Mathematical analysis

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Contents

Chapter 1

Basic Topology

1.1 Set

Notation.

- $\mathbb{N}_n = \{1, 2, \cdots, n\}$
- $A^n = \{(a_1, \cdots, a_n) : a_i \in A\}$

Definition 1 (Finite; Countable)**.** Let A be a set.

- A is *finite* if $A \sim \mathbb{N}_n$ for some n.
- A is *countable* if $A \sim \mathbb{N}$ or A is finite.
- A is *infinite* if A is not finite.
- A is *uncountable* if A is not countable.
- $*A$ is *infinite* if $\exists B \subset A,\ A \sim B$

Definition 2 (Cardinality)**.** Sets A, B have the same *Cardinality* denoted $A \sim B$ if there is a bijection $f : A \to B$.

Remark. The relation $\sim$ defined in Definition 2 is equivalent.

Definition 3 (Sequence)**.** *A sequence* $\{x_n\}$ *in* X is $f : \mathbb{N} \to X$, where:

- $x_n \in X$, $n \in \mathbb{N}$
- $f(n) = x_n$

Proposition 1. Every subset of a countable set is countable.
(No uncountable set can be subset of a countable set)

Proof. Let A be a countable set,
If $E \subset A$ is finite, then countable.
If $E \subset A$ is infinite, then we construct a sequence to obtain a 1-1 correspondence between J and E. $\square$

Lemma 1.

- If $\exists f : \mathbb{N} \to S$ that is subjection, then S is countable.
- If $\exists g : S \to \mathbb{N}$ that is injection, then S is countable.

Proposition 2. Let $\{E_n : n \in \mathbb{N}\}$ be a countable collection of countable sets.
Then
$$\bigcup_{n=1}^{\infty} E_n$$
is countable.

Proposition 3. Let A be a countable set, then A^n is countable.

Corollary. The set of all rational numbers is countable.

Proposition 4. Let A be the set of all sequence $\{x_n\}$ in $\{0, 1\}$, then A is uncountable.

Corollary.

- $\mathcal{P}(\mathbb{N})$ is uncountable.
- $\mathbb{R}$ is uncountable.

1.2 Metric space

Definition 4 (Metric space; Points; Distance)**.** Let X be a set. X is a metric space if:
$\exists d : X \times X \to \mathbb{R}$ such that $\forall p, q \in X$:

- $d(p, q) > 0$ if $p \neq q$;
  $d(p, p) = 0$
- $d(p, q) = d(q, p)$
- $d(p, q) \leq d(p, r) + d(r, q)$, $\forall r \in X$

where p, q are called *points*, and d is called a *distence function*.

Remark. Every subset of a metric space is still a metric space.

Example. The distance function on $\mathbb{R}^k$ is defined by

$$d(x, y) = |x - y| \qquad (x, y \in \mathbb{R}^k)$$

Definition 5 (Segment; Interval; k-cell)**.**

- *segment* $(a, b) := \{x : a < x < b\}$
- *interval* $[a, b] := \{x : a \leq x \leq b\}$
- Let $a_i < b_i$ for $i = 1, \cdots, k$. A *k-cell* is a set $\{(x_1, \cdots, x_k) \in \mathbb{R}^k : a_i \leq x_i \leq b_i\}$

Definition 6 (Ball)**.** Let (X, d) be a metric space.
A *open ball centered at* $\mathbf{x} \in X$ *with radius* $r > 0$ is a set $\{\mathbf{y} \in X : d(\mathbf{y} - \mathbf{x}) < r\}$. A *closed ball* contains the open ball and the boundary where $d(\mathbf{y} - \mathbf{x}) = r$.

Definition 7 (Convex)**.** A set $E \subseteq \mathbb{R}^k$ *convex* if

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in E$$

for all $\mathbf{x} \in E, \mathbf{y} \in E, 0 < \lambda < 1$.

Remark.
- Balls and k-cells are convex.

Geometrically, E is convex means for any two points in E, the line segment connects them is in E.

Definition 8. Let X be a metric space. Let $p, q \in X$, $E \subseteq X$.

- A *neighborhood* of a point p is an open ball centered at p with radius r.
- A point p is a *limit point* of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.
- If $p \in E$ and p is not a limit point of E, then p is called an *isolated point of* E.
- E is *closed* if every limit point of E is a point of E.
- A point p is an *interior* point of E if there is a neighborhood N of p such that $N \subseteq E$.
- E is *open* if every point of E is an interior point of E.

- The *complement* of E (denoted by E^c) is $\{p \in X : p \notin E\}$.
- E is *perfect* if E is closed and if every point of E is a limit point of E.
- E is *bounded* if there is a real number b, and a point $q \in X$ such that $d(p,q) < b$ for all $p \in E$.
- E is *dense* in X if every point of X is a limit point of E, or a point of E, or both.

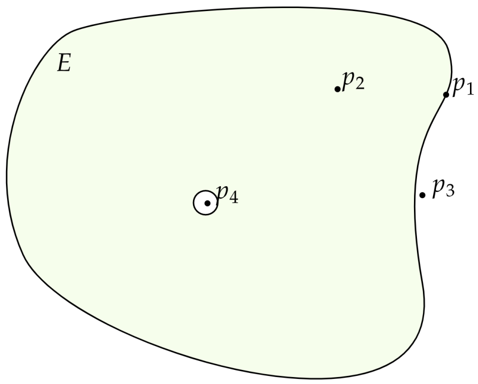

Figure 1.1: p_1, p_2 are limit points; p_3, p_4 are isolated points. Also, $p_1 \in E$ and p_1 is not interior point of E so E is not open. But E is closed.

