

## Exercise 1.

**(a)(i)** We prove that  $\delta$  is well defined by a contrapositive proof i.e. if  $\exists a \in \Sigma$   $[xa] \neq [ya]$ , then  $[x] \neq [y]$ . Suppose  $[xa] \neq [ya]$  for  $a \in \Sigma$ , then  $(xa, ya) \notin R_L$ . This means that  $\exists w \in \Sigma^*$  such that  $xaw \in L$  but  $yaw \notin L$  or vice versa. Now let  $z = aw$ , we have  $xz \in L \wedge yz \notin L$  or vice versa. Hence,  $(x, y) \notin R_L$  and  $[x] \neq [y]$ . We conclude by contrapositive that if  $[x] = [y]$  then  $[xa] = [ya]$  for all  $a \in \Sigma$ .

**(ii)** We prove a more general statement i.e.  $\forall x \in \Sigma^* \hat{\delta}([\epsilon], x) = [x]$ .

This implies  $\hat{\delta}([\epsilon], x) \in F$  iff  $x \in L$ . Since if  $x \in L$ , then by the definition of  $F$ ,  $[x] = \hat{\delta}([\epsilon], x) \in F$ . Conversely, if  $\hat{\delta}([\epsilon], x) \in F \Leftrightarrow [x] \in F$ , then by the definition of  $F$ ,  $x \in L$ .

Now we prove the lemma by an induction on  $|x|$ .

- **Basis case**  $x = \epsilon$ .  $\delta([\epsilon], \epsilon) = [\epsilon]$  by the definition of the transition function.
- **Step case** Assume that  $\hat{\delta}([\epsilon], x) = [x]$  for  $|x| < k$ . We prove for  $w = xa$  where  $x \in \Sigma^{k-1}, a \in \Sigma$ .

$$\begin{aligned} \hat{\delta}([\epsilon], xa) &= \delta(\hat{\delta}([\epsilon], x), a) && \text{(definition of } \hat{\delta}) \\ &= \delta([x], a) && \text{(IH)} \\ &= [xa] && \text{(definition of } \delta) \end{aligned}$$

Hence, we have proven the lemma.

**(b)** Denote the DFA in part (a) as  $D = (Q, \Sigma, \delta, q_0, F)$ . Let DFA  $A = (Q_A, \Sigma, \delta_A, q_{0_A}, F_A)$  with  $L(A) = L$  and no unreachable states. Construct  $f : Q_A \rightarrow Q$

$$f(\hat{\delta}_A(q_{0_A}, w)) = \hat{\delta}(q_0, w) \text{ for } \forall w \in \Sigma^*$$

$f$  is well defined since

- for each  $q \in Q_A$ , there exists a  $f(q) \in Q$ . Let  $q \in Q_A$ , then  $\exists w \in \Sigma^*$  such that  $q = \hat{\delta}_A(q_{0_A}, w)$  since  $q$  is a reachable state by construction. By def of the constructed  $f$ , we have  $f(q) = \hat{\delta}(q_0, w) = \hat{\delta}([\epsilon], w) \stackrel{\text{part(b)}}{=} [w] \in Q$
- and each  $q \in Q_A$  has a unique mapping  $f(q) \in Q$ . Suppose  $\hat{\delta}_A(q_{0_A}, x) = \hat{\delta}_A(q_{0_A}, y)$  for  $x \neq y$ , we show  $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$ . Let  $w \in \Sigma^*$ .  $xw \in L \Leftrightarrow \hat{\delta}_A(q_{0_A}, xw) \in F_A \Leftrightarrow \hat{\delta}_A(\hat{\delta}_A(q_{0_A}, x), w) \in F_A \Leftrightarrow \hat{\delta}_A(\hat{\delta}_A(q_{0_A}, y), w) \in F_A \Leftrightarrow \hat{\delta}_A(q_{0_A}, yw) \in F_A \Leftrightarrow yw \in L$ . Hence,  $\forall w \in \Sigma^*, xw \in L \Leftrightarrow yw \in L$ , i.e.  $(x, y) \in R_L \Leftrightarrow [x] = [y] \Leftrightarrow \hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$

If  $A$  has fewer states than  $D$ , then  $f$  means there exists at least one state  $[x] \in D$  such that for  $\forall w \in \Sigma^*$ ,  $f(\hat{\delta}_A(q_{0_A}, w)) \neq [x]$ . However,  $f(\hat{\delta}_A(q_{0_A}, x)) = \hat{\delta}(q_0, x) = [x]$ , this is a contradiction. Therefore,  $A$  must have at least as many states as  $D$ . For  $A$  with unreachable states, it must be more. Hence  $D$  is the minimal DFA for  $L$ .

