Real Analysis

In progress

Contents

Chapter 1 Preliminary

Definition 1.1 (Isometry)

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called **isometry** if for any $a, b \in X$,

$$d_X(a, b) = d_Y(f(a), f(b)).$$

♣

Definition 1.2 (Congruent)

Two subsets $E, F \subseteq \mathbb{R}^n$ are called **congruent** if there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $f(E) = F$. ♣

Intuition E congruent to F if E can be transformed into F by translations, rotations, and reflections.

Definition 1.3 (Indexed family)

Let I and X be sets and f a function s.t.

$$\begin{aligned} f :&I \to X \\ i &\mapsto x_i = f(i) \end{aligned}$$

(note that we denoted the image of i under f by x_i). We then call the image of I under f a **family of elements in X indexed by I**.

♣

Remark Similar definition goes to **indexed collection of nonempty sets**.

1.1 Topology

Proposition 1.1

1. Any open subset of $\mathbb{R}$ is a countable union of disjoint open intervals.
2. Any open interval is a countable union of closed intervals.

♠

1.2 Sequence and series

Definition 1.4

If X is an arbitrary set and $f : X \to [0, \infty]$, we define

$$\sum_{x \in X} f(x) = \sup\{\sum_{x \in F} f(x) : F \subseteq X, |F| < \infty\}$$

♣

Remark We define the sum to be the supremum of its finite partial sums so that it also works for uncountable set X.

Proposition 1.2

Given $f : X \to [0, \infty]$, let $A = \{x \in X : f(x) > 0\}$. If A is uncountable, then $\sum_{x \in X} f(x) = \infty$. If A is countably infinite, then $\sum_{x \in X} f(x) = \sum_1^\infty f(g(n))$ where $g : \mathbb{N} \to A$ is any bijection and the sum $\sum_1^\infty f(g(n))$ is an arbitrary infinite series.

♠

Proof We can write $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \{x \in X : f(x) > \frac{1}{n}\}$. If A is uncountable, then there exists A_k is uncountable, thus contains infinite many elements and $\sum_{x \in F} f(x) > |F|/n$ where $F \subset A_k$ is finite. As $|F| \to \infty$, $\sum_{x \in X} f(x) \to \infty$.

If A is countably infinite, then there exists a bijection $g : \mathbb{N} \to A$. For every finite subset F of A, we can always find a set $B_N = g(\{1, \cdots, N\})$ containing F with large enough N. Hence

$$\sum_{x \in F} f(x) \leq \sum_{1}^{N} f(g(n)) \leq \sum_{x \in X} f(x)$$

By taking the supremum over N, we have

$$\sum_{x \in F} f(x) \leq \sum_{1}^{\infty} f(g(n)) \leq \sum_{x \in X} f(x)$$

and then taking the supremum over F, we obtain the desired result.

Chapter 2 Measure Theory

2.1 Motivation

The motivation of measure theory is to measure the length, area, or volume of a region. Given a region $E \subseteq \mathbb{R}^n$, we would like to assign a non-negative number $\mu(E)$ to it. Intuitively, such function μ should satisfy the following:

1. For countable sequence of disjoint sets, $E_1, E_2, \cdots$ we have
$$\mu(E_1 \cup E_2 \cdots) = \mu(E_1) + \mu(E_2) + \cdots.$$
2. If E is congruent to F, then $\mu(E) = \mu(F)$.
3. $\mu(Q) = 1$, where Q is the unit cube
$$Q = \{x \in \mathbb{R}^n : 0 \leq x_j < 1 \text{ for } j = 1, \cdots, n\}$$

But unfortunately, there conditions are mutually inconsistent. Consider the following example:

Example 2.1

Let $n = 1$. Define an equivalence relation by the Axiom of Choice, we can build a set N that contains one member of each equivalent class. Let $R = \mathbb{Q} \cap [0, 1)$ (set of all rationals in $[0, 1)$), and for each $r \in R$, let
$$N_r = \{x + r : x \in N \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in N \cap [1 - r, 1)\},$$
Intuitively, we shifted the N to the right by r and then shifted the part sticks out beyond $[0, 1)$ one to the left.

