MAST90105 Lab and Workshop 5 Solutions

1. Suppose that 10^6 points are selected independently and at random from the unit square $\{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. Let W equal the number of points that fall in $A = \{(x, y) : x^2 + y^2 < 1\}$.

    a. How is W distributed? Namely, what is the name of the distribution of W if it has a name? And what is the pmf of W?

       - $W \overset{d}{=} b(10^6, \pi/4)$.
       - *The pmf of W is $f(w) = \binom{10^6}{w}(\pi/4)^w(1 - \pi/4)^{10^6-w}$, $w = 0, 1, \cdots, 10^6$.*

    b. Give the mean, variance and standard deviation of W.

       - $\mu = E(W) = 250000\pi$ *and* $\sigma^2 = \text{Var}(X) = 250000\pi(1 - \pi/4)$.

    c. What is the expected value of $W/250000$?

       - $E(W/250000) = E(W)/250000 = \pi$.

2. **The objective of this question is to use R to estimate the value of π using the idea and result of the previous question.** (*Note that everybody knows the meaning of π, but nobody can write down the exact and explicit decimal representation of π. So estimating the value of π is meaningful even though the method used here is clearly not the best.*)

   Note that selecting at random a million values of x from [0,1] can be done using `runif(10^6)` command in R.

    a. Using this information and the knowledge that you have learned about R so far, try to write up a few lines of R commands to generate an observation of W. Remember that your R commands must not involve the use of the true value of π.

       (Note that a command like `mean(rbinom(1000,10^6,pi/4))` can be used to estimate π. But this is not the one we want because it involves the use of the true value of π. Use Help inside R)

```
set.seed(1)
options(digits=6)
 # number of the points
n <- 10^6
# generate  x
x <- runif(n, min = 0, max = 1)
# generate  y
y <- runif(n, min = 0, max = 1)
# the number of points inside the unit circle
(W <-sum(x^2+y^2<1))
```

```
## [1] 785914
```

b. Once you get an observation of W, calculate $W/250000$ and see how the result is close to π.

```
( est <- W/250000)

## [1] 3.14366

 error <- abs(est - pi)
 print(paste0('Approximation error is: ', round(error,6)))

## [1] "Approximation error is: 0.002063"
```

c. Think about how you can improve the precision of your estimate of π.

One way of doing this would be to implement the R commands developed by you into an R function. Then use this function to generate a number of W observations. Then use the average of the generated W values divided by 250000 to estimate π.

```
rW <- function(size, n = 10^6){          #function to generate W
for(i in 1:size){
    x <- runif(n, min = 0, max = 1)      # generate  x
    y <- runif(n, min = 0, max = 1)      # generate  y
   W[[i]] <- sum(x^2+y^2<1)   # the number of points lie in the unit circle
}
return(W)
}

 W <- rW(50)
 (est <- mean(W)/250000)

## [1] 3.14164

 error <- abs(est-pi)
 print(paste0('Refined approximation error is: ', round(error,6)))

## [1] "Refined approximation error is: 4.3e-05"
```

3. **The objective of this question is to use R to produce plots of Gamma densities, introducing to the Graphics package, ggplot2, which is a standard for professional graphics.**

The idea of ggplot2 is to start with a base plot and add the other elements of your plot incrementally. A quick guide is in this link. The code is available in Lab5.R in the LMS Labs folder.

Here is the code used to produce one of the plots in Lectures:

```
p <- ggplot(data = data.frame(x= c(0,30)),aes(x=x)) + \newline
stat\_function(fun=function(x)dgamma(shape=0.25,x,scale=4),
aes(colour = "1"))
stat\_function(fun = function(x)dgamma(shape=1,x,scale=4),
aes(colour = "2"))
stat\_function(fun = function(x)dgamma(shape=2,x,scale=4),
aes(colour = "3"))
stat\_function(fun = function(x)dgamma(shape=3,x,scale=4),
aes(colour = "4"))
newcols <- c("1"="red","2"="blue","3"="darkgreen","4"="purple")
p +scale\_colour\_manual(values = newcols,name = "",
labels = c(expression("  " * alpha==0.25*"   "),
expression("  "*alpha==1*"  " ),
expression("  "*alpha==2*"  " ),
expression("  "*alpha==3*"  " )))
theme(legend.position="top",
text=element\_text(size=22),
panel.background =element\_rect(fill="white"),
axis.line = element\_line(colour = "black") )
ylim(0,0.25)
ylab("f(x)")
annotate("text",x=15,y=0.15, label="theta==4",parse=TRUE, size=8)
ggtitle("Gamma Probability Density Functions - Varying Shape")
```

a. Find out about the definition of data.frame and dgamma. How are these used in the code?