## Exercise 2.

(a) Pick  $n=7$ . We consider  $w = a^j c^k a^l b^m \in L$  with  $|w| \geq 7$  in two separate cases.

- $0 \leq j \leq 5, k \geq 2$  and  $l \neq m$

Divide  $w = xyz$  as  $x = a^j c$ ,  $y = c$ , and  $z = c^{k-2} a^l b^m$ .  $|xy| = j + 2 \leq 7$  since  $0 \leq j \leq 5$ ,  $|y| = 1 > 0$ , and by pumping  $y$ , we get

- $xy^0 z = a^j c c^{k-2} a^l b^m = a^j c^{k-1} a^l b^m$   
if  $k = 2$ , then  $k - 1 = 1$ ,  $l \neq m$  still means  $xy^0 z \in L$   
if  $k > 2$ , then  $k - 1 \geq 2$ ,  $l \neq m$  still holds, and  $xy^0 z \in L$
- $xy^i z = a^j c c^i c^{k-2} a^l b^m = a^j c^{i+k-1} a^l b^m$  for  $i > 1$   
 $i + k - 1 > 2$  for  $k \geq 2$ ,  $l \neq m$  still holds, so  $xy^i z \in L$  for  $i > 1$

Hence, all  $w$  when  $k \geq 2$  satisfies the pumping lemma.

- $0 \leq j \leq 5, k < 2$  and  $k, l, m \in \mathbb{N}$

Note that for  $w \in L$  with  $|w| \geq 7$ ,  $w$  should have at least one of  $l$  and  $m$  not being zero since  $j + k < 7$

- if  $l \neq 0 \Leftrightarrow l \geq 1$ , then we divide  $w = xyz$  such that  $x = a^j c^k$ ,  $y = a$ ,  $z = a^{l-1} b^m$ .  $|xy| = j + k + 1 \leq 7$ ,  $|y| = 1 > 0$ , and  $xy^i z = a^j c^k a^i a^{l-1} b^m = a^j c^k a^{i+l-1} b^m$  and  $i + l - 1 \geq 0$  for  $i \in \mathbb{N}$ , hence  $xy^i z \in L$  for  $\forall i \in \mathbb{N}$
- if  $l = 0$ , then  $m \neq 0 \Leftrightarrow m \geq 1$ . We divide  $w = xyz$  such that  $x = a^j c^k a^0$ ,  $y = b$ ,  $z = b^{m-1}$ .  $|xy| = j + k + 1 \leq 7$ ,  $|y| = 1 > 0$ , and  $xy^i z = a^j c^k a^0 b^i b^{m-1} = a^j c^k a^0 b^{i+m-1}$  and  $i + m - 1 \geq 0$  for  $i \in \mathbb{N}$ , hence  $xy^i z \in L$  for  $\forall i \in \mathbb{N}$

Hence, we have proven  $L$  satisfies the pumping lemma.

(b) We show that  $R_L$  has an infinite number of equivalence classes.

Consider  $u = a^j c^k a^l b^m$  and  $v = a^j c^k a^{l'} b^{m'}$  where  $0 \leq j \leq 5, k \geq 2, l > m$  and  $l' > m', l - m \neq l' - m'$ . Since  $k \geq 2 \wedge l \neq m \wedge l' \neq m'$ , by definition  $u, v \in L$ . Now pick  $w = b^{l-m}$ ,  $uw = a^j c^k a^l b^m b^{l-m} = a^j c^k a^l b^l \notin L$ , but  $vw = a^j c^k a^{l'} b^{m'} b^{l-m} = a^j c^k a^{l'} b^{m'+l-m} \in L$  since  $m' + l - m \neq l'$ . Hence,  $(u, v) \notin R_L \Leftrightarrow [u] \neq [v]$ . In other words, for any pair of  $(u, v) \in L$  with  $k \geq 2$  and distinct positive values of  $l - m$ , we have  $[u] \neq [v]$ . Since there is an infinite number of distinct positive values of  $l - m$  for  $l, m \in \mathbb{N}$ , there is an infinite number of distinct equivalence classes in  $R_L$ . By the Myhill-Nerode Theorem,  $L$  is not regular.

