Solutions to Problem Set 2

1.

i. There are 36 different outcomes when the dice is rolled twice. Let (X,Y) represent the different combinations of difference and sum of two outcomes. Then all possible (X, Y)s are presented in table 1 below:

	1	2	3	4	5	6
1	(0,2)	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)
2	(-1,3)	(0,4)	(1,5)	(2,6)	(3,7)	(4,8)
3	(-2,4)	(-1,5)	(0,6)	(1,7)	(2,8)	(3,9)
4	(-3,5)	(-2,6)	(-1,7)	(0,8)	(1,9)	(2,10)
5	(-4,6)	(-3,7)	(-2,8)	(-1,9)	(0,10)	(1,11)
6	(-5,7)	(-4,8)	(-3,9)	(-2,10)	(-1,11)	(0,12)

Table 1: Possible outcomes of sum and difference from two dice roll

The range means the list of all possible values at which the PMF is nonzero. Therefore, range of X is {−5,−4,...,5} and range of Y is {2,3,...,12}.

ii. Now, from the table we can see that

P(Y =8)=5/36, and,P(X|Y =8)= P(X=x & Y=8)/ P(Y=8) = (1/36)/(5/36) = 1/5 for x = {−4,−2,0,2,4} and 0 otherwise.

iii. P(Y|X≥4)=P(Y =y|X≥4)= 1/3 if y={6,7,8} and 0 else.

Now, E[Y|X≥4]= 6×1/3+7×1/3+8×1/3=7, using law of total probability.

2.

$$\mathrm{E}(x) = \int_0^2 \frac{1}{2}x \cdot x dx = \left[\frac{1}{6}x^3\right]_0^2 = \frac{8}{6} = \frac{4}{3}.$$

$$\mathrm{E}(x^2) = \int_0^2 \frac{1}{2}x \cdot \mathrm{x}^2 dx = \left[\frac{1}{8}x^4\right]_0^2 = 2.$$

$$\mathrm{Var}(x) = \mathrm{E}(x^2) - \{\mathrm{E}(x)\}^2 = 2 - \frac{16}{9} = \frac{2}{9}$$

3.

(i) We can write $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

$f(x) \geq 0$ for all x but $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2} \neq 1.$

Thus this function is not a valid probability density function because the integral's value is not 1.

(ii)

Note that $f(x) = \begin{cases} 1 - \dfrac{1}{2}x, & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

$f(x) \geq 0$ for all x

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 \left(1 - \frac{1}{2}x\right)\,dx = \left[x - \frac{x^2}{4}\right]_0^2 = 2 - 1 = 1$$

(Alternatively, the area of the triangle is $\frac{1}{2} \times 1 \times 2 = 1$)

This implies that $f(x)$ is a valid probability density function.

(iii)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 \left(x^2 - 4x + \frac{10}{3}\right)\,dx = \left[\frac{x^3}{3} - 2x^2 + \frac{10}{3}x\right]_0^3 = (9 - 18 + 10) = 1$$

but $f(x) < 0$ for $1 \leq x \leq 3$. Hence (iii) is not a pdf.

4..

(a) $p(x) = \frac{d}{dx}F(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x < 0 \end{cases}$

(b) $F(x > 2) = \int_2^{\infty} 2e^{-2u}du = -e^{-2u}|_2^{\infty} = e^{-4}$

Another method

By definition, $F(x \leq 2) = F(2) = 1 - e^{-4}$. Hence,

$$F(x > 2) = 1 - (1 - e^{-4}) = e^{-4}$$

(c) $F(-3 \leq x \leq 4) = \int_{-3}^{4} f(u)\,du$

$$= \int_{-3}^{0} 0\,du + \int_0^4 2e^{-2u}\,du$$

$$= -e^{-2u}|_0^4$$

$$= 1 - e^{-8}$$

Another method

$$
\begin{aligned}
F(-3 \le x \le 4) &= F(x \le 4) - F(x \le -3) \\
&= F(4) - F(-3) \\
&= (1 - e^{-8}) - (0) \\
&= 1 - e^{-8}
\end{aligned}
$$

5.

(a).

For $0 < x < 1$,

$$p(x) = \int_0^1 \left(\frac{3}{4} + xy\right) dy = \frac{3}{4} + \frac{x}{2}$$

and $$p(y|x) = \frac{p(x,y)}{p(x)} = \begin{cases} \frac{3+4xy}{3+2x} & 0 < x < 1,\ \ 0 < y < 1 \\ 0 & otherwise \end{cases}$$

For other values of $x, p(y|x)$ is not defined.

(b).

$$p(\mathrm{y}|x = 0.5) = \int_{\frac{1}{2}}^{\infty} p\left(y\middle|\frac{1}{2}\right) dy = \int_{\frac{1}{2}}^{1} \frac{3 + 2y}{4} dy = \frac{9}{16}$$