复几何

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本课程参考以下教材：

1. Demailly: Complex analytic and differential geometry.
2. Huybrechts: Complex geometry: an introduction.
3. Morrow, Kodaira: Complex manifolds.
4. Grauert, Remmert: Coherent analytic sheaves.
5. Hormander: An introduction to complex analysis in several variables.
6. Griffiths, Harris: Principles of algebraic geometry.

在五道口也要红专并进、理实交融呀 ∼

目录

第 1 章　多复变函数

1.1　多元全纯函数

首先快速回顾单复变函数的知识。我们通常用 Ω 来表示 $\mathbb{C}$ 的开子集，$z = x + iy$ 为 $\mathbb{C}$ 的坐标。对于 $z \in \mathbb{C}$ 以及实数 $R > 0$，我们令

$$\mathbb{D}(z, R) := \{w \in \mathbb{C} \mid |w - z| < R\}$$

为以 z 为圆心 R 为半径的开圆盘。

此外，我们有如下常用记号：

$$\begin{cases} \mathrm{d}z := \mathrm{d}x + i\mathrm{d}y \\ \mathrm{d}\bar{z} := \mathrm{d}x - i\mathrm{d}y \end{cases} \qquad \begin{cases} \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{cases}$$

对于函数 $f : \Omega \to \mathbb{C}$，称 f 是**全纯**（holomorphic）的，若在 Ω 中成立

$$\bar{\partial} f := \frac{\partial f}{\partial \bar{z}} \mathrm{d}\bar{z} = 0$$

我们知道，f 是全纯的当且仅当 f 在 Ω 处处能够局部地展开为收敛幂级数。

对于 $\mathbb{C}$ 中的紧致集 K，称函数 $f : K \to \mathbb{C}$ 是全纯的，如果存在 K 的开邻域 $\Omega \supseteq K$，使得 f 可延拓为 Ω 上的全纯函数。

单复变函数论中有如下重要结果：

定理 1.1.1. （柯西积分公式）设 $\mathbb{D} \subseteq \mathbb{C}$ 为 $\mathbb{C}$ 中的开圆盘，$f : \mathbb{D} \to \mathbb{C}$ 为 $\mathbb{D}$ 上的全纯函数，且在 $\partial\mathbb{D}$ 连续，则对于任意 $w \in \mathbb{D}$，成立

$$f(w) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(z)}{z - w} \mathrm{d}z$$

此定理能推导出单变量全纯函数理论的“almost everything”. 这里不再赘述。

我们开始考虑多变量全纯函数。

定义 1.1.2. 设 $\Omega \subseteq \mathbb{C}^n$ 为 $\mathbb{C}^n$ 的开子集，函数 $f : \Omega \to \mathbb{C}$ 称为（多变量）**全纯函数**，如果满足以下条件：

（*1*）f 是连续函数；

（*2*）对任意 $1 \le j \le n$，以及任意固定的 $z_1, ..., z_{j-1}; z_{j+1}, ..., z_n \in \mathbb{C}$，关于 z_j 的单变量函数

$$z_j \mapsto f(z_1, ..., z_{j-1}; z_j; z_{j+1}, ..., z_n)$$

是（单变量）全纯函数。

事实上，如果该定义中的（2）成立，那么能推出（1）成立，也就是说此定义中的（1）可以去掉。其证明比较复杂，我们承认之。

记号 1.1.3. 对于 $\mathbb{C}^n$ 的开子集 Ω，我们记

$$\mathcal{O}(\Omega) := \{f : \Omega \to \mathbb{C} | f \text{是 } \Omega \text{ 上的全纯函数}\}$$

容易知道 $\mathcal{O}(\Omega)$ 有显然的 $\mathbb{C}$-代数结构。

本节将说明，多变量全纯函数具有一些与单变量全纯函数类似的性质。

记号 1.1.4. 对于 $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$ 以及 $R = (R_1, R_2, ..., R_n) \in \mathbb{R}^n$，并且 $R_j > 0$ $(\forall 1 \le j \le n)$，则我们记

$$\mathbb{D}(z, R) := \mathbb{D}(z_1, R_1) \times \mathbb{D}(z_2, R_2) \times \cdots \times \mathbb{D}(z_n, R_n)$$

称为以 z 为中心，R 为半径的**多圆柱**（*polydisk*）。

对于多圆柱 $\mathbb{D}(z, R)$，我们记

$$\Gamma(z, R) := \partial\mathbb{D}(z_1, R_1) \times \partial\mathbb{D}(z_2, R_2) \times \cdots \times \partial\mathbb{D}(z_n, R_n)$$

称为 $\mathbb{D}(z, R)$ 的**特征边界**（*distinguished boundary*）。

特别注意特征边界 $\Gamma(z, R)$ 并不等于该多圆柱的边界 $\partial\mathbb{D}(z, R)$.

定理 1.1.5. （多变量全纯函数的柯西积分公式）

设 $f:\overline{\mathbb{D}(z,R)}\to\mathbb{C}$ 为全纯函数，则对任意的 $w\in\mathbb{D}(z,R)$，成立

$$f(w)=\frac{1}{(2\pi i)^n}\int_{\Gamma(z,R)}\frac{f(\xi)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_n}{(\xi_1-w_1)(\xi_2-w_2)\cdots(\xi_n-w_n)}$$

证明. 由多变量全纯函数的定义，反复使用单变量全纯函数的柯西积分公式即可。这是容易的。 □

与单复变函数完全类似，我们也有泰勒展开：

推论 1.1.6. （多元全纯函数的泰勒展开公式）

对于 $f\in\mathcal{O}(\Omega)$，其中 $\Omega\subseteq\mathbb{C}^n$ 为开子集，则对于任何多圆柱 $\mathbb{D}(z_0,R)$，如果 $\overline{\mathbb{D}(z_0,R)}\subseteq\Omega$，则对于任意 $w\in\mathbb{D}(z_0,R)$，成立

$$f(w)=\sum_{\alpha\in\mathbb{N}^n}a_\alpha(w-z_0)^\alpha$$

其中

$$a_\alpha=\frac{1}{(2\pi i)^n}\int_{\Gamma(z_0,R)}\frac{f(z)}{(z-z_0)^{\alpha+1}}\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_n=\frac{f^{(\alpha)}(z_0)}{\alpha!}$$

注意这里的 α 为多重指标，即 $\alpha=(\alpha_1,...,\alpha_n)$，其中每个 α_i 都为非负整数。我们记

$$\begin{aligned}
z^\alpha &:= z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n}\\
\alpha! &:= \alpha_1!\alpha_2!\cdots\alpha_n!\\
f^{(\alpha)} &:= (\partial_{z_1})^{\alpha_1}(\partial_{z_2})^{\alpha_2}\cdots(\partial_{z_n})^{\alpha_n}f\\
\alpha+1 &:= (\alpha_1+1,\alpha_2+1,...,\alpha_n+1)
\end{aligned}$$

其中 $z=(z_1,...,z_n)\in\mathbb{C}^n$，$f$ 为 n 元全纯函数。

证明. 与单复变函数的情形完全类似，可由柯西积分公式得到。 □

定理 1.1.7. （柯西不等式）对于 $\mathbb{C}^n$ 的开子集 Ω，若 $f\in\mathcal{O}(\Omega)$，多圆柱 $\overline{\mathbb{D}(z_0,R)}\subseteq\Omega$，则对任意多重指标 $\alpha\in\mathbb{N}^n$，成立

$$\left|f^{(\alpha)}(z_0)\right|\leq\frac{\alpha!}{R^\alpha}\sup_{z\in\Gamma(z_0,R)}|f(z)|$$

证明. 与单复变函数的情形完全类似。利用多元泰勒展开（推论1.1.6）即可。 □

推论 1.1.8. 设 $\Omega \subseteq \mathbb{C}^n$ 为**连通**开集，$f \in \mathcal{O}(\Omega)$ 满足 $\forall 1 \leq k \leq n$，$\frac{\partial f}{\partial z_k}$ 在 Ω 上恒为 0，则 f 在 Ω 上为常值函数。

推论 1.1.9. （刘维尔定理）设 $f \in \mathcal{O}(\mathbb{C}^n)$，并且满足

$$|f(z)| \leq A(1+|z|)^B$$

其中 A, B 为正实数，那么 f 必为次数不超过 B 的多项式函数。

这些性质于单变量全纯函数雷同，证明也是类似的。

推论 1.1.10. （*Montel* 定理）

设 Ω 为 $\mathbb{C}^n$ 的开子集，则 $\mathcal{O}(\Omega)$ 中的任何局部一致有界的全纯函数列都存在一致收敛的子列。

证明. 仍类似于单复变全纯函数的情形。使用柯西积分公式，再配合 Arzela-Ascoli 定理即可。从略。 □

第 2 章　层与层上同调

2.1　层的上同调

Today:

Sheaf cohomology

X a topological space, $\mathcal{F}$- sheaf (of abelian groups).

定义 2.1.1. *(resolution)*

(1)a resolution of $\mathcal{F}$ is an exact sequence

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F} \xrightarrow{d^0} \mathcal{F} \xrightarrow{d^1} \to \cdots$$

定义 2.1.2. *A sheaf $\mathcal{A}$ is called injective, if if for any injective morphism $j : \mathcal{A} \to \mathcal{B}$ and for any morphism $\varphi : \mathcal{A} \to \mathcal{S}$, there exists an extension $\psi : \mathcal{B} \to \mathcal{S}$,such that*

定理 2.1.3. *the category of sheaves of abelian sheaves have enough injective objects, i.e. any $\mathcal{F}$ can be embedded in some injective sheaf.*

定义 2.1.4. *Consider an injective resolution of $\mathcal{F}$, i.e. an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \mathcal{I}^2 \to \cdots$$

where every $\mathcal{I}^k (k \geq 0)$ is injective.

$\rightsquigarrow$*induces a sequence*

$$0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{I}^0) \xrightarrow{\mathrm{d}} \Gamma(X,\mathcal{I}^1) \xrightarrow{\mathrm{d}} \Gamma(X,\mathcal{I}^2) \to \cdots$$

Then

$$H^q(X,\mathcal{F}) := H^q(\Gamma(X,\mathcal{I}^\bullet))$$

then, $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$.