## Exercise 3.

(a) We prove a lemma i.e.  $f(\hat{\delta}(q, w)) = \hat{\delta}'(f(q), w)$  for  $\forall w \in \Sigma^*, \forall q \in Q$ .

Let  $q \in Q$ , by an induction on  $|w|$ ,

- **Base case**  $w = \epsilon$

$f(\hat{\delta}(q, \epsilon)) = f(q)$  by the def of  $\hat{\delta}$  and  $\hat{\delta}'(f(q), \epsilon) = f(q)$  by def of  $\hat{\delta}'$ , hence  $f(\hat{\delta}(q, \epsilon)) = \hat{\delta}'(f(q), \epsilon)$

- **Step case** Assume the claim for  $\forall w$  with  $|w| < n$ . Now we prove for  $w = xa$  where  $x \in \Sigma^{n-1}$ ,  $a \in \Sigma$

$$\begin{aligned}
f(\hat{\delta}(q, xa)) &= f(\delta(\hat{\delta}(q, x), a)) && (\text{def of } \hat{\delta}) \\
&= \delta'(f(\hat{\delta}(q, x)), a) && (\text{def (3) of } f) \\
&= \delta'(\hat{\delta}'(f(q), x), a) && (\text{IH since } |x| < n) \\
&= \hat{\delta}'(f(q), xa) && (\text{def of } \hat{\delta}')
\end{aligned}$$

Hence, we have proven the lemma.

Now we prove  $\mathcal{L}(P, q) = \mathcal{L}(P', f(q))$  for  $\forall q \in Q$ . Let  $q \in Q$ ,  $w \in \Sigma^*$

$$\begin{aligned}
w \in \mathcal{L}(P, q) &\Leftrightarrow \hat{\delta}(q, w) \in F \\
&\Leftrightarrow f(\hat{\delta}(q, w)) \in F' && (\text{def (2) of } f) \\
&\Leftrightarrow \hat{\delta}'(f(q), w) \in F' && (\text{lemma}) \\
&\Leftrightarrow w \in \mathcal{L}(P', f(q))
\end{aligned}$$

Hence the claim.

(b)

$$\begin{aligned}
L(P) &= \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F\} \\
&= \mathcal{L}(P, q_0) && (\text{def of } \mathcal{L}) \\
&= \mathcal{L}(P', f(q_0)) && (\text{part (a)}) \\
&= \mathcal{L}(P', q'_0) && (\text{def (1) of } f) \\
&= \{w \in \Sigma^* \mid \hat{\delta}'(q'_0, w) \in F'\} && (\text{def of } \mathcal{L}) \\
&= L(P')
\end{aligned}$$

#### Exercise 4.

1. We show validity of  $(r^*)^* = r^*$  by proving  $L((r^*)^*) = L(r^*)$ .

$L((r^*)^*) = L(r^*)^* = (L(r))^* = L(r)^* = L(r^*)$  where the third equality uses the algebraic law of Kleene- $*$ .

Now we prove for  $\forall L \subseteq \Sigma^*$ ,  $(L^*)^* = L^*$ . Let  $L \subseteq \Sigma^*$ ,

- $L^* \subseteq (L^*)^*$  since  $L^* = (L^*)^1 \subseteq (L^*)^*$

- $(L^*)^* \subseteq L^*$

Let  $w \in (L^*)^*$ , we can write  $w = w_1 w_2 \dots w_n$  for  $n \geq 0$  where each  $w_i \in L^*$ . We can also write each  $w_i = x_{i1} x_{i2} \dots x_{il_i}$  for  $l_i \geq 0$  where each  $x_{il_i} \in L$ . Then  $w = x_{11} x_{12} \dots x_{1l_1} \dots x_{n1} x_{n2} \dots x_{nl_n} \in L^{\sum_{i=1}^n l_i} \subseteq L^*$ . Hence,  $w \in L^*$ .

