)} \quad \text{on } \bigwedge^p T^*M \otimes E$$

$$\#^2 = (-1)^{p(n-1)} \quad \text{on } \bigwedge^p T^*M \otimes E$$

(正负号对吗?)

Recall: For all $s, t \in C^\infty(M, \bigwedge^p T^*M \otimes E)$, we have an inner product

$$\langle \langle s, t \rangle \rangle := \int_M \langle s, t \rangle dV$$

定理 5.3.5. Let D_E be an Hermite connection on E , acting on $\bigwedge^p T^*M \otimes E$, then

$$D_E^* := (-1)^{np+1} * D_E *$$

where D_E^* is the formal adjoint of D_E .

证明. Let $s \in C^\infty(M, \bigwedge^p T^*M \otimes E)$ and $t \in C^\infty(M, \bigwedge^{p+1} T^*M \otimes E)$. then

$$\langle \langle D_E s, t \rangle \rangle = \int_M \langle D_E s, t \rangle dV = \int_M \{D_E s, *t\}$$

Since D_E is Hermitian ,by definetion ,

$$d\{s, *t\} = \{D_E s, t\} + (-1)^p \{s, D_E(*t)\}$$

so,

$$\langle \langle D_E s, t \rangle \rangle = \int_M d\{s, *t\} + (-1)^{p+1} \{s, D_E *t\} = (-1)^{p+1} (-1)^{p(n_1)} \int_M \{s, *(D_E *t)\} = \langle \langle s, D_E^* t \rangle \rangle$$

so,

$$D_E^* t = (-1)^{np+1} * D_E *$$

□

定义 5.3.6.

$$\Delta_E = D_E D_E^* + D_E^* D_E : C^\infty(M, \bigwedge^p T^*M \otimes E) \rightarrow C^\infty(M, \bigwedge^p T^*M \otimes E)$$

例子 5.3.7. Let $M = \mathbb{R}^n$, $g = \sum_{i=1}^n dx_i^2$, $E = M \times \mathbb{C}$ trivial line bundle with $D_E = d$. then

$$\Delta_E u = (dd^* + d^*d)u = - \sum_{i=1}^n \left(\sum_{|I|=p} \frac{\partial^2 u_I}{\partial x_I^2} dx_I \right)$$

where

$$u = \sum_{|I|=p} u_I dx_I$$

性质 5.3.8. Δ_E is a self-adjoint elliptic operator. (i.e. $\Delta_E^* = \Delta_E$)

证明. $\Delta_E^* = \Delta_E$ be definition.

note that

$$e^{-tf} D_E (e^{tf} s) = t df \wedge s + D_E s$$

so,

$$\sigma_{D_E}(x, \xi) s = \xi \wedge s$$

$$\sum_{D_E^*} = -\overline{\sigma_{D_E}}^T$$

$$\sigma_{D_E^*}(x, \xi)s = -\tilde{\xi} \lrcorner s$$

where $\tilde{\xi}$ be the vector field dual to ξ . □

定义 5.3.9.

$$\Delta_E = D_E D_E^* + D_E^* D_E : C^\infty(M, \bigwedge^p T^* M \otimes E) \rightarrow C^\infty(M, \bigwedge^p T^* M \otimes E)$$

so,

$$\sigma_{\Delta_E}(x, \xi)s = \left(\sigma_{D_E} \sigma_{D_E^*}(x, \xi) + \sigma_{D_E^*} \sigma_{D_E}(x, \xi) \right) s$$

so, σ_{Δ_E} is injective if $\xi \neq 0$, so Δ_E is elliptic.

Harmonic forms and Hodge isomorphism.

定义 5.3.10. u is called harmonic if $\Delta_d u = 0$.

定理 5.3.11. M is a compact Riemannian manifold, then de Rham cohomology

$$H_{DR}^p(M, \mathbb{R}) \cong \ker(\Delta_d : C^\infty(M, \bigwedge^p T^* M))$$

证明. Δ_d self-adjoint elliptic, so by general result for elliptic operator,

$$C^\infty(M, \bigwedge^p T^* M) = \text{Im } \Delta_d \oplus \ker \Delta_d^* = \text{Im } \Delta_d \oplus \ker \Delta_d$$

Claim:

$$\text{Im } \Delta_d = \text{Im } d \oplus \text{Im } d^*$$

Recall $\Delta_d = dd^* + d^*d$, so

$$\text{Im } \Delta_d \subseteq \text{Im } d \oplus \text{Im } d^*$$

on the other hand,

$$\text{Im } d \oplus \text{Im } d^* \subseteq (\ker \Delta_d)^\perp = \text{Im } \Delta_d$$

so,

$$\text{Im } \Delta_d = \text{Im } d \oplus \text{Im } d^*$$

so,

$$C^\infty(M, \bigwedge^p T^*M) = \text{Im } d \oplus \text{Im } d^* \oplus \ker \Delta_d$$

so,

$$H_{DR}^p(M, \mathbb{R}) = \frac{\text{Im } d \oplus \ker \Delta_d}{\text{Im } d} = \ker \Delta_d$$

□

推论 5.3.12.

$$\dim H_{DR}^p(M, \mathbb{R}) = \dim \ker \Delta_d < +\infty$$

注记 5.3.13. Consider

$$u \mapsto \int_M (\langle u, u \rangle + \langle du, du \rangle + \langle d^*u, d^*u \rangle) dV$$

这个泛函的变分是什么鬼?

Harmonic forms and Hodge isomorphism

Recall: M is a compact Riemann manifold,

$$d : C^\infty(M, \bigwedge^* T^*M) \rightarrow C^\infty(M, \bigwedge^{*+1} T^*M)$$

adjoint d^* ,

$$\Delta_d = dd^* + d^*d$$

is a self-adjoint elliptic operator.

Hodge decomposition:

$$C^\infty(M, \bigwedge^p T^*M) = \ker \Delta_d \oplus \text{Im } d \oplus \text{Im } d^*$$

$$\mathcal{H}^p(M, \mathbb{R}) := \ker \Delta_d \quad \text{finite dimension}$$

$$\mathcal{H}^p(M, \mathbb{R}) \cong H_{DR}^p \cong H^p(M, \mathbb{R})$$

(Hodge isomorphism, and, de Rham-Weil)

Poincare duality

定理 5.3.14. The pairing

$$H_{DR}^p(M, \mathbb{R}) \times H_{DR}^{n-p}(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$(s, t) \mapsto \int_M s \wedge t$$

(is well defined) is non-degenerated. In particular, $H_{DR}^p(M, \mathbb{R})^* \cong H_{DR}^{n-p}(M, \mathbb{R})$

证明. the pairing factors through the pairing on

$$\mathcal{H}^p(M, \mathbb{R}) \times \mathcal{H}^{n-p}(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$(s, t) \mapsto \int_M s \wedge t$$

need to verify: (1) it is independent of the choice of representations. (Easy, check) (2) Pairing $\mathcal{H}^p \times \mathcal{H}^{n-p}$ is non-degenerated..

claim(Exercise): Hodge star $*$ s.t. $*\Delta_d = \Delta_d*$.

so, s is a harmonic p -form $\iff *s$ is a harmonic $(n-p)$ -form.

note that

$$s \wedge *s = \langle s, s \rangle dV = \int_M s \wedge *s = \int_M \langle s, s \rangle dV = \|s\|^2$$

□

推论 5.3.15.

$$\dim \mathcal{H}^p(M, \mathbb{R}) = \dim \mathcal{H}^{n-p}(M, \mathbb{R})$$

Generalization to flat bundle. M is a compact Riemannian manifold, $\dim_{\mathbb{R}} M = n$, $E \rightarrow M$ is a complex Hermitian vector bundle.

定义 5.3.16. $E \rightarrow X$ is called flat, if it admit a connection D_E s.t.

$$D_E^2 = 0$$

注记 5.3.17. E is flat $\iff E$ is given by a representation

$$\pi_1(M) \rightarrow GL(r, \mathbb{C})$$

(我们不证)

Consider the complex :

$$(C^\infty(M, \bigwedge^* T^*M \otimes E), D_E) \\ \rightsquigarrow H_{DR}^p(M, E) := \frac{\ker D_E}{\text{Im } D_E}$$

Exercise: we have decomposition

$$C^\infty(M, \bigwedge^p T^*M \otimes E) = \ker \Delta_{D_E} \oplus \text{Im } D_E \oplus \text{Im } D_E^*$$

$$H_{DR}^p(M, E) \cong \ker \Delta_{D_E}$$

and the pairing

$$H_{DR}^p(M, E) \times H_{DR}^{n-p}(M, E^*) \rightarrow \mathbb{C}$$

$$(s, t) \mapsto \int_M s \wedge t$$

is non-degenerate..

以上是实的 Hodge 理论。

5.4 Kähler 流形

定义 5.4.1. Let X be a complex manifold, $\dim_{\mathbb{C}} X = n$, X is called a Hermitian manifold, if X has a Hermitian metric, i.e. locally $h(z) := \sum_{1 \leq j, k \leq n} h_{jk}(z) dz_j \otimes d\bar{z}_k$, where (h_{jk}) is positive definition Hermitian matrix.

Check: the positivity of h is independent of the choice of holomorphic local coordinate

Rmk: Any complex manifold has a Hermitian metric...(Exercise)

Fundamental $(1, 1)$ -form associated to $h(z)$ is defined by

$$\omega := -\operatorname{Im} h = \frac{\sqrt{-1}}{2} \sum_{j, k} h_{jk} dz_j d\bar{z}_k$$

we also call ω is the Hermitian metric on X

Fact: ω is real (i.e. $\bar{\omega} = \omega$).

