Eigenvalue Equation in 2-Layer Wave Guide with a Curved Interface

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this work is not complete yet, as I am currently working on asymptotic solutions.

1 problem setup

The origional helmholtz equation is

$$\rho\left(\frac{1}{\rho}\psi_z\right)_z + \psi_{xx} + \kappa(z)^2\psi = 0 \tag{1}$$

When the waveguide is stratified, and the interface is curved, namely,

$$\kappa(z) = \begin{cases} \kappa_1 & \text{if } z < h(x) \\ \kappa_2 & \text{if } z > h(x) \end{cases}$$

$$\rho = \begin{cases} \rho_1 & \text{if } z < h(x) \\ \rho_2 & \text{if } z > h(x) \end{cases}$$

a othogonal transformation can be applied, so that under new coordinate $\hat{z}, \hat{x}$, the interface becomes flat.

$$\kappa(\hat{z}) = \begin{cases} \kappa_1 & \text{if } \hat{z} < 1 \\ \kappa_2 & \text{if } \hat{z} > 1 \end{cases}$$

$$\rho = \begin{cases} \rho_1 & \text{if } \hat{z} < 1 \\ \rho_2 & \text{if } \hat{z} > 1 \end{cases}$$

The stratified waveguide under new axis is illustrated in the figure(1). The new coordinate is divided into three layers along $\hat{z}$ axis, each span from $[0,1], [1,D], [D,D_1]$, and continuously connected at 2 interfaces $\hat{z}=1, \hat{z}=D$. In the third layer, z axis is not transformed, so $\hat{z}=z$ in this layer.

After that, we introduce a equation transformation to eliminate the $\psi_{\hat{x}}$ term: let

$$\psi(\hat{x},\hat{z}) = w(\hat{x},\hat{z})\phi(\hat{x},\hat{z}) \tag{2}$$

then, within each layer, the transformed Helmholtz equation becomes

$$\begin{aligned} &\phi_{\hat{x}\hat{x}} + \alpha(\hat{x},\hat{z})\phi_{\hat{z}\hat{z}} + \beta(\hat{x},\hat{z})\phi_{\hat{z}} + \eta(\hat{x},\hat{z})\phi = 0 \\ &\hat{z} \in [0,1] \text{ or } [1,D], \text{ or } [D,D_1] \end{aligned} \tag{3}$$

Recall that, $\hat{z}=z$ in the third layer of coordinate transformation. Since the refractive index does not vary along the original coordinate z axis in the third layer, it is also invariant along new axis $\hat{z}$. This is crucial for perfect matched layer(PML).

PML is used to truncate the open domain along $\hat{z}$ to a finite region. It absorbs wave's energy near the boundary to reduce the reflected waves.

It can be derived by a complex coordinate stretching,

$$\bar{z} = \hat{z} + i\int_0^{\hat{z}} \sigma(s)ds \tag{4}$$

$$\frac{d}{d\hat{z}}\phi \longrightarrow \frac{1}{1+i\sigma(\hat{z})}\frac{d}{d\hat{z}}$$

$$\sigma(s) = \begin{cases} 0 & \text{if } s < H \\ > 0 & \text{if } s \in [H, D_1] \end{cases}$$

We add the PML only within the third layer, which means $H > D$.

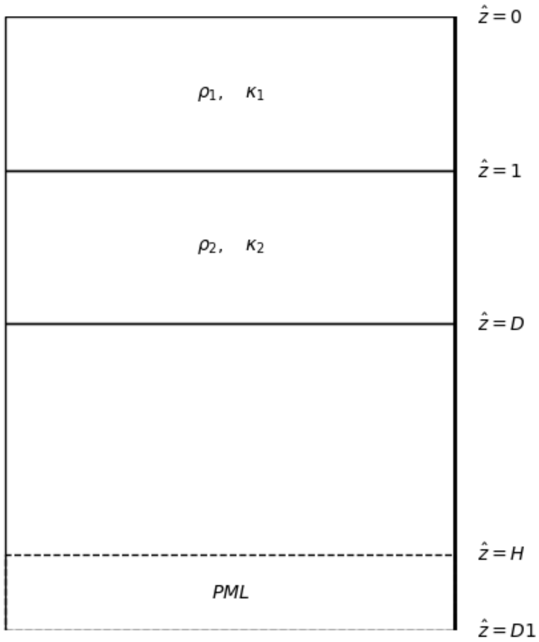

Figure 1: 3 layer coordinate transformed model

The new coordinate is divided into three layers along $\hat{z}$ axis, each span from $[0,1], \quad [1,D], \quad [D,D_1]$they are continuously connected at 2 interfaces $\hat{z}=1, \quad \hat{z}=D$.

After PML complex coordinate stretching, the eigenvalue problem for transformed Helmholtz equation (3) is

$$\begin{cases} \alpha \dfrac{d^2\phi}{d\bar{z}^2} + \beta \dfrac{d\phi}{d\bar{z}} + \eta\phi = \lambda\phi \\ \\ (w\phi)\Big|_{\hat{z}=1^-} = (w\phi)\Big|_{\hat{z}=1^+} \\ \dfrac{1}{\rho_1} w \left\{ \dfrac{1}{2}\left[h''(x) - 2\dfrac{h'(x)^2}{h(x)}\right]\phi - \dfrac{1+h'(x)^2}{h(x)}\phi_{\bar{z}} \right\}\Bigg|_{\hat{z}=1^-} \\ = \dfrac{1}{\rho_2} w \left\{ \dfrac{1}{2}\left[h''(x) + 2\dfrac{h'(x)^2}{D-h(x)}\right]\phi - \dfrac{D-1}{D-h(x)}\left[1+h'(x)^2\right]\phi_{\bar{z}} \right\}\Bigg|_{\hat{z}=1^+}, \\ \\ (w\phi)\Big|_{\hat{z}=D^-} = (w\phi)\Big|_{\hat{z}=D^+} \\ w\left(\dfrac{1-D}{h(x)-D}\phi_{\bar{z}}\right)\Bigg|_{\hat{z}=D^-} = w\phi_{\bar{z}}\Big|_{\hat{z}=D^+}, \\ \\ \phi|_{\hat{z}=0} = 0, \quad \phi|_{\hat{z}=D} = 0. \end{cases} \tag{5}$$

