MAST90105 Methods of Mathematical Statistics
Assignment 3, Semester 1 2019 **Solutions**

Note: You may use R and/or Mathematica for any questions but must include your commands and reasoning.

Problems:

1. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are repeated measurements of the nitrogen dioxide obtained by device A and device B, respectively. Measurement errors are different for these two devices and we assume that the measurements $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are all mutually independent and $X_i \sim N(\mu, \sigma^2)$, $Y_j \sim N(\mu, 2\sigma^2)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We want to estimate unknown parameters μ and σ^2 using the likelihood approach.

   (a) Write down the joint pdf of $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ and simplify it.

   - *Since X_i and Y_j are independent the joint pdf is*

$$
\begin{aligned}
f(\mathbf{x}, \mathbf{y}) &= \left[\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i-\mu)^2\right\}\right] \cdot \left[\prod_{j=1}^{m} \frac{1}{\sqrt{4\pi\sigma^2}} \exp\left\{-\frac{1}{4\sigma^2}(y_j-\mu)^2\right\}\right] \\
&= 2^{-\frac{n+2m}{2}} (\pi\sigma^2)^{-\frac{n+m}{2}} \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{1}{4\sigma^2}\sum_{j=1}^{m}(y_j-\mu)^2\right\}.
\end{aligned}
$$

[1]

   (b) Find the maximum likelihood estimators for μ and σ^2 (You are not required to demonstrate that the stationary points are maxima).

   - *The log-likelihood function is*

$$
\ell(\mu, \sigma^2) = C - \frac{n+m}{2} \ln \sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{1}{4\sigma^2}\sum_{j=1}^{m}(y_j-\mu)^2
$$

   up to a constant C not depending on the parameters.

   - *Differentiating $\ell(\mu, \sigma^2)$ with respect to μ and σ^2 and setting equal to 0 gives*

$$
\begin{aligned}
\frac{\partial \ell(\mu, \sigma^2)}{\partial \mu} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) + \frac{1}{2\sigma^2}\sum_{j=1}^{m}(y_j-\mu) = 0, \\
\frac{\partial \ell(\mu, \sigma^2)}{\partial \sigma^2} &= -\frac{m+n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2 + \frac{1}{4(\sigma^2)^2}\sum_{j=1}^{m}(y_j-\mu)^2 = 0,
\end{aligned}
$$

   .

   - *From the first equation,*

$$
\hat{\mu} = \frac{2}{2n+m}\sum_{i=1}^{n} x_i + \frac{1}{2n+m}\sum_{j=1}^{m} y_j.
$$

[1]

- *From the second equation,*

$$\hat{\sigma}^2 = \frac{1}{m+n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 + \frac{1}{2(m+n)}\sum_{j=1}^{m}(y_j - \hat{\mu})^2.$$

[1]

(c) Let $n = 10$ and $m = 8$. Use the following data to find estimates of μ and σ^2:

```
X:   3.39 2.19 2.18 1.57 1.30 3.52 2.41 2.00 2.87 3.17
Y:   1.01 1.97 3.51 1.53 1.88 2.34 1.14 1.29
```

```
x <- c(3.39, 2.19, 2.18, 1.57, 1.3, 3.52, 2.41, 2, 2.87, 3.17)
y <- c(1.01, 1.97, 3.51, 1.53, 1.88, 2.34, 1.14, 1.29)
(mu.hat <- (2/28) * sum(x) + (1/28) * sum(y))

## [1] 2.281071

(sigma2.hat <- sum((x - mu.hat)^2)/18 + sum((y - mu.hat)^2)/36)

## [1] 0.4778686
```

[1]

`Total marks = 4`

2. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density

$$f(x; a, b) = \begin{cases} ax^2 + bx + \frac{1}{2} - \frac{a}{3}, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the method of moments estimators for unknown parameters a and b.