注记 5.4.2. h is a Hermite structure on TX (holomorphic tangent bundle of X). locally,

$$\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle(z) = h_{ij}(z)$$

定义 5.4.3. (X, ω) is an Hermitian manifold, X is Kähler if $d\omega = 0$.

性质 5.4.4. Locally, $\omega = \frac{\sqrt{-1}}{2} \sum_{jk} h_{jk} dz_j \wedge d\bar{z}_k$ is Kaehler, $\iff \partial\omega = 0$ and $\bar{\partial}\omega = 0$, i.e.

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}$$

If (X, ω) is a compact Kaehler manifold, then

$$H^{2k}(X, \mathbb{R}) \neq 0$$

证明. $d\omega = 0$, so $\omega \in H^2(M, \mathbb{R})$. Claim:

$$0 \neq \omega^k \in H^{2k}(M, \mathbb{R})$$

proof of the claim:

$$[\omega^k][\omega^{n-k}] = \int_X \omega^k \wedge \omega^{n-k} = \int_X \omega^n$$

Since ω is positive, locally

$$\omega^n = n! \det(h_{j\bar{k}}) \bigwedge_{l=1}^n \left(\frac{\sqrt{-1}}{2} dz_l \wedge d\bar{z}_l \right) > 0$$

is a volume form. So,

$$[\omega^k][\omega^{n-k}] = \int_X \omega^n > 0$$

(Using Poincare dual)

□

例子 5.4.5. (Exists a complex manifold NOT Kaehler) (Hopf Surface)

$$X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

where discrete group $\Gamma := \{\lambda^n | n \in \mathbb{Z}\}$, $0 < \lambda < 1$ fixed.

Exercise: $X \cong S^1 \times S^3$ C^∞ homeomorphism.. and X is compact complex manifold.
and $H^2(X, \mathbb{R}) = H^2(S^1 \times S^3, \mathbb{R}) = 0$ by Künneth Formula...
So, X is non-Kähler...

例子 5.4.6. Examples of Kaehler manifold)

(1) Riemann surface must be Kaehler...(trivial)

(2) (complex torus) $X = \mathbb{C}^n/\Gamma$, Γ is a lattice. (this manifold may not compact...)

$$\omega = \sqrt{-1} \sum_{j,k} h_{j\bar{k}} dz_j \wedge d\bar{z}_k$$

is a Kähler metric on X if $(H_{j\bar{k}}) > 0$, $h_{j\bar{k}}$ are constant.

(3) Projective space \mathbb{CP}^n .

$$\omega := \sqrt{-1} \Theta_h(\mathcal{O}(1))$$

locally,

$$\omega = \sqrt{-1} \partial \bar{\partial} \log(1 + |z_1|^2 + \dots + |z_n|^2)$$

on Ω . This ω is a Kähler metric,

例子 5.4.7. Let (X, ω) is a Kahler manifold, then any complex submanifold $Y \subseteq X$ is also Kahler.

$$i : Y \hookrightarrow X$$

with the Kahler metric $i^*\omega$.

Exercise: Let $f : Y \rightarrow X$ be a holomorphic immersion, and assume X is Kahler, then Y is Kahler.

推论 5.4.8. Any projective manifold (i.e. $X \hookrightarrow \mathbb{CP}^N$) is Kähler.

(Algebraic Geometry.....)

性质 5.4.9. (Equivalent definition of Kaehler metrics) a Hermitian metric ω is Kahler, iff for all $x_0 \in X$, there exists a holomorphic chart (z_1, \dots, z_n) centered at x_0 , s.t.

$$\omega(z) = \sqrt{-1} \sigma_{jk} \delta_{jk} dz_j \wedge d\bar{z}_k + O(|z|^2)$$

(\Leftarrow is trivial...) (left to HW)

定理 5.4.10. (Exercise)

If (X, ω) is Kahler, then for all $x_0 \in X$, \exists holomorphic chart z_1, \dots, z_n centered at x_0 , s.t. assume

$$\omega = \sqrt{-1} h_{jk} dz_j \wedge d\bar{z}_k$$

then

$$h_{lm}(z) = \delta_{lm} - \sum_{j,k} c_{jk,lm} z_j \bar{z}_k + O(|z|^3)$$

where $c_{jk,lm}$ is the coefficients of the Chern curvature tensor,

$$\Theta(TX)_x := \sum c_{jk,lm} dz_j \wedge d\bar{z}_k \otimes \left(\frac{\partial}{\partial z_l}\right)^* \otimes \frac{\partial}{\partial z_m}$$

(查书)

5.5 紧复流形上的 Hodge 理论

(X, ω) is a compact Hermitian manifold, $E \rightarrow X$ is a homomorphic Hermitian vector bundle.

$$D_E := D'_E + D''_E$$

Chern connection, $D''_E = \bar{\partial}$.

定义 5.5.1.

$$\Delta_E := D_E D_E^* + D_E^* D_E$$

$$(D'_E)^* = - * D''_E *$$

$$(D''_E)^* = - * D'_E *$$

$$\Delta'_E = D'_E (D'_E)^* + \dots$$

$$\Delta''_E = \dots$$

Note that $(D''_E)^2 = 0$, consider the complex

$$\begin{aligned} C^\infty(X, \bigwedge^{p,q} \otimes E) &\xrightarrow{D''_E} C^\infty(X, \bigwedge^{p,q+1} \otimes E) \\ &\rightsquigarrow H_{D''_E}^{p,q}(X, E) \end{aligned}$$

Dolbeaut cohomology... it isom to $\ker \Delta''_E$

Hodge theory in compact complex manifold.

Let (X, ω) be a compact complex manifold of dimension n . $E \rightarrow X$ holomorphic Hermitian vector bundle, with Chern connection D_E , $D_E = D'_E + D''_E$ where $D''_E = \bar{\partial}$.

Recall: L^2 inner product: $u \in C^\infty(X, \bigwedge^{p,q} \otimes E)$,

$$\langle \langle u, v \rangle \rangle := \int_X \langle u, v \rangle \mathrm{dvol}$$

Hodge star operator $*$: $u, v \in C^\infty(X, \bigwedge^{p,q} \otimes E)$,

定义 5.5.2.

$$*: \bigwedge^{p,q} \otimes E \rightarrow \bigwedge^{n-q, n-p} \otimes E$$

s.t.

$$u \wedge *v = \langle u, v \rangle \mathrm{dvol}$$

(wedge product from $\bigwedge^{p,q}$, with inner product from E)

Exercise: Take a holomorphic chart (z_1, \dots, z_n) s.t.

$$\omega = \sqrt{-1} \sum_j dz_j \wedge d\bar{z}_j$$

at some point p . An orthonormal frame $\{e_1, \dots, e_r\}$, Let

$$u = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{\lambda=1}^r u_{IJ} dz_I \wedge d\bar{z}_J \otimes e_\lambda \in \bigwedge^{p,q} \otimes E$$

WHAT IS $*u$?

Formal adjoint of D_E, D'_E, D''_E ?

性质 5.5.3.

$$D_E^* = - * D_E *$$

$$(D'_E)^* = - * D''_E *$$

$$(D''_E)^* = - * D'_E *$$

定义 5.5.4.

$$\Delta_E := D_E D_E^* + D_E^* D_E$$

$$\Delta'_E := D'_E D_E'^* + D_E'^* D'_E$$

$$\Delta''_E := \dots$$

Check: $\Delta_E, \Delta'_E, \Delta''_E$ are self adjoint, elliptic operators.

Hodge theory w.r.t. Δ''_E .

定理 5.5.5. We have a decomposition

$$C^\infty(X, \bigwedge^{p,q} \otimes E) = \ker \Delta''_E \oplus \text{Im } D''_E \oplus \text{Im } D''_E^*$$

As a consequence, Dolbeault cohomology

$$H_{D''_E}^{p,q}(X, \mathbb{C}) \cong \ker \Delta''_E$$

推论 5.5.6.

$$\dim_{\mathbb{C}} H_{D_E''}^{p,q}(X, \mathbb{C}) < +\infty$$

Cohomology group

$$H_{D_E''}^{p,q}(X, \mathbb{C})$$

Ω^p : sheaf of holomorphic p -forms on X (i.e. a $(p, 0)$ -form φ is holomorphic if $\bar{\partial}\varphi = 0$).

$\mathcal{E}^{p,q}$: Sheaf of smooth (p, q) -forms on X .

Similarly, we have $\Omega^p(E)$ the sheaf of holomorphic p -forms with values in E , and $\mathcal{E}^{p,q}(E)$ the sheaf...smooth (p, q) -forms ...

we have an acyclic resolutions

$$0 \rightarrow \Omega^p(E) \xrightarrow{D_E''} \mathcal{E}^{p,1}(E) \xrightarrow{D_E''} \mathcal{E}^{p,2}(E) \xrightarrow{D_E''} \dots$$

(check, it is a resolution)

By de Rham-Weil theorem,

$$H^q(X, \Omega^p(E)) \cong D_{D_E''}^{p,q}(X, \mathbb{C}) \cong \mathcal{H}_{D_E''}^{p,q}(X, \mathbb{C}) := \ker \Delta_E''$$

定理 5.5.7. (*Serre duality*)

The pairing

$$H_{D_E''}^{p,q}(X, E) \times H_{D_E''}^{n-p, n-q}(X, E^*) \rightarrow \mathbb{C}$$

$$(s, t) \mapsto \int_X s \wedge t$$

is non-degenerate

证明. Define

$$\# : \bigwedge^{p,q} \otimes E \rightarrow \bigwedge^{n-p, n-q} \otimes E^*$$

by: for $u, v \in \bigwedge^{p,q} \otimes E$,

$$u \wedge \#v := \langle u, v \rangle \text{dvol}$$

Fact:

$$\Delta_{E^*}'' \# = \# \Delta_E''$$

□

Remark: take $E = X \times \mathbb{C}, D_E = d = d' + d'', (d' = \partial, d'' = \bar{\partial})$ then we have

$$\Delta' = d'd'^* + d'^*d'$$

$$\Delta'' = \dots$$

then

$$H_{d''}^{p,q}(X, \mathbb{C}) \cong \ker \Delta'' \hookrightarrow C^\infty(X, \bigwedge^{p,q})$$

the pairing

$$H^{p,q}(X, \mathbb{C}) \times H^{n-p, n-q}(X, \mathbb{C}) \rightarrow \mathbb{C}$$

is non-degenerate.

