Eigenvalue Equation in 2-Layer Wave Guide with a Curved Interface

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1 Problem Setup

The original Helmholtz equation is

$$\rho\left(\frac{1}{\rho}\psi_z\right)_z + \psi_{xx} + \kappa(z)^2\psi = 0 \tag{1}$$

For a stratified wave guide with a curved interface, we have:

$$\kappa(z) = \begin{cases} \kappa_1 & \text{if } z < h(x) \\ \kappa_2 & \text{if } z > h(x) \end{cases}, \quad \rho = \begin{cases} \rho_1 & \text{if } z < h(x) \\ \rho_2 & \text{if } z > h(x) \end{cases}$$

An orthogonal transformation is applied, and in the new coordinate system $\hat{z}, \hat{x}$, the interface is flattened:

$$\kappa(\hat{z}) = \begin{cases} \kappa_1 & \text{if } \hat{z} < 1 \\ \kappa_2 & \text{if } \hat{z} > 1 \end{cases}, \quad \rho = \begin{cases} \rho_1 & \text{if } \hat{z} < 1 \\ \rho_2 & \text{if } \hat{z} > 1 \end{cases}$$

The new stratified waveguide along the $\hat{z}$-axis is shown in Figure 1, where the coordinate is divided into three layers: $[0,1], [1,D], [D,D_1]$. The layers are connected at interfaces $\hat{z} = 1, \hat{z} = D$. In the third layer, z is unchanged, so $\hat{z} = z$.

Each layer is denoted using superscripts ℓ, where $\ell = 1, 2, 3$ represents the first, second, and third layer transformations, respectively.

Next, we transform the equation to remove the $\psi_{\hat{x}}$ term. Let:

$$\psi(\hat{x}, \hat{z}) = w(\hat{x}, \hat{z})\phi(\hat{x}, \hat{z}) \tag{2}$$

Then, in each layer, the transformed Helmholtz equation becomes:

$$\phi_{\hat{x}\hat{x}} + \alpha(\hat{x}, \hat{z})\phi_{\hat{z}\hat{z}} + \beta(\hat{x}, \hat{z})\phi_{\hat{z}} + \eta(\hat{x}, \hat{z})\phi = 0, \quad \hat{z} \neq 1, D$$

Recall that $\hat{z} = z$ in the third layer, and since the refractive index is constant along the original z-axis in this layer, it remains unchanged along the new axis $\hat{z}$. This property is key to applying the Perfectly Matched Layer (PML).

The PML is applied to truncate the open domain along $\hat{z}$, absorbing wave energy near the boundaries to minimize reflections. The complex coordinate stretching is given by:

$$\bar{z} = \hat{z} + i\int_0^{\hat{z}} \sigma(s)ds, \quad \sigma(s) = \begin{cases} 0 & \text{if } s < H \\ > 0 & \text{if } s \in [H, D_1] \end{cases} \tag{3}$$

leading to the transformation:

$$\frac{d}{d\hat{z}} \longrightarrow \frac{1}{1 + i\sigma(\hat{z})}\frac{d}{d\hat{z}}$$

The PML is applied only in the third layer, where $H > D$.

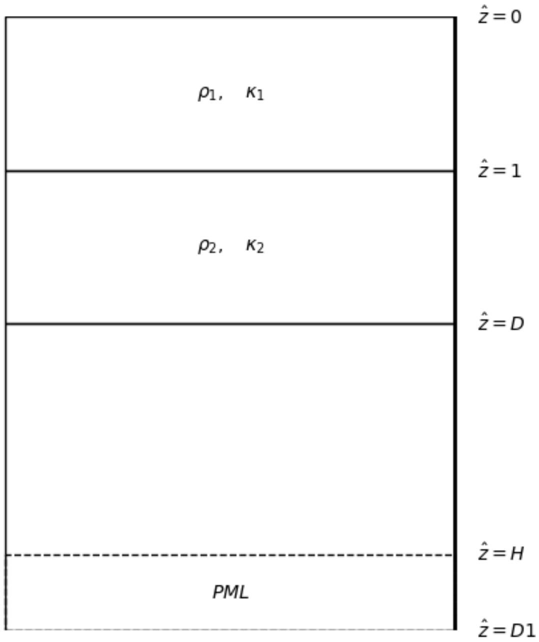

Figure 1: 3-layer coordinate transformed model
The new coordinate is divided into three layers along the $\hat{z}$ axis, each spanning $[0,1], [1,D], [D,D_1]$.

After PML stretching, the eigenvalue problem for the transformed Helmholtz equation (3) becomes:

$$\begin{cases} \alpha \dfrac{d^2\phi}{d\bar{z}^2} + \beta \dfrac{d\phi}{d\bar{z}} + \eta\phi = \lambda\phi, \quad \bar{z} \neq 1, D \\ (w\phi)\Big|_{\hat{z}=1^-} = (w\phi)\Big|_{\hat{z}=1^+} \\ \dfrac{1}{\rho_1} w \left\{ \dfrac{1}{2}\left[h''(x) - 2\dfrac{h'(x)^2}{h(x)}\right]\phi - \dfrac{1+h'(x)^2}{h(x)}\phi_{\bar{z}} \right\}\Bigg|_{\hat{z}=1^-} \\ = \dfrac{1}{\rho_2} w \left\{ \dfrac{1}{2}\left[h''(x) + 2\dfrac{h'(x)^2}{D-h(x)}\right]\phi - \dfrac{D-1}{D-h(x)}\left[1+h'(x)^2\right]\phi_{\bar{z}} \right\}\Bigg|_{\hat{z}=1^+}, \\ (w\phi)\Big|_{\hat{z}=D^-} = (w\phi)\Big|_{\hat{z}=D^+} \\ w\left(\dfrac{1-D}{h(x)-D}\phi_{\bar{z}}\right)\Bigg|_{\hat{z}=D^-} = w\phi_{\bar{z}}\Big|_{\hat{z}=D^+}, \\ \phi|_{\hat{z}=0} = 0, \quad \phi|_{\hat{z}=D} = 0. \end{cases} \tag{4}$$

To convert this eigenvalue problem to standard form, we introduce a new function $\hat{\phi}$ and define:

$$\phi(\bar{z}) = \hat{\phi}(\bar{z}) \cdot e^{\int_1^{\bar{z}} \mu(s)\, ds} \tag{5}$$

Substituting this into the first equation of (4) and canceling $e^{\int_1^{\bar{z}} \mu(s)\, ds}$, we get:

$$\hat{\phi}(\bar{z})\left[\alpha(\bar{z})\mu'(\bar{z}) + \alpha(\bar{z})\mu(\bar{z})^2 + \beta(\bar{z})\mu(\bar{z}) + \eta(\bar{z})\right] + \frac{d\hat{\phi}(\bar{z})}{d\bar{z}}\left(2\alpha(\bar{z})\mu(\bar{z}) + \beta(\bar{z})\right) + \alpha(\bar{z})\frac{d^2\hat{\phi}(\bar{z})}{d\bar{z}^2} = \lambda\hat{\phi}(\bar{z}) \tag{6}$$

To eliminate terms with $\frac{d\hat{\phi}(\bar{z})}{d\bar{z}}$, we set:

$$2\alpha(\bar{z})\mu(\bar{z}) + \beta(\bar{z}) = 0$$

yielding:

$$\mu(\bar{z}) = -\frac{\beta(\bar{z})}{2\alpha(\bar{z})} \tag{7}$$

Substituting $\mu(\bar{z})$ from (7) into (6), the eigenvalue equation for $\hat{\phi}(\bar{z})$ becomes:

$$\left(\frac{\frac{1}{2}(-\beta'(\bar{z})\alpha(\bar{z})+\beta(\bar{z})\alpha'(\bar{z}))-\frac{1}{4}\beta^2(\bar{z})}{\alpha(\bar{z})}+\eta(\bar{z})\right)\hat{\phi}(\bar{z})+\alpha(\bar{z})\frac{d^2\hat{\phi}(\bar{z})}{d\bar{z}^2}=\lambda\hat{\phi}(\bar{z}) \tag{8}$$

We denote this as $p(\bar{z})\hat{\phi}+\alpha(\bar{z})\hat{\phi}_{\bar{z}\bar{z}}=\lambda\hat{\phi}$

Together with the interface conditions for $\hat{\phi}$, which can be derived by substituting (5) into the boundary conditions in (4), the complete eigenvalue problem for the new function $\hat{\phi}$ is:

$$\begin{cases} p(\bar{z})\hat{\phi}+\alpha(\bar{z})\hat{\phi}_{\bar{z}\bar{z}}=\lambda\hat{\phi} \\ \left(w\hat{\phi}\right)\Big|_{\hat{z}=1^-}=\left(w\hat{\phi}\right)\Big|_{\hat{z}=1^+} \\ \dfrac{1}{\rho_1}w\left\{\left[\dfrac{1}{2}h''(x)-\dfrac{h'(x)^2}{h(x)}-\dfrac{1+h'(x)^2}{h(x)}\mu\right]\hat{\phi}-\dfrac{1+h'(x)^2}{h(x)}\hat{\phi}_{\bar{z}}\right\}\Big|_{\hat{z}=1^-} \\ =\dfrac{1}{\rho_2}w\left\{\left[\dfrac{1}{2}h''(x)+\dfrac{h'(x)^2}{D-h(x)}-\dfrac{D-1}{D-h(x)}\left(1+h'(x)^2\right)\mu\right]\hat{\phi}-\dfrac{D-1}{D-h(x)}\left(1+h'(x)^2\right)\hat{\phi}_{\bar{z}}\right\}\Big|_{\hat{z}=1^+}, \\ \left(w\hat{\phi}\right)\Big|_{\hat{z}=D^-}=\left(w\hat{\phi}\right)\Big|_{\hat{z}=D^+}, \\ w\left(\dfrac{1-D}{h(x)-D}\hat{\phi}_{\bar{z}}\right)\Big|_{\hat{z}=D^-}=w\hat{\phi}_{\bar{z}}\Big|_{\hat{z}=D^+}, \\ \hat{\phi}|_{\hat{z}=0}=0, \quad \hat{\phi}|_{\hat{z}=D_1}=0. \end{cases} \tag{9}$$

Note that the terms involving μ are eliminated at $\hat{z}=D$ because $\mu=0$ when $\hat{z}\geq D$.

