Eigenvalue Equation in 2-Layer Wave Guide with a Curved Interface

Qinghui

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1 problem setup

The original Helmholtz equation is

$$\rho\left(\frac{1}{\rho}\psi_z\right)_z + \psi_{xx} + \kappa(z)^2\psi = 0 \tag{1}$$

When the wave guide is stratified, and the interface is curved, namely,

$$\kappa(z) = \begin{cases} \kappa_1 & \text{if } z < h(x) \\ \kappa_2 & \text{if } z > h(x) \end{cases}, \quad \rho = \begin{cases} \rho_1 & \text{if } z < h(x) \\ \rho_2 & \text{if } z > h(x) \end{cases}$$

a othogonal transformation can be applied, so that under new coordinate $\hat{z}, \hat{x}$, the interface becomes flat.

$$\kappa(\hat{z}) = \begin{cases} \kappa_1 & \text{if } \hat{z} < 1 \\ \kappa_2 & \text{if } \hat{z} > 1 \end{cases}, \quad \rho = \begin{cases} \rho_1 & \text{if } \hat{z} < 1 \\ \rho_2 & \text{if } \hat{z} > 1 \end{cases}$$

The stratified waveguide under new axis is illustrated in the figure1. The new coordinate is divided into three layers along $\hat{z}$ axis, each span from $[0, 1], [1, D], [D, D_1]$. and continuously connected at 2 interfaces $\hat{z} = 1, \hat{z} = D$. In the third layer, z axis is not transformed, so $\hat{z} = z$ in this layer.

Different layers are denoted using superscripts in parentheses, (ℓ), where $\ell = 1, 2, 3$ corresponds to the first, second, third layer coordinate transformation.

After that, we introduce a equation transformation to eliminate the $\psi_{\hat{x}}$ term: let

$$\psi(\hat{x}, \hat{z}) = w(\hat{x}, \hat{z})\phi(\hat{x}, \hat{z}) \tag{2}$$

then, within each layer, the transformed Helmholtz equation becomes

$$\phi_{\hat{x}\hat{x}} + \alpha(\hat{x}, \hat{z})\phi_{\hat{z}\hat{z}} + \beta(\hat{x}, \hat{z})\phi_{\hat{z}} + \eta(\hat{x}, \hat{z})\phi = 0, \quad \hat{z} \neq 1, D$$

Recall that, $\hat{z} = z$ in the third layer of coordinate transformation. Since the refractive index does not vary along the original coordinate z axis in the third layer, it is also invariant along new axis $\hat{z}$. This is crucial for perfect matched layer(PML).

PML is used to truncate the open domain along $\hat{z}$ to a finite region. It absorbs wave's energy near the boundary to reduce the reflected waves.

It can be derived by a complex coordinate stretching,

$$\bar{z} = \hat{z} + i\int_0^{\hat{z}} \sigma(s)ds, \quad \sigma(s) = \begin{cases} 0 & \text{if } s < H \\ > 0 & \text{if } s \in [H, D_1] \end{cases} \tag{3}$$

the resulting effect is

$$\frac{d}{d\hat{z}} \longrightarrow \frac{1}{1 + i\sigma(\hat{z})}\frac{d}{d\hat{z}}$$

We add the PML only within the third layer, which means $H > D$.

After PML complex coordinate stretching, the eigenvalue problem for transformed Helmholtz equation (3)

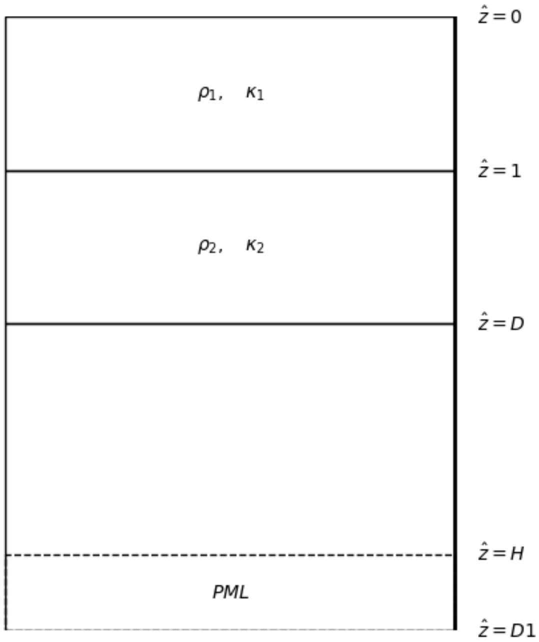

Figure 1: 3 layer coordinate transformed model
The new coordinate is divided into three layers along $\hat{z}$ axis, each span from $[0,1], [1,D], [D,D_1]$.

becomes

$$\begin{cases} \alpha \dfrac{d^2\phi}{d\bar{z}^2} + \beta \dfrac{d\phi}{d\bar{z}} + \eta\phi = \lambda\phi, \quad \bar{z} \neq 1, D \\ (w\phi)\Big|_{\hat{z}=1^-} = (w\phi)\Big|_{\hat{z}=1^+} \\ \dfrac{1}{\rho_1} w \left\{ \dfrac{1}{2}\left[h''(x) - 2\dfrac{h'(x)^2}{h(x)} \right]\phi - \dfrac{1+h'(x)^2}{h(x)}\phi_{\bar{z}} \right\}\Bigg|_{\hat{z}=1^-} \\ = \dfrac{1}{\rho_2} w \left\{ \dfrac{1}{2}\left[h''(x) + 2\dfrac{h'(x)^2}{D-h(x)} \right]\phi - \dfrac{D-1}{D-h(x)}\left[1+h'(x)^2\right]\phi_{\bar{z}} \right\}\Bigg|_{\hat{z}=1^+}, \\ (w\phi)\Big|_{\hat{z}=D^-} = (w\phi)\Big|_{\hat{z}=D^+} \\ w\left(\dfrac{1-D}{h(x)-D}\phi_{\bar{z}} \right)\Bigg|_{\hat{z}=D^-} = w\phi_{\bar{z}}\Big|_{\hat{z}=D^+}, \\ \phi|_{\hat{z}=0} = 0, \quad \phi|_{\hat{z}=D} = 0. \end{cases} \tag{4}$$

To further transform this eigenvalue problem to the standard form, we introduce a new function $\hat{\phi}$, and let

$$\phi(\bar{z}) = \hat{\phi}(\bar{z}) \cdot e^{\int_1^{\bar{z}} \mu(s)\, ds} \tag{5}$$

Plugging this into the first equation in (4) and canceling out the term $e^{\int_1^{\bar{z}} \mu(s)\, ds}$, we get:

$$\hat{\phi}(\bar{z})\left[\alpha(\bar{z})\mu'(\bar{z}) + \alpha(\bar{z})\mu(\bar{z})^2 + \beta(\bar{z})\mu(\bar{z}) + \eta(\bar{z})\right] + \frac{d\hat{\phi}(\bar{z})}{d\bar{z}}\left(2\alpha(\bar{z})\mu(\bar{z}) + \beta(\bar{z})\right) + \alpha(\bar{z})\frac{d^2\hat{\phi}(\bar{z})}{d\bar{z}^2} = \lambda\hat{\phi}(\bar{z}) \tag{6}$$

To let the terms with $\frac{d\hat{\phi}(\bar{z})}{d\bar{z}}$ to zero, $2\alpha(\bar{z})\mu(\bar{z}) + \beta(\bar{z}) = 0$. And the resulting $\mu(\bar{z})$ is:

$$\mu(\bar{z}) = -\frac{\beta(\bar{z})}{2\alpha(\bar{z})} \tag{7}$$

plug $\mu(\bar{z})$ from (7) into (6), the eigenvalue equation for $\hat{\phi}(\bar{z})$ becomes:

$$\left(\frac{\frac{1}{2}(-\beta'(\bar z)\alpha(\bar z)+\beta(\bar z)\alpha'(\bar z))-\frac{1}{4}\beta^2(\bar z)}{\alpha(\bar z)}+\eta(\bar z)\right)\hat\phi(\bar z)+\alpha(\bar z)\frac{d^2\hat\phi(\bar z)}{d\bar z^2}=\lambda\hat\phi(\bar z) \tag{8}$$

We denote it as $p(\bar z)\hat\phi+\alpha(\bar z)\hat\phi_{\bar z\bar z}=\lambda\hat\phi$

Together with the interface condition for $\hat\phi$, which can be derived by plugging (5) into the boundary conditions in (4), the complete eigenvalue problem for the new function $\hat\phi$ is

$$\begin{cases} p(\bar z)\hat\phi+\alpha(\bar z)\hat\phi_{\bar z\bar z}=\lambda\hat\phi \\ \left(w\hat\phi\right)\Big|_{\hat z=1^-}=\left(w\hat\phi\right)\Big|_{\hat z=1^+} \\ \frac{1}{\rho_1}w\left\{\left[\frac{1}{2}h''(x)-\frac{h'(x)^2}{h(x)}-\frac{1+h'(x)^2}{h(x)}\mu\right]\hat\phi-\frac{1+h'(x)^2}{h(x)}\hat\phi_{\bar z}\right\}\Big|_{\hat z=1^-} \\ =\frac{1}{\rho_2}w\left\{\left[\frac{1}{2}h''(x)+\frac{h'(x)^2}{D-h(x)}-\frac{D-1}{D-h(x)}\left(1+h'(x)^2\right)\mu\right]\hat\phi-\frac{D-1}{D-h(x)}\left(1+h'(x)^2\right)\hat\phi_{\bar z}\right\}\Big|_{\hat z=1^+}, \\ \left(w\hat\phi\right)\Big|_{\hat z=D^-}=\left(w\hat\phi\right)\Big|_{\hat z=D^+}, \\ w\left[\frac{1-D}{h(x)-D}\hat\phi_{\bar z}\right]\Big|_{\hat z=D^-}=w\hat\phi_{\bar z}\Big|_{\hat z=D^+}, \\ \hat\phi|_{\hat z=0}=0,\quad \hat\phi|_{\hat z=D_1}=0. \end{cases} \tag{9}$$

note that we eliminate terms with μ in the condition at $\hat z=D$, as $\mu=0$ when $\hat z\geq D$.