   From the documentation:

   - *The function data.frame() creates data frames, tightly coupled collections of variables which share many of the properties of matrices and of lists, used as the fundamental data structure by most of R's modeling software.*
   - *dgamma gives the density for the Gamma distribution with parameters shape and scale: dgamma(x, shape, rate = 1, scale = 1/rate, log = FALSE)*

b. Alter the code so it produces Gamma densities with shape parameters 0.1, 1.1, 5 and 10 and scale = 5. You will have to alter a number of parts of the code. Comment on the results in comparison to those in lectures.
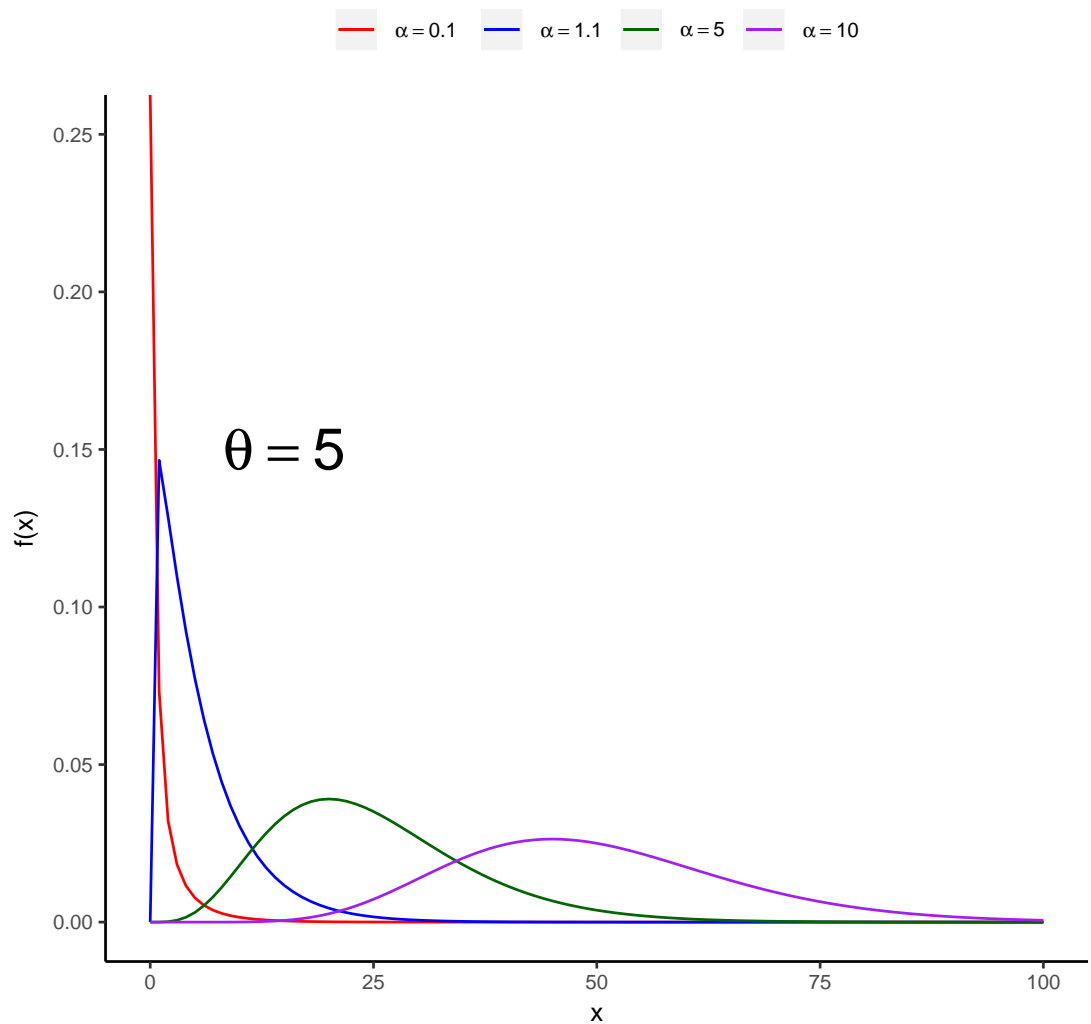
```
library("ggplot2")

p <- ggplot(data = data.frame(x= c(0,100)),aes(x=x)) +
  stat_function(fun=function(x)dgamma(shape=0.1,x,scale=5),
                aes(colour = "1")) +
  stat_function(fun = function(x)dgamma(shape=1.1,x,scale=5),
                aes(colour = "2")) +
```

```r
    stat_function(fun = function(x)dgamma(shape=5,x,scale=5),
                aes(colour = "3"))+
    stat_function(fun = function(x)dgamma(shape=10,x,scale=5),
                aes(colour = "4"))
newcols <- c("1"="red","2"="blue","3"="darkgreen","4"="purple")
p + scale_colour_manual(values = newcols,name = "",
            labels = c(expression("  " * alpha==0.1*"    "),
                    expression("  "*alpha==1.1*"  " ),
                    expression("  "*alpha==5*"  " ),
                    expression("  "*alpha==10*"  " ))) +
    theme(legend.position="top",
        text=element_text(size=10),  panel.background =

            element_rect(fill="white"),
        axis.line = element_line(colour = "black") ) +
    ylim(0,0.25)+
    ylab("f(x)") +
    annotate("text",x=15,y=0.15,
            label="theta==5",parse=TRUE, size=8)+
    ggtitle("Gamma Probability Density Functions - Varying Shape")
```

