MATH 595 ADT: Algebraic and Differential Topology in Data Analysis

By

Taught by Dr. Yuliy Baryshnikov

Notes Crowdsourced by Class

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Contents

Chapter 1

Lecture 1: Simplicial Complexes (Michels)

1.1 Simplicial Complexes

We begin by introducing the idea of a simplicial complex. A k-simplex can be thought of as the convex hull of $(k+1)$ points. You can think of a simplex as a generalization of triangles and tetrahedron to arbitrary dimensions. Note that a k-simplex is a complete graph of $(k+1)$ points and its convex hull, not all collections of $(k+1)$ points will be a k-simplex. Figure 1.1 illustrates a few basic simplices.

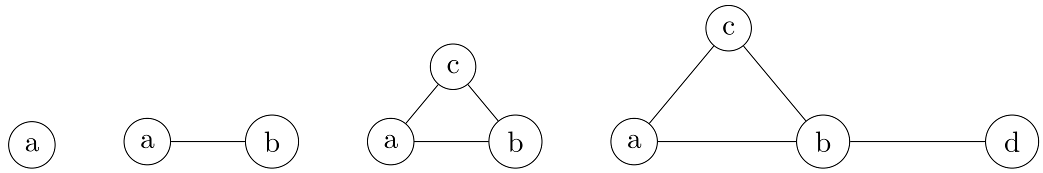


Figure 1.1: A 0-simplex or point (left), 1-simplex or line segment (center-left), 2-simplex or triangle (center-right), and simplicial complex (right).

Definition 1.1.1 (Simplicial Complex)**.** *A simplicial complex is a finite collection of simplices K such that $\sigma \in K$ and $\rho \leq \sigma$ implies $\rho \in K$ and $\sigma_\alpha, \sigma_\beta \in K$ implies $\sigma_\alpha \cap \sigma_\beta$ is either empty or a face of both simplices.*

In plain English, this definition means that a simplicial complex is a collection of simplices made up of simplicial complexes such that if any two simplices intersect, they must intersect in a reasonable way. More specifically, the intersection of any two simplices must also be a simplex. This is illustrated in the simplicial complex (right) in Figure 1.1, where there are 4 nodes, but we do not have a 3-simplex (tetrahedron), but rather a 2-simplex (left) and a 1-simplex (right) which intersect at a 0-simple (node "b"). Furthermore, the 2-simplex in Figure 1.1 contains three 1-simplices (the edges) which intersect to form 0-simplices (the nodes).

The simplices we have looked at so far have been illustrated in Euclidean space. This illustration is called the **(geometric) realization** of the simplicial complex. We can also discuss simplicial complexes without considering the realization in Euclidean space, or an abstract simplicial complex.

Definition 1.1.2 (Abstract Simplicial Complex)**.** *An abstract simplicial complex is a finite collection of sets A such that $\alpha \in A$ and $\beta \subset \alpha$ implies $\beta \in A$.*

In an abstract simplicial complex, the sets are simplices. For any simplex $\alpha \in A$ we denote the **dimension** of α to be the cardinality of α minus one. In notation, $\dim(\alpha) =$

$|\alpha| - 1$. The dimension of an abstract simplicial complex A is the maximum dimension of any of its simplices.

Example 1.1.1. *Consider the abstract simplicial complex*

$$A = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}\}$$

Each set $\alpha \in A$ can be thought of geometrically as a $\dim(\alpha)$-simplex. Thus we have a 2-simplex made up of three 1-simplices which intersect to form three 0-simplices. Our A can be geometrically realized a triangle.

This example shows that an abstract simplicial complex such as A can be realized in Euclidean space, a triangle in our example. brings us to the Geometric Realization Theorem:

Theorem 1.1.1 (Geometric Realization Theorem)**.** *Any abstract simplicial complex can be geometrically realized in Euclidean space. Further, an abstract simplicial complex of dimension d has a geometric realization in $\mathbb{R}^{2d+1}$.*

Example 1.1.2. *As we have seen, simple graphs are simplicial complexes constructed from nodes (0-simplices) and edges (1-simplices). However we know that only planar graphs can be realized in $\mathbb{R}^2$. As the Geometric Realization Theorem tells us, for some simple graph G, we require up to $2\dim(G) + 1 = 2 \cdot 1 + 1 = 3$ dimensions to realize a simple graph.*

A natural question at this point may be *"why should I care about abstract simplicial complexes?"* Abstract simplicial complexes arise naturally when using partially ordered sets or strict posets. A poset is a set P with a relation R usually denoted $\leq$ such that:

1. $a \leq a$ (reflexive)
2. if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetric)
3. if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitive)

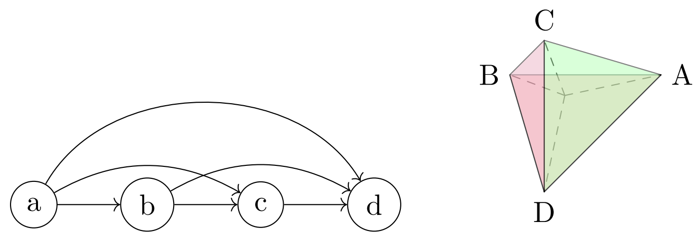


Figure 1.2: A poset and it's simplicial complex

Given any poset P we can produce a simplicial complex by representing related elements as simplices. Constructing a simplicial complex from some set or space X will be denoted by $\Delta(X)$.

Example 1.1.3. *Consider the set $P = \{a, b, c, d\}$ where $a \leq b \leq c \leq d$ as shown in Figure 1.2. Let's create $\Delta(P)$. Reflexivity gives us that each element can be represented as a 0-simplex. Further we can make six 1-simplices from the relations $a \leq b$, $a \leq c$, $a \leq d$, $b \leq c$, $b \leq d$, and $c \leq d$, four 2-simplices from the chains $a \leq b \leq c$, $a \leq b \leq d$, $a \leq c \leq d$ and $b \leq c \leq d$, and of course the 3-simplex from the chain $a \leq b \leq c \leq d$. The abstract simplicial complex obtained is then $\mathcal{P}(P)$ (the power set) which can be represented geometrically as a 3-simplex formed from four 2-simplices, six 1-simplices, and four 0-simplices.*

One or more topological spaces can be compared, and it is particularly important to understand when two spaces are equivalent.

Definition 1.1.3 (Homotopy)**.** *A homotopy between two continuous functions f and g from a topological space X to a topological space Y is a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ for all $x \in X$.*

This can be thought of H describing a continuous deformation through time ([0,1]) of f into g. If there is a homotopy, we say f and g are homotopic.

Definition 1.1.4. *Given two spaces X and Y, we say X and Y are homotopy equivalent* *(denoted $X \sim Y$) if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map $\mathbb{1}_X$ and $f \circ g$ is homotopic to $\mathbb{1}_Y$.*

We say a space X is **contractible** if it is homotopy equivalent to single point.

Using these tools, we can understand how a simplicial complex relates to the space it is produced from. First we will define how to construct a special kind of abstract simplicial complex called a *nerve*.

Definition 1.1.5 (Nerve). *Given an index set I, and open sets U_i contained in X, for all $J \subseteq I$, if $\bigcap_{j \in J} U_j \neq \emptyset$ then $J \in N$. All subsets of J are also contained in the nerve. A nerve is an abstract simplicial complex.*

Example 1.1.4. *Consider the open cover in Figure 1.1.4 and let R be the red oval, B be the red oval, and Y be the yellow oval. The non-empty intersections are $\{B, R\}$, $\{R, Y\}$, and $\{B, Y\}$. The nerve also contains the subsets of these non-empty intersections $\{R\}$, $\{B\}$, and $\{Y\}$. Note that $\{B, R, Y\} \notin N$ because they do not intersect. Thus our abstract simplicial complex is $\{\{B, R\}, \{B, Y\}, \{R, Y\}, \{B\}, \{R\}, \{Y\}\}$ meaning our complex has three 0-simplices and three 1-simplicies, as shown in the simplicial complex to the right of Figure 1.1.4.*

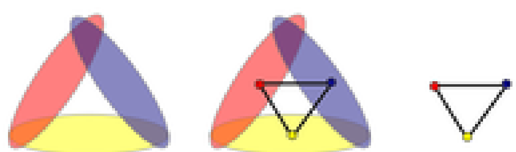


Figure 1.3: Nerve of an Open Cover

Nerves are extremely useful under certain conditions, leading to the Nerve Lemma:

Lemma 1.1.1 (Nerve Lemma). *Let X be a topological space and assume $\mathcal{U} = \{U_i\}_{i \in I}$ be a good cover of X. If each element of the nerve of $\mathcal{U}$ (denoted $N(\mathcal{U})$) is contractible, then the nerve $N(\mathcal{U})$ is homotopy equivalent to $\bigcap_{i \in I} U_i$.*

Why does this matter? The Nerve Theorem tells us that under certain conditions, nerves encode the homotopy type of a topological space.

