

Note 1: Naor-Yung 通用转化

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1.1 Preliminaries

In this section, we first introduce some useful notations and formal definitions.

Notation: We denote probabilistic polynomial Turing machine by PPT. We denote that two distribution is computationally indistinguishable by $\mathcal{D}_0 \approx_c \mathcal{D}_1$.

Algorithms: We use $x \xleftarrow{\$} \text{Alg}$ to present an algorithm randomly generating an output x , and $x := \text{Alg}$ to present an algorithm deterministically generating an output x . We use $\mathcal{A}^{\text{OAlg}(\cdot)}$, to present an algorithm with oracle access to OAlg .

Pseudo-code: We use **check** to check if the following condition is fulfilled; the algorithm aborts otherwise. We use **parse** $x := y$ to parse y into the variable x .

Negligible Function: We denote the negligible functions with respect to the security parameter λ by $\text{negl}(\lambda)$. We recall that a function f is negligible, if for all polynomial $p(\cdot)$, there exists a λ_0 such that

$$\forall \lambda > \lambda_0, f(\lambda) < \frac{1}{p(\lambda)}$$

Language: For any NP language \mathcal{L} , we denote a statement x in language \mathcal{L} with witness w by $x \in_w \mathcal{L}$.

1.1.1 Public Key Encryption scheme

Definition 1.1 (Public Key Encryption). A public key encryption scheme consists of three PPT algorithms $\text{PKE} = (\text{Setup}, \text{Enc}, \text{Dec})$ with the following syntax:

- $\text{Setup}(1^\lambda) \rightarrow (\text{pk}, \text{sk})$: takes the security parameter 1^λ as input, and returns a public key pk and a secret key sk .
- $\text{Enc}(\text{pk}, m; r) \rightarrow \text{ct}$: takes the public key pk , the message m , the randomness r as input, and returns a ciphertext ct .
- $\text{Dec}(\text{sk}, \text{ct}) \rightarrow m$: takes the secret key sk , the ciphertext ct as input, and returns a message m .

We also require the following properties:

- **Correctness:** For all messages $m \in \mathcal{M}$ in the message space, for all randomness $r \in \mathcal{R}$ in the randomness space, and for all key pairs $(\text{pk}, \text{sk}) \xleftarrow{\$} \text{Setup}(1^\lambda)$, we have

$$\text{Dec}(\text{sk}, \text{Enc}(\text{pk}, m; r)) = m$$

- **Semantic Security:** PKE is ε -IND-CPA secure, if for all two-stages PPT adversary $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ with an internal state st , we first define the security games $\text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CPA}_b}(\mathcal{A})$ as in Fig. 1.1. We say

Game $_{\text{PKE}, 1^\lambda}^{\text{IND-CPA}_b}(\mathcal{A}) :$

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01 (pk, sk)  $\xleftarrow{\$}$  Setup( $1^\lambda$ )
02 ( $m_0, m_1, \text{st}$ )  $\xleftarrow{\$}$   $\mathcal{A}_0(\text{pk})$ 
03  $r \xleftarrow{\$} \mathcal{R}$ ;  $\text{ct}_b := \text{Enc}(\text{pk}, m_b; r)$ 
04  $b' \xleftarrow{\$} \mathcal{A}_1(\text{st}, \text{ct}_b)$ 
05 return  $b'$ 

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Figure 1.1: This is the IND-CPA security game with bit $b \in \{0, 1\}$.

that the public-key encryption scheme PKE is IND-CPA secure, if and only if

$$\varepsilon = \left| \Pr \left[\text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CPA}_0}(\mathcal{A}) = 1 \right] - \Pr \left[\text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CPA}_1}(\mathcal{A}) = 1 \right] \right| \leq \text{negl}(\lambda).$$

- **IND-CCA1 Security:** PKE is ε -IND-CCA1 secure, if for all two-stages PPT adversary $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ with an internal state st , we first define the security games $\text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CCA1}_b}(\mathcal{A})$ as in Fig. 1.2.