Gamma Probability Density Functions – Varying Shape

The curve becomes more bell shaped as the shape paramter increases.

c. Alter the code so it produces Gamma densities with scale parameters 0.1, 1.1, 5 and 10 and shape = 5. You will have to alter a number of parts of the code. Comment on the results in comparison to those in lectures.

```
library("ggplot2")

p <- ggplot(data = data.frame(x= c(0,100)),aes(x=x)) +
  stat_function(fun=function(x)dgamma(shape=5,x,scale=0.1),
                aes(colour = "1")) +
  stat_function(fun = function(x)dgamma(shape=5,x,scale=1.1),
                aes(colour = "2")) +
  stat_function(fun = function(x)dgamma(shape=5,x,scale=5),
                aes(colour = "3"))+
  stat_function(fun = function(x)dgamma(shape=5,x,scale=10),
                aes(colour = "4"))
```

```r
newcols <- c("1"="red","2"="blue","3"="darkgreen","4"="purple")
p + scale_colour_manual(values = newcols,name = "",
               labels = c(expression("  " *theta==0.1*"   "),
                          expression("  "*theta==1.1*"  " ),
                          expression("  "*theta==5*"  " ),
                          expression("  "*theta==10*"  " ))) +
  theme(legend.position="top",
        text=element_text(size=10),  panel.background =

         element_rect(fill="white"),
        axis.line = element_line(colour = "black") ) +
  ylim(0,0.25)+
  ylab("f(x)") +
  annotate("text",x=15,y=0.15,
           label="alpha==4",parse=TRUE, size=8)+
  ggtitle("Gamma Probability Density Functions - Varying Scale")
```