定义 2.1.5. *A sheaf $\mathcal{S}$ is called a flabby (flasque ,in France) ,if for any open set $\Omega \subseteq X$, the morphism*

$$\mathcal{S}(X) \to \mathcal{S}(\Omega)$$

is surjective.

定义 2.1.6.

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \xrightarrow{d^0} \to \mathcal{F}^1$$

is an exact sequence is called a flabby resolution, if any $\mathcal{F}^k$ is flabby.

定义 2.1.7.

$$H^q(X,\mathcal{F}) := \text{...by flabby resolution...}$$

证明. Homological Algebra...omit. □

the two definitions of Sheaf Cohomology are isomorphic.

Godement's construction

$$God(\mathcal{F})(U) := \{f : U \to \bigcup_{x\in U} \mathcal{F}_x | f(y) \in \mathcal{F}_y, \forall y \in U\} := \prod_{x\in U} \mathcal{F}_x$$

$God(\mathcal{F})$ is a sheaf, and it is flabby. and there is a canonical morphism $\mathcal{F}(U) \to God(F)(U)$ by $x \mapsto (x \mapsto s_x)$ is injective.

$$\mathcal{F}^0 := God(\mathcal{F})$$

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{F}^0 \twoheadrightarrow \mathrm{coker}(j) = \mathcal{F}^0/\mathcal{F}$$

and consider

$$\mathcal{F}^1 := God(\mathrm{coker}(j))$$

......then construct by induction... this is a flabby resolution of $\mathcal{F}$.

定义 2.1.8. *(resolution by fine sheaves)*

$\mathcal{A}$ is a sheaf of ring, X is a paracompact topological space, $\mathcal{A}$ is called a fine sheaf, if for any open covering

$$X = \bigcup_{\alpha} V_\alpha \quad , \mathcal{V} := \{V_\alpha\}$$

there exists a partition of unit subordinate to $\mathcal{V}$, (i.e. $\exists f_\alpha \in \mathcal{A}(V_\alpha), supp(\alpha) := \overline{\{x \in V_\alpha | f_{\alpha,x} \neq 0\}} \subseteq V_\alpha$, and $\sum_\alpha f_\alpha = 1$(the sum is locally finite))

例子 2.1.9. *X is a differential manifold, $\mathcal{C}^\infty$ is the sheaf of smooth functions, then $\mathcal{C}^\infty$ is a fine sheaf.*

定理 2.1.10. *$\mathcal{S}$ is a sheaf of $\mathcal{A}$-modules, $\mathcal{A}$ is a fine sheaf. then for any $q \geq 1$,*

$$H^q(X, \mathcal{S}) = 0$$

证明. Consider a flabby(or injective) resolution

$$0 \to \mathcal{S} \xrightarrow{j} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \mathcal{I}^2 \cdots$$

where any $\mathcal{I}^k (k \geq 0)$ is a sheaf of $\mathcal{A}$-modules.

by definition,

$$H^q(X, m\mathcal{S}) := \frac{\ker \mathrm{d} : \Gamma(\mathcal{I}^q) \to \Gamma(\mathcal{I}^{q+1})}{\Im \mathrm{d} : \Gamma(\mathcal{I}^{q-1}) \to \Gamma(\mathcal{I}^q)}$$

Let $\alpha \in \ker\{\mathrm{d} : \Gamma(\mathcal{I}^q) \to \Gamma(\mathcal{I}^{q+1})\}$ by the exactness of resolution, $\exists$ an open covering $\mathcal{U} = (U_i)_i$, s.t. $\alpha|_{U_i} = \mathrm{d}\beta_i$ where $\beta_i \in \mathcal{T}^{q-1}(U_i)$. Let $(\beta_i)_i$ be the partition of unit w.r.t. $\mathcal{U}$. consider

$$\beta := \sum_i f_i \beta_i$$

(well defined). Then $\mathrm{d}\beta = \alpha$.... □

2.2 Cech 上同调

Cech cohomology

X- a topological space, $\mathcal{F}$- a sheaf of abelian group.

$$\mathcal{U} = (U_\alpha)_{\alpha \in I}$$

is an open covering.

notation:$U_{\alpha_1,\ldots,\alpha_q} := \bigcap_{i=1}^{q} U_{\alpha_i}$.

Cech q-chain w.r.t $\mathcal{U}$:

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{(\alpha_1,\ldots,\alpha_q) \in \mathcal{I}^{q+1}} \mathcal{F}(U_{\alpha_1,\ldots,\alpha_q})$$

$$c \in C^q(\mathcal{U}, \mathcal{F})$$

means that we have a family of sections $C_{\alpha_1,\ldots,\alpha_q} \in \mathcal{F}(U_{\alpha_1,\ldots,\alpha_q})$ with the relation

$$C_{\alpha_0,\ldots,\alpha_j,\ldots,\alpha_i,\ldots} = -C_{\ldots}$$

(C)ech differential:

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$$

$$\delta^q(c)_{\alpha_0,\ldots,\alpha_{q+1}} := \sum_{0 \leq k \leq q+1} (-1)^k c_{\ldots\hat{\alpha_k}\ldots}|_{U_{\alpha_0,\ldots,\alpha_{q+1}}}$$

性质 2.2.1.

$$\delta^q \circ \delta^q = 0$$

so, we have Cech cohomology

$$H^q(\mathcal{U}, \mathcal{F}) := \ker \delta^q \big/ \operatorname{Im} \delta^{q-1}$$

example:

$$C^0(\mathcal{U}, \mathcal{F}) := \prod_{\alpha \in I} \mathcal{F}(U_\alpha)$$

$$c = (c_\alpha)_{\alpha \in I} \in C^0(\mathcal{U}, \mathcal{F})$$

$$\delta^0 c = 0 \iff (\delta^0 c)_{\alpha_0 \alpha_1} := (c_{\alpha_1} - c_{\alpha_0})|_{U_{\alpha_0 \alpha_1}} = 0$$

so, $c_{\alpha_0} = c_{\alpha_1}$ on $U_{\alpha_0 \alpha_1}$.

$\rightsquigarrow H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

例子 2.2.2. *(1) consider* $X = \triangle \setminus \{0\}$*, where* $\triangle = \{(z_1, z_2) | |z_1| < 1, |z_2| < 1\}$*. Consider the covering*

$$\mathcal{U} = U_1 \cup U_2$$

where

$$U_1 := \{(z_1, z_2) \in \triangle | z_1 \neq 0\} = \mathbb{D}^* \times \mathbb{D}$$

$$U_2 := \{(z_1, z_2) \in \triangle | z_2 \neq 0\} = \mathbb{D} \times \mathbb{D}^*$$

then

$$U_1 \cap U_2 = \mathbb{D}^* \times \mathbb{D}^*$$

consider $H^0(X, \mathcal{O}) = \mathcal{O}(X) \cong \mathcal{O}(\triangle) = \{f : \triangle \to \mathbb{C} holomorphic\}$.

$$H^1(\mathcal{U}, \mathcal{O}) = \ker \delta^1 / \operatorname{Im} \delta^0$$

$$\delta^1 : C^1(\mathcal{U}, \mathcal{O}) \to C^2(\mathcal{U}, \mathcal{O}) \subseteq \prod_{\alpha_0, \alpha_1, \alpha_2} \mathcal{O}(U_{\alpha_0, \alpha_1, \alpha_2}) = 0$$

$$\ker \delta^1 = C^1(\mathcal{U}, \mathcal{O}) = \{c = c(\alpha_0, \alpha_1) | c_{\alpha_0, \alpha_1} \in \mathcal{O}(U_{\alpha_0 \alpha_1})\} = \{c \in \mathcal{O}(U_1 \cap U_2)\} = \{c = \sum_{m,n \in \mathbb{Z}} a_{mn} z_1^m z_2^n convergent\}$$

$$\delta^0 : C^0(\mathcal{U}, \mathcal{O}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{O})$$

$$(\delta^0 c)_{12} = (c_2 - c_1)|_{U_{12}}$$

where $c_2 \in \mathcal{O}(U_2)$ *and* $c_1 \in \mathcal{O}(U_1)$*. note that*

$$\mathcal{O}(U_1) = \{c(z_1, z_2) = \sum_{m \in \mathbb{Z}, n \geq 0} a_{mn} z_1^m z_2^n convergent\}$$

$$\mathcal{O}(U_2) = \{c(z_1, z_2) = \sum_{n \in \mathbb{Z}, m \geq 0} a_{mn} z_1^m z_2^n convergent\}$$

So, $H^1(\mathcal{U}, \mathcal{O}) = \{c(z_1, z_2) = \sum_{m,n<0} a_{mn} z_1^m z_2^n\}$

例子 2.2.3. *(complex projective space)*

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

$$(z_0, ..., z_n) \sim \lambda(z_0, ..., z_n)$$

for some $\lambda \in \mathbb{C}^*$.

$$\mathbb{C}P^n = \{[z_0, ..., z_n] | not\ all\ z_k = 0, z_i \in \mathbb{C}\} = \bigcup_{0 \leq p \leq n} V_k$$

where

$$V_k = \{[z_0, ..., z_n] | z_k \neq 0\} \cong \{(\frac{z_0}{z_k}, ..., 1, ..., \frac{z_n}{z_k}) | z_i \in \mathbb{C}, i \neq k, z_k \neq 0\} \cong \mathbb{C}^n$$

this is a holo chart.

$$\mathbb{C}P^1 = V_0 \cup V_1, \mathcal{V} = \{V_0, \mathcal{V}_1\}$$

HW: compute $H^q(\mathcal{V}, \mathcal{O})$.

