# MAST90105: Lab and Workshop Problems for Week 6

The Lab and Workshop this week cover problems arising from Module 3, Section 4 (Normal Distribution and Central Limit Theorem), Module 4 and Module 5, Section 1.

## 1 Lab

The objective of this week is to use Mathematica to do actual calculations on moment generating functions and probability density functions for sums of independent random variables.

1. Suppose $X_1 \stackrel{d}{=} N(\mu = 3, \sigma^2 = 4)$, $X_2 \stackrel{d}{=} N(3, 4)$, and $X_1$ and $X_2$ are independent.

   a. Use Mathematica to find the mgf of $X_1$ and $X_2$.

   b. Let $Y = 5X_1 - 2X_2 + 6$. Find the mgf of $Y$ using Mathematica.

   c. Name the distribution of $Y$ and give the values of the associated parameters.

2. Let $\bar{X}$ be the mean of a random sample of size 12 from the uniform distribution on the interval (0,1) (which has the mean 1/2 and variance 1/12). Use Mathematica to do the following:

   a. Find the pdf of $\bar{X}$. The command to get the pdf of the sum of 12 independent uniform random variables on (0,1) is `PDF[UniformSumDistribution[12], x]` and plot it - see Lab6.nb for some help on this - why does the help work?

   b. Find the probability $P(1/2 \leq \bar{X} \leq 2/3)$ based on the exact distribution of $\bar{X}$. Translate the required probability into one about the sum of the 12 independent uniform random variables

   c. By the CLT $\bar{X}$ approximately has a normal distribution with mean 1/2 and variance 1/144. Now approximate the probability $P(1/2 \leq \bar{X} \leq 2/3)$ based on this normal distribution. Compare the result with that in (b).

3. Let $Y = A + B + C + D$ be the sum of a random sample of size 4 from the distribution whose pdf is that of $U^{\frac{1}{3}}$ where $U \sim U(0, 1)$.

   a. The function $t1$ in the notebook Lab6.nb sets up the pdf of $U^{\frac{1}{3}}$. Plot it and find the mean and variance of $A$.

   b. Find the pdf of $Y$, and plot the pdf of $Y$.

   c. Calculate $P(0.3 \leq Y \leq 1.5)$ based on the pdf of $Y$.

   d. Approximate $P(0.3 \leq Y \leq 1.5)$ based on the normal distribution, and see how close it is to the result of (c).

4. Alter the code from Lab 4 Question 3 to investigate in Mathematica the relationship between the Gamma distribution with large values of $\alpha$ and the normal distribution. Take the scale parameter to be 1. How will you adapt the distance D?

# 2 Workshop

You can use Mathematica or R for numerical calculations as needed.

5. Suppose $X_1 \stackrel{d}{=} N(\mu = 3, \sigma^2 = 4)$, $X_2 \stackrel{d}{=} N(3, 4)$, and $X_1$ and $X_2$ are independent.

   a. Write down the mgf of $X_1$ and $X_2$.

   b. Let $Y = 5X_1 - 2X_2 + 6$. Find the mgf of $Y$.

   c. Name the distribution of $Y$ and give the values of the associated parameters.

6. The serum zinc level $X$ in micrograms per deciliter for males between ages 15 and 17 has a distribution which is approximately normal with $\mu = 90$ and $\sigma = 15$. Compute the conditional probability $P(X > 120 | X > 105)$.

7. Let $Z_1, Z_2, \cdots, Z_7$ be a random sample from the standard normal distribution $N(0, 1)$. Let $W = Z_1^2 + Z_2^2 + \cdots + Z_7^2$. Name the distribution of $W$ with associated parameter value. Justify your answer.

8. If $X_1, X_2, \cdots, X_{16}$ is a random sample of size $n = 16$ from the normal distribution $N(50, 100)$.

   a. What is the distribution of $\frac{1}{100}\sum_{i=1}^{16}(X_i - 50)^2$?

   b. What is the distribution of $\bar{X}$?