Gamma Probability Density Functions – Varying Scale
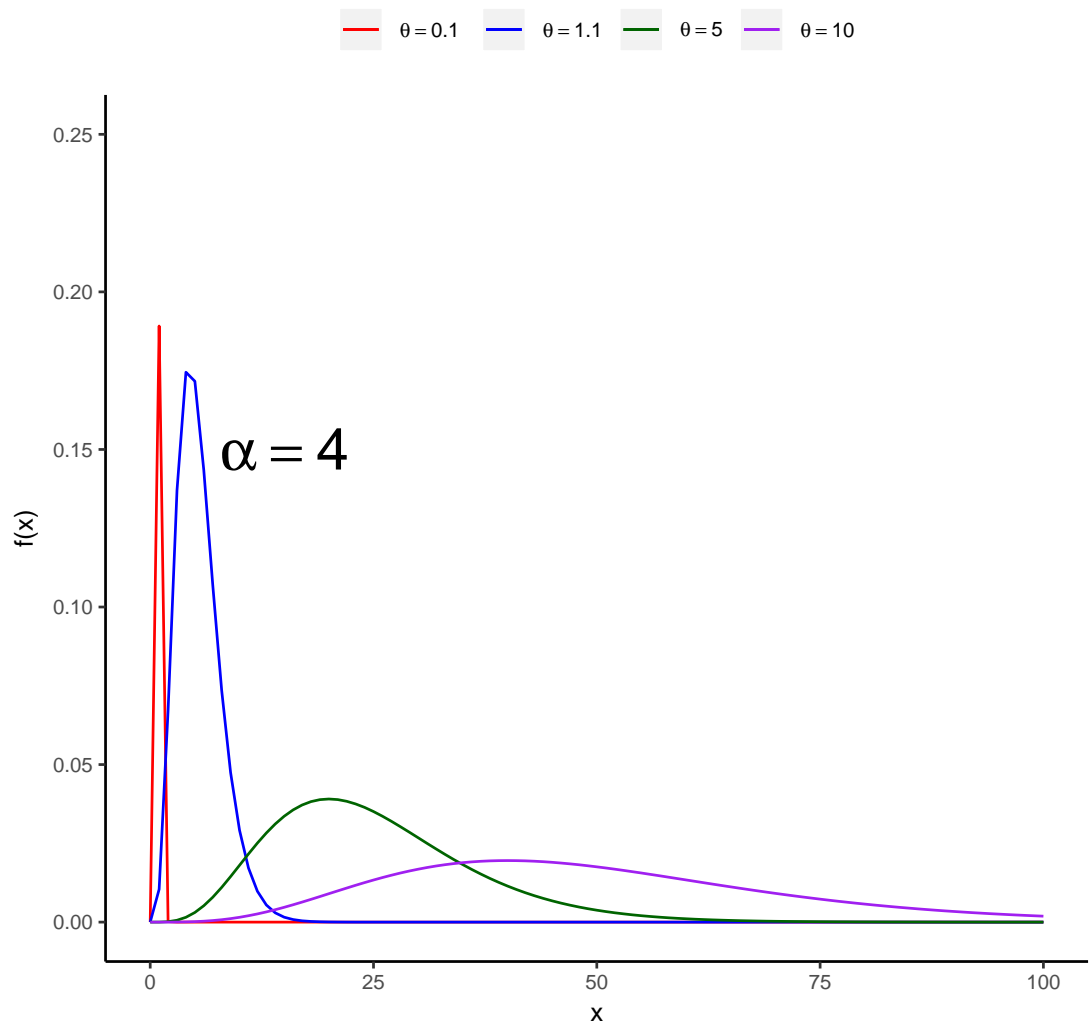
Unlike to shape parameter, the scale parameter of the gamma function will not change the shape of the curve,it only scales the curve horizontally and vertically.

d. Find out how the legends are plotted and write down which parts of the code are involved in the production of the legends.

```
scale_colour_manual(values = newcols,name = "",labels = c(expression("
" * alpha==0.25*" "),
expression(" "*alpha==1*" " ),
expression(" "*alpha==2*" " ),
expression(" "*alpha==3*" " ))) +
theme(legend.position="top",
text=element_text(size=22),
panel.background =element_rect(fill="white"),
axis.line = element_line(colour = "black") )
```

4. Let the random variable X have the pdf $f(x) = 2(1 - x)$, $0 \le x \le 1$, 0 elsewhere.

a. Sketch the graph of this pdf.
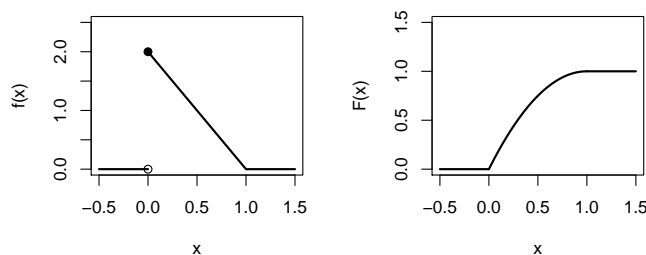
b. Determine and sketch the graph of the distribution function of X.

- $F(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ 1 - (1-x)^2, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$

c. Find

   i. $P(0 \leq X \leq 1/2)$,
- $P(0 \leq X \leq 1/2) = \int_0^{1/2} 2(1-x)dx = \frac{3}{4}$.

   ii. $P(1/4 \leq X \leq 3/4)$,
- $P(1/4 \leq X \leq 3/4) = \int_{1/4}^{3/4} 2(1-x)dx = \frac{1}{2}$.

   iii. $P(1/4 \leq X \leq 5/4)$,
- $P(1/4 \leq X \leq 5/4) = \int_{1/4}^{1} 2(1-x)dx = \frac{9}{16}$.

   iv. $P(X = 3/4)$,
- $P(X = 3/4) = 0$.

   v. $P(X \geq 3/4)$,
- $P(X \geq 3/4) = \int_{3/4}^{1} 2(1-x)dx = \frac{1}{16}$.

   vi. the value of μ,
- $\mu = \int_0^1 x \cdot 2(1-x)dx = (x^2 - \frac{2}{3}x^3)|_0^1 = \frac{1}{3}$.