- *The first moment is*

$$\begin{aligned} m_1 &= \int_{-1}^{1} x\left(ax^2 + bx + \frac{1}{2} - \frac{a}{3}\right)dx = \left(\frac{ax^4}{4} + \frac{bx^3}{3} + \left(\frac{1}{2} - \frac{a}{3}\right)\frac{x^2}{2}\right)\Bigg|_{-1}^{1} \\ &= \frac{2b}{3} = \bar{X} \quad \Rightarrow \quad \hat{b} = \frac{3\bar{X}}{2}. \end{aligned}$$

[1]

- *The second moment is*

$$\begin{aligned} m_2 &= \int_{-1}^{1} x^2\left(ax^2 + bx + \frac{1}{2} - \frac{a}{3}\right)dx = \left(\frac{ax^5}{5} + \frac{bx^4}{4} + \left(\frac{1}{2} - \frac{a}{3}\right)\frac{x^3}{3}\right)\Bigg|_{-1}^{1} \\ &= \frac{2a}{5} + \frac{2}{3}\left(\frac{1}{2} - \frac{a}{3}\right) = \frac{1}{3} + \frac{8a}{45} = \bar{X^2} \quad \Rightarrow \quad \hat{a} = \frac{45\bar{X^2}}{8} - \frac{15}{8}. \end{aligned}$$

[1]

(b) Are the method of moments estimators of a and b unbiased?

- *They are unbiased:*

$$E(\hat{a}) = \frac{45E(\bar{X}^2)}{8} - \frac{15}{8} = \frac{45m_2}{8} - \frac{15}{8} = a, \quad E(\hat{b}) = \frac{3E(\bar{X})}{2} = \frac{3m_1}{2} = b,$$

because $E(\bar{X}) = m_1 = \frac{2b}{3}$ *and* $E(\bar{X}^2) = m_2 = \frac{1}{3} + \frac{8a}{45}$. [1]

(c) Let $n = 10$. Use the following data to find method of moments estimates of a and b:

-0.77 0.33 -0.61 -0.27 -0.13 0.21 0.09 -0.46 0.85 0.43

```
x <- c(-0.77,0.33,-0.61,-0.27,-0.13,0.21,0.09,-0.46,0.85,0.43)
(a.hat <- 45*mean(x^2)/8 - 15/8)

## [1] -0.5616188

(b.hat <- 3*mean(x)/2)

## [1] -0.0495
```

[1]

Total marks = 4

3. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density:

$$f(x; c) = \begin{cases} c, & -1 < x < 0, \\ \frac{1-c}{3}, & 0 \leq x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the method of moments estimator for c. Is it an unbiased estimator?

- *The first moment is*

$$\begin{aligned} m_1 &= \int_{-1}^{1} x f(x; c) dx = \int_{-1}^{0} c x dx + \frac{1}{3} \int_{0}^{3} (1-c) x dx = c \frac{x^2}{2} \Big|_{-1}^{0} + (1-c) \frac{x^2}{6} \Big|_{0}^{3} \\ &= -0.5c + 1.5(1-c) = 1.5 - 2c = \bar{X} \quad \Rightarrow \quad \hat{c} = 0.75 - 0.5\bar{X}. \end{aligned}$$

[1]

- $E(\hat{c}) = 0.75 - 0.5E(\bar{X}) = 0.75 - 0.5m_1 = c$ *so* $\hat{c}$ *is an unbiased estimator.* [1]

(b) Find the variance of the method of moments estimator as a function of c.

- *The second moment is*

$$\begin{aligned} m_2 &= \int_{-1}^{1} x^2 f(x; c) dx = \int_{-1}^{0} c x^2 dx + \frac{1}{3} \int_{0}^{3} (1-c) x^2 dx \\ &= c \frac{x^3}{3} \Big|_{-1}^{0} + (1-c) \frac{x^3}{9} \Big|_{0}^{3} = c/3 + 3(1-c) = 3 - 8c/3. \end{aligned}$$

- *The variance is* $Var(X) = m_2 - m_1^2 = 3 - 8c/3 - (1.5 - 2c)^2 = 0.75 + 10c/3 - 4c^2$
- $Var(\hat{c}) = 0.25 Var(\bar{X}) = 0.25 Var(X)/n = \frac{1}{n}\left(\frac{3}{16} + \frac{5c}{6} - c^2\right)$. [1]

(c) Let M be the number of variables X_i such that $-1 < X_i < 0$ and $n - M$ be the number of variables X_i such that $0 \le X_i < 3$. Write down the likelihood function for these data and find the maximum likelihood estimator of c.