9. Let $\bar{X}$ be the mean of a random sample of size 12 from the uniform distribution on the interval (0,1) (which has the mean 1/2 and variance 1/12). Approximate the probability $P(1/2 \le \bar{X} \le 2/3)$ using the central limit theorem.

10. Let the distribution of $Y$ be $Binomial(25, 1/2)$ . Find the probability $P(10 \le Y \le 12)$ in two ways: using the binomial pmf formula, and using the normal approximation. Comment on any difference.

11. The number $X$ of flaws on a certain tape of length one yard follows a Poisson distribution with mean 0.3. We examine $n = 100$ such tapes and count the total number $Y$ of flaws.

    a. Assuming independence, what is the distribution of $Y$?

    b. Find the exact and approximate probabilities for $P(Y \le 25)$.

12. Let $Y = X_1 + X_2 + \cdots + X_{15}$ be the sum of a random sample of size 15 from the distribution whose pdf is $f(x) = (3/2)x^2$, $-1 < x < 1$. Approximate $P(-0.3 \le Y \le 1.5)$ using the central limit theorem.

13. Let the joint pmf of $X$ and $Y$ be defined by

$$f(x, y) = \frac{x + y}{32}, \quad x = 1, 2,\ y = 1, 2, 3, 4.$$

a. Find the marginal pmf of $X$.

b. Find the marginal pmf of $Y$.

c. Calculate $P(X > Y)$.

d. Calculate $P(Y = 2X)$.

e. Calculate $P(X + Y = 3)$.

f. Calculate $P(X \leq 3 - Y)$.

g. Are $X$ and $Y$ independent?

h. Find $E(X)$.

i. Find $E(X + Y)$.

14. Two construction companies make bids of $X$ and $Y$ (in \$100,000's) on a remodeling project. The joint pmf of $X$ and $Y$ is uniform on the space $2 < x < 2.5$, $2 < y < 2.3$. If $X$ and $Y$ are within 0.1 of each other, the companies will be asked to rebid; otherwise the lower bidder will be awarded the contract. What is the probability that they will be asked to rebid?

15. Let $f(x, y) = 2e^{-x-y}$, $0 \leq x \leq y < \infty$, be the joint pdf of $X$ and $Y$.

a. Find the marginal pdf $f_X(x)$ of $X$.

b. Find the marginal pdf $f_Y(y)$ of $Y$.

c. Compute $E(X)$ and $E(e^{-X-2Y})$.

d. Compute $P(X > \frac{1}{2})$.

e. Compute $P(X > \frac{1}{2}, Y > 2)$.

f. Compute $P(Y > 2 | X > \frac{1}{2})$.

16. Let the joint pmf of $X$ and $Y$ be

$$f(x, y) = \frac{1}{4}, \quad (x, y) \in S = \{(0, 0), (1, 1), (1, -1), (2.0)\}.$$

a. Represent the joint pmf by a table.

b. Are $X$ and $Y$ independent?

c. Calculate $\text{Cov}(X, Y)$ and $\rho$.

17. Consider continuous random variables $X$ and $Y$ which have the following joint pdf

$$f(x, y) = 24xy, \quad x > 0,\ y > 0,\ x + y < 1.$$

a. Sketch a graph of the support of $X$ and $Y$.

b. Find the probability $P(Y > 2X)$.

c. Find the marginal pdf $f_1(x)$ of $X$.

d. Find the mean $E(X)$.

e. Find the variance $\text{Var}(X)$.

f. Find the covariance $\text{Cov}(X, Y)$.

g. Find the correlation coefficient $\rho$ between $X$ and $Y$.

h. Find the conditional pdf $h(y|x)$ of $Y$ given $X = x$.

i. Find the condition probability $P(Y \leq \frac{1}{3}(1 - X)|X = x)$.

j. Find the conditional mean $E(Y|X = x)$.

18. Show that $\text{Cov}(aX + b, cX + d) = ac\text{Var}(X)$, where $X$ is a random variable and $a, b, c$ are deterministic constants.