We can write the interface condition into matrix form, denote

$$\begin{aligned} T_{1^-}&=\left[\begin{matrix} w & 0 \\ \frac{1}{\rho_1}w\left(\frac{1}{2}h''(x)-\frac{h'(x)^2}{h(x)}-\frac{1+h'(x)^2}{h(x)}\mu\right) & -\frac{1}{\rho_1}w\frac{1+h'(x)^2}{h(x)}\end{matrix}\right]\Bigg|_{\hat z=1^-} \\ T_{1^+}&=\left[\begin{matrix} w & 0 \\ \frac{1}{\rho_2}w\left(\frac{1}{2}h''(x)+\frac{h'(x)^2}{D-h(x)}-\frac{D-1}{D-h(x)}\left(1+h'(x)^2\right)\mu\right) & -\frac{1}{\rho_2}w\frac{D-1}{D-h(x)}\left(1+h'(x)^2\right)\end{matrix}\right]\Bigg|_{\hat z=1^-} \end{aligned} \tag{10}$$

Then, the interface condition at $\hat z=1$ can be expressed as

$$T_{1^-}\begin{bmatrix}\hat\phi\\ \hat\phi'\end{bmatrix}\Bigg|_{\hat z=1^-}=T_{1^+}\begin{bmatrix}\hat\phi\\ \hat\phi'\end{bmatrix}\Bigg|_{\hat z=1^+} \tag{11}$$

similarly, for the second interface at $\hat z=D$, denote

$$T_{D^-}=\begin{bmatrix} w & 0 \\ 0 & \frac{(D-1)w}{D-h(x)}\end{bmatrix}\Bigg|_{\hat z=D^-},T_{D^+}=\begin{bmatrix} w & 0\\ 0 & w\end{bmatrix}\Bigg|_{\bar z=D^+} \tag{12}$$

and the interface condition can be expressed as

$$T_{D^-}\begin{bmatrix}\hat\phi\\ \hat\phi'\end{bmatrix}\Bigg|_{\hat z=D^-}=T_{D^+}\begin{bmatrix}\hat\phi\\ \hat\phi'\end{bmatrix}\Bigg|_{\hat z=D^+} \tag{13}$$

2 dispersion relation Approximation

In the first and second layers $\hat z\in(0,1),(1,D_1)$, where there is no PML coordinate stretching, $\hat z=\bar z$, so we can use variable $\hat z$ instead in eigenvalue equation (8). Rearrange it then we get:

$$\hat\phi_{\hat z\hat z}(\hat z)+\frac{p(\hat z)-\lambda}{\alpha(\hat z)}\hat\phi(\hat z)=0.$$

$$\text{denote it as } \hat\phi_{\hat z\hat z}(\hat z)+K(\hat z)\hat\phi(\hat z)=0. \tag{14}$$

The function $K(\hat z)$ can be approximated by a piece wise polynomial of degree two.

We divide the interval $[0, 1]$ into m_1 sub intervals $[(j-1)l_1, jl_1], \quad (j = 1, 2, \ldots, m_1)$, and interval $[1, D]$ into m_2 sub intervals, $[1 + (j - m_1 - 1)l_2, 1 + (j - m_1)l_2], \quad (j = m_1 + 1, m_1 + 2, ..., m_1 + m_2)$ The sub intervals are denoted in order by $I_j = (\hat{z}_{j-1}, \hat{z}_j)$

$$\hat{z}_j = \begin{cases} jl_1, & \text{if } 1 \leq j \leq m_1, \\ 1 + (j - m_1)l_2, & \text{if } m_1 + 1 \leq j \leq m_1 + m_2. \end{cases}$$

On each subinterval, the function $K(\hat{z})$ can be interpolated by a polynomial of degree two with three nodes chosen to be the two endpoints and a midpoint. By approximating $K(\hat{z})$, (14) is now approximated by

$$\frac{d^2 y_j}{d\hat{z}^2} + (a_j \hat{z}^2 + b_j \hat{z} + c_j) y_j = 0, \quad \hat{z} \in I_j \tag{15}$$

where y_j is the approximation of the $\hat{\phi}$ on the interval I_j and the a, b, c are:
In the first layer:

$$\begin{cases} a_j = \dfrac{2K(t_0)}{l_1^2} - \dfrac{4K(t_1)}{l_1^2} + \dfrac{2K(t_2)}{l_1^2}, \\ b_j = \dfrac{(1-4j)K(t_0)}{l_1} + \dfrac{(8j-4)K(t_1)}{l_1} + \dfrac{(3-4j)K(t_2)}{l_1}, \\ c_j = \left(2j^2 - j\right) K(t_0) + \left(4j - 4j^2\right) K(t_1) + \left(2j^2 - 3j + 1\right) K(t_2) \\ t_0 = l_1(j-1), \quad t_1 = (j - 1/2)l_1, \quad t_2 = jl_1 \\ \; j = 1, 2, \ldots, m_1 \end{cases} \tag{16}$$

In the second layer:

$$\begin{cases} a_j = \dfrac{2K(t_0)}{l_2^2} - \dfrac{4K(t_1)}{l_2^2} + \dfrac{2K(t_2)}{l_2^2} \\ b_j = \dfrac{K(t_0)(-4l_2 j + 4l_2 m_1 + l_2 - 4)}{l_2^2} \\ \quad + \dfrac{K(t_1)(l_2(8j - 8m_1 - 4) + 8)}{l_2^2} \\ \quad + \dfrac{K(t_2)(l_2(-4j + 4m_1 + 3) - 4)}{l_2^2}, \\ c_j = \dfrac{K(t_0)\left(l_2^2\left(2j^2 - 4jm_1 - j + 2m_1^2 + m_1\right) + l_2(4j - 4m_1 - 1) + 2\right)}{l_2^2} \\ \quad + \dfrac{4K(t_1)(-l_2(j - m_1 - 1) - 1)(l_2(j - m_1) + 1)}{l_2^2} \\ \quad + \dfrac{K(t_2)(-l_2(j - m_1 - 1) - 1)(-2l_2 j + 2l_2 m_1 + l_2 - 2)}{l_2^2}, \\ t_0 = 1 + (j - m_1 - 1)l_2, \quad t_1 = 1 + (j - m_1 - 1/2)l_2, \quad t_2 = 1 + (j - m_1)l_2 \\ \; j = m_1 + 1, m_1 + 1, \ldots, m_2 + m_1 \end{cases} \tag{17}$$

The solution of (15) at each interval is given by confluent hypergeometric functions. Let $\{\ u_j(\hat{z}), v_j(\hat{z})\ \}$ be a fundamental pair on the jth interval, then

$$\begin{aligned} & y_j(\hat{z}) = A_j u_j(\hat{z}) + B_j v_j(\hat{z}), \quad \hat{z} \in I_j \\ & j = 1, 2, \ldots m_1 + m_2. \end{aligned} \tag{18}$$

In the third layer $\hat{z} \in [D, D_1]$, the eigenvalue equation(8) is simplified to

$$\alpha^{(3)} \hat{\phi}_{\bar{z}\bar{z}} + \eta^{(3)} \hat{\phi} = \lambda \hat{\phi} \tag{19}$$

as $\alpha^{(3)}, \eta^{(3)}$ are constants and $\beta^{(3)} = 0$. Denote$\gamma^{(3)} = \sqrt{\frac{\eta^{(3)} - \lambda}{\alpha^{(3)}}}$, and the solution of (19) is simply

$$\hat{\phi}(\hat{z}) = C_1 e^{-i\gamma^{(3)}\bar{z}} + C_2 e^{i\gamma^{(3)}\bar{z}} \quad \hat{z} \in [D, D_1] \tag{20}$$

To make the notation consistent, we can also denote (20) as

$$y_{m_1+m_2+1}(\hat{z}) = A_{m_1+m_2+1}u_{m_1+m_2+1}(\hat{z}) + B_{m_1+m_2+1}v_{m_1+m_2+1}(\hat{z})$$

notice we also use variable $\hat{z}$, instead of $\bar{z}$, in $u_{m_1+m_2+1}, v_{m_1+m_2+1}$.