To further transform this eigenvalue problem to a standard form, we introduce a new function $\hat{\phi}$, and let

$$\phi(\bar{z}) = \hat{\phi}(\bar{z}) \cdot e^{\int_1^{\bar{z}} \mu(s)\,ds} \tag{6}$$

Plugging this into the first equation in (5) and canceling out the term $e^{\int_1^{\bar{z}} \mu(s)\,ds}$, we get:

$$\begin{aligned} &\hat{\phi}(\bar{z})\left[\alpha(\bar{z})\mu'(\bar{z}) + \alpha(\bar{z})\mu(\bar{z})^2 + \beta(\bar{z})\mu(\bar{z}) + \eta(\bar{z})\right] \\ &+ \frac{d\hat{\phi}(\bar{z})}{d\bar{z}}\left(2\alpha(\bar{z})\mu(\bar{z}) + \beta(\bar{z})\right) \\ &+ \alpha(\bar{z})\frac{d^2\hat{\phi}(\bar{z})}{d\bar{z}^2} = \lambda\hat{\phi}(\bar{z}) \end{aligned} \tag{7}$$

By setting the terms with $\frac{d\hat{\phi}(\bar{z})}{d\bar{z}}$ to zero, we obtain the equation:

$$2\alpha(\bar{z})\mu(\bar{z}) + \beta(\bar{z}) = 0$$

The resulting $\mu(\bar{z})$ is:

$$\mu(\bar{z}) = -\frac{\beta(\bar{z})}{2\alpha(\bar{z})} \tag{8}$$

plug $\mu(\bar{z})$ from (8) into (7), the eigenvalue equation for $\hat{\phi}(\bar{z})$ becomes:

$$\left(\frac{\frac{1}{2}(-\beta'(\bar{z})\alpha(\bar{z}) + \beta(\bar{z})\alpha'(\bar{z})) - \frac{1}{4}\beta^2(\bar{z})}{\alpha(\bar{z})} + \eta(\bar{z}) \right) \hat{\phi}(\bar{z}) + \alpha(\bar{z}) \frac{d^2\hat{\phi}(\bar{z})}{d\bar{z}^2} = \lambda\hat{\phi}(\bar{z}) \tag{9}$$

Denote it as $p(\bar{z})\hat{\phi} + \alpha(\bar{z})\hat{\phi}_{\bar{z}\bar{z}} = \lambda\hat{\phi}$.

Together with the interface condition for $\hat{\phi}$, which can be derived by plugging (6) into the boundary conditions in (5), the complete eigenvalue problem for the new function $\hat{\phi}$ is

$$\begin{cases} p(\bar{z})\hat{\phi} + \alpha(\bar{z})\hat{\phi}_{\bar{z}\bar{z}} = \lambda\hat{\phi} \\ \left(w\hat{\phi}\right)\Big|_{\hat{z}=1^-} = \left(w\hat{\phi}\right)\Big|_{\hat{z}=1^+} \\ \frac{1}{\rho_1} w \left\{ \left[\frac{1}{2}h''(x) - \frac{h'(x)^2}{h(x)} - \frac{1+h'(x)^2}{h(x)}\mu \right] \hat{\phi} - \frac{1+h'(x)^2}{h(x)}\hat{\phi}_{\bar{z}} \right\} \Big|_{\hat{z}=1^-} \\ = \frac{1}{\rho_2} w \left\{ \left[\frac{1}{2}h''(x) + \frac{h'(x)^2}{D-h(x)} - \frac{D-1}{D-h(x)} \left(1+h'(x)^2\right)\mu \right] \hat{\phi} - \frac{D-1}{D-h(x)} \left(1+h'(x)^2\right)\hat{\phi}_{\bar{z}} \right\} \Big|_{\hat{z}=1^+}, \\ \left(w\hat{\phi}\right)\Big|_{\hat{z}=D^-} = \left(w\hat{\phi}\right)\Big|_{\hat{z}=D^+}, \\ w\left[\frac{1-D}{h(x)-D}\hat{\phi}_{\bar{z}} \right]\Big|_{\hat{z}=D^-} = w\hat{\phi}_{\bar{z}}\Big|_{\hat{z}=D^+}, \\ \hat{\phi}|_{\hat{z}=0} = 0, \quad \hat{\phi}|_{\hat{z}=D_1} = 0. \end{cases} \tag{10}$$

note that we eliminate terms with μ in the condition at $\hat{z} = D$, as $\mu = 0$ when $\hat{z} \geq D$.

We can write the interface condition into matrix form, denote

$$\begin{aligned} T_{1^-} &= \begin{bmatrix} w & 0 \\ \frac{1}{\rho_1} w \left(\frac{1}{2}h''(x) - \frac{h'(x)^2}{h(x)} - \frac{1+h'(x)^2}{h(x)}\mu \right) & -\frac{1}{\rho_1} w \frac{1+h'(x)^2}{h(x)} \end{bmatrix} \Bigg|_{\hat{z}=1^-} \\ T_{1^+} &= \begin{bmatrix} w & 0 \\ \frac{1}{\rho_2} w \left(\frac{1}{2}h''(x) + \frac{h'(x)^2}{D-h(x)} - \frac{D-1}{D-h(x)} \left(1+h'(x)^2\right)\mu \right) & -\frac{1}{\rho_2} w \frac{D-1}{D-h(x)} \left(1+h'(x)^2\right) \end{bmatrix} \Bigg|_{\hat{z}=1^-} \end{aligned} \tag{11}$$

Then, the interface condition at $\hat{z} = 1$ can be expressed as

$$T_{1^-} \begin{bmatrix} \hat{\phi} \\ \hat{\phi}' \end{bmatrix} \Bigg|_{\hat{z}=1^-} = T_{1^+} \begin{bmatrix} \hat{\phi} \\ \hat{\phi}' \end{bmatrix} \Bigg|_{\hat{z}=1^+} \tag{12}$$

similarly, for the second interface at $\hat{z} = D$, denote

$$T_{D^-} = \begin{bmatrix} w & 0 \\ 0 & \frac{(D-1)w}{D-h(x)} \end{bmatrix} \Bigg|_{\hat{z}=D^-}, T_{D^+} = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix} \Bigg|_{\bar{z}=D^+} \tag{13}$$

and the interface condition can be expressed as

$$T_{D^-} \begin{bmatrix} \hat{\phi} \\ \hat{\phi}' \end{bmatrix} \Bigg|_{\hat{z}=D^-} = T_{D^+} \begin{bmatrix} \hat{\phi} \\ \hat{\phi}' \end{bmatrix} \Bigg|_{\hat{z}=D^+} \tag{14}$$