1.2 Dowker's Theorem (Carmody)

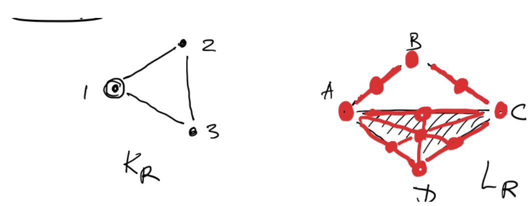
Definition 1.2.1. *Consider 2 finite sets K, L and a relation between them $R \subset K \times L$. Then K_R is a simplicial complex such that $(x_0, \ldots, x_k \in K_R \iff$ for some $y \in L, (x_0, y), \ldots, (x_k, y) \in R$. Similarly,*

$$L_R = \{(y_0, \ldots, y_l) \mid \text{for some } x \in K, (x, y_0), \ldots, (x, y_l) \in R\}$$

Example 1.2.1. *Say $K = \{1, 2, 3\}, L = \{A, B, C, D\}$. The relation is defined by the table*

	A	B	C	D
1	$*$	$*$		
2	$*$		$*$	$*$
3		$*$	$*$	

The simplex K_R is just a 1-dimensional simplex corresponding to the boundary of the



standard 2-simplex.

The simplex L_R is a triangle joined to a 2-simplex along an edge. Both are homotopy equivalent to S^1.

Theorem 1.2.1. *$K_R \sim L_R$, that is, K_R is homotopy equivalent to L_R.*

Proof. Take simplices σ_y which cover K i.e.

$$K = \bigcup_{y \in L} \sigma_y$$

Then $K \sim N(P_L)$ where P_L is the poset corresponding to this cover.

But $\Delta(P_L) \cong$ barycentric subdivision of $L_R \cong L_R$. Here we've used the fact that (the geometric realization of a) barycentric subdivision of a simplicial complex is homotopy equivalent to the (geometric realization of the) original simplicial complex. □

1.2.1 Applications

Early approaches by Atkins. Potential example: "Netflix complex": K = movies, L = viewers; $(x, y) \in R$ if y likes x.

Project: Poll students of the course about their preferences, compute K_R, L_R; analyze. Or: coauthorship networks.

[Other applications: directed networks Memoli].

1.3 Cech and Rips complexes

The Nerve Lemma leads to the widely popular notion of Cech complex:

Definition 1.3.1. *If $M = \bigcup_\alpha U_\alpha$ is a locally-finite cover of a topological space M by open sets $\{U_\alpha\}_{\alpha \in V}$, then $M \sim N(P(U))$, where P is the poset on the intersections of U's.*

It's important to emphasize that $\sim$, homotopy equivalence, is very different from homeomorphism. For example, dimension is not invariant under homotopy equivalence.

However, invariants like the number of path connected components or the homology groups are homotopy invariant.

If M is a metric space and $X \subset M$, we can cover X by balls of radius r centered at the points $x \in X$. However, there's a problem. In general, it's relatively hard to detect whether some collection of r-balls intersects. In $\mathbb{R}^D$ with Euclidean norm, have to solve a messy system of algebraic inequalities .

One way to deal with this issue is to introduce the Rips-Vietoris complex.

Definition 1.3.2. *Fix as above a subset $X \subset M$ of some metric space (M, d). $R(r) = \{(x_0, \dots, _k) \in X^k \mid d(x_i, x_j) \leq r \, \forall i, j\}$.*

This is easy to compute, but there's no analogue of the Nerve Lemma: there's no apriori equivalence between the Rips complex and the union of the r-balls around the $x \in X$.

However, there are inclusions

$$R(r) \subset \check{C}(r) \subset R(2r).$$

1.4 Basic setup of TDA

All data live in $\mathbb{R}^N$ along some unknown manifold which we want to discover.

Question 1.4.1. *Can one reconstruct a manifold M from observations given a dense enough sample from M.*

A pioneering paper: Niyogi, Smale, Weinberger

A quick reminder about manifolds: the regular value theorem tells us that if we have a function $F : \mathbb{R}^D \to R^N$, then the inverse image of a regular value is an embedded submanifold of $\mathbb{R}^D$.

Setup: Let $M \hookrightarrow \mathbb{R}^N$ a submanifold of dimension m. Draw points from M at random $X = \{x_1, \dots, x_n\}$. This is the point cloud from which we want to attempt to reconstruct M.

Potential problems: different pieces of M can be too close to each other. Take for example a parametrized curve representing a spiral. The way the metric changes via the map $[0, 1] \to \mathbb{R}^D$ becomes more and more extreme as we get close to 0.

To mitigate, introduce a geometric "condition number" $1/\tau$ where $\tau > 0$ is such that for any $x \in \mathbb{R}^N$, $d(x, M) < \tau$, there is a unique

$$\min_{y \in M} |x - y|^2.$$

Equivalently, $N_\tau M \hookrightarrow \mathbb{R}^D$ is an embedding. This is related to "medial skeleton" and local feature extraction.

Now take $X \subset M$, and define

$$X_\epsilon = \{y \in \mathbb{R}^n \mid d(y, X) \le \epsilon\} = \bigcup_{x_k \in X} B_\epsilon(x_k).$$

By the Nerve Lemma, the Cech complex of $\{B_\epsilon(x_k)\}_{k=1,\dots,n}$ is homotopy equivalent to X_ϵ.

Proposition 1.4.1. *If $M \subset X_{\epsilon/2}$ and $\epsilon < C_\tau$, where $C = \left(\frac{3}{5}\right)^{\frac{1}{2}}$, then X_ϵ is homotopy equivalent to M (actually deformation retracts onto M).*

Question 1.4.2. *How many randomly sampled points does one need to cover M by $\frac{\epsilon}{2}$-balls?*

This is a covering problem: consider random translates of an r-ball in $\mathbb{R}^m$. let N be the (random) number of such balls when all of a domain D is covered. Take

$$K(r) = \frac{\mathrm{vol}(D)}{\mathrm{vol}(B_r)} = \frac{\mathrm{vol}(D)}{\omega_m r^m}$$

where ω_m is the volume of the unit ball in $\mathbb{R}^m$. Then the number $N \sim K \log K + dK \log\log K + K$(some external value random variable).

You can reproduce this formula by looking at the Coupon collector problem. If you want to collect K coupons which occur randomly upon purchase of some food item, how many food items do you need to buy to collect all K coupons?

$\mathbb{P}$(after N steps there are coupons to be collected) $\le \mathbb{E}$(# of remaining coupons after N steps)

$$= K \cdot \mathbb{P}(\text{missing first coupon}) = K(1 - \frac{1}{K})^N \approx Ke^{-\frac{N}{K}}$$

when RHS $\leq \delta$, have $\log K - \frac{N}{K} \approx \log \delta$, and $N \approx K \log K + K(\log \delta)$.

To go from the coupon collector problem to the geometric covering problem, consider a $\beta\epsilon$-network in M. Rather than considering covers by B_ϵ, consider coverings by balls $B_{\alpha\epsilon}$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. If all points in $\beta\epsilon$ covering of M are hit by $\bigcup_{x_k \in X} B_{\alpha\epsilon}(x_k)$ then $M \subset \bigcup_{x_k} B_\epsilon(x_k)$.

Altogether, the proposition above and the covering estiamges give

Theorem 1.4.1. *(NSW) For $M^m \hookrightarrow \mathbb{R}^N$ a compact submanifold with condition number τ, and any $\epsilon < \tau/2$, an iid uniform sample from M gives the Cech complex $C(\epsilon)$ homotopy equivalent to M with probability at least $1 - \delta$ if the sample size is*

$$n \geq C_1(m, \tau)K(\epsilon)[\log(C_2(m, \tau)K(\epsilon)) + |\log \delta|]$$

where C_1, C_2 are constants depending on m, τ only, and

$$K(\epsilon) = \frac{\text{vol}(M)}{\omega_m \epsilon^m}$$

is the "volume of M measured in ϵ-balls".

Lemma[section]

1.5 Homotopy limit (Luo)

Recommended Paper: Cell Groups Reveal Structure of Stimulus Space [**?**]. This is an interesting paper that studies the neuron spike signals through creating simplicial complex.

Definition 1.5.1. Background of spaces

Given a partially order set X_α and a collection of maps $d_{\alpha\beta} : X_\alpha \to X_\beta$ for any α, define $P_{\leq\alpha} = \{\beta : \beta \leq \alpha | d_{\alpha\beta} X_\alpha = X_\beta\}$.

$P_{\leq\alpha}$ is the collection of β that α is mapped to. The following is a concrete example.

