Greedy Algorithms

Textbook Reading

Chapters 16, 17, 21, 23 & 24

Overview

Design principle:

Make progress towards a globally optimal solution by making locally optimal choices, hence the name.

Problems:

- Interval scheduling
- Minimum spanning tree
- Shortest paths
- Minimum-length codes

Proof techniques:

- Induction
- The greedy algorithm "stays ahead"
- Exchange argument

Data structures:

- Priority queue
- Union-find data structure

Interval Scheduling

Given:

A set of activities competing for time intervals on a certain resource
(E.g., classes to be scheduled competing for a classroom)

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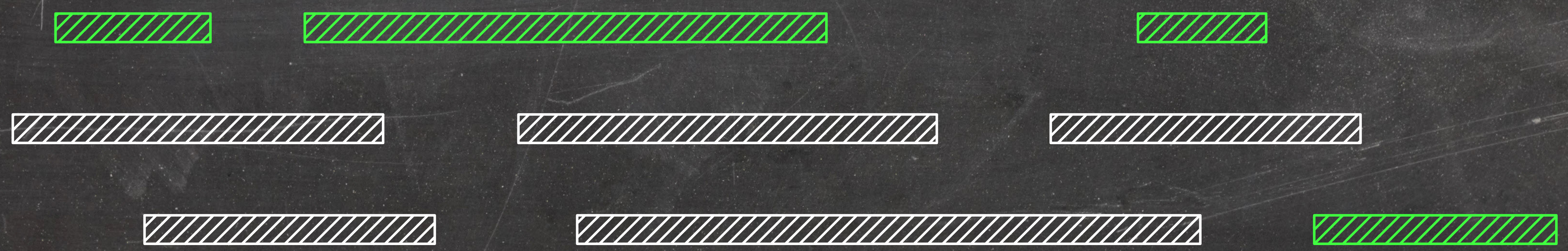

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A Greedy Framework for Interval Scheduling

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FindSchedule(S)
S′ = ∅
while S is not empty
    do pick an interval I in S
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Main questions:

- Can we choose an arbitrary interval I in each iteration?
- How do we choose interval I in each iteration?

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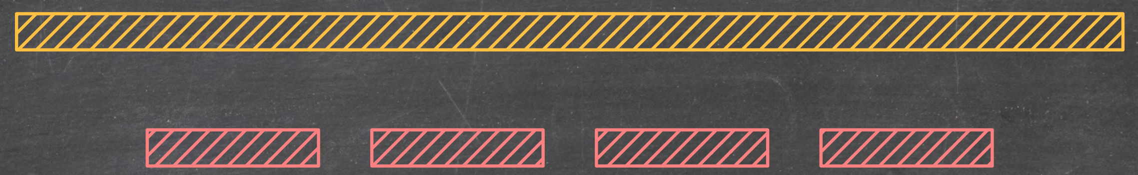

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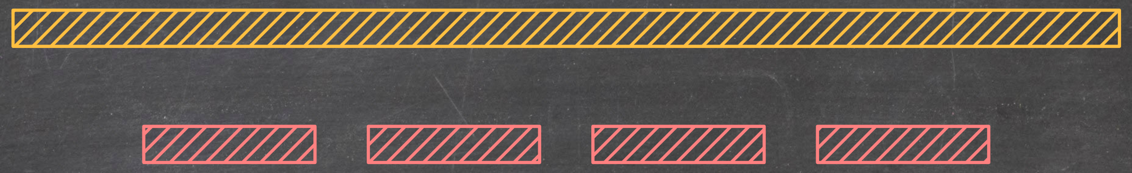

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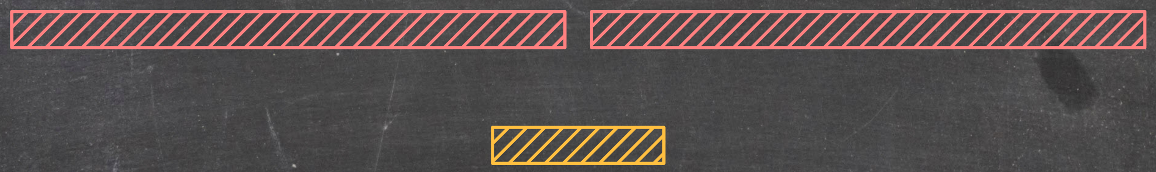

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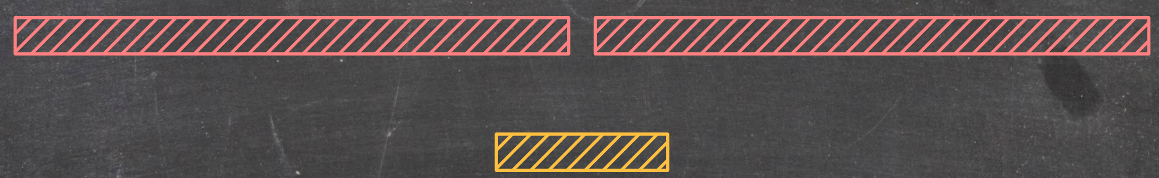

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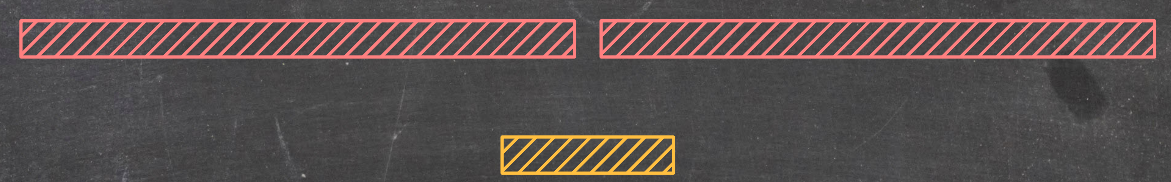

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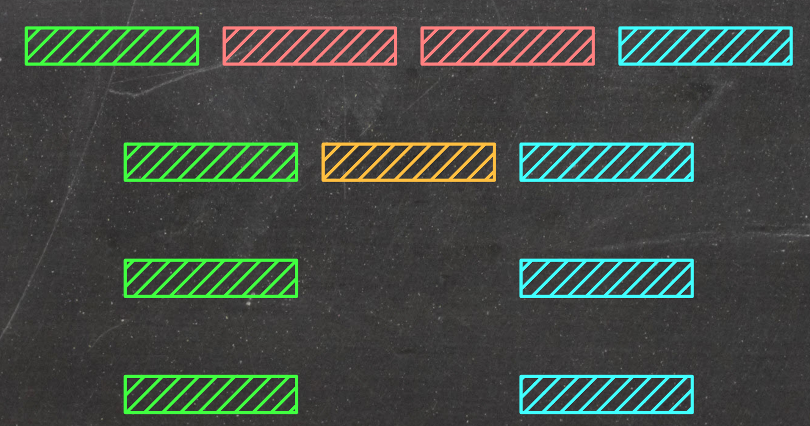

The Strategy That Works

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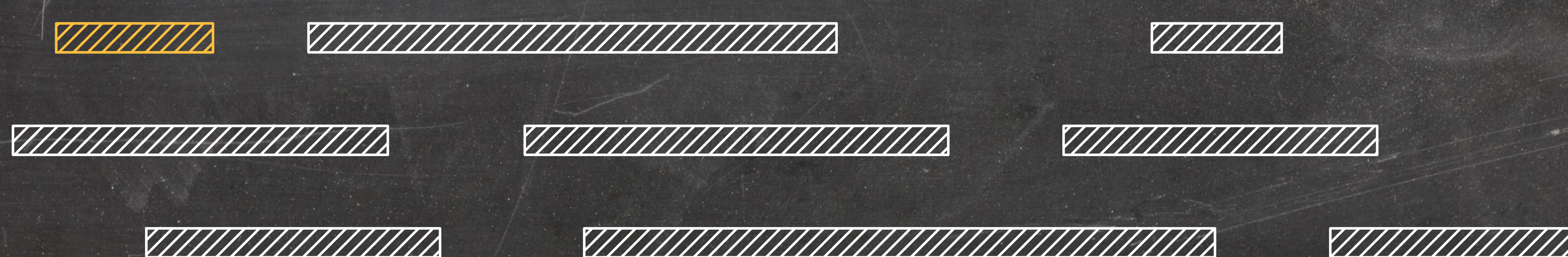

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Let $I_1 \prec I_2 \prec \cdots \prec I_k$ be the schedule we compute.

Let $O_1 \prec O_2 \prec \cdots \prec O_m$ be an optimal schedule.

Prove by induction on j that I_j ends no later than O_j.

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Prove by induction on j that I_j ends no later than O_j.

$\Rightarrow$ Since O_{j+1} starts after O_j ends, it also starts after I_j ends.

$\Rightarrow$ If $k < m$, FindSchedule inspects O_{k+1} after I_k and thus would have added it to its output, a contradiction.

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Lemma: FindSchedule finds a maximum-cardinality set of conflict-free intervals.

Proof by induction:

Base case(s): Verify that the claim holds for a set of initial instances.

Inductive step: Prove that, if the claim holds for the first k instances, it holds for the $(k + 1)$st instance.

The Greedy Algorithm Stays Ahead

Lemma: FindSchedule finds a maximum-cardinality set of conflict-free intervals.

Base case: I_1 ends no later than O_1 because both I_1 and O_1 are chosen from S and I_1 is the interval in S that ends first.

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$\Rightarrow$ O_{k+1} does not conflict with $I_1, I_2, \ldots, I_k$.

$\Rightarrow$ I_{k+1} ends no later than O_{k+1} because it is the interval that ends first among all intervals that do not conflict with $I_1, I_2, \ldots, I_k$.