2. We prove  $L((r+s)^*) = L((r^*s)^*r^*)$ . Denote  $R = L(r)$  and  $S = L(s)$  for clarity. By def of reg lang,

$$L((r+s)^*) = L(r+s)^* = (L(r) \cup L(s))^* = (R \cup S)^*$$

$$L((r^*s)^*r^*) = L((r^*s)^*)L(r^*) = L(r^*s)^*L(r)^* = (L(r^*)L(s))^*R^* = (L(r)^*S)^*R^* = (R^*S)^*R^*$$

In other words, we prove  $(R \cup S)^* = (R^*S)^*R^*$ .

- $(R \cup S)^* \subseteq (R^*S)^*R^*$

We prove the lemma that for  $\forall n \geq 0$ ,  $(R \cup S)^n \subseteq (R^*S)^*R^*$ , then by def of set union,  $(R \cup S)^* = \bigcup_{n \geq 0} (R \cup S)^n \subseteq (R^*S)^*R^*$ . By an induction on  $n$ ,

- $n = 0$   $(R \cup S)^0 = \{\epsilon\} = \{\epsilon\}\{\epsilon\} = (R^*S)^0R^0 \subseteq (R^*S)^*R^*$
- prove  $(R \cup S)^{n+1} \subseteq (R^*S)^*R^*$

$$\begin{aligned}
(R \cup S)^{n+1} &= (R \cup S)^n(R \cup S) && \text{(concat of lang)} \\
&= \{xa \in \Sigma^* \mid x \in (R \cup S)^n, a \in (R \cup S)\} && \text{(set notation of concat)} \\
&\subseteq \{xa \in \Sigma^* \mid x \in (R^*S)^*R^*, a \in (R \cup S)\} && \text{(IH)} \\
&= \{xa \in \Sigma^* \mid x \in (R^*S)^*R^*, a \in R\} \cup \{xa \in \Sigma^* \mid x \in (R^*S)^*R^*, a \in S\} \\
&= ((R^*S)^*R^*)R \cup ((R^*S)^*R^*)S && \text{(set def of concat)}
\end{aligned}$$

We show both subsets are in  $(R^*S)^*R^*$ . But first we prove some laws.

**L1.**  $\forall L \subseteq \Sigma^*, L^*L \subseteq L^*$

Proof.  $L^*L = (\bigcup_{n \geq 0} L^n)L = \{xa \in \Sigma^* \mid x \in \bigcup_{n \geq 0} L^n, a \in L\} = \bigcup_{n \geq 0} \{xa \in \Sigma^* \mid x \in L^n, a \in L\} = \bigcup_{n \geq 0} (L^nL) = \bigcup_{n \geq 0} L^{n+1} = \bigcup_{n \geq 1} L^n \subseteq L^*$

**L2.**  $\forall L, M, N \subseteq \Sigma^*, M \subseteq N \Rightarrow LM \subseteq LN$

Proof. Suppose  $M, N \subseteq \Sigma^*$  with  $M \subseteq N$ . Let  $w \in LM$ . Write  $w = xa$ , where  $x \in L, a \in M$ . Since  $M \subseteq N, a \in N$ . Then  $w = xa \in LN$ . In short,  $LM = \{xa \in \Sigma^* \mid x \in L, a \in M\} \subseteq \{xa \in \Sigma^* \mid x \in L, a \in N\} = LN$

Back to the proof

$$* ((R^*S)^*R^*)R \subseteq (R^*S)^*R^*$$

$$R^*R \subseteq R^* \tag{L1}$$

$$\Rightarrow (R^*S)^*(R^*R) \subseteq (R^*S)^*R^* \tag{L2}$$

$$\Leftrightarrow ((R^*S)^*R^*)R \subseteq (R^*S)^*R^* \tag{associativity}$$

$$* ((R^*S)^*R^*)S \subseteq (R^*S)^*R^*$$

$$(R^*S)^*(R^*S) \subseteq (R^*S)^* \tag{L1}$$

$$\Leftrightarrow ((R^*S)^*R^*)S \subseteq (R^*S)^* = (R^*S)^*\{\epsilon\} = (R^*S)^*R^0 \subseteq (R^*S)^*R^* \tag{assoc \& concat with \{\epsilon\}}$$

Hence the step case.