In the first sub interval, The zero boundary condition $\hat{\phi}(0) = 0$ gives the first relation:

$$A_1u_1(0) + B_1v_1(0) = 0$$

Suppose the approximation of $\hat{\phi}$ in each layer is continuous and has continuous first-order derivative. This requires that at the interface $\hat{z}_j$:

$$\begin{bmatrix} u_{j+1}(\hat{z}_j) & v_{j+1}(\hat{z}_j) \\ u'_{j+1}(\hat{z}_j) & v'_{j+1}(\hat{z}_j) \end{bmatrix} \begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \begin{bmatrix} u_j(\hat{z}_j) & v_j(\hat{z}_j) \\ u'_j(\hat{z}_j) & v'_j(\hat{z}_j) \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix} \tag{21}$$

Moreover,from (11).We have a relation at the interface $\hat{z} = 1$. Denote

$$T = T_{1^+}^{-1}T_{1^-}, \quad T = \begin{bmatrix} t_{11} & 0 \\ t_{21} & t_{22} \end{bmatrix} \tag{22}$$

then

$$\begin{bmatrix} u_{m_1+1}(1) & v_{m_1+1}(1) \\ u'_{m_1+1}(1) & v'_{m_1+1}(1) \end{bmatrix} \begin{bmatrix} A_{m_1+1} \\ B_{m_1+1} \end{bmatrix} = T \begin{bmatrix} u_{m_1}(1) & v_{m_1}(1) \\ u'_{m_1}(1) & v'_{m_1}(1) \end{bmatrix} \begin{bmatrix} A_{m_1} \\ B_{m_1} \end{bmatrix} \tag{23}$$

Similarly, from (13) denote

$$\overline{T} = T_{D^+}^{-1}T_{D^-}, \quad \overline{T} = \begin{bmatrix} \bar{t}_{11} & 0 \\ 0 & \bar{t}_{22} \end{bmatrix} \tag{24}$$

then

$$\begin{bmatrix} u_{m_1+m_2+1}(D) & v_{m_1+m_2+1}(D) \\ u'_{m_1+m_2+1}(D) & v'_{m_1+m_2+1}(D) \end{bmatrix} \begin{bmatrix} A_{m_1+m_2+1} \\ B_{m_1+m_2+1} \end{bmatrix} = \overline{T} \begin{bmatrix} u_{m_1+m_2}(D) & v_{m_1+m_2}(D) \\ u'_{m_1+m_2}(D) & v'_{m_1+m_2}(D) \end{bmatrix} \begin{bmatrix} A_{m_1+m_2} \\ B_{m_1+m_2} \end{bmatrix} \tag{25}$$

The zero boundary condition at D_1 in (8) implies

$$\begin{aligned} 0 &= A_{m_1+m_2+1}e^{-i\gamma^{(3)}\bar{D_1}} + B_{m_1+m_2+1}e^{i\gamma^{(3)}\bar{D_1}} \\ &\text{where } \bar{D_1} = D_1 + i\int_0^{D_1} \sigma(s)ds \end{aligned} \tag{26}$$

We can obtain a linear system of A_j, B_j. by putting together (21)(23)(25)(26)

$$\begin{cases} A_1u_1(0) + B_1v_1(0) = 0 \\ \\ u_{j+1}(\hat{z}_j)A_{j+1} + v_{j+1}(\hat{z}_j)B_{j+1} = u_j(\hat{z}_j)A_j + v_j(\hat{z}_j)B_j, \\ u'_{j+1}(\hat{z}_j)A_{j+1} + v'_{j+1}(\hat{z}_j)B_{j+1} = u'_j(\hat{z}_j)A_j + v'_j(\hat{z}_j)B_j. \\ j = 1, \ldots, m_1 - 1 \quad \text{and} \quad j = m_1 + 1, \ldots, m_1 + m_2 - 1, \\ \\ u_{m_1+1}(1)A_{m_1+1} + v_{m_1+1}(1)B_{m_1+1} = T_{11}\left[u_{m_1}(1)A_{m_1} + v_{m_1}(1)B_{m_1+1}\right], \\ u'_{m_1+1}(1)A_{m_1+1} + v'_{m_1+1}(1)B_{m_1+1} = \\ \left[T_{21}u_{m_1}(1) + T_{22}u'_{m_1}(1)\right]A_{m_1} + \left[T_{21}v_{m_1}(1) + T_{22}v'_{m_1}(1)\right]B_{m_1}, \\ \\ u_{m_1+m_2+1}(D)A_{m_1+m_2+1} + v_{m_1+m_2+1}(D)B_{m_1+m_2+1} = \\ \overline{T}_{11}\left[u_{m_1+m_2}(D)A_{m_1+m_2} + v_{m_1+m_2}(D)B_{m_1+m_2}\right], \\ \\ u'_{m_1+m_2+1}(D)A_{m_1+m_2+1} + v_{m_1+m_2+1}(D)'B_{m_1+m_2+1} = \\ \overline{T}_{22}\left[u'_{m_1+m_2}(D)A_{m_1+m_2} + v'_{m_1+m_2}(D)B_{m_1+m_2}\right], \\ \\ A_{m_1+m_2+1}e^{-i\gamma^{(3)}\bar{D_1}} + B_{m_1+m_2+1}e^{i\gamma^{(3)}\bar{D_1}} = 0 \end{cases} \tag{27}$$

The above linear system leads to an algebraic equation for λ, by eliminating all the A_j and B_j. Denote $R_j = \frac{A_j}{B_j}, (j = 1, 2, \ldots, m_1 + m_2 + 1)$. We can derive the dispersion relation of λ

$$R_1 = -\frac{v_1(0)}{u_1(0)} \tag{28}$$

$$R_{j+1} = \frac{R_j\left(u_j'(\hat{z}_j)v_{j+1}(\hat{z}_j) - u_j(\hat{z}_j)v_{j+1}'(\hat{z}_j)\right) - v_j(\hat{z}_j)v_{j+1}'(\hat{z}_j) + v_{j+1}(\hat{z}_j)v_j'(\hat{z}_j)}{R_j\left(u_{j+1}'(\hat{z}_j)u_j(\hat{z}_j) - u_j'(\hat{z}_j)u_{j+1}(\hat{z}_j)\right) + u_{j+1}'(\hat{z}_j)v_j(\hat{z}_j) - u_{j+1}(\hat{z}_j)v_j'(\hat{z}_j)},$$
$$j = 1, \dots, m_1 - 1 \quad \text{and} \quad j = m_1+1, \dots, m_1+m_2-1, \tag{29}$$

$$\begin{aligned}
&R_{m_1+1} = \\
&\Big\{R_{m_1}\left[u_{m_1}'(1)v_{m_1+1}(1)T_{22} + u_{m_1}(1)v_{m_1+1}(1)T_{21} - u_{m_1}(1)v_{m_1+1}'(1)T_{11}\right] + \dots \\
&v_{m_1}(1)v_{m_1+1}(1)T_{21} - v_{m_1}(1)v_{m_1+1}'(1)T_{11} + v_{m_1+1}(1)v_{m_1}'(1)T_{22}\Big\}\Big/ \\
&\Big\{R_{m_1}\left[u_{m_1+1}'(1)u_{m_1}(1)T_{11} - u_{m_1}'(1)u_{m_1+1}(1)T_{22} - u_{m_1}(1)u_{m_1+1}(1)T_{21}\right] + \dots \\
&u_{m_1+1}'(1)v_{m_1}(1)T_{11} - u_{m_1+1}(1)v_{m_1}(1)T_{21} - u_{m_1+1}(1)v_{m_1}'(1)T_{22}\Big\}
\end{aligned} \tag{30}$$

$$\begin{aligned}
&R_{m_1+m_2+1} = \\
&\Big\{R_{m_1+m_2}\left[\overline{T}_{22}\, v_{m_1+m_2+1}(D)\, u_{m_1+m_2}'(D) - \overline{T}_{11}\, u_{m_1+m_2}(D)\, v_{m_1+m_2+1}'(D)\right] - \dots \\
&\overline{T}_{11}\, v_{m_1+m_2}(D)\, v_{m_1+m_2+1}'(D) + \overline{T}_{22}\, v_{m_1+m_2+1}(D)\, v_{m_1+m_2}'(D)\Big\}\Big/ \\
&\Big\{R_{m_1+m_2}\left[\overline{T}_{11}\, u_{m_1+m_2}(D)\, u_{m_1+m_2+1}'(D) - \overline{T}_{22}\, u_{m_1+m_2+1}(D)\, u_{m_1+m_2}'(D)\right] + \dots \\
&\overline{T}_{11}\, v_{m_1+m_2}(D)\, u_{m_1+m_2+1}'(D) - \overline{T}_{22}\, u_{m_1+m_2+1}(D)\, v_{m_1+m_2}'(D)\Big\}
\end{aligned} \tag{31}$$

$$R_{m_1+m_2+1} = -e^{2i\gamma^{(3)}\bar{D}_1} \tag{32}$$

λ involved in each u_j, v_j. So $R_{m_1+m_2+1}$ is determined by λ recursively through (28)(29)(30)(31). On the other hand, $R_{m_1+m_2+1}$ is determined by λ in (32). Therefore, the dispersion relation has the form

$$g(\lambda) = R_{m_1+m_2+1} + e^{2i\gamma^{(3)}\bar{D}_1} = 0 \tag{33}$$

3 special case

when the wave guide interface is flat, with $h(x) \equiv 1$, the orthogonal coordinate transformation will be reduced to an identity mapping, namely $(\hat{x}, \hat{z}) = (x, z)$. And the equation transform (5) will be an identity transformation, as $w \equiv 1$. Therefore, in the transformed Helmholtz equation(3), $\alpha = 1, \beta = 0, \eta = \kappa^2$ And the eigenvalue problem (4) will be reduced to

$$\begin{cases}
\phi_{\bar{z}\bar{z}} + \kappa(z)^2\phi = \lambda\phi \\
\phi|_{z=1^-} = \phi|_{z=1^+} \\
\dfrac{1}{\rho_1}\phi_{\bar{z}}|_{z=1^-} = \dfrac{1}{\rho_2}\phi_{\bar{z}}|_{z=1^+} \\
\phi|_{z=D^-} = \phi|_{z=D^+} \\
\phi_{\bar{z}}|_{z=D^-} = \phi_{\bar{z}}|_{z=D^+} \\
\phi|_{z=0} = 0 \\
\phi|_{z=D_1} = 0
\end{cases} \tag{34}$$

Given that $\beta = 0$, from the definition of $\hat{\phi}$ (5) we have $\hat{\phi} = \phi$. so $\hat{\phi}$ should also satisfy the reduced eigenvalue problem(34).

