# CS3210: Solutions for Assignment 1

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## 1

### 1.1 Note

We use the following rules (you might need to memorize them), along with basic growth rates that we discussed in the class (such as *log* grows slower than polynomial etc.).

- $f = \Theta(g)$ and $g = \Theta(h)$ implies $f = \Theta(h)$
- $f_1 = \Theta(g_1)$ and $f_2 = \Theta(g_2)$ implies $f_1 + f_2 = \Theta(g_1 + g_2)$
- $f_1 = \Theta(g_1)$ and $f_2 = \Theta(g_2)$ implies $f_1 f_2 = \Theta(g_1\ g_2)$
- $f = \Theta(g)$ implies $f^b = \Theta(g^b)$ for any $b \neq 0$
- $f = \Theta(g)$ implies $\log_b(f) = \Theta(\log_b(g)$ for $b > 1$ if $g(n) \geq b$ for large enough $n$
- $f = o(g)$ implies $a^f = a^g$ for $a > 1$ (**Not true for big $O$**)

Some basic math (found at section 3.2 of CLRS) can also be very useful, such as

- $a = b^{\log_b a}$
- $a^{\log_b c} = c^{\log_b a}$

**When simplifying an equation to infere the order, always make sure the steps are reversible using the above rules, the trickiest one is taking logarithm, because the last rule is not correct for big O**

**Final Order** $1, 4, 3, 5, 2, 6, 7$

1 to 4: If we take *log* from both sides (are we allowed to?) then we are comparing $\sqrt(2)\sqrt(\log(n))$ with $\log(n) + 3\log(\log(n))$ or in other words $\sqrt(m)$ to $m + 3\log(m)$ if we let $m = \log(n)$.

4 to 3: If we divide both sides by $n$, we need to compare $log^3(n)$ to $n^{(1/3)}$ or further, $\log n$ to $n^{(1/9)}$ and *log* is always slower than polynomial (proved in class).

3 to 5: If we take log of both, we end up comparing $\log(n)$ and $\log^2(n)$ which obviously the later has a higher degree.

5 to 2: If we take log of both, we compare $\log^2(n)$ to $n$, and we know that *log* grows slower than polynomial.

2 to 6: We are comparing $2^n$ and $4^n$ and the later grows faster

6 to 7: We can take logarithm, and $2n$ grows slower than $n^2$

# 2

## 2.1

True. We need to prove there exist a $c_1, c_2$ and a $n_0$ such that for all $n > n_0$, $0 \leq c_1 n \lg n \leq 2n \lg n + 100 \lg^2 n \leq c_2 n \lg n$, or equivalently $0 \leq c_1 \leq 2 + 100\frac{\lg n}{n} \leq c_2$. We can let $n_0 = 2$, then note that $0 \leq \frac{\lg n}{n} \leq 1$, so obviously $2 \leq 2 + 100\frac{\lg 2}{2} \leq 102$, and therefore it suffices to let $c_1 \leq 2$ and $c_2 \geq 102$

## 2.2

True. We need to prove there exist a $c_1, c_2$ and a $n_0$ such that for all $n > n_0$, $0 \leq c_1 n \lg n \leq 2n \lg n - 100 \lg n + 100 \lg^2 n \leq c_2 n \lg n$, or equivalently $0 \leq c_1 \leq 2 - 100/n + 100\frac{\lg n}{n} \leq c_2$.

If we let $n_0 = 100$, then note that for the left side, $2 - 100/n \leq 2 - 100/n + 100\frac{\lg n}{n}$, and $2 - 100/n$ is always increasing after $n \geq 100$ (why?), so $2 - 100/n \geq 1$ and we can set $n_0 = 100$ and $c_1 \leq 1$. For the right side, $2 - 100/n + 100\frac{\lg n}{n} \leq 2 + 100\frac{\lg n}{n}$ and $\frac{\lg n}{n}$ is always decreasing (why?), so $2 + 100\frac{\lg n}{n} \leq 2 + 100 \lg 100/100 = 4$. So we can set $c_2 \geq 4$

## 2.3

We need to prove $0 \leq c_1 \lg f(n) \leq lg_b f(n) \leq c_2 lg f(n)$ for all $n \geq n_0$. We know that $\lg_b f(n) = \lg_b 2 \lg f(n)$, so what we need to prove becomes $0 \leq c_1 \lg f(n) \leq \lg_b 2 \lg f(n) \leq c_2 lg f(n)$ or simply $0 \leq \frac{c_1}{\lg_b 2} \lg f(n) \leq \lg f(n) \leq \frac{c_2}{\lg_b 2} \lg f(n)$ which is trivial if we set $c_1 \leq \lg_b 2$ and $c_2 \geq \lg_b 2$

## 2.4

False. Let $f(n) = 2n$ and $g(n) = n$, then $2^{f(n)} = 2^{2n}$ and $2^{g(n)} = 2^n$

## 2.5

False: let $f(n) = x(n) = n$ and $g(n) = y(n) = n^2$, **however, this is not what I wanted to ask for!** I meant to ask for the the following

$$f(n) + x(n) = O(g(n) + y(n)) \quad (1)$$

Which is True, because:

$f(n) = O(g(n))$ then $\exists c_1, n_1 : n \geq n_1, f(n) \leq c_1 g(n)$ and $x(n) = O(y(n))$ then $\exists c_2, n_2 : n \geq n_2, x(n) \leq c_2 y(n)$ . Let $n_0 = max(n_1, n_2)$ and $c = max(c_1, c_2)$. Then for all $n \geq n_0, f(n) \leq cg(n)$ and $x(n) \leq cy(n)$. Therefore, for all $n \geq n_0, f(n) + x(n) \leq c(g(n) + y(n))$

# 3

Generally False, let $f(n) = 2$ and $g(n) = 1$, then $\lg f(n) = 1$ but $\lg g(n) = 0$.

However, it is fixable: in Q2.3 we proved that $\lg_b f(n) = O(\lg f(n))$ for $b > 1$ (first condition) , so we only need to show that $\lg f(n) = O(\lg(g(n)))$ (do you remember why this entails the original question?)

We need to make sure that $g(n)$ is asymptotically greater than 2, i.e, $\exists n_1 : n \geq n_1, g(n) \geq 2$, in this case:

Since $f(n) \leq cg(n)$ for $n \geq n_0$, the $\lg f(n) \leq \lg cg(n)$, or, $\lg f(n) \leq \lg c + \lg g(n) \leq \lg c \lg g(n)$ for $n \geq \max(n_0, n_1)$