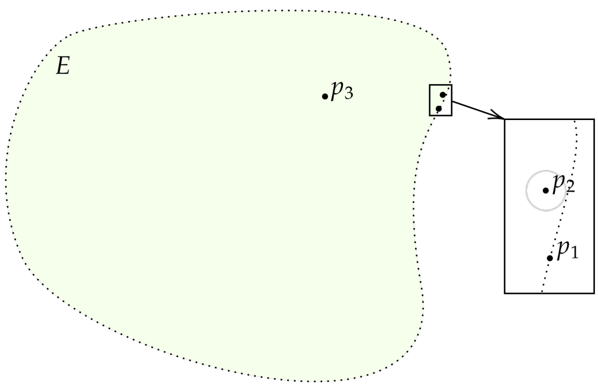

Figure 1.2: p_1 is a limit point but not in E; p_2 is an interior point; all three are limit points. E is open but not closed.

Example. Let the following be subsets of $\mathbb{R}^2$.

1. $\{z : z \in \mathbb{C}, |z| < 1\}$

2. $\{z : z \in \mathbb{C}, |z| \leq 1\}$

3. A finite set.

4. $\mathbb{Z}$

5. $\{\frac{1}{n} : n \in \mathbb{N}\}$

6. $\mathbb{C}$ or $\mathbb{R}^2$

7. The segment (a, b)

Example	**Closed**	**Open**	**Perfect**	**Bounded**
1	No	Yes	No	Yes
2	Yes	No	Yes	Yes
3	Yes	No	No	Yes
4	Yes	No	No	No
5	No	No	No	Yes
6	Yes	Yes	Yes	No
7	No	†	No	Yes

† Yes if $(a, b) \subset \mathbb{R}$, No if $(a, b) \subset \mathbb{R}^2$

Definition 9 (Relatively Open)**.** Let $E \subset Y \subset X$, where (X, d) is a metric space. Then E is *open relative to* Y if E is an open set of the metric space (Y, d).

Remark. The segment (a, b) is open relative to $\mathbb{R}$ but not $\mathbb{R}^2$.

Proposition 5.

1. Every neighborhood is an open set.

2. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

    2.1. A finite point set has no limit points.

3. A set E is open if and only if its complement is closed.

Proof.

1. .

2. .

3. If E is open, assume $\exists x \in E^c$, such that x is a limit point of E^c and $x \notin E^c$.
Then $x \in E$ and x is an interior point in E, implies there exists a neighborhood N of x, such that $N \subseteq E$, implies $N \nsubseteq E^c$, implies that x is not a limit point of E^c, which contradicts to the assumption. E^c is closed.

If E^c is closed, $\forall x \in E$, $x \notin E^c$, and x is not a limit point of E^c, implies there exists a neighbor N of x such that $N \not\subseteq E^c$, implies $N \subseteq E$. E is open.

□

Proposition 6.

1. For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.
2. For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.
3. For any finite collection $\{G_1, \cdots, G_n\}$ of open sets, $\bigcap_{i=1}^n G_i$ is open.
4. For any finite collection $\{F_1, \cdots, F_n\}$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

Proof.

1. Trivial.
2. Follow from 1, apply De Morgen's Law and **Proposition 5**-3.
3. Trivial.
4. Follow from 3 as in 2.

□

Definition 10 (Closure)**.** Let X be a metric space.
Let $E \subset X$, and E' be the set of all limit points of E, then the *closure* of E is $\bar{E} = E \cup E'$.

Proposition 7. Let X be a metric space and $E \subset X$, then

1. $\bar{E}$ is closed.
2. $E = \bar{E} \Leftrightarrow E$ is closed.
3. $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$. ($\bar{E}$ is the smallest closed set containing E)

Proof.

1. Let $x \in \bar{E}^c = E^c \cap E'^c$. Since $x \notin E$ and is not a limit point of E, so there exists a neighbor N of x, such that $N \cap E = \varnothing$, implies $N \cap E' = \varnothing$ (check this one using contradiction), implies $N \subset E^c \cap E'^c$. $\bar{E}^c$ is open.
2. Trivial.
3. Let $F \subset X$ be closed and $E \subset F$, then $F' \subset F$. Since $E' \subset F'$, so $E' \subset F \Rightarrow \bar{E} = E \cup E' \subset F$.

□

Proposition 8. Let E be a nonempty subset of $\mathbb{R}$ bounded above.
Then $\sup(E) \in \bar{E}$, and (E is closed) $\Rightarrow$ ($\sup(E) \in E$)

Proposition 9. Let $E \subset Y \subset X$ where (X, d) is a metric space.
E is open relative to Y if and only if $\exists G$, s.t. G is open relative to X and $E = G \cap Y$.

Proof. TODO □

1.2.1 Compactness in Metric Space

Definition 11 (Open cover)**.** Let X be a metric space.
An *open cover of* $E \subseteq X$ is $\{G_\alpha\}$, where G_α is an open subset of X, and $E \subseteq \bigcup_\alpha G_\alpha$.

Definition 12 (Compact)**.** Let X be a metric space, and K a subset of X.
K is *compact* if every open cover of K contains a finite subcover that covers K.

Proposition 10. Suppose $K \subseteq Y \subseteq X$. Then K is compact relative to Y if and only if K is compact relative to X.