Denote

$$\gamma^{(1)} = \sqrt{\kappa_1^2 - \lambda}, \quad \gamma^{(2)} = \sqrt{\kappa_2^2 - \lambda}$$

the 2 basis solution $\{u_i, v_j\}$ in each sub interval in the first 2 layer $[0,1], [1, D_1]$are simply

$$u_j = e^{-i\gamma^{(\ell)}z}, \quad v_j = e^{i\gamma^{(\ell)}z} \quad z \in I_j \tag{35}$$

$$\gamma^{(\ell)} = \begin{cases} \gamma^{(1)} & \text{if } j \le m_1 \\ \gamma^{(2)} & \text{if } j > m_1 \end{cases} \quad j = 1, 2, \dots m_1 + m_2$$

The basis solutions within one layer are continuous at each node $\hat{z}_j$ naturally:

$$\begin{aligned} &u_j'(z_j) = u_{j+1}'(z_j), \quad u_j(z_j) = u_{j+1}(z_j) \\ &v_j'(z_j) = v_{j+1}'(z_j), \quad v_j(z_j) = v_{j+1}(z_j) \\ &j = 1, \dots, m_1 - 1 \text{ and } j = m_1 + 1, \dots, m_1 + m_2 - 1 \end{aligned}$$

Thus, the iterative relation of R_j within a layer(29) simply becomes

$$\begin{aligned} &R_{j+1} = R_j \\ &j = 1, \dots, m_1 - 1 \text{ and } j = m_1 + 1, \dots, m_1 + m_2 - 1 \end{aligned} \tag{36}$$

The zero boundary condition $\hat{\phi}(0) = 0$ give rise to $R_1 = \frac{A_1}{B_1} = -1$ Therefore, in the first layer,$R_j \equiv -1, \quad j = 1, 2, \dots, m_1$

From the simplified interface condition (34), Interface condition matrices become

$$T_{1^-} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho_1} \end{bmatrix}, \quad T_{1^+} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho_2} \end{bmatrix} \tag{37}$$

T_{D^-} andT_{D^+} all reduced to identity matrix. Thus,

$$T = T_{1^+}^{-1} T_{1^-} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\rho_2}{\rho_1} \end{bmatrix}, \quad \overline{T} = T_{D^+}^{-1} T_{D^-} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{38}$$

The relation to transfer from the first layer to the second layer(30) becomes

$$\begin{aligned} &R_{m_1+1} = \\ &\left\{ -\left[\frac{\rho_2}{\rho_1} u_{m_1}'(1) v_{m_1+1}(1) - u_{m_1}(1) v_{m_1+1}'(1) \right] - v_{m_1}(1) v_{m_1+1}'(1) + v_{m_1+1}(1) v_{m_1}'(1) \frac{\rho_2}{\rho_1} \right\} \Big/ \\ &\left\{ -\left[u_{m_1+1}'(1) u_{m_1}(1) - \frac{\rho_2}{\rho_1} u_{m_1}'(1) u_{m_1+1}(1) \right] + u_{m_1+1}'(1) v_{m_1}(1) - \frac{\rho_2}{\rho_1} u_{m_1+1}(1) v_{m_1}'(1) \right\} \end{aligned}$$

plug in the exact expression for u, v in (35):

$$R_{m_1+1} = \frac{e^{i\gamma_2} \Big\{ -e^{-i\gamma_1} \left[-i\gamma_1 \frac{\rho_2}{\rho_1} - i\gamma_2 \right] + e^{i\gamma_1} \left[-i\gamma_2 + i\gamma_1 \frac{\rho_2}{\rho_1} \right] \Big\}}{e^{-i\gamma_2} \Big\{ -e^{-i\gamma_1} \left[-i\gamma_2 + i\gamma_1 \frac{\rho_2}{\rho_1} \right] + e^{i\gamma_1} \left[-i\gamma_2 - i\gamma_1 \frac{\rho_2}{\rho_1} \right] \Big\}}$$

simplify it

$$R_{m_1+1} = -e^{2i\gamma_2} \frac{e^{-i\gamma_1} \left[\gamma_1 \frac{\rho_2}{\rho_1} + \gamma_2 \right] + e^{i\gamma_1} \left[-\gamma_2 + \gamma_1 \frac{\rho_2}{\rho_1} \right]}{e^{-i\gamma_1} \left[-\gamma_2 + \gamma_1 \frac{\rho_2}{\rho_1} \right] + e^{i\gamma_1} \left[\gamma_2 + \gamma_1 \frac{\rho_2}{\rho_1} \right]} \tag{39}$$

As the second interface does not exist, the transfer relation from second to the third layer (31) is reduced to $R_{m_1+m_2+1} = R_{m_1+m_2}$ What's more,$R_j \equiv \text{Constant} j = m_1 + 1 \dots m_1 + m_2$, so we have

$$R_{m_1+m_2+1} = R_{m_1+m_2} = R_{m_1+1} = \text{ the right hand side of(39)}$$

In this case, $\gamma^{(3)} = \gamma^{(2)}$. so we substitute $\gamma^{(3)}$ with $\gamma^{(2)}$. The dispersion relation we derived (33) is reduced to

$$-e^{2i\gamma^{(2)}} \frac{e^{-i\gamma^{(1)}} \left[\gamma^{(1)} \frac{\rho_2}{\rho_1} + \gamma^{(2)} \right] + e^{i\gamma^{(1)}} \left[-\gamma^{(2)} + \gamma^{(1)} \frac{\rho_2}{\rho_1} \right]}{e^{-i\gamma^{(1)}} \left[-\gamma^{(2)} + \gamma^{(1)} \frac{\rho_2}{\rho_1} \right] + e^{i\gamma^{(1)}} \left[\gamma^{(2)} + \gamma^{(1)} \frac{\rho_2}{\rho_1} \right]} + e^{2i\gamma^{(2)} \bar{D}_1} = 0 \tag{40}$$

rearrange it, and expand $e^{i\gamma^{(1)}}, e^{-i\gamma^{(1)}}$ into trigonometric function, we have

$$\frac{i\gamma^{(1)} \frac{\rho_2}{\rho_1} cos(\gamma^{(1)}) - i\gamma^{(2)} sin(\gamma^{(1)})}{i\gamma^{(2)} sin(\gamma^{(1)}) + \gamma^{(1)} \frac{\rho_2}{\rho_1} cos(\gamma^{(1)})} = e^{2i\gamma^{(2)}(\bar{D}_1 - 1)} \tag{41}$$

multiply both the numerator and denominator of the left hand side by i, it will become the same as the dispersion relation in[1], this result theoretically verified new relation(33) we derived.

4 Asymptotic solution

First, we use a piece wise constant function to approximate $K(\hat{z})$ in each interpolate sub interval.

$$\hat{\phi}_{\hat{z}\hat{z}} + K(t_1)\phi = 0 \quad \hat{z} \in I_j \tag{42}$$

$K(t_1)$ is defined in (16)(17) Under this simplification, the basis solution at each interval becomes:

$$u_j = e^{-i\gamma_j z}, \quad v_j = e^{i\gamma_j z} \quad z \in I_j \tag{43}$$

$$\gamma_j = \sqrt{\frac{p(t_1) - \lambda}{\alpha(t_1)}}$$

The continuous condition within a layer becomes

$$\begin{cases} A_{j+1}e^{-i\gamma_{j+1}\hat{z}_j} + B_{j+1}e^{i\gamma_{j+1}\hat{z}_j} = A_{j+1}e^{-i\gamma_{j+1}\hat{z}_j} + B_{j+1}e^{i\gamma_{j+1}\hat{z}_j} \\ -i\gamma_{j+1}A_{j+1}e^{-i\gamma_{j+1}\hat{z}_j} + i\gamma_{j+1}B_{j+1}e^{i\gamma_{j+1}\hat{z}_j} = -i\gamma_{j+1}A_{j+1}e^{-i\gamma_{j+1}\hat{z}_j} + B_{j+1}e^{i\gamma_{j+1}\hat{z}_j} \\ j = 1, \ldots, m_1 - 1 \quad \text{and} \quad j = m_1 + 1, \ldots, m_1 + m_2 - 1, \end{cases} \tag{44}$$

We can write it as a matrix form, which gives an iteration relation of A_j, B_j.

