MAST90105 Assignment 1 Solutions

1. Five fair coins are tossed (i.e., the probability of tail and head equals 0.5 for each coin) and the number of tails, T is counted. Find the conditional probability that $T \geq 1$ given that at least one coin shows head.

   - *Define two events:*

     $$A_H = \{\texttt{at least one coin shows head}\},$$

     $$A_T = \{\texttt{at least one coin shows tail}\}.$$

     Using the formula of conditional probability, we get:

     $$\Pr\{A_T | A_H\} = \frac{\Pr\{A_T \cap A_H\}}{\Pr\{A_H\}}.$$

   - *The complement of the event* $B = A_T \cap A_H$ *is*

     $$B^c = \{\texttt{all 5 coins show tail or all 5 coins show head}\},$$

     and therefore $\Pr\{B^c\} = 2/2^5 = 1/16$ *so that* $\Pr\{A_T \cap A_H\} = 1 - \Pr\{B\} = 15/16.$

   - *Next, the complement of the event* A_H *is*

     $$A_H^c = \{\texttt{all 5 coins show tail}\},$$

     and therefore $\Pr\{A_H^c\} = 1/2^5 = 1/32$ *so that* $\Pr\{A_H\} = 1 - \Pr\{A_H^c\} = 31/32.$

   - *Finally,* $\Pr\{A_T | A_H\} = \frac{15/16}{31/32} = \frac{30}{31} \approx 0.9677.$

2. *Birthday paradox.*

   (a) Consider a group of 3 students. Each student has a birthday that can be any one of the days numbered 1, 2, 3, ..., 365. What is the probability that none of them have the same birthday with each other?

   - *Let A be the event that none of the three people have the same birthday. Thus A can be expressed as a set of outcomes where the 3 birthday numbers for each outcome are all different from each other.*
   - *Therefore* $P(A) = \frac{365 \times 364 \times 363}{365 \times 365 \times 365} = 0.9917858.$

   (b) Consider a group of 23 students. Each student has a birthday that can be any one of the days numbered 1, 2, 3, ..., 365.

   i. What is the probability that none of them have the same birthday with each other?

   - $P = \frac{365 \times 364 \times \cdots \times 343}{365^{23}} = 0.4927028.$

   ii. What is the probability that some of them have the same birthday with each other? Is this probability greater than 0.5?

- *$P = 1 - 0.4927028 = 0.5072972$.*

3. It is known from experience that in a certain industry 60 percent of all labor-management disputes are over wages, 15 percent are over working conditions, and 25 percent are over fringe issues. Also, 45 percent of the disputes over wages are resolved without strikes, 70 percent of the disputes over working conditions are resolved without strikes, and 40 percent of the disputes over fringe issues are resolved without strikes. What is the probability that a labour-management dispute in this industry will be resolved without a strike?

   - *Let A be the event that a labor-management dispute will be resolved without a strike, B_1 be the event that the dispute is over wages, B_2 be that over working conditions, and B_3 be that over fringe issues. Then*
     $P(B_1) = 0.6, \quad P(B_2) = 0.15, \quad P(B_3) = 0.25$
     $P(A|B_1) = 0.45, \quad P(A|B_2) = 0.7, \quad P(A|B_3) = 0.4.$
     Note that B_1, B_2 and B_3 are mutually exclusive and exhaustive events.
   - *By the law of total probability*
     $P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$
     $= P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$
     $= 0.6 \cdot 0.45 + 0.15 \cdot 0.7 + 0.25 \cdot 0.4 = 0.475.$

4. In a certain community, 8 percent of all adults have Type 2 diabetes. If a health service in this community correctly diagnoses 95 percent of adults with diabetes as having the disease and incorrectly diagnoses 2 percent adults without diabetes as having the disease, find the probabilities that

   (a) the community health service will diagnose an adult as having diabetes;

   - *Let D be the event that an adult has diabetes, C be the event that the community health service will diagnose an adult as having diabetes. Then*
     $P(D) = 0.08, \ \ P(D') = 1 - 0.08 = 0.92, \ \ P(C|D) = 0.95, \ \ P(C|D') = 0.02.$
   - *Using the law of total probability and the definition of conditional probability*
     $P(C) = P(C \cap D) + P(C \cap D') = P(D)P(C|D) + P(D')P(C|D')$
     $= 0.08 \cdot 0.95 + 0.92 \cdot 0.02 = 0.0944.$

   (b) an adult diagnosed by the health service as having diabetes actually has the disease.