Implementing The Algorithm

FindSchedule(S)

```
S′ = [ ]
sort the intervals in S by increasing finish times
S′.append(S[1])
f = S[1].f
for i = 2 to |S|
    do if S[i].s > f
            then S′.append(S[i])
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Lemma: A maximum-cardinality set of non-conflicting intervals can be found in $O(n \lg n)$ time.

Minimum Spanning Tree

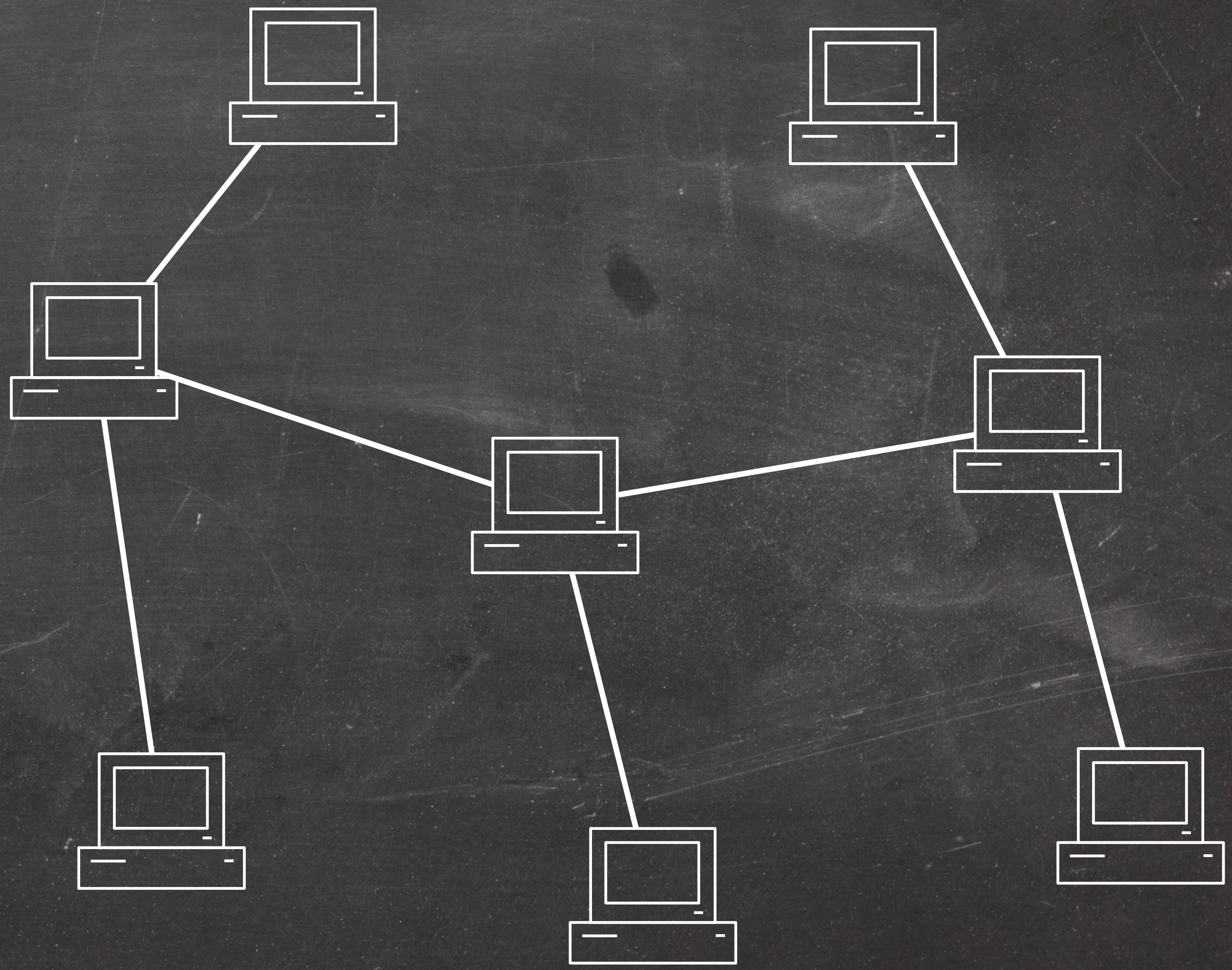
Given: n computers

Goal: Connect them so that every computer can communicate with every other computer.

We don't care whether the connection between any pair of computers is short.

We don't care about fault tolerance.

Every foot of cable costs us \$1.

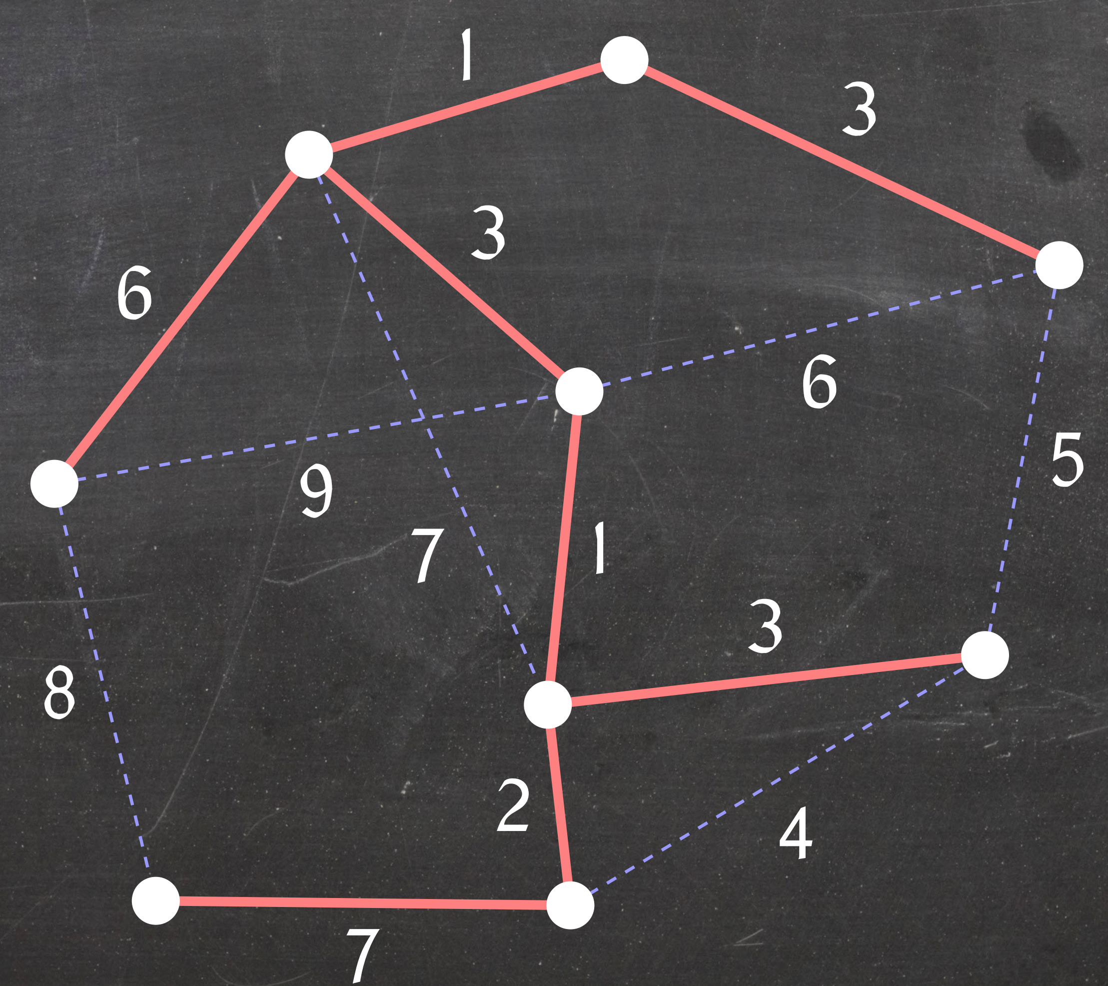


$\Rightarrow$ We want the cheapest possible network.

Minimum Spanning Tree

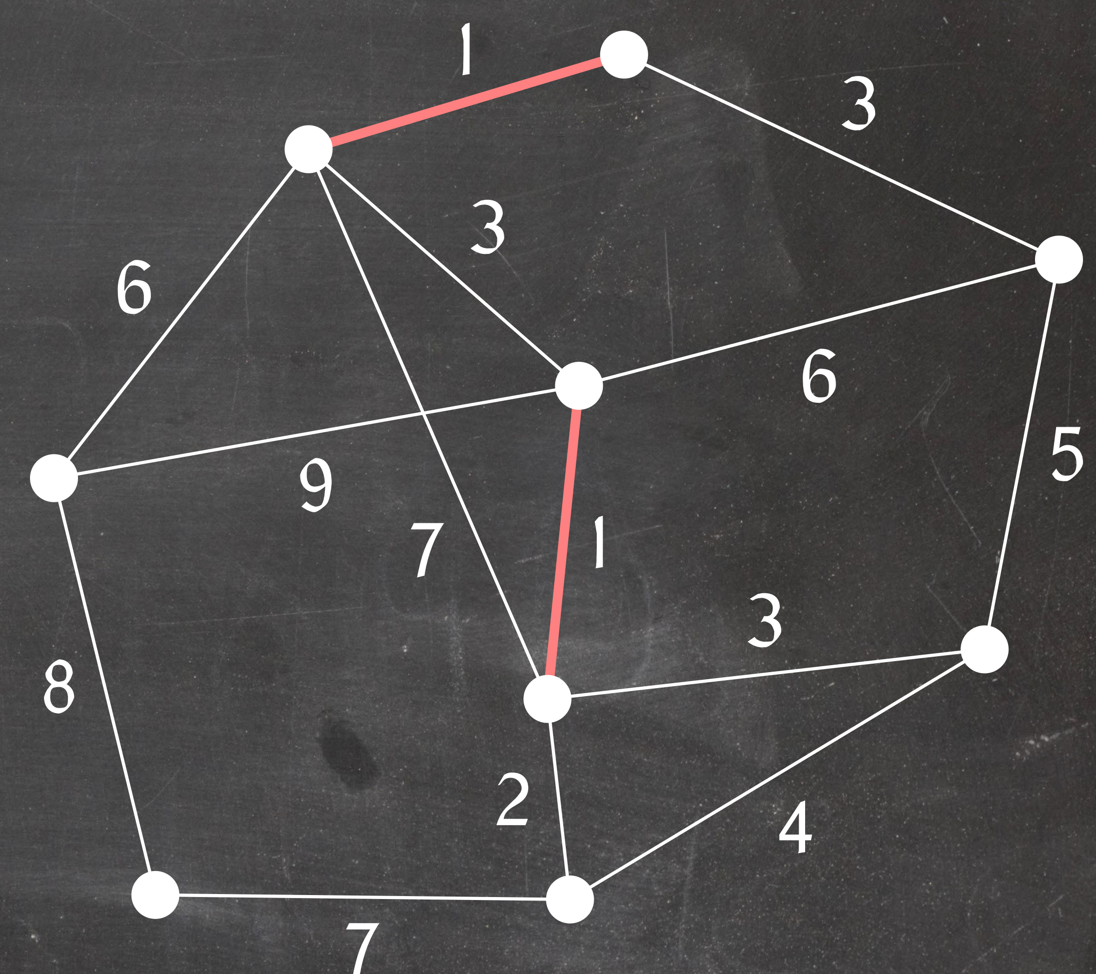
Given a graph $G = (V, E)$ and an assignment of weights (costs) to the edges of G, a minimum spanning tree (MST) T of G is a spanning tree with minimum total weight

$$w(T) = \sum_{e \in T} w(e).$$



Kruskal's Algorithm

Greedy choice: Pick the shortest edge



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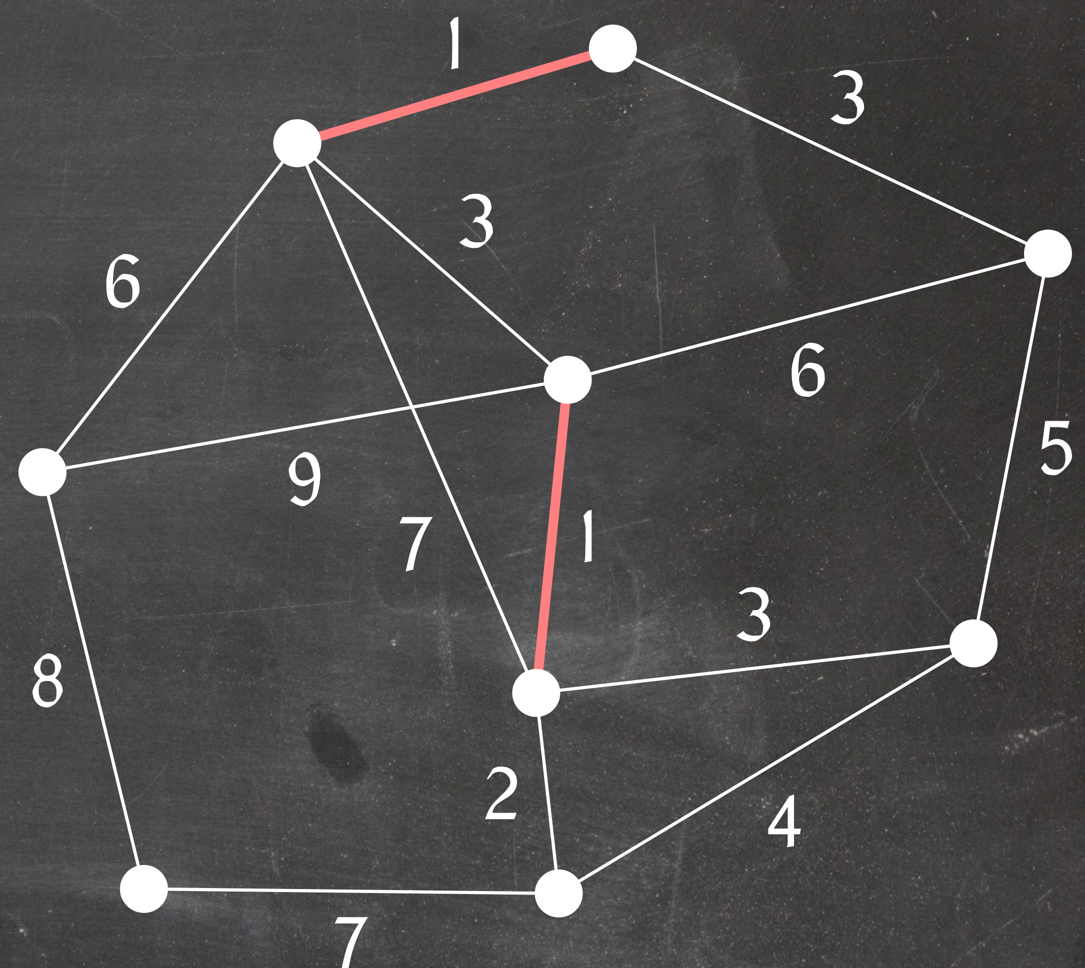

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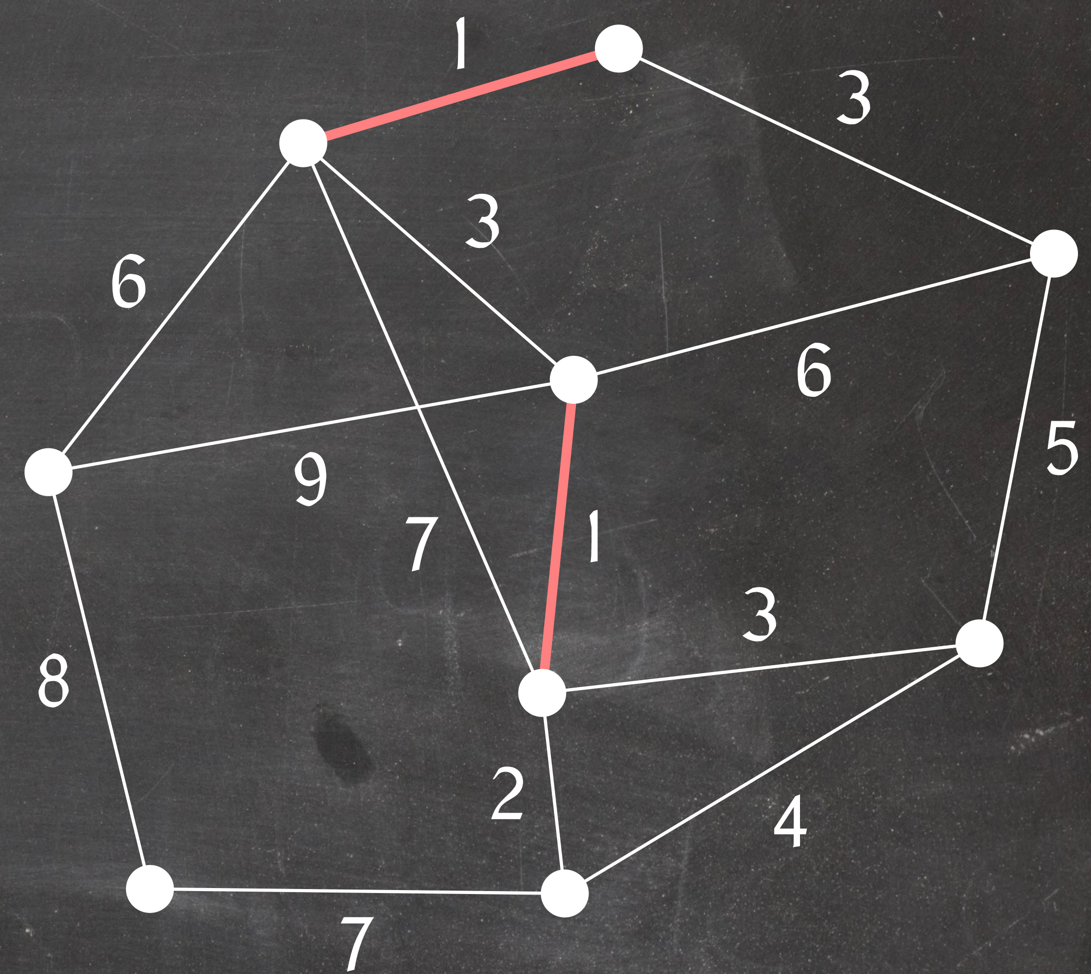

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T = (V, ∅)
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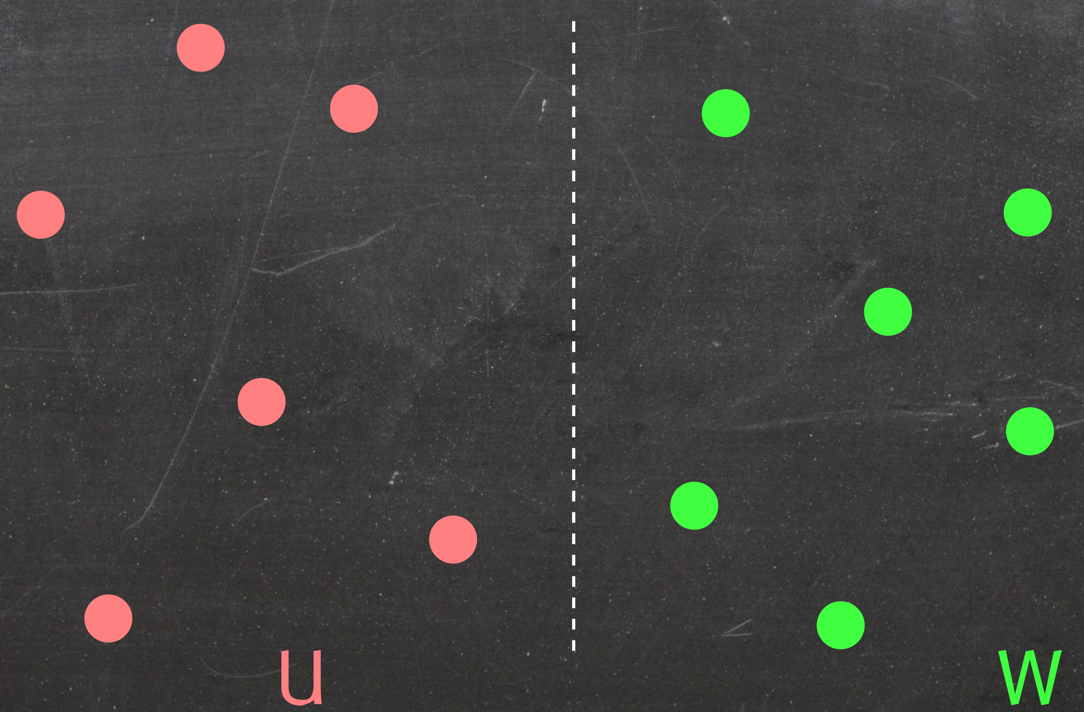
A Cut Theorem

A cut is a partition (U, W) of V into two non-empty subsets: $\emptyset \subset U \subset V$ and $W = V \setminus U$.



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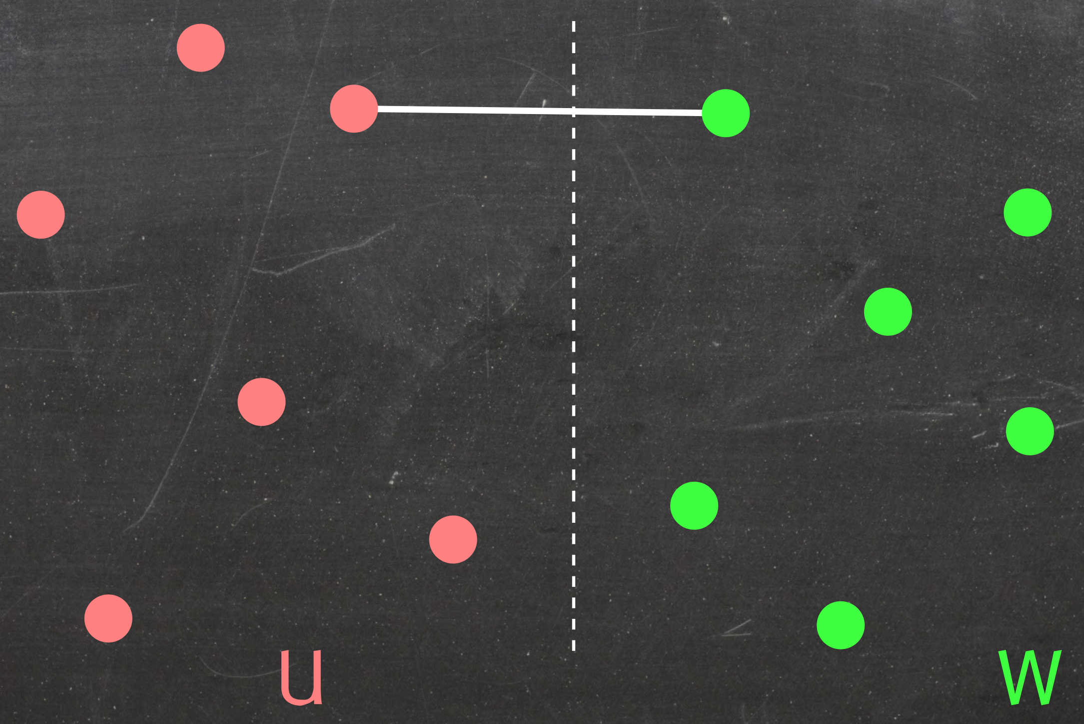

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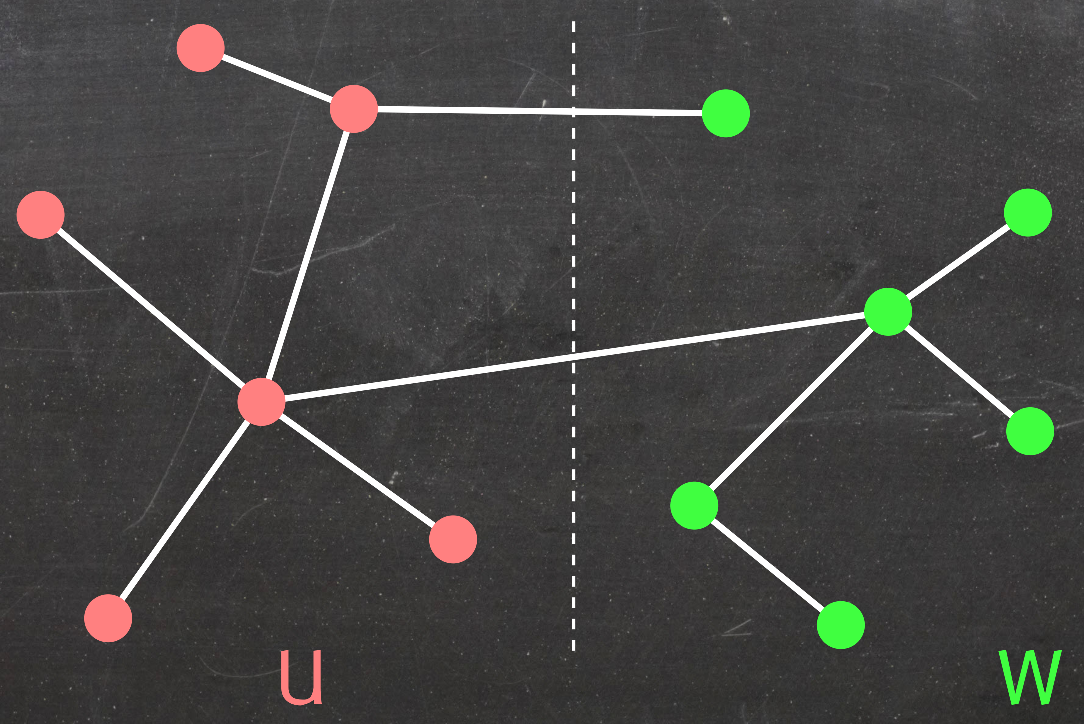
Theorem: Let T be a minimum spanning tree, let (U, W) be an arbitrary cut, and let e be the cheapest edge crossing the cut. Then there exists a minimum spanning tree that contains e and all edges of T that do not cross the cut.



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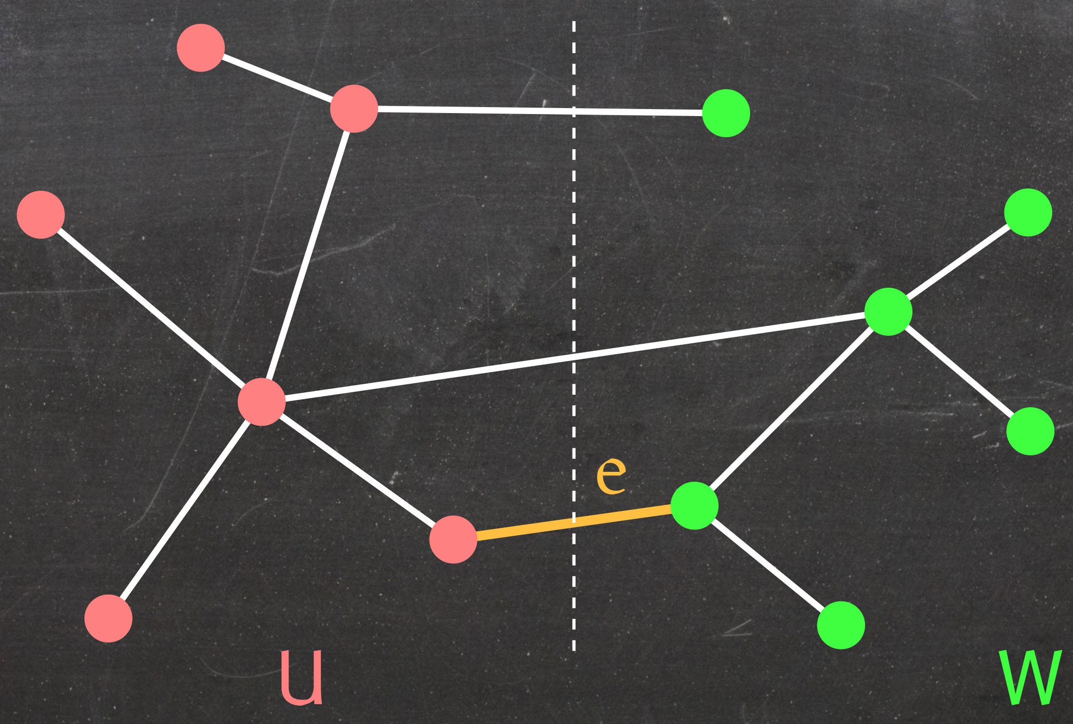

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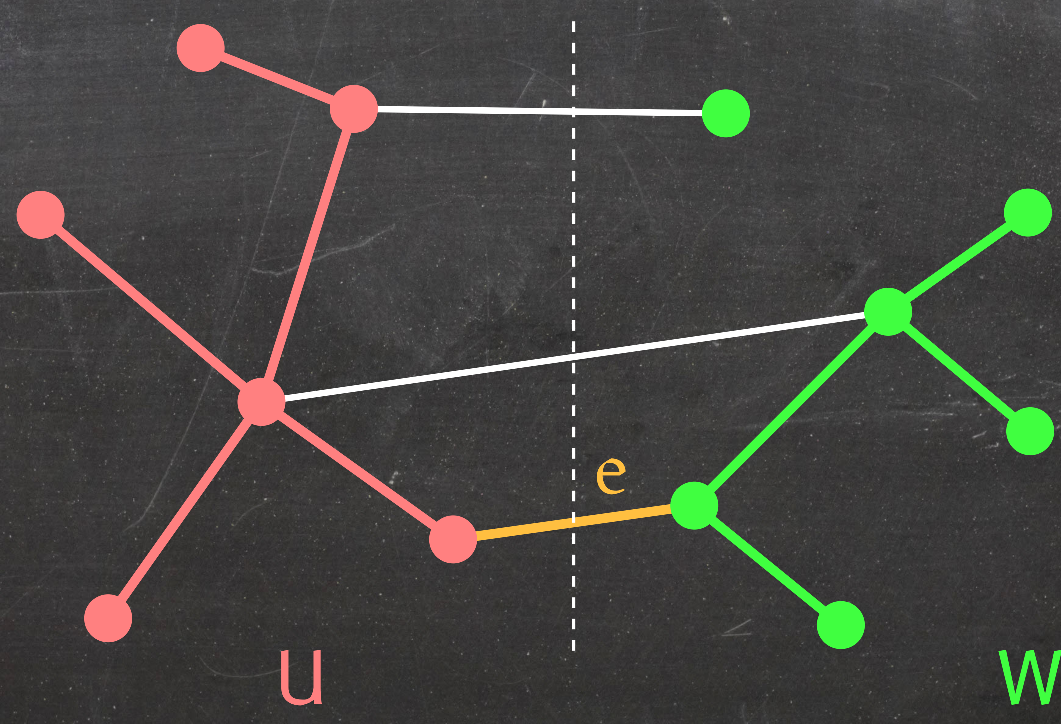

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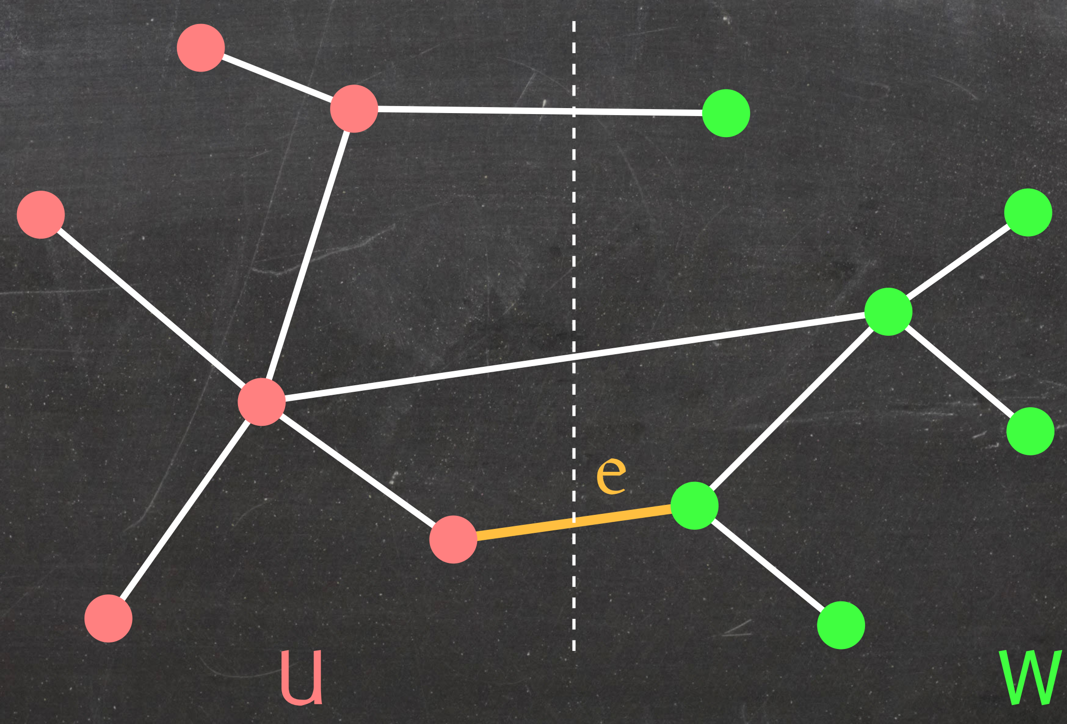

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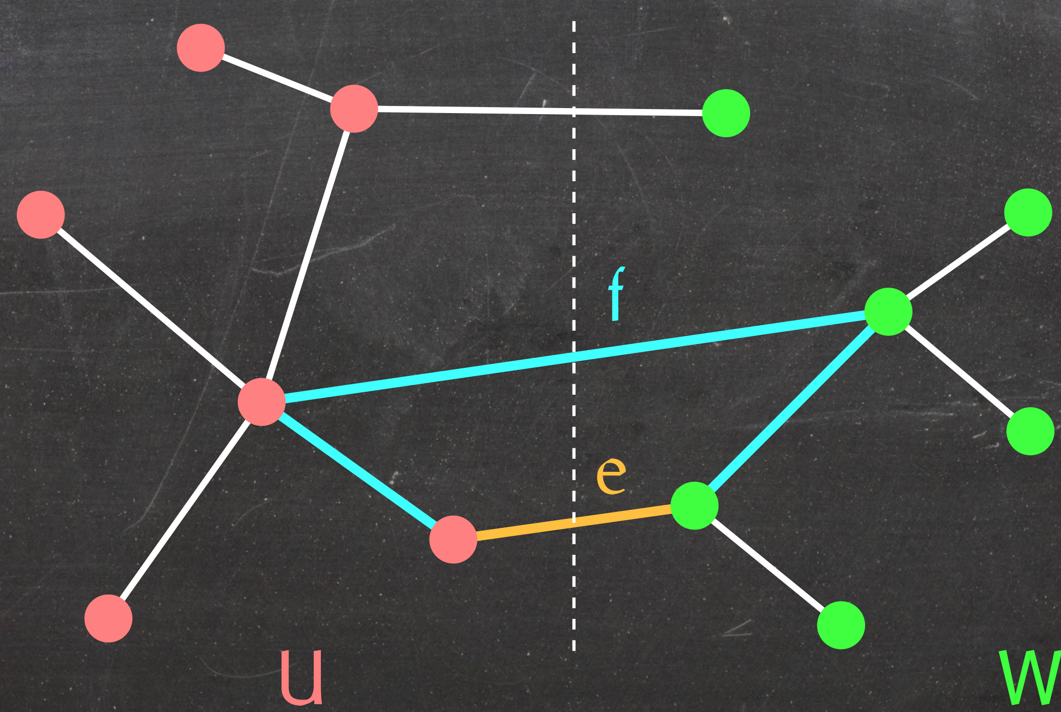

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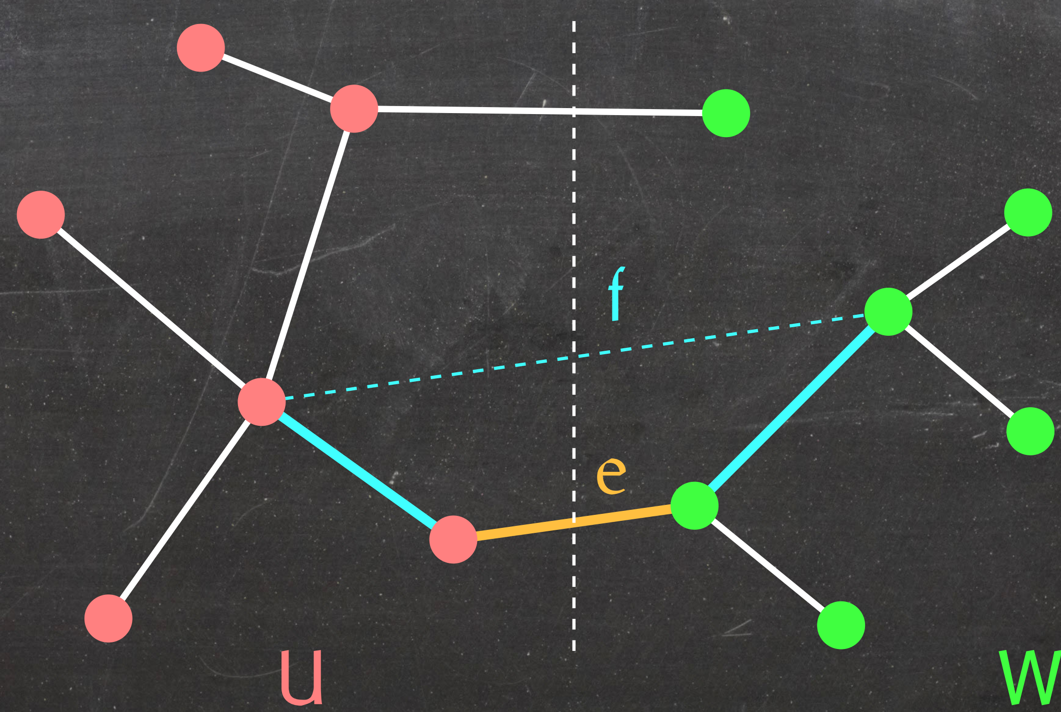

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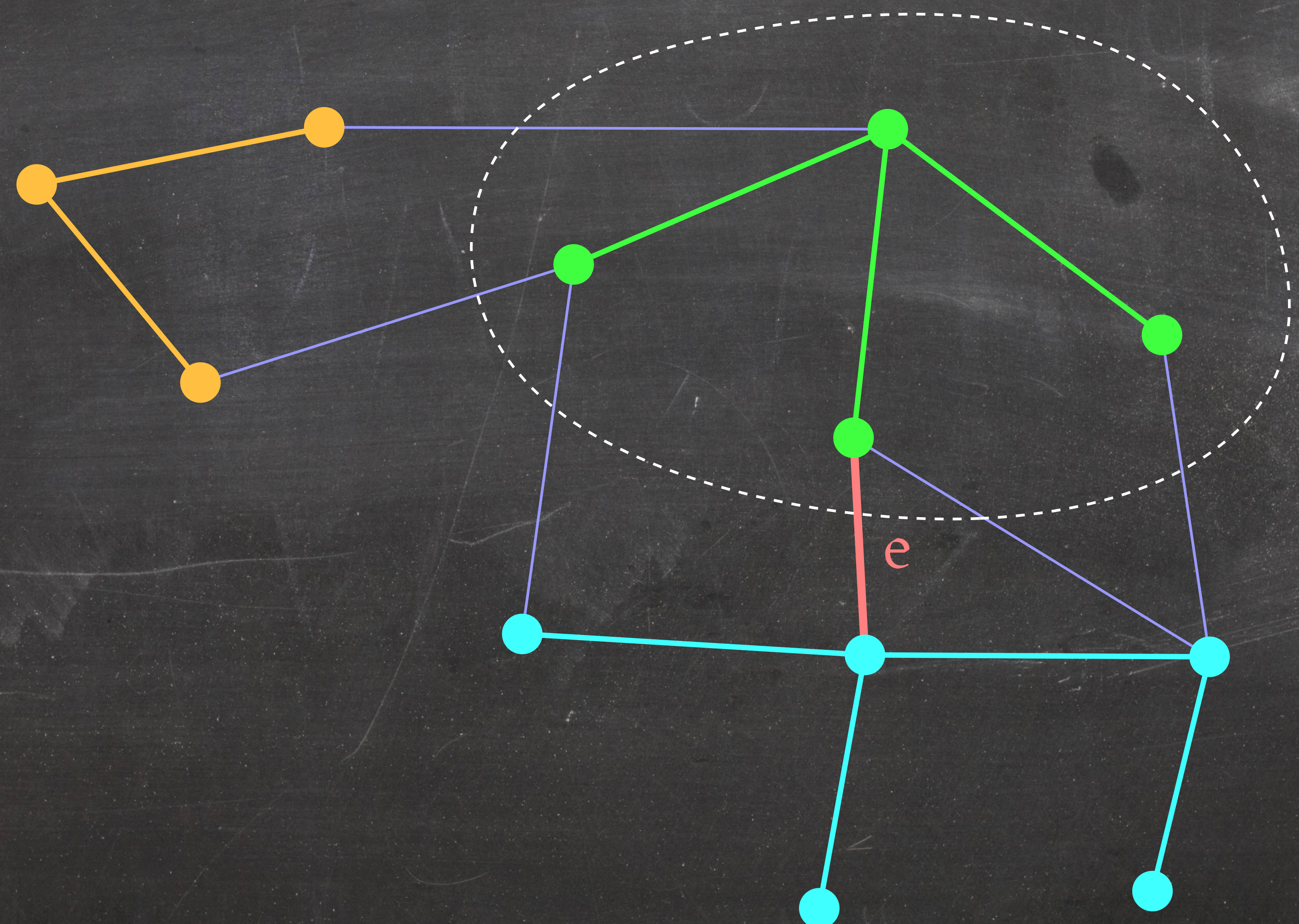

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$$\Downarrow$$

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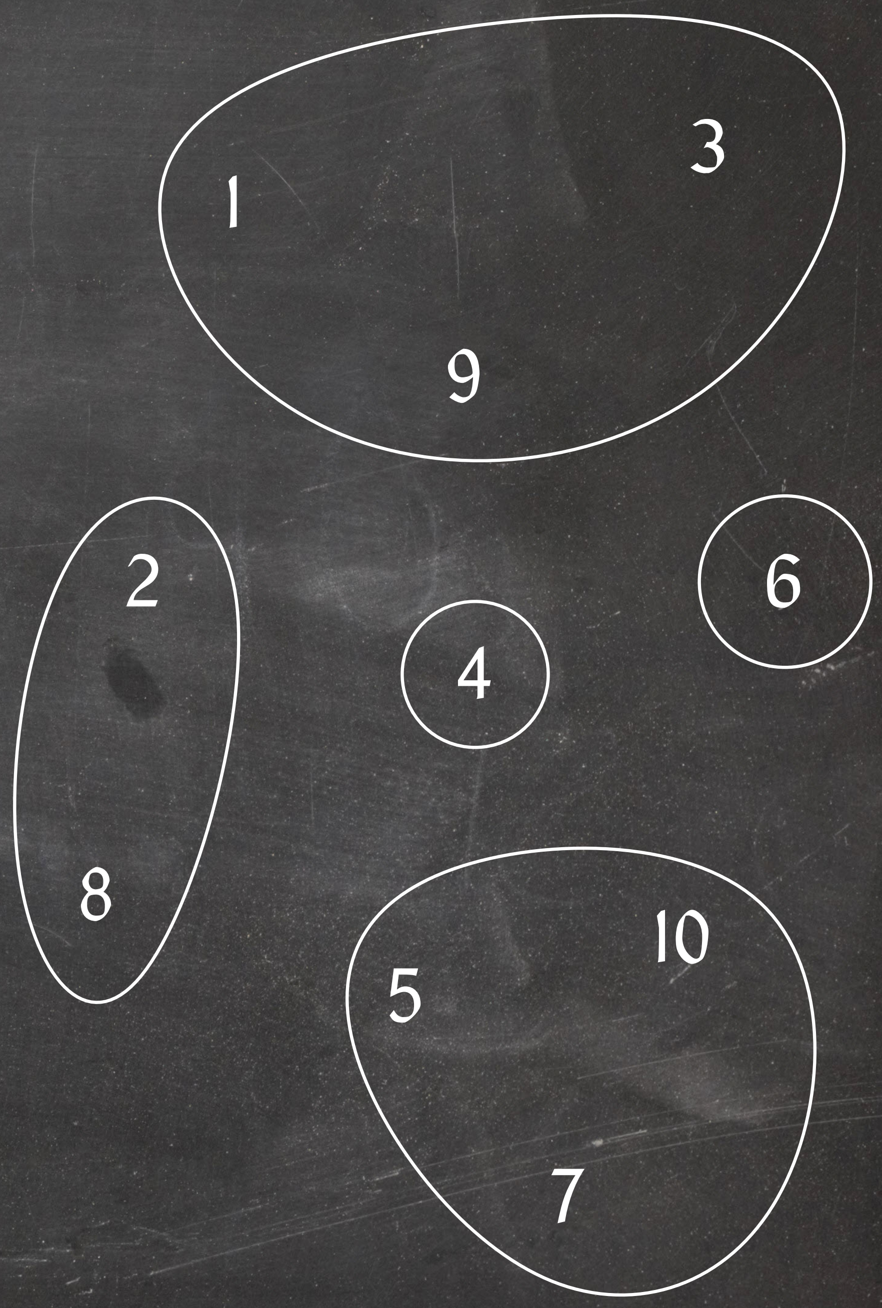
A Union-Find Data Structure

Given a set S of elements, maintain a partition of S into subsets $S_1, S_2, \ldots, S_k$.



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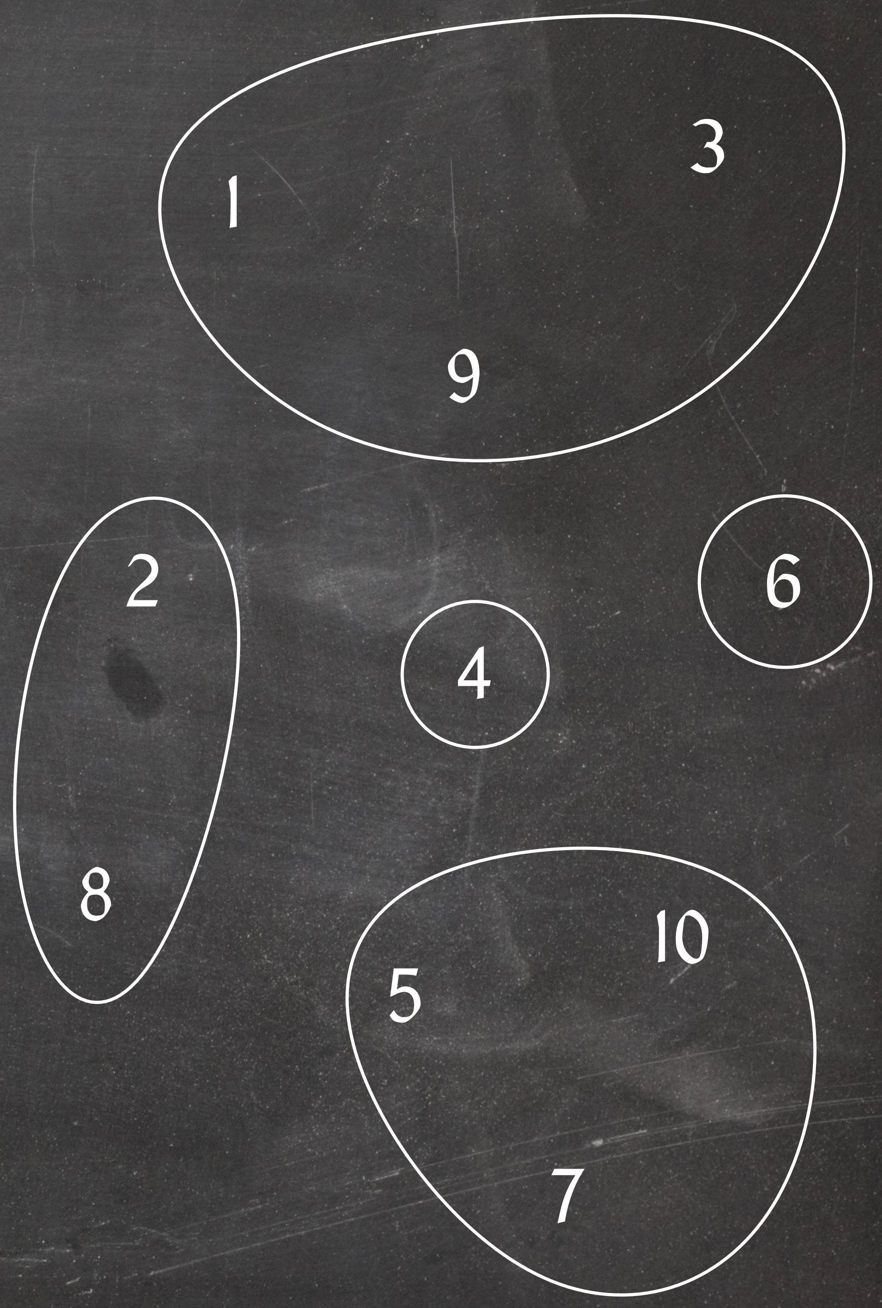

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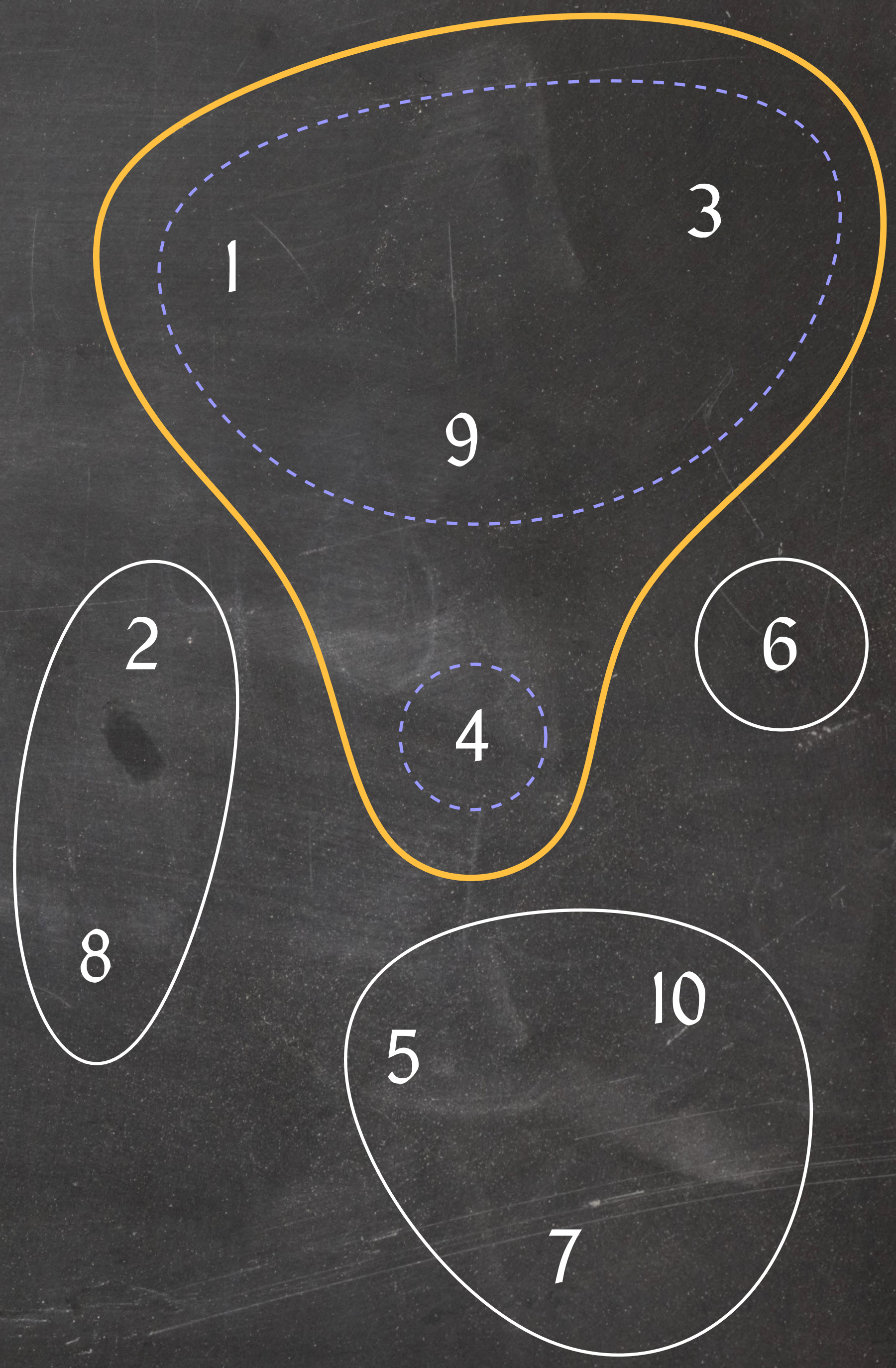

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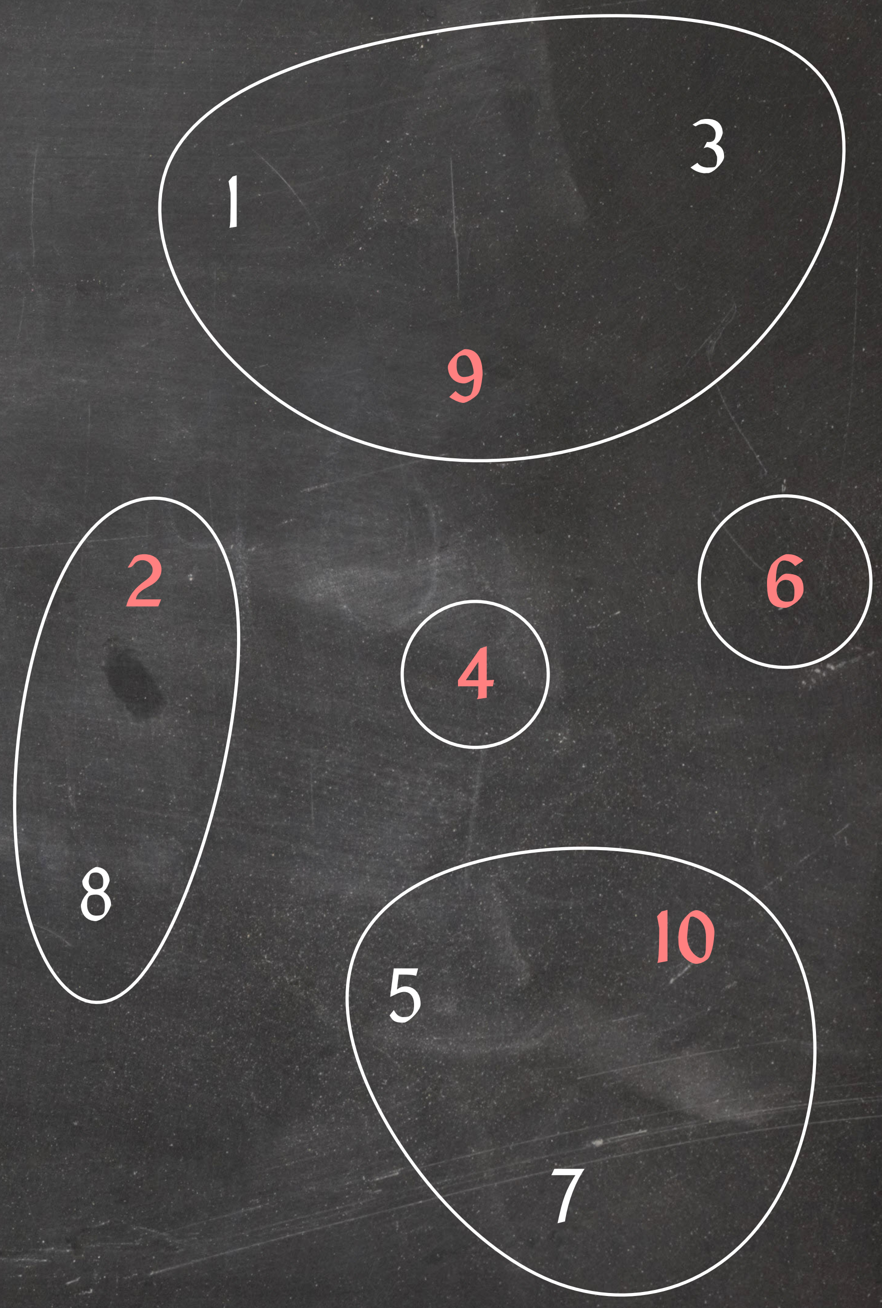

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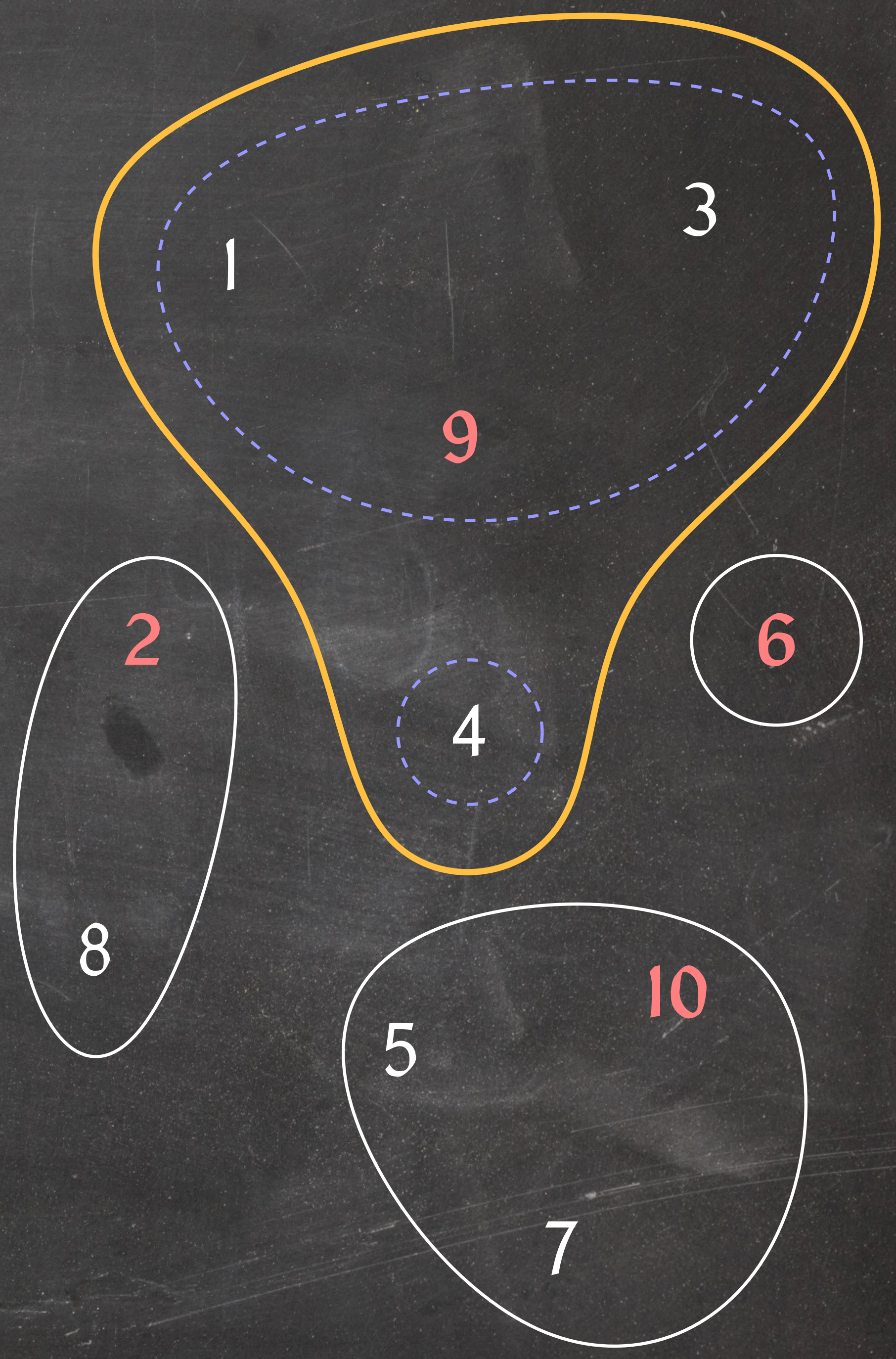

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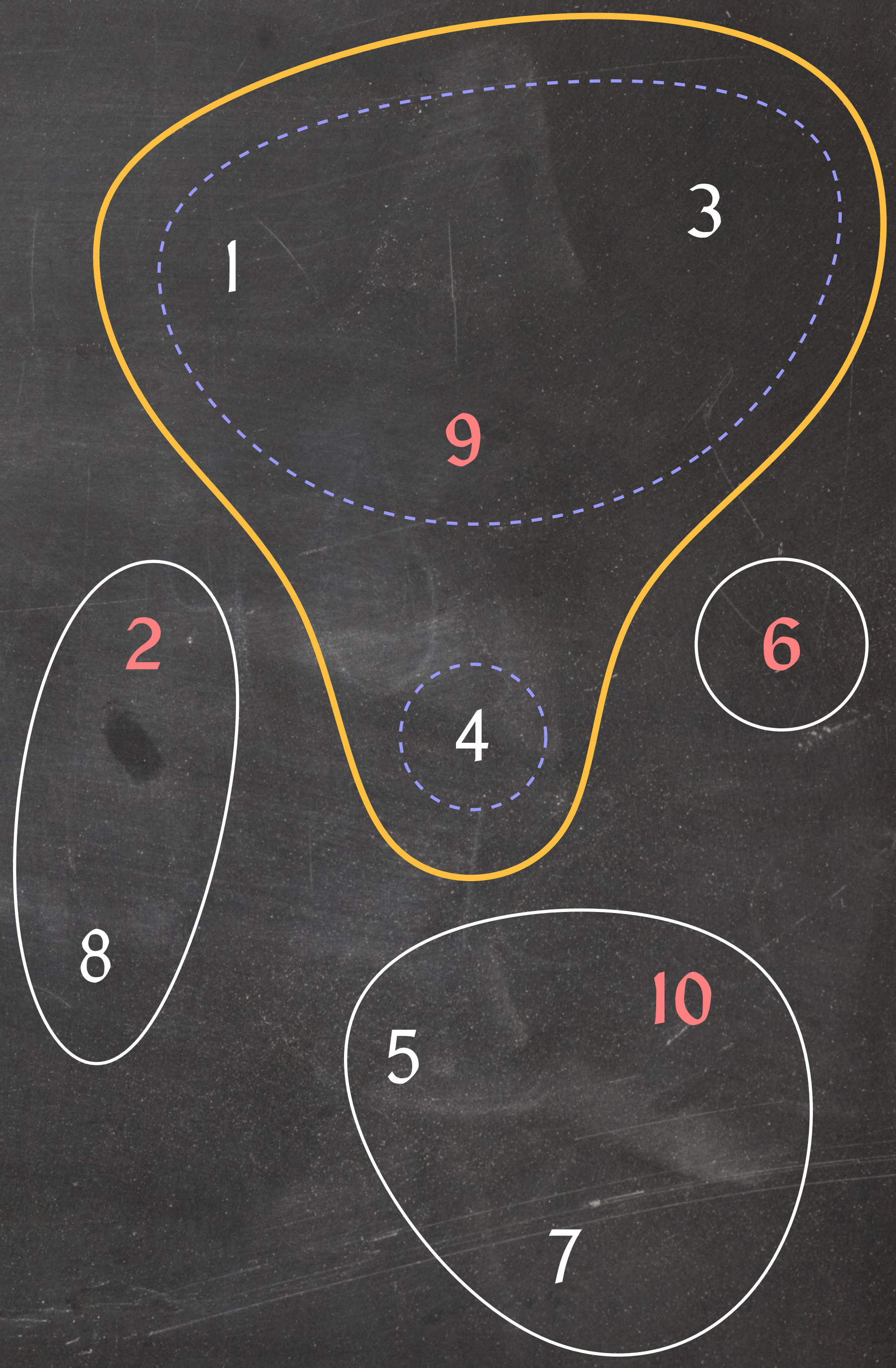
Find(x): Return a representative $r(S_i) \in S_i$ of the set S_i that contains x.

In particular, $\text{Find}(x) = \text{Find}(y)$ if and only if x and y belong to the same set.



Kruskal's Algorithm Using Union-Find

Idea: Maintain a partition of V into the vertex sets of the connected components of T.

Kruskal(G)

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initialize a union-find structure D for V with every vertex v ∈ V in its own set
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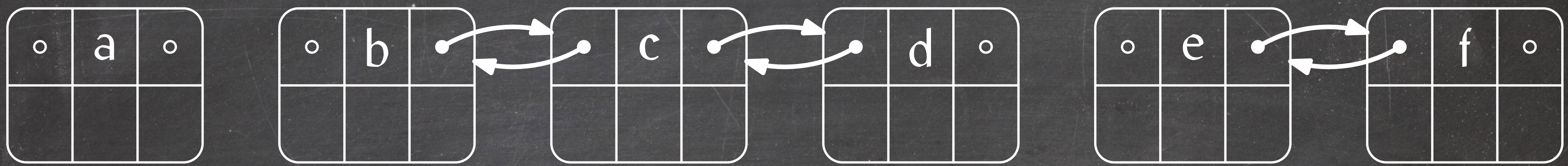
         then add (v, w) to T
              D.union(v, w)
return T
```