第6章 Lefschitz 分解

6.1 线性代数版本的 Lefschitz 算子

Three goals:

Kahler package

Lefschetz decomposition

Hodge-Riemann bilinear relations

Linear algebra(baby representation theory)(local case)

\mathbb{C}^n ,

$$\omega = \sqrt{-1} \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j$$

Kahler metric with constant coefficients.(i.e. h_{ij} is constant, (h_{ij}) is positive Hermite matrix)

W.L.O.G, by taking a linear transformation, we can assume

$$\omega = \sqrt{-1} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$$

记号 6.1.1. *An operator is of pure degree r if it transform a form of $\deg = k$ to as form of degree $k + r$.*

An operator ..of bi-degree (p, q) if $...(s, t) \rightarrow (s + p, t + q)$ (in this case, $\text{degree} = p + q$) if A, B with degree $\deg A, \deg B$, define

$$[A, B] := AB - (-1)^{\deg A \deg B} BA$$

定义 6.1.2.

$$L : \bigwedge^{p,q} \rightarrow \bigwedge^{p+1,q+1}$$
$$u \mapsto \omega \wedge u$$

is called Lefschetz operator.

Denote Λ to be the adjoint of L , adjointed by : Let $v \in \wedge^{p-1,q-1}$ and $u \in \wedge^{p,q}$

$$\langle Lv, u \rangle := \langle u, \Lambda v \rangle$$

The operator Λ is of bi-degree $(-1, -1)$.

性质 6.1.3. If

$$u = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ} dz_I \wedge d\bar{z}_J$$

then

$$Lu = \sqrt{-1} \sum_{\substack{|I|=p \\ |J|=q}} \sum_{m=1}^n u_{IJ} dz_m \wedge d\bar{z}_m \wedge dz_I \wedge d\bar{z}_J$$

$$\Lambda u = \sqrt{-1}(-1)^p \sum_{\substack{|I|=p \\ |J|=q}} \sum_{m=1}^n u_{IJ} \left(\frac{\partial}{\partial z_m} \lrcorner dz_I \right) \wedge \left(\frac{\partial}{\partial \bar{z}_m} \lrcorner d\bar{z}_J \right)$$

where " \lrcorner " is contraction.

推论 6.1.4. (Exercise) Let

$$\alpha = \sqrt{-1} \sum_{j=1}^n \alpha_j dz_j \wedge \bar{z}_j$$

then, (α is a operator of bi-degree $(1, 1)$)

$$[\alpha, \Lambda]u = \sum_{\substack{|I|=p \\ |J|=q}} \left(\sum_{i \in I} \alpha_i + \sum_{j \in J} \alpha_j - \sum_{k=1}^n \alpha_k \right) u_{IJ} dz_I \wedge d\bar{z}_J$$

where

$$u = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ} dz_I \wedge d\bar{z}_J$$

推论 6.1.5. if $u \in \wedge^{p,q}$, then

$$[L, \Lambda]u = (p + q - n)u$$

推论 6.1.6. Denote $B := [L, \lambda]$, then

$$[B, L] = 2L$$

$$[B, \Lambda] = -2\Lambda$$

证明. Take $u \in \bigwedge^{p,q}$, then

$$[B, L] = BLu - LBu = (p + q - n + 2)Lu - (p + q - n)Lu = 2Lu$$

the second is similar.. □

$\mathfrak{sl}(2, \mathbb{C})$ -representation

$$\mathfrak{sl}(2, \mathbb{C}) = \text{span}_{\mathbb{C}} l, \lambda, b$$

where

$$l = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have

$$[l, \lambda] = b \quad [b, l] = 2l \quad [b, \lambda] = -2\lambda$$

性质 6.1.7. There exists a natural action

$$\rho : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \text{End}\left(\bigoplus_{p,q} \bigwedge^{p,q}\right)$$

with

$$\rho(l) = L$$

$$\rho(\lambda) = \Lambda$$

$$\rho(b) = B$$

定理 6.1.8. (HL)

$$L^{n-k} : \bigwedge^k \rightarrow \bigwedge^{2n-k}$$

$$u \rightarrow \omega^{n-k} \wedge u$$

is an isomorphism.

$$L^{n-k} : \bigwedge^{p,q} \rightarrow \bigwedge^{n-k+p, n-k+q}$$

is also an isomorphism.

证明. Lemma:

$$[L^r, \Lambda]u = r(k - n + r - 1)L^{r-1}u$$

(induction, omit)

Assume $\alpha \in \bigwedge_{\mathbb{C}}^k$, $L^{n-k}\alpha = 0$, need to verify $\alpha = 0$.

Claim:

$$L^r : \bigwedge^k \rightarrow \bigwedge^{k+2r}$$

is injective whenever $r \leq n - k$.

proof of the claim:

claim is true when $k = 0$ or $k = 1$. (check)

Let $\alpha \in \bigwedge^k$ s.t. $L^r\alpha = 0$ ($r \leq n - k$). By the lemma,

$$L^r\Lambda\alpha - \Lambda L^r\alpha = r(k - n + r - 1)L^{r-1}\alpha$$

so,

$$L^{r-1}(L\Lambda\alpha - r(k - n + r - 1)\alpha) = 0$$

by the induction on r ,

$$L\Lambda\alpha = r(k - n + r - 1)\alpha$$

since $r(k - n + r - 1) \neq 0$, $\alpha = L\beta$ for some $\beta \in \bigwedge^{k-2}$. so, $L^r\alpha = L^{r+1}\beta = 0$, by induction on k , we have $\beta = 0$, so $\alpha = 0$.

The claim is proved. □

定义 6.1.9. (*Primitive form*)

$\alpha \in \bigwedge^k$ ($k \leq n$) is called primitive form, if

$$L^{n-k+1}\alpha = 0$$

推论 6.1.10. (*Lefschitz Decomposition*)(LD)

For any $\alpha \in \bigwedge^k$, ($1 \leq k \leq 2n$), we have a unique decomposition:

$$\alpha = \sum_{\gamma \geq (k-n)_+} L^\gamma \alpha_\gamma$$

$((k-n)_+ := \max\{k-n, 0\})$ with $\alpha_r \in \bigwedge^{k-2r}$ is primitive

证明. Existence: assume $k \leq n$, consider

$$L^{n-k+1}\alpha \in \bigwedge^{2n-k+2}$$

by HL, $\exists! \beta \in \bigwedge^{k-2}$ s.t. $L^{n-k+2}\beta = L^{n-k+1}\alpha$, so $L^{n-k+1}(\alpha - L\beta) = 0$, i.e. $\alpha_0 = \alpha - L\beta$ is primitive. $\alpha = \alpha_0 + L\beta$, then induction on degrees, we get the decomposition for α .

If $k > n$, we apply HL to reduce it to case 1.

Uniqueness: Next time..

□

Today: Continuous to Hard Lef decomposition, Hodge-Riemann bilinear relations.

Hard-Lefschitz: HL

Lefschitz decomposition :LD

Hodge-Riemann bilinear relations :HRR

Recall: $\mathbb{C}^n, \bigwedge^k = \bigoplus_{p+q=k} \bigwedge^{p,q}$, ω : a Kahler metric on \mathbb{C}^n with constant coefficient $\in \bigwedge_{\mathbb{R}}^{1,1}$.

Lefschitz operator : $Lu = \omega \wedge u$.

定理 6.1.11. (HL)

Assume $k \leq n, p+q \leq n$, then

$$L^{n-k} : \bigwedge^k \rightarrow \bigwedge^{2n-k}$$

is a linear isomorphism.

$$L^{n-k} : \bigwedge^{p,q} \rightarrow \bigwedge^{p+n-k, q+n-k}$$

is also a linear isomorphism.

Linear algebra..

定理 6.1.12. (LD) for any $u \in \bigwedge^k$, we have a unique decomposition

$$u = \sum_{r \geq (k-n)_+} L^r u_r$$

where $u_r \in \bigwedge_{\text{prim}}^{k-2r}$ is a primitive form.

Recall: a k -form $u \in \wedge^k (k \leq n)$ is called primitive, if $L^{n-k+1}(u) = 0$. When $k > n$, u is called primitive, $\Lambda(u) = 0$, where Λ is the adjoint of L .

证明. Existence: application of HL .