<p>Game$_{\text{PKE}, 1^\lambda}^{\text{IND-CCA1}_b}(\mathcal{A}) :$</p> <pre> 01 (pk, sk) $\xleftarrow{\\$}$ Setup(1^λ) 02 (m_0, m_1, st) $\xleftarrow{\\$}$ $\mathcal{A}_0^{\text{ODec}(\cdot)}(\text{pk})$ 03 $r \xleftarrow{\\$} \mathcal{R}$; $\text{ct}_b := \text{Enc}(\text{pk}, m_b; r)$ 04 $b' \xleftarrow{\\$} \mathcal{A}_1(\text{st}, \text{ct}_b)$ 05 return b' </pre>	<p>Oracle $\text{ODec}(\text{ct})$</p> <pre> 06 $m := \text{Dec}(\text{sk}, \text{ct})$ 07 return m </pre>
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Figure 1.2: This is the IND-CCA1 security game with bit $b \in \{0, 1\}$.

We say that the public-key encryption scheme PKE is IND-CCA1 secure, if and only if

$$\varepsilon = \left| \Pr \left[\text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CCA1}_0}(\mathcal{A}) = 1 \right] - \Pr \left[\text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CCA1}_1}(\mathcal{A}) = 1 \right] \right| \leq \text{negl}(\lambda).$$

1.1.2 Non-Interactive Zero-Knowledge Proof

Definition 1.2 (NIZK). Let \mathcal{L} be an NP language, an adaptive non-interactive zero-knowledge proof system consists of three PPT algorithms $\text{NIZK} = (\text{Setup}, \text{Prove}, \text{Ver})$ with the following syntax

- $\text{Setup}(1^\lambda) \rightarrow \text{crs}$: takes a security parameter 1^λ as input, and returns a common reference string crs .
- $\text{Prove}(\text{crs}, x, w) \rightarrow \pi$: takes a common reference string crs , a statement x , a witness w as input, and returns π .
- $\text{Ver}(\text{crs}, x, \pi) \rightarrow \{0, 1\}$: takes a common reference string crs , a statement x , and a proof π as input, and returns a result bit $b \in \{0, 1\}$.

We require the following properties:

- **Completeness:** For all statements $x \in_w \mathcal{L}$, for all honestly generated common reference string $\text{crs} \xleftarrow{\$} \text{Setup}(1^\lambda)$, we have

$$\text{Ver}(\text{crs}, x, \text{Prove}(\text{crs}, x, w)) = 1.$$

- **Soundness:** NIZK is ε_{snd} -sound, if for all PPT adversary \mathcal{A} , we have

$$\Pr \left[\begin{array}{l} \text{Ver}(\text{crs}, x, \pi) = 1 \\ \wedge x \notin \mathcal{L} \end{array} \mid \begin{array}{l} \varepsilon_{\text{snd}} = \text{crs} \xleftarrow{\$} \text{Setup}(1^\lambda) \\ \pi \xleftarrow{\$} \mathcal{A}(\text{crs}, x) \end{array} \right] \leq \text{negl}(\lambda).$$

- **Zero-Knowledge:** NIZK is ε_{zk} -zero-knowledge, if for all PPT adversary \mathcal{A} with running time t_{zk} , there exists a two-stage PPT algorithm $\text{Sim} = (\text{SimSetup}, \text{SimProve})$ with the following syntax:

- $\text{SimSetup}(1^\lambda) \rightarrow (\text{crs}, \text{td})$: takes a security parameter 1^λ as input, and returns a crs and a simulation trapdoor td .
- $\text{SimProve}(\text{crs}, x, \text{td}) \rightarrow \pi$: takes a crs, a statement x , and a trapdoor td as input, and returns a simulated proof π .