Lemma: Kruskal's algorithm takes $O(m \lg m)$ time plus the cost of $2m$ Find and $n - 1$ Union operations.

A Simple Union-Find Structure



List node:

- A set element
- Pointers to predecessor and successor
- Pointer to head of the list
- Pointer to tail of the list (only valid for head node)
- Size of the list (only valid for head node)

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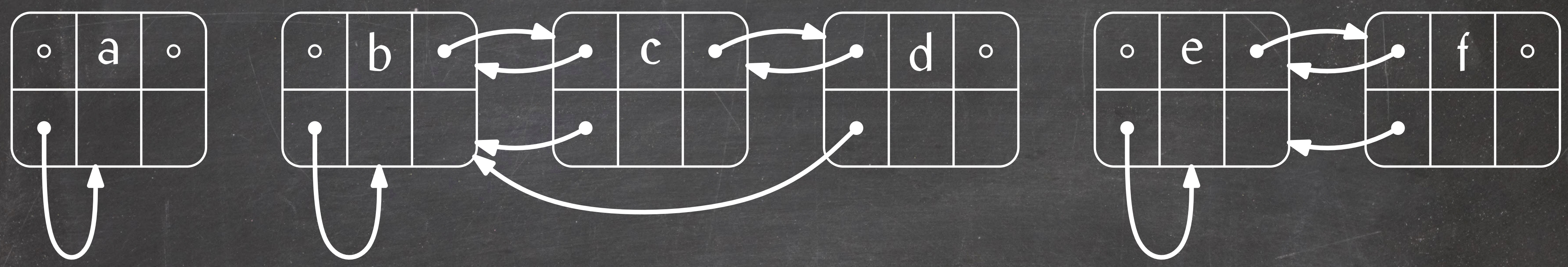

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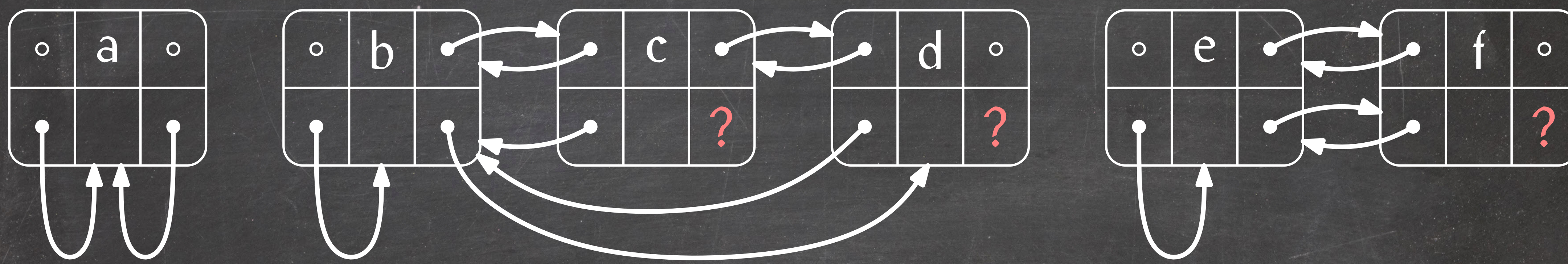

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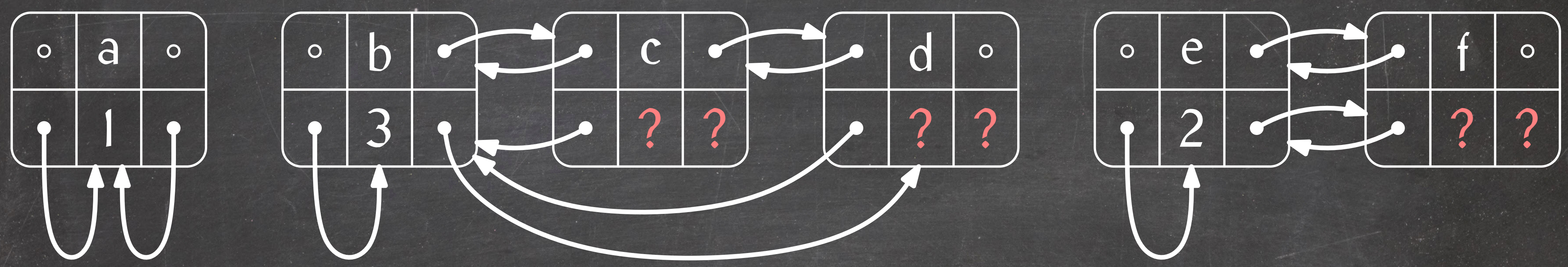

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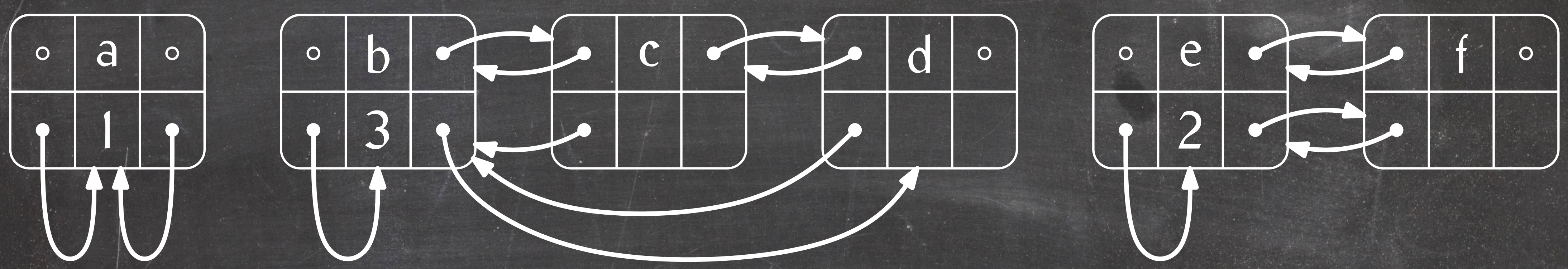
Find

D.find(x)

```
return x.head.key
```



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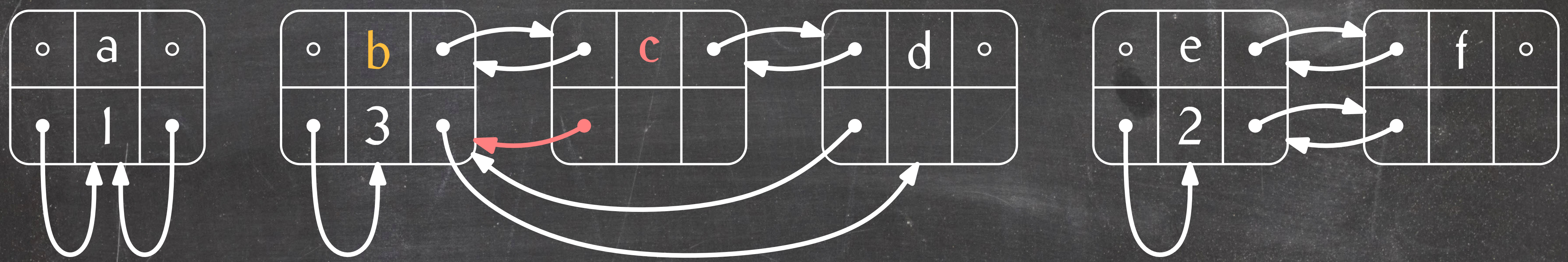

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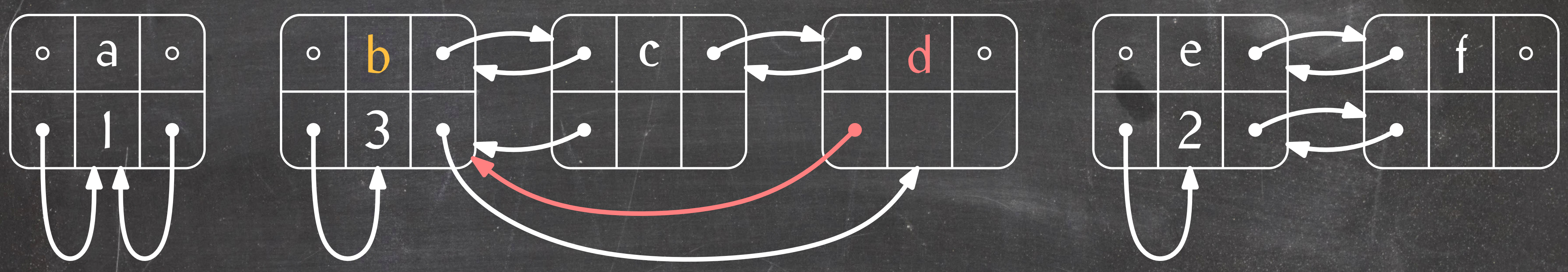

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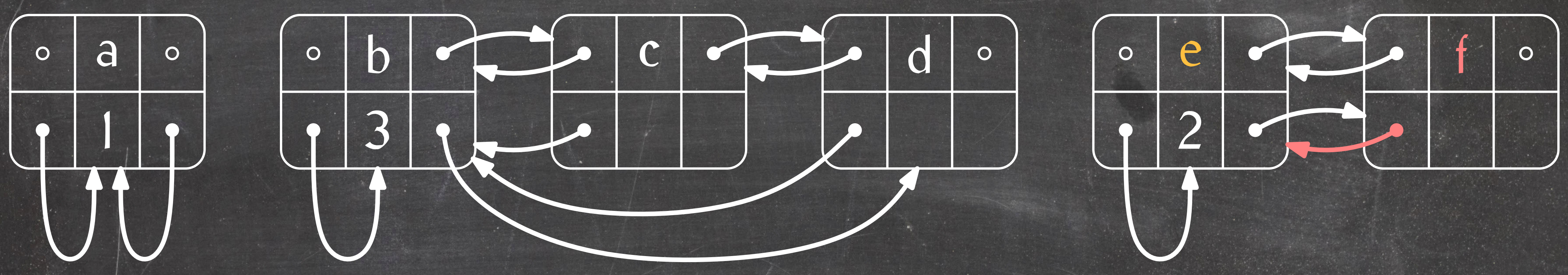