Example 1.5.1.

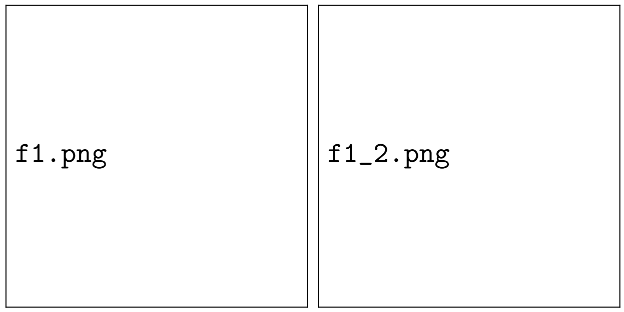


Figure 1.4: example of $\Delta(P_{\leq\alpha})$

Next, we consider glue different $P_{\leq\alpha}$ together. Let's consider each $\alpha > \beta$, $X_\alpha \times \Delta(P_{\leq\beta})$ can be mapped to both $X_\beta \times \Delta(P_{\leq\beta})$ and $X_\alpha \times \Delta(P_{\leq\alpha})$. The first one is done through mapping X_α to X_β through $d_{\alpha\beta}$ and use the identity map from $\Delta(P_{\leq\beta})$ to itself. The second one is done using identity from X_α to itself and embedding $\Delta(P_{\leq\beta})$ to $\Delta(P_{\leq\alpha})$. As a consequence, one can identity the same point for the two spaces. We define the **homotopy limit** $||\mathcal{D}(P)|| = \coprod X_\alpha \times \Delta(P_{\leq\alpha})/\sim$, where $\sim$ is introduced as above.

Example 1.5.2.

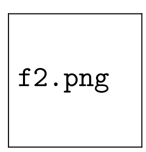


Figure 1.5: the 1st example of homotopy limit

Example 1.5.3.

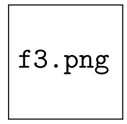


Figure 1.6: the 2nd example of homotopy limit

Example 1.5.4.

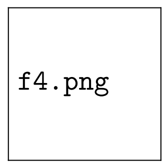


Figure 1.7: the 3rd example of homotopy limit

Arrangement $\{X_\alpha\}_{\alpha \in A}$ such that $X = \cup X_\alpha$ satisfies: (1) if $X_\alpha \cap X_\beta = X_\gamma$, then $\gamma \in A$. (2) if $X_\alpha \in X_\beta$, the inclusion map $X_\alpha \hookrightarrow X_\beta$ is a cofibration [**?**].

Lemma 1.5.1. *Projection Lemma[**?**]*

Let X be an arrangement in U with intersection partial order set P, then the natural collapsing map $||\mathcal{D}(P)|| \to X$ is a homotopy equivalence.

Lemma 1.5.2. *Homotopy Lemma[**?**]*

Consider two arrangement X and Y. If $f_\alpha : X_\alpha \to Y_\alpha$ is a homotopy equivalence for each α, then $||\mathcal{D}(P_X)|| \to ||\mathcal{D}(P_Y)||$ is a homotopy equivalence.

Lemma 1.5.3. *Nerve Lemma[**?**]*

If U is a finite collection of open contractible subsets of X with all non-empty interactions of subcollections of U contractible, then the nerve $\mathcal{N}(U)$ is homotopic to $\cup_\alpha U_\alpha$.

Lemma Definition Example

Chapter 2

2

2.1 Rips complex of dense sample (Phuong)

Question: What are the analogues of NSW (Niyogi-Smale-Weinberger paper) for dense sample from manifolds? Note that we don't have the automatic property of Rips complex $R_\epsilon \sim M$ for covering by $\epsilon-$ balls.

Theorem 1 (J-C Hausmann,99)**.** For a compact Riemannian manifold, there is $r > 0$ such that $R_\epsilon(M) \sim M$ for any $r > \epsilon > 0$.

The r mentioned is called **injection radius** of M.

Definition 2.1.1. Injective radius r is the largest r such that for any point $q \in M$, any two geodesic rays starting at q with unit speed won't collide at time r.

For example, for a sphere of radius R, $r = \pi R$. With Theorem.1, the Rips-Vietoris complex is still built on the whole manifold M, thus there will be unaccountably many simplices of all dimensions. As such, Hausmann's result is not very useful. However, there is a new result (better version) that deals with finite samples.

Theorem 2 (Latscher, 01)**.** For any closed Riemannian manifold of positive injective radius, there is $r > 0$ such that for any $\epsilon < r$, there is a $\delta > 0$ such that for δ-dense $Y \subset J$, $R_\epsilon(Y) \sim M$.

In particular, for compact manifolds, finite samples and their Rips-Vietoris complexes can be used to approximate topology of M. However, note that, like Theorem.1, this result is not very useful in practice for high dimensions for various reasons: the size of samples growing exponentially, etc.

2.2 Sketches of topological space

In this section, we will be studying three types of sketches and their applications.

2.2.1 Medial axes (skeleton)

Consider an open (bounded) subset U of $\mathbb{R}^N$(or any Riemannian manifold). The distance $d(q, U^{\mathsf{c}})$ from $q \in U$ to complement $U^{\mathsf{c}} = \mathbb{R}^N \setminus U$ is finite. The **Maxwell Stratum** is the closure of the set of q for which $d(q, U^{\mathsf{c}}) = \min_{p \in U^{\mathsf{c}}} d(q, p)$. In many respects, the medial axis(

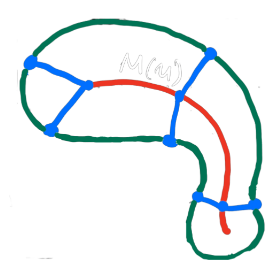


Figure 2.1: Maxwell Stratum Example

skeleton) is a reasionable sketch of U.

Proposition 1. $M(U) \sim U$

Proof. Run away from the boundary of U and reach U. This defines a deformation retraction from $U \to M(U)$. $\square$

Similarly, running towards the boundary defines a homotopy equivalence of $\mathbb{R}^{\mathbf{N}} \setminus \mathbf{M}(\mathbf{U}) \sim \mathbb{R} \setminus \mathbf{U}$, which is another important result.

This idea of medial axes is popular in computer graphics, and surprisingly it's also used in Mathematics. For example, **4 vertex theorem**: A vertex of a planar curve is a critical point of its curvature. The boundary of a domain in $\mathbb{R}^2$ has at least 4 vertices.

2.2.2 Reeb Graphs

Often the model intrinsically includes not just the underlying space X, but also a (real-valued) function. A sketch of such a datum would represent a morphism respecting homotopy equivalent and the mapping:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "sketch"] & R \arrow[d, "id"] \\ X_s \arrow[r, "f_s"] & R \end{tikzcd}$$

Definition 2.2.1. Let X be a path-connected space, and $f : X \to \mathbb{R}$ a continous function. Consider the set consisting of connected components of $f^{-1}(c)$, $c \in \mathbb{R}$ and impose the weakest topology with which the mapping from $f^{-1}(c) \to c$ is continuous.

This construction is actually pretty classical (Example shown in Figure.2.2).

The structure of the Reeb graph is especially transparent when X is a (smooth) manifold, and $f : X \to \mathbb{R}$ is a Morse function, which is very important in data analysis.

Definition 2.2.2. A function $f : M \to \mathbb{R}$ is Morse if:

1. f is C^2

2. its critical values are distinct and

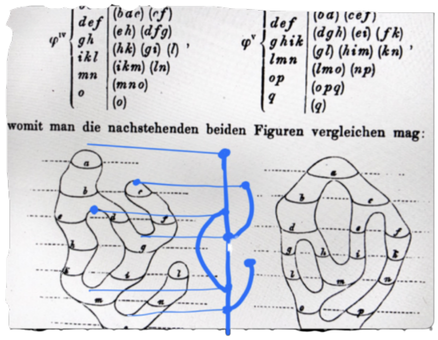


Figure 2.2: From A Mobius 1863'

3. nondegenerate: at critical points, the Hesse matrix $H = (\frac{\partial f}{\partial x_k \partial x_l})$ has $\det(H) \neq 0$.

Then, we know that $f^{-1}(c)$ does not change as c moves between critical points, and changes in connected components happen only at critical points of:

- index 0: which means a component is born.
- index m: a component dies.
- index 1: components might collide.
- index $m-1$: components might merge.

