Title

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January 1, 2023

Group Theory

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Question 1. There is a 1-1 correspondence between the set of left cosets of H in G and the set of right cosets of H in G.

Consider the function: $f(Ha) = a^{-1}H$.

Question 2. Center of Symmetric group is trivial for $n \geq 3$.

- Choose arbitrary $\pi(i) = j$, where $i \neq j$.
- Choose $\rho = (jk)$, where $k \neq i, j$.
- $\rho\pi\rho^{-1}(i) = k \neq j = \pi(i)$.
- Thus, for every element in S_n not equal to e, there exists another element does not commute with it.

Question 3. If $H < G$, let $N = \bigcap_{x \in G} xHx^{-1}$, then $N \triangleleft G$.

Question 4. U_n is cyclic if and only if $n = 1, 2, 4, p^k, 2p^k$ where p is an odd prime.

To prove it, there are some lemmas.

Remark.

1. If $n = p_1^{q_1} \cdots p_k^{q_k}$ be the prime factorization of n, then

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{q_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{q_k}\mathbb{Z}$$

2. For an abelian G, if $G \cong \mathbb{Z}_m \times \mathbb{Z}_n$, then G is cyclic if and only if $\gcd(m, n) = 1$.

Question 5. If G is a finite group, then the number of elements in the double coset AxB is

$$\frac{|A||B|}{|A \cap xBx^{-1}|}$$

$$\begin{aligned}
|AxB| &= |AxBx^{-1}| \text{ (cosets are of the same size)} \\
&= |A(xBx^{-1})| \text{ (association)} \\
&= \frac{|A||xBx^{-1}|}{|A \cap xBx^{-1}|} \\
&= \frac{|A||B|}{|A \cap xBx^{-1}|}
\end{aligned}$$

Question 6. If $H < G$ s.t. the product of two right cosets of H in G is again a right coset of H in G, then $H \lhd G$.

Question 7.

1. Let $H \lhd G$ and $N \lhd G$, then $H \cap N \lhd G$.
2. Let $H < G$ and $N \lhd G$, then $H \cap N \lhd H$.
3. Let $H \lhd G$ and $N \lhd G$, then $HN \lhd G$.

Question 8. If every subgroup of G is normal in G, then it is **not** necessary that G is Abelian.

Remark. A useful example is Quaternary group: $\{\pm 1, \pm i, \pm j, \pm k\}$, which has 4 non-trivial subgroups: ...

Question 9. Suppose H is the only subgroup of order $|H|$ in the finite group G, then $H \lhd G$.

Question 10. For dihedral group D_n, if n is odd, then $|Z(D_N)| = \{e\}$, otherwise is larger than $\{e\}$.

Chapter 2

Chapter 3