D.find(c) = b

D.find(d) = b

D.find(e) = e

Union

D.union(x, y)

```
if x.head.listSize < y.head.listSize
    then swap x and y
y.head.pred = x.head.tail
x.head.tail.succ = y.head
x.head.listSize = x.head.listSize + y.head.listSize
x.head.tail = y.head.tail
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while z ≠ null
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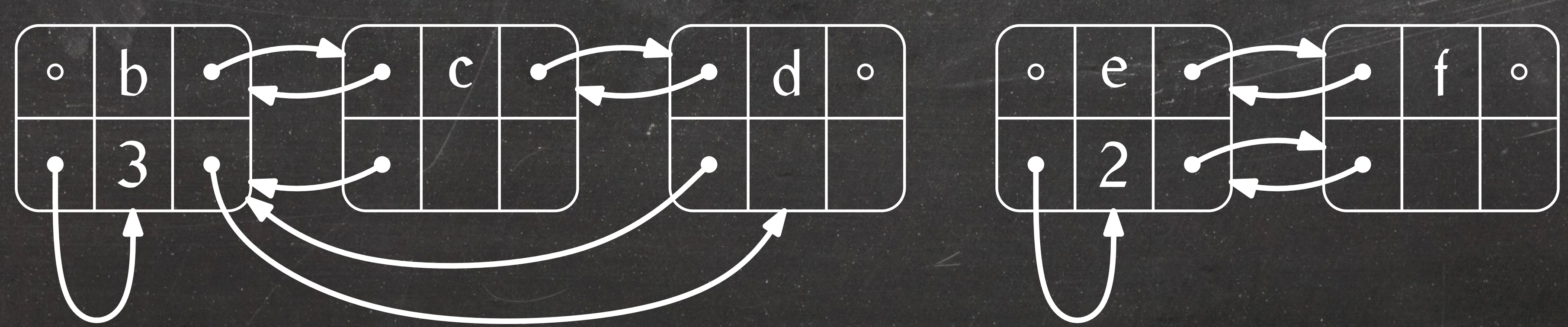

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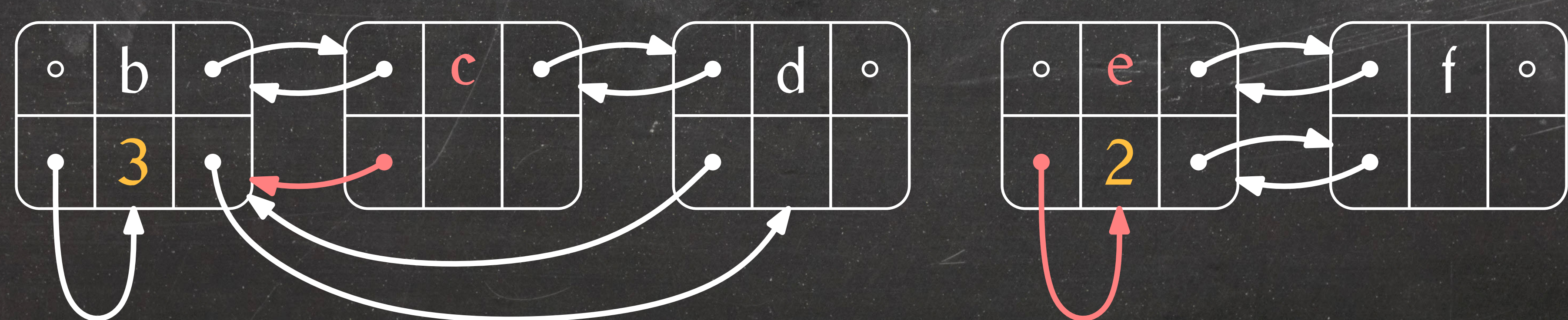

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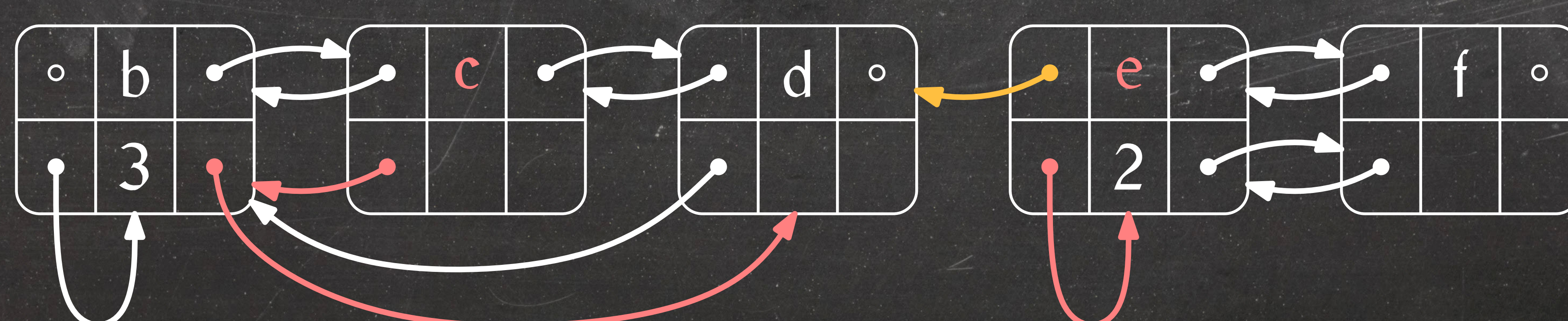

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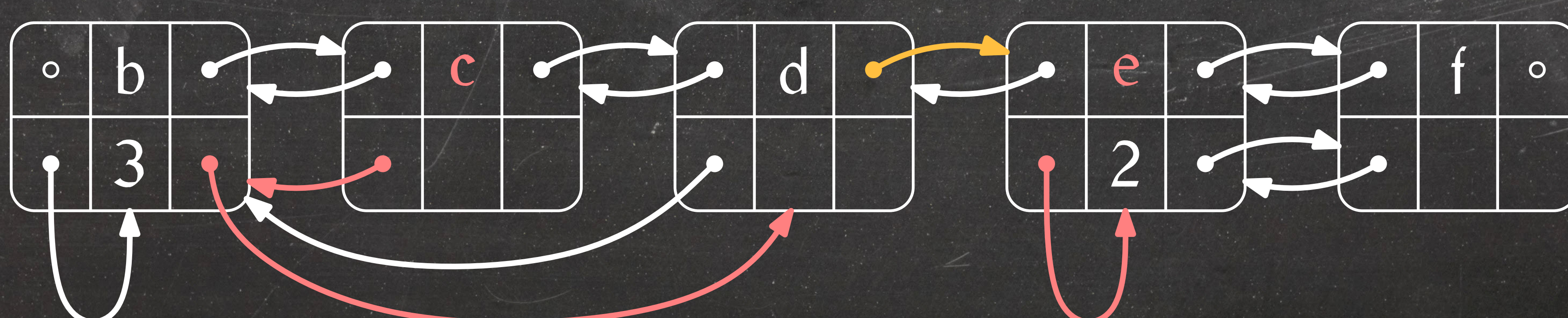

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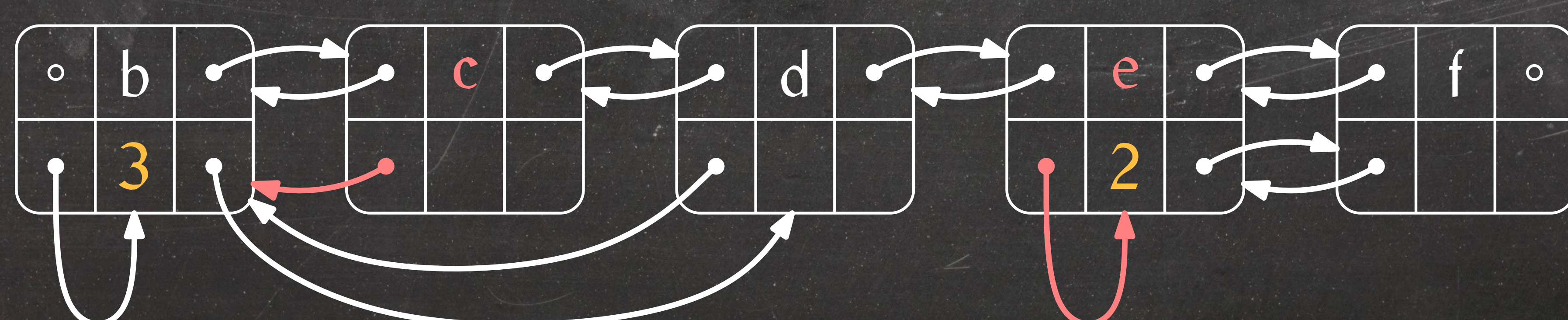

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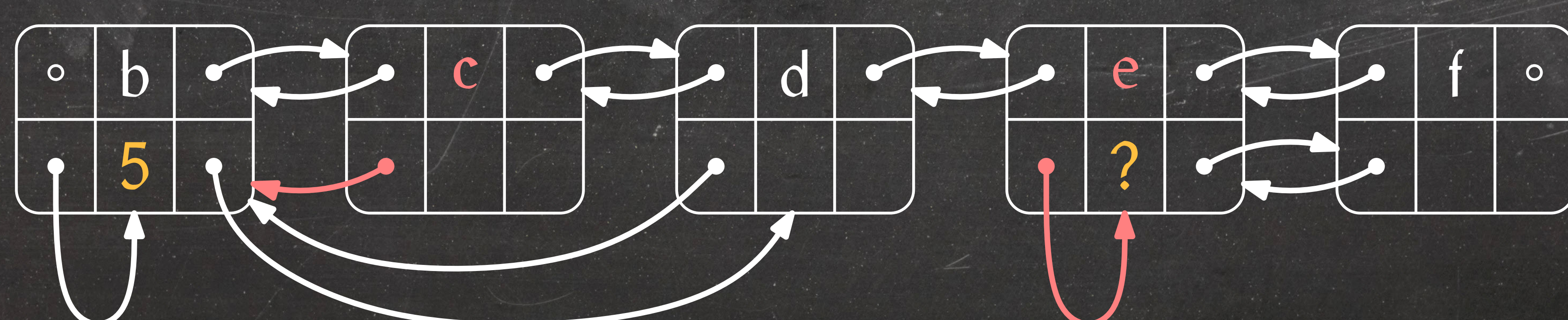

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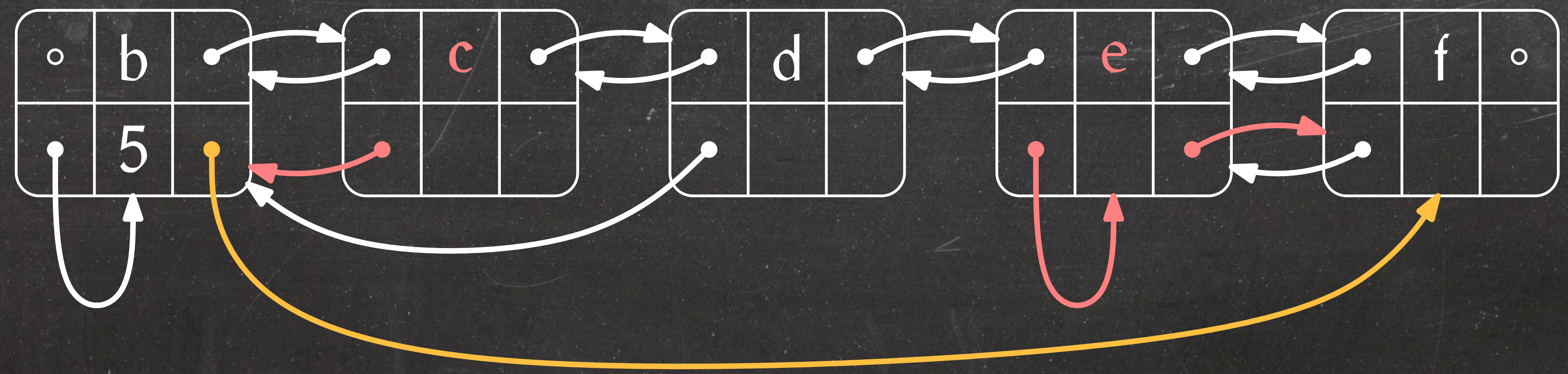

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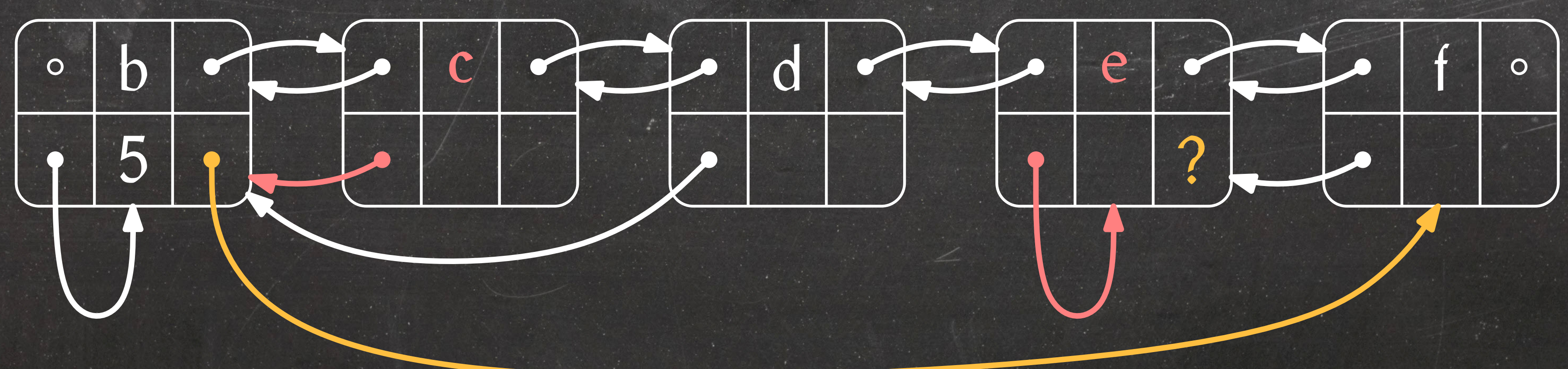

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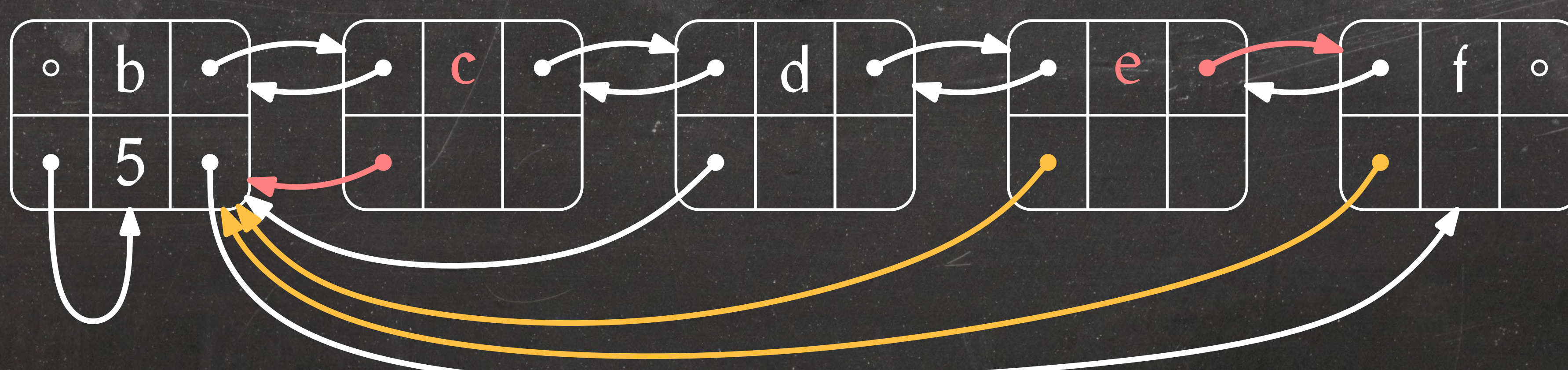

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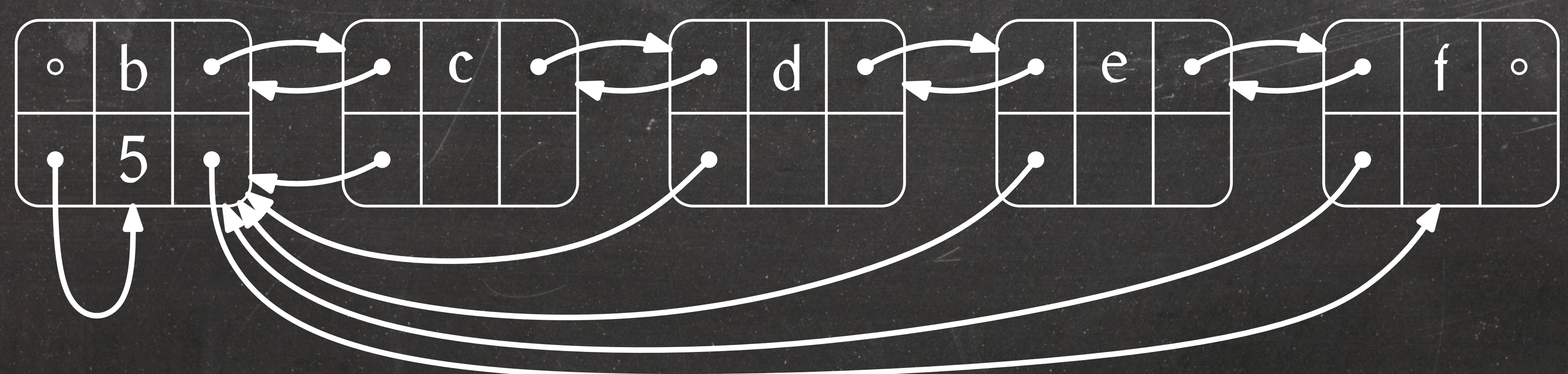

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Inductive step: $i > 0$.

- Consider the ith Union operation where x is in the smaller list.
- Let S_1 and S_2 be the two unioned lists and assume $x \in S_2$.
- Then $|S_1| \geq |S_2| \geq 2^{i-1}$.
- Thus, $|S_1 \cup S_2| \geq 2^i$.

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Corollary: $c(x) \leq \lg n$ for all $x \in S$.

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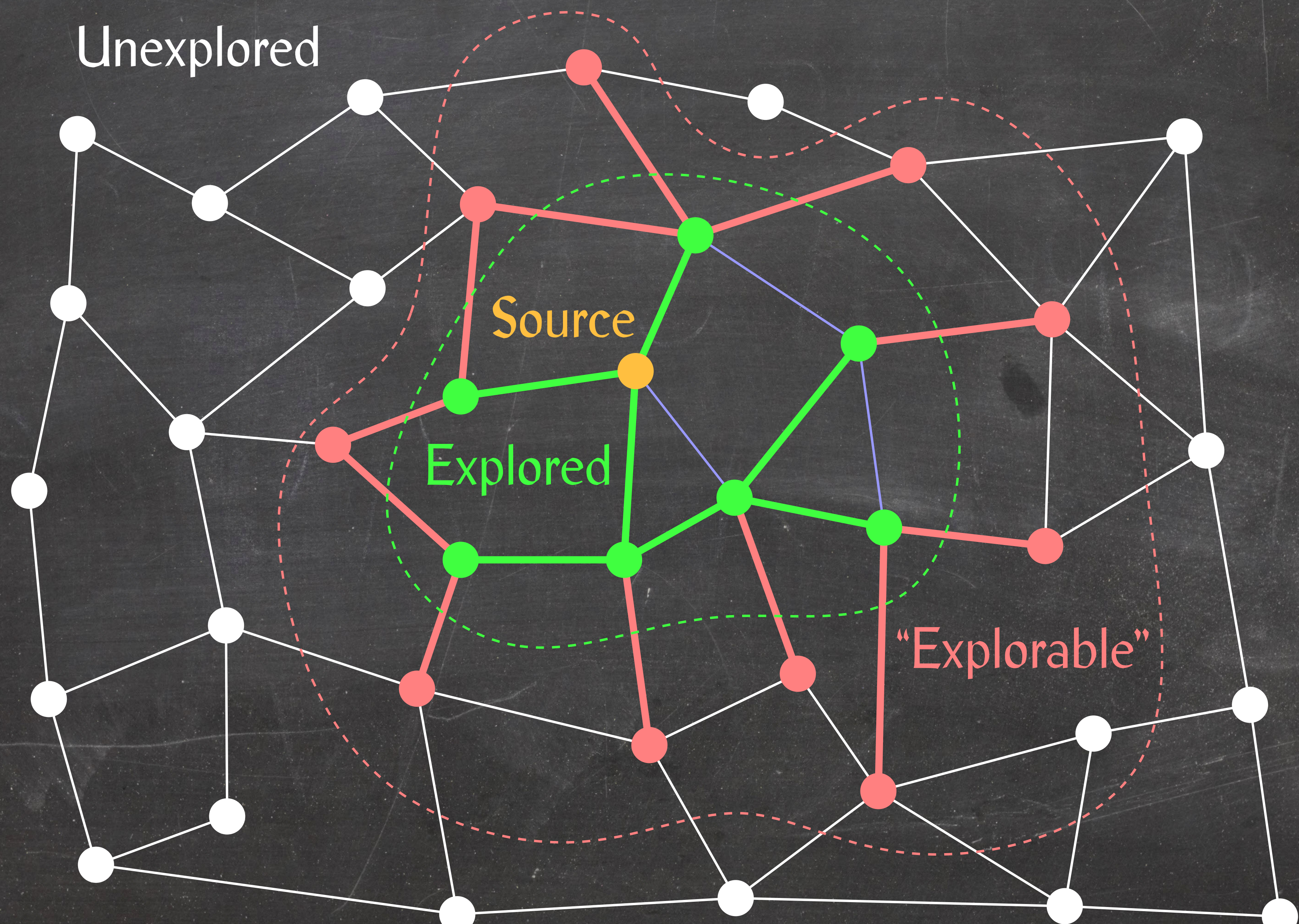
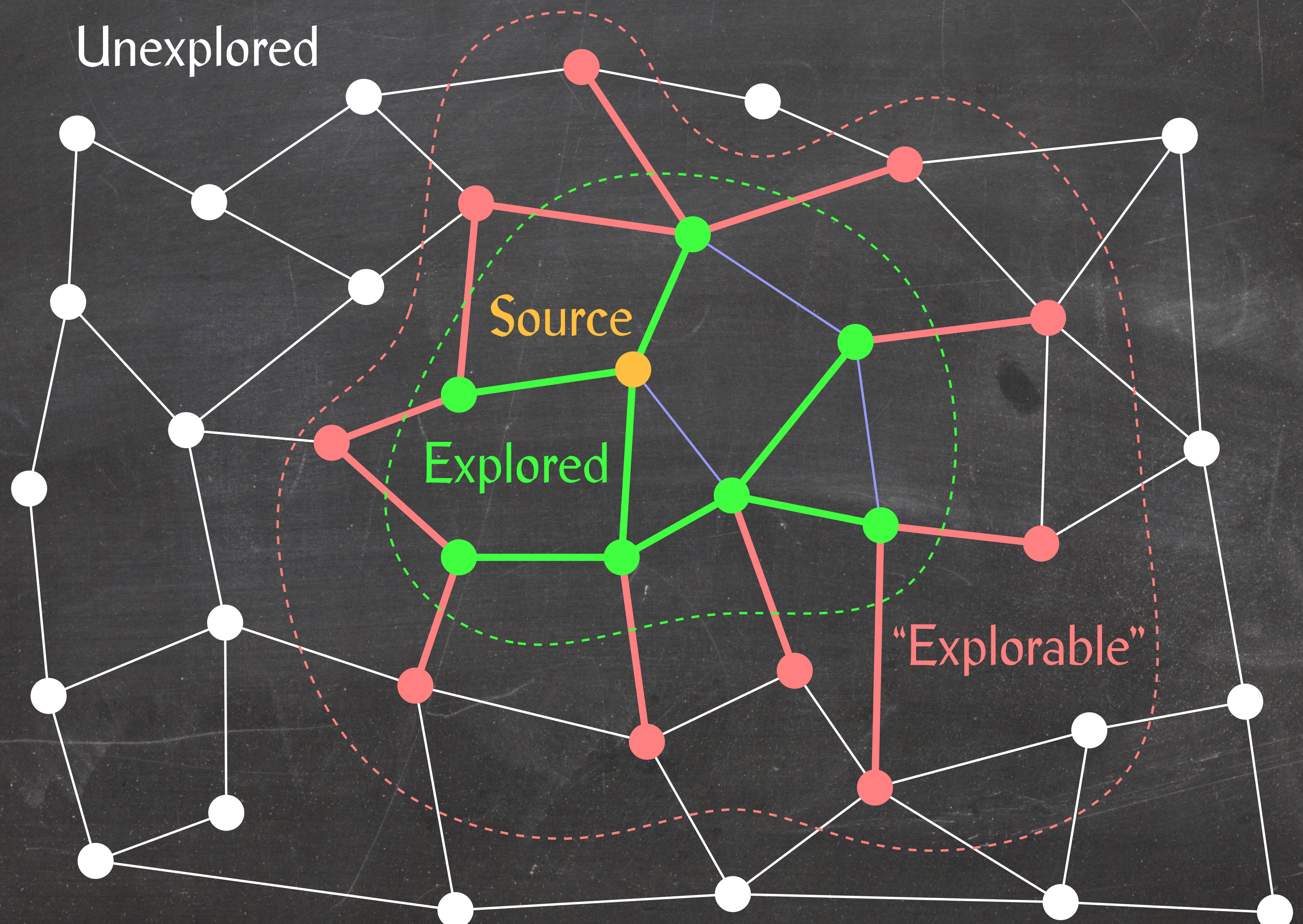
If the graph is connected, then $m \geq n - 1$, so the running time simplifies to $O(m \lg m)$.

The Cut Theorem And Graph Traversal



The Cut Theorem And Graph Traversal

If there exists an MST containing all green edges, then there exists an MST containing all green edges and the cheapest red edge.



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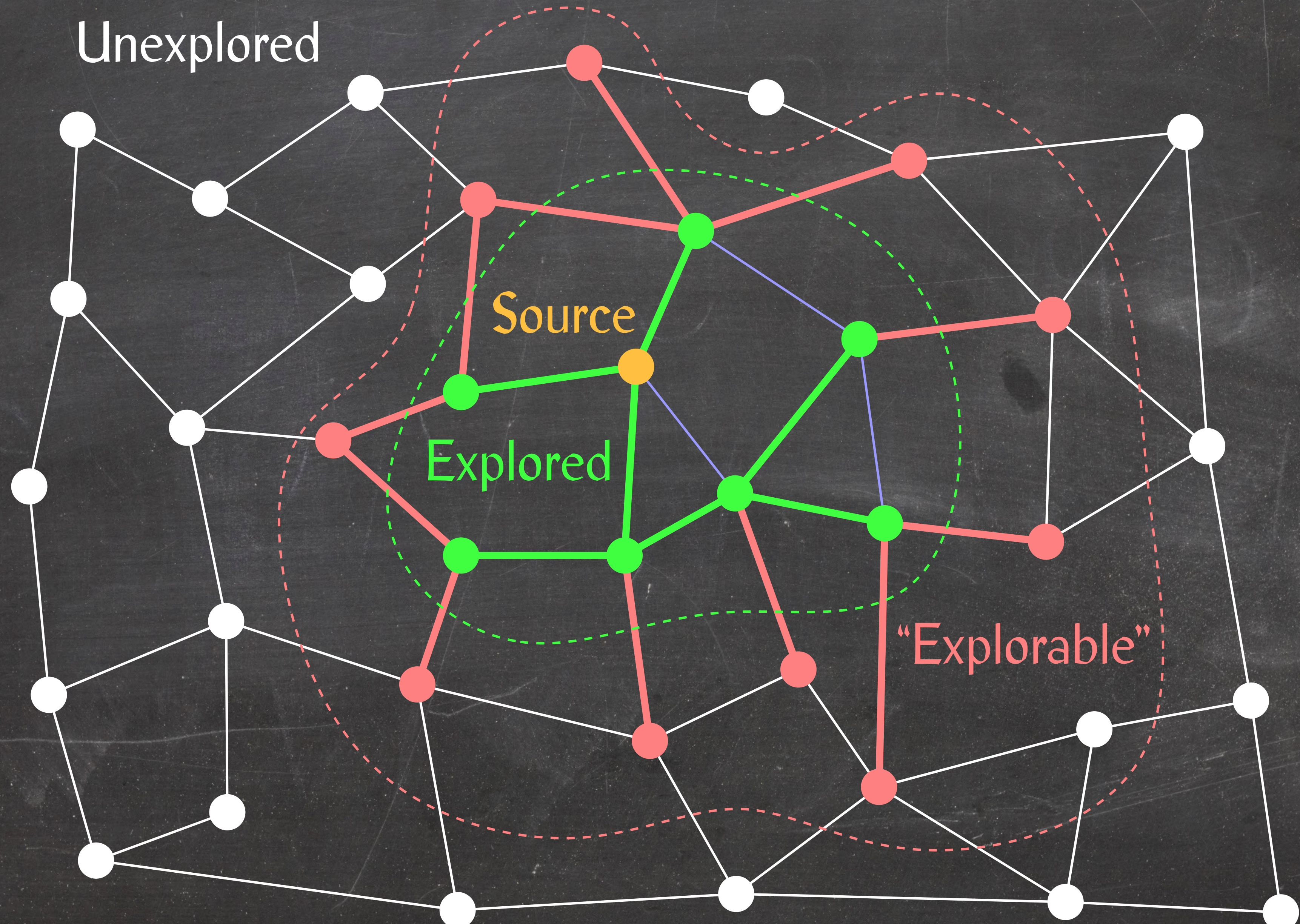

Cut: U = explored vertices, W = V \ U

Prim's Algorithm

Prim(G)

```
T = (V, ∅)
mark all vertices of G as unexplored
mark an arbitrary vertex s as explored
while not all vertices are explored
    do pick the cheapest edge e with exactly one unexplored endpoint v
       mark v as explored
       add e to T
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By induction on the number of edges in T, there exists an MST $T^* \supseteq T$.

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By induction on the number of edges in T, there exists an MST $T^* \supseteq T$.

Once T is connected, we have $T^* = T$.

The Abstract Data Type Priority Queue

Operations:

Q.insert(x, p):	Insert element x with priority p
Q.delete(x):	Delete element x
Q.findMin():	Find and return the element with minimum priority
Q.deleteMin():	Delete the element with minimum priority and return it
Q.decreaseKey(x, p):	Change the priority p_x of x to min(p, p_x)

Delete and DecreaseKey assume they're given a pointer to the place in Q where x is stored.

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Example: A binary heap is a priority queue supporting all operations in $O(\lg |Q|)$ time.

Prim's Algorithm Using A Priority Queue

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mark an arbitrary vertex s as explored
Q = an empty priority queue
for every edge (s, v) incident to s
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while not Q.isEmpty()
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                add edge (u, v) to T
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Prim's Algorithm Using A Priority Queue

Invariant: Q contains all edges with exactly one unexplored endpoint.

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This version of Prim's algorithm takes $O(m \lg m)$ time:

Every edge is inserted into Q once.

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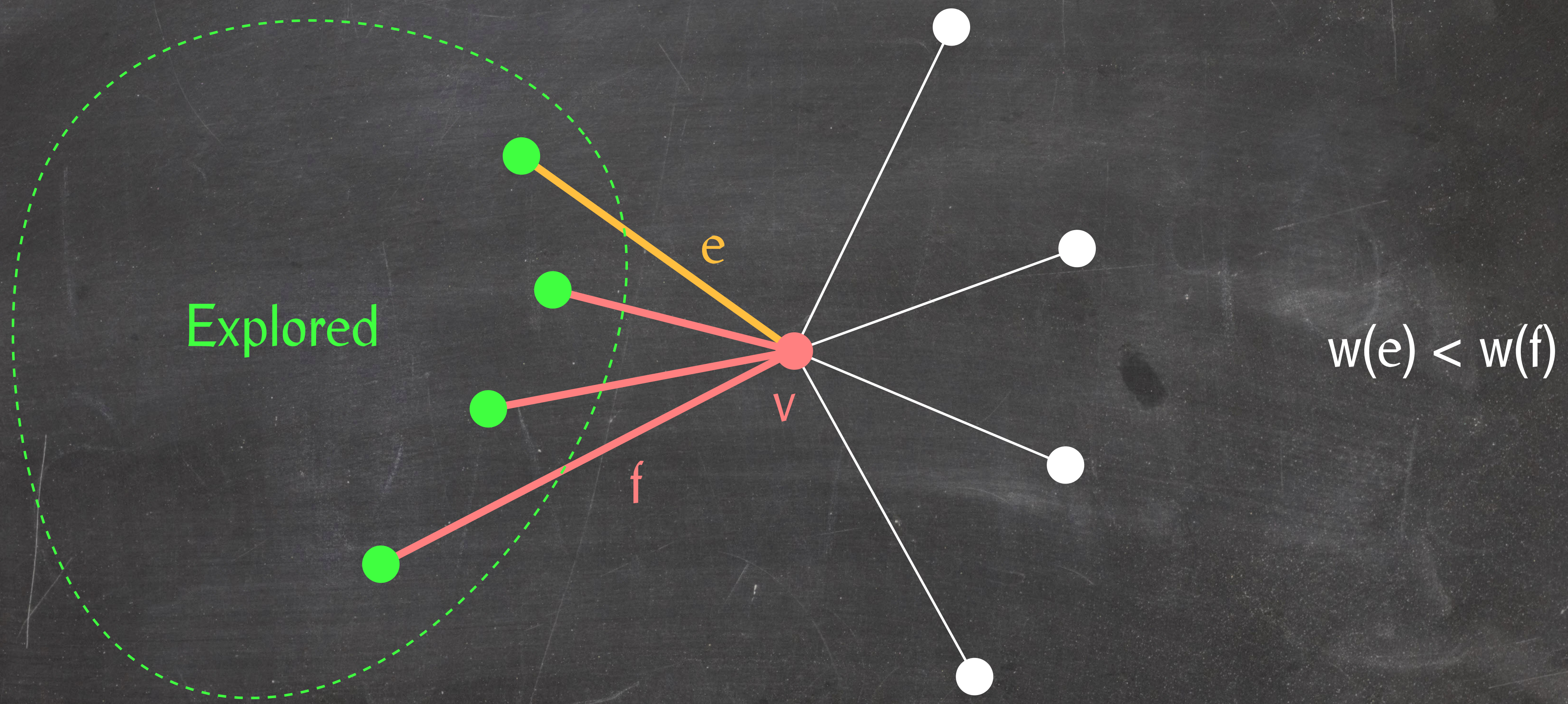
$\Rightarrow$ Every edge is removed from Q once.

$\Rightarrow$ 2m priority queue operations.

Most Edges In Q Are Useless

Observation: Of all the edges connecting an unexplored vertex to explored vertices only the cheapest has a chance of being added to the MST.



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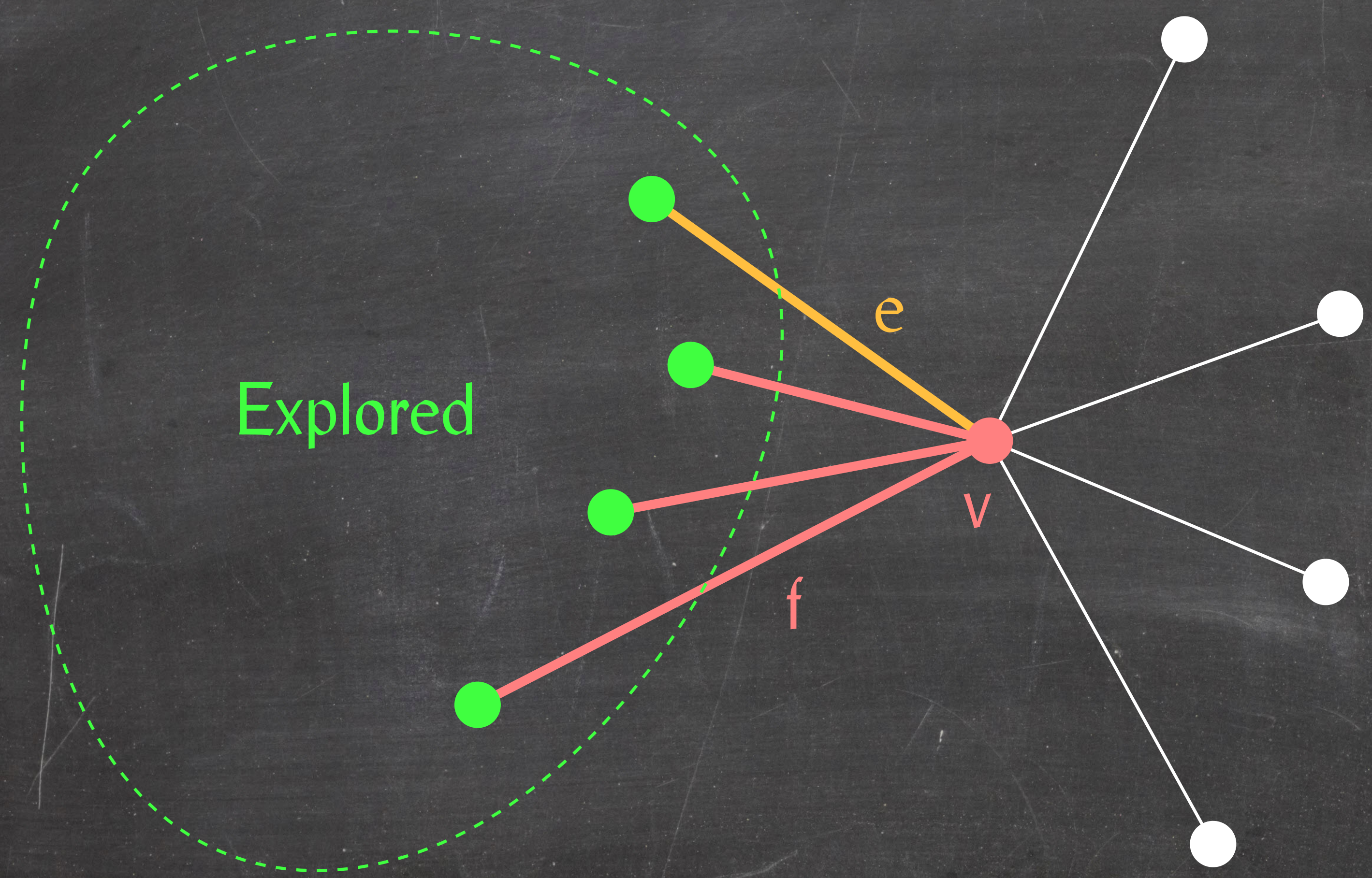

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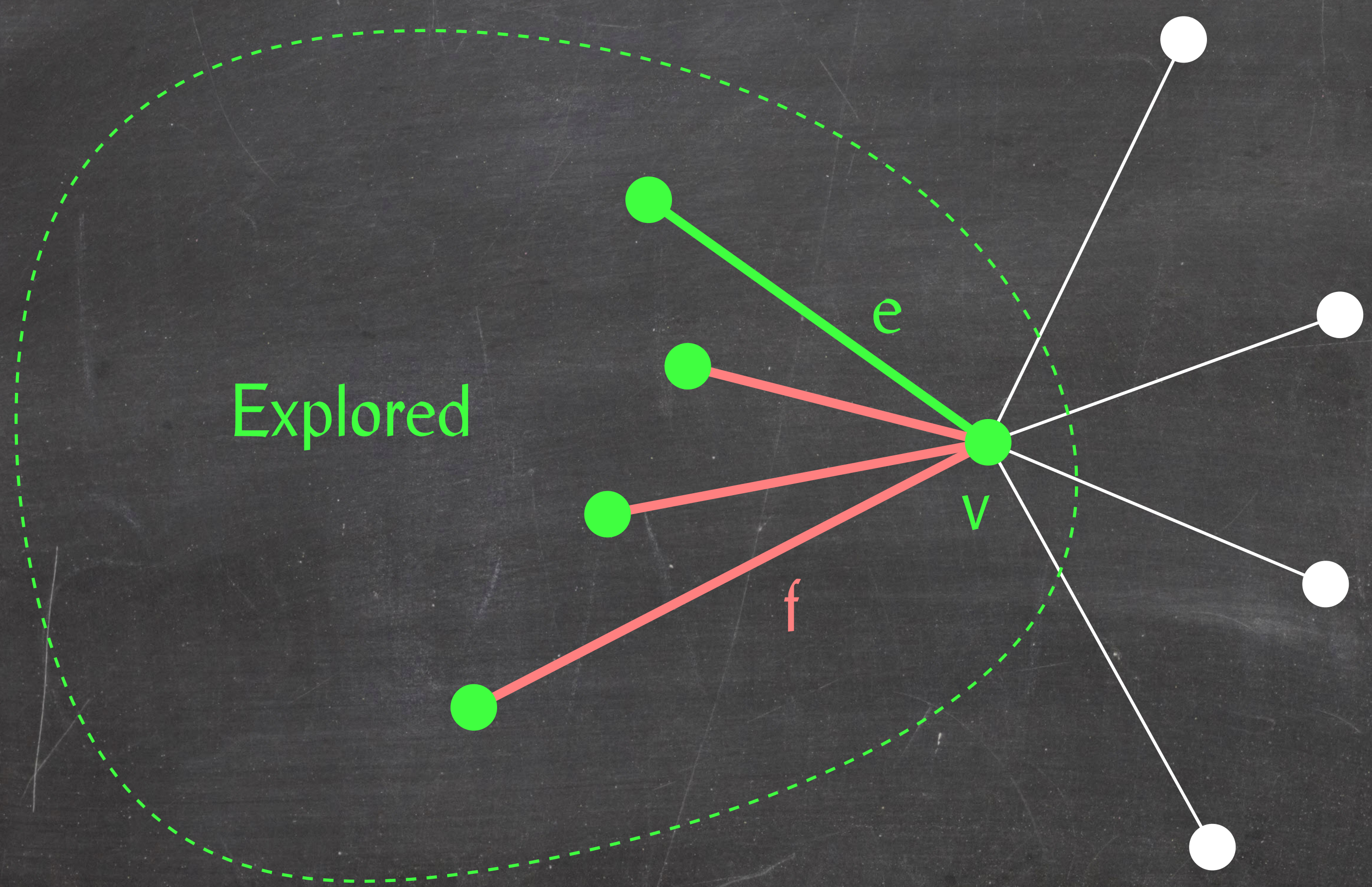

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While v is unexplored, all red and orange edges are in Q, so none of the red edges can be the first edge to be removed from Q.