We require that the simulated SimSetup and SimProve are indistinguishable from the real one for any PPT adversary. More formally, for all PPT adversaries, the following two games are indistinguishable: We require that the following requirement holds

$\text{Game}_{\text{PKE}, 1^\lambda}^{\text{Real}}(\mathcal{A}) :$	$\text{Game}_{\text{PKE}, 1^\lambda}^{\text{Sim}}(\mathcal{A}) :$
01 $\text{crs} \xleftarrow{\$} \text{Setup}(1^\lambda)$	03 $\text{crs} \xleftarrow{\$} \text{SimSetup}(1^\lambda)$
02 return $\mathcal{A}^{\text{OProve}(\text{crs}, \cdot, \cdot)}(\text{crs})$	04 return $\mathcal{A}^{\text{OSimProve}(\text{crs}, \cdot, \cdot)}(\text{crs})$

Figure 1.3: This is the indistinguishability game between the real and simulated worlds. Note that $\text{OSimProve}(\text{crs}, x, w)$ returns $\text{SimProve}(\text{crs}, x, \text{td})$ without using w .

$$\varepsilon_{\text{zk}} = \left| \Pr \left[\text{Game}_{\text{PKE}, 1^\lambda}^{\text{Real}}(\mathcal{A}) = 1 \right] - \Pr \left[\text{Game}_{\text{PKE}, 1^\lambda}^{\text{Sim}}(\mathcal{A}) = 1 \right] \right| \leq \text{negl}(\lambda).$$

1.2 Naor-Yung CCA1 construction

Let PKE be a ε -IND-CPA public-key encryption scheme, and NIZK be a $(\varepsilon_{\text{zk}}, \varepsilon_{\text{snd}})$ -adaptive non-interactive zero-knowledge proof system. We recalled the Naor-Yung CCA1 construction that we saw during the lecture.

As in [NY90], we give the detailed construction as in Fig. 1.4

Theorem 1.3 ([NY90]). *The public-key encryption scheme given in Fig. 1.4 is ε' -IND-CCA1 secure, with*

$$\varepsilon' \leq 2\varepsilon + 4\varepsilon_{\text{zk}} + 2\varepsilon_{\text{snd}}$$

Proof. We give the proof following a sequence of hybrid games $(\mathbf{G}_0, \dots, \mathbf{G}_k)$, in which $\mathbf{G}_0 = \text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CCA1}_0}$ and $\mathbf{G}_6 = \text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CCA1}_1}$. By arguing that $\mathbf{G}_i \approx_c \mathbf{G}_{i+1}$ for all $i \in \{0, \dots, 5\}$, we complete the proof.

We give the detailed hybrid game description as follows. We denote by pr_i the probability that the adversary outputs 1 in the game \mathbf{G}_i . Note that with the above notation, we only need to prove that $|\text{pr}_0 - \text{pr}_6| \leq \text{negl}(\lambda)$. We summarize all hybrid games in Fig. 1.5.

Alg Setup(1^λ) : 01 $(pk_0, sk_0) \xleftarrow{\$} \text{PKE.Setup}(1^\lambda)$ 02 $(pk_1, sk_1) \xleftarrow{\$} \text{PKE.Setup}(1^\lambda)$ 03 $crs \xleftarrow{\$} \text{NIZK.Setup}(1^\lambda)$ 04 $pk := (pk_0, pk_1, crs); sk := sk_0$ Alg Dec(sk, ct) : 05 parse $sk_0 =: sk; (ct_0, ct_1, \pi) =: ct$ 06 check $\text{NIZK.Ver}(crs, (ct_0, ct_1), \pi) = 1$ 07 $m := \text{PKE.Dec}(sk_0, ct_0)$ 08 return m	Alg Enc(pk, m) : 09 parse $(pk_0, pk_1, crs) =: pk$ 10 $r_0, r_1 \xleftarrow{\$} \mathcal{R}$ 11 $ct_0 \xleftarrow{\$} \text{PKE.Enc}(pk_0, m; r_0)$ 12 $ct_1 \xleftarrow{\$} \text{PKE.Enc}(pk_1, m; r_1)$ 13 $\pi \xleftarrow{\$} \text{Prove}(crs, (ct_0, ct_1), (m, r_0, r_1))$ 14 return (ct_0, ct_1, π)
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Figure 1.4: This is CCA1 Naor-Yung construction.

\mathbf{G}_0 : This is the initial security game with the challenge bit $b = 0$.