The interface conditions can be written in matrix form. Denote:

$$\begin{aligned} T_{1^-}&=\left[\begin{matrix} w & 0 \\ \frac{1}{\rho_1}w\left(\frac{1}{2}h''(x)-\frac{h'(x)^2}{h(x)}-\frac{1+h'(x)^2}{h(x)}\mu\right) & -\frac{1}{\rho_1}w\frac{1+h'(x)^2}{h(x)} \end{matrix}\right]\Bigg|_{\hat{z}=1^-} \\ T_{1^+}&=\left[\begin{matrix} w & 0 \\ \frac{1}{\rho_2}w\left(\frac{1}{2}h''(x)+\frac{h'(x)^2}{D-h(x)}-\frac{D-1}{D-h(x)}\left(1+h'(x)^2\right)\mu\right) & -\frac{1}{\rho_2}w\frac{D-1}{D-h(x)}\left(1+h'(x)^2\right) \end{matrix}\right]\Bigg|_{\hat{z}=1^+} \end{aligned} \tag{10}$$

Thus, the interface condition at $\hat{z}=1$ is:

$$T_{1^-}\begin{bmatrix}\hat{\phi}\\ \hat{\phi}'\end{bmatrix}\Bigg|_{\hat{z}=1^-}=T_{1^+}\begin{bmatrix}\hat{\phi}\\ \hat{\phi}'\end{bmatrix}\Bigg|_{\hat{z}=1^+} \tag{11}$$

Similarly, for the second interface at $\hat{z}=D$, define:

$$T_{D^-}=\begin{bmatrix} w & 0 \\ 0 & \frac{(D-1)w}{D-h(x)}\end{bmatrix}\Bigg|_{\hat{z}=D^-}, \quad T_{D^+}=\begin{bmatrix} w & 0 \\ 0 & w\end{bmatrix}\Bigg|_{\bar{z}=D^+} \tag{12}$$

And the interface condition can be expressed as:

$$T_{D^-}\begin{bmatrix}\hat{\phi}\\ \hat{\phi}'\end{bmatrix}\Bigg|_{\hat{z}=D^-}=T_{D^+}\begin{bmatrix}\hat{\phi}\\ \hat{\phi}'\end{bmatrix}\Bigg|_{\hat{z}=D^+} \tag{13}$$

2 Dispersion Relation Approximation

In the first and second layers, $\hat{z}\in(0,1)$ and $\hat{z}\in(1,D_1)$, where there is no PML coordinate stretching, $\hat{z}=\bar{z}$, so we can directly use the variable $\hat{z}$ in the eigenvalue equation (8). Rearranging it gives:

$$\hat{\phi}_{\hat{z}\hat{z}}(\hat{z})+\frac{p(\hat{z})-\lambda}{\alpha(\hat{z})}\hat{\phi}(\hat{z})=0,$$

which we denote as:

$$\hat{\phi}_{\hat{z}\hat{z}}(\hat{z})+K(\hat{z})\hat{\phi}(\hat{z})=0, \tag{14}$$

where $K(\hat{z})=\frac{p(\hat{z})-\lambda}{\alpha(\hat{z})}$.

The function $K(\hat{z})$ can be approximated by a piecewise polynomial of degree two. We divide the interval $[0,1]$ into m_1 sub-intervals $[(j-1)l_1, jl_1]$ for $j = 1, 2, \ldots, m_1$, and the interval $[1, D]$ into m_2 sub-intervals $[1+(j-m_1-1)l_2, 1+(j-m_1)l_2]$ for $j = m_1+1, m_1+2, \ldots, m_1+m_2$. These sub-intervals are denoted as $I_j = (\hat{z}_{j-1}, \hat{z}_j)$, where:

$$\hat{z}_j = \begin{cases} jl_1, & \text{if } 1 \le j \le m_1, \\ 1+(j-m_1)l_2, & \text{if } m_1+1 \le j \le m_1+m_2. \end{cases}$$

On each sub-interval, $K(\hat{z})$ is interpolated using a polynomial of degree two with three nodes chosen at the endpoints and the midpoint.

By approximating $K(\hat{z})$, equation (14) becomes:

$$\frac{d^2 y_j}{d\hat{z}^2} + (a_j \hat{z}^2 + b_j \hat{z} + c_j) y_j = 0, \quad \hat{z} \in I_j \tag{15}$$

where y_j approximates $\hat{\phi}$ on the interval I_j, and the coefficients a_j, b_j, and c_j are defined for each sub-interval.

For the first layer:

$$\begin{cases} a_j = \dfrac{2K(t_0)}{l_1^2} - \dfrac{4K(t_1)}{l_1^2} + \dfrac{2K(t_2)}{l_1^2}, \\ b_j = \dfrac{(1-4j)K(t_0)}{l_1} + \dfrac{(8j-4)K(t_1)}{l_1} + \dfrac{(3-4j)K(t_2)}{l_1}, \\ c_j = \left(2j^2 - j\right) K(t_0) + \left(4j - 4j^2\right) K(t_1) + \left(2j^2 - 3j + 1\right) K(t_2), \\ t_0 = l_1(j-1), \quad t_1 = (j-1/2)l_1, \quad t_2 = jl_1, \\ j = 1, 2, \ldots, m_1. \end{cases} \tag{16}$$

In the second layer:

$$\begin{cases} a_j = \dfrac{2K(t_0)}{l_2^2} - \dfrac{4K(t_1)}{l_2^2} + \dfrac{2K(t_2)}{l_2^2} \\ b_j = \dfrac{K(t_0)(-4l_2 j + 4l_2 m_1 + l_2 - 4)}{l_2^2} \\ \quad + \dfrac{K(t_1)(l_2(8j - 8m_1 - 4) + 8)}{l_2^2} \\ \quad + \dfrac{K(t_2)(l_2(-4j + 4m_1 + 3) - 4)}{l_2^2}, \\ c_j = \dfrac{K(t_0)\left(l_2^2\left(2j^2 - 4jm_1 - j + 2m_1^2 + m_1\right) + l_2(4j - 4m_1 - 1) + 2\right)}{l_2^2} \\ \quad + \dfrac{4K(t_1)(-l_2(j-m_1-1)-1)(l_2(j-m_1)+1)}{l_2^2} \\ \quad + \dfrac{K(t_2)(-l_2(j-m_1-1)-1)(-2l_2 j + 2l_2 m_1 + l_2 - 2)}{l_2^2}, \\ t_0 = 1+(j-m_1-1)l_2, \quad t_1 = 1+(j-m_1-1/2)l_2, \quad t_2 = 1+(j-m_1)l_2 \\ j = m_1+1, m_1+1, \ldots, m_2+m_1 \end{cases} \tag{17}$$

The solution to equation (15) for each interval is given by confluent hypergeometric functions. Let $\{u_j(\hat{z}), v_j(\hat{z})\}$ be the fundamental solutions on the j-th interval, then:

$$y_j(\hat{z}) = A_j u_j(\hat{z}) + B_j v_j(\hat{z}), \quad \hat{z} \in I_j,$$

for $j = 1, 2, \ldots, m_1+m_2$. In the third layer, $\hat{z} \in [D, D_1]$, the eigenvalue equation (8) simplifies to:

$$\alpha^{(3)} \hat{\phi}_{\bar{z}\bar{z}} + \eta^{(3)} \hat{\phi} = \lambda \hat{\phi}, \tag{18}$$

where $\alpha^{(3)}$ and $\eta^{(3)}$ are constants, and $\beta^{(3)} = 0$. Denoting $\gamma^{(3)} = \sqrt{\frac{\eta^{(3)} - \lambda}{\alpha^{(3)}}}$, the solution to (18) is simply:

$$\hat{\phi}(\hat{z}) = C_1 e^{-i\gamma^{(3)}\bar{z}} + C_2 e^{i\gamma^{(3)}\bar{z}}, \quad \hat{z} \in [D, D_1]. \tag{19}$$

To maintain consistent notation, we can also express (19) as:

$$y_{m_1+m_2+1}(\hat{z}) = A_{m_1+m_2+1} u_{m_1+m_2+1}(\hat{z}) + B_{m_1+m_2+1} v_{m_1+m_2+1}(\hat{z}),$$

where we use $\hat{z}$ instead of $\bar{z}$ in $u_{m_1+m_2+1}$ and $v_{m_1+m_2+1}$.