After marking v as explored, both endpoints of red edges are explored, so they cannot be added to T either.

A Faster Version Of Prim's Algorithm

Prim(G)

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T = (V, ∅)
mark every vertex of G as unexplored
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for every edge (s, v) incident to s
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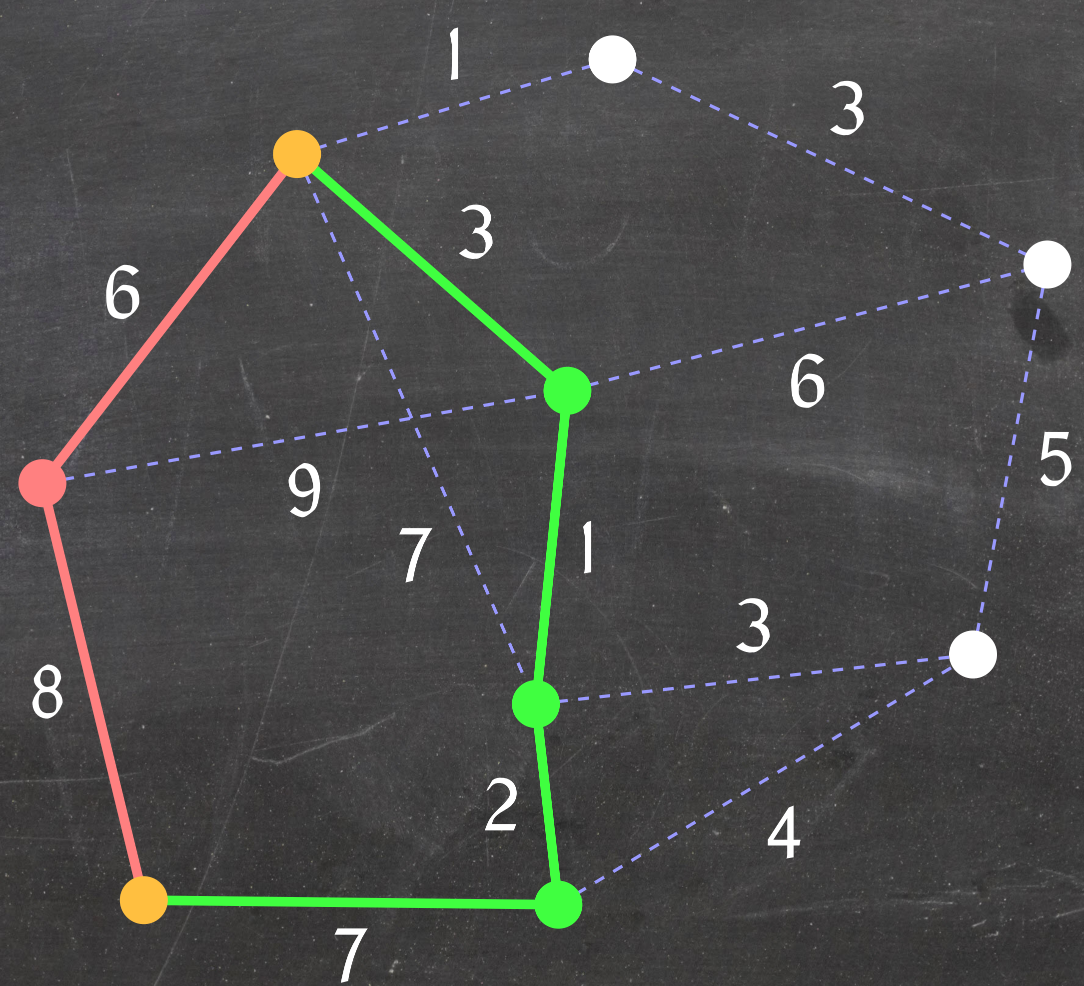
Given a graph $G = (V, E)$ and an assignment of weights (costs) to the edges of G, a shortest path from u to v is a path from u to v with minimum total edge weight among all paths from u to v.



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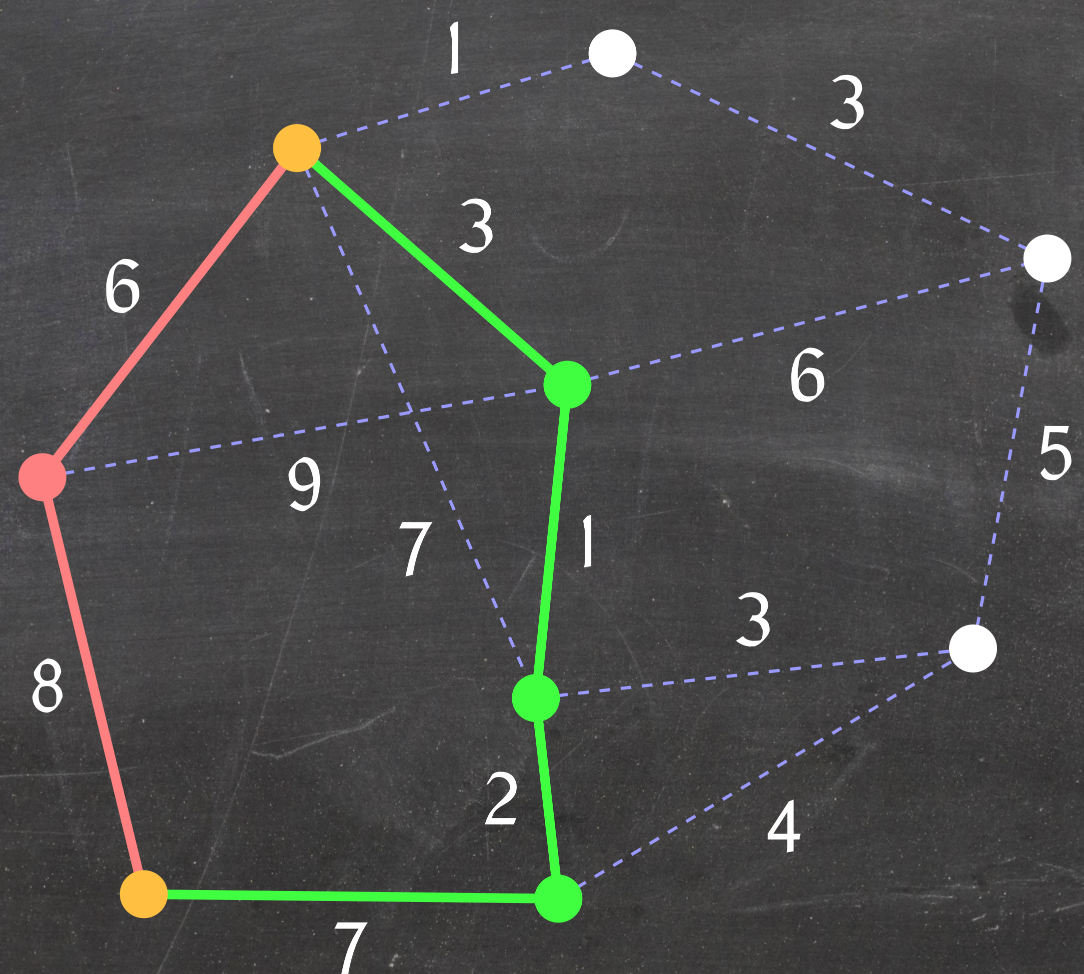

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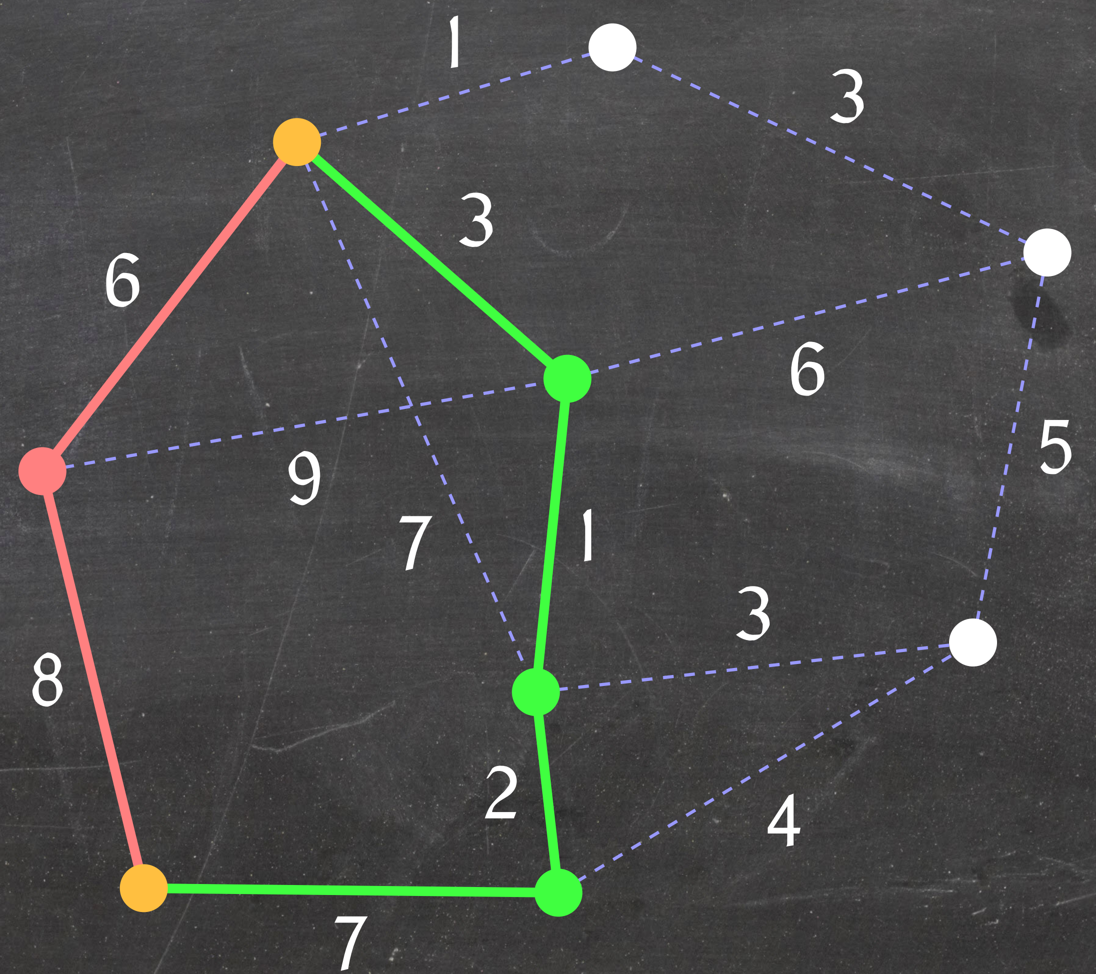

This is well-defined only if there is no negative cycle (cycle with negative total edge weight) that has a vertex on a path from u to v.

Optimal Substructure of Shortest Paths

For a path P and two vertices u and w in P, let P[u, w] be the subpath of P from u to w.

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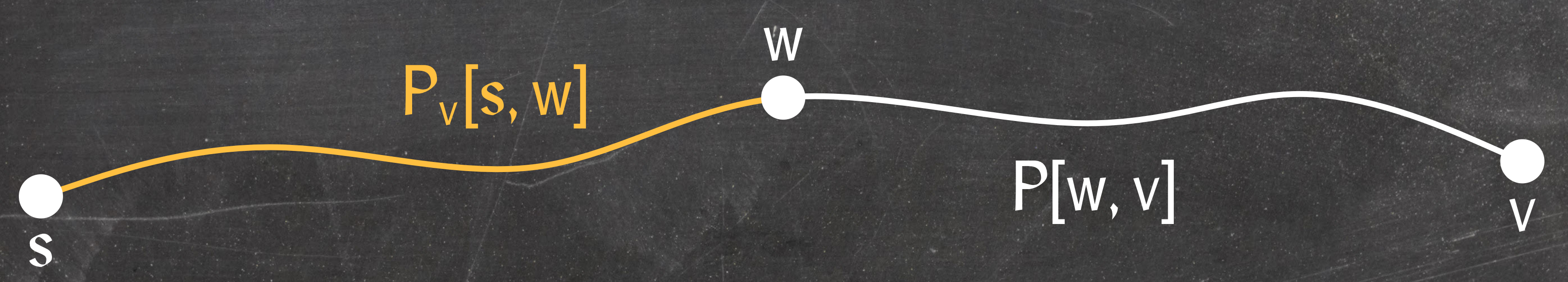

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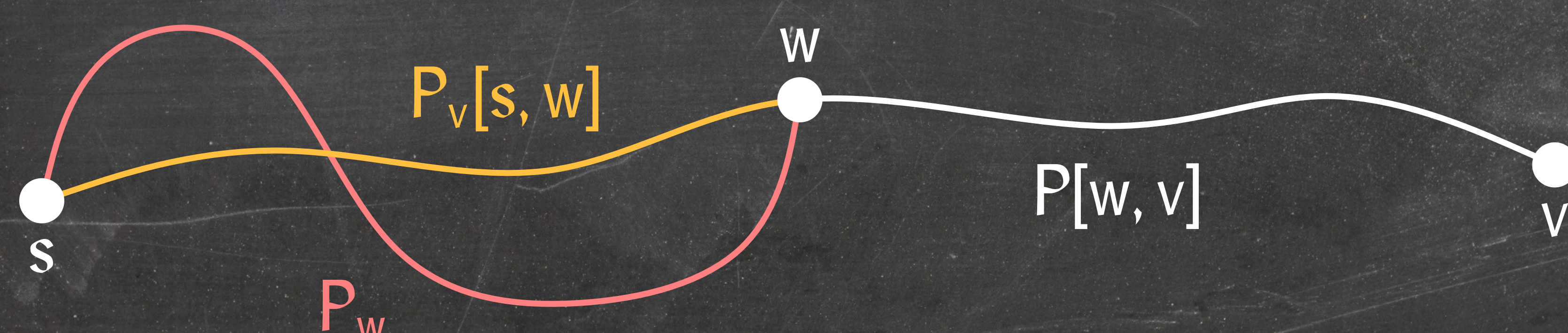

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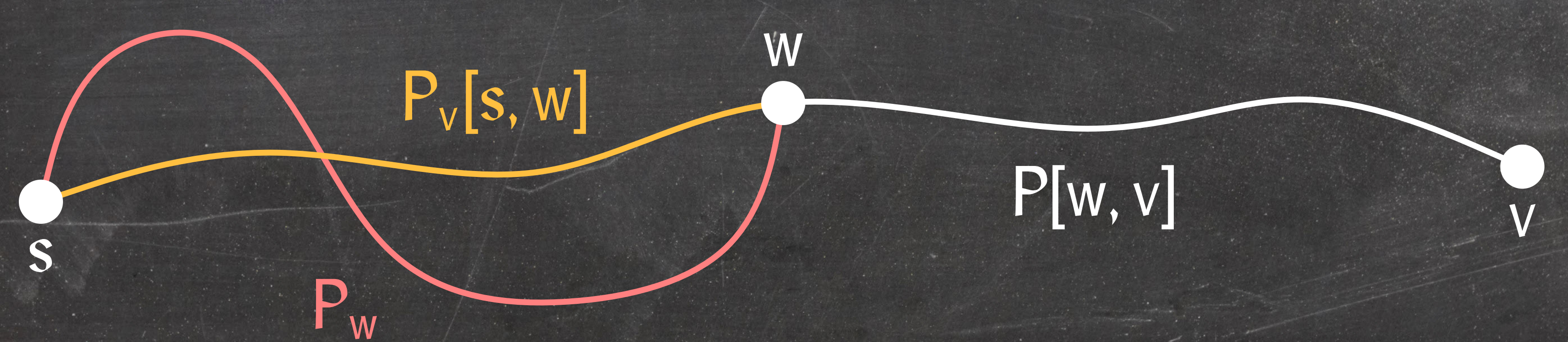

Then $w(P_w \circ P_v[w, v]) < w(P_v[s, w] \circ P_v[w, v]) = w(P_v)$, a contradiction because P_v is a shortest path from s to v.

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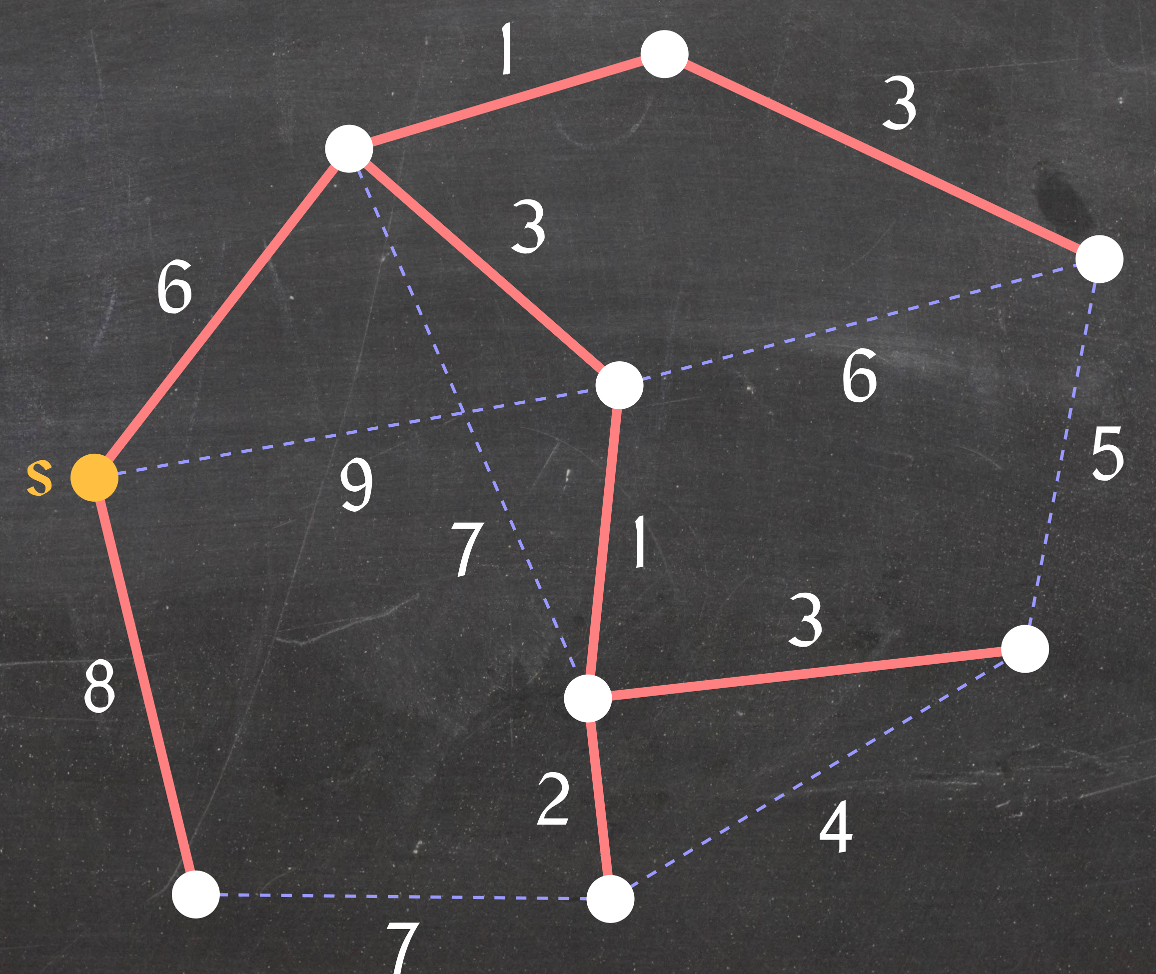
For a vertex $s \in G$, let $R(s)$ be the set of vertices reachable from s: for every vertex $v \in R(s)$, there exists a path from s to v.

Lemma: For every node $s \in G$, there exists a collection of paths $\mathcal{S} = \{P_v \mid v \in R(s)\}$ such that P_v is a shortest path from s to v and $\bigcup_{v \in R(s)} P_v$ is a tree.



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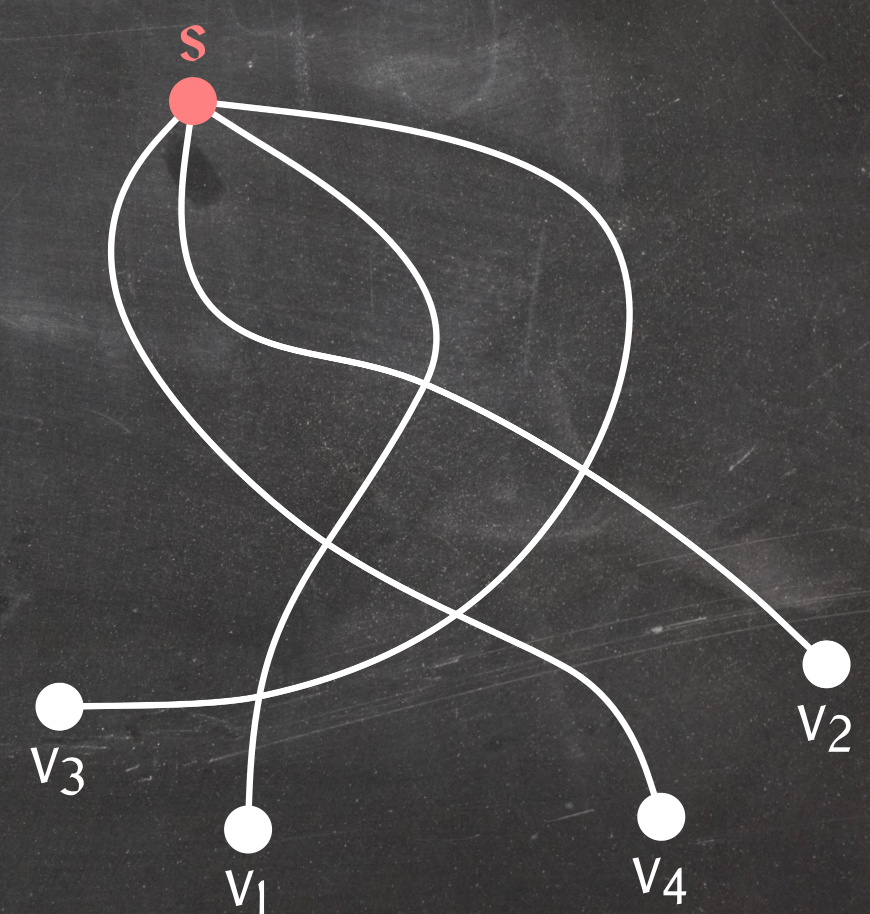

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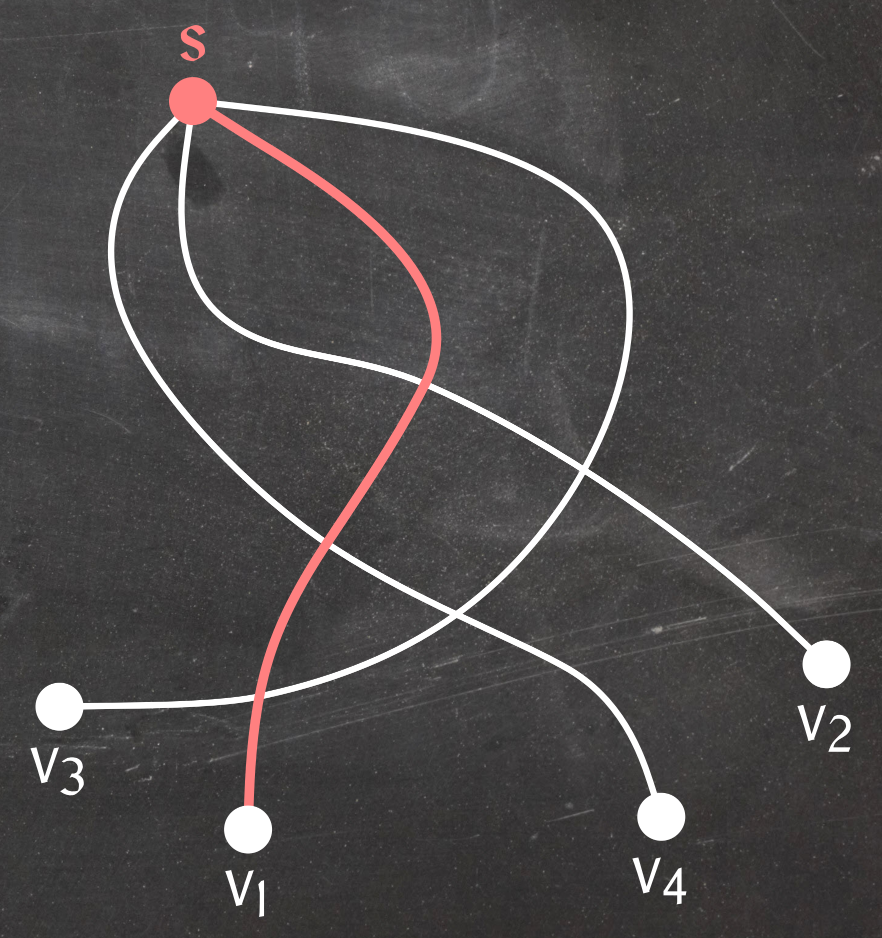

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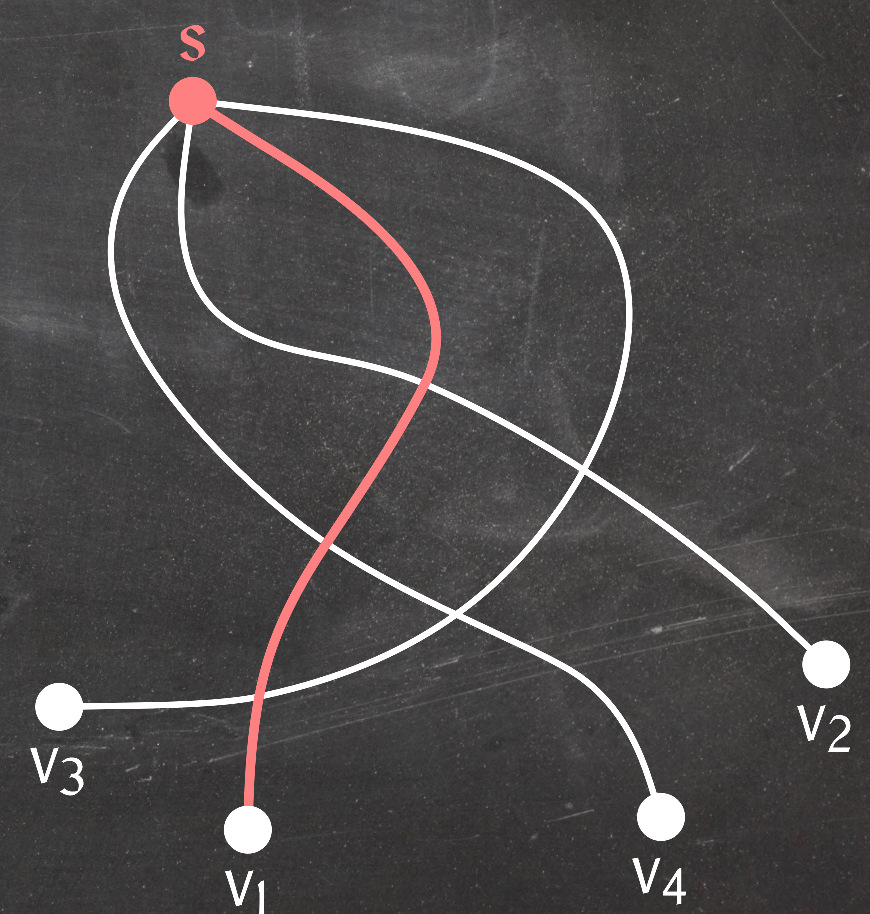

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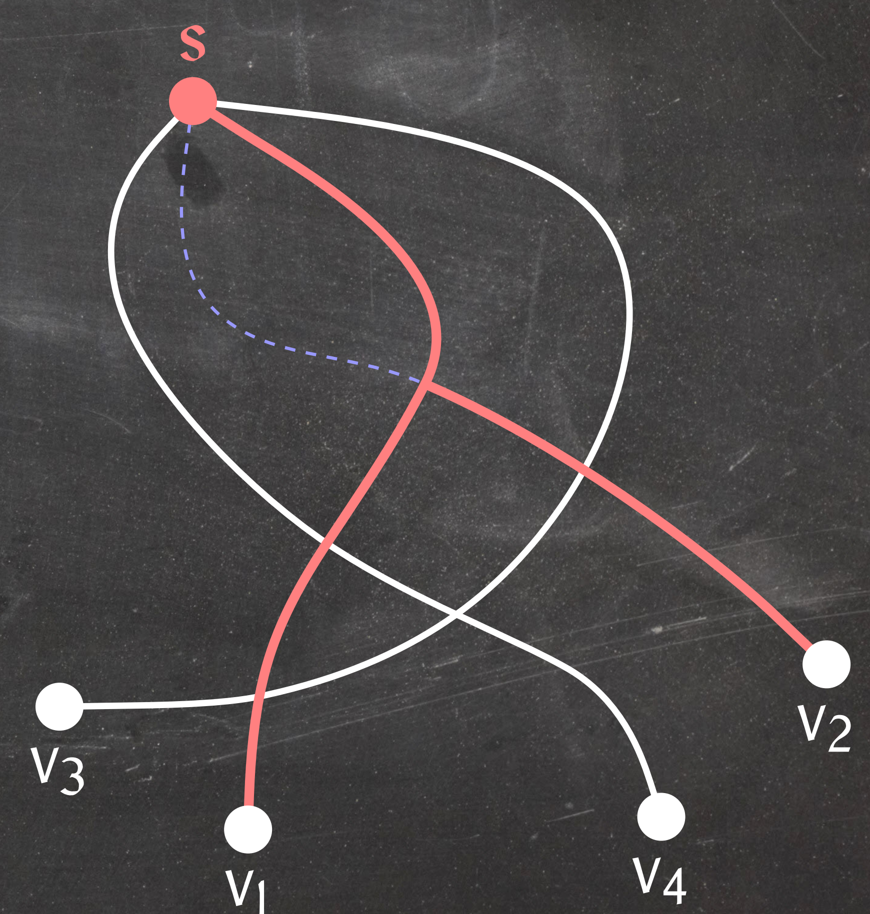

