The Bisection Method

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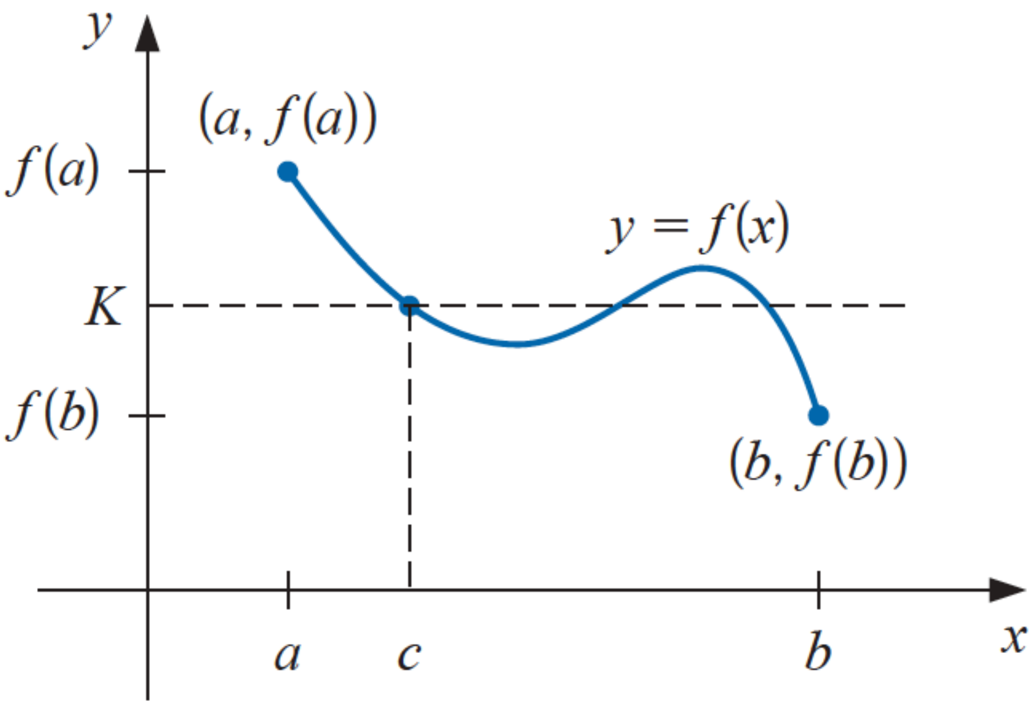
CSCE 440/840 Numerical Analysis
Fall 2016

Intermediate Value Theorem

If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) for which $f(c) = K$. ■



Intermediate Value Theorem
(with f(a) and f(b) having opposite signs)

Suppose a continuous function f , defined on [a, b] is given with f(a) and f(b) of opposite sign. Then there exists a point p in (a, b) for which f(p) = 0.

Show that $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a root in the interval $[0, 1]$

$f(0) = -1 < 0$ and $f(1) = 1 > 0$

The Bisection Method

The Bisection Method repeatedly applies the Intermediate Value Theorem. When we call the method, we assume that the function is continuous on [a,b] and f(a) and f(b) have opposite signs.

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BisectionMethod(f, a, b)
Let p = (a + b)/2.
if   stopping criterion = True then
        Return(p).
if  f(a) and f(p) have opposite signs  then
        BisectionMethod(f, a, p).
else
        BisectionMethod(f, p, b).
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Bisection Method Stopping Criteria

Let TOL be an error tolerance value. Then the iterations are stopped when one of the following conditions hold:

- $|p_N - p_{N-1}| < \text{TOL}$ (1)
- $|p_N - p_{N-1}| / |p_N| < \text{TOL}$ (2)
- $|f(p_N)| < \text{TOL}$ (3)

Bisection Method Characteristics

The Bisection Method has two significant drawbacks.

- First, it is very slow to converge in that N may become quite large before $p - p_N$ becomes sufficiently small.
- Second, it is possible that a good intermediate approximation may be inadvertently discarded.

The Bisection Method has one advantage:

- It will always converge to a solution. For this reason, it is often used to provide a good initial approximation for a more efficient procedure.

Bisection Method Example

Example: Find a root of the polynomial

$$f(x) = 3(x+1)(x - 0.5)(x - 1)$$

within the region [-2, 1.5].

Solution: Using Bisection Method with stop condition (3) and TOL = 0.01.