In the first sub-interval, the zero boundary condition $\hat{\phi}(0) = 0$ gives the first relation:

$$A_1 u_1(0) + B_1 v_1(0) = 0.$$

Assuming the approximation of $\hat{\phi}$ is continuous across layers and has a continuous first-order derivative, the interface conditions at each $\hat{z}_j$ become:

$$\begin{bmatrix} u_{j+1}(\hat{z}_j) & v_{j+1}(\hat{z}_j) \\ u'_{j+1}(\hat{z}_j) & v'_{j+1}(\hat{z}_j) \end{bmatrix} \begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \begin{bmatrix} u_j(\hat{z}_j) & v_j(\hat{z}_j) \\ u'_j(\hat{z}_j) & v'_j(\hat{z}_j) \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix}. \tag{20}$$

From the interface condition at $\hat{z} = 1$ in (11), we have:

$$T = T_{1^+}^{-1} T_{1^-}, \quad T = \begin{bmatrix} t_{11} & 0 \\ t_{21} & t_{22} \end{bmatrix}. \tag{21}$$

Thus, the relation at the interface $\hat{z} = 1$ is:

$$\begin{bmatrix} u_{m_1+1}(1) & v_{m_1+1}(1) \\ u'_{m_1+1}(1) & v'_{m_1+1}(1) \end{bmatrix} \begin{bmatrix} A_{m_1+1} \\ B_{m_1+1} \end{bmatrix} = T \begin{bmatrix} u_{m_1}(1) & v_{m_1}(1) \\ u'_{m_1}(1) & v'_{m_1}(1) \end{bmatrix} \begin{bmatrix} A_{m_1} \\ B_{m_1} \end{bmatrix}. \tag{22}$$

Similarly, from the interface condition at $\hat{z} = D$ in (13), we denote:

$$\overline{T} = T_{D^+}^{-1} T_{D^-}, \quad \overline{T} = \begin{bmatrix} \bar{t}_{11} & 0 \\ 0 & \bar{t}_{22} \end{bmatrix}. \tag{23}$$

At the interface $\hat{z} = D$, we have the relation:

$$\begin{bmatrix} u_{m_1+m_2+1}(D) & v_{m_1+m_2+1}(D) \\ u'_{m_1+m_2+1}(D) & v'_{m_1+m_2+1}(D) \end{bmatrix} \begin{bmatrix} A_{m_1+m_2+1} \\ B_{m_1+m_2+1} \end{bmatrix} = \overline{T} \begin{bmatrix} u_{m_1+m_2}(D) & v_{m_1+m_2}(D) \\ u'_{m_1+m_2}(D) & v'_{m_1+m_2}(D) \end{bmatrix} \begin{bmatrix} A_{m_1+m_2} \\ B_{m_1+m_2} \end{bmatrix}. \tag{24}$$

Finally, the zero boundary condition at D_1 implies:

$$0 = A_{m_1+m_2+1} e^{-i\gamma^{(3)}\bar{D}_1} + B_{m_1+m_2+1} e^{i\gamma^{(3)}\bar{D}_1}, \tag{25}$$

where $\bar{D}_1 = D_1 + i\int_0^{D_1} \sigma(s)ds$.

By combining equations (20), (22), (24), and (25), we derive a linear system of A_j and B_j. This system can be solved to find the dispersion relation of λ.

Denote $R_j = \frac{A_j}{B_j}$ for $j = 1, 2, \ldots, m_1 + m_2 + 1$. We can iteratively solve for R_j as follows:

$$R_1 = -\frac{v_1(0)}{u_1(0)}, \tag{26}$$

$$R_{j+1} = \frac{R_j \left(u'_j(\hat{z}_j) v_{j+1}(\hat{z}_j) - u_j(\hat{z}_j) v'_{j+1}(\hat{z}_j) \right) - v_j(\hat{z}_j) v'_{j+1}(\hat{z}_j) + v_{j+1}(\hat{z}_j) v'_j(\hat{z}_j)}{R_j \left(u'_{j+1}(\hat{z}_j) u_j(\hat{z}_j) - u'_j(\hat{z}_j) u_{j+1}(\hat{z}_j) \right) + u'_{j+1}(\hat{z}_j) v_j(\hat{z}_j) - u_{j+1}(\hat{z}_j) v'_j(\hat{z}_j)}, \tag{27}$$

for $j = 1, \ldots, m_1 - 1$, and $j = m_1 + 1, \ldots, m_1 + m_2 - 1$. For the interval around $\hat{z} = 1$, from (22), we have:

$$\begin{aligned} R_{m_1+1} = \\ & \Big\{ R_{m_1} \left[u'_{m_1}(1) v_{m_1+1}(1) t_{22} + u_{m_1}(1) v_{m_1+1}(1) t_{21} - u_{m_1}(1) v'_{m_1+1}(1) t_{11} \right] + \ldots \\ & v_{m_1}(1) v_{m_1+1}(1) t_{21} - v_{m_1}(1) v'_{m_1+1}(1) t_{11} + v_{m_1+1}(1) v'_{m_1}(1) t_{22} \Big\} \Big/ \\ & \Big\{ R_{m_1} \left[u'_{m_1+1}(1) u_{m_1}(1) t_{11} - u'_{m_1}(1) u_{m_1+1}(1) t_{22} - u_{m_1}(1) u_{m_1+1}(1) t_{21} \right] + \ldots \\ & u'_{m_1+1}(1) v_{m_1}(1) t_{11} - u_{m_1+1}(1) v_{m_1}(1) t_{21} - u_{m_1+1}(1) v'_{m_1}(1) t_{22} \Big\} \end{aligned} \tag{28}$$

At the second interface $\hat{z} = D$, from (24), we get:

$$\begin{aligned} R_{m_1+m_2+1} = \\ & \Big\{ R_{m_1+m_2} \left[\bar{t}_{22}\, v_{m_1+m_2+1}(D)\, u'_{m_1+m_2}(D) - \bar{t}_{11}\, u_{m_1+m_2}(D)\, v'_{m_1+m_2+1}(D) \right] - \ldots \\ & \bar{t}_{11}\, v_{m_1+m_2}(D)\, v'_{m_1+m_2+1}(D) + \bar{t}_{22}\, v_{m_1+m_2+1}(D)\, v'_{m_1+m_2}(D) \Big\} \Big/ \\ & \Big\{ R_{m_1+m_2} \left[\bar{t}_{11}\, u_{m_1+m_2}(D)\, u'_{m_1+m_2+1}(D) - \bar{t}_{22}\, u_{m_1+m_2+1}(D)\, u'_{m_1+m_2}(D) \right] + \ldots \\ & \bar{t}_{11}\, v_{m_1+m_2}(D)\, u'_{m_1+m_2+1}(D) - \bar{t}_{22}\, u_{m_1+m_2+1}(D)\, v'_{m_1+m_2}(D) \Big\} \end{aligned} \tag{29}$$

The boundary condition at $\hat{z} = D_1$ gives us:

$$R_{m_1+m_2+1} = -e^{2i\gamma^{(3)}\bar{D_1}}, \tag{30}$$

which relates the final $R_{m_1+m_2+1}$ to λ.

The values of u_j and v_j depend on λ, so the dispersion relation is expressed as a function of λ. Recursively solving for $R_{m_1+m_2+1}$ through (26), (27), (28), and (29), we get the final dispersion relation:

$$g(\lambda) = R_{m_1+m_2+1} + e^{2i\gamma^{(3)}\bar{D_1}} = 0. \tag{31}$$

3 Special Case: Flat Interface

When the waveguide interface is flat, with $h(x) \equiv 1$, the orthogonal coordinate transformation reduces to an identity mapping, i.e., $(\hat{x}, \hat{z}) = (x, z)$. The equation transformation (5) becomes an identity transformation with $w \equiv 1$. Therefore, the transformed Helmholtz equation (3) reduces to:

$$\alpha = 1, \beta = 0, \eta = \kappa^2.$$

In this case, the eigenvalue problem simplifies to:

$$\begin{cases} \phi_{\bar{z}\bar{z}} + \kappa(z)^2\phi = \lambda\phi, \\ \phi|_{z=1^-} = \phi|_{z=1^+}, \\ \dfrac{1}{\rho_1}\phi_{\bar{z}}|_{z=1^-} = \dfrac{1}{\rho_2}\phi_{\bar{z}}|_{z=1^+}, \\ \phi|_{z=D^-} = \phi|_{z=D^+}, \\ \phi_{\bar{z}}|_{z=D^-} = \phi_{\bar{z}}|_{z=D^+}, \\ \phi|_{z=0} = 0, \quad \phi|_{z=D_1} = 0. \end{cases} \tag{32}$$

Given that $\beta = 0$, from the definition of $\hat{\phi}$ in (5), we have $\hat{\phi} = \phi$. Therefore, $\hat{\phi}$ satisfies the reduced eigenvalue problem (32).