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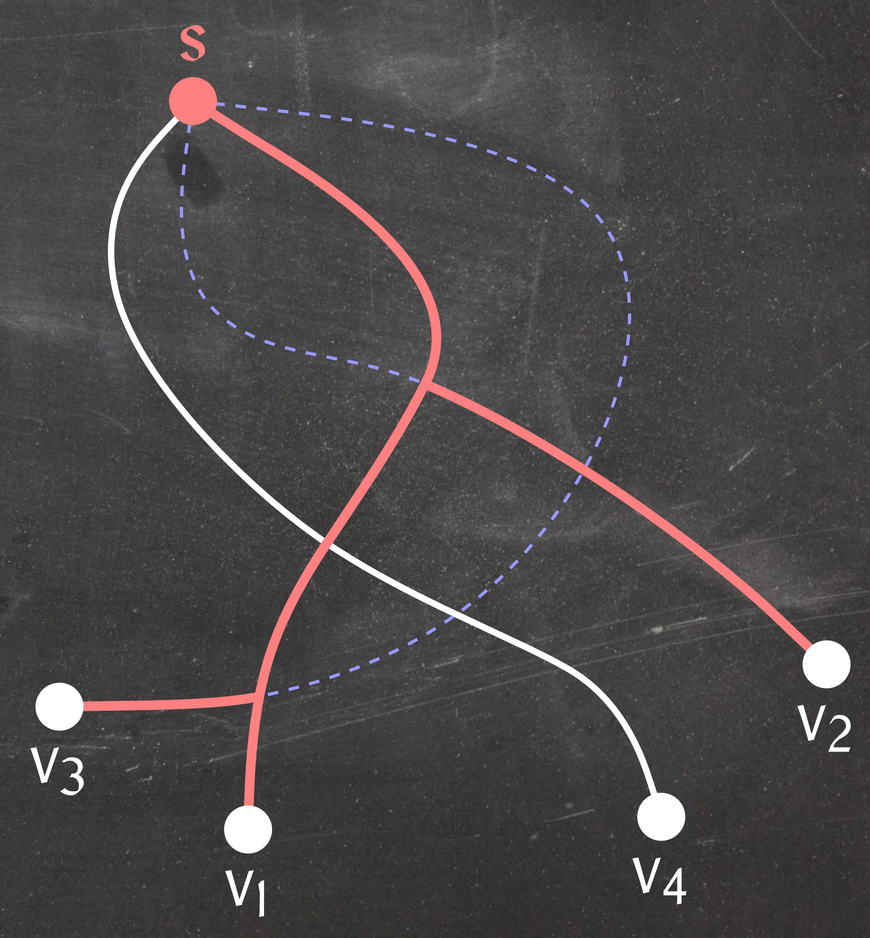

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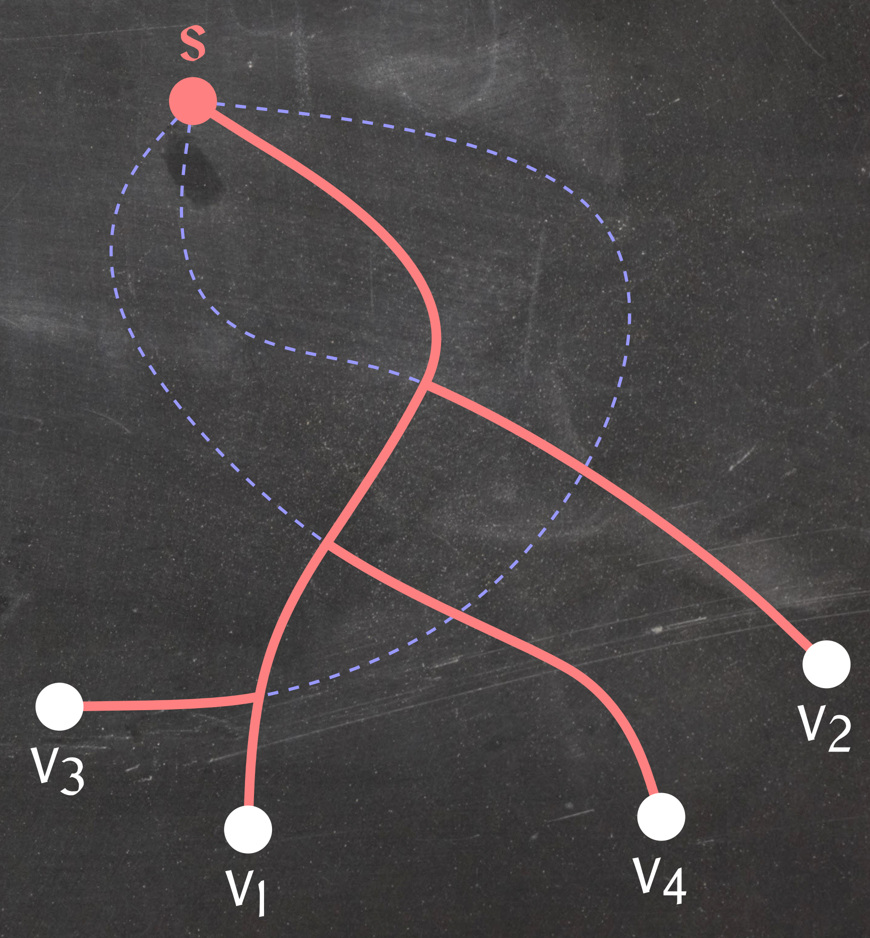

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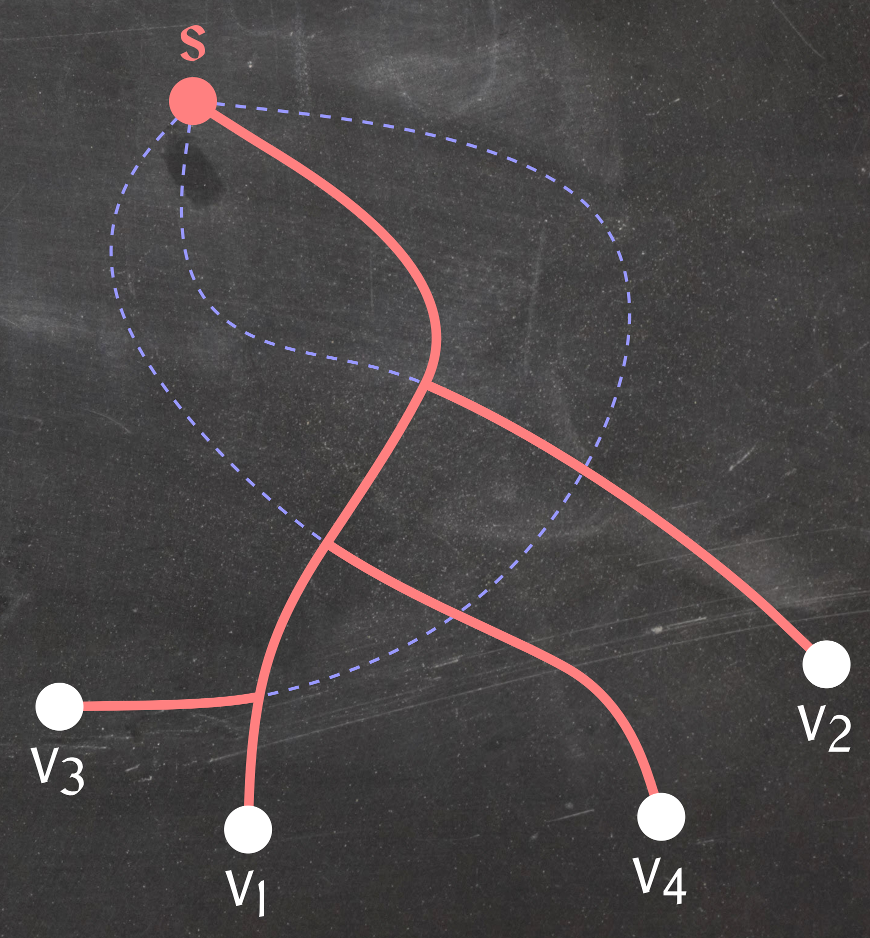

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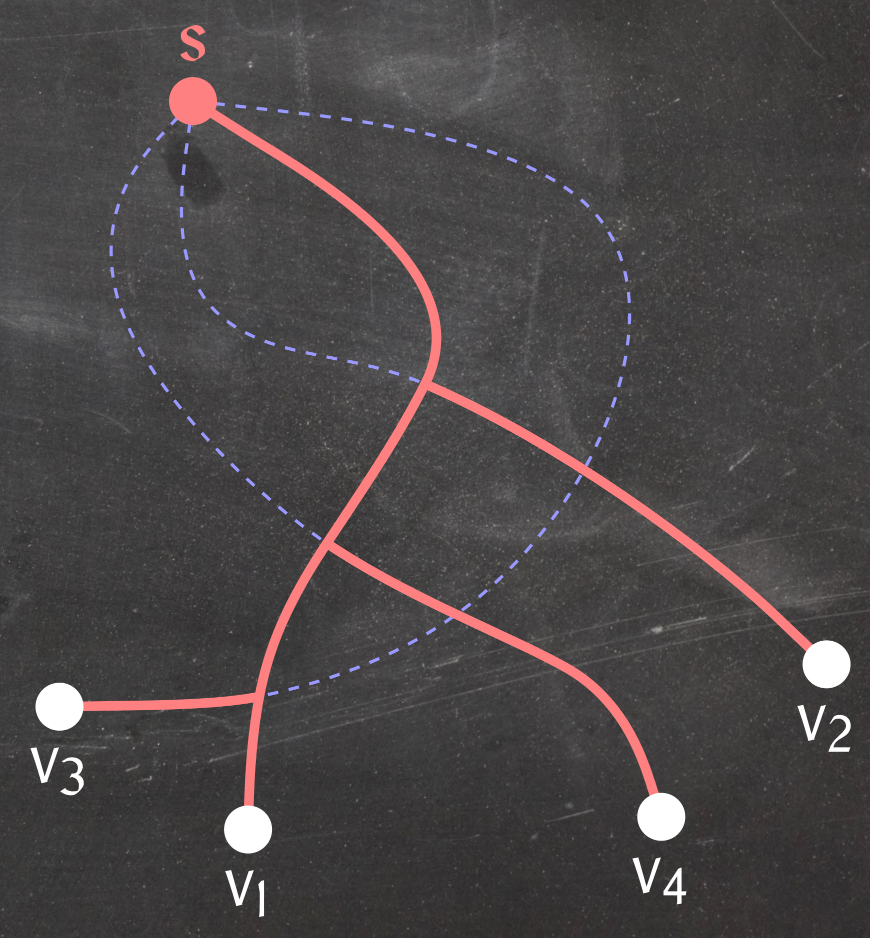
T_t is a tree:

- T_1 is a tree.
- T_i is obtained by adding a path to T_{i-1} that shares only one vertex with T_{i-1}.
- To create a cycle, the added path would have to share two vertices with T_{i-1}.



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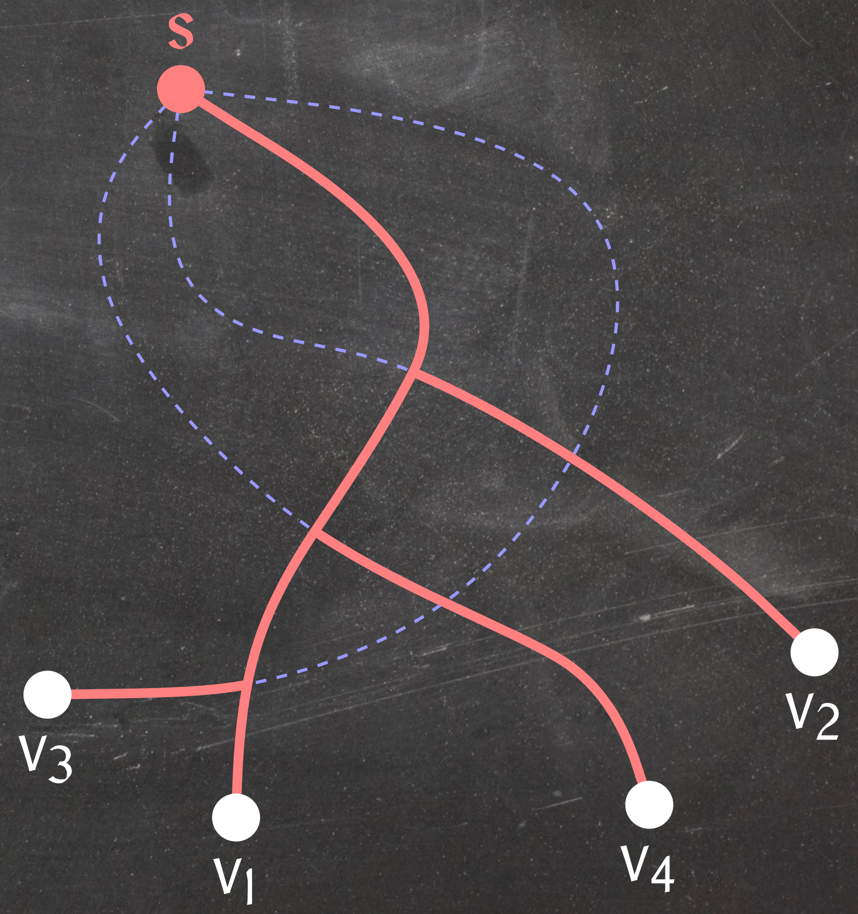

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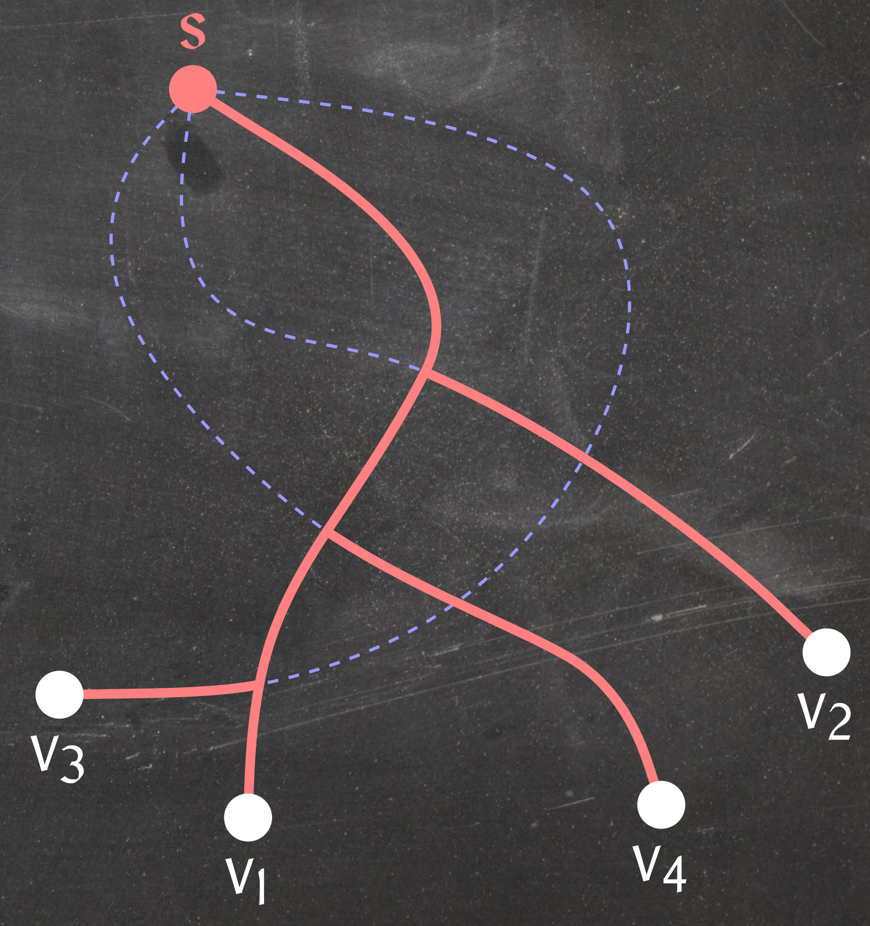
Prove by induction on i that $T_i[s, v]$ is a shortest path from s to v, for all $v \in T_i$.



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P_v is a shortest path from s to v, for all $v \in R(s)$.

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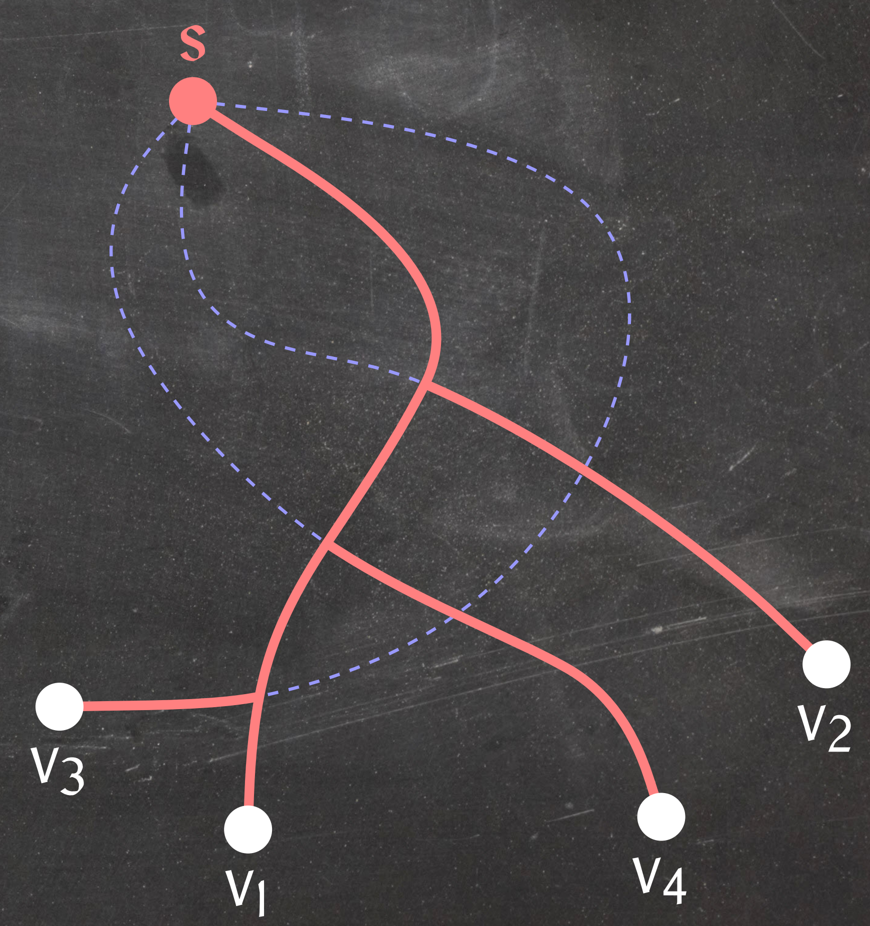

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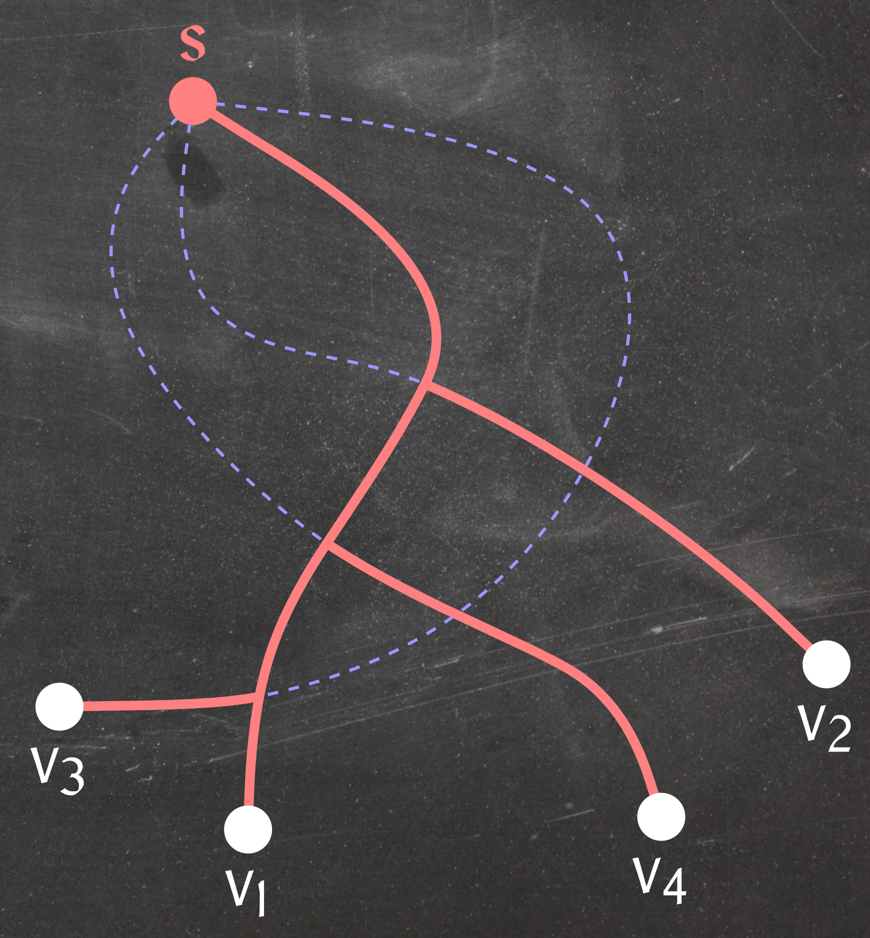

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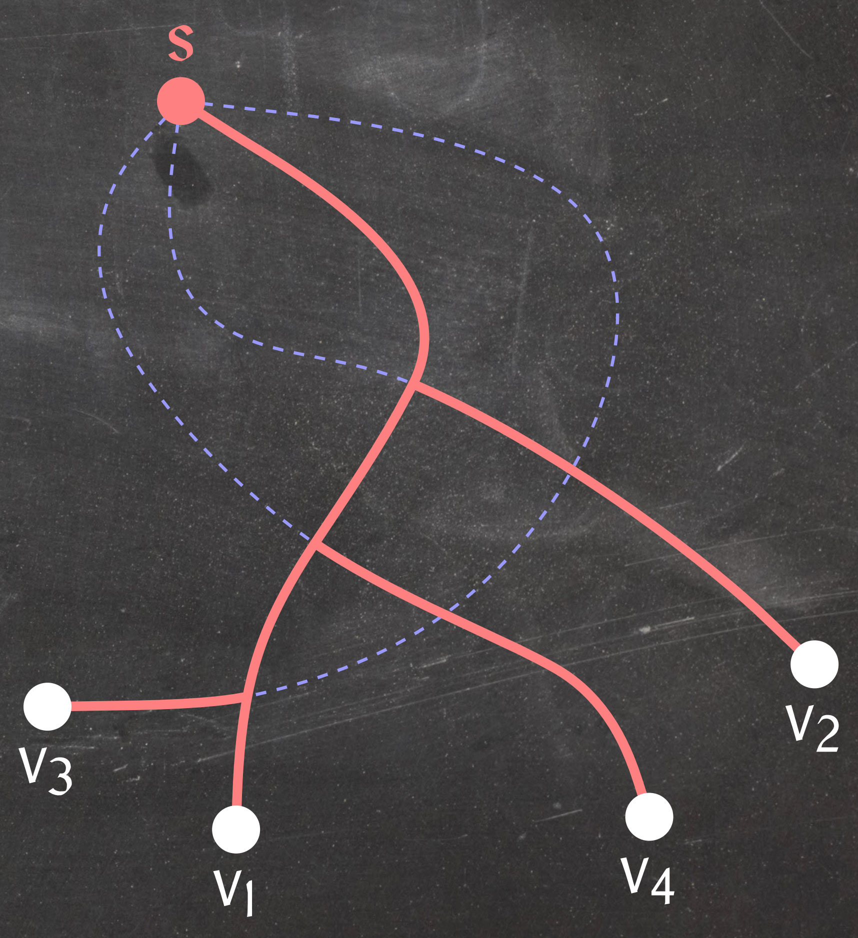

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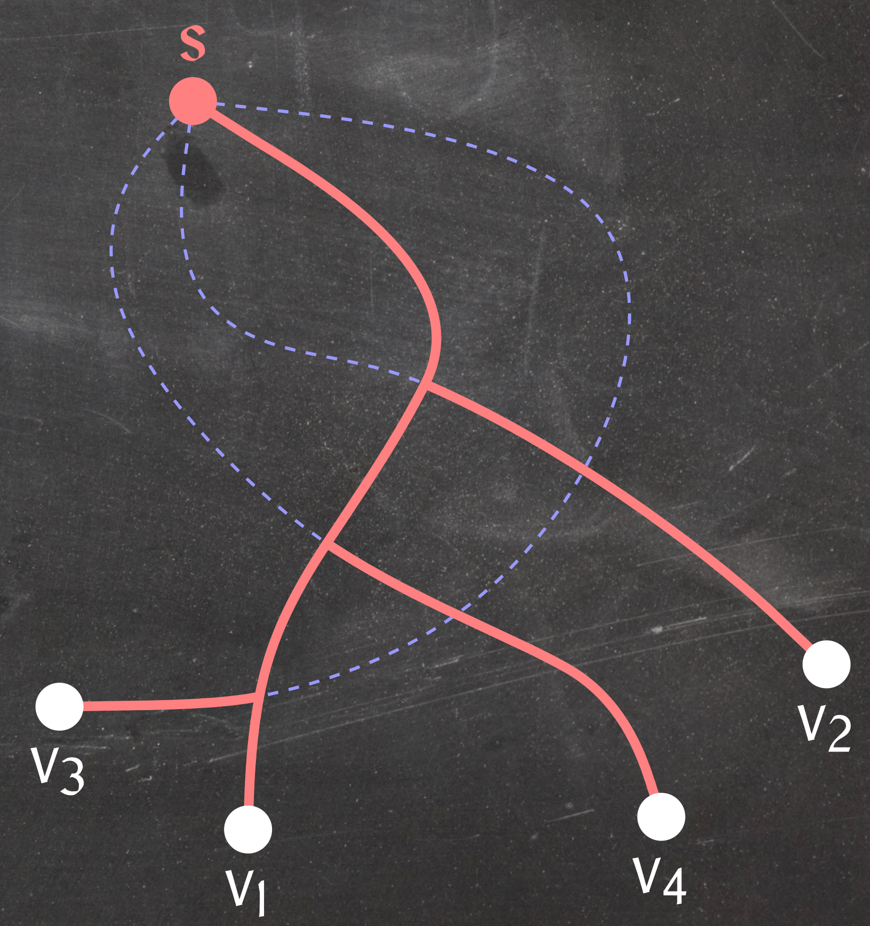

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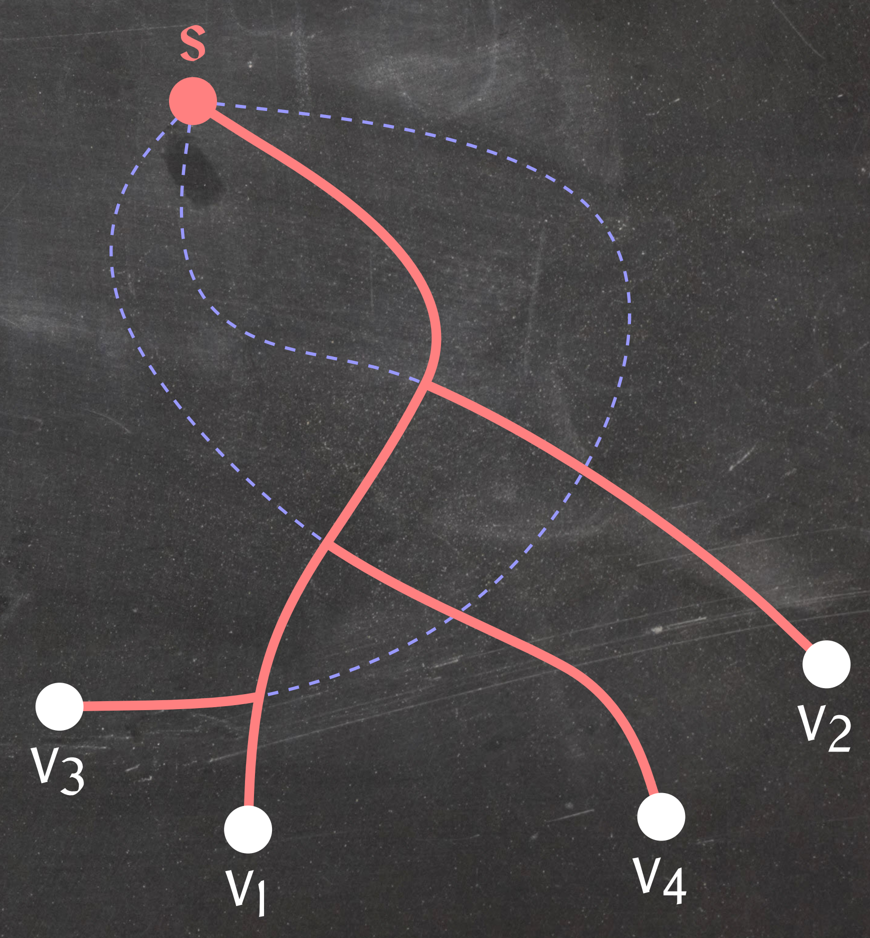

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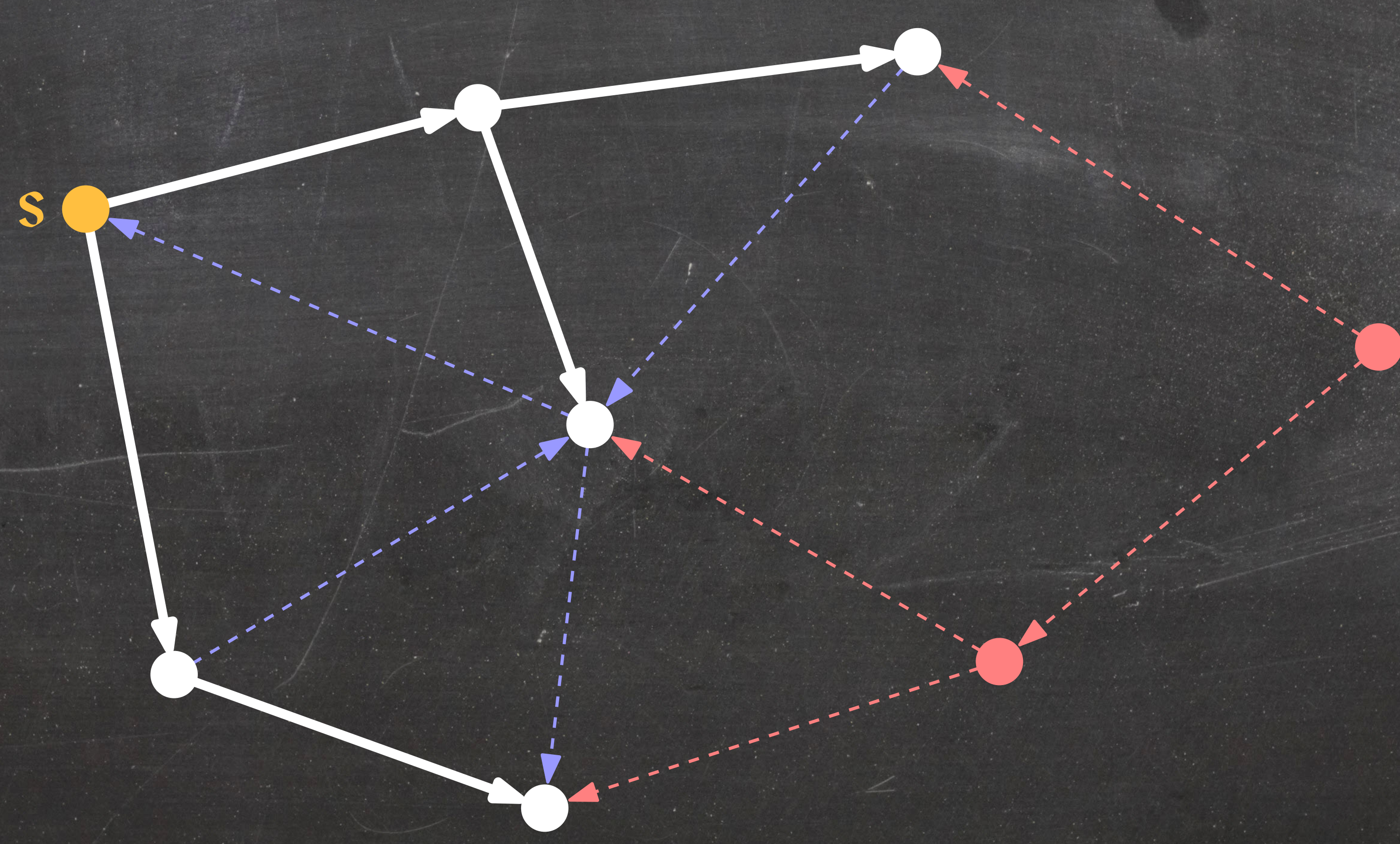