$$\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \begin{bmatrix} \frac{e^{-i(\gamma_j-\gamma_{j+1})}(\gamma_j+\gamma_{j+1})}{2\gamma_{j+1}} & \frac{e^{i(\gamma_j+\gamma_{j+1})}(\gamma_{j+1}-\gamma_j)}{2\gamma_{j+1}} \\ \frac{e^{-i(\gamma_j+\gamma_{j+1})}(\gamma_{j+1}-\gamma_j)}{2\gamma_{j+1}} & \frac{e^{i(\gamma_j-\gamma_{j+1})}(\gamma_j+\gamma_{j+1})}{2\gamma_{j+1}} \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix} \tag{45}$$

denote $\hat{z}_{j+1} = \hat{z}_j + \Delta z$

$$\frac{\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} - \begin{bmatrix} A_j \\ B_j \end{bmatrix}}{\Delta z} = \frac{\begin{bmatrix} \frac{e^{-i(\gamma_j-\gamma_{j+1})}(\gamma_j+\gamma_{j+1})-2\gamma_{j+1}}{2\gamma_{j+1}} & \frac{e^{i(\gamma_j+\gamma_{j+1})}(\gamma_{j+1}-\gamma_j)}{2\gamma_{j+1}} \\ \frac{e^{-i(\gamma_j+\gamma_{j+1})}(\gamma_{j+1}-\gamma_j)}{2\gamma_{j+1}} & \frac{e^{i(\gamma_j-\gamma_{j+1})}(\gamma_j+\gamma_{j+1})-2\gamma_{j+1}}{2\gamma_{j+1}} \end{bmatrix}}{\Delta z} \begin{bmatrix} A_j \\ B_j \end{bmatrix} \tag{46}$$

When Δz approaches 0 the limit of (46) is:

$$\frac{d}{d\hat{z}} \begin{bmatrix} A \\ B \end{bmatrix} = \frac{\gamma'}{2\gamma} \begin{bmatrix} -1 + 2i\gamma\hat{z} & e^{2i\gamma\hat{z}} \\ e^{-2i\gamma\hat{z}} & -1 - 2i\gamma\hat{z} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \quad \hat{z} \neq 1, D \tag{47}$$

Here, we assume $p(\hat{z})$ and $\alpha(\hat{z})$ have first order derivative, thus γ' exist.

If $p(\hat{z})$ and $\alpha(\hat{z})$ varies gradually (when the interface $h(x)$ varies gradually), and$|\lambda|$is large we have $\left|\frac{\gamma'}{\gamma}\right| \approx 0$. (47) can be approximated as

$$\frac{d}{d\hat{z}} \begin{bmatrix} A \\ B \end{bmatrix} \approx \begin{bmatrix} i\gamma'\hat{z} & 0 \\ 0 & -i\gamma'\hat{z} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \tag{48}$$

The solution of (48) is:

$$\text{denote } U^{(\ell)} = \begin{bmatrix} i\gamma^{(\ell)\,\prime}\hat{z} & 0 \\ 0 & -i\gamma^{(\ell)\,\prime}\hat{z} \end{bmatrix} \tag{49}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=1^-} \approx e^{\int_0^1 U^{(1)}\,d\hat{z}} \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=0}, \quad \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=D^-} \approx e^{\int_1^D U^{(2)}\,d\hat{z}} \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=1^+} \tag{50}$$

We denote $T_{0\to 1^-} = e^{\int_0^1 U^{(1)}\,d\hat{z}}$ and $T_{1^+\to D^-} = e^{\int_1^D U^{(2)}\,d\hat{z}}$. Since $\gamma^{(3)}$ is a constant, the transfer matrix from $D^+ to D_1$ is an identity matrix, so we omit it.

The transfer matrices across the interfaces $\hat{z} = 1$ and D are listed below. We denote $\gamma_{11} = \gamma^{(1)}(1), \quad \gamma_{21} = \gamma^{(2)}(1), \quad \gamma_{2D} = \gamma^{(2)}(D)$

$$\begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=1^+} = T_{1^-\to 1^+} \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=1^-} \quad \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=D^+} = T_{D^-\to D^+} \begin{bmatrix} A \\ B \end{bmatrix}\Bigg|_{z=D^-} \tag{51}$$

$$T_{1^-\to 1^+} = \begin{bmatrix} \frac{e^{-i(\gamma_{11}-\gamma_{21})}(\gamma_{11}t_{22}+\gamma_{21}t_{11}+it_{21})}{2\gamma_{21}} & \frac{e^{i(\gamma_{11}+\gamma_{21})}(-\gamma_{11}t_{22}+\gamma_{21}t_{11}+it_{21})}{2\gamma_{21}} \\ \frac{e^{-i(\gamma_{11}+\gamma_{21})}(-\gamma_{11}t_{22}+\gamma_{21}t_{11}-it_{21})}{2\gamma_{21}} & \frac{e^{i(\gamma_{11}-\gamma_{21})}(\gamma_{11}t_{22}+\gamma_{21}t_{11}-it_{21})}{2\gamma_{21}} \end{bmatrix} \tag{52}$$

$$T_{D^-\to D^+} = \begin{bmatrix} \frac{e^{-i(\gamma_{2D}-\gamma^{(3)})}(\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} & \frac{e^{i(\gamma_{2D}+\gamma^{(3)})}(-\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} \\ \frac{e^{-i(\gamma_{2D}+\gamma^{(3)})}(-\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} & \frac{e^{i(\gamma_{2D}-\gamma^{(3)})}(\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} \end{bmatrix} \tag{53}$$

The t_{ij} and $\bar{t}_{ij}$ are defined in (22)(24). When $|\lambda|$ is large (52)can be approximated by:

$$T_{1^- \to 1^+} \approx \begin{bmatrix} \frac{e^{-i(\gamma_{11}-\gamma_{21})}(\gamma_{11}t_{22}+\gamma_{21}t_{11})}{2\gamma_{21}} & \frac{e^{i(\gamma_{11}+\gamma_{21})}(-\gamma_{11}t_{22}+\gamma_{21}t_{11})}{2\gamma_{21}} \\ \frac{e^{-i(\gamma_{11}+\gamma_{21})}(-\gamma_{11}t_{22}+\gamma_{21}t_{11})}{2\gamma_{21}} & \frac{e^{i(\gamma_{11}-\gamma_{21})}(\gamma_{11}t_{22}+\gamma_{21}t_{11})}{2\gamma_{21}} \end{bmatrix} \tag{54}$$

What's more, when the derivatives of the interface $h(x)$ are small, we can show that $\gamma_{2D}\bar{t}_{22} \approx \gamma^{(3)}\bar{t}_{11}$, the analysis is in appendix(A). Thus, (53) can be approximated as

$$T_{D^- \to D^+} \approx \begin{bmatrix} \frac{e^{-i(\gamma_{2D}-\gamma^{(3)})}(\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} & 0 \\ 0 & \frac{e^{i(\gamma_{2D}-\gamma^{(3)})}(\gamma_{2D}\bar{t}_{22}+\gamma^{(3)}\bar{t}_{11})}{2\gamma^{(3)}} \end{bmatrix} \tag{55}$$

From the boundary condition $\hat{\phi}(0) = 0$, we can set $A(0) = 1, B(0) = -1$, and another boundary condition $\hat{\phi}(\bar{D_1}) = 0$, gives $A(D^+)e^{-i\gamma^{(3)}\bar{D_1}} + B(D^+)e^{i\gamma^{(3)}\bar{D_1}} = 0$. The dispersion relation is derived:

$$\begin{bmatrix} e^{-i\gamma^{(3)}\bar{D_1}}, & e^{i\gamma^{(3)}\bar{D_1}} \end{bmatrix} T_{D^- \to D^+} T_{1^+ \to D^-} T_{1^- \to 1^+} T_{0 \to 1^-} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \tag{56}$$

plug in the approximated transfer matrices(50)(54)(55) and expand, we derived the approximated dispersion relation:

$$\frac{\gamma_{21}t_{11}\left(-1+e^{2i\int_0^1 \gamma^{(1)}(z)\,dz}\right) - \gamma_{11}t_{22}\left(1+e^{2i\int_0^1 \gamma^{(1)}(z)\,dz}\right)}{\gamma_{11}t_{22}\left(1+e^{2i\int_0^1 \gamma^{(1)}(z)\,dz}\right) + \gamma_{21}t_{11}\left(-1+e^{2i\int_0^1 \gamma^{(1)}(z)\,dz}\right)} + e^{2i(\gamma^{(3)}(\hat{D_1}-D)+\int_1^D \gamma^{(2)}(z)\,dz)} \approx 0 \tag{57}$$

4.1 leaky mode

$|e^{2i\gamma^{(3)}(\bar{D_1}-D)}| \approx 0$, therefore (57) is reduced to:

$$\frac{\gamma_{21}t_{11} + \gamma_{11}t_{22}}{\gamma_{21}t_{11} - \gamma_{11}t_{22}} \approx e^{2i\int_0^1 \gamma^{(1)}(z)\,dz} \tag{58}$$