Define the following parameters:

$$\gamma^{(1)} = \sqrt{\kappa_1^2 - \lambda}, \quad \gamma^{(2)} = \sqrt{\kappa_2^2 - \lambda}.$$

The two basis solutions $\{u_j, v_j\}$ in each sub-interval of the first two layers $[0, 1]$ and $[1, D_1]$ are simply:

$$u_j = e^{-i\gamma^{(\ell)}z}, \quad v_j = e^{i\gamma^{(\ell)}z}, \quad z \in I_j, \tag{33}$$

where $\ell = 1$ or 2 for the respective layers. The continuity conditions at each interface $\hat{z}_j$ give rise to the relations:

$$\begin{aligned} u_j'(\hat{z}_j) &= u_{j+1}'(\hat{z}_j), \quad u_j(\hat{z}_j) = u_{j+1}(\hat{z}_j), \\ v_j'(\hat{z}_j) &= v_{j+1}'(\hat{z}_j), \quad v_j(\hat{z}_j) = v_{j+1}(\hat{z}_j), \end{aligned} \tag{34}$$

for $j = 1, \ldots, m_1 - 1$ and $j = m_1 + 1, \ldots, m_1 + m_2 - 1$.

Thus, the recursive relations for R_j within each layer simplify to:

$$R_{j+1} = R_j, \quad j = 1, \ldots, m_1 - 1 \quad \text{and} \quad j = m_1 + 1, \ldots, m_1 + m_2 - 1. \tag{35}$$

From the zero boundary condition $\hat{\phi}(0) = 0$, we obtain:

$$R_1 = \frac{A_1}{B_1} = -1.$$

Thus, in the first layer, we have:

$$R_j \equiv -1, \quad j = 1, 2, \ldots, m_1.$$

From the interface condition in (32), the interface condition matrices simplify to:

$$T_{1^-} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho_1} \end{bmatrix}, \quad T_{1^+} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho_2} \end{bmatrix}.$$

Similarly, both T_{D^-} and T_{D^+} reduce to the identity matrix. Hence,

$$T = T_{1^+}^{-1} T_{1^-} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\rho_2}{\rho_1} \end{bmatrix}, \quad \overline{T} = T_{D^+}^{-1} T_{D^-} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The transfer relation from the first layer to the second layer, based on (28), is:

$$R_{m_1+1} = \frac{-\left[\frac{\rho_2}{\rho_1} u'_{m_1}(1) v_{m_1+1}(1) - u_{m_1}(1) v'_{m_1+1}(1)\right] - v_{m_1}(1) v'_{m_1+1}(1) + v_{m_1+1}(1) v'_{m_1}(1) \frac{\rho_2}{\rho_1}}{-\left[u'_{m_1+1}(1) u_{m_1}(1) - \frac{\rho_2}{\rho_1} u'_{m_1}(1) u_{m_1+1}(1)\right] + u'_{m_1+1}(1) v_{m_1}(1) - \frac{\rho_2}{\rho_1} u_{m_1+1}(1) v'_{m_1}(1)}. \tag{36}$$

Substituting the expressions for u_j and v_j from (33), we get:

$$R_{m_1+1} = -e^{2i\gamma^{(2)}} \frac{e^{-i\gamma^{(1)}}\left[\gamma^{(1)} \frac{\rho_2}{\rho_1} + \gamma^{(2)}\right] + e^{i\gamma^{(1)}}\left[-\gamma^{(2)} + \gamma^{(1)} \frac{\rho_2}{\rho_1}\right]}{e^{-i\gamma^{(1)}}\left[-\gamma^{(2)} + \gamma^{(1)} \frac{\rho_2}{\rho_1}\right] + e^{i\gamma^{(1)}}\left[\gamma^{(2)} + \gamma^{(1)} \frac{\rho_2}{\rho_1}\right]}. \tag{37}$$

At the second interface $\hat{z} = D$, the transfer relation simplifies since both T_{D^-} and T_{D^+} are identity matrices:

$$R_{m_1+m_2+1} = R_{m_1+m_2}.$$

Furthermore, since R_j is constant for $j = m_1+1, \ldots, m_1+m_2$, we have:

$$R_{m_1+m_2+1} = R_{m_1+m_2} = R_{m_1+1}.$$

In this case, $\gamma^{(3)} = \gamma^{(2)}$, so we substitute $\gamma^{(3)}$ with $\gamma^{(2)}$ in (37). The dispersion relation (31) is thus reduced to:

$$-e^{2i\gamma^{(2)}} \frac{e^{-i\gamma^{(1)}}\left[\gamma^{(1)} \frac{\rho_2}{\rho_1} + \gamma^{(2)}\right] + e^{i\gamma^{(1)}}\left[-\gamma^{(2)} + \gamma^{(1)} \frac{\rho_2}{\rho_1}\right]}{e^{-i\gamma^{(1)}}\left[-\gamma^{(2)} + \gamma^{(1)} \frac{\rho_2}{\rho_1}\right] + e^{i\gamma^{(1)}}\left[\gamma^{(2)} + \gamma^{(1)} \frac{\rho_2}{\rho_1}\right]} + e^{2i\gamma^{(2)}\bar{D}_1} = 0. \tag{38}$$

Rearranging and expanding $e^{i\gamma^{(1)}}$ and $e^{-i\gamma^{(1)}}$ into trigonometric functions, we obtain:

$$\frac{i\gamma^{(1)} \frac{\rho_2}{\rho_1} \cos(\gamma^{(1)}) - i\gamma^{(2)} \sin(\gamma^{(1)})}{i\gamma^{(2)} \sin(\gamma^{(1)}) + \gamma^{(1)} \frac{\rho_2}{\rho_1} \cos(\gamma^{(1)})} = e^{2i\gamma^{(2)}(\bar{D}_1 - 1)}. \tag{39}$$

Multiplying both the numerator and the denominator of the left-hand side by i, we arrive at the same dispersion relation as in [1]. This result confirms the theoretical validity of the dispersion relation we derived.

4 Asymptotic Solution

We first use a piecewise constant function to approximate $K(\hat{z})$ in each interpolated subinterval:

$$\hat{\phi}_{\hat{z}\hat{z}} + K(t_1)\phi = 0 \quad \hat{z} \in I_j \tag{40}$$

where $K(t_1)$ is defined in (16)(17). Under this simplification, the basis solution at each interval is:

$$u_j = e^{-i\gamma_j \hat{z}}, \quad v_j = e^{i\gamma_j \hat{z}} \quad \hat{z} \in I_j \tag{41}$$

$$\gamma_j = \sqrt{\frac{p(t_1) - \lambda}{\alpha(t_1)}}$$

The continuity condition within a layer becomes:

$$\begin{cases} A_{j+1} e^{-i\gamma_{j+1}\hat{z}_j} + B_{j+1} e^{i\gamma_{j+1}\hat{z}_j} = A_{j+1} e^{-i\gamma_{j+1}\hat{z}_j} + B_{j+1} e^{i\gamma_{j+1}\hat{z}_j} \\ -i\gamma_{j+1} A_{j+1} e^{-i\gamma_{j+1}\hat{z}_j} + i\gamma_{j+1} B_{j+1} e^{i\gamma_{j+1}\hat{z}_j} = -i\gamma_{j+1} A_{j+1} e^{-i\gamma_{j+1}\hat{z}_j} + i\gamma_{j+1} B_{j+1} e^{i\gamma_{j+1}\hat{z}_j} \\ j = 1, \ldots, m_1 - 1 \quad \text{and} \quad j = m_1+1, \ldots, m_1+m_2-1, \end{cases} \tag{42}$$

This can be written in matrix form, giving the recurrence relation for A_j and B_j:

$$\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \begin{bmatrix} \frac{e^{-i(\gamma_j - \gamma_{j+1})}(\gamma_j + \gamma_{j+1})}{2\gamma_{j+1}} & \frac{e^{i(\gamma_j + \gamma_{j+1})}(\gamma_{j+1} - \gamma_j)}{2\gamma_{j+1}} \\ \frac{e^{-i(\gamma_j + \gamma_{j+1})}(\gamma_{j+1} - \gamma_j)}{2\gamma_{j+1}} & \frac{e^{i(\gamma_j - \gamma_{j+1})}(\gamma_j + \gamma_{j+1})}{2\gamma_{j+1}} \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix} \tag{43}$$

Let $\hat{z}_{j+1} = \hat{z}_j + \Delta z$. Then:

$$\frac{\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} - \begin{bmatrix} A_j \\ B_j \end{bmatrix}}{\Delta z} = \frac{\begin{bmatrix} \frac{e^{-i(\gamma_j - \gamma_{j+1})}(\gamma_j+\gamma_{j+1})-2\gamma_{j+1}}{2\gamma_{j+1}} & \frac{e^{i(\gamma_j+\gamma_{j+1})}(\gamma_{j+1}-\gamma_j)}{2\gamma_{j+1}} \\ \frac{e^{-i(\gamma_j+\gamma_{j+1})}(\gamma_{j+1}-\gamma_j)}{2\gamma_{j+1}} & \frac{e^{i(\gamma_j-\gamma_{j+1})}(\gamma_j+\gamma_{j+1})-2\gamma_{j+1}}{2\gamma_{j+1}} \end{bmatrix}}{\Delta z} \begin{bmatrix} A_j \\ B_j \end{bmatrix} \tag{44}$$

When Δz approaches 0, the limit of (44) becomes:

$$\frac{d}{d\hat{z}} \begin{bmatrix} A \\ B \end{bmatrix} = \frac{\gamma'}{2\gamma} \begin{bmatrix} -1+2i\gamma\hat{z} & e^{2i\gamma\hat{z}} \\ e^{-2i\gamma\hat{z}} & -1-2i\gamma\hat{z} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \quad \hat{z} \neq 1, D \tag{45}$$

Here, we assume $p(\hat{z})$ and $\alpha(\hat{z})$ have first-order derivatives, thus γ' exists.