   vii. the value of σ^2, and
- $E(X^2) = \int_0^1 x^2 \cdot 2(1-x)dx = (\frac{2}{3}x^3 - \frac{2}{4}x^4)|_0^1 = \frac{1}{6}$.
- So $\sigma^2 = E(X^2) - [E(X)]^2 = \frac{1}{6} - (\frac{1}{3})^2 = \frac{1}{18}$.
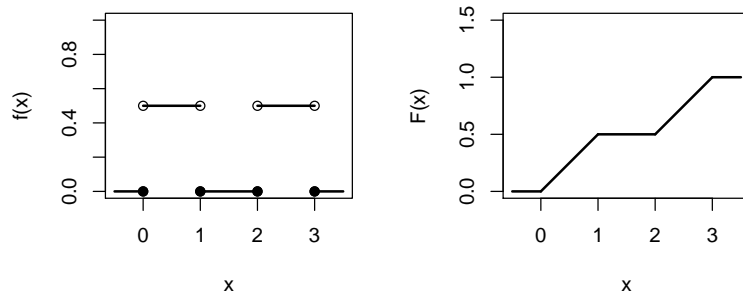
   viii. the 36th percentile $\pi_{0.36}$ of X.
- $F(\pi_{0.36}) = 0.36$. *Hence* $1 - (1 - \pi_{0.36})^2 = 0.36$ *and* $\pi_{0.36} = 0.2$.

5. The pdf of X is $f(x) = c/x^2$, $1 < x < \infty$.

a. Find the value of c so that $f(x)$ is a pdf.
- $\int_1^{\infty} \frac{c}{x^2}dx = c$.

- *Also $\int_1^\infty \frac{c}{x^2}dx = 1$ because $f(x) = \frac{c}{x^2}$, $1 < x < \infty$, is a pdf. Thus $c = 1$.*

b. Show that $E(X)$ is not finite.

- $E(X) = \int_1^\infty xf(x)dx = \int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}dx = \lim_{b\to\infty} \ln b = \infty$.

6. Let $f(x) = 1/2$, $0 < x < 1$ or $2 < x < 3$, 0 elsewhere, be the pdf of X.

a. Sketch the graph of this pdf.

b. Define cdf of X and sketch its graph.

- $F(x) = \int_{-\infty}^x f(t)dt = \begin{cases} 0, & x \leq 0 \\ \frac{1}{2}x, & 0 < x < 1 \\ \frac{1}{2}, & 1 \leq x \leq 2 \\ \frac{1}{2}(x-1), & 2 < x < 3 \\ 1, & x \geq 3 \end{cases}$

c. Find $q_1 = \pi_{0.25}$.

- $F(q_1) = 0.25$, *thus $q_1 = 0.5$.*

d. Find $m = \pi_{0.50}$. Is it unique?

- $m =$ *any value in [1,2] because for any value $a \in [1,2]$, $F(a) = 0.5$.*

e. Find the value of $E(X)$.

- $E(X) = \int_{-\infty}^\infty xf(x)dx = \int_0^1 x \cdot \frac{1}{2}dx + \int_2^3 x \cdot \frac{1}{2}dx = \frac{1}{4} + \frac{5}{4} = \frac{3}{2}$.

7. Let $F(x) = 1 - (\frac{1}{2}x^2 + x + 1)e^{-x}$, $0 < x < \infty$ be the cdf of X.

a. Find the mgf $M(t)$ of X.

- *First find the pdf of X: $f(x) = F'(x) = \frac{1}{2}x^2e^{-x}$, $0 < x < \infty$ and 0 elsewhere.*
- *Then $M(t) = E(e^{tX}) = \int_0^\infty e^{tx} \cdot \frac{1}{2}x^2e^{-x}dx = \int_0^\infty \frac{1}{2}x^2e^{(t-1)x}dx$*
  $= \frac{1}{(1-t)^3} \frac{1}{2} \int_0^\infty y^2e^{-y}dy$ *if we denote $y = (1-t)x$ and assume $t < 1$.*

- *By integration by parts it can be found that $\int_0^\infty y^2 e^{-y} dy = 2! = 2$.*
  Therefore $M(t) = \frac{1}{(1-t)^3}$, $t < 1$.

b. Find the values of μ and σ^2.