Since f is continuous in [-2, 1.5] and $f(-2) < 0$ and $f(1.5) > 0$, we call:

BisectionMethod(f, -2, 1.5).

$p_1 = -0.25$

Since $f(-0.25) = 2.109375$, $|f(-0.25)| > TOL$ and we have to continue.

Since $f(-0.25) > 0$, we call:

BisectionMethod(f, -2, -0.25)

$P_2 = -2.25/2 = -1.125$

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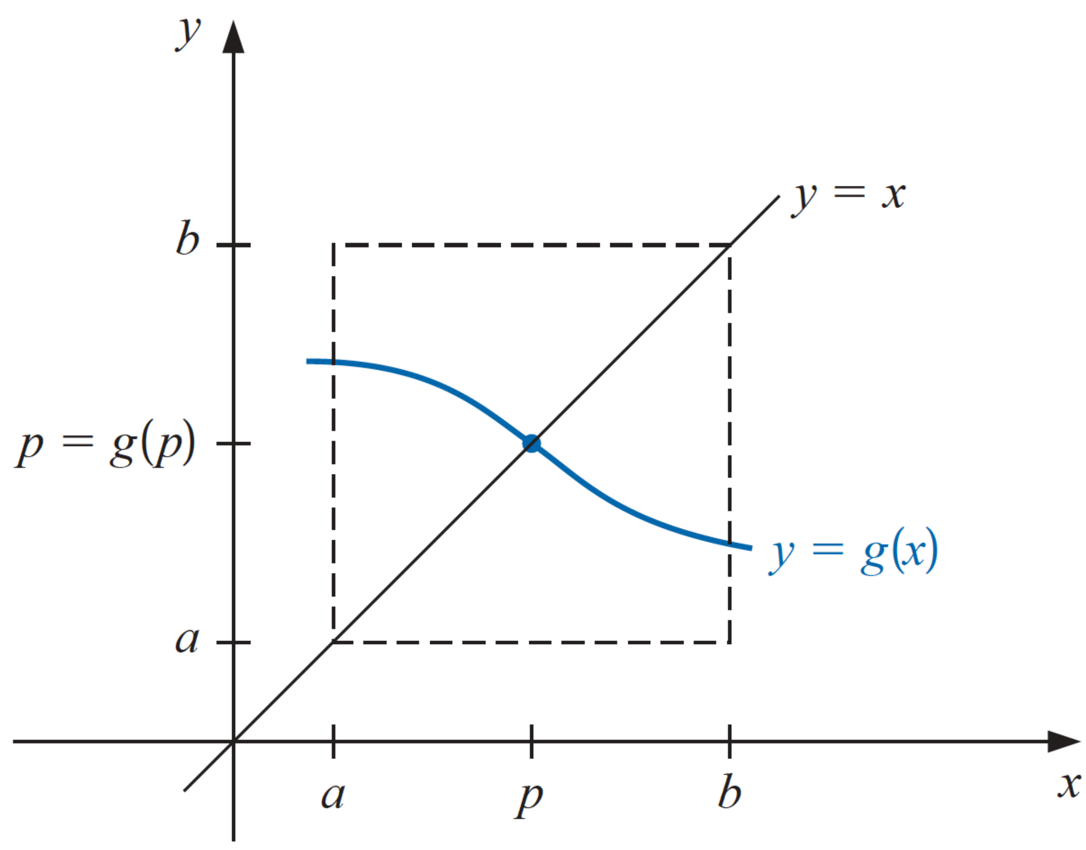
Fixed Point of a Function

Let g(x) be a function. Then the value p is a fixed point of g(x) if g(p) = p.



Finding the Fixed Point of a Function

Finding the fixed point of a function g(x) can be reduced to the problem of finding the root of the function f(x) = g(x) – x.

Example: Find the fixed point of the function

$$g(x) = x^2 - 2$$

within the region [-3, 0].

Solution: Use the Bisection Method to find the root of the function

$$f(x) = g(x) - x = x^2 - x - 2.$$

Since f(-3) = 10 > 0 and f(0) = -2 < 0, we call:

BisectionMethod(f, -3, 0)

$p_1 = -3 / 2 = -1.5$

Since f(-1.5) = 1.75, |f(-1.5)| > TOL and we have to continue.

Since f(-1.5) > 0, we call:

BisectionMethod(f, -1.5, 0)

$P_2 = -1.5 / 2 = -0.75$

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