2 dispersion relation Approximation

In the first and second layers $\hat{z} \in [0,1], [1, D_1]$, where there is no PML coordinate stretching, $\hat{z} = \bar{z}$, we can change the derivative variable, as well as the function variable from $\bar{z}$ to $\hat{z}$ in eigenvalue equation (9). Rearrange it then we get:

$$\hat{\phi}_{\hat{z}\hat{z}}(\hat{z}) + \frac{p(\hat{z}) - \lambda}{\alpha(\hat{z})}\hat{\phi}(\hat{z}) = 0. \tag{15}$$

Denote it as

$$\hat{\phi}_{\hat{z}\hat{z}}(\hat{z}) + K(\hat{z})\hat{\phi}(\hat{z}) = 0. \tag{16}$$

the function $K(\hat{z})$ can be approximated by a piece wise polynomial of degree two.

We divide the interval $[0,1]$ into m_1 sub intervals $[(j-1)l_1, jl_1], \quad (j = 1, 2, \ldots, m_1)$ interval $[1, D1]$ to into m_2 sub intervals, $[1 + (j - m_1 - 1)l_2, 1 + (j - m_1)l_2], \quad (j = m_1 + 1, m_1 + 2, ..., m_1 + m_2)$ The subintervals are denoted in order by $I_j = (\hat{z}_{j-1}, \hat{z}_j)$

$$\hat{z}_j = \begin{cases} jl_1, & \text{if } 1 \leq j \leq m_1, \\ 1 + (j - m_1)l_2, & \text{if } m_1 + 1 \leq j \leq m_1 + m_2. \end{cases}$$

On each subinterval, the function $K(\hat{z})$ can be interpolated by a polynomial of degree two with three nodes chosen to be the two endpoints and the midpoint. By approximating $K(\hat{z})$, (16) is now approximated by

$$\frac{d^2 y_j}{d\hat{z}^2} + (a_j\hat{z}^2 + b_j\hat{z} + c_j)y_j = 0, \quad \hat{z} \in I_j \tag{17}$$

where y_j is the approximation of the $\hat{\phi}$ on the interval $[\hat{z}_{j-1}, \hat{z}_j]$ and the a, b, c are:

In the first layer:

$$\begin{cases} a_j = \frac{2K(t_0)}{l_1^2} - \frac{4K(t_1)}{l_1^2} + \frac{2K(t_2)}{l_1^2}, \\ b_j = \frac{(1-4j)K(t_0)}{l_1} + \frac{(8j-4)K(t_1)}{l_1} + \frac{(3-4j)K(t_2)}{l_1}, \\ c_j = \left(2j^2 - j\right)K(t_0) + \left(4j - 4j^2\right)K(t_1) + \left(2j^2 - 3j + 1\right)K(t_2) \\ t_0 = l_1(j-1), \quad t_1 = (j - 1/2)l_1, \quad t_2 = jl_1 \\ j = 1, 2, \ldots, m_1 \end{cases}$$

In the second layer:

$$\begin{cases} a_j = \dfrac{2K(t_0)}{l_2^2} - \dfrac{4K(t_1)}{l_2^2} + \dfrac{2K(t_2)}{l_2^2} \\ b_j = \dfrac{K(t_0)(-4l_2j + 4l_2m_1 + l_2 - 4)}{l_2^2} \\ \qquad + \dfrac{K(t_1)(l_2(8j - 8m_1 - 4) + 8)}{l_2^2} \\ \qquad + \dfrac{K(t_2)(l_2(-4j + 4m_1 + 3) - 4)}{l_2^2}, \\ c_j = \dfrac{K(t_0)\left(l_2^2\left(2j^2 - 4jm_1 - j + 2m_1^2 + m_1\right) + l_2(4j - 4m_1 - 1) + 2\right)}{l_2^2} \\ \qquad + \dfrac{4K(t_1)(-l_2(j - m_1 - 1) - 1)(l_2(j - m_1) + 1)}{l_2^2} \\ \qquad + \dfrac{K(t_2)(-l_2(j - m_1 - 1) - 1)(-2l_2j + 2l_2m_1 + l_2 - 2)}{l_2^2}, \\ t_0 = 1 + (j - m_1 - 1)l_2, \quad t_1 = 1 + (j - m_1 - 1/2)l_2, \quad t_2 = 1 + (j - m_1)l_2 \\ \ j = m_1 + 1, m_1 + 1, \ldots, m_2 + m_1 \end{cases}$$

The solution of (17) at each interval is given by confluent hypergeometric functions. Let $\{\ u_j(\hat{z}), v_j(\hat{z})\ \}$ be a fundamental pair on the jth interval, then

$$\begin{aligned} &y_j(\hat{z}) = A_j u_j(\hat{z}) + B_j v_j(\hat{z}), \quad \hat{z} \in I_j \\ &j = 1, 2, \ldots m_1 + m_2. \end{aligned} \tag{18}$$

In the third layer $\hat{z} \in [D, D_1]$, the eigenvalue equation(9) is simplified to

$$\alpha \hat{\phi}_{\bar{z}\bar{z}} + \eta \hat{\phi} = \lambda \hat{\phi} \tag{19}$$

as α, η are constants and $\beta = 0$ in that layer.

Denote$\gamma_3 = \sqrt{\frac{\eta - \lambda}{\alpha}}$, and the solution of (19) is simply

$$\hat{\phi}(\hat{z}) = C_1 e^{-i\gamma_3 \bar{z}} + C_2 e^{i\gamma_3 \bar{z}} \quad \hat{z} \in [D, D_1] \tag{20}$$

To make the notation consistent, we can also denote (20) as

$$y_{m_1+m_2+1}(\hat{z}) = A_{m_1+m_2+1} u_{m_1+m_2+1}(\hat{z}) + B_{m_1+m_2+1} v_{m_1+m_2+1}(\hat{z})$$

notice we also use variable $\hat{z},$ instead of $\bar{z}$, in $u_{m_1+m_2+1}, v_{m_1+m_2+1}$.

