Probability Problem Set 3

Q1. Suppose that a factory produces 3 types of coins. – **a.** 50% of the coins produced are perfectly balanced i.e. with a bias $\theta = 0.5$**, b.** 25% of the of the coins produced have a bias $\theta = 0.25$ and **c.** 25% of the of the coins produced have a bias $\theta = 0.75$**.** If a random coin was chosen from the factory:

What is the probability that the coin will produce 50 heads and 50 tails from 100 throws?

If the coin produced 50 heads and 50 tails from 100 throws, what is the probability that the coin was fair?

Q2. Suppose that a factory produces coins whose bias θ follow a Beta distribution with parameters $\alpha = 5$ and $\beta = 10$**.** If a random coin was chosen from the factory:

What is the probability that the coin will produce 50 heads and 50 tails from 100 throws?

If the coin produced 50 heads and 50 tails from 100 throws, what is the probability that the coin was fair?

If the coin produced 50 heads and 50 tails from 100 throws, what is the probability that bias of the coin was between **0.4** to **0.6** i.e. $0.4 \leq \theta \leq 0.6$

Q3. Compute the expected value i.e.$E[\theta]$of the posterior of the binomial distribution under the beta prior. See the lecture slides for explanation of this distribution.

If the coin was thrown$(c_1 + c_2)$times giving$c_1$heads and $c_2$tails, explain in plain English, what does$E[\theta]$mean?

Q4. Suppose that $X_1, \dots, X_n$ are independent identically distributed samples from a univariate Gaussian distribution with mean μ and variance σ^2.

a. Find the ML estimate of the mean μ
b. Find the MAP estimate of the mean μ when the mean μ is distributed according to a Gaussian prior with mean α and variance σ^2

Comment on the difference.

Q5. Prove that if there are n successes from N Bernoulli trials, then the maximum likelihood estimator (**MLE**) of the probability of success is $p = n/N$.

Q6. In **Q5,** suppose that the prior for the probability p follows a Beta distribution

Find the **MAP** estimate of p**.**

Q7. Suppose that $X_1, \dots, X_n$ are independent identically distributed samples from a Poisson distribution. Each random variable X_i is distributed according to the Poission distribution whose **pdf** is given by:

$p(X_i = k \mid \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$ where the parameter λ is known as the **mean** of the Poission distribution.

a. Determine the maximum likelihood estimate (**MLE**) of the **mean** λ**.**
b. Work out the **pdf** of sum of the random variables $X_1 + \dots + X_n$
c. Determine the maximum likelihood estimate (**MLE**) of the **mean**.

Q8. Two players **A** and **B** are competing at a trivia quiz game involving a series of questions. On any individual question, the probabilities that **A** and **B** give the correct answer are α and β respectively, for all questions, with outcomes for different questions being independent. The game finishes when a player wins by answering a question correctly.
Compute the probability that **A** wins if
(a) **A** answers the first question
(b) **B** answers the first question.

Q9. Patients are recruited onto the two arms (**0 - Control, 1 - Treatment**) of a clinical trial. The probability that an adverse outcome occurs on the control arm is p_0 and is p_1 for the treatment arm. Patients are allocated alternately onto the two arms, and their outcomes are independent.

What is the probability that the first adverse event occurs on the control arm?