We shall show that

Corollary 2.1

There is no such $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ exists that satisfies the aforementioned three conditions. ♡

Proof We proof by contradiction. First show that $\{N_r\}$ partitions $[0, 1)$:

- For all $y \in N$, if $y \in [x]$, then $y \in N_r$ where $r = x - y$ or $r = x - y + 1$.
- Let $y \in [0, 1)$, if $y \in N_r \cap N_s$ $(r \neq s)$, then $y - r$ (or $y - r + 1$) and $y - s$ (or $y - s + 1$) would belong to two distinct equivalent classes, while their difference is a rational, so is a contradiction.

Suppose there exists a $\mu : \mathcal{P}(\mathbb{R}) \to [0, 1)$ satisfies the aforementioned three conditions, then by condition 1 and 2, we have $\mu(N) = \mu(N_s)$.

Also since R is countable, so by condition 1, we have $\mu([0, 1)) = \sum\limits_{r \in R} \mu(N_r) = |R|\mu(N_r)$, but $|R|\mu(N_r)$ is either 0 (if $\mu(N_r) = 0$) and ∞ (if $\mu(N_r) > 0$), which contradicts condition 3. ■

Remark

- Even weaken the condition 1 so that it only allows the additively holds for finite sequences it will still lead to contradiction when dimension $n \geq 3$. [1]
- Such contradiction also arises in probability theory, just consider $X \sim \text{Uniform}[0, 1]$, and for $E \subseteq X$, let $\mu(E) = \mathbb{P}(E)$.

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[1] See Complementary 1

2.2 σ-algebras

Motivation Before measuring subsets, we first define the collection of subsets we want to give the measure to, that is, the collection of measurable sets. As we have seen it is not possible to give measure to the power set all the time.

Definition 2.1 (Semialgebras)

Let X be a nonempty set. A **semialgebra** of X is a nonempty collection $\mathcal{A}$ of subsets of X that is

1.

♣

Definition 2.2 (Algebra)

Let X be a nonempty set. An **algebra** of X is a nonempty collection $\mathcal{A}$ of subsets of X that is

1. closed under *finite unions*: If $E_1, \cdots, E_n \in \mathcal{A}$, then $E_1 \cup \cdots \cup E_n \in \mathcal{A}$,
2. and closed under complements: If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

♣

Definition 2.3 (σ-algebra)

A **σ-algebra** is an algebra that is closed under *countable unions*:
If $E_1, E_2, \cdots \in \mathcal{A}$, then $E_1 \cup E_2 \cup \cdots \in \mathcal{A}$.

♣

Corollary 2.2

1. Algebra (σ-algebra) is closed under finite (countable) intersections.
2. If $\mathcal{A}$ is an algebra, then $\varnothing \in \mathcal{A}$, and $X \in \mathcal{A}$.
3. If an algebra is closed under countable *disjoint* unions, then it is a σ-algebra.
4. Let $\mathcal{A}_i$ be a family of σ-algebra, then $\bigcap_i \mathcal{A}$ is also a σ-algebra.

♡

By Corollary 2.2.4, we can define the following

Definition 2.4

Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and let $\sigma(\mathcal{E})$ be the intersection of all σ-algebra containing $\mathcal{E}$, then we call $\sigma(\mathcal{E})$ the σ-alegbra **generated** by $\mathcal{E}$, in another word, $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$.

♣

Lemma 2.1

If $\mathcal{E} \subseteq \sigma(\mathcal{F})$, then $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{F})$.

♡

Definition 2.5 (Borel σ-algebra)

Let X be a metric space, the σ-alegbra generated by the collection of all open sets (or, equivalently, by the collection of all closed sets) in X is called **Borel σ-algebra**, denoted by $\mathcal{B}_X$, and its members are called **Borel sets**.