If $p(\hat{z})$ and $\alpha(\hat{z})$ vary gradually (when the interface $h(x)$ changes gradually), and $|\lambda|$ is large, we have $\left|\frac{\gamma'}{\gamma}\right| \approx 0$. Therefore, (45) can be approximated as:

$$\frac{d}{d\hat{z}} \begin{bmatrix} A \\ B \end{bmatrix} \approx \begin{bmatrix} i\gamma'\hat{z} & 0 \\ 0 & -i\gamma'\hat{z} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \tag{46}$$

The solution to (46) is:

$$\text{Let } U^{(\ell)} = \begin{bmatrix} i\gamma^{(\ell)\,\prime}\hat{z} & 0 \\ 0 & -i\gamma^{(\ell)\,\prime}\hat{z} \end{bmatrix} \tag{47}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=1^-} \approx e^{\int_0^1 U^{(1)}\,d\hat{z}} \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=0}, \quad \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=D^-} \approx e^{\int_1^D U^{(2)}\,d\hat{z}} \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=1^+} \tag{48}$$

We define $T_{0\to1^-} = e^{\int_0^1 U^{(1)}\,d\hat{z}}$ and $T_{1^+\to D^-} = e^{\int_1^D U^{(2)}\,d\hat{z}}$. Since $\gamma^{(3)}$ is constant, the transfer matrix from D^+ to D_1 is the identity matrix, so we omit it.

The transfer matrices across the interfaces $\hat{z} = 1$ and D are given below. We denote $\gamma_{11} = \gamma^{(1)}(1), \gamma_{21} = \gamma^{(2)}(1), \gamma_{2D} = \gamma^{(2)}(D)$:

$$\begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=1^+} = T_{1^-\to1^+} \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=1^-}, \quad \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=D^+} = T_{D^-\to D^+} \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=D^-} \tag{49}$$

The transfer matrix across $z = 1$ is:

$$T_{1^-\to1^+} = \begin{bmatrix} \frac{e^{-i(\gamma_{11}-\gamma_{21})}(\gamma_{11}t_{22}+\gamma_{21}t_{11}+it_{21})}{2\gamma_{21}} & \frac{e^{i(\gamma_{11}+\gamma_{21})}(-\gamma_{11}t_{22}+\gamma_{21}t_{11}+it_{21})}{2\gamma_{21}} \\ \frac{e^{-i(\gamma_{11}+\gamma_{21})}(-\gamma_{11}t_{22}+\gamma_{21}t_{11}-it_{21})}{2\gamma_{21}} & \frac{e^{i(\gamma_{11}-\gamma_{21})}(\gamma_{11}t_{22}+\gamma_{21}t_{11}-it_{21})}{2\gamma_{21}} \end{bmatrix} \tag{50}$$

The transfer matrix across $z = D$ is:

$$T_{D^-\to D^+} = \begin{bmatrix} \frac{e^{-i(\gamma_{2D}-\gamma^{(3)})}(\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} & \frac{e^{i(\gamma_{2D}+\gamma^{(3)})}(-\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} \\ \frac{e^{-i(\gamma_{2D}+\gamma^{(3)})}(-\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} & \frac{e^{i(\gamma_{2D}-\gamma^{(3)})}(\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} \end{bmatrix} \tag{51}$$

The values of t_{ij} and $\bar{t}_{ij}$ are defined in (21)(23).

When $|\lambda|$ is large, the matrix in (50) can be approximated as:

$$T_{1^-\to1^+} \approx \begin{bmatrix} \frac{e^{-i(\gamma_{11}-\gamma_{21})}(\gamma_{11}t_{22}+\gamma_{21}t_{11})}{2\gamma_{21}} & \frac{e^{i(\gamma_{11}+\gamma_{21})}(-\gamma_{11}t_{22}+\gamma_{21}t_{11})}{2\gamma_{21}} \\ \frac{e^{-i(\gamma_{11}+\gamma_{21})}(-\gamma_{11}t_{22}+\gamma_{21}t_{11})}{2\gamma_{21}} & \frac{e^{i(\gamma_{11}-\gamma_{21})}(\gamma_{11}t_{22}+\gamma_{21}t_{11})}{2\gamma_{21}} \end{bmatrix} \tag{52}$$

Additionally, when the derivatives of the interface $h(x)$ are small, we can show that $\gamma_{2D}\bar{t}_{22} \approx \gamma^{(3)}\bar{t}_{11}$, as demonstrated in Appendix (A). Therefore, the matrix in (51) can be approximated as:

$$T_{D^-\to D^+} \approx \begin{bmatrix} \frac{e^{-i(\gamma_{2D}-\gamma^{(3)})}(\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} & 0 \\ 0 & \frac{e^{i(\gamma_{2D}-\gamma^{(3)})}(\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} \end{bmatrix} \tag{53}$$

From the boundary condition $\hat{\phi}(0) = 0$, we set $A(0) = 1$ and $B(0) = -1$. The other boundary condition $\hat{\phi}(\bar{D}_1) = 0$ gives $A(D^+)e^{-i\gamma^{(3)}\bar{D}_1} + B(D^+)e^{i\gamma^{(3)}\bar{D}_1} = 0$. Thus, the dispersion relation is derived:

$$\begin{bmatrix} e^{-i\gamma^{(3)}\bar{D}_1}, & e^{i\gamma^{(3)}\bar{D}_1} \end{bmatrix} T_{D^-\to D^+} T_{1^+\to D^-} T_{1^-\to1^+} T_{0\to1^-} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \tag{54}$$

Substituting the approximated transfer matrices (48), (52), and (53), and expanding the terms, we obtain the approximated dispersion relation:

$$\frac{\gamma_{21}t_{11}\left(-1+e^{2i\int_0^1\gamma^{(1)}(z)\,dz}\right)-\gamma_{11}t_{22}\left(1+e^{2i\int_0^1\gamma^{(1)}(z)\,dz}\right)}{\gamma_{11}t_{22}\left(1+e^{2i\int_0^1\gamma^{(1)}(z)\,dz}\right)+\gamma_{21}t_{11}\left(-1+e^{2i\int_0^1\gamma^{(1)}(z)\,dz}\right)}+e^{2i(\gamma^{(3)}(\hat{D}_1-D)+\int_1^D\gamma^{(2)}(z)\,dz)}\approx 0 \tag{55}$$

4.1 Leaky Mode

For $|e^{2i\gamma^{(3)}(\bar{D}_1-D)}|\approx 0$, the dispersion relation (55) simplifies to:

$$\frac{\gamma_{21}t_{11}+\gamma_{11}t_{22}}{\gamma_{21}t_{11}-\gamma_{11}t_{22}}\approx e^{2i\int_0^1\gamma^{(1)}(z)\,dz} \tag{56}$$

Taking the logarithm on both sides, we get:

$$\log\left(\frac{\gamma_{21}t_{11}+\gamma_{11}t_{22}}{\gamma_{21}t_{11}-\gamma_{11}t_{22}}\right)+2m\pi i\approx 2i\int_0^1\gamma^{(1)}(z)\,dz,\quad m\in\mathbb{Z} \tag{57}$$

Assuming $\int_0^1\gamma^{(1)}(z)\,dz=\gamma^{(1)}(\hat{z}_*^{(1)})$, where $\hat{z}_*^{(1)}$ can be found by setting $\lambda=0$ and minimizing $\left|\int_0^1\sqrt{\frac{p^{(1)}(z)}{\alpha^{(1)}(z)}}\,dz-\sqrt{\frac{p^{(1)}(\hat{z})}{\alpha^{(1)}(\hat{z})}}\right|$. We define $\gamma_{1*}=\gamma^{(1)}(\hat{z}_*^{(1)})$.