Hence the lemma.

- $(R^*S)^*R^* \subseteq (R \cup S)^*$

**L3.** for  $\forall M, N \in \Sigma^*, M \subseteq N \Rightarrow M^* \subseteq N^*$

Proof. Suppose  $M \subseteq N$ . Let  $w \in M^*$ , then we can write  $w = w_1w_2...w_n$  where each  $w_i \in M$ . Since  $M \subseteq N$ , hence each  $w_i \in N$ . Then  $w = w_1w_2...w_n \in N^*$ . Hence,  $M^* \subseteq N^*$ .

**L4.** for  $\forall M, N, X, Y \in \Sigma^*, M \subseteq N \wedge X \subseteq Y \Rightarrow MX \subseteq NY$

Proof. Suppose  $M \subseteq N \wedge X \subseteq Y$ . Let  $w \in MX$ , then  $w = ax$  where  $a \in M$ ,  $x \in X$ . Since  $M \subseteq N$  and  $X \subseteq Y$ , then  $a \in N$  and  $x \in Y$ . Hence,  $w \in NY$ .  $MX \subseteq NY$ .

Now we can prove  $(R^*S)^*R^* \subseteq (R \cup S)^*$ .

Since  $R \subseteq (R \cup S)$ , by L3 we have  $R^* \subseteq (R \cup S)^*$ . Also,  $S \subseteq (R \cup S)$ , by L4  $R^*S \subseteq (R \cup S)^*(R \cup S) \stackrel{L1}{\subseteq} (R \cup S)^*$ . By L3 again,  $(R^*S)^* \subseteq ((R \cup S)^*)^* \stackrel{part1}{=} (R \cup S)^*$ . By L4,  $(R^*S)^*R^* \subseteq (R \cup S)^*(R \cup S)^* = ((R \cup S)^*)^2 \subseteq ((R \cup S)^*)^* \stackrel{part1}{=} (R \cup S)^*$ .

Hence, we have formally proven  $L((r + s)^*) = L((r^*s)^*r^*)$ , thus the validity of  $(r + s)^* = (r^*s)^*r^*$ .

**3.** Let  $R = L(r)$ ,  $S = L(s)$ .

$$L((rs)^*) = L(rs)^* = (L(r)L(s))^* = (RS)^*$$

$$L(\epsilon + r(sr)^*s) = L(\epsilon) \cup L(r(sr)^*s) = \{\epsilon\} \cup L(r(sr)^*)L(s) = \{\epsilon\} \cup L(r)L((sr)^*)S = \{\epsilon\} \cup RL(sr)^*S = \{\epsilon\} \cup R(L(s)L(r))^*S = \{\epsilon\} \cup R(SR)^*S$$

We prove  $(RS)^* = \{\epsilon\} \cup R(SR)^*S$  by equality of the subsets, i.e.,  $(RS)^0 = \{\epsilon\}$  and  $(RS)^n = R(SR)^{n-1}S$  for  $\forall n \geq 1$ .

- $(RS)^0 = \{\epsilon\}$  by definition
- we prove by induction on  $n$  that  $(RS)^n = R(SR)^{n-1}S$  for  $\forall n \geq 1$ .
  - **Base case**  $n = 1$ .  $(RS)^1 = RS = (R\{\epsilon\})S = (R(SR)^0)S = R(SR)^0S$
  - **Step case**  $(RS)^{n+1} = (RS)^n(RS) \stackrel{IH}{=} (R(SR)^{n-1}S)(RS) \stackrel{assoc}{=} ((R(SR)^{n-1}S)R)S \stackrel{assoc}{=} (R(SR)^{n-1}(SR))S = (R(SR)^n)S = R(SR)^nS$