In the first sub interval, The zero boundary condition $\hat{\phi}(0) = 0$ gives the first relation:

$$A_1 u_1(0) + B_1 v_1(0) = 0$$

Suppose the approximation of $\hat{\phi}$ is continuous and has continuous first-order derivative. This requires that at the interface$\hat{z}_j$:

$$\begin{bmatrix} u_{j+1}(\hat{z}_j) & v_{j+1}(\hat{z}_j) \\ u'_{j+1}(\hat{z}_j) & v'_{j+1}(\hat{z}_j) \end{bmatrix} \begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \begin{bmatrix} u_j(\hat{z}_j) & v_j(\hat{z}_j) \\ u'_j(\hat{z}_j) & v'_j(\hat{z}_j) \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix} \tag{21}$$

Moreover,from (12).We have a relation at the interface $\hat{z} = 1$. Denote

$$T = T_{1^+}^{-1} T_{1^-}, \quad T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \tag{22}$$

then

$$\begin{bmatrix} u_{m_1+1}(1) & v_{m_1+1}(1) \\ u'_{m_1+1}(1) & v'_{m_1+1}(1) \end{bmatrix} \begin{bmatrix} A_{m_1+1} \\ B_{m_1+1} \end{bmatrix} = T \begin{bmatrix} u_{m_1}(1) & v_{m_1}(1) \\ u'_{m_1}(1) & v'_{m_1}(1) \end{bmatrix} \begin{bmatrix} A_{m_1} \\ B_{m_1} \end{bmatrix} \tag{23}$$

Similarly, from (14) denote

$$\overline{T} = T_{D^+}^{-1} T_{D^-}, \quad \overline{T} = \begin{bmatrix} \overline{T}_{11} & 0 \\ 0 & \overline{T}_{22} \end{bmatrix}$$

then

$$\begin{bmatrix} u_{m_1+m_2+1}(D) & v_{m_1+m_2+1}(D) \\ u'_{m_1+m_2+1}(D) & v'_{m_1+m_2+1}(D) \end{bmatrix} \begin{bmatrix} A_{m_1+m_2+1} \\ B_{m_1+m_2+1} \end{bmatrix} = \overline{T} \begin{bmatrix} u_{m_1+m_2}(D) & v_{m_1+m_2}(D) \\ u'_{m_1+m_2}(D) & v'_{m_1+m_2}(D) \end{bmatrix} \begin{bmatrix} A_{m_1+m_2} \\ B_{m_1+m_2} \end{bmatrix} \tag{24}$$

The zero boundary condition at D_1 in (9) implies

$$\begin{aligned} &0 = A_{m_1+m_2+1} e^{-i\gamma_3 \bar{D_1}} + B_{m_1+m_2+1} e^{i\gamma_3 \bar{D_1}} \\ &\text{where } \bar{D_1} = D_1 + i\int_0^{D_1} \sigma(s) ds \end{aligned} \tag{25}$$

We can obtain a linear system of A_j, B_j. by putting together (21)(23)(24)(25)

$$\begin{cases} A_1 u_1(0) + B_1 v_1(0) = 0 \\ \\ u_{j+1}(\hat{z_j})A_{j+1} + v_{j+1}(\hat{z_j})B_{j+1} = u_j(\hat{z_j})A_j + v_j(\hat{z_j})B_j, \\ u'_{j+1}(\hat{z_j})A_{j+1} + v'_{j+1}(\hat{z_j})B_{j+1} = u'_j(\hat{z_j})A_j + v'_j(\hat{z_j})B_j. \\ j = 1, \dots, m_1 - 1 \quad \text{and} \quad j = m_1 + 1, \dots, m_1 + m_2 - 1, \\ \\ u_{m_1+1}(1)A_{m_1+1} + v_{m_1+1}(1)B_{m_1+1} = T_{11}\left[u_{m_1}(1)A_{m_1} + v_{m_1}(1)B_{m_1+1}\right], \\ u'_{m_1+1}(1)A_{m_1+1} + v'_{m_1+1}(1)B_{m_1+1} = \\ \left[T_{21}u_{m_1}(1) + T_{22}u'_{m_1}(1)\right]A_{m_1} + \left[T_{21}v_{m_1}(1) + T_{22}v'_{m_1}(1)\right]B_{m_1}, \\ \\ u_{m_1+m_2+1}(D)A_{m_1+m_2+1} + v_{m_1+m_2+1}(D)B_{m_1+m_2+1} = \\ \overline{T}_{11}\left[u_{m_1+m_2}(D)A_{m_1+m_2} + v_{m_1+m_2}(D)B_{m_1+m_2}\right], \\ \\ u'_{m_1+m_2+1}(D)A_{m_1+m_2+1} + v_{m_1+m_2+1}(D)'B_{m_1+m_2+1} = \\ \overline{T}_{22}\left[u'_{m_1+m_2}(D)A_{m_1+m_2} + v'_{m_1+m_2}(D)B_{m_1+m_2}\right], \\ \\ A_{m_1+m_2+1}e^{-i\gamma_3\bar{D}_1} + B_{m_1+m_2+1}e^{i\gamma_3\bar{D}_1} = 0 \end{cases} \tag{26}$$