Uniqueness: Omit. □

性质 6.1.13. Assume $\alpha \in \wedge_{prim}^{p,q}$, and $p + q \leq n$. (i.e. $L^{n-p-q+1}\alpha = 0$), then

$$*\alpha = (-1)^{\frac{(p+q)(p+q-1)}{2}} (\sqrt{-1})^{p-q} \frac{1}{(n-p-q)!} L^{n-p-q}\alpha$$

证明. See [Humphreys, Prop 1.2.31] □

定理 6.1.14. (HRR) Define the bilinear form Q on $\wedge^k (k \leq n)$ as follows:

$$Q(\alpha, \beta) := L^{n-k} \wedge \alpha \wedge \bar{\beta}$$

Then

$$(\sqrt{-1})^{p-q} (-1)^{\frac{(p+q)(p+q-1)}{2}} Q(u, u) \geq 0$$

for any $u \in \wedge_{prim}^{p,q}$, $p + q = k \leq n$, and equal holds

$$\iff u = 0$$

(i.e. $Q|_{\wedge_{prim}^{p,q}}$ is positive definite up to a factor)

证明. Take $u \in \wedge_{prim}^{p,q}$,

$$Q(u, u) = L^{n-k} \wedge u \wedge \bar{u} = *u \wedge \bar{u} = \langle \bar{u}, \bar{u} \rangle dVol = |u|^2 dVol \geq 0$$

(up to a factor!)

(We apply the following result: $\overline{* \varphi} = * \bar{\varphi}$, i.e. $*$ is a real operator) □

Summary: $\wedge^\bullet = \bigoplus_{1 \leq k \leq n} \wedge_{\mathbb{C}}^k$, where $\wedge_{\mathbb{C}}^k = \bigoplus_{p+q=k} \wedge_{\mathbb{C}}^{p,q}$.

Lefschitz operator $L \rightsquigarrow HL, LD, HRR$.

6.2 紧 Kahler 流形的上同调群

The analogue of compact Kahler manifolds,

$$H_{DR}^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H_{Dol}^{p,q}(X, \mathbb{C})$$

ω : A Kahler metric $\in H_{Dol}^{1,1}(X, \mathbb{R})$.

Denote $L \hookrightarrow H_{DR}^k(X, \mathbb{C})$,

$$L(u) = [\omega, u] = [\omega] \wedge u$$

Commutative relations on Kahler manifolds

$$(\mathbb{C}^n, \omega = \sqrt{-1} \sum_{j=1}^n dz_j \wedge d\bar{z}_j)$$

$u \in C^\infty(\mathbb{C}^n, \wedge^{p,q})$, locally

$$u = \sum_{|I|=p, |J|=q} u_{I,J} dz_I \wedge d\bar{z}_J, \quad v = \sum_{|I|=p, |J|=q} v_{I,J} dz_I \wedge d\bar{z}_J$$

$$\langle \langle u, v \rangle \rangle = \int_{\mathbb{C}^n} \sum_{|I|=p, |J|=q} u_{I,J} \overline{v_{I,J}} dVol$$

$$d = d' + d'', \quad d' = \partial, \quad d'' = \bar{\partial}.$$

$$d'u = \sum_{I,J} \sum_k \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\bar{z}_J$$

$$d''u = \dots$$

定理 6.2.1.

$$(d'')^* u = - \sum_{I,J} \sum_k \frac{\partial u_{I,J}}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_k} \lrcorner (dz_I \wedge d\bar{z}_J)$$

$$(d')^* u = - \sum_{I,J} \sum_k \frac{\partial u_{I,J}}{\partial \bar{z}_k} \frac{\partial}{\partial z_k} \lrcorner (dz_I \wedge d\bar{z}_J)$$

性质 6.2.2.

$$[(d'')^*, L] = \sqrt{-1} d'$$

证明. Exercise. □

定理 6.2.3. *Let X be a Kahler manifold (may not compact), with Kahler metric ω , then we have*

$$[(d'')^*, L] = \sqrt{-1}d'$$

证明. Only need to verify $u \in C_c^\infty(X, \wedge^{p,q})$ with compact support in a holomorphic chart at x .

Assume the holomorphic chart near x is choosen s.t.

$$\omega(z) = \sqrt{-1} \sum_{1 \leq j \leq n} dz_j \wedge d\bar{z}_j + O(|z|^2)$$

$$u \in \sum_{I,J} u_{I,J} dz_I \wedge \bar{z}_J$$

is a (p,q) -form, v is also...

$$\langle u, q \rangle = u_{I,J} \overline{v_{M,N}} \langle dz_I, dz_M \rangle \langle d\bar{z}_J, d\bar{z}_N \rangle = u_{IJ} \overline{V_{ij}} + a_{IJMN}(z) u_{IJ} \overline{V_{MN}}$$

where $a_{IJMN} = O(|z|^2)$.

So,

$$(d'')^* u = - \sum_{I,jk} \frac{\partial u_{IJ}}{\partial z_k} \frac{\partial}{\partial \bar{z}_k} \lrcorner (dz_I \wedge d\bar{z}_J) + \sum_{IJMN} b_{IJMN} u_{IJ} dz_M \wedge d\bar{z}_N$$

where $b_{IJMN}(z) = O(|z|)$. So,

$$[(d'')^*, L]u(x) = \sqrt{-1}d'u(x)$$

$$\implies [(d'')^*, L] = \sqrt{-1}d'$$

□

性质 6.2.4. *In Kahler manifold,*

$$[(d')^*, L] = -\sqrt{-1}d''$$

$$[\Lambda, d''] = -\sqrt{-1}(d')^*$$

$$[\Lambda, d'] = \sqrt{-1}(d'')^*$$

推论 6.2.5. (X, ω) is a Kahler manifold, then

$$\Delta_d = 2\Delta_{d'} = 2\Delta_{d''}$$

证明. For example, $\Delta_d = 2\Delta_{d''}$,

$$\Delta_d = (d' + d'')(d' + d'')^* + (d' + d'')^*(d' + d'') = (d' + d'')(d'^* - \sqrt{-1}[\Lambda, d']) + (d'^* - \sqrt{-1}[\Lambda, d'])(d' + d'')$$

然后暴力展开, 12 项??? ...

从略。

□

推论 6.2.6. If (X, ω) is a Kahler manifold, then

$$\Delta_d : C^\infty(C, \bigwedge^{p,q}) \rightarrow C^\infty(C, \bigwedge^{p,q})$$

证明. Since $\Delta_d = 2\Delta_{d'}$, $\Delta_{td'}$ preserves the bi-degree.

□

推论 6.2.7. If (X, ω) is a compact Kahler manifold, u is a Δ_d -harmonic k -form. Assume

$$u = \sum_{p+q=k} u^{p,q}$$

$$u^{p,q} \in C^\infty(X, \bigwedge^{p,q})$$

then each $u^{p,q}$ is also harmonic.

定理 6.2.8. (Hodge decomposition)

X is a compact Kahler manifold, then we have a decomposition

$$H_d^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H_{d''}^{p,q}(X, \mathbb{C})$$

Equivalently, (sheaf cohomology)

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega^p)$$

证明. take a Kahler metric ω , we can define $\Delta_d, \Delta_{td'}, \Delta_{d''}$, then

$$\ker \Delta_d := \mathcal{H}^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} \mathcal{H}_{d''}^{p,q}(X, \mathbb{C})$$

then \implies the decomposition for $H_d^k(X, \mathbb{C})$

the decomposition for $H_d^k(X, \mathbb{C})$ is independent of the choice of ω (Next time) \square

Recall: Hodge decomposition,

X compact Kahler manifold, $\dim_{\mathbb{C}} X = n$,

Thm:(Hodge decomposition)

$$H_{DR}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H_{d''}^{p,q}(X, \mathbb{C})$$

where

$$H^{p,q}(X, \mathbb{C}) = \{[\alpha] \in H_{DR}^k(X, \mathbb{C}) | \alpha \text{ is a } d\text{-closed s.m. } (p, q)\text{-form}\}$$

Proof: take a Kahler metric ω ,

$$H_{DR}^k(X, \mathbb{C}) \cong \mathcal{H}_d^k(X, \mathbb{C}) = \bigoplus \mathcal{H}_d^{p,q}(X, \mathbb{C}) = \bigoplus \mathcal{H}_{d''}^{p,q}(X, \mathbb{C})$$

性质 6.2.9. *There is a canonical isomorphism*

$$H_d^{p,q}(X, \mathbb{C}) \xrightarrow{\sim} H_{d''}^{p,q}(X, \mathbb{C})$$

$$[\alpha]_d \mapsto [\alpha]_{d''}$$

where $d\alpha = 0, \alpha$ is a (p, q) -form. $\implies d''\alpha = 0$

证明. Check: this map is well defined. Need to verify: if $\alpha = d\beta$ is a (p, q) -form, then $[\alpha]_{d''} = 0$, i.e. α is also d'' -exact.

α is a (p, q) -form,

$$\implies \alpha = d'\beta^{p-1,q} + d''\beta^{p,q-1}$$

we have $d''d'\beta^{p-1,q} = 0, d'd''\beta^{p,q-1} = 0$

We need a very important lemma: \square

引理 6.2.10. ($\partial\bar{\partial}$ -lemma)

Let X is a Kahler manifold, α is a smooth form which is d' and d'' closed. Then, if α is d or d' or d'' -exact, then $\alpha = d'd''\gamma$ for some γ .

Using $\partial\bar{\partial}$ -lemma, this map is well-defined.

Now, notice that the two space has the same dimension. So, we need to show the map is injective(or, surjective). Claim : this map is injective. If α is a d -closed with $[\alpha]_{d''} = 0$, i.e. $\alpha = d''\beta^{p,q-1}$. α is d -closed $\Rightarrow d'd''\beta^{p,q-1} = 0$, $\partial\bar{\partial}$ -lemma applying to $d''\beta^{p,q-1}$, we have

$$d''\beta^{p,q-1} = d'd''\gamma = d(d''\gamma)$$

for some γ .