Answer:

$$H^0 \cong \mathbb{C}, H^1 \cong 0$$

Correction:

$\mathcal{A}$: Sheaf of rings (with unit)

X: paracompact topological space,

定义 2.2.4. *$\mathcal{A}$ is called fine, if for any open covering $\mathcal{U} = (V_\alpha)_{\alpha\in\mathcal{I}}$,there exist $s_\alpha \in \mathcal{A}(X)$ such that such that $supp(s_\alpha) \subseteq V_\alpha$,*

$$\sum_\alpha s_\alpha = 1$$

(this is a locally finite sum)

注记 2.2.5. *we call $\mathcal{A}$ is a **soft sheaf**, if for any closed set $K \subseteq X$, the morphism*

$$\mathcal{A}(X) \to \mathcal{A}(K)$$

is surjective. where $\mathcal{A}(K) := \Gamma(K, \mathcal{A}|_K)$

fact: $\mathcal{A}$ is fine if and only if $\mathcal{H}om(\mathcal{A}, \mathcal{A})$ is soft. (omit)

Recall:

Cech cohomology: X topological space, $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{I}}$,

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0<\ldots<\alpha_q} \mathcal{F}(\alpha_1, \ldots, \alpha_q)$$

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$$

fact: $H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Today:

定义 2.2.6. *Let $\mathcal{V} = (V_\beta)_{\beta\in J}$ be another open covering, then $\mathcal{V}$ is called a refinement of $\mathcal{U}$, if there exists a map*

$$\rho : \mathcal{J} \to \mathcal{I}$$

such that

$$V_\beta \subseteq U_{\rho(\beta)}$$

性质 2.2.7. *Let $\mathcal{V}$ be a refinement of $\mathcal{U}$, then ρ induces a map*

$$\rho^q : C^q(\mathcal{U},\mathcal{F}) \to C^q(\mathcal{V},\mathcal{F})$$

$$(\rho^q C)_{\beta_0,\dots,\beta_q} \mapsto C_{\rho(\beta_0),\dots,\rho(\beta_q)}|_{V_{\beta_0,\dots,\beta_q}}$$

ρ is a morphism of complexes.

so, ρ induces a map

$$H^q(\rho) : H^q(\mathcal{U},\mathcal{F}) \to H^q(\mathcal{V},\mathcal{F})$$

Let $\tilde{\rho} : \mathcal{J} \to \mathcal{I}$ be another refinement of $\mathcal{U}$

(induces $H^q(\tilde{\rho}) : H^q(\mathcal{U},\mathcal{F}) \to H^q(\mathcal{V},\mathcal{F})$) then $\rho, \tilde{\rho}$ are homotopic (chain homotopy$\rightsquigarrow H^q(\rho) = H^q(\tilde{(\rho)})$)

so, if $\rho : \mathcal{J} \to \mathcal{I}$ is refinement, then

$$H^q(\rho)$$

is independent of the refinement.

定义 2.2.8.

$$\check{H}^q(X,\mathcal{F}) := \varinjlim_{\mathcal{U}} H^q(\mathcal{U},\mathcal{F})$$

i.e. $a \in H^q(\mathcal{U},\mathcal{F}) \sim\in H^q(\mathcal{V},\mathcal{F})$ iff $\exists$ a refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$ such that a, b have the same image in $H^q(\mathcal{W},\mathcal{F})$

注记 2.2.9.

$$\check{H}^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$$

Exercise: For $q = 1$, if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then

$$H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F})$$

is injective.

so ,for any open cover $\mathcal{U}$,

$$H^1(\mathcal{U},\mathcal{F}) \to \check{H}^1(X,\mathcal{F})$$

is injective.

Homological Algebra recall: let $(K^\bullet, \mathrm{d}_k), (L^\bullet, \mathrm{d}_l)$ and $(M^\bullet, \mathrm{d}_M)$, if we have a short exact sequence

$$0 \to K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} M^\bullet \to 0$$

then it induces a long exact sequence :

$$\cdots \to H^q(K^\bullet) \to H^q(L^\bullet) \to H^q(M^\bullet) \to H^{q+1}(K^\bullet) \to \cdots$$

analogy of Cech cohomology: X is a topological space, $\mathcal{U}$ is an open covering of X. $\mathcal{A}$ and $\mathcal{B}$ sheaves on X, Let

$$\varphi : \mathcal{A} \to \mathcal{B}$$

be a morphism, then it induces

$$\varphi^\bullet : C^\bullet(\mathcal{U}, \mathcal{A}) \to C^\bullet(\mathcal{U}, \mathcal{B})$$

Let

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

be an exact sequence of sheaves, then we have: for any open set Ω,

$$0 \to \mathcal{A}(\Omega) \to \mathcal{B}(\Omega) \to \mathcal{C}(\Omega)$$

left exact.

Example: consider

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{exp} \to 0$$

is exact on $bbC^\times := \mathbb{C} \setminus \{0\}$

but we have :

$$0 \to \mathcal{A}(\Omega) \xrightarrow{\psi} \mathcal{B}(\Omega) \to \operatorname{Im} \psi(\Omega) \to 0$$

is exact.

First we have the following exact sequence

$$C^q(\mathcal{U}, \mathcal{A}) \to C^q(\mathcal{U}, \mathcal{B}) \to C^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \to 0$$

where $C^q_{\mathcal{B}}$ is the image of ...

then we get an exact sequence

$$0 \to (C^\bullet(\mathcal{U}, \mathcal{A}), \delta) \to (C^\bullet(\mathcal{U}, \mathcal{B}), \delta) \to (C^\bullet_{\mathcal{B}}(\mathcal{U}, \mathcal{C}), \delta) \to 0$$

it induces a long exact sequence

$$\cdots \to H^q(\mathcal{U}, \mathcal{A}) \to H^q(\mathcal{U}, \mathcal{B}) \to H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \to H^{q+1}(\mathcal{U}, \mathcal{A}) \to \cdots$$

定理 2.2.10. *If X is paracompact,*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a sheaf exact sequence. Then there is a long exact sequence

$$\cdots \to \check{H}^q(X, \mathcal{A}) \to \check{H}^q(X, \mathcal{B}) \to \check{H}^q(X, \mathcal{C}) \to \check{H}^{q+1}(X, \mathcal{Z}) \to \cdots$$

证明. Key lemma: need to prove

$$\lim_{\vec{\mathcal{U}}} H^q(\mathcal{U}, \mathcal{C}) = \lim_{\vec{\mathcal{U}}} H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C})$$

if X is paracompact.

Omit. □

if

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

exact,

recall:(cohomology by resolutions)

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$

flabby resolution. then it induces

$$0 \to \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \cdots$$

then define the sheaf cohomology...

we have a long exact sequence

$$\cdots \to H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B}) \to H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A}) \to \cdots$$

it is homological algebra...

定理 2.2.11. *(Leray's acyclic theorem) Let $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{I}}$ be an open covering of X, ($\mathcal{F}$ is a sheaf on X), if satisfying*

$$H^k(U_{\alpha_0, \ldots, \alpha_q}) = 0$$

for any $k \geq 1$,then

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{(H)}^q(X, \mathcal{F})$$

and if X is paracompact ,we also have

$$H^q(\mathcal{U}, \mathcal{F}) \cong \check{(H)}^q(X, \mathcal{F}) \cong H^q(X, \mathcal{F})$$

(this $\mathcal{U}$ is called acyclic covering)

de Rham- Weil theorem

定义 2.2.12. *$\mathcal{F}$ is a sheaf on X, Ω is an open set of X, then $\mathcal{F}$ is called* ***acyclic sheaf*** *if*

$$H^q(\Omega, \mathcal{F}) = 0$$

for any $q \geq 1$.

定理 2.2.13. *Let*

$$0 \to \mathcal{F} \to (L^\bullet, \mathrm{d})$$

be an acyclic resolution of $\mathcal{F}$ (i.e. $L^q$is acyclic on X) then

$$H^q(X, \mathcal{F}) \cong H^q(\Gamma(X, L^\bullet), \mathrm{d})$$

for any $q \geq 0$.

(先看例子)

例子 2.2.14. *Let X be a differential manifold, $\mathcal{E}^p$:sheaf of smooth p-forms, then we have a resolution (de Rham complex)*

$$0 \to \mathbb{R} \hookrightarrow \mathcal{E}^0 \xrightarrow{\mathrm{d}} \mathcal{E}^1 \xrightarrow{\mathrm{d}} \mathcal{E}^2 \xrightarrow{\mathrm{d}} \mathcal{E}^3 \to \cdots$$

where d *differential operators. (Why it is a resolution? because of Poincare lemma...locally solvable..)*

Note that

$$\mathcal{E}^0 = \mathcal{C}^\infty$$

$\mathcal{E}^p$ is a sheaf of C^∞-modules..

then we have

$$H^q(X, \mathcal{E}^p) = 0$$

for all $q \geq 1$

and then

$$H^q(X, \mathbb{R}) \cong \frac{\ker(\mathrm{d} : \Gamma(X, \mathcal{E}^q) \to \Gamma(X, \mathcal{E}^{q+1}))}{\mathrm{Im}(\mathrm{d} : \Gamma(X, \mathcal{E}^{q-1}) \to \Gamma(X, \mathcal{E}^q))} = H^q_{DR}(X, \mathcal{R})$$

例子 2.2.15. *Let X be a complex manifold, $\mathcal{E}^{p,q}$ sheaf of smooth (p,q) forms, Ω^p is the sheaf of holomorphic p-forms (i.e. $(p,0)$-form φ with $\bar{\partial}\varphi = 0$).*

Then we have resolution

$$0 \to \Omega^p \xrightarrow{j} \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \to \cdots$$

(Why it is a resolution? because of the Dolbeault lemma),remain to Exercise...

$$H^q(X,\Omega^p) \cong H^{p,q}_{\bar\partial}(X,\mathbb{C})$$

Today：de Rham-Weil Isomorphism Thm

定理 2.2.16. *Let X be a topological space, $\mathcal{F}$ be a sheaf of abelian groups on X,*

$$0 \to \mathcal{F} \to (\mathcal{L}^\bullet, \mathrm{d})$$

be an acyclic resolution, i.e.