Reeb graph behaviors at critical points are summarized in Figure.2.3

In dimension 2, Reeb graphs describe the structure of M quite precisely. Recall that an orientable 2-dim manifold is a sphere with g handles, where g is called "genus". For example, S^2 is sphere with 0 handles, torus is sphere with one handle, etc. Some examples of Reeb graphs are include in Figure 2.4 and 2.5:

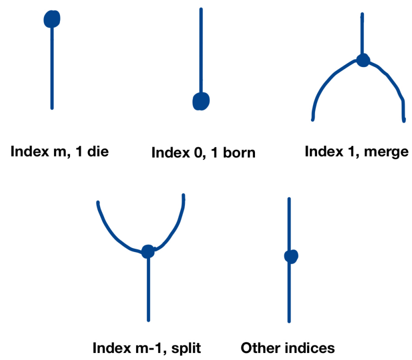


Figure 2.3: Reeb graph behaviors at critical points

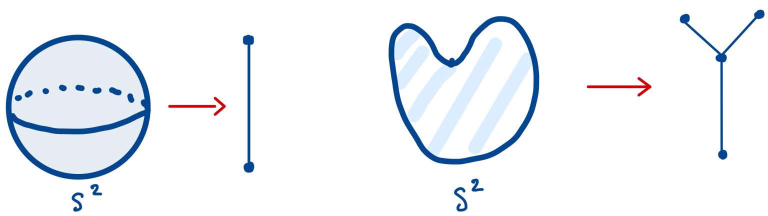


Figure 2.4: Reeb graphs of S^2, both don't have hole

Proposition 2. Reeb graph S_g has g loops.

The introduction of Euler characteristics (of simplicial complexes) χ is essential for understanding the proof.

Definition 2.2.3. For a simplicial complex K, $\chi(K) = \sum\limits_{\sigma \in K} (-1)^{\dim \sigma}$

Note that Euler characteristic is independent of triangulation for every 2-manifold, and if $X \sim Y$, then $\chi(X) = \chi(Y)$

Proof. Consider S_g a sphere with g handles, we know that $\chi(S^2) = \chi(\text{tetrahedron}) = 2$. To attach a handle to S^2, we need to cut out one pair of triangle, whose Euler charactistic is 1.

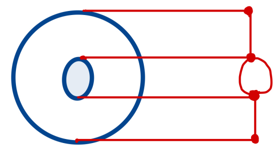


Figure 2.5: Reeb graph of Torus, with one hole

Also, a handle contributes 0 to the total sum, thus $\chi(S_g) = \#$ max + # min - # saddle $= 2-2g$. On the other hand, $\chi(\text{Reeb}(S_g)) = V - E = 1 - \#\text{loops}$, where V, E is the number of vertices and edges, respectively. Let denote the number of maxima and minima M, and the number of saddles S. The total number of vertices equals the number of minima, maxima, and saddles, and number of edges can also be counted through these points also. Thus $\chi(\text{Reeb}(S_g)) = M + S - \frac{M}{2} - \frac{3S}{2} = \frac{M-S}{2} = \frac{2-2g}{2} = \frac{1}{2}\chi(S_g)$. $\square$

Application of Reeb trees: Mapper (paper is linked)

2.3 Merge Trees

Merge tree is similar to Reeb tree, but it deals with the connected component of $M_{\leq c} = \{q \in M : f(q) \leq c\}$, i.e we are working with sublevel sets instead. Increasing family $M_{\leq c}$ forms a **filtration**.

Again, we consider the components of $M_{\leq c}$ as sets and the laziest topology. We will encounter more the of merge trees later. For now, for any open decreasing filtration, the topology is right semi-continuous. For any c, the topology of the set is the same as some interval right of c as at c. Note that the transitions where a component die or components branch cannot happen with merge trees.

Chapter 3

3

3.1 Computing Persistent Homology (Assif)

The aim of this lecture is to present an algorithm for computing the persistent homology of a given filtration of a finite simplicial complex. We assume that the simplices contained in the complex are $\{\sigma_1, \sigma_2, ..., \sigma_K\}$ and they are ordered in such a way that:

1. If σ_j is born after σ_i then $j > i$,
2. If σ_i is a face of σ_j then $j > i$.

It is possible to order the simplices in this way because we have assumed in the definition of a filtration that if a new simplex is added to the existing complex, all of it's faces are already contained in the complex. We can now think of the boundary operator ∂ as an element of $M(K \times K, k)$ where k is the field in which the homology is computed. The i, j-th element of ∂ is defined as

$$\partial[i,j] = \begin{cases} 1 & \text{if } \sigma_i \text{ is a face of } \sigma_j, \\ -1 & \text{if } -\sigma_i \text{ is a face of } \sigma_j, \\ 0 & \text{otherwise.} \end{cases}$$

Since all the faces of σ_j appear before j, $\partial[i,j] = 0$ if $i \geq j$, and so ∂ is an upper triangular matrix. For each column $\alpha \in \{1, 2, ..., K\}$ we define the height $L(\alpha)$ to be the largest index β such that $\partial[\beta, \alpha]$ is nonzero. The following algorithm reduces the matrix ∂ to a form from which the persistent homologies can be readily computed. We will denote column α of ∂ by $\partial[\cdot, \alpha]$.

Algorithm 1: Reduction of ∂ for computing PHs

```
Result: R
R = ∂;
α = 1;
L(i) = Height of column i in ∂;
while α ≤ K do
    β = 1;
    while β < α do
        if L(α) = L(β) then
            if R[L(α), α] = R[L(β), β] then
                R[·, α] = R[·, α] − R[·, β]
            else
                R[·, α] = R[·, α] + R[·, β]
            end
            Update L(α);
        else
            β = β + 1;
        end
    end
    α = α + 1;
end
```

Observe that at each step of the algorithm in which we modify $R[\cdot, \alpha]$, we are essentially multiplying R on the right by an upper triangular matrix with 1s on its diagonal and $\pm$ at the $[\beta, \alpha]$ position. Therefore the end result is a matrix. Therefore the end result of the algorithm is a matrix R which will be of the form $R = \partial V$ where V is an upper triangular matrix with all diagonal elements equal to 1. In addition, R has the property that each nonzero column of R has a different height L. The kernel of R can be readily computed now as the span of the basis vectors whose corresponding column in R is zero. Since V is invertible, we can say that the $\ker \partial = V. \ker R$ and that $\dim Z_p = \big|\{$ zero columns in R corresponding to simplices of dimension p $\}\big|$. Similarly the dimensions of B_p can be computed as the number of nonzero columns in R whose height is the index of a simplex of dimension p.

For computing persistent homologies from R we need to understand what happens when a p-simplex is added at a particular step of the filtration. Geometrically speaking, one of two things can happen:

1. The p-simplex generates a new nontrivial p-cycle increasing the dimension of the pth homology by 1,
2. The p-simplex fits in as the boundary of an existing nontrivial $(p-1)$-cycle decreasing the dimension of the $(p-1)$th homology by 1.

In other words, addition of the simplex σ_j to the simplicial complex $\{\sigma_1, \sigma_2, ..., \sigma_{j-1}\}$ either increases the pth betti number by 1 or decreases the $(p-1)$th betti number by 1. In terms of the matrix R, if σ_j increases the p-th betti number by 1 then during the reduction of column j in the algorithm one should get a new zero column at j which leads to this increase in $\dim Z_p$. On the other hand, if σ_j kills a $(p-1)$-cycle then during the reduction of column j in the algorithm one should get a new nonzero column at j whose height corresponds to the index of a $(p-1)$-simplex which leads to the increase in $\dim B_{p-1}$. Therefore, during the execution of the matrix reduction algorithm, we can mark instant j as either the birth

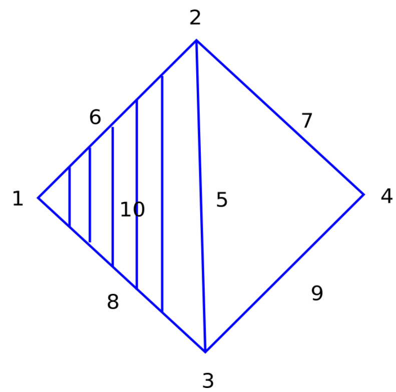


Figure 3.1: Simplicial complex with ordered vertices.

of a p-bar or death of a $p-1$ bar depending on whether or not we get a zero or nonzero matrix after reducing the column. This procedure can be done systematically keeping track of births and deaths to generate the persistence diagram. We will next look at an example of this procedure.