Proof of $\partial\bar{\partial}$ -lemma:

证明. Assume α is d'' exact, i.e. $\alpha = d''\beta$, write

$$\beta = H(\beta) + \Delta_d \gamma$$

where $H(\beta)$ is Δ_d -harmonic, so

$$\alpha = d''H(\beta) + d''\Delta_d \gamma - 2d''\Delta_{d'} \gamma$$

(Since $\Delta_d = 2\Delta_{d''}$)

$$\Rightarrow \alpha = 2d''(d'd'^* + d'^*d') = 2d''d; d'^* \gamma - 2d'^*d''d' \gamma$$

By the assumption, $d'\alpha = 0$, so $d'^*d''d' \gamma = 0$

$$\alpha = -2d'd''d'^* \gamma$$

□

注记 6.2.11. (*Deligne-Griffiths-Morrora*)

If \hat{X} is bimeromorphic to X , where X is a compact Kahler, then \hat{X} is also satisfies the $\partial\bar{\partial}$ -lemma. X is a kahler manifold, then

$$H_d^{p,q}(X, \mathbb{C}) \cong H_{d''}^{p,q}(X, \mathbb{C}) \cong H^{p,q}X, \mathbb{C}$$

X is a compact complex manifold, define

$$H_{BC}^{p,q} := \frac{d\text{-closed } (p,q)}{d'd''\text{-exact}}$$

Bott-Chern cohomology

Exercise" If X is Kahler, then $H_{BC}^{p,q} = H_d^{p,q}$

$$H_A^{p,q}(X, \mathbb{C}) := \frac{d'd''\text{closed}}{(d')\text{-exact} + \{d''\text{exact}\}}$$

(Appeli cohomology)

denote

$$h_{BC}^k := \sum_{p+q=k} \dim_{\mathbb{C}} H_{BC}^{p,q}$$

$$h_A^k := \sum_{p+q=k} \dim_{\mathbb{C}} H_A^{p,q}$$

定理 6.2.12. X satisfies $\partial\bar{\partial}$ -lemma \iff

$$h_B^k + h_A^k = 2b_k$$

where

$$b_k = \dim_{\mathbb{C}} H_{DR}^k(X, \mathbb{C})$$

定理 6.2.13. (Hard Lef)

X is a compact Kahler, $\dim_{\mathbb{C}} X = n$, denote $L = \{\omega\} \curvearrowright H_{DR}^k(X, \mathbb{C})$, ω is a Kahler metric, Then we have:

$$L^{n-k} : H_{DR}^k(X, \mathbb{C}) \cong H_{DR}^{2n-k}(X, \mathbb{C})$$

$$H^{p,q}(X, \mathbb{C}) \cong H^{p+n-k, q+n-k}(X, \mathbb{C})$$

where $k \leq n$, $p+q \leq n$.

证明. For a Kahler metric ω ,

$$L^{n-k} : H_{DR}^k \rightarrow H_{DR}^{2n-k}$$

($\cong \mathcal{H}_d^k, \cong \mathcal{H}_d^{2n-k}$ respectively) (there is a commutative diagram...)

need to proof: For any $\varphi \in \mathcal{H}_d^k$, then

$$L^{n-k}(\varphi) = \omega^{n-k} \wedge \varphi$$

is also harmonic. □

引理 6.2.14.

$$[\triangle_d, L] = 0$$

证明.

$$[\Delta_d, L] = 2[\Delta_{d'}, L] = 2([d'd'^*, L] + [d'^*d', L]) = 2(d'[d'^*, L] + [d'^*, L]d')$$

(check: $[L, d'] = 0$) So,

$$= -2\sqrt{-1}(d'd'' + d''d') = 0$$

□

Exercise: Complex tori

$$\mathbb{T}^n := \mathbb{C}^n / \Gamma$$

where $\Gamma = \mathbb{Z}^n$. \mathbb{T}^n is a compact Kahler manifold. Then

$$H^{1,1}(\mathbb{T}^n, \mathbb{C}) \cong \bigwedge_{\mathbb{C}}^{1,1}$$

the space of $(1,1)$ -forms on \mathbb{C}^n with constant coefficient, in particular,

$$\dim_{\mathbb{C}} H^{1,1}(\mathbb{T}^n, \mathbb{C}) = n^2$$

Exercise: the set of all the Kahler class on $\mathbb{T}^n \subseteq H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$ is equal to the set of $n \times n$ positive definite Hermitian metrics.

(Hint: using Hodge theory)

定理 6.2.15. (*Lefschitz decomposition*)

Define a class $\alpha \in H_{DR}^k(X, \mathbb{C})$ to be positive if

$$L^{n-k+1}(\alpha) = 0$$

if $k \leq n$.

(When $\alpha \in H_{DR}^k(X, \mathbb{C})$, $k > n$, we call α positive)

Then $\forall \varphi \in H_{DR}^k(X, \mathbb{C})$, exist unique decomposition

$$\varphi = \sum_{\gamma \geq (k-n)_+} L^\gamma \varphi_\gamma$$

where $\varphi_\gamma \in H_{prim}^{k-2\gamma}(X, \mathbb{C})$.

Similarly,

$$H^{p,q}(X, \mathbb{C}) = \bigoplus_{r \geq (p+q-n)_+} H_{prim}^{p-r, q-r}(X, \mathbb{C})$$

证明. Exercise.

□

定理 6.2.16. (HRR)

X compact Kahler, $\dim_{\mathbb{C}} X = n$, ω is Kahler metric, define

$$Q(\alpha, \beta) = L^{n-k} \alpha \wedge \bar{\beta}$$

where $\alpha, \beta \in H^{p,q}(X, \mathbb{C})$, and $p + q = k$.

Then $Q|_{H_{\text{prim}}^{p,q}}$ is positive defined (up to a factor).

证明. Exercise. □

Exercise: Consider X -compact Kahler, $\dim_{\mathbb{C}} X = n$, ω -Kahler metric, Then $\forall \alpha, \beta \in H^{1,1}(X, \mathbb{R}) = H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$, Then

$$(\{\omega^{n-2}\} \cdot \alpha \cdot \beta)^2 \geq (\{\omega^{n-2}\} \cdot \alpha^2) (\{\omega^{n-2}\} \cdot \beta^2)$$

with equality if and only if $\alpha = \lambda \beta$ for some $\lambda \in \mathbb{R}$

Eg: \mathbb{C}^2 , α, β real $(1,1)$ -forms,

$$(\alpha, \beta)^2 \geq \alpha^2 \beta^2$$

Hint: Using HRR, and Lefschitz decomposition... "Alg-Geom-inequality over Kahler manifold".

性质 6.2.17. X is a compact Kahler, then

$$\overline{H^{p,q}(X, \mathbb{C})} = H^{q,p}(X, \mathbb{C})$$

证明. Use harmonic form.. and \triangle_d is a real operator... □

Summary X -compact Kahler with a Kahler metric ω , then define Lefschitz operator $L = [\omega] \wedge$, then:

Hodge decomposition:

$$H^k = \bigoplus_{p+q=k} H^{p,q}$$

$$\overline{H^{p,q}} = H^{q,p}$$

Hard Lefschitz:

$$L^{n-k} : H^{p,q} \cong H^{p+n-k, q+n-k}$$

where $p + q = k$

Lefschitz decomposition:

$$H^{p,q} = \bigoplus_{r \geq (p+q-1)_+} L^r H_{prim}^{p-r, q-r}$$

HRR:...

References Kahler pairing in other settings..

Adiprasito-Huh-Katz: Hodge theory in combinatorial geometries

McMullen: On simple polytopes

Deligne: Weil II

Beilinson-Bernstein-Deligne-Gabber: Faisceaux Pervers

Adiprasito: Combinatorial Lefschetz theorem beyond positivity, 2018

Recall: Kahler pairing: X-compact Kahler manifold of complex dimension n , ω -Kahler metric.

Lefschitz operator

$$L = \{\omega\} \curvearrowright H^\bullet$$

Hodge decomposition

$$H^k = \bigoplus_{p+q=k} H^{p,q}, \quad \overline{H^{p,q}} = H^{q,p}$$

(Corollary: if k is odd, then $b_k := \dim_{\mathbb{C}} H^k(X, \mathbb{C})$ is even.)

Rmk: if X is compact complex surface ($\dim_{\mathbb{C}} = 2$), X is Kahler $\iff b_1$ is even. (The proof of " \Leftarrow " we not given...Ref: Kodaira&Siu, Lamari 1999)

Hard Lef. ($p+q=k$)

$$L^{n-k} : H^{p,q} \xrightarrow{\sim} H^{p+n-k, q+n-k}$$

Lef. decomposition:

$$H^{p,q} = \bigoplus_{r \geq (k-n)_+} L^r H_{prim}^{p-r, q-r}$$

Denote $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$, "Hodge number". Cor:

$$h^{p,q} = \begin{cases} h_{prim}^{p,q} + h_{prim}^{p-1, q-1} + \dots & p+q \leq n \\ h_{prim}^{n-q, n-p} + h_{prim}^{n-q-1, n-p-1} + \dots & p+q \geq n \end{cases}$$

(Using the property of L^r)

If $p+q \leq n$, $h^{p,q} \geq h^{p-1, q-1} \Rightarrow b_k \geq b_{k-2}$ if $k \leq n$.

If $p+q \geq n$, $h^{p,q} \leq h^{p-1, q-1} \Rightarrow b_k \leq b_{k-2}$ if $k \geq n$.