Taking logarithm both sides

$$\log\left(\frac{\gamma_{21}t_{11} + \gamma_{11}t_{22}}{\gamma_{21}t_{11} - \gamma_{11}t_{22}}\right) + 2m\pi i \approx 2i\int_0^1 \gamma^{(1)}(z)\,dz, \quad m \in \mathbb{Z} \tag{59}$$

assumes $\int_0^1 \gamma^{(1)}(z)\,dz = \gamma^{(1)}(\hat{z}_*^{(1)})$ (in practice, $\hat{z}_*^{(1)}$ can be obtained by setting $\lambda = 0$, and searching for $\hat{z}_*^{(1)}$ that minimizes $\left|\int_0^1 \sqrt{\frac{p^{(1)}(z)}{\alpha^{(1)}(z)}}\,dz - \sqrt{\frac{p^{(1)}(\hat{z})}{\alpha^{(1)}(\hat{z})}}\right|$). We denote $\gamma_{1*} = \gamma^{(1)}(\hat{z}_*^{(1)})$

To represent γ_{21}, γ_{11} in terms of γ_{1*}, we denote

$$\sigma_{21} = \sqrt{\frac{\alpha^{(1)}(\hat{z}_*^{(1)})}{\alpha^{(2)}(1)}}, \quad \sigma_{11} = \sqrt{\frac{\alpha^{(1)}(\hat{z}_*^{(1)})}{\alpha^{(1)}(1)}}$$

$$\delta_{21} = \frac{p^{(1)}(\hat{z}_*^{(1)}) - p^{(2)}(1)}{\alpha(\hat{z}_*^{(1)})}, \quad \delta_{11} = \frac{p^{(1)}(\hat{z}_*^{(1)}) - p^{(1)}(1)}{\alpha^{(1)}(\hat{z}_*^{(1)})}$$

then,

$$\gamma_{21} = \sigma_{21}\gamma_{1*}\sqrt{1 - \frac{\delta_{21}}{(\gamma_{1*})^2}}, \quad \gamma_{11} = \sigma_{11}\gamma_{1*}\sqrt{1 - \frac{\delta_{11}}{(\gamma_{1*})^2}}$$

They can be expand into inverse power series of γ_{1*}. Plug in (59), then expand the left hand side into inverse power series of γ_{1*}:

$$2i\gamma_{1*} \approx A_0 + \frac{A_2}{(\gamma_{1*})^2} + \frac{A_4}{(\gamma_{1*})^4} \cdots \tag{60}$$

where

$$A_0 = 2im\pi + \log\left(\frac{\sigma_{21}t_{11} + \sigma_{11}t_{22}}{\sigma_{21}t_{11} - \sigma_{11}t_{22}}\right)$$

$$A_2 = \frac{(-\delta_{11} + \delta_{21})\sigma_{11}\sigma_{21}t_{11}t_{22}}{(\sigma_{21}t_{11} - \sigma_{11}t_{22})(\sigma_{21}t_{11} + \sigma_{11}t_{22})}$$

$$A_4 = \frac{(\delta_{11} - \delta_{21})\sigma_{11}\sigma_{21}t_{11}t_{22}\left(-(\delta_{11} + 3\delta_{21})\sigma_{21}^2 t_{11}^2 + (3\delta_{11} + \delta_{21})\sigma_{11}^2 t_{22}^2\right)}{4\left(\sigma_{21}^2 t_{11}^2 - \sigma_{11}^2 t_{22}^2\right)^2}$$

asymptotic solutions of different orders $\tilde{\gamma}_{1*}^n$ are:

$$\begin{aligned}\tilde{\gamma}_{1*}^0 &= A_0/2i \\ \tilde{\gamma}_{1*}^2 &= \left(A_0 + \frac{A_2}{\left(\tilde{\gamma}_{1*}^0\right)^2}\right)/2i \\ \tilde{\gamma}_{1*}^4 &= \left(A_0 + \frac{A_2}{\left(\tilde{\gamma}_{1*}^2\right)^2} + \frac{A_4}{\left(\tilde{\gamma}_{1*}^2\right)^4}\right)/2i\end{aligned} \tag{61}$$

And the eigenvalue approximations $\tilde{\lambda}^i$ can by solved from $\tilde{\gamma}_{1*}^n$ by $\tilde{\lambda}_n = p^{(1)}(\hat{z}_*^{(1)}) - \alpha^{(1)}(\hat{z}_*^{(1)})\,(\tilde{\gamma}_{1*}^n)^2, \quad n = 0, 2, 4$

4.2 Berenger mode

Berenger modes would let $e^{-2i\gamma^{(l)}} \approx 0$ dispersion relation (57) can be reduced to

$$-\frac{\gamma_{21}t_{11} - \gamma_{11}t_{22}}{\gamma_{21}t_{11} + \gamma_{11}t_{22}} \approx e^{2i(\gamma^{(3)}(\hat{D}_1 - D) + \int_1^D \gamma^{(2)}(z)\,dz)} \tag{62}$$

Similar from the discussion in Leaky modes, we assumes $\int_1^D \gamma^{(2)}(z)\,dz = (D-1)\gamma(\hat{z}_*^{(2)})$, and denote $\gamma_{2*} = \gamma(\hat{z}_*^{(2)})$. And we proceed to expand the $\gamma_{21}, \gamma_{11}, \gamma_{2*}$ into inverse power series of $\gamma^{(3)}$: Firstly, denote

$$\begin{aligned}&\tau_{21} = \sqrt{\frac{\alpha_{(3)}}{\alpha^{(2)}(1)}}, \quad \tau_{11} = \sqrt{\frac{\alpha_{(3)}}{\alpha^{(1)}(1)}}, \quad \tau_{2*} = \sqrt{\frac{\alpha_{(3)}}{\alpha(\hat{z}_*^{(2)})}}, \\ &\epsilon_{21} = \frac{p^{(3)} - p^{(2)}(1)}{\alpha_{(3)}}, \quad \epsilon_{11} = \frac{p^{(3)} - p^{1}(1)}{\alpha_{(3)}}, \quad \epsilon_{2*} = \frac{p^{(3)} - p(\hat{z}_*^{(2)})}{\alpha_{(3)}},\end{aligned} \tag{63}$$

In particular, as we analysed in appendixA, ϵ_{2*} is negligible when the derivative of interface function $h(x)$ are small. As a result, zero order expansion for γ_{2*} is enough: $\gamma_{2*} \approx \tau_{2*}\gamma^{(3)}$. After logarithm and inverse power series expansion (62) is approximated by

$$2i\gamma^{(3)}\left[(\hat{D}_1 - D) + \tau_{2*}(D-1)\right] \approx B_0 + \frac{B_2}{\left(\gamma^{(3)}\right)^2} + \frac{B_4}{\left(\gamma^{(3)}\right)^4} + \dots \qquad m \in \mathbb{Z} \tag{64}$$

$$\begin{aligned}B_0 &= 2im\pi + \log\left(\frac{t_{22}\tau_{11} - t_{11}\tau_{21}}{t_{22}\tau_{11} + t_{11}\tau_{21}}\right) \\ B_2 &= \frac{(-\epsilon_{11} + \epsilon_{21})t_{11}t_{22}\tau_{11}\tau_{21}}{(t_{22}\tau_{11} - t_{11}\tau_{21})(t_{22}\tau_{11} + t_{11}\tau_{21})} \\ B_4 &= \frac{(\epsilon_{11} - \epsilon_{21})t_{11}t_{22}\tau_{11}\tau_{21}\left(-(3\epsilon_{11} + \epsilon_{21})t_{22}^2\tau_{11}^2 + (\epsilon_{11} + 3\epsilon_{21})t_{11}^2\tau_{21}^2\right)}{4\left(t_{22}^2\tau_{11}^2 - t_{11}^2\tau_{21}^2\right)^2}\end{aligned}$$

Asymptotic solutions of different orders are:

$$\begin{aligned}\tilde{\gamma}^{(3)\,0} &= B_0/2i\left[(\hat{D}_1 - D) + \tau_{2*}(D-1)\right] \\ \tilde{\gamma}^{(3)\,2} &= \left(B_0 + \frac{B_2}{\left(\tilde{\gamma}^{(3)\,0}\right)^2}\right)\Big/2i\left[(\hat{D}_1 - D) + \tau_{2*}(D-1)\right] \\ \tilde{\gamma}^{(3)\,4} &= \left(B_0 + \frac{B_2}{\left(\tilde{\gamma}^{(3)\,2}\right)^2} + \frac{B_4}{\left(\tilde{\gamma}^{(3)\,2}\right)^4}\right)\Big/2i\left[(\hat{D}_1 - D) + \tau_{2*}(D-1)\right]\end{aligned} \tag{65}$$

And the eigenvalue approximations $\tilde{\lambda}^i$ can be solved by $\tilde{\lambda}_n = p^{(3)} - \alpha^{(3)}\left(\tilde{\gamma}^{(3)\,n}\right)^2 \quad n = 0, 2, 4$

5 Numerical example

In this example, we set $D = 2, D_1 = 4$, and the stratified wave guide profile is

$$\kappa = \begin{cases} 16 & z < h(x) \\ 14.4 & z > h(x) \end{cases} \qquad \rho = \begin{cases} 1 & z < h(x) \\ 1.7 & z > h(x) \end{cases}$$