Consider the ordered simplicial complex given in Figure 3.1. A straightforward computation shows that the corresponding boundary matrix ∂ is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We observe what happens as the algorithm gets executed. The first column is a zero column and the corresponding simplex is 0 dimensional. This indicates the birth of a 0-bar at time 1. When we get to the second column, a new zero column is added which means that a new 0-bar is born at time 2. This makes sense from the picture since the line connecting the simplices 1 and 2 hasn't appeared yet, the vertex 2 generates a new connected component at this point. The same thing happens when we go through columns 3 and 4 also, a new 0-bar is born at instants 3 and 4. When we reach column 5, we first observe the $L(5) = 3$. No column before 5 has height 3, and so the column remains unchanged. This means that a new nonzero column is added, which implies addition of the 1-simplex 5 kills one of the 0-bars. Since $L(5) = 3$, it is precisely the 0-bar born at 3 that is killed. A similar situation occurs when we encounter columns 6 and 7 which leads to the deaths of the bars born at instants 2 and 4 respectively. This makes sense geometrically, since the lines 5, 6, 7 each connect previously disconnected vertices with each other. In the case of column 8, $L(8) = L(5) = 3$, which means we have to add column 5 to column 8 before determining the effect it has on homology. This addition results in the change of $L(8)$ to 2 which is equal to $L(6)$. So we further add column 6 to column 8 which makes column 8 zero. This means addition of the 1-simplex 8 leads to the birth of a new 1-bar. We can see clearly in the picture that the cycle $1 + 6 + 8$ is nontrivial and this is what leads to the birth of the 1-bar. Going forward, we see that $L(9) = 4 = L(7)$. Similar to the previous case, we end up adding columns 7 and 5 to 9 turning column 9 to zero. Therefore a new 1-bar is born at instant 9. Finally at instant 10, $L(10) = 8$ which means the simplex 10 kills the 1-bar born at instant 8. Putting all this together we can say that the persistence diagram of the simplicial complex given in Figure 3.1 is

$$PH_0 = \{(1, \infty), (2, 6), (3, 5), (4, 7)\},$$

$$PH_1 = \{(8, 10), (9, \infty)\}.$$

The performance of this algorithm is $O(K^3)$ which might be a problem since a single D-dimensional simplex has 2^D number of faces all of which have to go into ∂ causing the dimension K to explode quickly. However if one is only interested in computing homologies upto dimension p only the simplices of dimension at most $p+1$ have to be considered.

There are many software packages that implement persistent homology computations and we provide a non-exhaustive list here:

1. Gudhi: https://gudhi.inria.fr/,
2. Eirene: https://github.com/Eetion/Eirene.jl,
3. Perseus: http://people.maths.ox.ac.uk/nanda/perseus/.

3.2 Stability

We discuss the notion of stability of persistence diagrams in this section. Stability is a very important feature of persistent homology without which its applicability to the real world would be severely limited. To motivate the study of stability, we go to the earlier problem of determining the homology of a metric space M given only a finite point cloud X from it. We defined $f : M \to \mathbb{R}$ to be

$$f(q) = \min_{x \in X} d(q, x)$$

and studied the persistent homologies generated by the filtration $X_s = \{f \leq s\}$. It is natural to ask what happens to these persistence diagrams if we perturb the point cloud X by a small amount, since one has to deal with a certain amount of noise in any real world data. Towards this end, we first recall the defintion of Hausdorff distance between sets which gives an appropriate notion of a distance between point clouds. Let X and X' be subsets of a metric space M, then the Hausdorff distance $d_H(X, X')$ between them is defined as

$$d_H(X, X') = \inf\{\epsilon \geq 0 | X \subset X'_\epsilon \text{ and } X' \subset X_\epsilon\}.$$

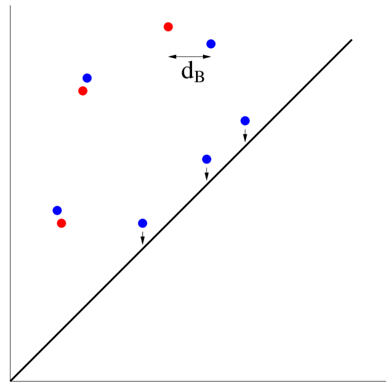


Figure 3.2: Bottleneck distance between the red and blue collections indicated by d_B.

We also define $f'(q) = \min_{x \in X'} d(q,x)$. A straightforward computation shows that

$$d_H(X, X') \leq \epsilon \implies \|f - f'\|_{C^0} \leq \epsilon$$

. Due to this observation, we forget point clouds for the time being, and ask the following question: If two function f and f' close in the supremum norm, are their persistence diagrams also close in some sense? The answer turns out to be positive.

Definition 3.2.1 (Bottleneck distance)**.** *Given two collections of points Q, Q' in $\{(b,d) \in \mathbb{R}^2 | b < d\}$, we define*

$$W_\infty(Q, Q') := \inf_P \sup_{(q,q') \in P} \|q - q'\|_\infty$$

where $P \subset (Q \cup \Delta) \times (Q' \cup \Delta)$ is a pairing of points in Q, Q' such that each point q from Q and q' from Q' appear in exactly one pair in P.

If the bottleneck distance between two persistence diagrams is small, then any point far away from the diagonal in one of the diagrams can be matched with a nearby point in the other, and points that are close to the diagonal may get paired to a point on the diagonal. This can be observed in Figure 6.1, where even though there are blue points that cant be matched to any red points, they lie close to the diagonal line and are coupled away with the diagonal bringing the bottleneck distance down to d_B. The points away from the diagonal in the persistence diagram are precisely the long bars, the features that persist over a long range of time. The following theorem is the key result of this section which says that long bars are preserved under small perturbations of f.

Theorem 3.2.1. *The bottleneck distance*

$$W_\infty\left(PH_k(f), PH_k(f')\right) \leq \|f - f'\|_\infty.$$

3.3 Lecture 8: Morse Theory and Bi-parametric persistence (Wu)

Let's first recall some classical results about Morse theory, which also know as theory of critical points, a classical textbook reference for this topic is the Morse theory by John Milnor.

3.3.1 Morse Theory

Given a manifold $\mathcal{M}$, where a Morse function is a smooth real-valued function with no degenerate critical points. A fundamental fact in Morse theory is nearly all functions are Morse functions, in particularly:

Theorem 3.3.1. *On a compact $\mathcal{C}^2$ manifold, Morse functions form a open everywhere dense subset.*

Another classical result of Morse theory is the behavior of level sets when passing through a critical point. Denote the sub-level set of function f at point a : $M_a = f^{-1}(-\infty, a] = \{x \in \mathcal{M}; f(x) \leq a\}$. We have the following results:

Theorem 3.3.2. *Let f be a Morse function on $\mathcal{M}$, $a < b$ and suppose the set $f^{-1}[a, b]$ is compact, and contains no critical points of f. Then M_a is diffeomorphic to M_b, further, the inclusion map of M_a to M_b is a homotopy equivalence .*

Theorem 3.3.3. *Let f be a Morse function, and p be a non-degenerate critical point of index k, and $f(p) = c$. Suppose $f^{-1}[c - \epsilon, c + \epsilon]$ is compact, and contains no other critical points than p, then for small $\epsilon > 0$, $M_{c+\epsilon}$ has a homotopy type of $M_{c-\epsilon}$ with a k-simplex attached.*

Based on those classical results, we can see the behavior of 0-dim persistent homology when passing through a non-degenerate critical point of index k, which either generate a k-dim cycle or a (k-1)-dim boundary. In particular, in term of Euler characteristic,

$$\chi(M_c) = \sum_q (-1)^{index(q)} \tag{3.1}$$

where q is critical points with $f(q) \leq c$. In general, as we defined the persistent homology, where some cycles are born and killed at some time, actually this gives a coupling of critical points of the Morse function. The decomposition of those coupled critical points was known before persistent homology (Barannikov 96), and those decomposition has many application in PDEs and large deviation theory in probability.

3.3.2 Bi-parametric Persistence

As we studied in previous lectures about persistent homology, the persistence and the Morse theory are connected through the study of filtrations generated by a single function $f : \mathcal{M} \to R$, so it's natural to generalize the above situation to multi-parametric persistence, where the filtrations generated by several functions on the manifold $\mathcal{M}$. However, there is no some elementary quivers representations of the analogue bars as in 1-d case. On the other side, we can try to reformulate Morse theory of multiple functions, and this relies on the local structure of singularities.

Let $\mathcal{M}$ be a compact manifold, and $\mathcal{H} = (f, g) : \mathcal{M} \to R^2$ be a smooth map, the critical point of this smooth map is defined as $\Sigma(H) = \{p \in \mathcal{M}; rank(D) < 2\}$, where D is the corresponding differential map between tangent spaces: $D_p : T\mathcal{M}_p \to (TR^2)_{\mathcal{H}(p)}$. Let's see the following example:

Example 3.3.1. *$\mathcal{M} = \{(x, y, z) \in \mathbb{R}^3; (x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2\}$, and $\mathcal{H} : \mathcal{M} \to \mathbb{R}^2$ is identified by $(x, y, z) \to (x, y)$, i.e the projection to x-y plane. Then the critical points $\Sigma(\mathcal{H})$ will be the equator as shown in the picture. Since the projection will be 1 dimensional object based on the definition of critical set above.*

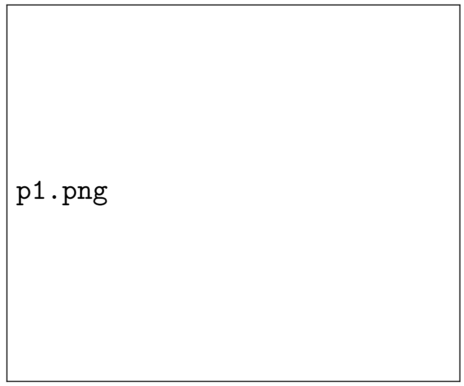


Figure 3.3: Example of multi-parametric map

Here is a theorem dealing with the critical set for generic smooth map $\mathcal{H}$.