(Hodge-Frolicher spectral sequence)

X-compact Kahler, then Hodge decomposition

$$\Rightarrow b_k = \sum_{p+q=k} h^{p,q}$$

Question: X compact complex manifold, relation between b_k and $\sum_{p+q=k} h^{p,q}$?

定理 6.2.18. (Hodge-Frolicher inequality) X compact complex manifold, then

$$b_k \leq \sum_{p+q=k} h^{p,q}$$

Spectral sequence: $(K^{p,q}, d = d' + d'')$ a double complex of modules.

$$K^{p,q} \xrightarrow{d'} K^{p+1,q} \quad K^{p,q} \xrightarrow{d''} K^{p,q+1}$$

with $d'^2 = 0, d''^2 = 0, d^2 = 0$.

Assume $K^{p,q} = 0$ if $p \leq 0$ or $q \leq 0$.

\rightsquigarrow total complex (K^\bullet, d) where

$$K^l := \bigoplus_{p+q=l} K^{p,q}$$

\exists a natural filtration

$$F_p K^l := \bigoplus_{l \geq i \geq p} K^{i,l-i}$$

F induces a filtration on $H^\bullet(K^\bullet)$.

$$F_p H^l(K^\bullet) = \text{Im}(H^l(F_p K^\bullet) \rightarrow H^l(K^\bullet)) = \frac{F_p Z^l}{F_p B^l}$$

where $Z^l = \ker d \cap K^l$ and $B^l = \text{Im } d \cap K^{l-1}$

Denote $G_p H^l(K^\bullet) = F_p H^l / F_{p+1} H^l$.

定理 6.2.19. *There exists a sequence*

$$\{E_r, d_r\}_{r \geq 0}$$

satisfying:

- (1) $E_r = \bigoplus_{p,q \geq 0} E_r^{p,q}$
- (2) $d_r : E_r^{p,q} \rightarrow E_r^{p+r,q+r-1}, d_r^2 = 0$.
- (3) $E_{r+1} = H^\bullet((E_r, d_r))$.

$$E_0^{p,q} = \frac{F_p K^{p+q}}{F_{p+1} K^{p+q}} = K^{p,q}$$

d_0 induced by d .

$$E_1^{p,q} = H^q((K^{p,\bullet}, d''))$$

d_1 induced by d .

查任何一本同调代数的书。

定义 6.2.20. We call the sequence E_r converges at E_{r_0} , if $E_{r+1} = E_r$ for any $r \geq r_0$, ($\iff d_r = 0$ for any $r \geq r_0$) then we denote $E_\infty = E_{r_0}$

In our setting, $E_\infty^{p,q} = G_p H^{p+q}(K^\bullet)$

Application: X compact complex manifold,

$$K^{p,q} = C^\infty(X, \bigwedge^{p,q}) \quad d = d' + d''$$

$$\rightsquigarrow E_0^{p,q} = K^{p,q}, E_1^{p,q} = H^{p,q}(X, \mathbb{C}).$$

推论 6.2.21.

$$E_\infty^{p,q} = G_p H^{p+q}(X, \mathbb{C})$$

定理 6.2.22. X is a compact complex manifold of complex dimension n , then

$$b_l = \dim_{\mathbb{C}} H^l(X, \mathbb{C}) = \sum_{p+q=l} \dim_{\mathbb{C}} E_\infty^{p,q} \leq \sum_{p+q=l} \dim_{\mathbb{C}} E_1^{p,q} = \sum_{p+q=l} h^{p,q}$$

with equality holds if and only if $d_1 = 0$ (i.e. $\{E_r\}$ converges at E_1 .)

定理 6.2.23. X compact Kahler $\Rightarrow \{E_r\}$ converges at E_1 ($\iff b_l = \sum_{p+q=l} h^{p,q}$)

Remark: algebraic proof by Deligne-Illusie 1987.

Relèvement module p^2 et décomposition du complexe de de Rham

remark: Assume X is bimeromorphic to a compact Kahler manifold, then we still have the convergence of $\{E_r\}$ (\iff Hodge decomposition)

(Deligne-Griffiths-Morgan)

Picard group $H^1(X, \mathcal{O}^*)$.

Recall:

$$\{\text{isomorphic class of holomorphic line bundle}\} \xrightarrow{1-1} H^1(X, \mathcal{O}^*)$$

Consider the sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{e^{2\pi\sqrt{-1}}} \mathcal{O}^* \rightarrow 0$$

$$\rightsquigarrow 0 \rightarrow H^0(X, \mathbb{Z}) \rightarrow H^0(X, \mathcal{O}) \rightarrow H^0(X, \mathcal{O}^*) \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}^*) \rightarrow \dots$$

Assume X is a compact complex manifold, then

$$H^0(X, \mathcal{O}) = \mathbb{C}$$

$$H^0(X, \mathcal{O}^*) = \mathbb{C}^*$$

$\Rightarrow H^0(X, \mathcal{O}) \rightarrow H^0(X, \mathcal{O}^*)$ is surjective,

$\Rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O})$ is injective.

So we have an exact sequence

$$0 \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

so we have an isomorphism

$$\ker\{c_1 : H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z})\} \cong H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$$

定义 6.2.24. (*Irregularity of X*)

$$q(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}) = h^{0,1}$$

if X is also complex Kahler, then $h^{0,1} = h^{1,0}$.

Assume X is compact Kahler:

引理 6.2.25. $H^1(X, \mathbb{Z})$ is also a lattice in $H^1(X, \mathcal{O})$ of

$$\text{rank}_{\mathbb{Z}} H^1(X, \mathbb{Z}) = 2q$$

$\Rightarrow H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$ is a compact torus of $\dim_{\mathbb{C}} = q$.

$$H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) := \ker\{c_1 : H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z})\}$$

is called **Jacobian variety** ($Jac(X)$) or **Picard variety** ($Pic^\circ(X)$)

Denote $NS(X)_{\mathbb{Z}} = \text{Im}(c_1 : H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z}))$ the Neron-Severi group of X ,

$$\rightsquigarrow 0 \rightarrow Pic^\circ(X) \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{c_1} NS(X, \mathbb{Z}) \rightarrow 0$$

proof of the lemma. $\mathbb{Z} \rightarrow \mathcal{O}$ can be decomposed : $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathcal{O}$. It induces a sequence

$$H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathbb{R}) \rightarrow H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathcal{O})$$

$H^1(X, \mathbb{R}) \rightarrow H^1(X, \mathcal{O})$ is an isomorphism.

Consider the diagram

then $H^1(X, \mathbb{R}) \rightarrow H^1(X, \mathcal{O})$ corresponds to

$$H_{DR}^1(X, \mathbb{R}) \hookrightarrow H_{DR}^1(X, \mathbb{C}) \twoheadrightarrow H^{0,1}(X, \mathbb{C})$$

$H^1(X, \mathbb{Z})$ is a lattice in $H^1(X, \mathbb{R})$ of $rank_{\mathbb{Z}} = 2q$

□

Albanese map, Albanese torus

X -compact Kahler \Rightarrow any holomorphic p -forms are d -closed.

(Exercise!!)

Special case: holo 1-forms is d -closed.

$$Alb(X) := H^0(X, \Omega^1)^* / \text{Im}(H_1(X, \mathbb{Z}))$$

where $H^1(X, \mathbb{Z})$ is mapped to $H^0(X, \Omega^1)^*$ in the following way:

$$[\gamma] \mapsto (\alpha \in H^0(X, \Omega^1) \mapsto \int_{\gamma} \alpha)$$

(Fact: $\int_{\gamma} \alpha$ depends only on the class on $[\gamma]$)

Then $Alb(X)$ is compact complex of $\dim_{\mathbb{C}} = q(X)$. More precisely, we have a map:

$$alb : X \rightarrow Alb(X)$$

Fix a base point $x_0 \in X$, then

$$alb(x) = \left(u \mapsto \int_{x_0}^x u \right) \mod \Lambda$$

where

$$\Lambda := \left\{ \left(\int_{\gamma} u_1, \dots, \int_{\gamma} u_q \right) \mid [\gamma] \in H_1(X, \mathbb{Z}) \right\}$$

$\{u_1, \dots, u_q\}$ is a basis of $H^0(X, \Omega^1)$. Then Λ is a lattice of $rank_{\mathbb{Z}} = 2q$.

The map

$$alb : X \rightarrow Alb(X)$$

is holomorphic.

6.3 正性与消灭定理

positivity and vanishing theorem

X-Kahler manifold, i.e. \exists Hermitian metric ω s.t. $d\omega = 0$, $d = d' + d''$, $d' = \partial$, $d'' = \bar{\partial}$.

$$\Delta_d = [d, d^*] = dd^* + d^*d$$

$$\Delta_{d'} = [d', d'^*]$$

$$\Delta_{d''} = [d'', d''^*]$$

$d \curvearrowright C^\infty(X, \bigwedge^{p,q})$.

Fact: ω is Kahler $\iff \Delta_{d'} = \Delta_{d''} = \frac{1}{2}\Delta_d$.

Let $\underline{\mathbb{C}} := X \times \mathbb{C}$ be the trivial line bundle, d can be regraded as the Chern connection on $\underline{\mathbb{C}}$.

(E, h) -Hermitian holomorphic vector bundle over (X, ω) , with Chern connection $D_E = D'_E + D''_E$. ($D''_E = \bar{\partial}$).

$$C^\infty(X, \bigwedge^{p,q} \otimes E)$$

has an inner product induced by ω, h . \rightsquigarrow adjoint operators $D_E^* = D'^*_E + D''^*_E$.