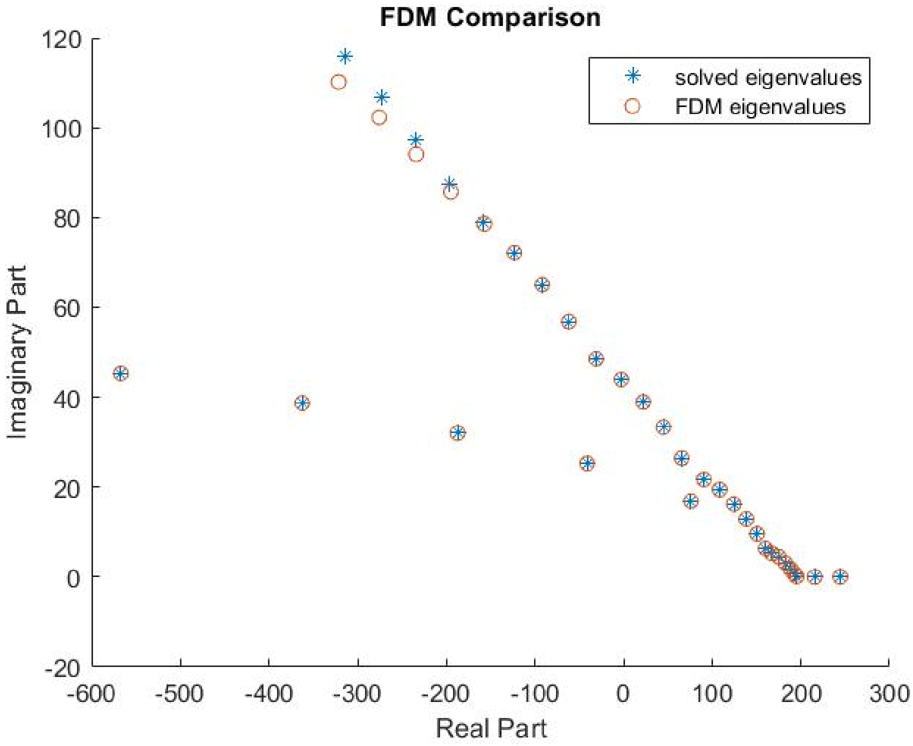

Figure 2: comparing our method with FDM

we chose the curve interface

$$h(x) = 1 - 0.2e^{-10(x/10-0.5)^2}$$

and compute the eigenvalues at $x = 6$.

The PML parameter is

$$\sigma(\hat{z}) = \begin{cases} 0, & 0 < \hat{z} \leq H \\ 5t^3/(1+t^2), & H < \hat{z} \leq D_1 \end{cases}$$

$$t = \frac{\hat{z} - H}{D_1 - H}, \quad H = 3.5$$

We first compare our method with Finite Difference Methods(FDM). In FDM, we divide $[0, D_1]$ to 1000 equally spaced grids. In our new method, The first and second layer $[0, 1], [1, D]$ are both divided to 50 sub intervals for interpolation. Müller's method is applied to solve the relation we derived (33),by using eigenvalues from FDM as the first initial guess. From figure2, our method agrees well with FDM.

Next, we verified asymptotic solutions we derived (61)(65), by using it as initial guesses to solve eigenvalues from our new relation.Figure3 shows the Leaky modes, and figure4 listed a few Berenger modes. The numerical experiments verified the asymptotic solutions can serve as reasonable initial guesses.

A Appendix

We will proof the therm:

$$\text{if } \max_{x \in [0,L]} \{|h'(x)|, |h''(x)|\} < \delta \tag{66}$$

$$\text{then} |\gamma^{(2)}(D)\bar{t}_{22} - \gamma^{(3)}\bar{t}_{11}| = O(\delta) \tag{67}$$

where $O(\delta)$ is a small term with the same order as δ.

This is therm used to simplify the transfer matrix across the second interface(53)

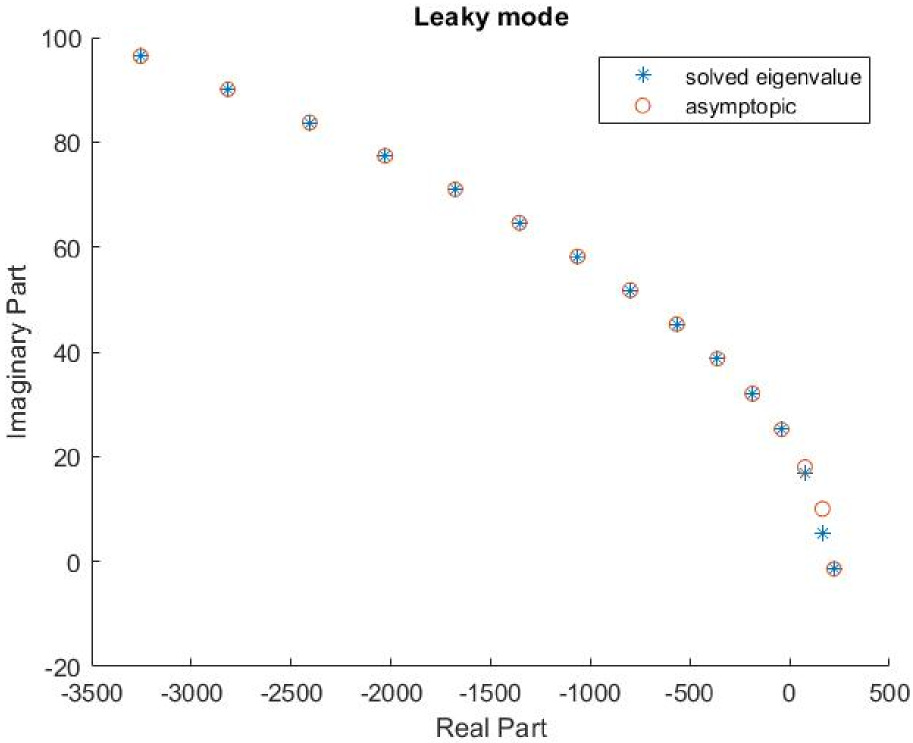

Figure 3: Leaky modes

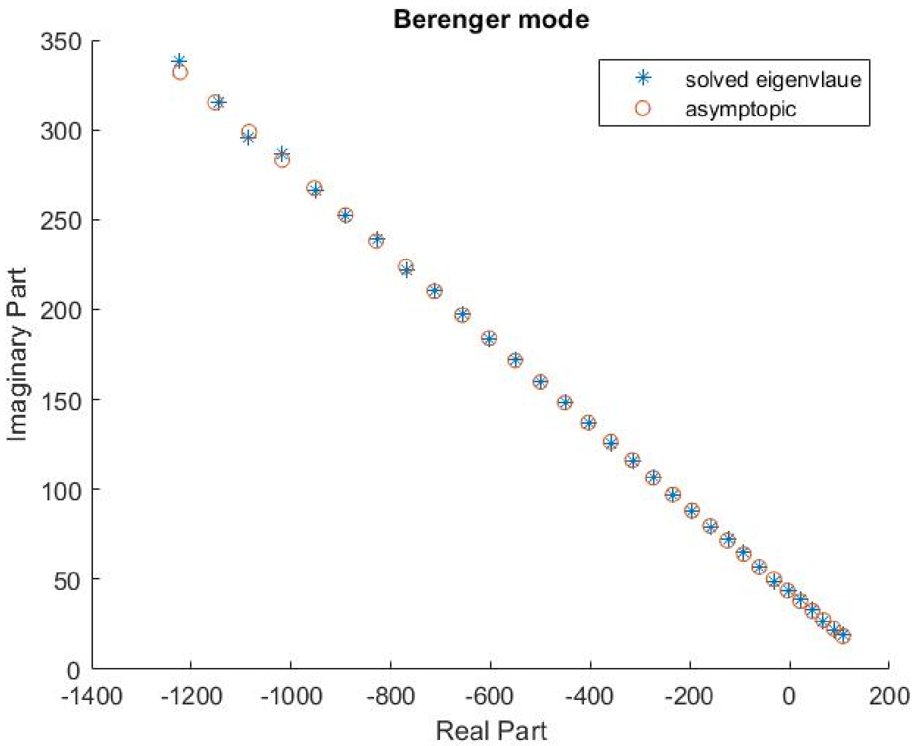

Figure 4: Berenger modes

Firstly, I will show $\eta^{(2)}(D)$ and $\eta^{(3)}$ from (3) are close when (66) is satisfied. Their expressions are:

$$
\begin{aligned}
\eta^{(2)}(\hat{x},\hat{z}) = \Bigg\{ & \tfrac{[(D-h(x))/h'(x)]^2+(D-z)^2}{[D-h(x^*)]^2} \Bigg[\tfrac{h^2(x^*)}{h^2(\hat{x})} \Bigg(-\tfrac{3}{4}\tfrac{\left(h'(\hat{x})\right)^4}{h^2(\hat{x})} \\
& + \tfrac{\left(h'(\hat{x})\right)^4 h''(\hat{x})}{h(\hat{x})} + \tfrac{1}{4}\left(h''(\hat{x})\right)^2 - \tfrac{1}{2} h'(\hat{x}) h_i'''(\hat{x}) \Bigg) + \tfrac{D\left(h'(x^*)\right)^2}{h(x^*)(D-h(x^*))\left[1+(h'(x^*))^2\right]^2} \\
& \times \Bigg(\tfrac{3}{4}\tfrac{\left(h'(x^*)\right)^2(D-2h(x^*))}{h(x^*)(D-h(x^*))} - \tfrac{h''(x^*)}{1+(h'(x^*))^2} \Bigg) \Bigg] + \tfrac{2\left(h'(x)\right)^2}{(D-h(x))^2} + \tfrac{2h''(x)}{D-h(x)} + \tfrac{h'''(x)}{2h'(x)} \\
& -\tfrac{1}{4}\tfrac{\left(h''(x)\right)^2}{(h'(x))^2} + \kappa^2(x,z) \Bigg\} \tfrac{(h(\hat{x}))^2(D-h(x^*))^2\left(h'(x)\right)^2}{(h(x^*))^2(h'(\hat{x}))^2\left[(D-h(x))^2+(h'(x))^2(D-z)^2\right]}.
\end{aligned} \tag{68}
$$