$$H^k(X,\mathcal{L}^q) = 0$$

for all $k \geq 1$ and $q \geq 0$. Then,

$$H^q(X,\mathcal{F}) \cong H^q((\Gamma(\mathcal{L}^\bullet), \mathrm{d}))$$

证明. Since

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{\mathrm{d}^0} \mathcal{L}^1 \xrightarrow{\mathrm{d}^1} \mathcal{L}^2 \to \cdots$$

be an exact sequence, denote

$$\mathcal{Z}^q := \ker \mathrm{d}^q$$

then we have short exact sequences

$$0 \to \mathcal{Z}^q \to \mathcal{L}^q \to \mathcal{Z}^{q+1} \to 0$$

for any q. They induce long exact sequence of cohomology groups:

$$\cdots \to H^k(X,\mathcal{Z}^q) \to H^k(X,\mathcal{L}^q) \to H^k(X,\mathcal{Z}^{q+1}) \xrightarrow{\partial} H^{k+1}(X,\mathcal{L}^q) \to H^{q+1}(X,\mathcal{L}^q) \to \cdots$$

For any $k \geq 1$, since $\mathcal{L}^q$ are acyclic on X,

$$H^k(X,\mathcal{Z}^{q+1}) \cong H^{k+1}(X,\mathcal{Z}^q)$$

and for $k = 0$, we have

$$0 \to H^0(X,\mathcal{Z}^q) \to H^0(X,\mathcal{L}^q) \to H^0(X,\mathcal{Z}^{q+1}) \to H^1(X,\mathcal{Z}^q) \to H^1(X,\mathcal{L}^q) = 0 \to \cdots$$

so,

$$H^1(X,\mathcal{Z}^q) \cong H^0(X,\mathcal{Z}^{q+1}) \big/ \operatorname{Im} \mathrm{d}^q \cong H^{q+1}((\Gamma(\mathcal{L}^\bullet), \mathrm{d}))$$

$$H^{q+1}(\Gamma(\mathcal{L}^\bullet)) \cong H^1(X,\mathcal{Z}^q) \cong H^2(X,\mathcal{Z}^{q-1}) \cong \cdots H^{q+1}(X,\mathcal{Z}^0) = H^{q+1}(X,\mathcal{F})$$

□

$$0 \to \mathbb{R} \to \mathcal{E}^0 \xrightarrow{\mathrm{d}} \mathcal{E}^1 \xrightarrow{\mathrm{d}} \mathcal{E}^2 \to \cdots$$

(de Rham resolution) then we have

$$H^k(X, \mathcal{R}) \cong H^k_{DR}(X; \mathcal{R})$$

(if X is compact ,then by Hodge theory, it also isomorphic to $\ker(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})$)

Another example:X is a complex manifold, then

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2} \to \cdots$$

then

$$H^q(X, \Omega^p) \cong H^{p,q}_{\bar{\partial}}(X, \mathbb{C})$$

(RHS= Dolbeault cohomology)

X be a smooth manifold, we define

$$C_q(X, \mathbb{Z}) := \text{the free abelian group generated by continuous map}$$

$$\phi : \triangle_q := \{(t_1, ..., t_{q+1}) \in [0,1]^{q+1} | \sum_{i=1}^{n} t_i = 1\}$$

and we define (for $\phi \in C_q(X, \mathbb{Z})$)

$$\partial\phi := \sum_{i=1}^{q+1} (-1)^q \phi|_{\triangle_{q,i}}$$

$$\triangle_{q,i} := \{t \in \triangle_q | t_i = 0\}$$

we define

$$(C^\bullet_{sing}, \partial)$$

be the dual complex of $(C^{sing}_\bullet), \partial$.

(These are all Basic Algebraic Topology)

For any open $U \subseteq X$, we have

$$U \to C^q_{sing}(U, \mathbb{Z})$$

we get a sheaf

$$\mathcal{C}^q_{sing}$$

FACT: $(\mathcal{C}^\bullet_{sing}, \partial)$ is a flabby resolution of $\mathbb{Z}$. (check!)So,

$$H^q_{sing}(X, \mathbb{Z}) = H^q(\Gamma(\mathcal{C}^\bullet_{sing}), \partial) \cong H^q(X, \mathbb{Z})$$

第 3 章　Hermite 向量丛

3.1　联络与曲率

Recall: X is a smooth manifold, E is a vector bundle of rank r, if

(1)$\pi : E \to X$ is smooth map,

(2)for any $x \in X$, $E_x := \pi^{-1}(x)$ is a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of dimension r.

(3)there an open covering $\mathcal{U} = (\mathcal{U}_\alpha)_{\alpha \in I}$ and trivializations

$$\theta_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

and for any intersection $U_\alpha \cap U_\beta$, we have

注记 3.1.1.

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$$

(cocycle condition)

Special Case: line bundle rank E=1.

then $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, \mathbb{K}^*) = \mathcal{E}^*(U_{\alpha\beta})$ invertible smooth function on $U_{\alpha\beta}$. then, Cech cohomology,

$$(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = 1$$

so,

$$(g_{\alpha,\beta}) \in \mathcal{Z}^1(\mathcal{U}, \mathcal{E}^*) \twoheadrightarrow H^1(\mathcal{U}, \mathcal{E}^*) \hookrightarrow \check{H}^1(X, \mathcal{E}^*)$$

we get a map

$$\{\text{line bundles}\} \to \check{H}^1(X, \mathcal{E}^*)$$

actually, we have

$$\{\text{isomorphic classes of line bundles}\} \longleftrightarrow H^1(X, \mathcal{E}^*)$$

1-1 correspondence.

Now, X be a complex manifold, a complex vector bundle E is called homomorphic, if ... the transition matrix $g_{\alpha\beta}$ is holomorphic...

Holomorphic line bundles :

$$g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$$

$\mathcal{O}^*$:sheaf of invertible holomorphic functions...

FACT: there is a map

$$\{\text{holomorphic line bundle}\} \to \check{H}^1(X, \mathcal{O}^*)$$

例子 3.1.2. *trivial vector bundle* $X \times \mathbb{K}^r$

例子 3.1.3. *Tangent bundle* TX. *(transition matrix* $g_{\alpha\beta}$ *are given by Jacobi matrix..)*

定义 3.1.4. *(Local frame of vector bundles)*

$$\theta_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{K}^r$$

be a trivialization, we define

$$e_\lambda(x) := \theta_\alpha^{-1}(x, \begin{pmatrix} 0 \\ \cdots \\ 1(\leftarrow ith) \\ \cdots \\ 0 \end{pmatrix})$$

then, $\{e_1, ..., e_r\}$ *be a local smooth section* $s \in \Gamma(U_\alpha, E)$ *can be written as*

$$s(x) = \sum \sigma_\lambda(x)$$

where $\sigma_\lambda \in C^\infty(U_\alpha, \mathbb{K})$.

(Connection)

记号 3.1.5. *For* X *be a smooth manifold,* E *is a vector bundle(real or complex), denote*

$$C_p^k(\Omega, E) := C^k(\Omega, \bigwedge^p T^*M \otimes E)$$

is the space of k*-differential* p*-forms with values in* E.

Locally, consider a trivialization of E,

$$\theta_\alpha E|_{U_\alpha} \cong U_\alpha \times \mathbb{K}^r$$

($\rightsquigarrow$ *frame* $(e_1, ...e_r)$*)*

$$s \in \sum \varphi_\lambda(x) \otimes e_\lambda(x)$$

where φ_λ *is a* p*-form.*

定义 3.1.6. *a (linear) connection on E is a linear differential operator of order* 1 *acting on* $C^\infty_\bullet(X, E)$*:*

$$D : C^\infty_p(X, E) \to C^\infty_{p+1}(X, E)$$

$$D(f \wedge x) := \mathrm{d}f \wedge s + (-1)^p f \wedge Ds$$

where $f \in C^\infty(X, \bigwedge^p T^*M)$, $s \in C^\infty(X, E)$.

Locally, consider a local trivialization

$$\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

with a frame $\{e_1, ..., e_r\}$. any section $t \in C^\infty_p(\Omega, E)$ can be written as

$$t = \sum_{1 \le \lambda \le r} \sigma_\lambda \otimes e_\lambda$$

$$Ds = \sum_{\lambda=1}^{r} \mathrm{d}\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge De_\lambda$$

where

$$De_\lambda \in C^\infty_1(\Omega, E)$$

can be written as

$$De_\lambda = \sum_{\mu=1}^{r} a_{\mu\lambda} \otimes e_\mu$$

where "$a_{\mu\lambda}$" is called the coefficients of D with respect to frame $\{e_1, ..., e_r\}$.

so,

$$D(t) = \sum_{\lambda,\mu} \mathrm{d}\sigma_\lambda \wedge e_\lambda + (-1)^p \sigma_\lambda \wedge a_{\mu\lambda} \wedge e_\mu = \sum_\mu \sum_\lambda \left(\mathrm{d}\sigma_\mu + a_{\mu\lambda} \wedge \sigma_\lambda\right)$$

$$Dt = \mathrm{d}\sigma + A \wedge \sigma$$

where $A = (a_{\mu\lambda})$.

RMK: connection always exists!

Recall: for any (connected) smooth manifold, $E \to X$ is a smooth vector bundle,

Connection:

$$D : C^\infty_p(X, E) \to C^\infty_{p+1}(X, E)$$

where $C^\infty_p(X, E) := C^\infty(X, \wedge^p T^*M \otimes E)$

$$D(f \wedge s) = \mathrm{d}f \wedge s + (-1)^{\deg f} f \wedge Ds$$

Essentially,

$$D : C^\infty(X, E) \to C_1^\infty(X, E)$$

Locally, consider a trivialization $\theta : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$, and a local frame $(e_1, ..., e_r)$ where $e_k(x) = \theta^{-1}(x, \begin{pmatrix} 0 \\ \vdots \\ 1(k^{th}) \\ \vdots \\ 0 \end{pmatrix})$.