Theorem 3.3.4. *For generic(modulo some small closed nowhere dense set, check first theorem in this lecture) $\mathcal{H} = (f, g)$, the critical set $\Sigma(\mathcal{H})$ is a smooth curve in $\mathcal{M}$.*

From the definition, we know that $\Sigma(\mathcal{H})$ contains the critical points where $rank(D(\mathcal{H})) < 2$. For ones with $rank(D(\mathcal{H})) = 1$, we name it fold points. While $rank(D(\mathcal{H})) < 1$, it is called pleat points. Let's see some examples to those two notions

Example 3.3.2. *(Fold points) If $p \in \Sigma(\mathcal{H})$, from the picture we can see $rank(D(\mathcal{H})|_{T\Sigma}) = 1$. In this case, we call the critical points fold points, where the mapping $\mathcal{H}$ has some local representation as $(x_1, x_2, \ldots, x_m) \mapsto (x_1, \sum_{i=2}^{m} x_i^2)$ in this example with appropriate coordinates chart chosen. Actually, we can check this local behavior from the following picture:*

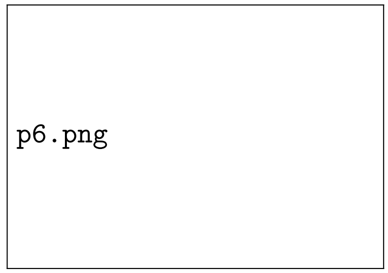


Figure 3.4: Example of fold points

Example 3.3.3. *(Pleat Points) Take* $\mathcal{H} : (x_1, x_2) \mapsto (x_1^3 + x_1 x_2, x_2)$*, then*

$$D(\mathcal{H}) = \begin{pmatrix} 3x_1^2 + x_2 & x_1 \\ 0 & 1 \end{pmatrix}$$

and $\Sigma(\mathcal{H}) = \{(x_1, x_2); x_2 = -3x_1^2\}$ *then*

$$D(\mathcal{H})|_{\Sigma} = \begin{pmatrix} -6x_1^2 & -6x_1 \\ 0 & 0 \end{pmatrix}$$

and the rank is 0 at point $(x_1, x_2) = (0, 0)$

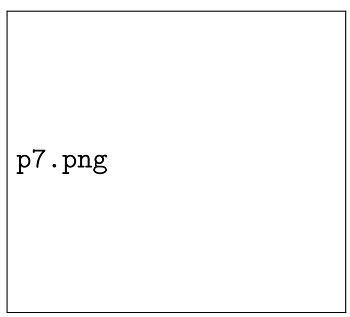


Figure 3.5: Example of pleat point

From the picture, we can see that the pleat points are indeed on a smooth curve of the manifold, but when you apply the projection map, there is a singularity point appears. This fact in some sense characterizes the pleat points.

Theorem 3.3.5. *(Whitney) For generic map $\mathcal{H} : \mathcal{M} \to \mathbb{R}^2$, there is a finite set of pleat points on a smooth curve of fold points, all the other points are regular, i.e $rank(D(\mathcal{H})) = 2$*

Now we are ready to handle the bi-parametric persistence, given a smooth map, $\mathcal{H} : \mathcal{M} \to \mathbb{R}^2$, consider the bi-parametric filtration $M_{s,t} = \{f \leq s, g \leq t\}$, thus we can define $\epsilon((s,t),(s',t')) : M_{s,t} \to M_{s',t'}$ for $s \leq s', t \leq t'$, we also define the visible contour as the set of critical values of $\mathcal{H}$

p2.png p3.png

Figure 3.6: Caption

the above picture basically describes what happens on the topology of corresponding sub-level sets when we changes s ,t values. Where the yellow circle is the visible contour, when we study the corresponding sublevel sets, we can see that the topology does not change while not crossing the visible contour, but indeed changes while crossing the visible contour. Those results are the high dimensional analogue of theorem 1.8.2 and 1.8.3.

Let's check another example, which is the cloud gate in Chicago Millennium Park. The upper right figure is the visible contour, and the one below that are used to study how to topology of sub-level sets (or slices) change with respect to s,t.

p8.png p4.png

p5.png

Figure 3.7: Caption

For fold points and cusp points, we can see that there are well defined tangents, but the difference is when we draw the slices at those points, then the slices will cross the visible contour, in this case, the topology changes. However, there are also some points, named as pareto points, where we do slices, but the slices are fully supported inside of the visible contour, and the topology will not change in this case. With all those information, how can we turn to compute the 1-d persistent homology? What we have now is the 2-parameter(s,t) slice information, but if we can get a glimpse of the usual 1-parameter filtration, then PH_1 could be computed based on the corresponding persistence diagram.

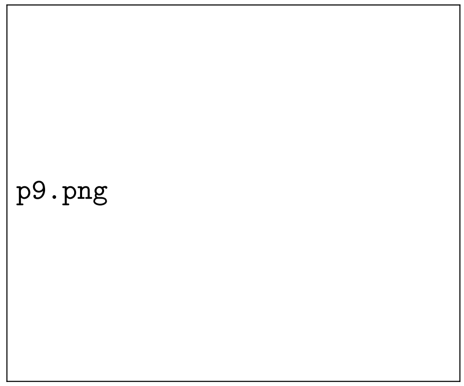


Figure 3.8: Caption

The idea is to take the increasing curves with respect to both parameters s,t. For example, let's take the blue curve in the above picture, when it's passing the visible contour first time, we got first component, then PH_0 appears, then move forward a bit, crossing the curve between cusps, a new bar added, and PH_1 will be born and killed with time progressing as shown in the above picture. For the increasing curves(green ones), we can do a similar thing. Then we can summarize the above results in the following theorem

Theorem 3.3.6. *If γ, γ' are 2 increasing curves in (s, t) plane, which can be deformed on into the other such that*

- *they remain increasing*
- *Not hitting cusps or quasi-cusps or contour self- intersections*

then PH_γ is a deformation of $PH_{\gamma'}$ that there is a unique lift of the deformation into the space of persistence diagrams, while not collapsing bars into the diagonal.

This theorem tells us that there is a unique lift of the deformation into the space of persistence diagrams, while not collapsing bars into the diagonal. So the next question will be what are the homotopy equivalence class among increasing curves avoiding a finite set of points on the plane? The answer of this question is phrased in the next theorem.

Theorem 3.3.7. *The classes of homotopy equivalent curves are the chains of obstacles with respect to vector ordering.*

Remark 3.3.1. *In fact, those classes are the vertices in the NPC cubical complexes.*

5.1 Euler Characteristic (Laskowski)

We defined Euler Characteristic in two fashions: first, as an alternating sum of numbers of simplicies (cells) of different dimensions for (finite) simplicial complexes; secondly, as the alternating sum of ranks of homotopy groups:

$$\chi(x) = \sum (-1)^k \mathrm{rk}\big(H_k(X)\big)$$

The second definition is invariant with respect to homotopy. The first one is additive: if X, Y are simplicial subcomplexes of an ambient complex, then

$$\chi(X \cup Y) + \chi(X \cap Y) = \chi(X) + \chi(Y)$$

(in other words, the inclusion-exclusion formula holds).

An important fact is that additivity stops working once one considers non-compact complexes, but still wants to preserve homotopy invariance:

$$\chi(\bullet\!\!-\!\!\bullet) = 1, \text{ but } \chi(\bullet) = 1 \ \& \ \chi(-) = 1 \text{ if homotopy invariant}$$

Here we sacrifice homotopy invariance. This means, in particular, that we need to be careful about spaces we are working with: just simplicial complexes is to narrow, so typically one works with a class of sets in $\mathbb{R}^4$, $n = 0, 1, \ldots$ fitting the o-minimal structure properties:

- closed under Cartesian products
- closed under projections $\mathbb{R}^n \mapsto \mathbb{R}^{n-1)}$

- contains sets $\{P = 0\}$, for P $\in$ real polynomials
- O_1 = finite unions of points intervals

We call sets in an O-minimal structure *definable*.