$\rightsquigarrow \Delta_E = [D_E, D_E^*] = D_E D_E^* + D_E^* D_E$, and Δ'_E, Δ''_E . (self adjoint, elliptic operators)

Question: relation between Δ'_E and Δ''_E ?

定理 6.3.1. (*Bochner-Kodaira-Nakano identity*)

$$\Delta''_E - \Delta'_E = [\sqrt{-1}\Theta_E, \Lambda]$$

where Θ_E is the Chern curvature of D_E .

Recall: $\Theta_E = D_E^2$, when D_E is Chern connectoin, we have

$$D_E'^2 = 0 \quad D_E''^2 = 0$$

i.e. $\Theta_E = [D'_E, D''_E]$.

Remark: E is flat (i.e. $D_E^2 = 0$) $\iff \Delta'_E = \Delta''_E$.

证明. based on following identities:

$$[D_E''^*, L] = \sqrt{-1}D'_E$$

$$[D_E'^*, L] = -\sqrt{-1}D''_E$$

$$[\Lambda, D'_E] = -\sqrt{-1}D_E'^*$$

$$[\Lambda, D_E''] = \sqrt{-1} D_E''^*$$

then (by super Jacobi identity):

$$\begin{aligned} \Delta_E'' = [D_E'', D_E''^*] &= -\sqrt{-1} [D_E'', [\Lambda, D_E']] = -\sqrt{-1} ([\Lambda, [D_E', D_E'']] + [D_E', [D_E'', \Lambda]]) \\ &= -\sqrt{-1} ([\Lambda, \Theta_E] + [D_E', \sqrt{-1} D_E'^*]) \end{aligned}$$

so,

$$\Delta_E'' - \Delta_E' = [\sqrt{-1} \Theta_E, \Lambda]$$

□

引理 6.3.2. (normal frame)

Let X be a complex manifold, then for any $x_0 \in X$, and any holomorphic chart (z_1, \dots, z_n) centered at x_0 , there exists a holomorphic frame $\{e_\lambda\}_{\lambda=1}^{r:=\text{rank} E}$ of E near x_0 such that

$$\langle e_\lambda(z), e_\mu(z) \rangle = \delta_{\lambda,\mu} - \sum_{1 \leq j, k \leq n} C_{jk\lambda\mu} z_j \bar{z}_k + O(|z|^3)$$

where $(C_{jk\lambda\mu})$ are the coefficients of the Chern curvature

$$\Theta_E(x_0) = \sum_{\substack{1 \leq j, k \leq n \\ 1 \leq \lambda, \mu \leq r}} C_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

need to verify: $\forall s \in C^\infty(X, \bigwedge^{p,q} \otimes E), x_0 \in X$,

$$[D_E''^*, L]s(x_0) = \sqrt{-1} D_E' s(x_0)$$

w.r.t the normal frame $(e_\lambda)_{\lambda=1}^r$ near x_0 , assume

$$s = \sum_{\lambda=1}^n \sigma_\lambda \otimes e_\lambda$$

then

$$D_E s(z) = \sum_{\lambda=1}^n d\sigma_\lambda \otimes e_\lambda + O(|z|)$$

$$D_E^* s(z) = \sum_{\lambda=1}^n d^* \sigma_\lambda \otimes e_\lambda + O(|z|)$$

$$D_E''^* = \sum_{\lambda=1}^r d''^* \sigma_\lambda \otimes e_\lambda + O(|z|)$$

$$\Rightarrow [D_E''^*, L]s = D_E''^* (\sum \omega \wedge \sigma_\lambda \otimes e_\lambda) - \omega \wedge \left(\sum_{\lambda=1}^r d''^* \sigma_\lambda \otimes e_\lambda + O(|z|) \right) = \sum_{\lambda=1}^r [d''^*, L] \sigma_\lambda \otimes e_\lambda + O(|z|)$$

Similarly,

$$D_E' s = \sum_{\lambda=1}^r d' \sigma_\lambda \otimes e_\lambda + O(|z|)$$

we have:

$$[d''^*, L] = \sqrt{-1} d'$$

(because ω is Kahler)

...

(E, h) hermitian holomorphic vector bundle over Kahler manifold (X, ω) . we have BKN identity

$$\Delta_E'' - \Delta_E' = [\sqrt{-1} \Theta_E, \Lambda]$$

Recall: L^2 -Hodge theory. X compact manifold, then

$$H^{p,q}(X, E) := \frac{\ker D_E''}{\text{Im } D_E''} \cong \ker \Delta_E''$$

(harmonic form)

Take $u \in C^\infty(X, \wedge^{(p,q)} \otimes E)$, applying BKN identity to u ,

$$\Delta_E'' u - \Delta_E' u = [\sqrt{-1} \Theta_E, \Lambda] u$$

note that

$$\begin{aligned} \langle \Delta_E' u, u \rangle &= \|D_E' u\|^2 + \|D_E'^* u\|^2 \geq 0 \\ \Rightarrow \|D_E'' u\|^2 + \|D_E''^* u\|^2 &\geq \langle [\sqrt{-1} \Theta_E, \Lambda] u, u \rangle \end{aligned}$$

i.e.

$$\|D_E'' u\|^2 + \|D_E''^* u\|^2 \geq \int_X \langle [\sqrt{-1} \Theta_E, \Lambda] u, u \rangle dVol$$

Observation: if $u \in \ker \Delta_E''$, and $[\sqrt{-1} \Theta_E, \Lambda]$ has "positivity", then $LHS = 0$. So, $H^{p,q}(X, E) = 0$.

定义 6.3.3. (Positivity)

We call $[\sqrt{-1} \Theta_E, \Lambda]$ is positive at $x_0 \in X$, if for any $0 \neq v \in (\wedge^{p,q} \otimes E)_{x_0}$, we have

$$\langle [\sqrt{-1} \Theta_E, \Lambda] v, v \rangle > 0$$

....positive on X , if ... at each point

定理 6.3.4. *If $[\sqrt{-1}\Theta_E, \Lambda]$ is positive on X , then*

$$H^{p,q}(X, E) = 0$$

Special case: E is a holomorphic line bundle, with Hermitian metric h ,

$$\Theta_E = -d'd'' \log h$$

$\Rightarrow \sqrt{-1}\Theta_E$ is a real d -closed $(1,1)$ -form on X .

locally,

$$\alpha = \sqrt{-1} \sum_{1 \leq i, j \leq n} a_{ij} dz_i \wedge d\bar{z}_j$$

α is real $\iff \alpha = \bar{\alpha}$, (i.e. locally (a_{ij}) is an hermitian matrix)

定义 6.3.5. *a real $(1,1)$ -form α is called positive, if $(a_{ij})_{ij}$ is positive definite.*

引理 6.3.6. *If $\sqrt{-1}\Theta_E$ is positive, then $\omega := \sqrt{-1}\Theta_E$ gives a Kahler metric on X .*

引理 6.3.7. *If $\omega = \sqrt{-1}\Theta_E > 0$, and Λ is the adjoint of $L = \omega \wedge$, then*

$$[\sqrt{-1}\Theta_E, \Lambda]$$

is positive on $\bigwedge^{p,q} \otimes E$ whenever $p + q \geq n + 1$.

引理 6.3.8. *Let α be a real $(1,1)$ -form, ω a Kahler metric, assume the eigenvalue of α at x_0 is $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$, then (in the coordinate chart (z_1, z_2, \dots, z_n) , and $u = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ} dz_I \wedge d\bar{z}_J$)*

$$[\alpha, L] = \sum_{I, J} \left(\sum_{i \in I} \alpha_i + \sum_{j \in J} \alpha_j - \sum_{k=1}^n \alpha_k \right) u_{IJ} dz_I \wedge d\bar{z}_J$$

推论 6.3.9. $\alpha = \omega$, then

$$[\omega, \Lambda]u = (p + q - n)u$$

推论 6.3.10. Take an orthonormal frame e of E , then for any $u = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ} dz_I \wedge d\bar{z}_J \otimes e$, we have

$$\langle [\sqrt{-1}\Theta_E, \Lambda]u, u \rangle = (p + q - n)|u|^2$$

定理 6.3.11. If $[\sqrt{-1}\Theta_E, \Lambda]$ is positive on X , then

$$H^{p,q}(X, E) = 0$$

定理 6.3.12. If E is a holomorphic line bundle with a smooth hermitian metric h s.t. $\sqrt{-1}\Theta_{(E,h)} \geq 0$, then $H^{p,q}(X, E) = 0$ whenever $p + q \geq n + 1$.

de Rham-Weil... $\cong H^q(X, \Omega^p \otimes E)$.

定义 6.3.13. (canonical bundle)

$$K_X = \det T^*X$$

determinate bundle of cotangent bundle, is called canonical bundle. $(\mathcal{O}(K_X) = \Omega_X^n)$

定义 6.3.14. X is called Fano, if $K_X^* = \det(TX)$ has a metric with positive curvature.

X is called Calabi-Yau, if K_X has a metric with vanishing curvature.

X is of general type, if K_X has a metric with positive curvature.

推论 6.3.15. (Kodaira vanishing theorem) E is a positive line bundle, then

$$H^q(X, K_X \otimes E) = 0$$

for any $q \geq 1$.