\mathbf{G}_1 : This game is the same as in \mathbf{G}_0 except that the challenger uses Sim for simulating the proof instead of honestly generating the zero-knowledge proofs.

Since the only difference is whether using the simulator to generate the proofs, we have

$$|\text{pr}_0 - \text{pr}_1| \leq \varepsilon_{\text{zk}}.$$

\mathbf{G}_2 : In this game, we change the generation of ct_1 . In \mathbf{G}_2 , ct_1 is an encryption of m_1 instead of m_0 .

Notice that, the adversary \mathcal{A} has only access of $\text{ODec}(\cdot)$ which uses only sk_0 . Therefore, any adversary \mathcal{B} which can distinguish \mathbf{G}_2 from \mathbf{G}_1 can also break the IND-CPA security of the underlying encryption scheme. Thus, we have

$$|\text{pr}_2 - \text{pr}_1| \leq \varepsilon.$$

\mathbf{G}_3 : In \mathbf{G}_3 , we switch the the decryption key from sk_0 to sk_1 .

To analyze the probability of distinguishing \mathbf{G}_2 from \mathbf{G}_3 , we define a bad event Bad. Bad happens when the adversary submits a ciphertext $ct = (ct_0, ct_1, \pi)$ to the decryption oracle with $\text{Dec}(sk_0, ct_0) \neq \text{Dec}(sk_1, ct_1)$ and $\text{Ver}(crs, (ct_0, ct_1), \pi) = 1$. Our first observation is that the adversary's view is different in \mathbf{G}_2 and \mathbf{G}_3 only if Bad happens in \mathbf{G}_2 . Therefore, we have $|\text{pr}_2 - \text{pr}_3| \leq \Pr[\text{Bad}]$. Our second observation is that Bad can also happen in \mathbf{G}_0 , \mathbf{G}_1 and \mathbf{G}_2 , we denote these event by Bad_i with $i \in \{0, 1, 2\}$. We can have the following analysis:

- In \mathbf{G}_0 the crs is generated by an honest NIZK.Setup algorithm, we have $\text{Bad}_0 \leq \varepsilon_{\text{snd}}$.
- Since the probability of distinguishing \mathbf{G}_0 and \mathbf{G}_1 is bounded by ε_{zk} , and Bad can be detected by the adversary himself, we have

$$\Pr[\text{Bad}_1] \leq \Pr[\text{Bad}_0] + \varepsilon_{\text{zk}} = \varepsilon_{\text{snd}} + \varepsilon_{\text{zk}}$$

- The change in \mathbf{G}_2 happens after all decryption queries. Therefore, we have $\Pr[\text{Bad}_2] = \Pr[\text{Bad}_1] \leq \varepsilon_{\text{snd}} + \varepsilon_{\text{zk}}$.

In summary, we have

$$|\text{pr}_3 - \text{pr}_2| \leq \varepsilon_{\text{snd}} + \varepsilon_{\text{zk}}.$$

\mathbf{G}_4 : In the game \mathbf{G}_4 , we change the message m_0 in ct_0 to ct_1 .

Similar to the argument in \mathbf{G}_2 , now the decryption oracle does not use sk_0 , thus we can bound the probability of distinguishing \mathbf{G}_3 and \mathbf{G}_4 by the IND-CPA security of PKE. We have

$$|\text{pr}_4 - \text{pr}_3| \leq \varepsilon.$$

\mathbf{G}_5 : In the game \mathbf{G}_5 , we change back the decryption oracle using sk_0 . Similarly to \mathbf{G}_3 , we have

$$|\text{pr}_5 - \text{pr}_4| \leq \varepsilon_{\text{snd}} + \varepsilon_{\text{zk}}.$$