The above linear system leads to an algebraic equation for λ, by eliminating all the A_j and B_j. Denote $R_j = \frac{A_j}{B_j}, (j = 1, 2, \dots, m_1 + m_2 + 1)$. We can derive the dispersion relation of λ

$$R_1 = -\frac{v_1(0)}{u_1(0)} \tag{27}$$

$$R_{j+1} = \frac{R_j\left(u'_j(\hat{z}_j)v_{j+1}(\hat{z}_j) - u_j(\hat{z}_j)v'_{j+1}(\hat{z}_j)\right) - v_j(\hat{z}_j)v'_{j+1}(\hat{z}_j) + v_{j+1}(\hat{z}_j)v'_j(\hat{z}_j)}{R_j\left(u'_{j+1}(\hat{z}_j)u_j(\hat{z}_j) - u'_j(\hat{z}_j)u_{j+1}(\hat{z}_j)\right) + u'_{j+1}(\hat{z}_j)v_j(\hat{z}_j) - u_{j+1}(\hat{z}_j)v'_j(\hat{z}_j)},$$
$$j = 1, \dots, m_1 - 1 \quad \text{and} \quad j = m_1 + 1, \dots, m_1 + m_2 - 1, \tag{28}$$

$$\begin{aligned} &R_{m_1+1} = \\ &\Big\{R_{m_1}\left[u'_{m_1}(1)v_{m_1+1}(1)T_{22} + u_{m_1}(1)v_{m_1+1}(1)T_{21} - u_{m_1}(1)v'_{m_1+1}(1)T_{11}\right] + \dots \\ &v_{m_1}(1)v_{m_1+1}(1)T_{21} - v_{m_1}(1)v'_{m_1+1}(1)T_{11} + v_{m_1+1}(1)v'_{m_1}(1)T_{22}\Big\}\Big/ \\ &\Big\{R_{m_1}\left[u'_{m_1+1}(1)u_{m_1}(1)T_{11} - u'_{m_1}(1)u_{m_1+1}(1)T_{22} - u_{m_1}(1)u_{m_1+1}(1)T_{21}\right] + \dots \\ &u'_{m_1+1}(1)v_{m_1}(1)T_{11} - u_{m_1+1}(1)v_{m_1}(1)T_{21} - u_{m_1+1}(1)v'_{m_1}(1)T_{22}\Big\} \end{aligned} \tag{29}$$

$$\begin{aligned} &R_{m_1+m_2+1} = \\ &\Big\{R_{m_1+m_2}\left[\overline{T}_{22}\, v_{m_1+m_2+1}(D)\, u'_{m_1+m_2}(D) - \overline{T}_{11}\, u_{m_1+m_2}(D)\, v'_{m_1+m_2+1}(D)\right] - \dots \\ &\overline{T}_{11}\, v_{m_1+m_2}(D)\, v'_{m_1+m_2+1}(D) + \overline{T}_{22}\, v_{m_1+m_2+1}(D)\, v'_{m_1+m_2}(D)\Big\}\Big/ \\ &\Big\{R_{m_1+m_2}\left[\overline{T}_{11}\, u_{m_1+m_2}(D)\, u'_{m_1+m_2+1}(D) - \overline{T}_{22}\, u_{m_1+m_2+1}(D)\, u'_{m_1+m_2}(D)\right] + \dots \\ &\overline{T}_{11}\, v_{m_1+m_2}(D)\, u'_{m_1+m_2+1}(D) - \overline{T}_{22}\, u_{m_1+m_2+1}(D)\, v'_{m_1+m_2}(D)\Big\} \end{aligned} \tag{30}$$

$$R_{m_1+m_2+1} = -e^{2i\gamma_3\bar{D}_1} \tag{31}$$

λ involved in each u_j, v_j. So $R_{m_1+m_2+1}$ is determined by λ recursively through (27)(28)(29)(30). On the other hand, $R_{m_1+m_2+1}$ is determined by λ in (31). Therefore, the dispersion relation has the form

$$g(\lambda) = R_{m_1+m_2+1} + e^{2i\gamma_3\bar{D}_1} = 0 \tag{32}$$

3 special case

when the wave guide interface is flat, with $h(x) \equiv 1$, the orthogonal coordinate transformation will be reduced to an identity mapping, namely $(\hat{x}, \hat{z}) = (x, z)$. And the equation transform (6) will be an identity transformation, as $w \equiv 1$. Therefore, in the transformed Helmholtz equation(3), $\alpha = 1, \beta = 0, \eta = \kappa^2$ And the eigenvalue problem (5) will be reduced to

$$\begin{cases} \phi_{\bar{z}\bar{z}} + \kappa(z)^2\phi = \lambda\phi \\ \phi|_{z=1^-} = \phi|_{z=1^+} \\ \dfrac{1}{\rho_1}\phi_{\bar{z}}|_{z=1^-} = \dfrac{1}{\rho_2}\phi_{\bar{z}}|_{z=1^+} \\ \phi|_{z=D^-} = \phi|_{z=D^+} \\ \phi_{\bar{z}}|_{z=D^-} = \phi_{\bar{z}}|_{z=D^+} \\ \phi|_{z=0} = 0 \\ \phi|_{z=D_1} = 0 \end{cases} \tag{33}$$

Given that $\beta = 0$, from the definition of $\hat{\phi}$ (6) we have $\hat{\phi} = \phi$. so $\hat{\phi}$ should also satisfy the reduced eigenvalue problem(33). Denote

$$\gamma_1 = \sqrt{\kappa_1^2 - \lambda}, \quad \gamma_2 = \sqrt{\kappa_2^2 - \lambda}$$

the 2 basis solution $\{u_i, v_j\}$ in each sub interval in the first 2 layer $[0,1],[1,D_1]$are simply

$$u_j = e^{-i\gamma_j z}, \quad v_j = e^{i\gamma_j z} \quad z \in I_j \tag{34}$$

$$\gamma_j = \begin{cases} \gamma_1 & \text{if } j \le m_1 \\ \gamma_2 & \text{if } j > m_1 \end{cases} \quad j = 1, 2, \ldots m_1 + m_2$$

The basis solutions within one layer are the same, so they are continuous at each end point of sub interval naturally:

$$\begin{aligned} &u_j'(z_j) = u_{j+1}'(z_j), \quad u_j(z_j) = u_{j+1}(z_j) \\ &v_j'(z_j) = v_{j+1}'(z_j), \quad v_j(z_j) = v_{j+1}(z_j) \\ &j = 1, \ldots, m_1 - 1 \text{ and } j = m_1 + 1, \ldots, m_1 + m_2 - 1 \end{aligned}$$

Thus, the iterative relation of R_j within a layer(28) simply becomes

$$\begin{aligned} &R_{j+1} = R_j \\ &j = 1, \ldots, m_1 - 1 \text{ and } j = m_1 + 1, \ldots, m_1 + m_2 - 1 \end{aligned} \tag{35}$$