♣

Proposition 2.1

$\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:

1. the open intervals: $\mathcal{E}_1 = \{(a, b) : a < b\}$.
2. the clased intervals: $\mathcal{E}_2 = \{[a, b] : a < b\}$.
3. the half-open intervals: $\mathcal{E}_3 = \{(a, b] : a < b\}$ or $\mathcal{E}_4 = \{[a, b) : a < b\}$
4. the open rays: $\mathcal{E}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, b) : a \in \mathbb{R}\}$.

5. the closed rays: $\mathcal{E}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$.

Proof Idea To prove this proposition, we apply Proposition 1.1 and Lemma 2.1. Note that half intervals can be build by the intersection of a ray and the complement of another ray.

2.3 Product σ-algebra

Given a collection of nonempty set $\{X_\alpha\}$ and their σ-algebra $\mathcal{M}_\alpha$, we want to obtain a σ-algebra of the Cartesian product space $X_1 \times \cdots$, so we define:

Definition 2.6 (product σ-algebra)

Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, $X = \Pi_{\alpha \in A} X_\alpha$ the Cartesian product space, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each α, the **product σ-algebra** on X is the σ-algebra generated by:

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$$

we denote this σ-algebra by $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$.

Proposition 2.2

If the index set A is countable, then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is the σ-algebra generated by

$$\{\Pi_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$$

Proposition 2.3

Proposition 2.4

Corollary 2.3

2.4 Measures

Definition 2.7 (Measure)

Let X be a set equipped with a σ-algebra $\mathcal{M}$, we call $(X, \mathcal{M})$ **measurable space**, and $\mathcal{M}$ a collection of **measurable sets**. A **measure** on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\varnothing) = 0$,
2. (**countable additivity**) if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$.

We call $(X, \mathcal{M}, \mu)$ the measure space.

Also, for μ satisfies **finite additively** instead of countable additively we call it **finite additive measure**.

Theorem 2.1

1. (**Monotonicity**) If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. (**Subadditivity**) If $\{E_j\}_1^\infty \subseteq \mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$.
3. (**Continuity from below**) If $\{E_j\}_1^\infty \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.
4. (**Continuity from above**) If $\{E_j\}_1^\infty \subseteq \mathcal{M}$, $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_1) < \infty$, then $\mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

♡

Proof skip for now

Definition 2.8

Let $(X, \mathcal{M}, \mu)$ be a measure space,

- If $\mu(X) < \infty$, μ is called **finite**.
- If $E = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j, the set E is said to be σ-**finite** for μ, and if $E = X$ we call μ to be σ-**finite**.
- If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, μ is called **semifinite**.

♣

Intuition If the measure of X is finite then μ called finite; if X is the union of countable subsets with finite measure then μ called σ-finite.

Example 2.2 Consider $(X, \mathcal{M}, \mu)$:

- X is any nonempty set,
- $\mathcal{M} = \mathcal{P}(X)$,
- $\mu(E) = \sum_{x\in E} f(x)$, where f any function from X to $[0, \infty]$

then

1. μ is semifinite iff $f(x) < \infty$ for every $x \in X$,
2. μ is σ-finite iff μ is semifinite and $\{x \in X : f(x) > 0\}$ is countable.

Explaination

1. If $f(x) = \infty$ for some $x \in X$, then $\mu(\{x\}) = \infty$ and no subset of $\{x\}$ has finite nonzero measure. On the other hand, if $f(x) < \infty$ for every $x \in X$, then for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there always exists a finite subset of E with finite measure,
2. If μ is σ-finite, then for every $x \in X$, $x \in E_j$ for some j (use Definite 2.7), since $\mu(E_j) < \infty$, so $f(x) = \mu(\{x\}) < \mu(E_j) < \infty$. Assume that $\{x \in X : f(x) > 0\}$ is uncountable, then some E_k is uncountable thus by Proposition 1.2, $\mu(E_k) = \infty$ $\lightning$. On the other hand, if μ is semifinite and $S = \{x \in X : f(x) > 0\}$ is countable, then exists a bijection h from $\mathbb{N}$ to S and let $E = \bigcup_1^\infty E_j$ where each E_j contains exact one element with positive measure: $h(j)$, then $\mu(E_j) < \infty$ since all other elements in E_j are either 0 or negative.