Let $s \in C^\infty(\Omega, E)$, i.e.

$$s = \sum_{i=1}^{r} \sigma_i e_i$$

where σ_i are smooth functions.

$$Ds = \mathrm{d}\sigma + A \wedge \sigma$$

where

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \end{pmatrix} \quad A = a_{ij}$$

consider another trivialization

$$\tilde{\theta} : E|_\Omega \xrightarrow{\sim} \Omega \times \mathbb{K}^r$$

$\rightsquigarrow$ a local frame $(\tilde{e_1}, ..., \tilde{e_r})$. Then there exists a invertible linear transform s.t.

$$\tilde{e_k} = g_k^m e_m$$

assume

$$De_k = a_k^l e_l \qquad D\tilde{e_k} = \tilde{a}_k^l \tilde{e}_l$$

we have

$$\begin{aligned} \mathrm{d}g_k^n e_n + g_k^m a_m^n e_n &= \tilde{a}_k^l g_l^n e_n \\ \rightsquigarrow \quad \tilde{a}_k^l g_l^n (g^{-1})_n^p &= \mathrm{d}g_k^n (g^{-1})_n^p + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \quad \tilde{a}_l^p &= \mathrm{d}g_k^n (g^{-1})_n^p) + g_k^m a_m^n (g^{-1})_n^p \\ \rightsquigarrow \quad \tilde{A} &= \mathrm{d}g \cdot g^{-1} + g \cdot A \cdot g^{-1} \end{aligned}$$

Curvature

$$H_D := D^2$$

locally,

$$D^2 s = D(\mathrm{d}\sigma + A \wedge \sigma) = \mathrm{d}(\mathrm{d}\sigma + A \wedge \sigma) + A \wedge (\mathrm{d}\sigma + A \wedge \sigma)$$

$$= \mathrm{d}A \wedge \sigma - A \wedge \mathrm{d}\sigma + A \wedge \mathrm{d}\sigma + A \wedge A \wedge \sigma = (\mathrm{d}A + A \wedge A) \wedge \sigma$$

so we have

$$H = \mathrm{d}A + A \wedge A$$

Similarly to $\tilde{A}, A$ we have

Exercise:

$$\tilde{H} = gHg^{-1}$$

曲率在不同平凡化下的表达式。where

$$\tilde{e} = ge$$

$\rightsquigarrow$ H can be considered as a section of $C_2^\infty(X, \mathrm{Hom}(E, E))$. because

$$\tilde{H}\tilde{e} = gHg^{-1}\tilde{e} = gHe$$

independent of the choice of local frames.

3.2 向量丛的构造

定义 3.2.1. *(dual of vector bundles) $E \to X$, and $g_{\alpha\beta}$:transition matrix of E, the dual is given by $(g_{\alpha\beta})^{-1}$.* （用转移函数来定义向量丛）

定义 3.2.2. *direct sum of two vector bundles $(E, F) \to E \oplus F$. locally,*

$$(g_{\alpha,\beta}) \oplus (h_{\alpha\beta})$$

direct sum of transition matrices.

定义 3.2.3. *tensor product of two vector bundles.*

locally, tensor product of two transition matrices.

fact: let D_E be a connection on E,then it induces a connection D_{E^*}. Let u be a local section of E^*, s local section of E, then we define

$$\mathrm{d}\langle u, s\rangle = \langle D_{E^*}u, s\rangle + \langle u, D_E s\rangle$$

Exercise:

$$H(D_{E^*}) = -H(D_E)^T$$

and for two vector bundles E, F, connections D_E, D_F, then

$$D_{E\oplus F} := D_E \oplus D_F$$

$$H(E \oplus F) = H_E \oplus H_F$$

as for tensor product, we define $D_{E\otimes F}$ as follows:

$$D_{E\otimes F}(s \otimes t) = D_E s \otimes t + s \otimes D_F t$$

check the curvature

$$H_{E\otimes F} = H_E \otimes id_F + id_E \otimes H_F$$

注记 3.2.4. *we can also consider wedge product of vector bundles. Consider vector bundles $E_1, ..., E_k$, with connections $D_{E_1}, ..., D_{E_k}$, let $s_i \in C^\infty_{p_i}(X, E^i)$ then*

$$D_{E_1\wedge,...,\wedge E_k}(s_1 \wedge ... \wedge s_k) = \sum_{i=1}^{k} (-1)^{p_1+...+p_{i-1}} s_1 \wedge ... \wedge D_{E_i} s_i \wedge ... \wedge s_k$$

Let E be a vector bundle of rank r, then $\bigwedge^r E$ is a line bundle, with transition matrix by $\det(g_{\alpha\beta})$. this bundle is denoted by $\det E$.(Det-bundle)

Let $s_1, ..., s_r$ be local sections of E, then we have

$$D_{\det E}(s_1 \wedge \cdots \wedge s_r) = tr(H_E) s_1 \wedge \cdots \wedge s_r$$

3.3 陈省身示性类

chern classes (defined by curvature).

Let $E \to X$ be a smooth complex vector bundle of rank r, where X be a complex manifold.

(Chern-Weil theory)

V be a complex vector space, $f : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{C}$ be a symmetric multi-linear form of degree k.

$\rightsquigarrow f(v) := f(v, v, ..., v)$ is a homogeneous polynomial of degree k.

定义 3.3.1. *assume G is a group (left) acting on V, s.t.*

$$f(g(v_1), ..., g(v_k)) = f(v_1, ..., v_k)$$

for any $g \in G, v_i \in V$, then we say f is G-invariant.

Special case: $G = GL(r, \mathbb{C})$ and $V = LieG = \mathfrak{gl}r, \mathbb{C}$ be the Lie algebra of G. the action is

$$(g, M) \mapsto gMg^{-1}$$

Consider

$$\det(I + \frac{i}{2\pi}tm) = I + tf_1(M) + t^2 f_2(M) + \cdots t^r f_r(M)$$

$\rightsquigarrow \forall 1 \leq k \leq r$, f_k is G-invariant.

Let $E \to X$ complex vector bundle on a complex manifold, let D_E be a connection, curvature $H_E \in C_2^\infty(X, \mathrm{Hom}(E,E))$. Let $f \in GL(r, \mathbb{C})$- invariant "k-form",then

(1)Let H_α, H_β be the curvature forms of E in different trivialization, then $f(H_\alpha) = f(H_\beta)$, so we get a globally defined $2k$-form.

assume $H_\alpha = gH_\beta g^{-1}$, then

$$f(H_\alpha) = f(gH_\beta g^{-1}) = f(H_\beta)$$

(2) we also have

$$\mathrm{d}f(H) = 0$$

locally , $H = H_\alpha = \mathrm{d}a_\alpha + A_\alpha \wedge A_\alpha$,then

$$\mathrm{d}f(H) = \mathrm{d}f(H_\alpha, H_\alpha, ..., H_\alpha) = \sum_{i=1}^{k} f(H_\alpha, ..., \underbrace{\mathrm{d}H_\alpha,}_{i} ..., \alpha)$$

$$= \sum_{i=1}^{k} f(H_\alpha, ..., \mathrm{d}A_\alpha \wedge A_\alpha - A_\alpha \wedge \mathrm{d}A_\alpha, ..., H_\alpha)$$

Fact:(in Riemannian geometry) For any $x \in X$, we always can find a local frame s.t. $A_\alpha(x) = 0$. so, choose this frame,

$$\mathrm{d}f(H) = 0$$

So, $[f(H)] \in H^{2k}(X, \mathbb{C})$

(3) Claim : the class $[f(H)]$ is independent of the choice of the connections D_E.

Let D_0, D_1 be two connections, consider

$$D_t = (1-t)D_0 + tD_1$$

$t \in [0,1]$, curvature H_t

Fact: $\alpha := A_1 - A_0$ is globally defined, and in $C_1^\infty(X, \mathrm{Hom}(E,E))$.

Fact:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(H_t) = k\mathrm{d}f(\alpha, H_t, H_t, ..., H_t)$$

So,

$$f(H_1) - f(H_0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(H_t)\mathrm{d}t = \mathrm{d}\int_0^1 f(\alpha, H_t, H_t, ..., H_t)\mathrm{d}t$$

So,

$$[f(H_1)] - [f(H_0)]$$

定义 3.3.2. *the k-th Chern class of E*

$$c_k(E) := [f_k(\Theta_E)] \in H^{2k}(X, \mathbb{C})$$

Recall: Chern Class

X complex manifold, $E \to X$ is a smooth complex vector bundle of rank r. D is a connection, curvature $\Theta(D) \in C_2^\infty(X, \mathrm{Hom}(E, E))$.

linear algebra:

$$\det(I + \frac{i}{2\pi} tM) = I + tf_1(M) + t^2 f_2(M) + \cdots + t^r f_r(M)$$

Chern class $\{f_k(\Theta)\} \in H_{DR}^{2k}(X, \mathbb{C})$ is independent of choice of connection.

Today:

Special case: E is a complex line bundle. Let D_0 be a connection on E, locally $D_0 e = A_0 e$, A_0 is 1-form. curvature

$$\Theta(D_0) = D_0^2 = \mathrm{d}A_0 + A_0 \wedge A_0 = \mathrm{d}A_0$$

so, curvature is d-exact, so $\mathrm{d}\Theta(D_0) = 0$.

$$\det(I + \frac{i}{2\pi} tM) = I + \frac{i}{2\pi} tM$$

so, the first Chern class of line bundle is

$$c_1(E) = \{\frac{i}{2\pi}\Theta(D_0)\}$$

Let D_1 be another connection, locally $D_1 e = A_1 e$, so $\Theta(D_1) = \mathrm{d}A_1$.so,

$$\Theta(D_1) - \Theta(D_0) = \mathrm{d}(A_1 - A_0)$$

where

$$A_1 - A_0 \in C_1^\infty(X, \mathrm{Hom}(E, E))$$

(when E is line bundle ,$\mathrm{Hom}(E, E) \cong E^* \otimes E$ is trivial bundle)

so, $A_1 - A_0$ is a globally defined smooth function on X. So,

$$\{\Theta(D_1)\} = \{\Theta(D_0)\} \in H^2(X, \mathbb{C})$$

independent of the choice of connection.