The zero boundary condition $\hat{\phi}(0) = 0$ give rise to $R_1 = \frac{A_1}{B_1} = -1$ Therefore, in the first layer,

$$R_j \equiv -1, \quad j = 1, 2, \ldots, m_1$$

From the simplified interface condition (33), Interface condition matrices become

$$T_{1^-} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho_1} \end{bmatrix}, \quad T_{1^+} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho_2} \end{bmatrix} \tag{36}$$

T_{D^-} andT_{D^+} all reduced to identity matrix. Thus,

$$T = T_{1^+}^{-1} T_{1^-} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\rho_2}{\rho_1} \end{bmatrix}, \quad \overline{T} = T_{D^+}^{-1} T_{D^-} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{37}$$

The relation to transfer from the first layer to the second layer(29) becomes

$$\begin{aligned} &R_{m_1+1} = \\ &\left\{ -\left[\frac{\rho_2}{\rho_1} u_{m_1}'(1) v_{m_1+1}(1) - u_{m_1}(1) v_{m_1+1}'(1)\right] - v_{m_1}(1) v_{m_1+1}'(1) + v_{m_1+1}(1) v_{m_1}'(1) \frac{\rho_2}{\rho_1} \right\} \Big/ \\ &\left\{ -\left[u_{m_1+1}'(1) u_{m_1}(1) - \frac{\rho_2}{\rho_1} u_{m_1}'(1) u_{m_1+1}(1)\right] + u_{m_1+1}'(1) v_{m_1}(1) - \frac{\rho_2}{\rho_1} u_{m_1+1}(1) v_{m_1}'(1) \right\} \end{aligned} \tag{38}$$

plug in the exact expression for u, v in (34):

$$R_{m_1+1} = \frac{e^{i\gamma_2}\Big\{ -e^{-i\gamma_1}\left[-i\gamma_1\frac{\rho_2}{\rho_1} - i\gamma_2\right] + e^{i\gamma_1}\left[-i\gamma_2 + i\gamma_1\frac{\rho_2}{\rho_1}\right] \Big\}}{e^{-i\gamma_2}\Big\{ -e^{-i\gamma_1}\left[-i\gamma_2 + i\gamma_1\frac{\rho_2}{\rho_1}\right] + e^{i\gamma_1}\left[-i\gamma_2 - i\gamma_1\frac{\rho_2}{\rho_1}\right] \Big\}} \tag{39}$$

simplify it

$$R_{m_1+1} = -e^{2i\gamma_2} \frac{e^{-i\gamma_1}\left[\gamma_1\frac{\rho_2}{\rho_1} + \gamma_2\right] + e^{i\gamma_1}\left[-\gamma_2 + \gamma_1\frac{\rho_2}{\rho_1}\right]}{e^{-i\gamma_1}\left[-\gamma_2 + \gamma_1\frac{\rho_2}{\rho_1}\right] + e^{i\gamma_1}\left[\gamma_2 + \gamma_1\frac{\rho_2}{\rho_1}\right]} \tag{40}$$

The transfer relation from second to the third layer (30) is reduced to

$$R_{m_1+m_2+1} = R_{m_1+m_2}$$

From (35), R_j are also constant within the second layer, thus$R_{m_1+m_2} = R_{m_1+1}$. Therefore, we have

$$R_{m_1+m_2+1} = R_{m_1+m_2} = R_{m_1+1} = \text{ the right hand side of(40)}$$

as layer 2 and layer3 becomes homogeneous in this case, thus $\gamma_3 = \gamma_2$. The new dispersion relation we derived (32) is reduced to

$$-e^{2i\gamma_2} \frac{e^{-i\gamma_1}\left[\gamma_1\frac{\rho_2}{\rho_1} + \gamma_2\right] + e^{i\gamma_1}\left[-\gamma_2 + \gamma_1\frac{\rho_2}{\rho_1}\right]}{e^{-i\gamma_1}\left[-\gamma_2 + \gamma_1\frac{\rho_2}{\rho_1}\right] + e^{i\gamma_1}\left[\gamma_2 + \gamma_1\frac{\rho_2}{\rho_1}\right]} + e^{2i\gamma_2\bar{D_1}} = 0 \tag{41}$$

rearrange it, and expand $e^{i\gamma_1}, e^{-i\gamma_1}$ into trigonometric function, we have

$$\frac{i\gamma_1\frac{\rho_2}{\rho_1}cos(\gamma_1) - i\gamma_2 sin(\gamma_1)}{i\gamma_2 sin(\gamma_1) + \gamma_1\frac{\rho_2}{\rho_1}cos(\gamma_1)} = e^{2i\gamma_2(\bar{D_1}-1)} \tag{42}$$

multiply both the numerator and denominator of the left hand side by i, it will become the same as the dispersion relation in[1], this result theoretically verified new relation(32) we derived.