To express γ_{21} and γ_{11} in terms of γ_{1*}, we define:

$$\sigma_{21}=\sqrt{\frac{\alpha^{(1)}(\hat{z}_*^{(1)})}{\alpha^{(2)}(1)}},\quad \sigma_{11}=\sqrt{\frac{\alpha^{(1)}(\hat{z}_*^{(1)})}{\alpha^{(1)}(1)}}$$

$$\delta_{21}=\frac{p^{(1)}(\hat{z}_*^{(1)})-p^{(2)}(1)}{\alpha(\hat{z}_*^{(1)})},\quad \delta_{11}=\frac{p^{(1)}(\hat{z}_*^{(1)})-p^{(1)}(1)}{\alpha^{(1)}(\hat{z}_*^{(1)})}$$

Then, we have:

$$\gamma_{21}=\sigma_{21}\gamma_{1*}\sqrt{1-\frac{\delta_{21}}{(\gamma_{1*})^2}},\quad \gamma_{11}=\sigma_{11}\gamma_{1*}\sqrt{1-\frac{\delta_{11}}{(\gamma_{1*})^2}}$$

Expanding them into inverse power series of γ_{1*} and substituting into (57), the left-hand side can be expanded as:

$$2i\gamma_{1*}\approx A_0+\frac{A_2}{(\gamma_{1*})^2}+\frac{A_4}{(\gamma_{1*})^4}+\dots \tag{58}$$

where:

$$\begin{aligned}
A_0&=2im\pi+\log\left(\frac{\sigma_{21}t_{11}+\sigma_{11}t_{22}}{\sigma_{21}t_{11}-\sigma_{11}t_{22}}\right)\\
A_2&=\frac{(-\delta_{11}+\delta_{21})\sigma_{11}\sigma_{21}t_{11}t_{22}}{(\sigma_{21}t_{11}-\sigma_{11}t_{22})(\sigma_{21}t_{11}+\sigma_{11}t_{22})}\\
A_4&=\frac{(\delta_{11}-\delta_{21})\sigma_{11}\sigma_{21}t_{11}t_{22}\left(-(\delta_{11}+3\delta_{21})\sigma_{21}^2t_{11}^2+(3\delta_{11}+\delta_{21})\sigma_{11}^2t_{22}^2\right)}{4\left(\sigma_{21}^2t_{11}^2-\sigma_{11}^2t_{22}^2\right)^2}
\end{aligned}$$

The asymptotic solutions for different orders $\tilde{\gamma}_{1*}^n$ are:

$$\begin{aligned}
\tilde{\gamma}_{1*}^0&=A_0/2i\\
\tilde{\gamma}_{1*}^2&=\left(A_0+\frac{A_2}{(\tilde{\gamma}_{1*}^0)^2}\right)/2i\\
\tilde{\gamma}_{1*}^4&=\left(A_0+\frac{A_2}{(\tilde{\gamma}_{1*}^2)^2}+\frac{A_4}{(\tilde{\gamma}_{1*}^2)^4}\right)/2i
\end{aligned} \tag{59}$$

The eigenvalue approximations $\tilde{\lambda}^i$ can be solved from $\tilde{\gamma}_{1*}^n$ by:

$$\tilde{\lambda}_n=p^{(1)}(\hat{z}_*^{(1)})-\alpha^{(1)}(\hat{z}_*^{(1)})\,(\tilde{\gamma}_{1*}^n)^2,\quad n=0,2,4$$

4.2 Berenger Mode

For Berenger modes, where $e^{-2i\gamma^{(3)}} \approx 0$, the dispersion relation (55) reduces to:

$$-\frac{\gamma_{21}t_{11} - \gamma_{11}t_{22}}{\gamma_{21}t_{11} + \gamma_{11}t_{22}} \approx e^{2i(\gamma^{(3)}(\hat{D}_1 - D) + \int_1^D \gamma^{(2)}(z)\,dz)} \tag{60}$$

Similar to the discussion of Leaky modes, we assume $\int_1^D \gamma^{(2)}(z)\,dz = (D-1)\gamma(\hat{z}_*^{(2)})$, where $\gamma_{2*} = \gamma(\hat{z}_*^{(2)})$. Expanding $\gamma_{21}, \gamma_{11}, \gamma_{2*}$ into inverse power series of $\gamma^{(3)}$, we proceed by defining the following parameters:

$$\tau_{21} = \sqrt{\frac{\alpha_{(3)}}{\alpha^{(2)}(1)}}, \quad \tau_{11} = \sqrt{\frac{\alpha_{(3)}}{\alpha^{(1)}(1)}}, \quad \tau_{2*} = \sqrt{\frac{\alpha_{(3)}}{\alpha(\hat{z}_*^{(2)})}}$$

$$\epsilon_{21} = \frac{p^{(3)} - p^{(2)}(1)}{\alpha_{(3)}}, \quad \epsilon_{11} = \frac{p^{(3)} - p^{(1)}(1)}{\alpha_{(3)}}, \quad \epsilon_{2*} = \frac{p^{(3)} - p(\hat{z}_*^{(2)})}{\alpha_{(3)}}$$

In particular, as analyzed in Appendix (A), ϵ_{2*} is negligible when the derivatives of the interface function $h(x)$ are small. Therefore, the zero-order expansion for γ_{2*} is sufficient, giving $\gamma_{2*} \approx \tau_{2*}\gamma^{(3)}$.

After taking the logarithm and expanding into an inverse power series, the dispersion relation (60) becomes:

$$2i\gamma^{(3)}\left[(\hat{D}_1 - D) + \tau_{2*}(D-1)\right] \approx B_0 + \frac{B_2}{\left(\gamma^{(3)}\right)^2} + \frac{B_4}{\left(\gamma^{(3)}\right)^4} + \dots \qquad m \in \mathbb{Z} \tag{61}$$

where:

$$B_0 = 2im\pi + \log\left(\frac{t_{22}\tau_{11} - t_{11}\tau_{21}}{t_{22}\tau_{11} + t_{11}\tau_{21}}\right)$$
$$B_2 = \frac{(-\epsilon_{11} + \epsilon_{21})t_{11}t_{22}\tau_{11}\tau_{21}}{(t_{22}\tau_{11} - t_{11}\tau_{21})(t_{22}\tau_{11} + t_{11}\tau_{21})}$$
$$B_4 = \frac{(\epsilon_{11} - \epsilon_{21})t_{11}t_{22}\tau_{11}\tau_{21}\left(-(3\epsilon_{11} + \epsilon_{21})t_{22}^2\tau_{11}^2 + (\epsilon_{11} + 3\epsilon_{21})t_{11}^2\tau_{21}^2\right)}{4\left(t_{22}^2\tau_{11}^2 - t_{11}^2\tau_{21}^2\right)^2}$$

The asymptotic solutions for different orders are:

$$\tilde{\gamma}^{(3)\,0} = \frac{B_0}{2i\left[(\hat{D}_1 - D) + \tau_{2*}(D-1)\right]}$$
$$\tilde{\gamma}^{(3)\,2} = \left(B_0 + \frac{B_2}{\left(\tilde{\gamma}^{(3)\,0}\right)^2}\right) \Big/ 2i\left[(\hat{D}_1 - D) + \tau_{2*}(D-1)\right]$$
$$\tilde{\gamma}^{(3)\,4} = \left(B_0 + \frac{B_2}{\left(\tilde{\gamma}^{(3)\,2}\right)^2} + \frac{B_4}{\left(\tilde{\gamma}^{(3)\,2}\right)^4}\right) \Big/ 2i\left[(\hat{D}_1 - D) + \tau_{2*}(D-1)\right] \tag{62}$$

The eigenvalue approximations $\tilde{\lambda}^i$ can be solved by:

$$\tilde{\lambda}_n = p^{(3)} - \alpha^{(3)}\left(\tilde{\gamma}^{(3)\,n}\right)^2 \qquad n = 0, 2, 4$$

5 Numerical Example

In this example, we set $D = 2$, $D_1 = 4$, and the stratified waveguide profile is given by:

$$\kappa(z) = \begin{cases} 16 & \text{if } z < h(x) \\ 14.4 & \text{if } z > h(x) \end{cases} \qquad \rho(z) = \begin{cases} 1 & \text{if } z < h(x) \\ 1.7 & \text{if } z > h(x) \end{cases}$$

We choose the curved interface as:

$$h(x) = 1 - 0.2e^{-10(x/10 - 0.5)^2}$$

and compute the eigenvalues at $x = 6$.

