Newton's Method

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Taylor Polynomial

Suppose that $f \in C^2[a,b]$. Let p_0 be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small". Using the first Taylor polynomial of $f(x)$ for $x = p$ expanded about p_0 as

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p-p_0)^2}{2} f''(\xi(p))$$

where $\xi(p)$ lies between p and p_0

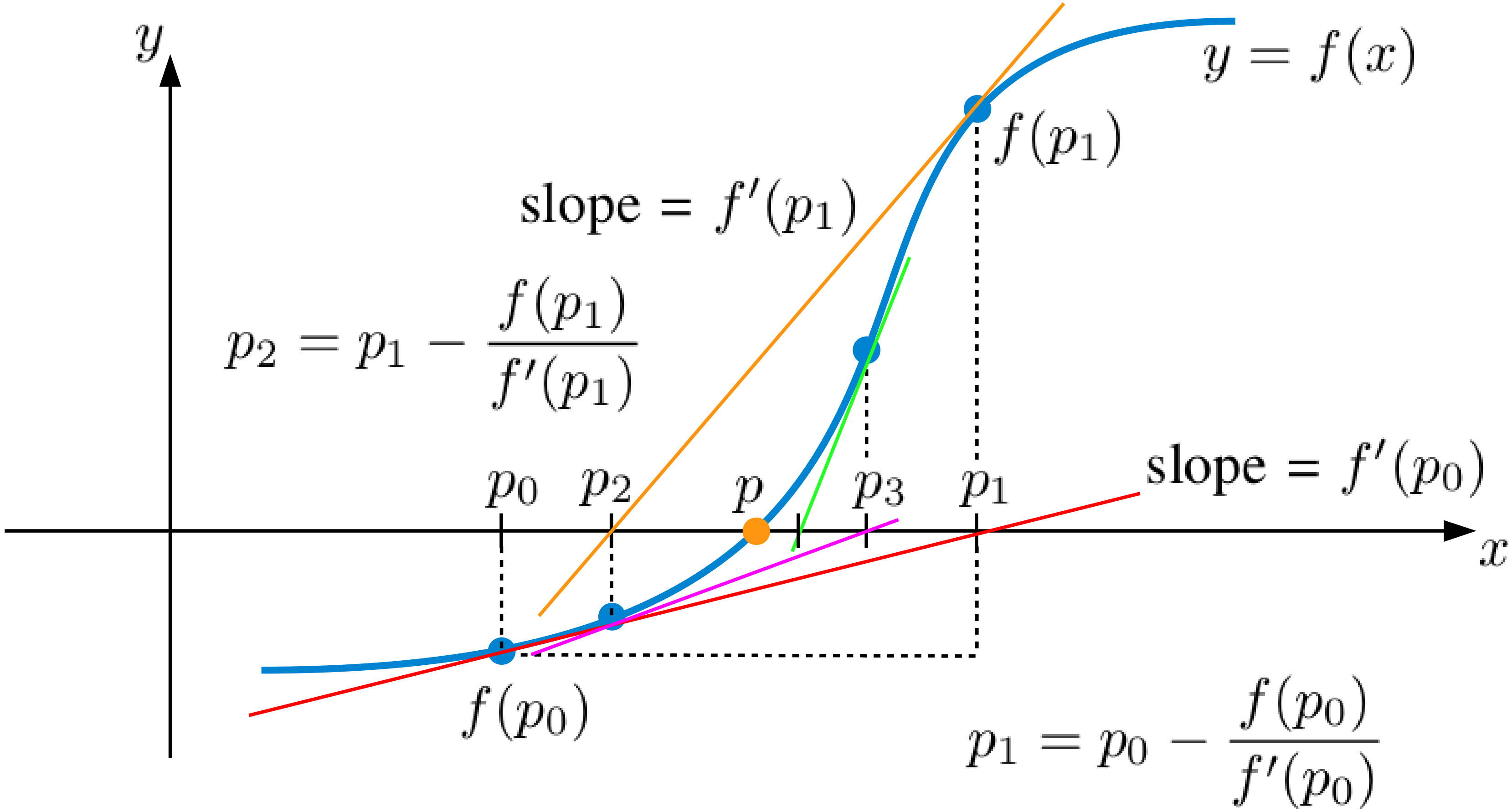
$|p - p_0|$ is "small" $\Rightarrow (p - p_0)^2$ is much smaller, and thus

$$0 \approx f(p_0) + (p - p_0)f'(p_0) \quad \Rightarrow p \approx p_0 - \frac{f(p_0)}{f'(p_0)}$$

$p_0 - f(p_0) / f'(p_0)$ is better than p_0 in approximating p.