- *The likelihood function is*

$$L(c) = \prod_{i=1}^{n} f(X_i; c) = \prod_{i:X_i<0} f(X_i; c) \cdot \prod_{i:X_i\ge 0} f(X_i; c) = c^M \cdot (1-c)^{n-M}.$$

[1]

- *The log-likelihood function is*

$$\ell(c) = M \ln c + (n-M)\ln(1-c), \quad \frac{\partial \ell(c)}{\partial c} = \frac{M}{c} - \frac{n-M}{1-c} = 0, \quad \hat{c} = \frac{M}{n},$$

$$\frac{\partial^2 \ell(c)}{\partial c^2} = -\frac{M}{c^2} - \frac{n-M}{(1-c)^2} < 0 \quad \Rightarrow \hat{c} \textit{ gives the maximum of } \ell(c).$$

[1]

(d) Is the maximum likelihood estimator of c unbiased? Find the variance of this estimator as a function of c. *Hint:* M follows a Binomial distribution $\texttt{Binom}(n, p)$ with probability of success $p = \Pr(X < 0)$ where the pdf of X is $f(x; c)$.

- $E(\hat{c}) = E(M)/n = np/n = p$ *where*

$$p = \Pr(X < 0) = \int_{-1}^{0} f(x; c)dx = c\int_{-1}^{0} dx = c,$$

so $\hat{c}$ *is an unbiased estimator.*
- $Var(\hat{c}) = Var(M)/n^2 = np(1-p)/n^2 = c(1-c)/n$. [1]

(e) Does the exact variance of the method of moments and maximum likelihood estimators of c attain its Cramer-Rao lower bound for $0 < c < 1$? Why or why not?

- *Cramer-Rao lower bound is* $v = -\left\{E(\frac{\partial^2 \ell(c)}{\partial c^2})\right\}^{-1} = \left(\frac{E(M)}{c} + \frac{n-E(M)}{1-c}\right)^{-1} = \left(\frac{nc}{c^2} + \frac{n-nc}{(1-c)^2}\right)^{-1} = \frac{c(1-c)}{n}$. [1]
- *The exact variance of the maximum likelihood estimator attains the lower bound. The exact variance of the method of moments estimator does not attain this bound:*

$$\frac{1}{n}\left(\frac{3}{16} + \frac{5c}{6} - c^2\right) = v + \frac{1}{n}\left(\frac{3}{16} - \frac{c}{6}\right) > v \quad \textit{for} \quad 0 < c < 1.$$

[1]

`Total marks = 8`

4. Consider the cumulative distribution function (cdf)

$$F(x;\theta) = \begin{cases} 0, & x \leq 1, \\ 1 - x^{-\theta}, & x > 1. \end{cases}$$

Assume that the prior density of the unknown parameter $\theta > 0$ is

$$f(\theta) = \begin{cases} 0.2e^{-0.2\theta}, & \theta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Let X_1, X_2, X_3 be a random sample from $F(x;\theta)$. The observed data are $x_1 = 1.5, x_2 = 1.2, x_3 = 2.0$. Find the posterior distribution of θ given the observed data. *Hint:* $x^k = e^{k \ln x}$.

- *The pdf of the data is* $f(x;\theta) = F'(x;\theta) = \theta x^{-\theta-1}$ *for* $x > 1$. `[1]`
- *The likelihood function for the observed data is*

$$f(x_1, x_2, x_3|\theta) = \prod_{i=1}^{3} f(x_i;\theta) = \theta \cdot 1.5^{-\theta} \cdot \theta \cdot 1.2^{-\theta} \cdot \theta \cdot 2.0^{-\theta} = \theta^3 \cdot 3.6^{-\theta}.$$

`[1]`

- *The posterior density is (up to a multiplicative constant* C*)*

$$\begin{aligned} f(\theta|x_1, x_2, x_3) &= \frac{f(\theta) \cdot f(x_1, x_2, x_3|\theta)}{\int_\theta f(\theta) \cdot f(x_1, x_2, x_3|\theta)} \\ &= C \cdot e^{-0.2\theta} \cdot \theta^3 e^{-\ln(3.6)\theta} = C \cdot \theta^3 \cdot e^{-(0.2+\ln(3.6))\theta}. \end{aligned}$$

- *The posterior distribution is the Gamma distribution with rate* $\lambda = 0.2 + \ln(3.6) = 1.481$ *and shape* $\alpha = 4$. `[1]`

(b) Find the posterior mean of θ. Find the posterior probability $\Pr(2 < \theta < 5)$.

```
alpha <- 0.2 + log(3.6)
(post.mean <- 4/alpha)

## [1] 2.700998

(prob <- pgamma(5, 4, rate = alpha) - pgamma(2, 4, rate = alpha))

## [1] 0.5928139
```

`[1]`

`Total marks = 4`