3.4 Hermite 向量丛

定义 3.4.1. *a complex vector bundle $E \to X$ of rank r is called a Hermitian vector bundle, if we have an inner product on E, i.e. locally, consider a local frame $\{e_1, ..., e_r\}$, we have*

$$\{e_i(x), e_j(x)\} = h_{ij}(x)$$

s.t. $(h_{ij}(x))$ is a positive definite Hermitian matrix depending smoothly on x.

注记 3.4.2. *For any complex vector bundle, Hermitian structure always exists.*

证明与黎曼几何类似。（黎曼度量的存在性）

定义 3.4.3. *(Hermitian connection)*

A connection D on E is called Hermitian, if

$$\mathrm{d}\{e_i, e_j\} = \{De_i, e_j\} + \{e_i, De_j\}$$

More generally, let $t \in C_p^\infty(X, E)$, $s \in C_q^\infty(X, Y)$,

$$\mathrm{d}\{s, t\} = \{\mathrm{d}t, s\} + (-1)^p\{t, Ds\}$$

性质 3.4.4. *D is a Hermitian connection ,then the curvature*

$$\Theta(D)^* = -\Theta(D)$$

(where $(-)^*$ is conjugate transpose of matrix)

it means that, $i\Theta(D) \in C_2^\infty(X, \mathrm{Herm}(E, E))$

证明.

$$0 = \mathrm{d}^2\{e_i, e_j\} = \mathrm{d}\{De_i, e_j\} + \mathrm{d}\{e_i, De_j\}$$

$$= \{D^2e_i, e_j\} - \{De_i, De_j\} + \{De_i, De_j\} + \{e_i, D^2e_j\} = \{(\Theta + \Theta^*)e_i, e_j\}$$

□

注记 **3.4.5.** *E is a Hermitian line bundle, D is a Hermitian connection, then $i\Theta(D)$ is a real 2-form , $c_1(E) \in H^2(X, \mathbb{R})$.*

(Chern connection)

定义 **3.4.6.** *Let X be a complex manifold. D' is called a connection of type $(1,0)$ on E, if for any section $s \in C^\infty_{p,q}(X, E)$, we have $D's \in C^\infty_{p+1,q}(X, E)$.*

A connection D'' is called a connection of type $(0,1)$, if ... $D''s \in C^\infty_{p,q+1}(X, E)$.

注记 **3.4.7.** *Let $E \to X$ be a vector bundle. Let D be a connection on E, locally*

$$Ds \xrightarrow{\sim} \mathrm{d}\sigma + A \wedge \sigma$$

$$\mathrm{d}\sigma = \partial\sigma + \bar{\partial}\sigma$$

so,let A' be the $(1,0)$-part of A,...,

$$Ds = \partial\sigma + A' \wedge \sigma + (\bar{\partial}\sigma + A'' \wedge \sigma) =: D's + D''s$$

性质 **3.4.8.** *E:Hermitian vector bundle, D is a Hermitian connection, locally, take a C^∞-frame $e_1, ..., e_r$ which is orthonomal (i.e. $\{e_i(x), e_j(x)\} = \delta_{ij}$), then the connection coefficient $A = A' + A''$ satisfies*

$$(A')^* = -A''$$

($\iff \overline{(iA)} = iA$)

证明. because

$$0 = \mathrm{d}e_i, e_j = \{De_i, e_j\} + \{e_i, De_j\} = \{a_i^k e_k, e_j\} + \{e_i, a_j^l e_l\} = a_i^j + \overline{a_j^i}$$

so, $A^* = -A$. □

推论 **3.4.9.** *$E \to X$ is a Hermitian vector bundle, D_0'' is a connection of type $(0,1)$ on E. Then exists a unique Hermitian connection D such that $D'' = D_0''$.*

证明. Let $A'' = A_0''$ and $A' = -(A_0'')^* \rightsquigarrow A = A' + A''$, and D is given by A. □

Let $E \to X$ is a holomorphic Hermitian vector bundle, observe that $\overline{\partial}$ defines a connection of type $(0,1)$ on E(check!)

assume E is a holomorphic line bundle, take a section $s \in C_p^\infty(X,E)$, i.e. we have a family of p-forms (s_α) such that $s_\alpha = g_{\alpha\beta}s_\beta$ where $g_{\alpha,\beta}$ is the holomorphic transition matrix.

$$\overline{\partial}s \xrightarrow{\sim} \overline{\partial}s_\beta$$

then

$$\overline{\partial}s_\alpha = g_{\alpha,\beta}\overline{\partial}s_\beta$$

(so, $\overline{\partial}$ is a connection of $(0,1)$)

this connection is called the canonical connection of type $(0,1)$.

定义 3.4.10. *Let $E \to X$ holomorphic Hermitian vector bundle, the connection D on E is called Chern connection if*

$$D'' = \overline{\partial}$$

Curvature of Chern connection

$E \to X$ is holomorphic Hermite vector bundle , D is the Chern connection, Locally let $\{e_1, \ldots, e_r\}$ be a holomorphic frame, and two local sections

$$s, t \in C^\infty(\Omega, E)$$

where

$$s = \sum_{i=1}^{r} \sigma_i e_i$$

$$t = \sum_{i=1}^{r} t_i e_i$$

Since D is Hermitian ,

$$\mathrm{d}\{s,t\} = \mathrm{d}((\sigma_1, \ldots, \sigma_r)H\begin{pmatrix} t_1 \\ \vdots \\ t_r \end{pmatrix}) = (\mathrm{d}\sigma)^T Ht + \sigma^T(\mathrm{d}H)t + \sigma^T H\mathrm{d}(t)$$

so, we have

$$\{Ds,t\} + \{s,Dt\} = (\mathrm{d}\sigma + \overline{H}^{-1}\partial\overline{H} \wedge \sigma)^T \wedge H\bar{t} + \sigma^T \wedge H\overline{(\mathrm{d}t + \overline{H}^{-1}\partial\overline{H} \wedge t)}$$

so ,

$$Ds = \mathrm{d}\sigma + \overline{H}^{-1}\partial\overline{H} \wedge \sigma$$

$$D's = \partial\sigma + \overline{H}^{-1}\partial\overline{H}\wedge\sigma = \overline{H}^{-1}\partial(\overline{H}\sigma)$$
$$D''s = \overline{\partial}\sigma$$

so,

$$(D')^2 s = \overline{H}^{-1}\partial(\overline{H}(\overline{H}^{-1}\partial(\overline{H}\sigma))) = \cdots = 0$$

$$(D'')^2 s = \cdots = 0$$

So we have

$$\Theta(D) = (D' + D'')^2 = D'D'' + D''D'$$

Locally ,

$$\Theta s = D'D''s + D''D's = \overline{H}^{-1}\partial(\overline{H}\overline{\partial}\sigma) + \overline{\partial}(\overline{H}^{-1}\overline{\partial}(\overline{H}\sigma)) = \cdots = \overline{H}^{-1}\partial\overline{H}\wedge\overline{\partial}\sigma + \overline{\partial}(\overline{H}^{-1})\sigma$$
$$= \overline{\partial}(\overline{H}^{-1}\partial\overline{H})\sigma$$

So, Chern curvature

$$\Theta_D = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})$$

Last time: $E \to X$ is a holomorphic vector bundle with a Hermitian metric H. Then there is a unique connection D_Es.t. ... called Chern connection.

Curvature of Chern Connection:

$$\Theta(D_E) = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})$$

so,

$$i\Theta(D_E) \in C^\infty_{1,1}(X, \mathrm{Hom}(E,E))$$

例子 3.4.11. *(Special case: E is a holomorphic line bundle)*

locally, let e be ha holomorphic frame, $\langle e, e\rangle = h$ is the metric. then,

$$\Theta = \overline{\partial}(h^{-1}\partial h) = \overline{\partial}\partial\log h$$

so,

$$i\Theta(E) = -i\partial\overline{\partial}\log h$$

if $h = e^{-2\varphi}$ where φ is a smooth function, then

$$i\Theta(E) = 2i\partial\overline{\partial}\varphi = 2\sqrt{-1}\sum_{k,l}\frac{\partial^2\varphi}{\partial z_k\partial\overline{z_l}}\mathrm{d}z_k\wedge\mathrm{d}\overline{z_l}$$

Question: let s be a local holomorphic section of E,

$$-i\partial\overline{\partial}\log|s|_h^2 = ?$$

(Hint: $\frac{i}{\pi}\partial\overline{\partial}\log z = ?$ 单复变，按分布意义下求导. 等于狄拉克测度 2333333) 可能是期末题目？

例子 3.4.12. *$\mathcal{O}(-1)$ on $\mathbb{C}P^n$, tautological line bundle. (Recall: $\mathbb{C}P^n$ is a compact complex manifold with holomorphic charts*

$$\Omega_j := \{[z_0; z_1; ...; z_n] | z_j \neq 0\} \to \left(\frac{z_0}{z_j}, \cdots, \hat{1}, \cdots \frac{z_n}{z_j}\right) \in \mathbb{C}^n$$

)

Let V be a complex vector space, $\dim_{\mathbb{C}} V = n+1$. Denote the projective space by

$$\mathbb{P}(V) = (V \setminus \{0\})/\mathbb{C}^*$$

Let $\underline{V} := \mathbb{P}(V) \times V$ be the trivial vector bundle, define

$$\mathcal{O}(-1) := \{([x], \xi) | \xi \in \mathbb{C} \cdot x\}$$

性质 3.4.13. *$\mathcal{O}(-1)$ is a holomorphic line bundle on $\mathbb{P}(V)$.*

证明. $\mathcal{O}(-1)|_{\Omega_j}$ has a non-vanishing holomorphic section $\mathcal{E}_j$ defined by

$$\mathcal{E}_j([x]) = \frac{x}{x_j}$$

for $0 \leq j \leq n$. □

Assume V has a Hermitian inner product, then $\mathcal{O}(-1)$ has an Hermitian structure induced from V.