- $E(X) = \int_{-\infty}^\infty x f(x) dx = \int_0^\infty \frac{1}{2} x^3 e^{-x} dx = (-\frac{1}{2} x^3 e^{-x})|_0^\infty + \int_0^\infty \frac{3}{2} x^2 e^{-x} dx$

  $= (-\frac{3}{2} x^2 e^{-x})|_0^\infty + \int_0^\infty \frac{6}{2} x e^{-x} dx = (-\frac{6}{2} x e^{-x})|_0^\infty + \int_0^\infty \frac{6}{2} e^{-x} dx = (-3 e^{-x})|_0^\infty = 3$.
- $E(X^2) = \int_{-\infty}^\infty x^2 f(x) dx = \int_0^\infty \frac{1}{2} x^4 e^{-x} dx = 12$ *by integration by parts.*
- *So $\sigma^2 = 12 - 3^2 = 3$.*

8. The life X in years of a voltage regulator of a car has the pdf

$$f(x) = \frac{3x^2}{7^3} e^{-(x/7)^3}, \quad 0 < x < \infty.$$

a. What is the probability that this regulator will last at least 7 years?

- $P(X \geq 7) = \int_7^\infty \frac{3x^2}{7^3} e^{-(x/7)^3} dx = -e^{-(x/7)^3}|_7^\infty = e^{-1}$.

b. Given that it has lasted at least 7 years, what is the conditional probability it will last at least another 3.5 years?

- $P(X \geq 10.5 | X \geq 7) = \frac{P(\{X \geq 10.5\} \cap \{X \geq 7\})}{P(X \geq 7)} = \frac{P(X \geq 10.5)}{P(X \geq 7)}$

  $= \frac{\int_{10.5}^\infty \frac{3x^2}{7^3} e^{-(x/7)^3} dx}{\int_7^\infty \frac{3x^2}{7^3} e^{-(x/7)^3} dx} = \frac{e^{-1.5^3}}{e^{-1}} = e^{-2.375}$.

A function is given as $f(x) = (3/16)x^2$, $-c < x < c$.

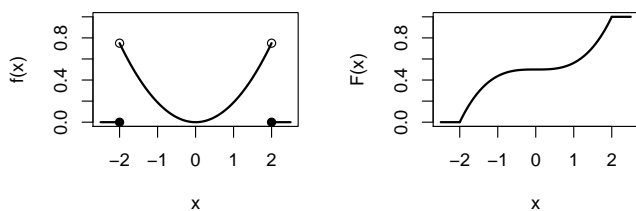a. Find the constant c so that $f(x)$ is a pdf of a random variable X.

- $1 = \int_{-c}^c \frac{3}{16} x^2 dx = \frac{1}{16} x^3|_{-c}^c = \frac{1}{8} c^3$.
- *Thus $c = 2$.*

b. Find the cdf $F(x) = P(X \leq x)$.

- $F(x) = P(X \leq x) = \int_{-2}^x \frac{3}{16} t^2 dt = \frac{1}{16} x^3 + \frac{1}{2}$, *if $-2 < x < 2$.*

- *Thus $F(x) = \begin{cases} 0, & x \leq -2 \\ \frac{1}{16} x^3 + \frac{1}{2}, & -2 < x \leq 2 \\ 1, & x > 2. \end{cases}$*

c. Sketch graphs of the pdf $f(x)$ and the cdf $F(x)$.

9. A function is given as $f(x) = 4x^c$, $0 \le x \le 1$.

    a. Find the constant c so that $f(x)$ is a pdf of a random variable X.

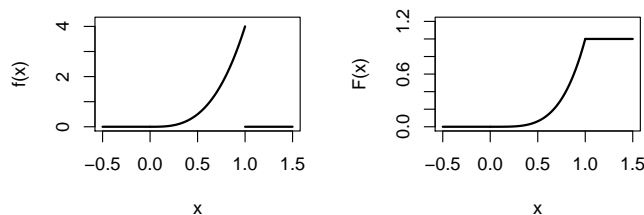   - $1 = \int_0^1 4x^c dx = \frac{4}{c+1}x^{c+1}\big|_0^1 = \frac{4}{c+1}$.
       - *Thus* $c = 3$.

    b. Find the cdf $F(x) = P(X \le x)$.

       - $F(x) = P(X \le x) = \int_0^x 4t^3 dt = x^4$, *if* $0 < x < 1$.
       - *Thus*

       $$F(x) = \begin{cases} 0, & x \le 0 \\ x^4, & 0 < x \le 1 \\ 1, & x > 1. \end{cases}$$

    c. Sketch graphs of the pdf $f(x)$ and the cdf $F(x)$.

10. Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10-minute period, let X equal the time within the 10 minute-period that the customer arrived. Assuming X is $U(0, 10)$, find:

    a. the pdf of X,

       -

       $$f(x) = \begin{cases} \frac{1}{10}, & 0 < x < 10, \\ 0, & elsewhere. \end{cases}$$

    b. $P(X \ge 8)$,

       - $P(X \ge 8) = \int_8^{10} \frac{1}{10}dx = 0.2$.

    c. $P(2 \le X < 8)$,

       - $P(2 \le X < 8) = \int_2^8 \frac{1}{10}dx = 0.6$.

    d. $E(X)$, and

       - $E(X) = \int_0^{10} x\frac{1}{10}dx = \frac{10}{2} = 5$.

    e. $\text{Var}(X)$.

       - $\text{Var}(X) = \int_0^{10} x^2\frac{1}{10}dx - 5^2 = \frac{25}{3}$.

11. Let X have an exponential distribution with a mean of $\theta = 20$. Compute

   a. $P(10 < X < 30)$,

   - $P(10 < X < 30) = \int_{10}^{30} \frac{1}{20} e^{-x/20} dx = [-e^{-x/20}]_{10}^{30} = e^{-1/2} - e^{-3/2}$.

   b. $P(X > 30)$.

   - $P(X > 30) = \int_{30}^{\infty} \frac{1}{20} e^{-x/20} dx = [-e^{-x/20}]_{30}^{\infty} = e^{-3/2}$.

   c. $P(X > 40 | X > 10)$.

   - $P(X > 40) = \int_{40}^{\infty} \frac{1}{20} e^{-x/20} dx = e^{-2}$. *Similarly, $P(X > 10) = e^{-1/2}$*

   - *So* $P(X > 40 | X > 10) = \frac{P(X>40)}{P(X>10)} = e^{-3/2} = P(X > 30)$.

   d. What are the variance and the mgf of X?

   - $\text{Var}(X) = \sigma^2 = \theta^2 = 400$, $M(t) = (1 - 20t)^{-1}$, $t < 1/20$.

   e. Find the 80th percentile of X.

   - $0.8 = \int_0^{\pi_{0.8}} \frac{1}{20} e^{-x/20} dx = 1 - e^{-\pi_{0.8}/20}$.

   - *So* $\pi_{0.8} = -20 \ln(1 - 0.8) = 20 \ln(5) = 32.19$.

12. What are the pdf, the mean, and the variance of X if the mgf of X is given by the following?

   a. $M(t) = (1 - 3t)^{-1}$, $t < 1/3$.

   - $X \overset{d}{=} exponential(\theta = 3)$; *pdf* $f(x) = \frac{1}{3} e^{-x/3}$, $x > 0$; $\mu = 3$; $\sigma^2 = 9$.

   b. $M(t) = \frac{3}{3-t}$, $t < 3$.

   - $X \overset{d}{=} exponential(\theta = 1/3)$; *pdf* $f(x) = 3e^{-3x}$, $x > 0$; $\mu = 1/3$; $\sigma^2 = 1/9$.

13. Let X_t equal the number of flawed recordings in each length t measured in billions of records. Assume that X_t is a Poisson process with rate 2.5 per billion records (so t is treated as continuous). Let W be the length of records before the first bad record is found.

   a. Give the mean number of flaws per billion records.

   - $\lambda = 2.5$.

   b. How is W distributed?

   - $W \overset{d}{=} exponential(\theta = 1/2.5 = 0.4)$.

   c. Give the mean and variance of W.

   - $E(W) = 0.4$ *and* $\text{Var}(W) = 0.4^2 = 0.16$.

14. Let random variable X have the pdf

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2}, \quad -\infty < x < \infty.$$

(The distribution of such X is known as the *logistic distribution*.)

a. Write down the cdf of X.

- $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{e^{-t}}{(1+e^{-t})^2}dt = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}, \quad -\infty < x < \infty.$

b. Find the mean and variance of X.

- $\mu = \int_{-\infty}^{\infty} x \frac{e^{-x}}{(1+e^{-x})^2}dx = 0$ *because* $x\frac{e^{-x}}{(1+e^{-x})^2}$ *is an odd function.*

- $\sigma^2 = \int_{-\infty}^{\infty} x^2 \frac{e^{-x}}{(1+e^{-x})^2}dx - 0^2 = \frac{1}{3}\pi^2. = 2\int_0^\infty \frac{x^2 e^{-x}}{(1+e^{-x})^2}dx = 2\int_0^\infty \frac{x^2 e^x}{(1+e^x)^2}dx =$
  $4\int_0^\infty \frac{x}{1+e^x}dx = 4\int_1^\infty \frac{\ln y}{(1+y)y}dy = 4\int_1^\infty \ln y \, d\ln\frac{y}{1+y} = 4\int_1^\infty \frac{\ln(1+\frac{1}{y})}{y}dy = 4\int_0^1 \frac{\ln(1+t)}{t}dt =$
  $4\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{1}{3}\pi^2.$
  Without the help of Maple, it would be very difficult to calculate this integral.

c. Find $P(3 < X < 5)$.