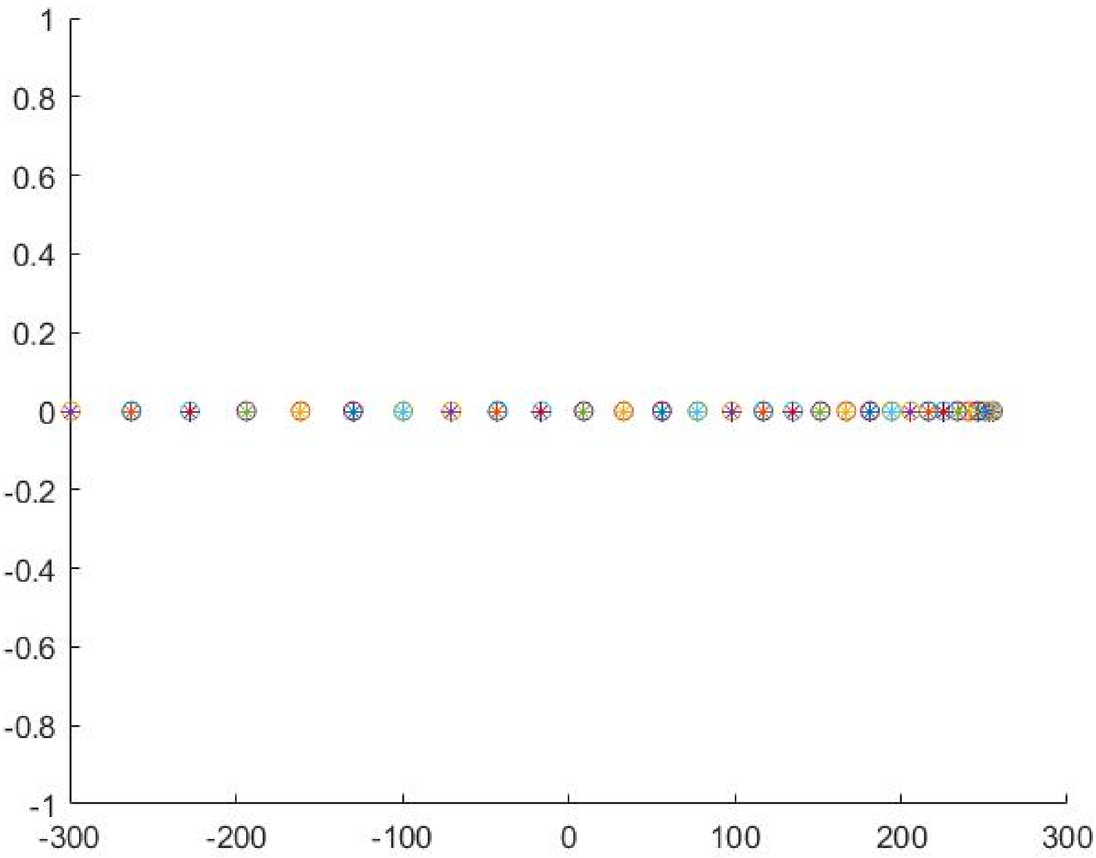

Figure 2: example 1
homogenious waveguide, and without PML. $\kappa = 16$, $\quad D_1 = 4$,'o' denotes the exact eigenvalue from (44), '*' denotes the solution of (32)

4 numerical example

To verify the dispersion relation (32), we compared the solution of (32) with solutions obtained by other methods under several simplified cases.

example 1:
The wave guide is homogeneous. and without PML. thus, the eigenvalue problem is simplified to

$$\begin{cases} \hat{\phi}_{zz} + \kappa^2 \phi = \lambda \hat{\phi} \\ \hat{\phi}(0) = 0 \\ \hat{\phi}(D_1) = 0 \end{cases} \tag{43}$$

The exact solution also has a simple form:

$$\hat{\phi} = 2isin(\frac{k\pi z}{D_1}), \quad k = 1, 2, 3, ...$$

and the eigenvalues are

$$\lambda = \kappa^2 - (\frac{k\pi}{D_1})^2, \quad k = 1, 2, 3... \tag{44}$$

we chose $\kappa = 16$, $D_1 = 4$, and calculate 30 eigenvalues for $k = 1, 2, ...30$ from (44). We use Müller's method solve (32). The Müller's method requires three initial guesses. We use the exact solution from (44) as the first guess λ_0^*. And the other 2 initial guesses λ_1^*, λ_2^* are obtained by applying a deviation to λ_0^*. namely,

$$\lambda_{1,2}^* = \lambda_0^* \pm 0.01$$

We terminate the Müller's method's iteration when $|g(\lambda)| \leq 10^{-20}$

We plot the exact eigenvalues and the solution of (32) in figure(2) Since exact solutions are obtained in this case, for almost all eigenvalues, the Müller's method stops at the first iteration.

Example 2:

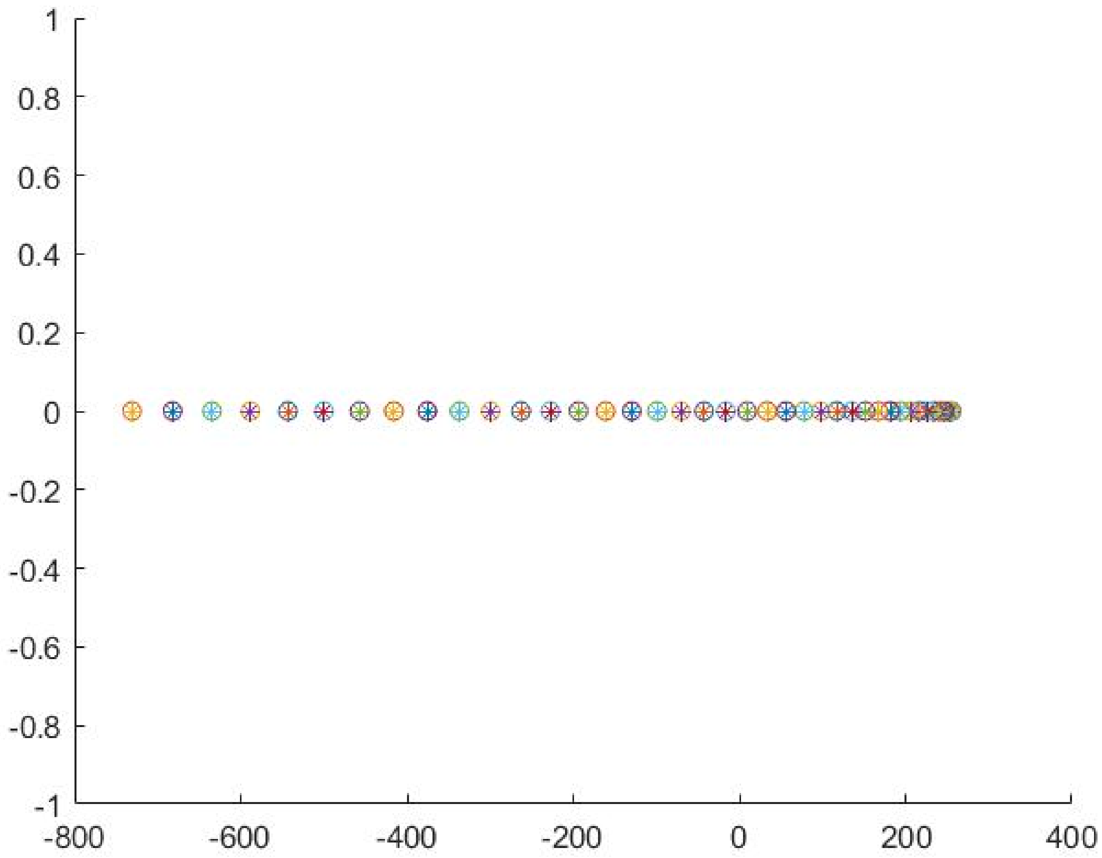

Figure 3: example 2
stratified wave guide with flat interface,$\kappa_1 = 16, \quad \kappa_2 = 14.4, \quad \rho_1 = 1, \quad \rho_2 = 1.7$, without PML. 'o' denotes eigenvalue obtained from FDM, '*' denotes the solution of (32)

