Exercise 1.

(a)(i) We prove that δ is well defined by proving its contrapositive statement i.e.

$$\text{if } \exists a \in \Sigma \ [xa] \neq [ya], \text{ then } [x] \neq [y]$$

Now suppose $[xa] \neq [ya]$ for $a \in \Sigma$, then by the definition of the equivalence classes of R_L, $(xa, ya) \notin R_L$.

$$(xa, ya) \notin R_L \Leftrightarrow \exists w \in \Sigma^* \ (xaw \in L \wedge yaw \notin L \text{ (or vice versa)})$$

Let $z = aw$, we now have $xz \in L \wedge yz \notin L$ (or vice versa).

$$(xz \in L \wedge yz \notin L) \Leftrightarrow (x, y) \notin R_L \Leftrightarrow [x] \neq [y]$$

Hence, we have proven the contrapositive statement.
(ii) We prove a more general statement i.e.

$$\forall x \in \Sigma^* \ \hat{\delta}([\epsilon], x) = [x]$$

This implies the statement we want to prove since, if $x \in L$, then by the definition of F, $\hat{\delta}([\epsilon], x) \in F$; and conversely, if $\hat{\delta}([\epsilon], x) \in F \Leftrightarrow [x] \in F$, then by the definition of F, $x \in L$.
Now we prove the lemma by an induction on $|x|$.

- **Basis case** $x = \epsilon$. $\delta([\epsilon], \epsilon) = [\epsilon]$ by the definition of the transition function.
- **Step case** Assume that $\hat{\delta}([\epsilon], x) = [x]$ for $|x| < k$. We prove for $w = xa$ where $x \in \Sigma^{k-1}, a \in \Sigma$.

$$\begin{aligned} \hat{\delta}([\epsilon], xa) &= \delta(\hat{\delta}([\epsilon], x), a) && \text{(definition of } \hat{\delta}) \\ &= \delta([x], a) && \text{(IH)} \\ &= [xa] && \text{(definition of } \delta) \end{aligned}$$

Hence, we have proven the lemma.
(b).