\mathbf{G}_6 : In \mathbf{G}_6 , we use the (Setup, Prove) instead of (SimSetup, SimProve). Similar to \mathbf{G}_1 , we have

$$|\text{pr}_6 - \text{pr}_5| \leq \varepsilon_{\text{zk}}.$$

We can notice that \mathbf{G}_6 is exactly the same as $\text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CCA1}_1}$. By the triangle inequality we have

$$\begin{aligned} |\text{pr}_6 - \text{pr}_0| &\leq |\text{pr}_6 - \text{pr}_5| + |\text{pr}_5 - \text{pr}_4| + |\text{pr}_4 - \text{pr}_3| + |\text{pr}_3 - \text{pr}_2| + |\text{pr}_2 - \text{pr}_1| + |\text{pr}_1 - \text{pr}_0| \\ &\leq 2\varepsilon + 4\varepsilon_{\text{zk}} + 2\varepsilon_{\text{snd}}. \end{aligned}$$

$\text{Game}_{\text{PKE}, 1^\lambda}^{\text{IND-CCA1}}(\mathcal{A})$		Oracle $\text{ODec}(\text{sk}, \text{ct})$:
01 $(\text{pk}_0, \text{sk}_0) \xleftarrow{\$} \text{PKE.Setup}(1^\lambda)$		17 parse $\text{sk}_0 =: \text{sk}; (\text{ct}_0, \text{ct}_1, \pi) =: \text{ct}$
02 $(\text{pk}_1, \text{sk}_1) \xleftarrow{\$} \text{PKE.Setup}(1^\lambda)$		18 check $\text{NIZK.Ver}(\text{crs}, (\text{ct}_0, \text{ct}_1), \pi) = 1$
03 $\text{crs} \xleftarrow{\$} \text{NIZK.Setup}(1^\lambda)$	// $\mathbf{G}_{0,6}$	19 $m := \text{PKE.Dec}(\text{sk}_0, \text{ct}_0)$ // $\mathbf{G}_{0-2,5-6}$
04 $(\text{crs}, \text{td}) \xleftarrow{\$} \text{NIZK.SimSetup}(1^\lambda)$	// \mathbf{G}_{1-5}	20 $m := \text{PKE.Dec}(\text{sk}_1, \text{ct}_0)$ // \mathbf{G}_{3-4}
05 $\text{pk} := (\text{pk}_0, \text{pk}_1, \text{crs}); \text{sk} := \text{sk}_0$		21 return m
06 $(m_0, m_1, \text{st}) \xleftarrow{\$} \mathcal{A}_0^{\text{ODec}(\cdot)}(\text{pk})$		
07 $r_0, r_1 \xleftarrow{\$} \mathcal{R}$		
08 $\text{ct}_0 \xleftarrow{\$} \text{PKE.Enc}(\text{pk}_0, m_0; r_0)$	\mathbf{G}_{0-3}	
09 $\text{ct}_0 \xleftarrow{\$} \text{PKE.Enc}(\text{pk}_0, m_1; r_0)$	\mathbf{G}_{4-6}	
10 $\text{ct}_1 \xleftarrow{\$} \text{PKE.Enc}(\text{pk}_1, m_0; r_1)$	// \mathbf{G}_{0-1}	
11 $\text{ct}_1 \xleftarrow{\$} \text{PKE.Enc}(\text{pk}_1, m_1; r_1)$	// \mathbf{G}_{2-6}	
12 $\pi \xleftarrow{\$} \text{Prove}(\text{crs}, (\text{ct}_0, \text{ct}_1), (m, r_0, r_1))$	// $\mathbf{G}_{0,6}$	
13 $\pi \xleftarrow{\$} \text{SimProve}(\text{crs}, \text{td}, (\text{ct}_0, \text{ct}_1))$	// \mathbf{G}_{1-5}	
14 $\text{ct} := (\text{ct}_0, \text{ct}_1, \pi)$		
15 $b' \xleftarrow{\$} \mathcal{A}_1(\text{st}, \text{ct})$		
16 return b'		

Figure 1.5: This is a summary of all hybrid games. The code line ends with $//\mathbf{G}_i$ only appears in security game \mathbf{G}_i .

□

References

- [NY90] Moni Naor and Moti Yung. Public-key cryptosystems provably secure against chosen ciphertext attacks. In *22nd Annual ACM Symposium on Theory of Computing*, pages 427–437, Baltimore, MD, USA, May 14–16, 1990. ACM Press.