Here we consider a stratified wave guide, with flat interface, and also without PML. The interface is at $z = 1$ The eigenvalue problem becomes

$$\begin{cases} \hat{\phi}_{zz} + \kappa(z)^2\phi = \lambda\hat{\phi} \\ \dfrac{1}{\rho_1}\dfrac{d\hat{\phi}}{dz}\Big|_{z=1^-} = \dfrac{1}{\rho_2}\dfrac{d\hat{\phi}}{dz}\Big|_{z=1^+} \\ \hat{\phi}(0) = 0 \\ \hat{\phi}(D_1) = 0 \end{cases} \tag{45}$$

We chose

$$\kappa(z) = \begin{cases} 16, & 0 < z < 1 \\ 14.4, & z > 1 \end{cases}$$

$$\rho_1 = 1, \quad 0 < z < 1$$

$$\rho_2 = 1.7, \quad z > 1$$

An explicit expression for eigenvalues is hard to derive, so and we use finite difference method(FDM) to approximate the exact eigenvalue. To implement FDM, We divide $[0, 1], [1, D]$ and $[D, D_1]$ into 200, 200, 400 grids respectively.

We adopt the same scheme as example 1 to solve (32). Again, We compared the solution on the largest 30 eigenvalues.

The results in figure(3) shows the solutions comply with each other well. this further verified the dispersion relation(32)

Example 3:

Finally, We add a PML to example 2. The PML parameter is

$$\sigma(\hat{z}) = \begin{cases} 0, & 0 < \hat{z} \leq H \\ 10t^3/(1+t^2), & H < \hat{z} \leq D_1 \end{cases}$$

where $t = \frac{\hat{z}-H}{D_1-H}$. We chose $H = 3.5, D_1 = 4$

We also use FDM to compute the initial value for Müller's method's. Note that eigvenvalues are complex numbers in this case. so the deviation for getting another 2 initial guesses are modified to

$$\lambda^*_{1,2} = \lambda^*_0 \pm 0.01(1 + 1i)$$

The comparison between the solution of (32) and eigenvalues obtained from FDM are shown in figure(4)

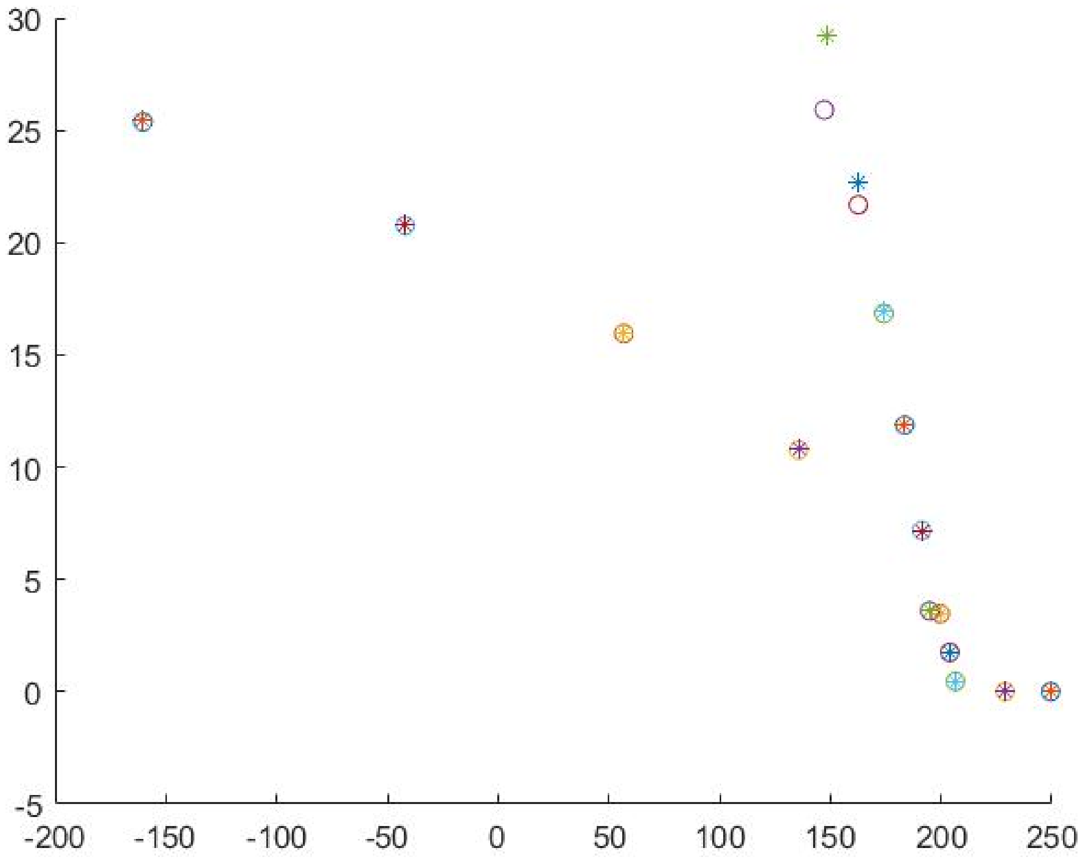

Figure 4: example 3
stratified wave guide with flat interface,$\kappa_1 = 16, \quad \kappa_2 = 14.4, \quad \rho_1 = 1, \quad \rho_2 = 1.7$, A PML is added below $H = 3.5$

eigenvalues from 2 methods match well when the modulus of eigenvalue is small. But When the modulus is large, Müller's method's do not converge near the initial values given by FDM. This might be that FDM does not give an accurate enough initial value.

References

[1] Jianxin Zhu and Ying Zhang. Cross orthogonality between eigenfunctions and conjugate eigenfunctions of a class of modified helmholtz operator for pekeris waveguide. *Journal of Theoretical and Computational Acoustics*, 27(02):1850048, 2019.