The Perfectly Matched Layer (PML) parameter is:

$$\sigma(\hat{z}) = \begin{cases} 0, & 0 < \hat{z} \le H \\ 5t^3/(1+t^2), & H < \hat{z} \le D_1 \end{cases}$$

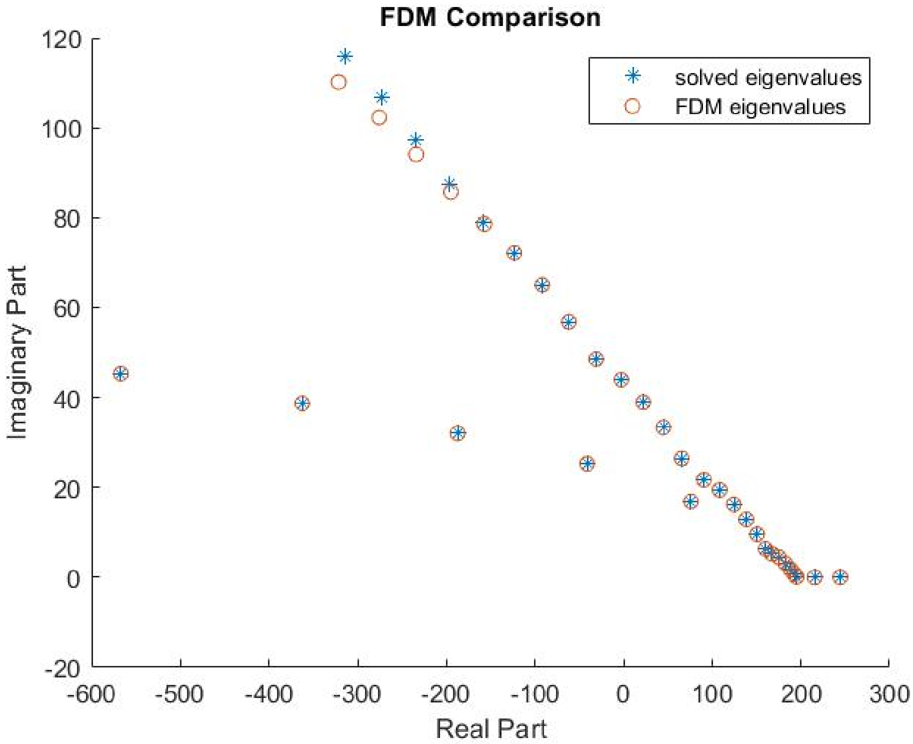

Figure 2: Comparison of our method with FDM

where $t = \frac{\hat{z}-H}{D_1-H}$ and $H = 3.5$.

We first compare our method with Finite Difference Methods (FDM). In FDM, we divide $[0, D_1]$ into 1000 equally spaced grids. In our new method, both the first and second layers $[0, 1]$ and $[1, D]$ are divided into 50 subintervals for interpolation. Müller's method is applied to solve the dispersion relation we derived (55), using eigenvalues from FDM as the initial guess.

From Figure 2, we observe that our method agrees well with FDM.

Next, we verify the asymptotic solutions derived in (59) and (62) by using them as initial guesses to solve eigenvalues using our new relation. Figure 3 shows the Leaky modes, and Figure 4 lists a few Berenger modes. The numerical experiments confirm that the asymptotic solutions serve as reasonable initial guesses.

A Appendix

We will prove the following theorem:

$$\text{If } \max_{x\in[0,L]} \{|h'(x)|, |h''(x)|\} < \delta \tag{63}$$

$$\text{then} |\gamma^{(2)}(D)\bar{t}_{22} - \gamma^{(3)}\bar{t}_{11}| = O(\delta) \tag{64}$$

where $O(\delta)$ represents a small term of the same order as δ.

This theorem is used to simplify the transfer matrix across the second interface (51).

Firstly, we will show that $\eta^{(2)}(D)$ and $\eta^{(3)}$ from (3) are close when the condition (63) is satisfied. Their expressions are:

$$\begin{aligned}\eta^{(2)}(\hat{x}, \hat{z}) = &\Bigg\{ \frac{[(D-h(x))/h'(x)]^2+(D-z)^2}{[D-h(x^*)]^2} \Bigg[\frac{h^2(x^*)}{h^2(\hat{x})} \Bigg(-\frac{3}{4}\frac{\big(h'(\hat{x})\big)^4}{h^2(\hat{x})} \\ &+\frac{\big(h'(\hat{x})\big)^4 h''(\hat{x})}{h(\hat{x})} + \frac{1}{4}\left(h''(\hat{x})\right)^2 - \frac{1}{2}h'(\hat{x})h_i'''(\hat{x}) \Bigg) + \frac{D\big(h'(x^*)\big)^2}{h(x^*)(D-h(x^*))\big[1+(h'(x^*))^2\big]^2} \\ &\times \Bigg(\frac{3}{4}\frac{\big(h'(x^*)\big)^2(D-2h(x^*))}{h(x^*)(D-h(x^*))} - \frac{h''(x^*)}{1+(h'(x^*))^2} \Bigg) \Bigg] + \frac{2\big(h'(x)\big)^2}{(D-h(x))^2} + \frac{2h''(x)}{D-h(x)} + \frac{h'''(x)}{2h'(x)} \\ &-\frac{1}{4}\frac{\big(h''(x)\big)^2}{(h'(x))^2} + \kappa^2(x,z) \Bigg\} \frac{(h(\hat{x}))^2(D-h(x^*))^2\big(h'(x)\big)^2}{(h(x^*))^2(h'(\hat{x}))^2\big[(D-h(x))^2+(h'(x))^2(D-z)^2\big]}.\end{aligned} \tag{65}$$