Example 3.3.4.
- *PL polyhedra*
- *semialgebraic sets*
- *subanalytic sets*

A definable set is homeomorphic (1-to-1 with a definable graph) to a collection of simplicies in a simplicial complex.

In particular, one can form a Euler characteristic (and prove it is independent of triangulation).

Example 3.3.5. $\chi(\mathring{D}^n) = (-1)^n$, $\chi(D^n) = 1$, $\chi(S^n) = 1 + (-1)^n$.

chi examples.jpg

Theorem 3.3.8. *For definable X, dim(X) and $\chi(X)$ are the only definable quantities.*

5.2 Integrals with respect to χ

Definition 3.3.1. *Say that $f : X \to \mathbb{R}$ is constructible if rank is finite and $f^{-1}(a)$ are definable.*

Definition 3.3.2.

$$\int_X d\chi = \sum_{a \in Im(f)} a \cdot \chi\big(f^{-1}(a)\big)$$

Remark 3.3.2. *This is a parallel to Lebesgue integrals. In particular, if $f = 1_x$, then $\int 1 d\chi = \chi(x)$.*

```
overlap example.jpg
```

The Euler characteristics of the "1" and "2" regions are 0, and by the above definition $\int_X d\chi$, which is $= 1 \cdot \chi(..) + 2 \cdot \chi(..) + 3 \cdot \chi(..)$ where only the third component contributes, so $\chi(\text{example}) = 3$.

Definition 3.3.3. *Properties*

$\int (f \pm g) d\chi = \int f d\chi \pm \int g d\chi$

In particular, if $f : X \to \mathbb{Z}_+$, then

$$\int f d\chi = \sum_{n=1}^{\infty} \chi(f \geq n)$$

In addition, the parallel of Fubini's theorem holds:

$$\int f(a, y) d\chi = \int \left[\int_Y f(a, y) d\chi \right] d\chi$$

5.3 Application: Target enumeration

Consider a domain $\mathcal{D} \subset \mathbb{R}^n$ and a collection of objects $\mathcal{O}_\alpha \subset \mathcal{D}$. Assume a dense field of sensors that returns additive counts of objects crossing the sensor. If all objects have the same area, integrating the measurements to compute the total number of objects would be easy: if $f = \sum 1_{\mathcal{O}_\alpha}$

$$N = \frac{1}{|\mathcal{O}} \int f d\chi$$

However, if the object footprints are different in shape and area, one can still do the integral as long as it is additive and all $\int \nVdash_{\mathcal{O}}$ are the same. Say, if we interpret $d\chi$ and $f(\mathcal{O}_\alpha = 1$, then

$$N = \int f d\chi$$

Example 3.3.6. *Consider planar fields:*

$$\int f d\chi = \sum_{n \geq 1} \chi(f \geq n)$$

planar (2).jpg

Of course, massive difficulties arise when $\chi(0) = 0$;

Example 3.3.7. cross example (2).jpg

The number of objects is not very well defined here.

3.4 Data Aggregation (Charles Ross)

Finding the average is one of the key proceudres in data analysis. yet little attention is paid to when an average exists at all in the first place.

Definition: Let X be a topological space A_n. n-averaging is a mapping associating to a tuple its average $m_n(x_1, ..., x_n)$. Requirements

- $m_n(x_1, ..., x_n) = x$
- m is a S_n equivalent for any position
- m is continuous

Topology of space can be an obstacle. Example:

Two points on circle. Fix one point and vary other. Take X=S' Then for a fixed x_*, x-¿ $m_2(x_*, x)$ maps S'-¿S' There is a middle point that between two points. As x traverses circle, image of the point also traverses circle, so there is a mapping between circle and itself. The degree of the mapping of the manifold onto itself is determined by the number of wrappings of circle onto itself. However, considering the diagonal, the degree of the mapping should be equal to the sums of the degrees, which in 2D cannot be 1.

So, no 2 averaging on S' or any sphere. Turns out, existence of n-means (for all $n > 2$) means that X is contractible.

Theorem: If X is topologically equivalent to finite spherical complex (or, finitely many simplices) and there exists n-averaging for any $n > 2$ then X is contractible

The proof is essentially taking the homologies and establishing, using the reasoning above, that H_k is n-divisble for each n. This is possible only if $H_k(X, Z)$ is like Q or is 0.

Now, if all homologies are zero, then all homotropic groups are 0, which implies that X is contractible.

We should note that one can construct quite easily some pathological no-contractible spaces. Simplest example is "2-solenoid", defined as infinite sequence of poitns in S' such that $\phi = 2\phi_{n+1}$ for al n-1,2,...

Conversely, if X is contractible, it admits n-averageing for all n¿2. Some examples are easy: R^n has a natural averaging etc.

3.5 Spaces of non-positive curvature

We now will be constructing means– require that X is a path metric space (will be assuming completeness), ie we have a distance function d(x,y) and we can always find a halfway point (and these are resconstructed geodesic, the shortest line connecting a pair of points).

Among path metric spaces non-positive curvature spaces: for any geodesic triangle ABC, and pt D on the geodesic BC, the distance l is less than or equal what it should have been in Euclidean space.

Corolleries:

Distances in linear homotropies along geodesics are convex
Metric balls are convex (i.e. round sphere are not non-positive curvature)

Examples:
Hyperbolic spaces
Product of NP spaces
Gluing along convex subsets

Properties of NPC spaces:

-any two points are connected by unique geodesic line

-geodesic (as a mapping I-¿X) depends continuously on its endpoints

3.5.1 Barycenters

For NPC spaces, $d^2(x)$ is convex. For any collection of points define it barycenter as the unique point b(q) which minimizes

$$x- > \int d^2(x,z)dq(z)$$

Easy to see $q = a_o\,()\,x + a_1\,()\,x$ is on geodesic between two points.

3.5.2 Phylogenetic trees

Phylogenetic trees are metric trees with roots, labelled leaves and same distance from root to leaves.

Theorem Space of phloygenetic trees is NPC

Proof: If the link of any cube is a cubical complex it is a flag complex, then the space is NPC. Links in the space of trees correspond to the comptaible families of edges, thus they can be varied simultaneously and are a flag complex.

3.6 Around Arrow (VESTO)

We begin with an introduction to the conventional implementation of **Arrow's Impossibility Theorem**. This begins with the following setup:

- A – the set of alternatives with $|A| \geq 3$
- P – the set of preferences or ranking on A

  Typically defined with a weak order (a $\geq b$ which is transitive, reflexive and complete) and depict $a > b$ as $^{a}_{b}$.

- V – the set of voters with $|V| \geq 2$
- P^V – the set of profiles or societal views for all voters.
- $R : P^V \to P$ – the social aggregation rule

Define the following axioms:

1. Unanimity – there exists Δ such that $P \xrightarrow{\Delta} P^V \xrightarrow{R} P = \mathrm{id}_P$.
2. Independence of Irrelevant Alternatives (IIA) – the social preference depends between alternatives depends only on the individual preference on the same alternatives.

Definition 3.6.1. *Say that $C \subset V$ is an **assertive** coalition over a pair k, l if:*

$$\text{profile} \to \begin{array}{c} C \\ k \\ l \end{array} \Bigg| \begin{array}{c} C^c \\ l \\ k \end{array} \Bigg\| \begin{array}{c} \\ k \\ l \end{array} \leftarrow \text{outcome} \tag{3.2}$$

Lemma 3.6.1. *If C is assertive over k, l then it is assertive over any pair*

Proof. If C is assertive on $1, 2$ then 3 can be inserted such that:

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \Bigg| \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \Bigg\| \begin{array}{c} 1 \\ 2 \end{array} \tag{3.3}$$

By unanimity, the outcome should have $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ and by transivity:

$$\begin{array}{c} 1 \\ 3 \end{array} \Big| \begin{array}{c} 3 \\ 1 \end{array} \Big\| \begin{array}{c} 1 \\ 3 \end{array} \tag{3.4}$$

Therefore C is assertive on $1, 3$. If instead we place 3 above $1, 2$ then:

$$\begin{array}{c} 3 \\ 1 \\ 2 \end{array} \Bigg| \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \Bigg\| \begin{array}{c} 1 \\ 2 \end{array} \tag{3.5}$$

which implies:

$$\begin{array}{c} 3 \\ 2 \end{array} \Big| \begin{array}{c} 2 \\ 3 \end{array} \Big\| \begin{array}{c} 3 \\ 2 \end{array} \tag{3.6}$$

This logic is repeated for all pairs. $\square$

Definition 3.6.2. *Say that $C \subset V$ is **deciding** if:*

$$\begin{array}{c|c||c} 1 & * & 1 \\ 2 & * & 2 \end{array} \tag{3.7}$$

or equivalently that C' is assertive for any $C' \supset C$.