Newton's Method

- An iterative method $$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$



Newton's Method

Algorithm 2: Newton's Method Root Finding

input : initial approximation p_0, tolerance ϵ, max number of iterations N_{max}
output: approximate solution p or message of failure

$i \leftarrow 1$; // init the counter;
while $i \leq N_{max}$ **do**
    $p = p_0 - f(p_0)/f'(p_0)$; // estimate the root;
    if $|p - p_0| < \epsilon$ **then**
        return (p); // "root" with sufficient accuracy;
    end
    $p_0 \leftarrow p$; // update the current estimation;
    $i \leftarrow i + 1$; // enter the next iteration;
end
echo ("Failed to find the root after N_{max} iterations");

An electric power cable is suspended (at points of equal height) from two towers 100 meters apart. The cable is allowed to dip 10 meters in the middle. How long is this cable?

$$f(l) = l \cdot cosh(\frac{50}{l}) - l - 10 = 0$$

Bisection

```
N = 50;
TOL = 1e-4;
i = 1;
a = 100;
b = 150;
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Iteration 1, p = 125.000000, a = 100.000000, b = 150.000000, fp = 0.134046
....
Iteration 19, p = 126.632404, a = 126.632309, b = 126.632500, fp = 0.000003

Root p = 126.632404 (TOL = 0.0001) was found after 19 iterations
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Newton's

$$cosh'(x) = sinh(x) = \frac{e^x - e^{-x}}{2} \qquad f'(l) = cosh(\frac{50}{l}) - \frac{50}{l} sinh(\frac{50}{l}) - 1$$

```
Iteration 1, pi = 200.000000, p = 82.880200
Iteration 2, pi = 82.880200, p = 110.763009
Iteration 3, pi = 110.763009, p = 124.561248
Iteration 4, pi = 124.561248, p = 126.597254
Iteration 5, pi = 126.597254, p = 126.632426
Iteration 6, pi = 126.632426, p = 126.632436