Let $e_0, ..., e_n$ be an orthonormal basis of V, then $\mathcal{O}(-1)|_{\Omega_0}$ has a non-vanishing holomorphic section:

$$\mathcal{E}_0(z_1, ..., z_n) = e_0 + z_1 e_1 + ... + z_n e_n$$

where

$$\Omega_0 = \{[1; z_1; ...; z_n] | z_j \in \mathbb{C}\} \cong \mathbb{C}^n$$

then,

$$|\mathcal{E}_0|_h^2 = 1 + |z_1|^2 + ... + |z_n|^2$$

so the Chern curvature of $\mathcal{O}(-1)$ on Ω_0 is given by

$$\Theta = \bar{\partial}\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2)$$

Denote $\mathcal{O}(1) := \mathcal{O}(-1)^*$, then

$$\Theta(\mathcal{O}(1)) = -\bar{\partial}\partial \log(1 + |z_1|^2 + ... + |z_n|^2)$$

on Ω_0.

$$i\Theta(\mathcal{O}(1)) = i\partial\overline{\partial}\log(1+|z_0|^2+...+|z_n|^2) = \sqrt{-1}\sum_{1\le k,l\le n} c_{k,l}\mathrm{d}z_k\wedge \mathrm{d}\overline{z_l}$$

Exercise: (c_{kl}) is a positive definite Hermitian matrix.

"Fubini-Study metric" on $\mathbb{P}(V).\mathcal{O}(1)$ is "hyperplane line bundle of $\mathbb{P}(V)$".

Exercise: calculate

$$\int_{\mathbb{P}(V)}\left(\frac{i}{2\pi}\Theta(\mathcal{O}(1))\right)^{\wedge n} = ?$$

(Hint: $\mathbb{P}(V)\setminus\Omega_0$ is a zero-measure set)

$E\to X$: holomorphic line bundle, D_E is a Chern connection.

$$c_1(E) = \{\frac{i}{2\pi}\Theta(D_E)\}\in H^2_{DR}(X,\mathbb{R})$$

Exercise: 60% 的概率出现于期末试题

Consider the sequence

$$0\to\mathbb{Z}\to\mathcal{O}\xrightarrow{e^{2\pi i *}}\mathcal{O}^*\to 0$$

it induces a long exact sequence

$$\cdots\to H^1(X,\mathcal{O})\to H^1(X,\mathcal{O}^*)\xrightarrow{\delta} H^2(X,\mathbb{Z})\to H^2(X,\mathcal{O})\to\cdots$$

prove: Consider E as an element of $H^1(X,\mathcal{O}^*)$, then the image of $\delta(E)$ in $H^2(X,\mathbb{R})\cong H^2_{DR}(X,\mathbb{R})$ is $c_1(E)$.

Exercise: E is a holomorphic line bundle, denote $\theta := \frac{i}{2\pi}\Theta(D_E)$ real $(1,1)-form$, where D_E is Chern connection with a metric h. Prove: for any smooth function $f\in C^\infty(X,\mathbb{R})$, there exists a Hermitian metric h_f s.t.

$$\frac{i}{2\pi}\Theta_{E,h_f} = \theta + i\partial\overline{\partial}f$$

第 4 章　L^2 Hodge 理论

4.1　向量丛上的微分算子

Differential operators on vector bundles.

Let X is a (connected) smooth manifold of ($\mathbb{R}$-)dimension n. E, F : $\mathbb{K}$-vector bundle of rank r, r' respectively.

定义 4.1.1. *a linear differential operator of degree k from E to F is a $\mathbb{K}$-linear map*

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

$$u \mapsto Pu$$

locally given by

$$Pu(x) = \sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u(x)$$

where $a_\alpha(x) = (a_{afa,\lambda\mu}(x))$ be a $r' \times r$ matrix.

$$u(x) = (u_1(x), ..., u_r(x))^T$$

Let $t \in \mathbb{K}$, $f \in C^\infty(M, \mathbb{K})$, $u \in C^\infty(M, E)$, then

$$e^{-tf(x)} P(e^{tf(x)} u(x)) = t^k \sigma_P(x, \mathrm{d}f(x)) u(x) + \text{terms } c_j(x)^{t_j} \quad (j < k)$$

定义 4.1.2.

$$\sigma_P : T^*M \to \mathrm{Hom}(E, F)$$

is called the principal symbol of P, which is a polynomial on T^*M.

locally,

$$\sigma_P(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

$(\xi^\alpha := \xi_1^{\alpha_1}...\xi_n^{\alpha_n})$

例子 4.1.3. *Consider* $\mathrm{d} : C^\infty(M,\mathbb{K}) \to C^\infty(M, T^*M)$. *then*

$$\mathrm{d}u = \sum_{j=1}^{n} \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \frac{\partial u}{\partial x^i}$$

i.e.

$$\sigma_d(x,\xi) = \sum_{j=1}^{n} \begin{pmatrix} 0 \\ \vdots \\ 1(j^{th}) \\ \vdots \\ 0 \end{pmatrix} \xi_j$$

定义 4.1.4. *P is called elliptic, if* $\forall x \in M, \xi \in T_x^*M \setminus \{0\}$,

$$\sigma_P(x,\xi) \in \mathrm{Hom}(E_x, E_x)$$

is injective.

For example, d is elliptic.

L^2-inner product

Let M be an oriented C^∞-manifold with a smooth volume form, locally

$$\mathrm{d}V(x) = \gamma(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

$\gamma(x) > 0$. Assume E has a Euclidean(or Hermitian) structure...

Let $u, v \in C^\infty(M,E)$,define

$$\langle\langle u,v\rangle\rangle := \int_M \langle u,v\rangle \mathrm{d}V(x)$$

define $L^2(M,E)$:= space of sections with measurable coefficients with are L^2 w.r.t $\langle\langle,\rangle\rangle$.

定义 4.1.5. *Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be a differential operator, E, F have Euclidean (or Hermitian) structure, then there exists unique differential operator*

$$P^* : C^\infty(M,F) \to C^\infty(M,E)$$

s.t.

$$\langle\langle Pu, v\rangle\rangle = \langle\langle u, P^*v\rangle\rangle$$

for all u, v s.t. $Supp u \cap Supp v \subset\subset M$(relative compact...)

P^* is called the formal adjoint of P.

证明. Existence: Assume that $SuppU, Suppv \subset\subset$ some coordinate chart Ω with coordinates $(x_1, ..., x_n)$, then

$$\ll Pv, u \gg = \int_\Omega \sum_{\alpha,\lambda,\mu} a_{\alpha,\lambda\mu}(x) D^\alpha u_\mu(x) \overline{v_\lambda(x)} \gamma(x) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

integration by parts, it

$$= \int_\Omega \sum_{\alpha,\lambda,\mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha(\gamma(x)\overline{a_{\alpha,\lambda\mu}} v_\lambda(x))} \mathrm{d}x_1..\mathrm{d}x_n$$

Locally,

$$P^*v = \sum_{|\alpha|\le k} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha(\gamma(x) \overline{a_\alpha(x)}^T v(x))$$

Uniqueness: use the density of C^∞-section with compact support in $L^2(M, -)$. □

推论 4.1.6. *If $\sigma_P(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$,then*

$$\sigma_{P^*} = (-1)^k \overline{\sigma_P(x,\xi)}^T$$

推论 4.1.7. *If rank E = rank$F$, P is differential operator, then P^* is elliptic $\iff$ P^* is elliptic.*

4.2 椭圆算子的基本性质

Fundamental results of elliptic operators

M is a compact (oriented) C^∞-manifold, $\dim_{\mathbb{R}} M = n$, with a smooth volume form $\mathrm{d}V$.

E is an Hermite vector bundle, $rank_{\mathbb{C}} E = r$.

Sobolev space:$W^k(M, E)$:= the space of section $s : M \to E$ whose derivations up to order $= k$, := the completion of space of smooth sections w.r.t W^k-norm.

$(\Omega_j)_{j\in I}$: a finite open covering of M, $E|_{\Omega_j}$ trivial, Let $(\rho_j)_{j\in I}$ be a partition of unity w.r.t. $(\Omega_j)_{j\in I}$, s.t. $\sum_j \rho_j^2 = 1$. locally, choose an orhtonormal frame $(e_{j,\lambda})_{1\leq\lambda\leq r}$ on Ω_j, then $u = \sum_{\lambda=1}^r u_{j,\lambda} e_{j,\lambda}$ on Ω_j. Define

$$||u||_k^2 := \sum_{j,\lambda} ||e_j u_{j,\lambda}||_k^2$$

where

$$||e_j u_{j,\lambda}||_k^2 := \int_{\Omega_j} \sum_{|\alpha|\leq k} |D^\alpha(e_j u_{j,\lambda})|^2 \mathrm{d}V(x)$$

注记 4.2.1. *On a compact manifold, the equivalence of class of $||\cdot||_k$ is independent of the choice of : partition of unity, local trivialization, holomorphic covering...*

引理 4.2.2. *(Sobolev lemma)*

For $k > l + \frac{n}{2}$, then we have

$$W^k(M, E) \subseteq C^l(M, E)$$

引理 4.2.3. *(Rellich lemma)*

For any $k \in \mathbb{Z}_{\geq 0}$, the inclusion

$$W^{k+1}(M, E) \hookrightarrow W^k(M, E)$$

is a compact operator.

引理 4.2.4. *(Garding inequality)*

If

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

is elliptic, and rankE =rank F, $\tilde{P}$:the extension of P to sections with distribution coefficients, then : for all $u \in W^0(M, E)$, if $\tilde{P}u \in W^k(M, F)$, then $u \in W^{k+d}(M, E)$, where $d = \deg P$, and

$$||u||_{k+d} \leq C_k \left(||\tilde{P}u||_k + ||u||_0\right)$$

where C_k depending on k, M.

证明. Reference: `Kodaira: deformation of complex structures (Appendix)` □

推论 4.2.5. *If $u \in \ker \tilde{P} \cap W^0(M, E)$, then $u \in C^\infty(M, E)$.*

引理 4.2.6. *(Finiteness theorem)*

Setting M be a compact manifold, rank$E$ =rank$F$,

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

elliptic,then:

(1) $\ker P$ is of finite dimension

(2) $P(C^\infty(M, E))$ is closed and of finite codimension in $C^\infty(M, F)$. If P^* is the formal adjoint of P, then $\exists$ decomposition

$$C^\infty(M, F) = P(C^\infty(M, E)) \oplus \ker P^*$$

which is orthogonal in $W^0(M, F) = L^2(M, F)$

证明. 椭圆算子的一般结果，分析的东西 233333333. 可以参考小平邦彦复流形与复结构形变的附录。 □

Hodge theory in compact Riemannian manifold

Hodge star operator.

