MAST90105 Methods of Mathematical Statistics

Assignment 3, Semester 1 2019

Due: Sunday 12 May, end of day. Please submit a scanned or other electronic copy of your work via the Learning Management System - see [this link for instructions]()

Note: You may use R and/or Mathematica for any questions but must include your commands and reasoning.

Problems:

1. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are repeated measurements of the nitrogen dioxide obtained by device A and device B, respectively. Measurement errors are different for these two devices and we assume that the measurements $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are all mutually independent and $X_i \sim N(\mu, \sigma^2)$, $Y_j \sim N(\mu, 2\sigma^2)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We want to estimate unknown parameters μ and σ^2 using the likelihood approach.

   (a) Write down the joint pdf of $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ and simplify it.

   (b) Find the maximum likelihood estimators for μ and σ^2 (You are not required to demonstrate that the stationary points are maxima).

   (c) Let $n = 10$ and $m = 8$. Use the following data to find estimates of μ and σ^2:

   ```
   X:   3.39 2.19 2.18 1.57 1.30 3.52 2.41 2.00 2.87 3.17
   Y:   1.01 1.97 3.51 1.53 1.88 2.34 1.14 1.29
   ```

2. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density

$$f(x; a, b) = \begin{cases} ax^2 + bx + \frac{1}{2} - \frac{a}{3}, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the method of moments estimators for unknown parameters a and b.

   (b) Are the method of moments estimators of a and b unbiased?

   (c) Let $n = 10$. Use the following data to find method of moments estimates of a and b:

   ```
   -0.77 0.33 -0.61 -0.27 -0.13 0.21 0.09 -0.46 0.85 0.43
   ```

3. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density:

$$f(x; c) = \begin{cases} c, & -1 < 0 < x, \\ \frac{1-c}{3}, & 0 \le x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the method of moments estimator for c. Is it an unbiased estimator?

(b) Find the variance of the method of moments estimator as a function of c.

(c) Let M be the number of variables X_i such that $-1 < X_i < 0$ and $n - M$ be the number of variables X_i such that $0 \leq X_i < 3$. Write down the likelihood function for these data and find the maximum likelihood estimator of c.

(d) Is the maximum likelihood estimator of c unbiased? Find the variance of this estimator as a function of c. *Hint:* M follows a Binomial distribution $\texttt{Binom}(n, p)$ with probability of success $p = \Pr(X < 0)$ where the pdf of X is $f(x; c)$.

(e) Does the exact variance of the method of moments and maximum likelihood estimators of c attain its Cramer-Rao lower bound for $0 < c < 1$? Why or why not?

4. Consider the cumulative distribution function (cdf)

$$F(x; \theta) = \begin{cases} 0, & x \leq 1, \\ 1 - x^{-\theta}, & x > 1. \end{cases}$$

Assume that the prior density of the unknown parameter $\theta > 0$ is

$$f(\theta) = \begin{cases} 0.2e^{-0.2\theta}, & \theta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Let X_1, X_2, X_3 be a random sample from $F(x; \theta)$. The observed data are $x_1 = 1.5, x_2 = 1.2, x_3 = 2.0$. Find the posterior distribution of θ given the observed data. *Hint:* $x^k = e^{k \ln x}$.

(b) Find the posterior mean of θ. Find the posterior probability $\Pr(2 < \theta < 5)$.