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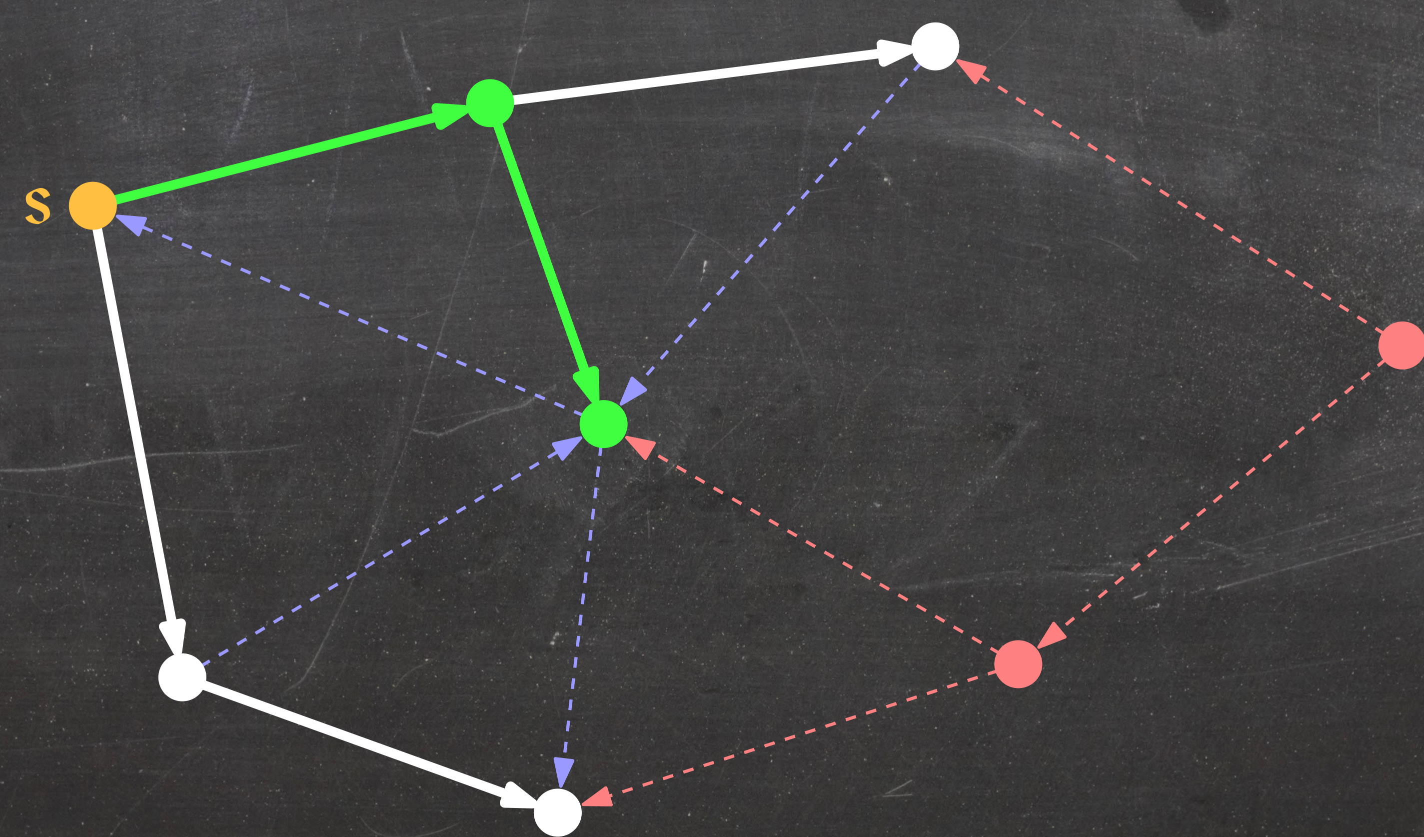

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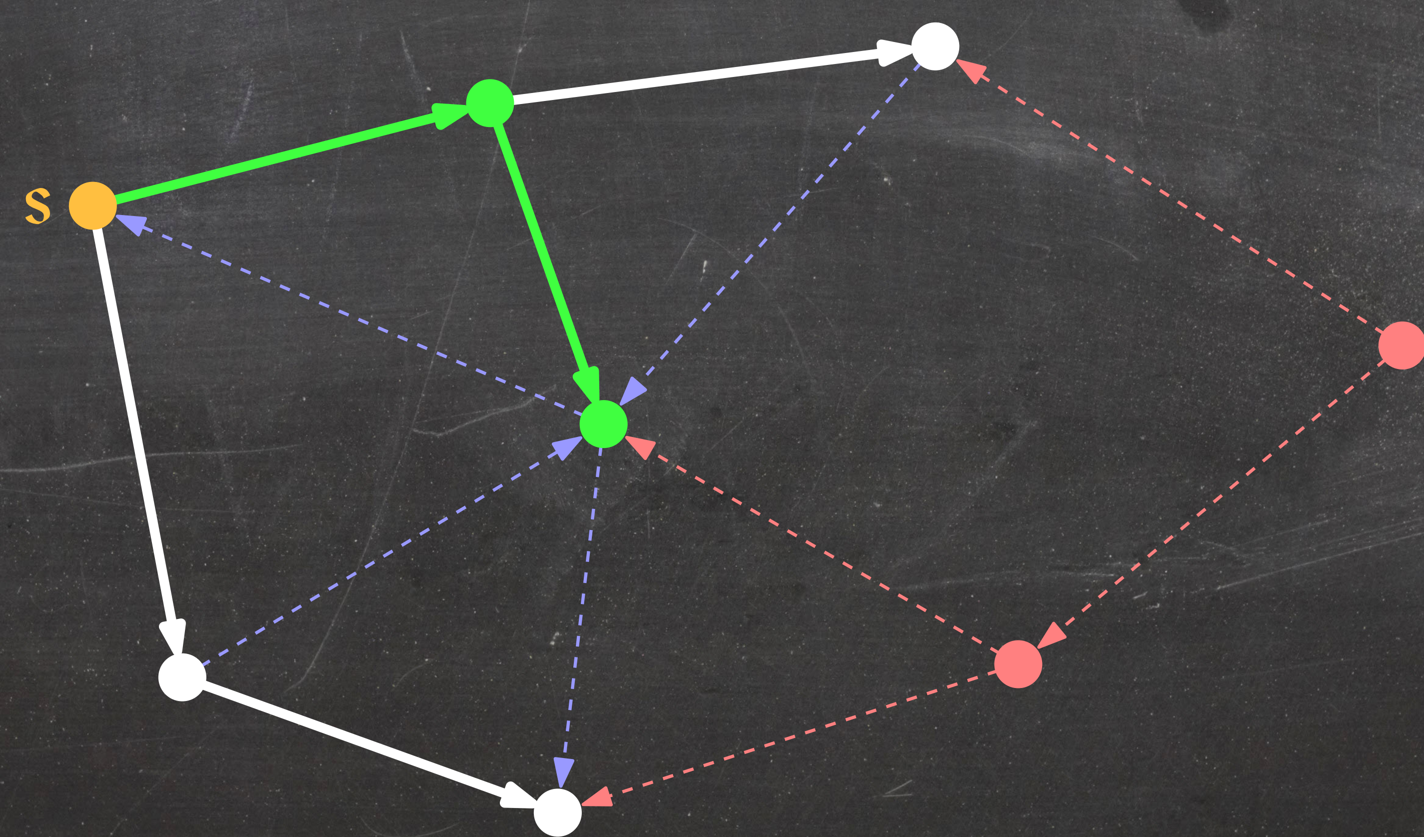

