MAST90105 Workshop Solutions

1. Box A contains one red and one black marbles. Box B contains one green and one white marbles. Consider the experiment of drawing a marble at random from Box A, transferring it to Box B and then drawing a marble from Box B at random.

   (a) Write down the associated sample space.

   - $S = \{RR, RG, RW, BB, BG, BW\}$.

   (b) List the sample point(s) that comprise the event that a red ball is selected from Box B.

   - $\{RR\}$.

   (c) List the sample point(s) that comprise the event that a white ball is selected from Box B.

   - $\{RW, BW\}$.

   (d) List the sample points that comprise the event that a green ball is not selected from Box B.

   - $\{RR, RW, BB, BW\}$.

2. Let $A_n = [0, 1 + \frac{1}{n})$, $n = 1, 2, \cdots$, be a sequence of events (i.e. intervals). Find $A_1 \cap A_2$, $A_1 \cap A_2 \cap A_3$ and $A_1 \cap A_2 \cap \cdots \cap A_{100}$. Then guess the result of $\cap_{n=1}^{\infty} A_n$.

   - $A_1 \cap A_2 = A_2 = [0, 1.5)$,
   - $A_1 \cap A_2 \cap A_3 = A_3 = [0, 1\frac{1}{3})$,
   - $A_1 \cap A_2 \cap \cdots \cap A_{100} = A_{100} = [0, 1.01)$,
   - $\cap_{n=1}^{\infty} A_n = [0, 1]$.

3. A six-sided die is to be rolled. However, it is known that the die is loaded so that a 1 and 6 are equally likely and are three times as likely to occur as any of the other sides. What are the probabilities for each of the six sides to be observed?

   - *Let $p_1, p_2, \cdots, p_6$ be the probabilities for the six sides of the dice.*
   - *Then $p_1 = p_6 = 3p_2 = 3p_3 = 3p_4 = 3p_5$.*
   - *Since $\sum_{i=1}^{6} p_i = 1$, we have $p_1 = p_6 = 0.3$ and $p_2 = p_3 = p_4 = p_5 = 0.1$.*

4. Each day a machine produces 100 items of certain product. Assume that 10 of these item are defective on a particular day. What is the probability that a random sample of 5 items to be selected from the outputs will contain 3 defectives?

   - *Let A be the event that the random sample will contain 3 defectives.*
   - *Then $P(A) = \frac{\binom{10}{3}\binom{100-10}{5-3}}{\binom{100}{5}} = \frac{120 \times 4005}{75287520} \approx 0.00638$.*

5. If we had a choice of two airlines, we would possibly choose the airline with the better "on time performance". So consider Alaska and America West using data reported by Arnold Barnett, "How Numbers Can Trick You," in *Technology Review*, 1994.

Airline Destination	Alaska Airlines Relative Frequency On Time	America West Relative Frequency On Time
Los Angeles	$\frac{497}{559} = 0.889$	$\frac{694}{811} = 0.856$
Phonex	$\frac{221}{233} = 0.948$	$\frac{4840}{5255} = 0.921$
San Diego	$\frac{212}{232} = 0.914$	$\frac{383}{448} = 0.855$
San Francisco	$\frac{503}{605} = 0.831$	$\frac{320}{449} = 0.713$
Seattle	$\frac{1841}{2146} = 0.858$	$\frac{201}{262} = 0.767$
Five-city Total	$\frac{3274}{3775} = 0.867$	$\frac{6438}{7225} = 0.891$

(a) For each of the five cities listed, which airline has the better on time performance?

- *Alaska Airlines has the better performance for each city.*

(b) Combining the results, which airline has the better on time performance?

- *America West has the better performance when the factor "Destination" is dropped.*

(c) Interpret your results.

- *This is an example of Simpson's Paradox, where the overall result hides the truth that Alaska Airlines is doing better in each city. The reason that this occurs is that Alaska Airlines flies more flights into San Francisco and Seattle where both airlines have worse on-time performance. Thinking about these ratios as probabilities in an experiment of picking a random flight, it is important to condition on the Destination event, to avoid misleading the public. It could well be that the worse performance in San Fransisco and Seattle is for reasons outside the control of either airline.*

6. (a) Calculate $\sum_{r=0}^{n}\binom{n}{r}$ for $n = 0, 1$ and 2. Then guess the result of $\sum_{r=0}^{n}\binom{n}{r}$ for general n and prove it.

- *From the binomial expansion* $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}$ *it follows that*
  $\sum_{r=0}^{n}\binom{n}{r} = (1+1)^n = 2^n$
  by taking $a = b = 1$.

(b) Calculate $\sum_{r=0}^{n}(-1)^r\binom{n}{r}$ for $n = 1, 2$ and 3. Then guess the result of $\sum_{r=0}^{n}(-1)^r\binom{n}{r}$ for general n and prove it.

- *From the binomial expansion* $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}$ *it follows that*
  $0 = (-1+1)^n = \sum_{r=0}^{n}(-1)^r\binom{n}{r}$
  by taking $a = -1$ *and* $b = 1$.