- $P(3 < X < 5) = F(5) - F(3) = \frac{1}{1+e^{-5}} - \frac{1}{1+e^{-3}}.$

d. Find the 85-th percentile of X.

- *Solve* $0.85 = F(\pi_{0.85}) = \frac{1}{1+e^{-\pi_{0.85}}}$. *We get* $\pi_{0.85} = \ln(17/3)$.

e. Let $Y = \frac{1}{1+e^{-X}}$. Find the cdf of Y.
   Can you tell the name of the distribution of Y?

- *First the support of* Y *is* $0 < y < 1$.
- $G(y) = P(Y \leq y) = P(\frac{1}{1+e^{-X}} < y) = P(X \leq \ln(\frac{y}{1-y})) = F(\ln(\frac{y}{1-y})) =$
  $\frac{1}{1+e^{-\ln[y/(1-y)]}} = y, \quad 0 < y < 1.$
- *Therefore,* Y *has a Uniform(0,1) distribution.*

15. Telephone calls enter a university switchboard at a mean rate of $2/3$ call per minute according to a Poisson process. Let X denote the waiting time until the 10th call arrives.

a. What is the pdf of X?.

- $X \overset{d}{=} Gamma(\theta = 3/2, \alpha = 10)$.
- $f(x) = \frac{1}{\Gamma(10)(3/2)^{10}}x^9 e^{-2x/3}, \quad 0 \leq x < \infty.$

b. What are the mgf, mean and variance of X?

- $M(t) = (1 - \theta t)^{-\alpha} = (1 - \frac{3}{2}t)^{-10}, \quad t < \frac{2}{3}.$
- $\mu = \alpha\theta = 15$ *and* $\sigma^2 = \alpha\theta^2 = 22.5$.

16. If X has a gamma distribution with scale parameter $\theta = 4$ and $\alpha = 2$, find $P(X < 5)$.

- *First let* Y *be a Poisson random variable with mean* $\lambda x = (1/\theta)x = 5/4$.
- *Then by the relationship between gamma and Poisson distributions,*
  $P(X < 5) = P(Y \geq \alpha) = P(Y \geq 2) = 1 - [P(Y = 0) + P(Y = 1)]$
  $= 1 - [e^{-5/4} + (5/4)e^{-5/4}] = 0.35536$

17. Use an argument similar to the one used for the exponential distribution, find the moment generating function of the gamma distribution with general non-integer scale parameter. Use the moment-generating function of a gamma distribution to show that $E(X) = \alpha\theta$ and $\mathrm{Var}(X) = \alpha\theta^2$.

    - $M(t) = (1 - \theta t)^{-\alpha}, \quad t < \theta^{-1}$.
    - *So $M'(t) = \alpha\theta(1 - \theta t)^{-\alpha-1}$, and $E(X) = M'(0) = \alpha\theta$*
    - *Also $M''(t) = (\alpha + 1)\alpha\theta^2(1 - \theta t)^{-\alpha-2}$. Thus $M''(0) = (\alpha + 1)\alpha\theta^2$.*
      Therefore $\mathrm{Var}(X) = M''(0) - [M'(0)]^2 = (\alpha + 1)\alpha\theta^2 - \alpha^2\theta^2 = \alpha\theta^2$.

18. Let X have a $\chi^2(2)$ distribution. Find constants a and b such that

$$P(a < X < b) = 0.90, \quad \text{and} \quad P(X < a) = 0.05.$$

    - *The distribution of X is also gamma with $\theta = 2$ and $\alpha = 2/2 = 1$.*
    - *Thus the distribution of X is further exponential with $\theta = 2$.*
    - $0.05 = P(X < a) = \int_0^a \frac{1}{2}e^{-x/2}dx = 1 - e^{-a/2}$.
      So $a = -2\ln(0.95) = 0.1026$.
    - $P(X < b) = P(X < a) + P(a < X < b) = 0.95$.
      So $0.95 = P(X < b) = \int_0^b \frac{1}{2}e^{-x/2}dx = 1 - e^{-b/2}$.
      Hence $b = -2\ln(0.05) = 5.9915$.