Lemma 3.6.2. *If C is assertive then it is deciding.*

Proof. Using pair $1, 2$, we can insert 3 such that:

$$\begin{array}{c|c|c||c} C & C^c \cap C' & & \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 3 & 1 & 3 \end{array} \tag{3.8}$$

Then it holds:

$$\begin{array}{c|c|c||c} 1 & 1 & 3 & 1 \\ 3 & 3 & 1 & 3 \end{array} \tag{3.9}$$

and since the first two profiles make up C' then it holds that C' is assertive. $\square$

Lemma 3.6.3. *If C is deciding and $C = A \amalg B$, then either A or B is deciding.*

Proof. If C is deciding then:

$$\begin{array}{c|c|c||c} A & B & & \\ 1 & 1 & * & 1 \\ 2 & 2 & * & 2 \end{array} \tag{3.10}$$

Then if we can insert 3 such that:

$$\begin{array}{c|c|c||c} 1 & 2 & 3 & 1 \\ 2 & 3 & * & 2 \\ 3 & 1 & * & \end{array} \tag{3.11}$$

Then it follows:

$$\begin{array}{c|c|c||c} 1 & 3 & 3 & 1 \\ 3 & 1 & 1 & 3 \end{array} \tag{3.12}$$

Then A is assertive and therefore deciding. All other cases are performed similarly.

$\square$

Then the set of deciding coalitions satisfies:

- Non-empty

- If C is deciding, so is $C' > C$
- If $A \amalg B = C$, then either A or B is deciding

A set which satisfies these is called an ultrafilter.

Remark 3.6.1. *If V is finite, then any ultrafilter on V is principal, i.e. consists of all sets containing v, a very special voter.*

Theorem 3.6.1. ***Arrow's Impossibility Theorem**: If a system satisfies N unanimity, IIA, and $|A| \geq 3$ then P is a projection, i.e. the system is a dictatorship with v deciding all outcomes.*

Arrow's Impossibility theorem lead to an explosion of research in social choice. Note, that this is not a predictive model but rather a very good metaphor.

3.6.1 Topological Angle

In the 1970s, Graciela Chichilniskey started to introduce topological tools into social choice theory. She (with Geoffrey Heal) rediscovered a form of Eckmann's result on contractibilitiy of spaces admitting aggregation with continuous, anonymous (equivalent) and unanimous rules.

She proposed the following alternative model with the space of preferences $P = S^{d-1}$ i.e. linear functions in $(\mathbb{R}^d)^*/\mathbb{R}_*$.

Pareto condition: If all $p_1, \ldots, p_n$ agree on a pair of points $x, y \in \mathbb{R}^d$ then $m(p_1, \ldots, p_n)|_{x,y}$ is the same, unanimously on pairs.

Geometrically this means that $m(p_1, \ldots, p_n) \in \mathrm{conv}(p_1, \ldots, p_n)$.

Theorem 3.6.2. *In dimension $d \leq 2$ or for $n \leq 3$, any continuous aggregation rule satisfying the Pareto condition can be homotopied to a dictorial rule in the class of Pareto rules. For other value of (d, n) there exists Pareto rules not homotopic to dictorial ones.*

This has the following interpretation: the democracies with enough choices ($d > 2$) and voters ($n \geq 4$), there are rules that resists the gradual slide (continuous deformation) into dictatorship. They tend to be cumbersome and hard to construct.

Another relevant notion is of a (strategic) manipulator who has a preference which they strategically do not declare.

Definition 3.6.3. *We say that voter l is a **manipulator** if for any desired outcome, p_l and any complementary profile: $p_1, p_2, \ldots, p_{l-1}, p_{l+1}, \ldots, p_n$, there exists a clamed profile $\hat{p}_l$, such that:*

$$m_n(p_1, p_2, \ldots, p_{l-1}, \hat{p}_l, p_{l+1}, \ldots, p_n) = p_l. \tag{3.13}$$

Proposition 3.6.1. *l is a manipulator iff for some $p \in S^{d-1}$,*

$$m_n(p, p, \ldots, p, \overbrace{-p}^{l}, p, \ldots, p) = -p \tag{3.14}$$

where the negative entry is the l-th input.

Proposition 3.6.2. *Let d_l be the degree of the map:*

$$x \mapsto m(p, p, \ldots, p, \overbrace{x}^{l}, p, \ldots, p) \tag{3.15}$$

where the x occurs on the l-th input. Then $d_l = 1$ if l is a manipulator and 0 otherwise.

Theorem 3.6.3. *(Chichilniskey) For a Pareto social choice rule m, there exists unique manipulator.*

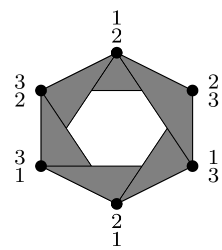
Proof. In $(d-1)$-dimensional homology of $S^{d-1} \times \ldots S^{d-1}$, the class of diagonal Δ is such of classes of $S^{d-1} \times p_* \times \cdots \times p_*$, $p_* \times S^{d-1} \times \cdots \times p_*$, $\ldots$.

Then $H_m : H_{d-1}((S^{d-1})^n) \to H_{d-1}(S^{d-1}) \cong \mathbb{Z}$ induced by m sends generators to 1 or 0, and their sum to 1. ∎

3.6.2 Back to Arrow's Theorem

The axiom of IIA hints at a simplicial complex that can be constructed from pairwise comparisons. Namely, consider $d(d-1)$ pairs $\genfrac{}{}{0pt}{}{a}{b}$, $a, b \in \{1, \ldots, d\}$, the set of alternatives and define the simplicial complex as collections of pairwise comparisons without cycles (i.e. such that can be satisfied by an ordering of alternatives).

Example 3.6.1. *With $d = 3$*



has the homology of a circle.

Proposition 3.6.3. *The resulting simplicial complex is homotopy equivalent to S^{d-1}.*

Proof. Consider the family of halfspaces $H(\genfrac{}{}{0pt}{}{a}{b}) = \{x_a > x_b\}$, $1 \leq a \neq b \leq d$. A collection of these halfspaces intersect iff the are compatible with an ordering, so the nerve is homotopy equivalent to our simplicial complex. The union:

$$\bigcup H(\genfrac{}{}{0pt}{}{a}{b}) = \mathbb{R} - \Delta \sim S^{d-1} \tag{3.16}$$

with diagonal $\Delta = \{(x, x, \ldots, x) : x \in \mathbb{R}\}$. $\square$

The space of profiles is constructed the same way. One takes $2^n \binom{d}{n}$ profiles of pairwise preferences and form simplices whenever combinations of pairwise profiles are consistent (for each voter).

In this setting, IIA implies that social aggregation rule is a simplicial mapping between simplicial complexes.

The next two steps are expectable:

Lemma 3.6.4. H_{d-2}(simplicial complex of profiles) $= \mathbb{Z}^n$.

This can be proven, essentially, just like for a single voter.

The group H_{d-2}(profiles) has a natural basis $\{[e_l]\}_{l=1,\dots,n}$, one for each voter.

Lemma 3.6.5. *$H_m : [e_l] \to [e]$, the generator of H_{d-1}(preferences) if l is a dictator, and 0 otherwise.*

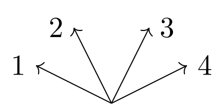
The proof is somewhat combinatorial.

Also, trivially, H_m take the class of the diagonal into the generator of H_{d-1}(preferences).

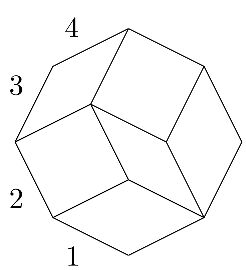
Corollary 3.6.1. *There exists a unique dicatator.*

One of the branches of the social choice theory deals with the domain restriction, reducing the set of allowable preferences so that Arrow paradox disappears. There are many alternatives. One geometric approaches deals with zomotopal tilings. Take d vectors in $\mathbb{R}^2$, pointing up, ordered clockwise; take the Minkowski sum. The resulting polygon can be tiled by $\binom{d}{2}$ parallelograms, with edges parallel to e_k, e_l, $1 \le l, k \le d$.

Example 3.6.2. *For $d = 4$ vectors:*

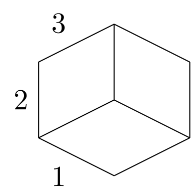


the resulting polygon is:

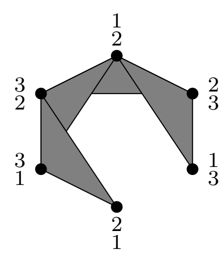


Each path in the polygon corresponds to a permutation (ordering). remarkably, on this subset of orderings, averaging exists. And it is easy to see that the corresponding simplicial complex is contractible.

Example 3.6.3. *For $d = 3$, a polygon has form:*



which corresponds to simplicial complex:



which is contractable.