Definition 2.9

Given $(X, \mathcal{M}, \mu)$ defined in Example 2.2,

- if $f(x) = 1$ for all x, μ is called **counting measure**,
- if for some $x_0 \in X$, f is defined by $f(x_0) = 1$ and $f(x) = 0$ $(x \neq x_0)$, μ is called the **point mass**.

♣

Definition 2.10 (Complete measure)

Let $(X, \mathcal{M}, \mu)$ be a measure space, a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a **null set**.
μ is called **complete** if its domain, $\mathcal{M}$, contains all the subsets of null sets. ♣

We defined complete measure because it is natural for a subset of null set to have measure 0.

Theorem 2.2

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let

$$\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$$

and

$$\overline{M} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subsetneq N \text{ for some } N \in \mathcal{N}\}.$$

Then $\overline{M}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{M}$. ♡

Proof Idea set $\overline{\mu}(E \cup F) = \mu(E)$.

2.5 Outer Measures

Motivation Recall we approximate the area of a region using outer rectangles in calculus, now we develop an abstract way of describing it.

Definition 2.11 (Outer Measure)

An **Outer Measure** on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies:

1. $\mu^*(\varnothing) = 0$,
2. (Monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subsetneq B$,
3. (Subadditivity) $\mu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$. ♣

To give an outer measure on any subset, we define a collection of subsets $\mathcal{E}$ that is easy to measure. For example to measure the area of any regions in $\mathbb{R}^2$, we could define $\mathcal{E}$ to be rectangles on $\mathbb{R}^2$, and use countably many rectangles that covers the region to approximate the area, the area will be the infimum of all possible approximations.

Proposition 2.5

Let $\mathcal{E} \subsetneq \mathcal{P}(X)$ be such that $\varnothing \in \mathcal{E}$, $X \in \mathcal{E}$, and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\rho(\varnothing) = 0$. For any $A \subsetneq X$, define

$$\mu^*(A) = \inf\{\sum_{j=1}^\infty \rho(E_j) : E_j \in \mathcal{E} \text{ and } A \subsetneq \bigcup_{j=1}^\infty E_j\}$$

Then μ^* is an outer measure. ♠

Proof skip for now.

To obtain a complete measure from an outer measure we need to restrict the collection a bit, and it turns out to be that if we restrict $\mathcal{P}(X)$ to μ^*-measurable sets as defined follow, μ^* becomes a complete measure.

Definition 2.12 (μ^*-measurable)

If μ^* is an outer measure on X, a set $A \subseteq X$ is called μ^***-measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subseteq X$$

♣

Theorem 2.3 (Caratheodory Extension Theorem)

If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

♡

Now we can extend measures from algebra to σ-algebra .

Definition 2.13 (Premeasure)

If $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra, a function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called **premeasure** if

1. $\mu_0(\varnothing) = 0$,
2. if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty A_j \in \mathcal{A}$, then $\mu_0(\bigcup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$.

♣

2.6 Borel Measures

Motivation Now we want to measure the subsets of $\mathbb{R}$ based on the idea that the measure of an interval is its length.

Complementary 1: The Axiom of Choice

Axiom 2.1 (The Axiom of Choice [AC])

For any collection X of nonempty sets, there exists a choice function $f : X \to \bigcup X$, such that for every $A \in X$, $f(A) \in A$.

Intuition AC tells us that for any given collection of non-empty sets, we can pick one element from each set and form a new set.

Corollary 2.4 (Zorn's lemma)

Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

Proof Idea Assume the contrary, we can build a very long chain that has no upper bound in P.

Banach–Tarski paradox

Corollary 2.5 (strong form of the Banach–Tarski paradox)

Given any two bounded subsets A and B of a Euclidean space in at least three dimensions, both of which have a nonempty interior, there are partitions of A and B into a finite number of disjoint subsets, $A = A_1 \cup \cdots \cup A_k$, $B = B_1 \cup \cdots \cup B_k$ (for some integer k), such that for each (integer) i between 1 and k, the sets A_i and B_i are congruent.