So, if X is Fano, ($\iff K_X^*$) positive, $K_X \otimes K_X^* = \underline{\mathbb{C}}$, $\Rightarrow H^1(X, \mathcal{O}) = 0, \Rightarrow H^1(X, \mathbb{R}) = 0$,

Recall: BKN-inequality.

holomorphic Hermitian vector bundle $(E, h) \rightarrow (X, \omega)$, ω is Kahler. For any $u \in C^\infty(X, \bigwedge^{p,q} \otimes E)$, we have

$$\|D''u\|^2 + \|D''^*u\|^2 \geq \int_X \langle [\sqrt{-1}\Theta_E, \Lambda_\omega]u, u \rangle dVol$$

Recall: If $[\sqrt{-1}\Theta_E, \Lambda_\omega]$ is positive on $C^\infty(X, \bigwedge^{p,q} \otimes E)$, then $H^{p,q}(X, E) = 0$.

定理 6.3.16. (Kodaira-Nakano vanishing theorem)

If E is a holomorphic line bundle with a smooth metric h s.t. $\sqrt{-1}\Theta_{(E,h)} > 0$, then $[\sqrt{-1}\Theta_E, \Lambda_\omega]$ is positive on $C^\infty(X, \bigwedge^{p,q} \otimes E)$ whenever $p + q \geq n + 1$.

$\Rightarrow H^{p,q}(X, E) = 0$ when $p + q \geq n + 1$.

(Last time)

Today:

定理 6.3.17. (Girbau vanishing theorem, 1976)

E is a holomorphic line bundle over compact Kahler manifold, with smooth metric h s.t. $\sqrt{-1}\Theta_{(E,h)} \geq 0$, and has at least $n - s + 1$ positive eigenvalues at every points of X , then

$$H^{p,q}(X, E) = 0$$

if $p + q \geq n + s$.

α : a **real** $(1,1)$ -form on X , locally $\alpha = \sqrt{-1} \sum \alpha_{ij} dz_i \wedge d\bar{z}_j$. then we have a matrix $M(\alpha) = (\alpha_{ij})_{n \times n}$, (α is real \Rightarrow) a hermite matrix.

we call α has at least k positive eigenvalues at x , if $M(\alpha)(x)$ has k positive eigenvalues. (Remark: It is well defined)

证明. Claim: there exists some Kahler metric ω s.t. $[\sqrt{-1}\Theta, \Lambda]$ is positive.

Fix a Kahler metric ω , for $p \in X$, choose a holomorphic chart (z_1, \dots, z_n) , s.t. $\omega(p) = \sqrt{-1} \sum dz_j \wedge d\bar{z}_j$ and $\sqrt{-1}\Theta_E(p) = \sqrt{-1} \sum_{j=1}^n \gamma_j dz_j \wedge d\bar{z}_j$. WLOG, $0 \leq \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n$, and for any $j \geq s$, $\gamma_j > 0$.

Consider

$$\omega_\varepsilon := \varepsilon\omega + \sqrt{-1}\Theta_E$$

for $\varepsilon > 0$, then ω_ε is a Kahler metric. $\omega_\varepsilon(p) = \sqrt{-1} \sum_j (\varepsilon + \gamma_j) dz_j \wedge d\bar{z}_j$.

\Rightarrow the eigenvalue of $\sqrt{-1}\Theta$ with respect to $\omega_\varepsilon(p)$ is given by

$$\gamma_{j,\varepsilon} = \frac{\gamma_j}{\varepsilon + \gamma_j} = \frac{1}{1 + \frac{\varepsilon}{\gamma_j}}$$

Claim: $[\sqrt{-1}\Theta, \Lambda_{\omega_\varepsilon}]$ is positive on $\wedge^{p,q} \otimes E$ when $p + q \geq n + s$, $0 < \varepsilon \ll 1$.

Take $u = \sum u_{IJ} dw_T \wedge d\bar{w}_J \otimes e$, then

$$\langle [\sqrt{-1}\Theta_E, \Lambda_{\omega_\varepsilon}], u \rangle = \sum_{\substack{|I|=p \\ |J|=q}} \left(\sum_{i \in I} \gamma_{i,\varepsilon} + \sum_{j \in J} \gamma_{j,\varepsilon} + \sum_{k=1}^n \gamma_{k,\varepsilon} \right) |u_{IJ}|^2 \geq (\gamma_{1,\varepsilon} + \dots + \gamma_{p,\varepsilon} - \gamma_{q+1,\varepsilon} - \dots - \gamma_{n,\varepsilon}) |u|^2$$

note that $\gamma_{j,\varepsilon} \geq 1 - \frac{\varepsilon}{\gamma_s}$ if $j \geq s$, $\gamma_{j,\varepsilon} \in [0, 1)$ for all j . it

$$\geq \left((q + s - 1) \left(1 - \frac{\varepsilon}{\gamma_s}\right) - (n - p) \right) |u|^2 > 0$$

if $p + q \geq n + s$ and $0 < \varepsilon \ll 1$. □

注记 6.3.18. (*Kawamata-Viewheg vanishing theorem*)

$E \rightarrow (X, \omega)$ is a holomorphic line bundle over a compact Kahler manifold.

Definition: E is called positive, ... (positive = "ample" in AG). numerically effective (nef) if for any $\varepsilon > 0$, there is a smooth metric h_ε s.t. $\sqrt{-1}\Theta_{h_\varepsilon} \geq -\varepsilon\omega$.

Theorem: If E is nef, and $\int_X c_1(E)^n > 0$, then $H^q(X, K_X \otimes E) = 0$ for $q \geq 1$.

Positivity concept of vector bundles (rank > 1)

$(E, h) \rightarrow (X, \omega)$ Hermitian vector bundle of rank r , over a complex manifold (may not Kahler).

Denote (e_1, \dots, e_r) a local orthonormal frame of E , (z_1, \dots, z_n) local holomorphic chart, Chern curvature of (E, h) :

$$\Theta_{(E,h)} = \sum_{\substack{1 \leq j, k \leq n \\ 1 \leq \lambda, \mu \leq r}} c_{ik\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

Fact: $\sqrt{-1}\Theta_E$ induces a Hermitian operator θ_E on $TX \otimes E$.

Let u, v be local sections of $TX \otimes E$,

$$u = \sum_{\substack{1 \leq j \leq n \\ 1 \leq \lambda \leq r}} u_{k\mu} \frac{\partial}{\partial z_k} \otimes e_\mu$$

$$\theta_E(u, v) := \sum_{\substack{1 \leq j, k \leq n \\ 1 \leq \lambda, \mu \leq r}} c_{jk\lambda\mu} u_{j\lambda} \overline{v_{k\mu}}$$

定义 6.3.19. We call E Nakano positive, if θ_E is positive. (i.e for any non-zero local section $u \in TX \otimes E$, $\theta_E(u, u) > 0$)

We call E Griffith positive, if for any $0 \neq \xi \in T_x X$, $s \in E_x, s \neq 0$,

$$\theta_E(\xi \otimes s, \xi \otimes s) > 0$$

注记 6.3.20. By definition, Nakano positivity \Rightarrow Griffith positivity.

If E is line bundle, Nakano positivity \iff Griffith positivity. (and \iff positivity of lines bundles)

定理 6.3.21. (Demailly-Skoda, 1979)

E is Griffith positive $\Rightarrow E \otimes \det E$ is Nakano positive.

证明. Omit. Non-trivial. □

Notation: $E >_{Nak} 0$ (E is Nakano positive). Similarly, $E >_{Giff} 0 \dots$

性质 6.3.22. (1) E is Griffith positive if and only if E^* is Griffith negative.

(2) Consider an exact sequence of holomorphic vector bundles:

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$$

then if E is Griffith positive, then Q is Griffith positive. If E is Griffith negative, then S is Griffith negative. If E is Nakano negative, then S is Nakano negative.

证明. Omit. Compute curvature... □

Remark: In general, E is Nakano positive, $\nRightarrow Q$ is Nakano positive.

定理 6.3.23. (*Nakano vanishing theorem*)

(X, ω) is compact Kahler of dimension n , (E, h) is a Nakano positive holomorphic Hermitian vector bundle, then

$$H^{n,q}(X, E) = 0 \quad \forall q \geq 1$$

证明. E is Nakano positive, check:

$$[\sqrt{-1}\Theta_E, \Lambda_\omega]$$

is positive on $\bigwedge^{n,q} E$ for $(q \geq 1)$ □

Ampleness

$E \rightarrow X$, E : holomorphic line bundle of rank r , X : complex manifold.

定义 6.3.24. (*Jet vector bundle*)

$$J^k E = \bigcup_{x \in X} (J^k E)_x$$

where

$$(J^k E)_x = \mathcal{O}_x(E) / \mathfrak{m}_x^{k+1} \mathcal{O}_x(E)$$

$\mathfrak{m}_x \subseteq \mathcal{O}_x$ be the maximal ideal of \mathcal{O}_x .

In local coordinate,

$$(J^k E)_x = \left\{ \sum_{\substack{1 \leq \lambda \leq r \\ |\alpha| \leq k}} C_{\lambda\alpha} (z - x)^\alpha e_\lambda(z) \right\}$$

性质 6.3.25. $J^k E$ is a holomorphic vector bundle of rank $= r \binom{n+k}{n}$.

证明. Exercise. □

定义 6.3.26. E is called very ample, if the following maps:

$$H^0(X, E) \rightarrow (J^1 E)_x$$

$$H^0(X, E) \rightarrow E_x \oplus E_y$$

are surjective, for all $x, y \in X$, $x \neq y$.

E is called ample, if $S^m E := \text{Sym}^m E$ is very ample for some $m \in \mathbb{N}$.

(ample: ” 足够多的全纯截面”)

定理 6.3.27. (*Kodaira*)

L -holomorphic line bundle, X is a compact complex manifold. Then L is positive if and only if L is ample.

$$\mathcal{F}_x \otimes_{\mathcal{A}_x} \mathfrak{g}^\vee$$

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