$$
\begin{aligned}
\eta_{(3)}(\hat{x},\hat{z}) = & \tfrac{h''(x)}{h(\hat{x})} - \tfrac{3}{4}\left[\tfrac{h'(\hat{x})}{h(\hat{x})}\right]^2 + \tfrac{1}{4}\left[\tfrac{h''(\hat{x})}{h'(\hat{x})}\right]^2 - \tfrac{1}{2}\tfrac{h'''(\hat{x})}{h'(\hat{x})} \\
& + \left\{ \tfrac{h'(x^*)h(\hat{x})}{\left[1+(h'(x^*))^2\right]h'(\hat{x})h(x^*)} \right\}^2 \times \tfrac{D}{h(x^*)[D-h(x^*)]} \Bigg\{ \tfrac{3}{4}\tfrac{[D-2h(x^*)]\left[h'(x^*)\right]^2}{h(x^*)[D-h(x^*)]} \\
& - \tfrac{h''(x^*)}{1+[h'(x^*)]^2} \Bigg\} + \left\{ \tfrac{h'(x)h(\hat{x})[D-h(x^*)]}{h'(\hat{x})h(x^*)[D-h(x)]} \right\}^2 \times \Bigg\{ \kappa^2(x,z) + \tfrac{h''(x)}{2[D-h(x)]} \\
& + \tfrac{2h'''(x)h'(x)-\left[h''(x)\right]^2}{4[h'(x)]^2} + \tfrac{2h''(x)[D-h(x)]+3\left[h'(x)\right]^2}{4[D-h(x)]^2} \Bigg\}.
\end{aligned} \tag{69}
$$

When the condition (66) is satisfied, $\hat{x} \approx x \approx x^*$. So i assume they are the same x. At $\hat{z} = D$, the long expressions can be simplified to

$$
\eta^{(2)}(D) \approx -\frac{3}{4}\frac{h'^2}{h^2} + \frac{h'^2 h''}{h} + \frac{D}{h(D-h)(1+h'^2)^2}\left(\frac{3}{4}\frac{(D-2h)h'^2}{h(D-h)} - \frac{h''}{1+h'^2}\right) + \left(\frac{2h'}{D-h}\right)^2 + \frac{2h''}{D-h} + \kappa^2 \tag{70}
$$

$$
\eta^{(3)} \approx \frac{h''}{h} - \frac{3}{4}\frac{h'^2}{h^2} + \frac{D}{h(D-h)(1+h'^2)^2}\left(\frac{3}{4}\frac{(D-2h)h'^2}{h(D-h)} - \frac{h''}{1+h'^2}\right) + \kappa^2 + \frac{h''}{(D-h)} + \frac{3h'}{4(D-h)^2} \tag{71}
$$

where

$$
h = h(x), h' = h'(x), h'' = h''(x), \kappa = \kappa_2
$$

subtract them both, the remaining terms are all the same order as $h'(x), h''(x)$, so by condition (66),

$$
|\eta^{(3)} - \eta^{(2)}(D)| = O(\delta) \tag{72}
$$

In addition, at $\hat{z} = D$, β, α', β' in equation (8) are all zero. Therefore, $p^{(2)}(D) = \eta^{(2)}(D), p^{(3)} = \eta^{(3)}$. By the definition of γ, we have $\gamma^{(\ell)}(D) = \sqrt{\frac{\eta^{(\ell)}(D)-\lambda}{\alpha^{(\ell)}(D)}}, \ell = 2,3$.

Next, we will show $\frac{\bar{t}_{11}}{\sqrt{\alpha^{(3)}}} = \frac{\bar{t}_{22}}{\sqrt{\alpha^{(2)}(D)}}$.Firstly, Denote $w_D^{(2)} = w|_{D^-}, w^{(3)} = w|_{D^+}$, by the definition of $\overline{T}$ and (12),

$$
\bar{t}_{11} = \frac{w_D^{(2)}}{w^{(3)}}, \quad \bar{t}_{22} = \frac{(D-1)w_D^{(2)}}{(D-h(x))w^{(3)}}
$$

plug in the expression for w:

$$
w_D^{(2)} = \sqrt{\frac{h(\hat{x})}{h'(\hat{x})} \cdot \frac{D-h(x^*)}{h(x^*)} \cdot \frac{h'(x)}{[D-h(x)]^2}}
$$

$$
w^{(3)} = \sqrt{\frac{h(\hat{x})}{h'(\hat{x})} \cdot \frac{D-h(x^*)}{h(x^*)} \cdot \frac{h'(x)}{[D-h(x)]}}
$$

we have

$$
\bar{t}_{11} = \frac{1}{\sqrt{D-h(x)}}, \quad \bar{t}_{22} = \frac{D-1}{(D-h(x))^{\frac{3}{2}}} \tag{73}
$$

What's more, by the expressions for $\alpha^{(2)}$ and $\alpha^{(3)}$:

$$
\alpha^{(2)} = (D-1)^2 \frac{[D-h(x^*)]^2 [h'(x)]^2 [h(\hat{x})]^2}{[D-h(x)]^4 [h'(\hat{x})]^2 [h(x^*)]^2}
$$

$$
\alpha^{(3)} = \frac{[D-h(x^*)]^2 [h'(x)]^2 [h(\hat{x})]^2}{[D-h(x)]^2 [h'(\hat{x})]^2 [h(x^*)]^2}
$$

We can see

$$
\sqrt{\alpha^{(2)}(D)} = \frac{D-1}{D-h(x)}\sqrt{\alpha^{(3)}} \tag{74}
$$

From (73)(74), it is easy to see that

$$\frac{\bar{t}_{11}}{\sqrt{\alpha^{(3)}}} = \frac{1}{\sqrt{(D-h(x))\alpha^{(3)}}} = \frac{\bar{t}_{22}}{\sqrt{\alpha^{(2)}(D)}} \tag{75}$$

We are now ready to show the main therm(67)

$$|\gamma^{(2)}(D)\bar{t}_{22} - \gamma^{(3)}\bar{t}_{11}| = \left| \frac{\bar{t}_{22}}{\sqrt{\alpha^{(2)}(D)}}\sqrt{\eta_{2D}-\lambda} - \frac{\bar{t}_{11}}{\sqrt{\alpha^{(3)}}}\sqrt{\eta^{(3)}-\lambda} \right|$$

By the relation(75) and (72), the right hand side of the above expression can be written as

$$\left| \frac{1}{\sqrt{(D-h(x))\alpha^{(3)}}} \left(\sqrt{\eta^{(3)} + O(\delta) - \lambda} - \sqrt{\eta^{(3)} - \lambda} \right) \right|$$

Expand the $O(\delta)$ to zero order: $\sqrt{\eta^{(3)} + O(\delta) - \lambda} = \sqrt{\eta^{(3)} - \lambda} + O(\delta)$, and plug this into the above expression, we have

$$\left|\gamma^{(2)}(D)\bar{t}_{22} - \gamma^{(3)}\bar{t}_{11}\right| = \left| \frac{1}{\sqrt{(D-h(x))\alpha^{(3)}}} O(\delta) \right| = O(\delta)$$

which proofed the therm(67)

With this therm, we can provide a brief explanation to $|p^{(3)} - p(\hat{z}_*^{(2)})| = O(\delta)$: We can proof the derivatives of $\alpha^{(2)}, \beta^{(2)}, \eta^{(2)}$ are bounded by the derivatives of $h(x)$ namely, δ. Thus $p^{(2)\prime}(\hat{z}) = O(\delta)$. We assumes $p^{(2)\prime}(\hat{z})$ is continuous in $(1^+, D^-)$, therefore, $|p(\hat{z}) - p(\hat{z}_*^{(2)})| \leq \underset{\hat{z}\in(1^+,D^-)}{Sup} \{|p^{(2)\prime}(\hat{z})|\}|\hat{z} - \hat{z}_*^{(2)}| = O(\delta)$. substitute $\hat{z}$ with D^- gives $|p^{(2)}(D^-) - p^{(2)}(\hat{z}_*^{(2)})| = O(\delta)$. Apply the result of (67), $|p_{3D} - p(\hat{z}_*^{(2)})| \leq |p_{3D} - p_{2D}| + |p_{2D} - p(\hat{z}_*^{(2)})| = O(\delta)$

References

[1] Jianxin Zhu and Ying Zhang. Cross orthogonality between eigenfunctions and conjugate eigenfunctions of a class of modified helmholtz operator for pekeris waveguide. *Journal of Theoretical and Computational Acoustics*, 27(02):1850048, 2019.