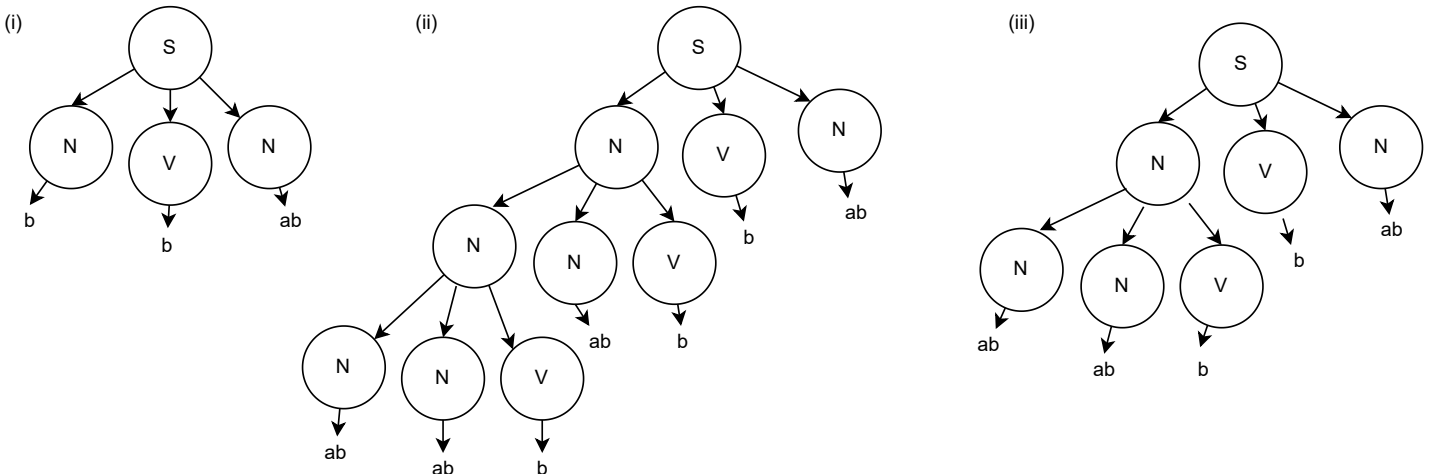
$$\text{Hence } \cup_{n \geq 1} (RS)^n = \cup_{n \geq 1} R(SR)^{n-1}S = \cup_{n \geq 0} R(SR)^nS$$

From above,  $(RS)^* = \cup_{n \geq 0} (RS)^n = (RS)^0 \cup (\cup_{n \geq 1} (RS)^n) = \{\epsilon\} \cup (\cup_{n \geq 0} R(SR)^nS) = \{\epsilon\} \cup R(SR)^*S$ .

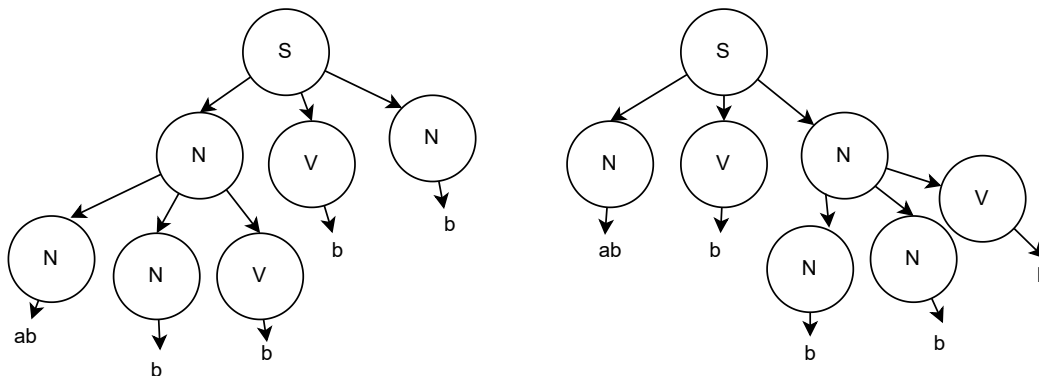
Hence,  $(rs)^* = \epsilon + r(sr)^*s$  is valid.

## Exercise 5.

(a) As below



(b)  $G$  is ambiguous because there exists  $w \in L(G)$  that has two parse trees. Consider  $w = abbbbb \in L(G)$ . The following two parse trees generate the same string  $abbbbb$ .



(c)  $G'$  is defined by the following productions:

$$\begin{aligned}
 S &\rightarrow AVN \\
 A &\rightarrow ANV \mid ab \mid a \\
 N &\rightarrow NNV \mid ab \mid b \\
 V &\rightarrow b
 \end{aligned}$$