Root p = 126.632436 (TOL = 0.0001) was found after 6 iterations
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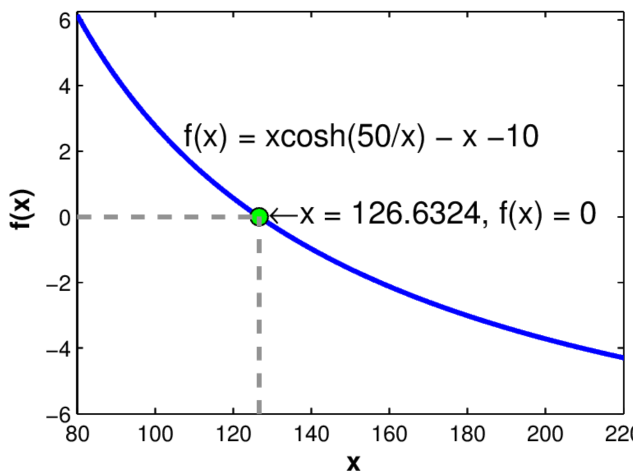
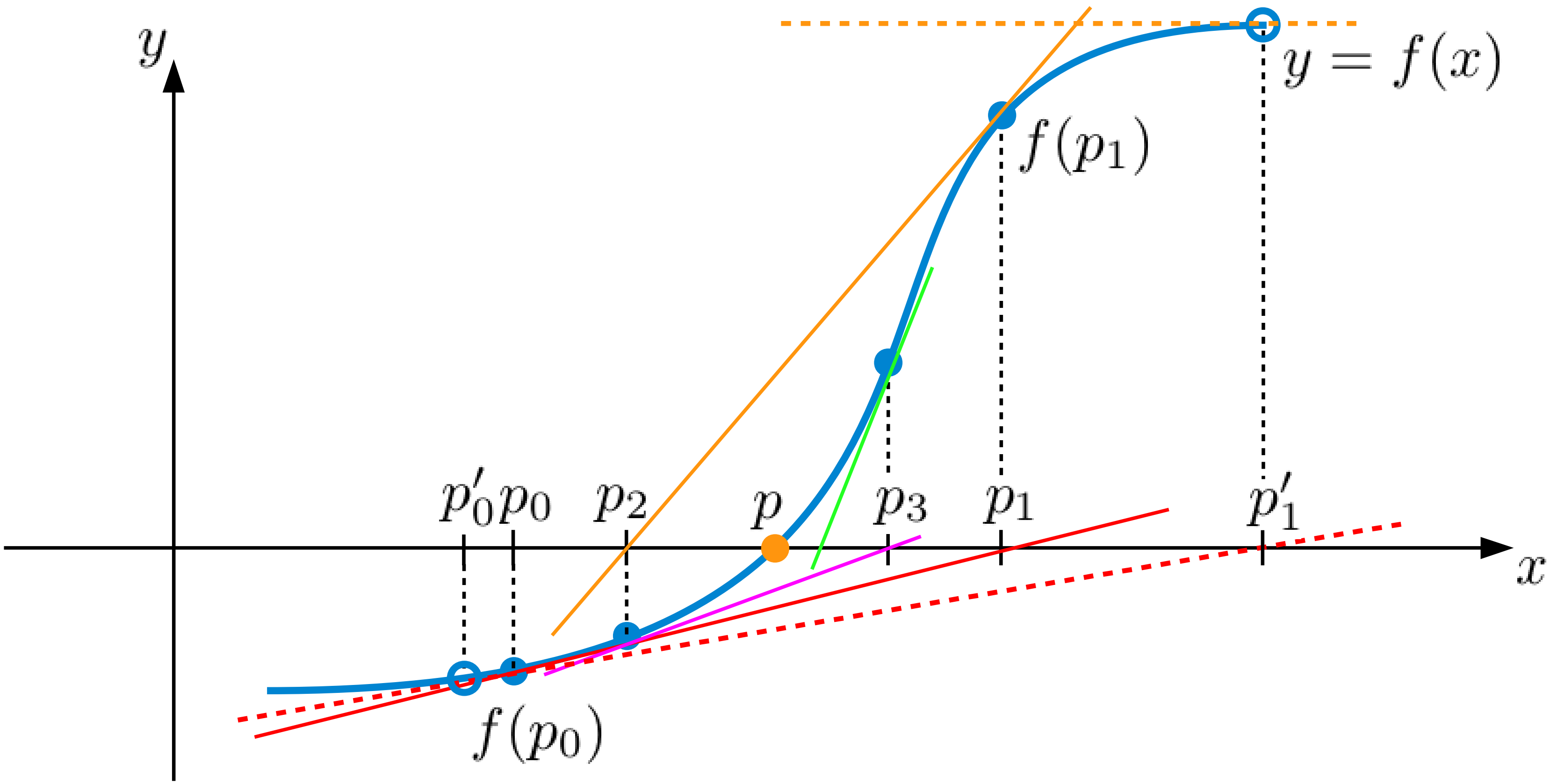


Newton's Method fails when derivative is zero

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p))$$



Exercise 1 on p. 75 of Textbook, 9th Edition

Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's Method to find p_2.

Solution: Here $f'(x) = 2x$. Hence

$$\begin{aligned}
p_0 &= 1 \\
p_1 &= p_0 - \tfrac{f(p_0)}{f'(p_0)} = 1 - \tfrac{-5}{2} = 3.5 \\
p_2 &= p_1 - \tfrac{f(p_1)}{f'(p_1)} = 3.5 - \tfrac{6.25}{7} = \tfrac{18.25}{7} \approx 2.6071429
\end{aligned}$$

Exercise 2 on p. 75 of Textbook, 9th Edition

Let $f(x) = -x^3 - cos(x)$ and $p_0 = -1$. Use Newton's Method to find p_2.

Solution: Here $f'(x) = -3x^2 + sin(x)$. Hence

$$
\begin{aligned}
p_0 &= -1 \\
p_1 &= p_0 - \frac{f(p_0)}{f'(p_0)} = -1 - \frac{-(-1)^3 - cos(-1)}{-3(-1)^2 + sin(-1)} = -1 - \frac{0.4597}{-3.8415} \approx -0.8803 \\
p_2 &= p_1 - \frac{f(p_1)}{f'(p_1)} = -08803 - \frac{-(-0.8803)^3 - cos(-0.8803)}{-3(-0.8803)^2 + sin(-0.8803)} = -0.8803 - \frac{0.0452}{-3.0957} \approx -0.8657
\end{aligned}
$$