19. Let the pmf of $X$ be $f_1(x) = \frac{1}{10}$, $x = 0, 1, 2, \cdots, 9$, and the conditional pmf of $Y$ given $X = x$ be $h(y|x) = \frac{1}{10-x}$, $y = x, x + 1, \cdots, 9$. Find

    a. the joint pmf $f(x, y)$ of $X$ and $Y$.

    b. The marginal pmf $f_2(y)$ of $Y$.

    c. $E(Y|x)$.

20. The marginal distribution of $X$ is $U(0, 1)$. The conditional distribution of $Y$, given $X = x$, is $U(0, e^x)$.

    a. Determine $h(y|x)$, the conditional pdf of $Y$, given $X = x$.

    b. Find $E(Y|x)$.

    c. Find the joint pdf of $X$ and $Y$. Sketch the region where $f(x, y) > 0$.

    d. Find $f_2(y)$, the marginal pdf of $Y$.

    e. Find $g(x|y)$, the conditional pdf of $X$, given $Y = y$.

21. An obstetrician does ultrasound examinations on her patients between their 16th and 25th weeks of pregnancy to check on the growth of the unborn child. Let $X$ equal the widest diameter of the head and $Y$ be the length of the femur, both in mm. Assume that $X$ and $Y$ have a bivariate normal distribution with $\mu_X = 60.6$, $\sigma_X = 11.2$, $\mu_Y = 46.8$, $\sigma_Y = 8.4$, $\rho = 0.94$

    a. Find $P(40.5 < Y < 48.9)$

    b. Find $P(40.5 < Y < 48.9|X = 68.6)$

22. Karl Pearson carried out a famous study on the resemblances between fathers and sons. He found that the distribution from his sample of 1078 pairs of fathers and sons had a mean and standard deviation of height for fathers of 69 and 2 (in.), whereas for sons it was 70 and 2 (in.). The correlation between fathers' and sons' heights was 0.5. Assuming that the distribution can of the pairs of heights can be modelled as bivariate normal with the sample means, sds and correlation, what is

a. the expected height for the son of a father who is 74 in. tall,

b. the chance that the son's height is more than 1 in. from the expected height in (a)?

c. the expected height of a father whose son is 72.5 in?

d. the chance that both a father and son are above average height?

23. Scores in the two exams in a double weight course have means 65 and 60, standard deviations 18 and 20 and a correlation of 0.75. Assuming they can be modelled as bivariate normal, what is the expected score in the second exam for a student how is above average on the first exam?

24. Suppose $(X, Y)$ has a standard bivariate normal density with correlation $\rho$. For $a, b$, find $E(Y|a < X < b)$.

25. Suppose $(X, Y)$ are random variables and $a, b, c, d$ are constants. Show

$$Cov(aX + bY, cX + dY) = acVar(X) + bdVar(Y) + (bc + da)Cov(X, Y).$$

26. Suppose that $X \sim N(0, 1)$ and that $Z$ is an independent random variable which takes the values $1, -1$ each with probability $\frac{1}{2}$.

a. Show that $X, -X$ have the same distribution.

b. What is the distribution of $ZX$?

c. Find $Cov(X, ZX), Corr(ZX, X)$.

d. Are $X$ and $ZX$ independent? (Hint: consider $P(ZX \leq -1|X \leq -1)$.)

e. Is $(X, Y)$ bivariate normal?

27. Suppose $X$ has a uniform pdf on [0,1]. Find the pdf of $-ln(1 - U)$.

28. Suppose $X$ has the Laplace pdf, $f$, given by $f(x) = \frac{1}{2}e^{-|x|}$ for all real x.

a. Find the cdf of $X$(hint: consider negative $x$ first and compare with the exponential density).

b. Find the cdf of $X^2$.

c. Find the pdf of $X^2$.

d. Find the cdf of $X^4$.

e. Find the pdf of $X^4$.