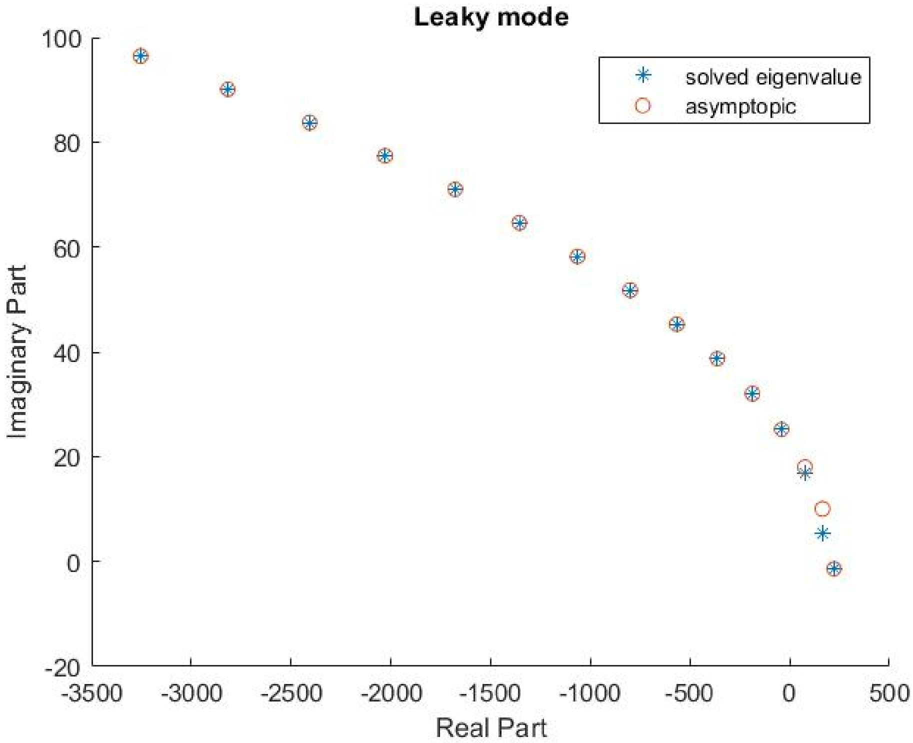

Figure 3: Leaky modes

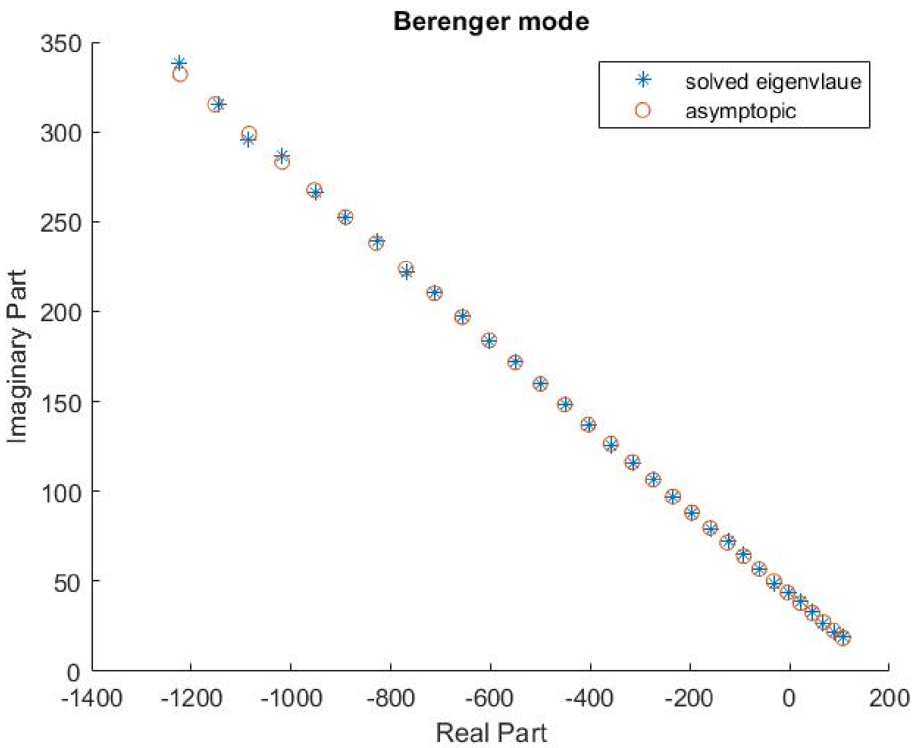

Figure 4: Berenger modes

$$
\begin{aligned}
\eta_{(3)}(\hat{x},\hat{z}) = &\tfrac{h''(x)}{h(\hat{x})} - \tfrac{3}{4}\left[\tfrac{h'(\hat{x})}{h(\hat{x})}\right]^2 + \tfrac{1}{4}\left[\tfrac{h''(\hat{x})}{h'(\hat{x})}\right]^2 - \tfrac{1}{2}\tfrac{h'''(\hat{x})}{h'(\hat{x})} \\
&+ \left\{\tfrac{h'(x^*)h(\hat{x})}{\left[1+(h'(x^*))^2\right]h'(\hat{x})h(x^*)}\right\}^2 \times \tfrac{D}{h(x^*)[D-h(x^*)]}\left\{\tfrac{3}{4}\tfrac{[D-2h(x^*)]\left[h'(x^*)\right]^2}{h(x^*)[D-h(x^*)]}\right. \\
&\left.-\tfrac{h''(x^*)}{1+[h'(x^*)]^2}\right\} + \left\{\tfrac{h'(x)h(\hat{x})[D-h(x^*)]}{h'(\hat{x})h(x^*)[D-h(x)]}\right\}^2 \times \left\{\kappa^2(x,z) + \tfrac{h''(x)}{2[D-h(x)]}\right. \\
&\left.+\tfrac{2h'''(x)h'(x)-\left[h''(x)\right]^2}{4[h'(x)]^2} + \tfrac{2h''(x)[D-h(x)]+3\left[h'(x)\right]^2}{4[D-h(x)]^2}\right\}.
\end{aligned}
\tag{66}
$$

When the condition (63) is satisfied, $\hat{x} \approx x \approx x^*$, and so we assume they are the same x. At $\hat{z} = D$, the expressions for $\eta^{(2)}(D)$ and $\eta^{(3)}$ can be simplified:

$$
\eta^{(2)}(D) \approx -\frac{3}{4}\frac{h'^2}{h^2} + \frac{h'^2 h''}{h} + \frac{D}{h(D-h)(1+h'^2)^2}\left(\frac{3}{4}\frac{(D-2h)h'^2}{h(D-h)} - \frac{h''}{1+h'^2}\right) + \left(\frac{2h'}{D-h}\right)^2 + \frac{2h''}{D-h} + \kappa^2 \tag{67}
$$

$$
\eta^{(3)} \approx \frac{h''}{h} - \frac{3}{4}\frac{h'^2}{h^2} + \frac{D}{h(D-h)(1+h'^2)^2}\left(\frac{3}{4}\frac{(D-2h)h'^2}{h(D-h)} - \frac{h''}{1+h'^2}\right) + \kappa^2 + \frac{h''}{(D-h)} + \frac{3h'}{4(D-h)^2} \tag{68}
$$

where:

$$
h = h(x), \quad h' = h'(x), \quad h'' = h''(x), \quad \kappa = \kappa_2
$$

and by subtracting the two, the remaining terms are all of the same order as $h'(x)$ and $h''(x)$. Therefore, we can conclude:

$$
|\eta^{(3)}(D) - \eta^{(2)}(D)| = O(\delta) \tag{69}
$$

Additionally, at $\hat{z} = D$, the terms β, α', β' in equation (8) are all zero. Therefore, $p^{(2)}(D) = \eta^{(2)}(D)$ and $p^{(3)} = \eta^{(3)}(D)$. By the definition of γ, we have:

$$
\gamma^{(\ell)}(D) = \sqrt{\frac{\eta^{(\ell)}(D) - \lambda}{\alpha^{(\ell)}(D)}}, \quad \ell = 2, 3
$$

Next, we will show that $\frac{\bar{t}_{11}}{\sqrt{\alpha^{(3)}}} = \frac{\bar{t}_{22}}{\sqrt{\alpha^{(2)}(D)}}$. Denote $w_D^{(2)} = w|_{D^-}, w^{(3)} = w|_{D^+}$. By the definition of $\overline{T}$ in (12):

$$
\bar{t}_{11} = \frac{w_D^{(2)}}{w^{(3)}}, \quad \bar{t}_{22} = \frac{(D-1)w_D^{(2)}}{(D-h(x))w^{(3)}}
$$

Substituting the expression for w, we get:

$$
w_D^{(2)} = \sqrt{\frac{h(\hat{x})}{h'(x)} \cdot \frac{D-h(x^*)}{h(x^*)} \cdot \frac{h'(x)}{(D-h(x))^2}}, \quad w^{(3)} = \sqrt{\frac{h(\hat{x})}{h'(x)} \cdot \frac{D-h(x^*)}{h(x^*)} \cdot \frac{h'(x)}{(D-h(x))}}
$$

Thus, we have:

$$
\bar{t}_{11} = \frac{1}{\sqrt{D-h(x)}}, \quad \bar{t}_{22} = \frac{D-1}{(D-h(x))^{\frac{3}{2}}} \tag{70}
$$

Furthermore, from the expressions for $\alpha^{(2)}$ and $\alpha^{(3)}$, we have:

$$
\alpha^{(2)} = (D-1)^2 \frac{[D-h(x^*)]^2\,[h'(x)]^2\,[h(\hat{x})]^2}{(D-h(x))^4\,[h'(x)]^2\,[h(x^*)]^2}, \quad \alpha^{(3)} = \frac{[D-h(x^*)]^2\,[h'(x)]^2\,[h(\hat{x})]^2}{(D-h(x))^2\,[h'(x)]^2\,[h(x^*)]^2}
$$

It is straightforward to verify:

$$
\sqrt{\alpha^{(2)}(D)} = \frac{D-1}{D-h(x)}\sqrt{\alpha^{(3)}} \tag{71}
$$

From equations (70) and (71), it is easy to see that:

$$
\frac{\bar{t}_{11}}{\sqrt{\alpha^{(3)}}} = \frac{1}{\sqrt{(D-h(x))\alpha^{(3)}}} = \frac{\bar{t}_{22}}{\sqrt{\alpha^{(2)}(D)}} \tag{72}
$$

We are now ready to prove the main theorem (64):

$$
|\gamma^{(2)}(D)\bar{t}_{22} - \gamma^{(3)}\bar{t}_{11}| = \left|\frac{\bar{t}_{22}}{\sqrt{\alpha^{(2)}(D)}}\sqrt{\eta_{2D} - \lambda} - \frac{\bar{t}_{11}}{\sqrt{\alpha^{(3)}}}\sqrt{\eta^{(3)} - \lambda}\right|
$$

By the relation (72) and equation (69), the right-hand side of the above expression becomes:

$$\left|\frac{1}{\sqrt{(D-h(x))\alpha^{(3)}}}\left(\sqrt{\eta^{(3)}+O(\delta)-\lambda}-\sqrt{\eta^{(3)}-\lambda}\right)\right|$$

Expanding $O(\delta)$ to the first order gives $\sqrt{\eta^{(3)}+O(\delta)-\lambda}=\sqrt{\eta^{(3)}-\lambda}+O(\delta)$. Substituting this into the above expression, we obtain:

$$\left|\gamma^{(2)}(D)\bar{t}_{22}-\gamma^{(3)}\bar{t}_{11}\right|=\left|\frac{1}{\sqrt{(D-h(x))\alpha^{(3)}}}O(\delta)\right|=O(\delta)$$

This proves the theorem (64).

With this theorem, we can provide a brief explanation for $|p^{(3)}-p(\hat{z}_*^{(2)})|=O(\delta)$: We can prove that the derivatives of $\alpha^{(2)}$, $\beta^{(2)}$, and $\eta^{(2)}$ are bounded by the derivatives of $h(x)$, namely δ. Therefore, ${p^{(2)}}'(\hat{z})=O(\delta)$. Assuming ${p^{(2)}}'(\hat{z})$ exist in $[1,D]$, we have $|p(\hat{z})-p(\hat{z}_*^{(2)})|\leq\sup_{\hat{z}\in[1^+,D^-]}|{p^{(2)}}'(\hat{z})||\hat{z}-\hat{z}_*^{(2)}|=O(\delta)$. Substituting $\hat{z}$ with D^-, we get $|p^{(2)}(D^-)-p^{(2)}(\hat{z}_*^{(2)})|=O(\delta)$. Applying the result of (64), we conclude $|p_{3D}-p(\hat{z}_*^{(2)})|\leq|p_{3D}-p_{2D}|+|p_{2D}-p(\hat{z}_*^{(2)})|=O(\delta)$.

References

[1] Jianxin Zhu and Ying Zhang. Cross orthogonality between eigenfunctions and conjugate eigenfunctions of a class of modified helmholtz operator for pekeris waveguide. *Journal of Theoretical and Computational Acoustics*, 27(02):1850048, 2019.