M compact Riemannian manifold, $\dim_{\mathbb{R}} = n$, E is a Hermitian vector bundle. Assume $(\xi_1, ..., \xi_n), (e_1, ..., e_n)$ be orthonormal frame of TM, E on some local chart Ω, denote $(\xi_1^*, ..., \xi_n^*), (e_1^*, ..., e_n^*)$ be the co-frame of T^*M, T^*E.

$\wedge^\bullet T^*M$ is endowed with an inner product frame from TM. locally,

$$\langle u_1 \wedge \cdots \wedge u_p, u_1 \wedge \cdots \wedge u_p \rangle := \det(\langle u_i, v_j \rangle)$$

for $u_i, v_j \in T^*M$. Then , get an inner product on $\wedge^p T^*M$.

Assume

$$U = \sum_{\substack{|I|=p \\ i_1 \leq ... \leq i_p}} u_I \xi_I^*$$

$$V = \sum_{\substack{|I|=p \\ i_1 \leq ... \leq i_p}} v_I \xi_I^*$$

be p-forms, then

$$\langle u, v\rangle = \sum_{|I|=p} u_I v_I$$

i.e. $\{\xi_T^*\}$ is an orthonormal basis of $\wedge^p T^*M$.

$\wedge^* T^*M \otimes E$ has an inner product induced from $\wedge^* T^*M, E$,

定义 4.2.7. *the Hodge star operator*

$$^* : \wedge^p T^*M \to \wedge^{n-p} T^*M$$

is defined by

$$u \wedge *v = \langle u, v\rangle \mathrm{d}V$$

Locally, let

$$U = \sum_{|I|=p} u_I \xi_I^*,\ V = \sum_{|I|=p} v_I \xi_I^*$$

assume

$$*V = \sum_{|J|=n-p} a_J \xi_J^*$$

then

$$U \wedge * \sum u_I a_{I^c} \xi_I^* \wedge \xi_{I^c}^* = \sum u_I a_{I^c} \varepsilon(I, I^c) \xi_1^* \wedge \cdots \wedge \xi_n^*$$
$$\langle u, v\rangle \mathrm{d}V = \sum_{|I|=p} u_I v_I \xi_1^* \wedge \cdots \wedge \xi_n^*$$

so, we have

$$*V = \sum_{|I|=p} \varepsilon(I, I^c) V_I \xi_{I^c}^* \in \bigwedge^{n-p} T^*M$$

定义 4.2.8.

$$* : \bigwedge^p T^*M \otimes E \to \bigwedge^{n-p} T^*M \otimes E$$

is defined by

$$\{s, *t\} := \langle s, t\rangle \mathrm{d}V$$

Locally, assume

$$t = \sum_{\substack{|I|=p \\ 1\le\lambda\le r}} t_{I,\lambda} \xi_I^* \otimes e_\lambda$$

then

$$*t = \sum_{\substack{|I|=p \\ 1\leq\lambda\leq r}} \varepsilon(I, I^c) t_{I,\lambda} \xi^*_{I^c} \otimes e_\lambda$$

定义 4.2.9.

$$\# : \bigwedge^p T^*M \otimes E \to \bigwedge^{n-p} T^*M \otimes E^*$$

is defined by: for any $s, t \in \bigwedge^p T^*M \otimes E$,*such that*

$$s \wedge \# t := \langle s, t \rangle \mathrm{d}V$$

wedge product+ pairing of $E^* \times E \to \mathbb{C}$.

Locally: assume

$$t = \sum_{\substack{|I|=p \\ 1\leq\lambda r}} t_{I,\lambda} \xi^*_T \otimes e_\lambda$$

then,

$$\# t = \sum_{|I|=p,\lambda} \varepsilon(I, I^c) t_{I,\lambda} \xi^*_c I \otimes e^*_\lambda$$

性质 4.2.10.

$$*^2 = (-1)^{p(n-1)} \quad on \quad \bigwedge^p T^*M \otimes E$$

$$\#^2 = (-1)^{p(n-1)} \quad on \quad \bigwedge^p T^*M \otimes E$$

（正负号对吗？）

Recall: For all $s, t \in C^\infty(M, \bigwedge^p T^*M \otimes E)$, we have an inner product

$$\langle\langle s, t \rangle\rangle := \int_M \langle s, t \rangle \mathrm{d}V$$

定理 4.2.11. *Let* D_E *be an Hermite connection on* E*, acting on* $\bigwedge^p T^*M \otimes E$*, then*

$$D_E^* := (-1)^{np+1} * D_E *$$

where D_E^* *is the formal adjoint of* D_E.

证明. Let $s \in C^\infty(M, \bigwedge^p T^*M \otimes E)$ and $t \in C^\infty(M, \bigwedge^{p+1} T^*M \otimes E)$. then

$$\langle\langle D_E s, t\rangle\rangle = \int_M \langle D_E s, t\rangle \mathrm{d}V = \int_M \{D_E s, *t\}$$

Since D_E is Hermitian ,by defination ,

$$\mathrm{d}\{s, *t\} = \{D_E s, t\} + (-1)^p\{s, D_E(*t)\}$$

so,

$$\langle\langle D_E s, t\rangle\rangle = \int_M \mathrm{d}\{s, *t\} + (-1)^{p+1}\{s, D_E * t\} = (-1)^{p+1}(-1)^{p(n_1)}\int_M \{s, *(*D_E * t)\} = \langle\langle s, D_E^* t\rangle\rangle$$

so,

$$D_E^* t = (-1)^{np+1} * D_E *$$

□

定义 4.2.12.

$$\triangle_E = D_E D_E^* + D_E^* D_E : C^\infty(M, \bigwedge^p T^*M \otimes E) \to C^\infty(M, \bigwedge^p T^*M \otimes E)$$

例子 4.2.13. *Let* $M = \mathbb{R}^n$, $g = \sum\limits_{i=1}^n \mathrm{d}x_i^2$, $E = M \times \mathbb{C}$ *trivial line bundle with* $D_E = \mathrm{d}$. *then*

$$\triangle_E u = (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})u = -\sum_{i=1}^n \left(\sum_{|I|=p} \frac{\partial^2 u_I}{\partial x_I^{2^2}} \mathrm{d}x_I\right)$$

where

$$u = \sum_{|I|=p} u_I \mathrm{d}x_I$$

性质 4.2.14. $\triangle_E$ *is a self-adjoint elliptic operator. (i.e.* $\triangle_E^* = \triangle_E$*)*

证明. $\triangle_E^* = \triangle_E$ be definition.

note that

$$e^{-tf} D_E(e^{tf} s) = t\mathrm{d}f \wedge s + D_E s$$

so,

$$\sigma_{D_E}(x, \xi)s = \xi \wedge s$$

$$\sum_{D_E^*} = -\overline{\sigma_{D_E}}^T$$

$$\sigma_{D_E^*}(x,\xi)s = -\tilde{\xi} \lrcorner s$$

where $\tilde{\xi}$ be the vector field dual to ξ. $\square$

定义 4.2.15.

$$\triangle_E = D_E D_E^* + D_E D_E^* : C^\infty(M, \bigwedge^p T^*M \otimes E) \to C^\infty(M, \bigwedge^p T^*M \otimes E)$$

so,

$$\sigma_{\triangle_E}(x,\xi)s = \left(\sigma_{D_E}\sigma_{D_E^*}(x,\xi) + \sigma_{D_E^*}\sigma_{D_E}(x,\xi)\right) s$$

so, $\sigma_{\triangle_E}$ is injective if $\xi \neq 0$, so $\triangle_E$ is elliptic.

Harmonic forms and Hodge isomorphism.

定义 4.2.16. *u is called harmonic if $\triangle_{\mathrm{d}} u = 0$.*

定理 4.2.17. *M is a compact Riemannian manifold, then de Rham cohomology*

$$H^p_{DR}(M, \mathbb{R}) \cong \ker(\triangle_d : C^\infty(M, \bigwedge^p T^*M))$$

证明. $\triangle_d$ self-adjoint elliptic, so by general result for elliptic operator,

$$C^\infty(M, \bigwedge^p T^*M) = \operatorname{Im} \triangle_d \oplus \ker \triangle_d^* = \operatorname{Im} \triangle_d \oplus \ker \triangle_d$$

Claim:

$$\operatorname{Im} \triangle_{\mathrm{d}} = \in \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^*$$

Recall $\triangle_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, so

$$\operatorname{Im} \triangle_{\mathrm{d}} \subseteq \operatorname{Im} \mathrm{d} \oplus \in \mathrm{d}^*$$

on the other hand,

$$\operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^* \subseteq (\ker \triangle_{\mathrm{d}})^\perp = \operatorname{Im} \triangle_{\mathrm{d}}$$

so,

$$\operatorname{Im} \triangle_{\mathrm{d}} = \operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^*$$

so,

$$C^\infty(M, \bigwedge^p T^*M) = \operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^* \oplus \ker \triangle_{\mathrm{d}}$$

so,

$$H^p_{DR}(M, \mathbb{R}) = \frac{\operatorname{Im} \mathrm{d} \oplus \ker \triangle_{\mathrm{d}}}{\operatorname{Im} \mathrm{d}} = \ker \triangle_{\mathrm{d}}$$

□

推论 4.2.18.

$$\dim H^p_{DR}(M, \mathbb{R}) = \dim \ker \triangle_{\mathrm{d}} < +\infty$$

注记 4.2.19. *Consider*

$$u \mapsto \int_M (\langle u, u\rangle + \langle \mathrm{d}u, \mathrm{d}u\rangle + \langle \mathrm{d}^*u, \mathrm{d}^*u\rangle)\mathrm{d}V$$

这个泛函的变分是什么鬼?

术语索引