   - *Using Bayes's theorem*
     $P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{P(D)P(C|D)}{P(C)} = \frac{0.08 \cdot 0.95}{0.0944} = \frac{0.76}{0.0944} = 0.805.$

5. A bag contains 5 coins, one of which has a head on both sides, another one has a tail on both sides while the other 3 coins are normal (with equal probability of tail and head). A coin is chosen at random from the bag and tossed 2 times. The number of heads obtained is a random variable, say X.

   (a) What are the possible values of X? Also tabulate the pmf of X.

(*Hint*: The coin chosen is either normal (a) or with head (b) or tail (c) on both sides. Find the conditional probability of X in each of these three cases (a), (b) and (c). Then use the law of total probability and multiplication rule to find the pmf of X.)

- *Define*

$$\begin{aligned} A &= \{\textit{the selected coin is a normal one}\} \\ B &= \{\textit{the selected coin has head on both sides}\} \\ C &= \{\textit{the selected coin has tail on both sides}\} \end{aligned}$$

  Then $P(A) = 3/5$ *and* $P(B) = P(C) = 1/5$.

- *The possible values for* X *are* $\{0, 1, 2\}$. *We find:*

$$\begin{aligned} P(X=0) &= P(X=0|A)P(A) + P(X=0|B)P(B) + P(X=0|C)P(C) \\ &= \left(\frac{1}{2}\right)^2 \cdot \frac{3}{5} + 0 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5} = \frac{7}{20} \\ P(X=1) &= P(X=1|A)P(A) + P(X=1|B)P(B) + P(X=1|C)P(C) \\ &= 2 \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{3}{5} + 0 \cdot \frac{1}{5} + 0 \cdot \frac{1}{5} = \frac{6}{20} \\ P(X=2) &= P(X=2|A)P(A) + P(X=2|B)P(B) + P(X=2|C)P(C) \\ &= \left(\frac{1}{2}\right)^2 \cdot \frac{3}{5} + 1 \cdot \frac{1}{5} + 0 \cdot \frac{1}{5} = \frac{7}{20} \end{aligned}$$

  Therefore the pmf of X *is*

x	*0*	*1*	*2*
$f(x)$	*7/20*	*6/20*	*7/20*

(b) Calculate $E(X)$ and $\mathrm{Var}(X)$.

- $E(X) = 0 \times \frac{7}{20} + 1 \times \frac{6}{20} + 2 \times \frac{7}{20} = \frac{20}{20} = 1.$
- $E(X^2) = 0^2 \times \frac{7}{20} + 1^2 \times \frac{6}{20} + 2^2 \times \frac{7}{20} = 1.7.$
- $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 1.7 - 1^2 = 0.7.$

6. Among the 16 applicants for a job, 10 have university degrees. A sample of 3 applicants are to be randomly chosen for interviews. Let X be the number of applicants in the sample who have university degrees.

(a) Give the name to the distribution of X if it has a name. Also specify the values of all parameters involved in this distribution.

- *X has hypergeometric distribution with parameters* $N_1 = 10$, $N_2 = 6$ *and* $n = 3$.

(b) Find the probability that exactly 1 applicant in the sample has a university degree.

- $$P(X=1) = \frac{\binom{10}{1}\binom{6}{2}}{\binom{16}{3}} = \frac{15}{56} = 0.268.$$

(c) Find the probability that at most 1 applicant in the sample has a university degree.

- $$P(X \le 1) = P(X=0) + P(X=1) = \frac{\binom{10}{0}\binom{6}{3}}{\binom{16}{3}} + \frac{\binom{10}{1}\binom{6}{2}}{\binom{16}{3}}$$
$$= \frac{15}{56} + \frac{1}{28} = \frac{17}{56} = 0.304.$$

7. A moment-generating function of X is given by $M(t) = e^t/(2-e^t)$.

(a) Find the values of the mean, μ, and variance, σ^2, for X.

- *X is a Geometric random variable with parameter $p = 0.5$. We find:*

$$\mu = 1/p = 2, \quad \sigma^2 = (1-p)/p^2 = 2.$$

(b) Calculate $\Pr\{X \ge 4\}$.

- *For the Geometric random variable X with $p = 1/2$,*

$$\Pr\{X = k\} = (1-p)^{k-1}p = 1/2^k, \quad k = 1, 2, \ldots.$$

We find:

$$P\{X \ge 4\} = 1 - \Pr(X \le 3) = 1 - \sum_{k=1}^{3} \frac{1}{2^k} = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{1}{8} = 0.125.$$