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Dijkstra(G, s)

```
T = ({s}, ∅)
while some vertex in T has an out-neighbour not in T
    do choose an edge (u, v) such that
         • u ∈ T,
         • v ∉ T, and
         • d_T(u) + w(u, v) is minimized.
       add v and (u, v) to T
return T
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T = (V, ∅)
mark every vertex of G as unexplored
set d(v) = +∞ and e(v) = nil for every vertex v ∈ G
mark s as explored and set d(v) = 0
Q = an empty priority queue
for every edge (s, v) incident to s
    do Q.insert(v, w(s, v))
       d(v) = w(s, v)
       e(v) = (s, v)
while not Q.isEmpty()
    do u = Q.deleteMin()
       mark u as explored
       add e(u) to T
       for every edge (u, v) incident to u
           do if v is unexplored and (v ∉ Q or d(u) + w(u, v) < d(v))
                 then d(v) = d(u) + w(u, v)
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$\Rightarrow$ Dijkstra's algorithm takes $O(n \lg n + m)$ time.

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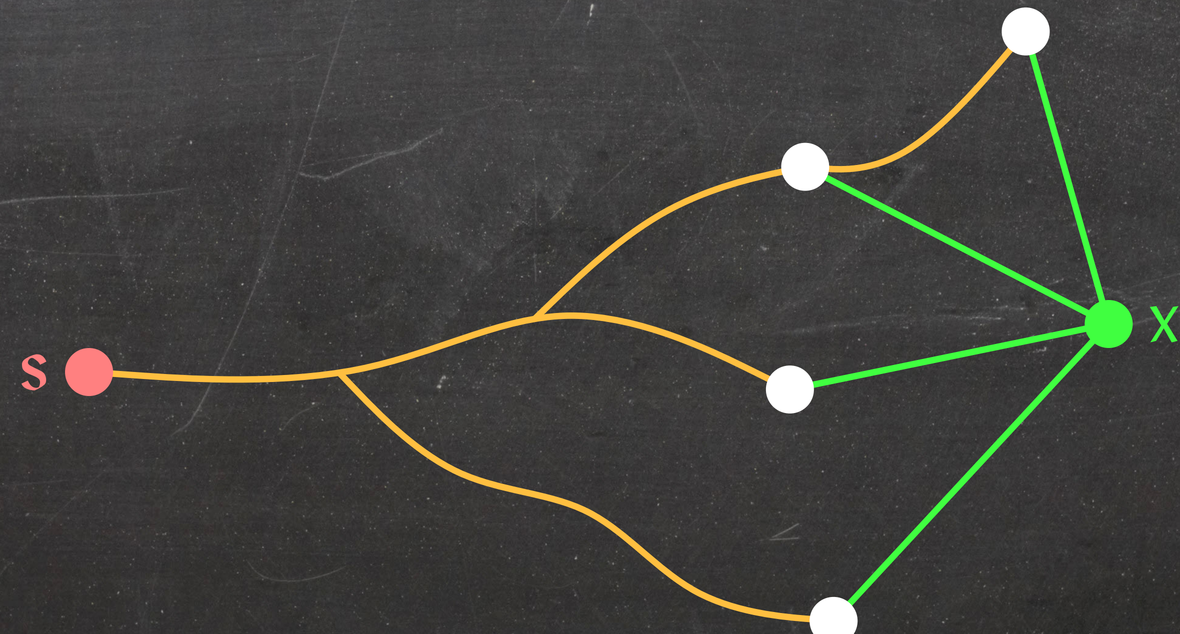
For every vertex $x \notin T$, we have

$$d(x) = \min_{\substack{(u,x)\in E\\ u\in T}} d(u) + w(u, x) = \min_{\substack{(u,x)\in E\\ u\in T}} \mathrm{dist}(s, u) + w(u, x).$$



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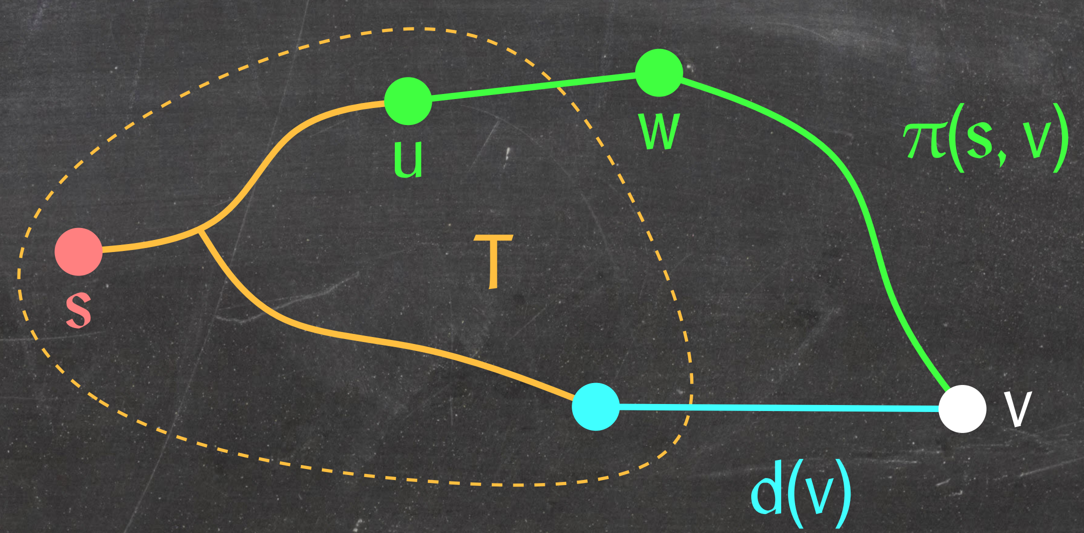

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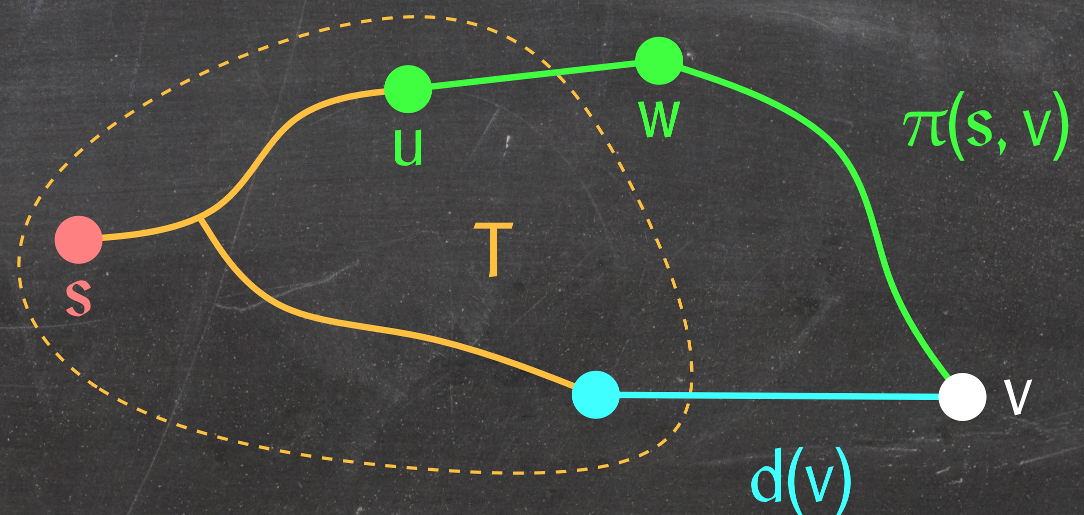

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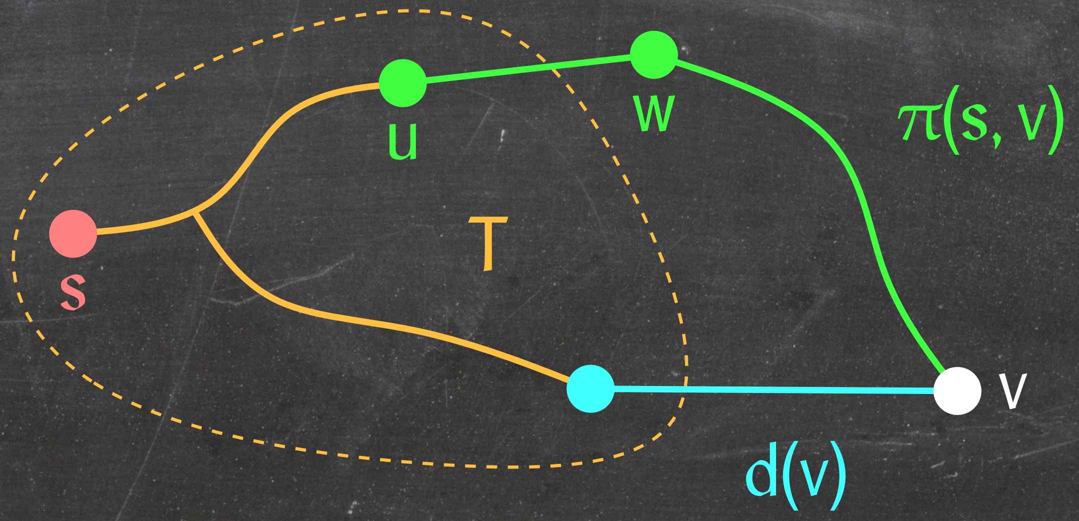
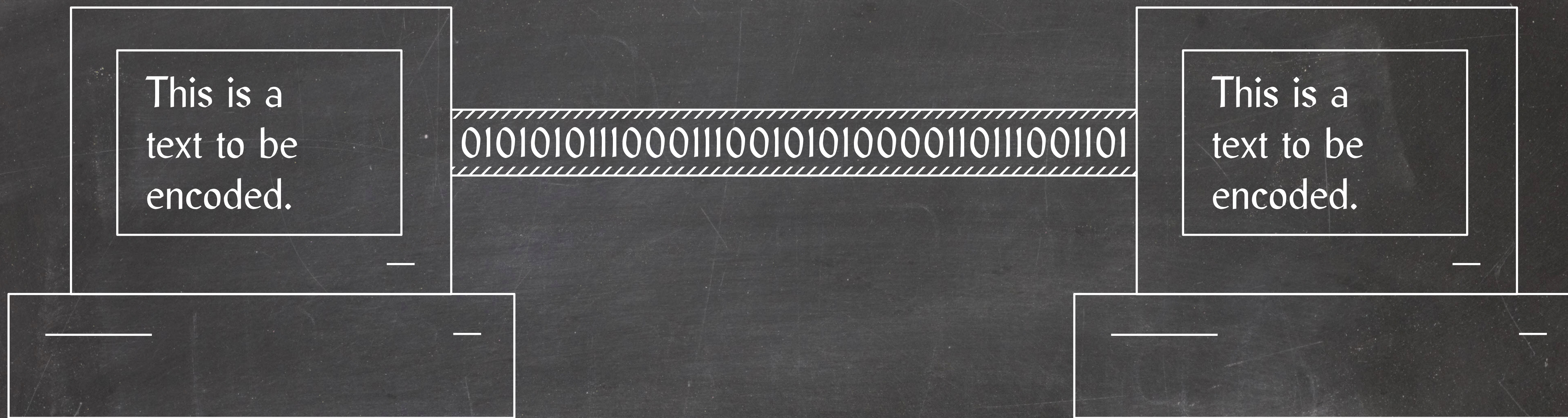
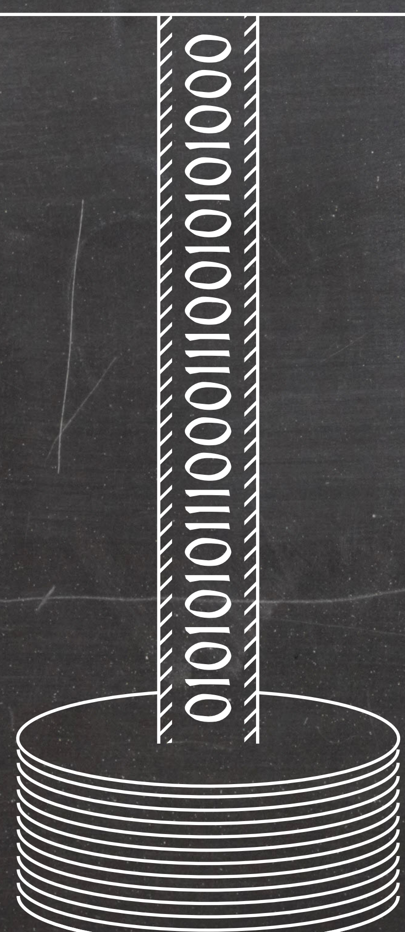


$\Rightarrow$ $d(w) \leq \text{dist}(s, u) + w(u, w) = \text{dist}(s, w) \leq \text{dist}(s, v) < d(v)$.

$\Rightarrow$ v is not the next vertex we add to T, a contradiction.

Minimum Length Codes





Goal:

- Encode a given text using as few bits as possible:
  - Limit amount of disk space required to store the text.
  - Send the text over a potentially slow network.
  - …

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Non-prefix-free codes cannot always be decoded uniquely!

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$\Rightarrow$ $C(\langle x_1, x_2, \ldots, x_{i-1} \rangle) = C(\langle y_1, y_2, \ldots, y_{i-1} \rangle)$ and
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$C(T)$	$C(\langle x_1, x_2, \ldots, x_{i-1} \rangle)$	$C(x_i)$	$C(\langle x_{i+1}, x_{i+2}, \ldots, x_m \rangle)$
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Assume w.l.o.g. that $|C(x_i)| \leq |C(y_i)|$.

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Assume w.l.o.g. that $|C(x_i)| \leq |C(y_i)|$.

Since both $C(x_i)$ and $C(y_i)$ are prefixes of $C(\langle x_i, x_{i+1}, \ldots, x_m \rangle)$, $C(x_i)$ must be a prefix of $C(y_i)$, a contradiction.

$C(T)$	$C(\langle x_1, x_2, \ldots, x_{i-1} \rangle)$	$C(x_i)$	$C(\langle x_{i+1}, x_{i+2}, \ldots, x_m \rangle)$
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Prefix Codes and Binary Trees

Observation: Every prefix-free code $C(\cdot)$ can be represented as a binary tree $\mathcal{T}_C$ whose leaves correspond to the letters in the alphabet.

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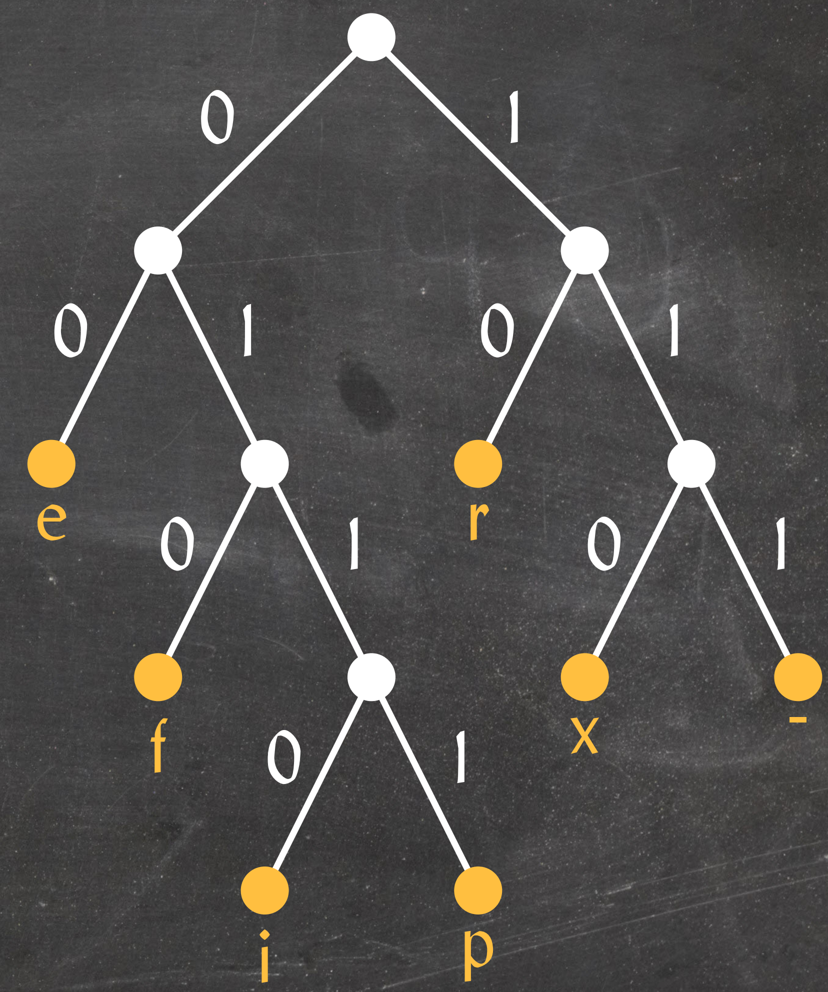

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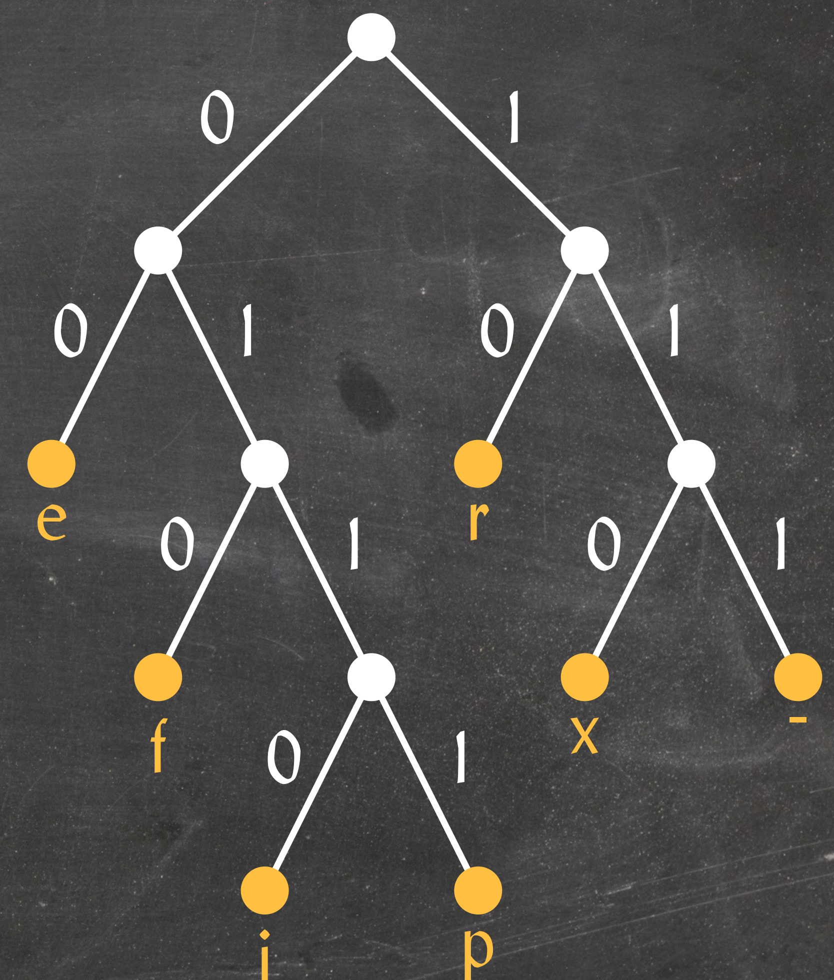

The depth of character x in $\mathcal{T}_C$ is the number of bits $|C(x)|$ used to encode x using $C(\cdot)$.

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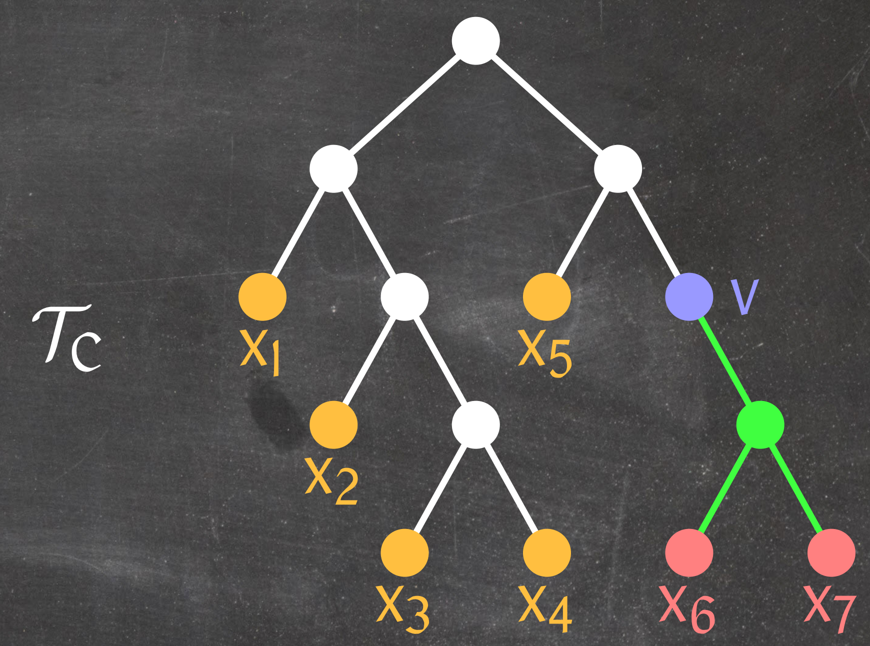
Lemma: For every text T, there exists an optimal prefix-free code C(·) such that every internal node in $\mathcal{T}_C$ has two children.



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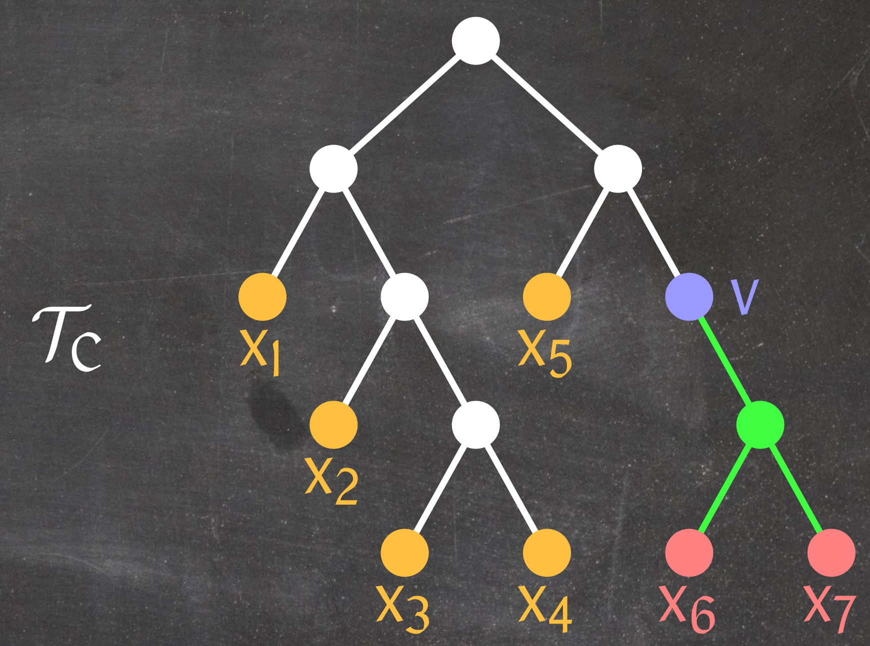
Choose C(·) so that $\mathcal{T}_C$ has as few internal nodes with only one child as possible among all optimal prefix-free codes for T.



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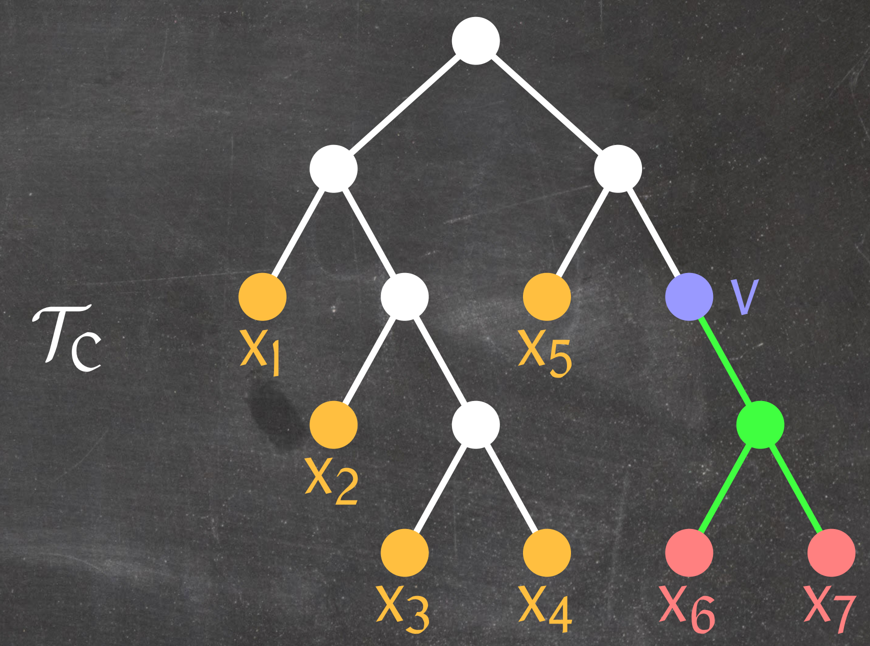

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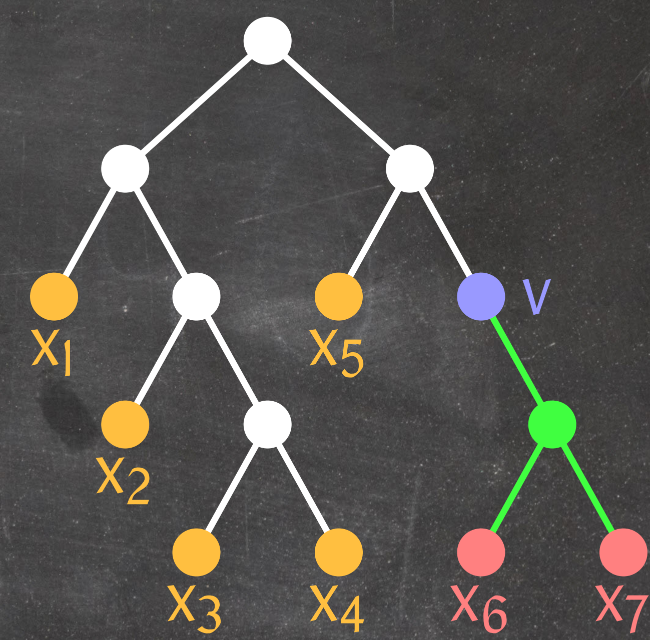
If $\mathcal{T}_C$ has no internal node with only one child, the lemma holds.

Otherwise, choose an internal node v with only one child w and contract the edge (v, w).

$\mathcal{T}_C$



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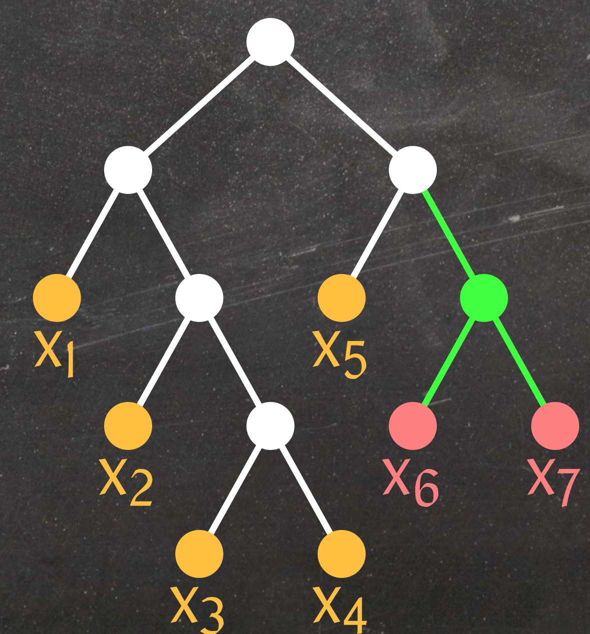

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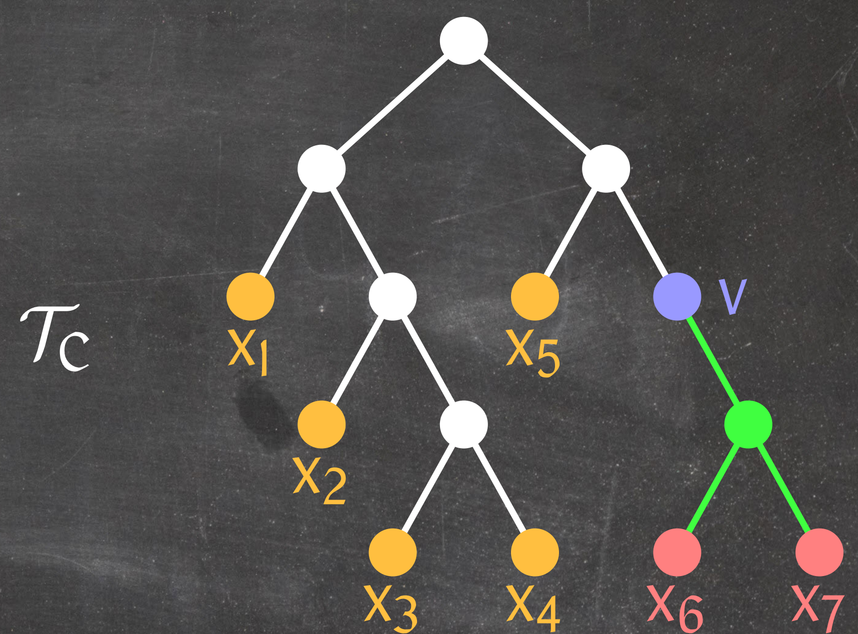

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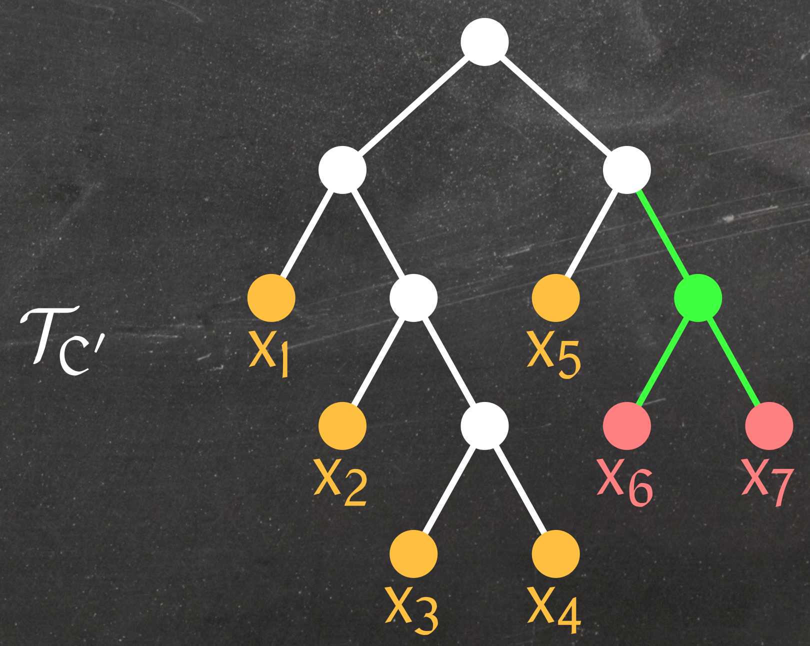
The resulting tree $\mathcal{T}_{C'}$ has one less internal node with only one child and represents a prefix-free code $C'(\cdot)$ with the property that $|C'(x)| \le |C(x)|$ for every character x.



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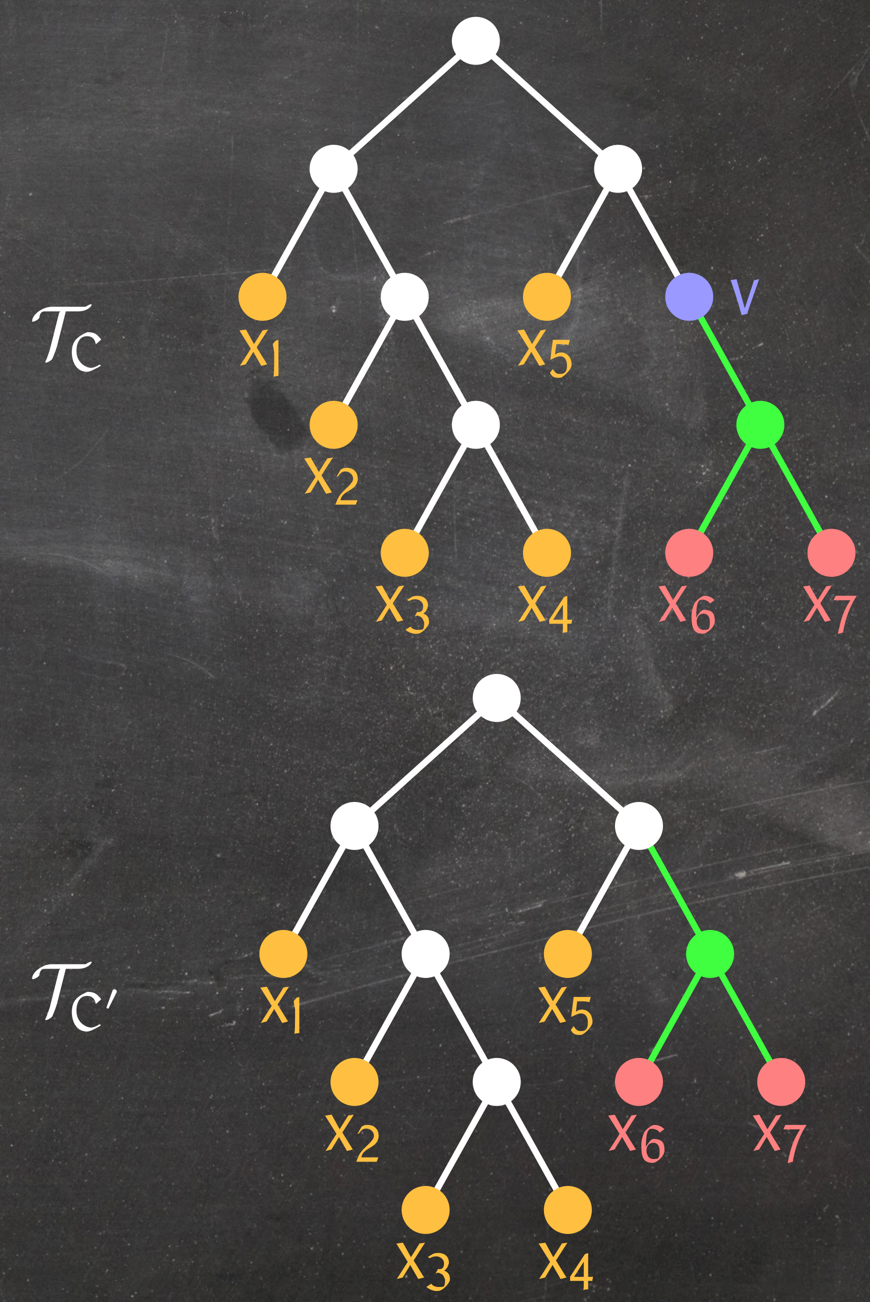
$\Rightarrow$ $|C'(T)| \le |C(T)|$, contradicting the choice of C.



A Greedy Choice for Optimal Prefix Codes

We can build binary trees by starting with each leaf in its own tree, joining two trees under a common parent, and repeating this until only one tree is left.

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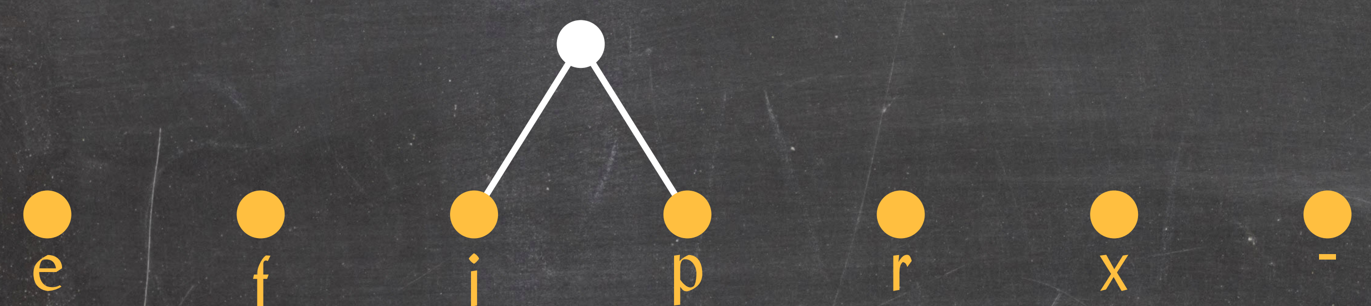

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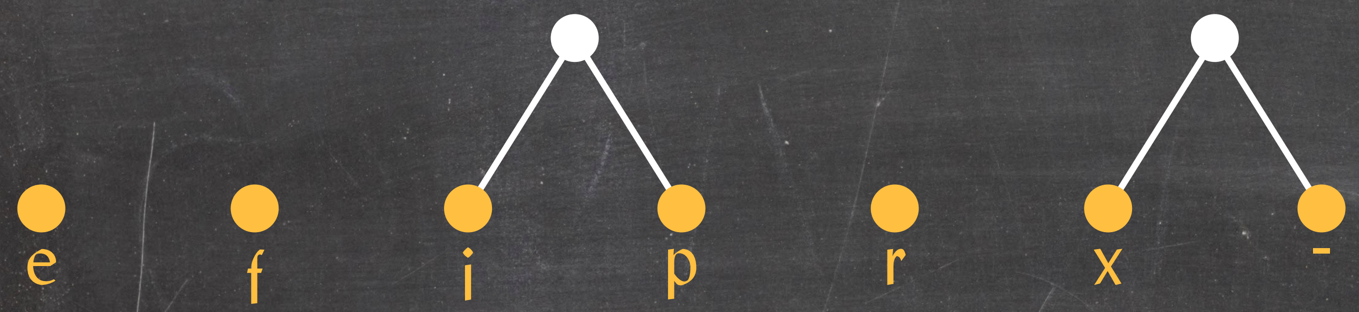

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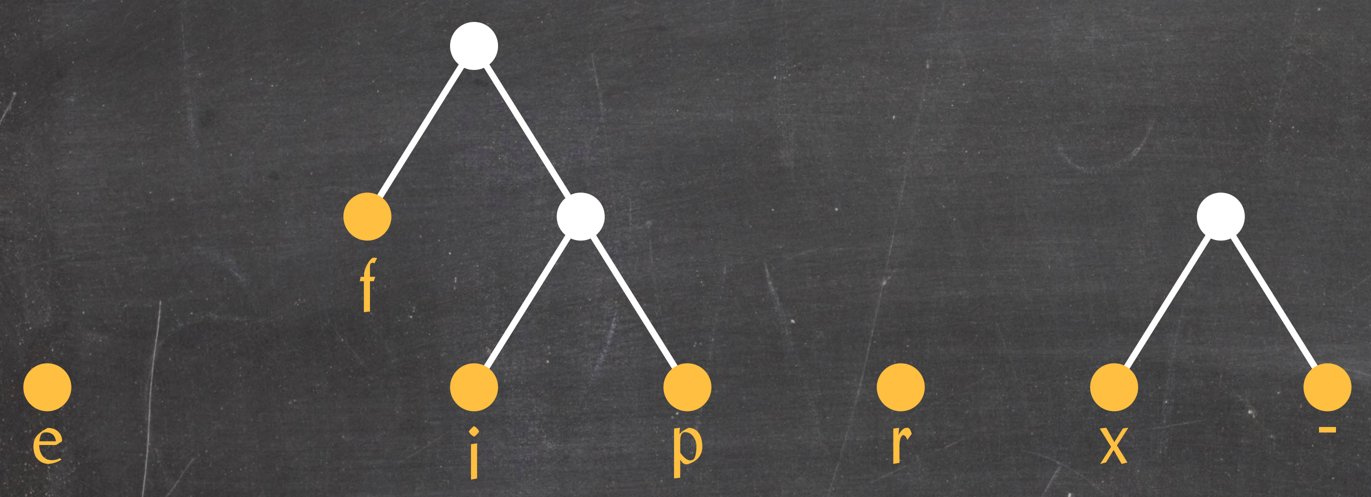

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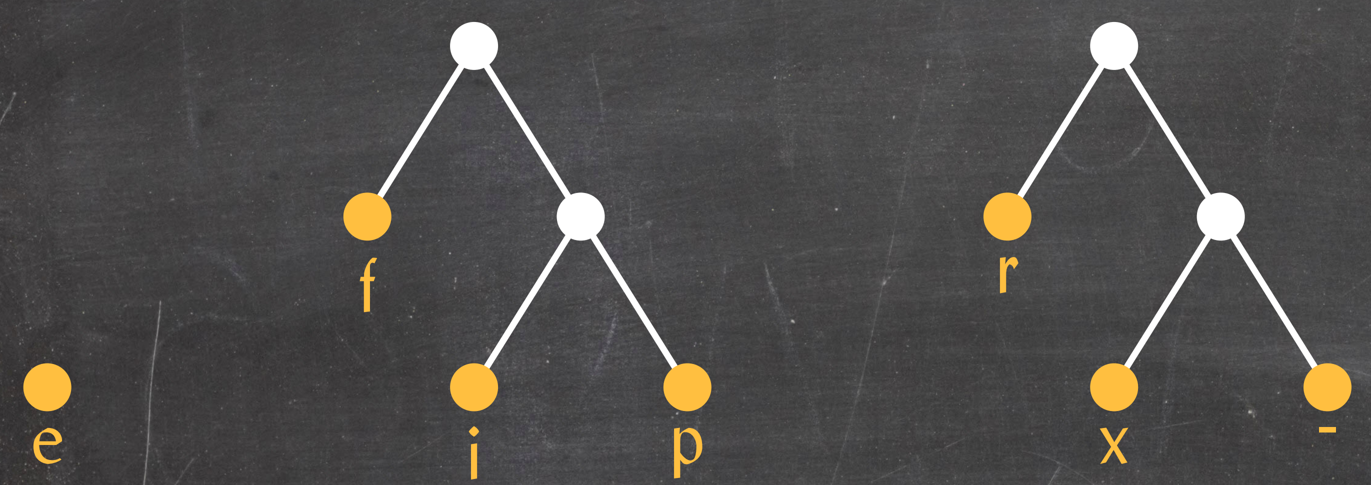

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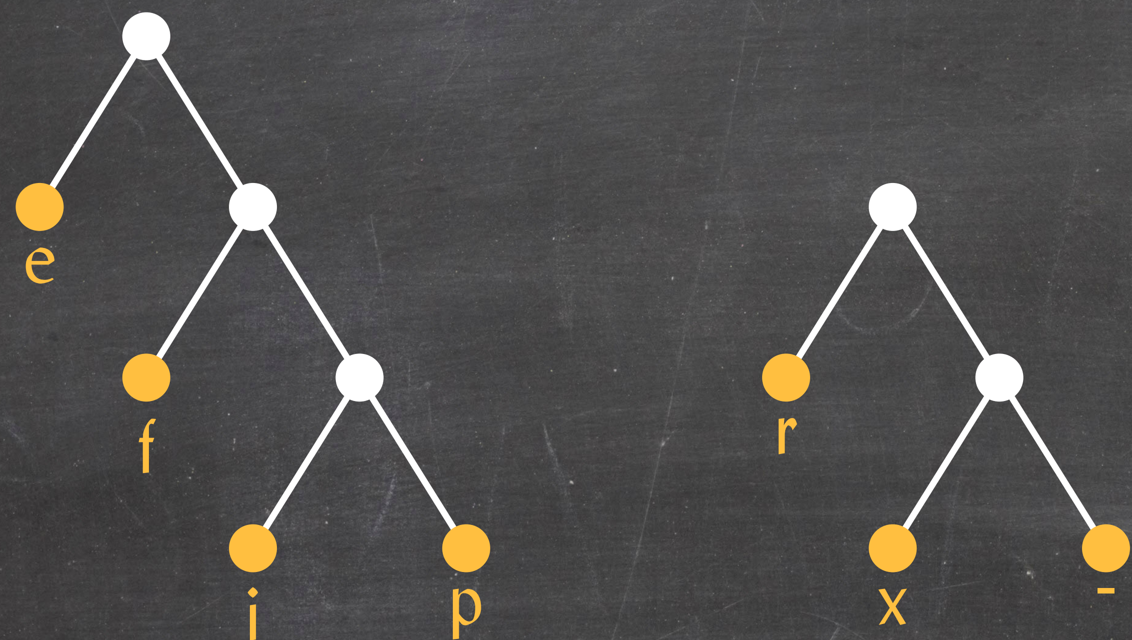

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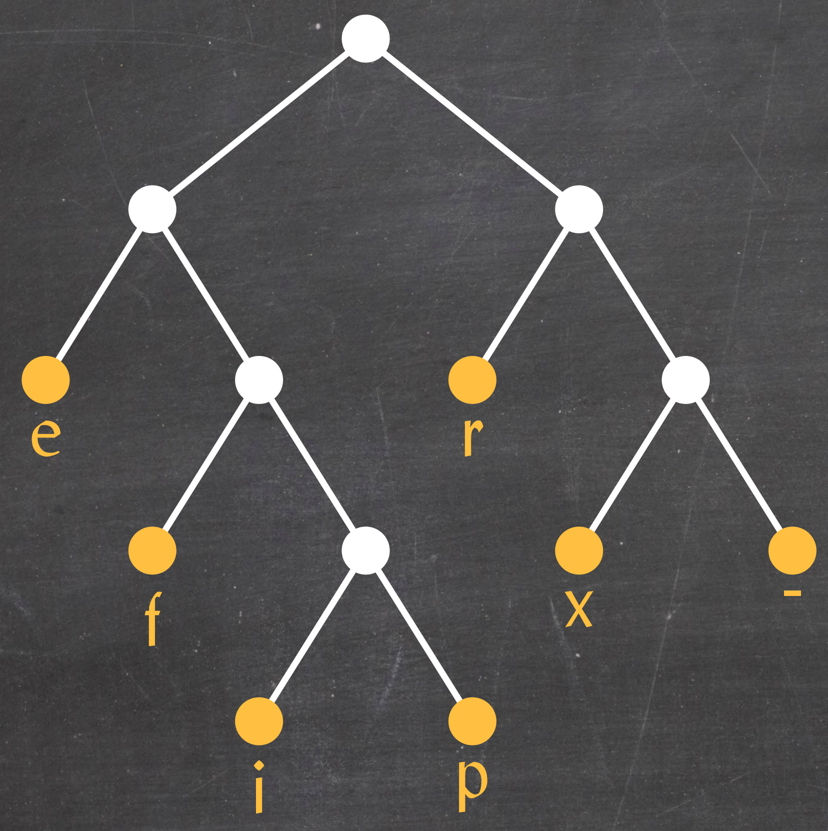

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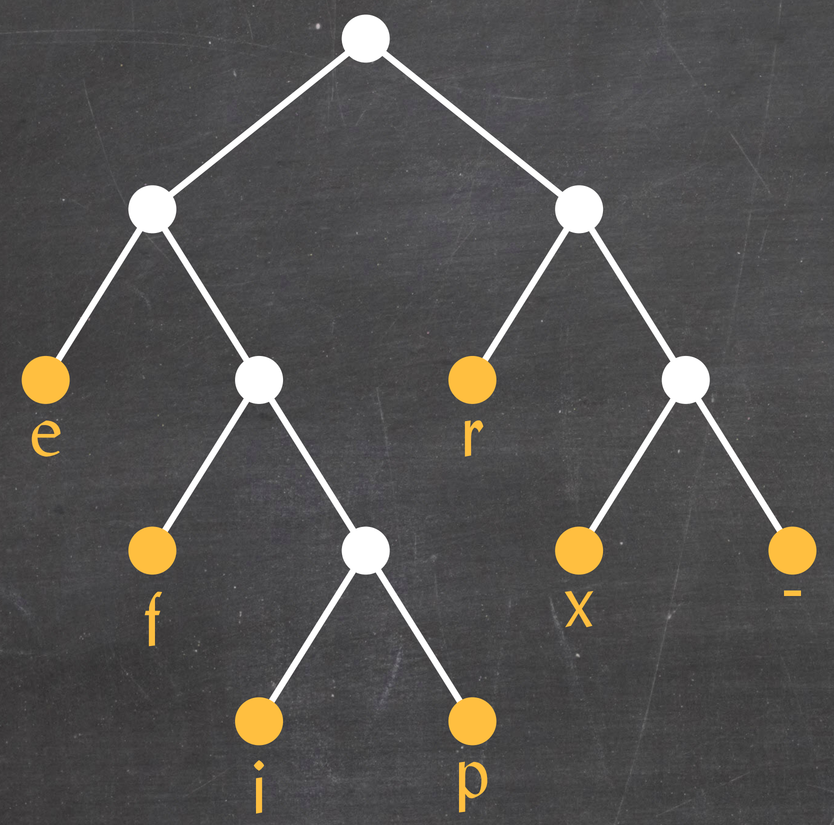

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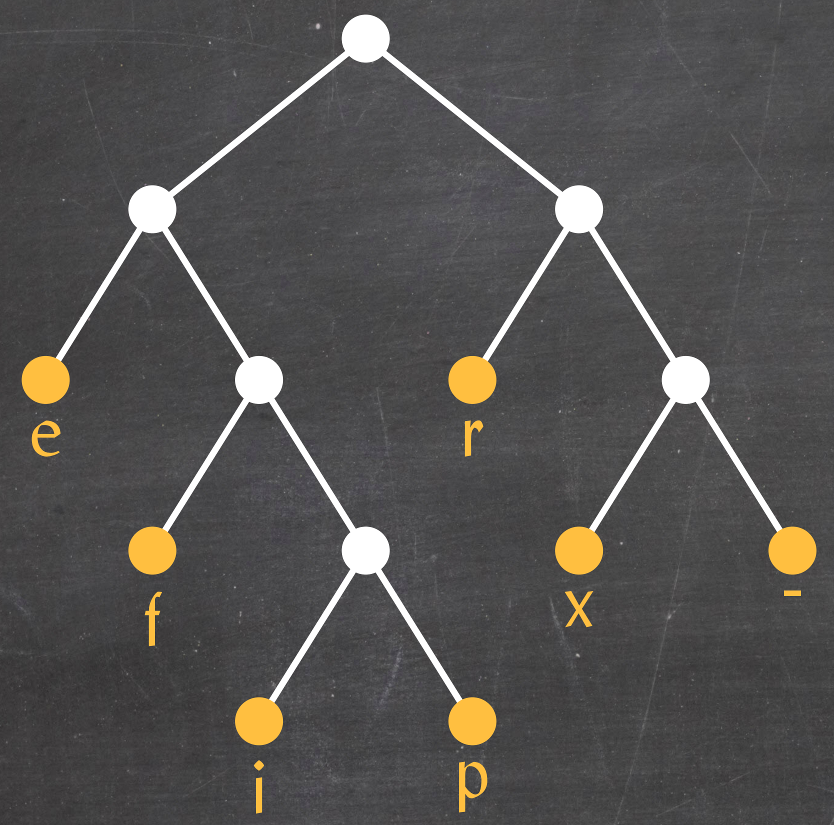

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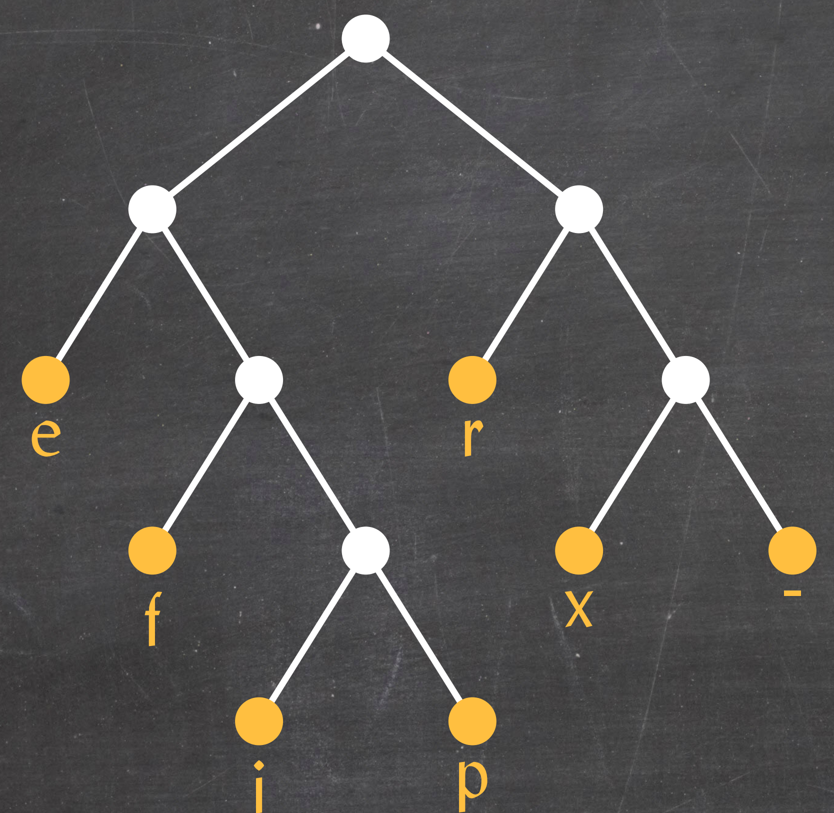

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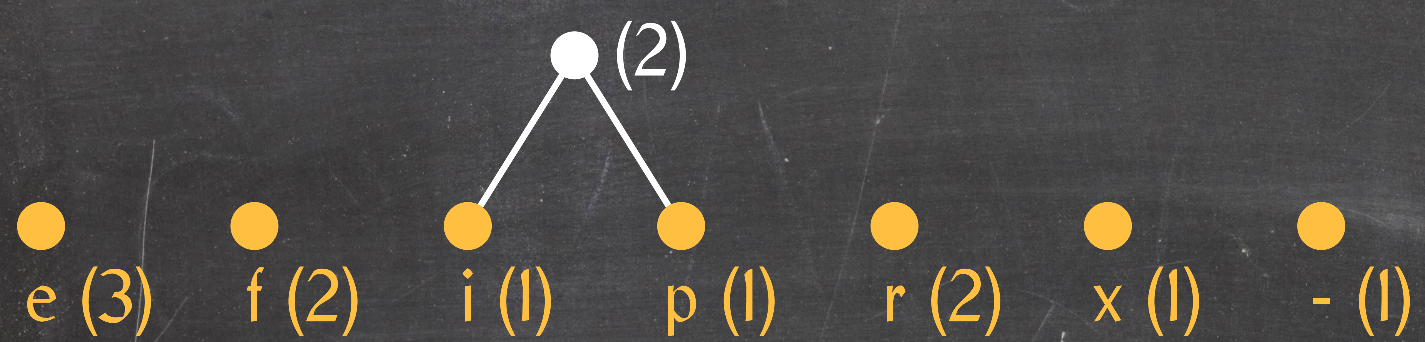

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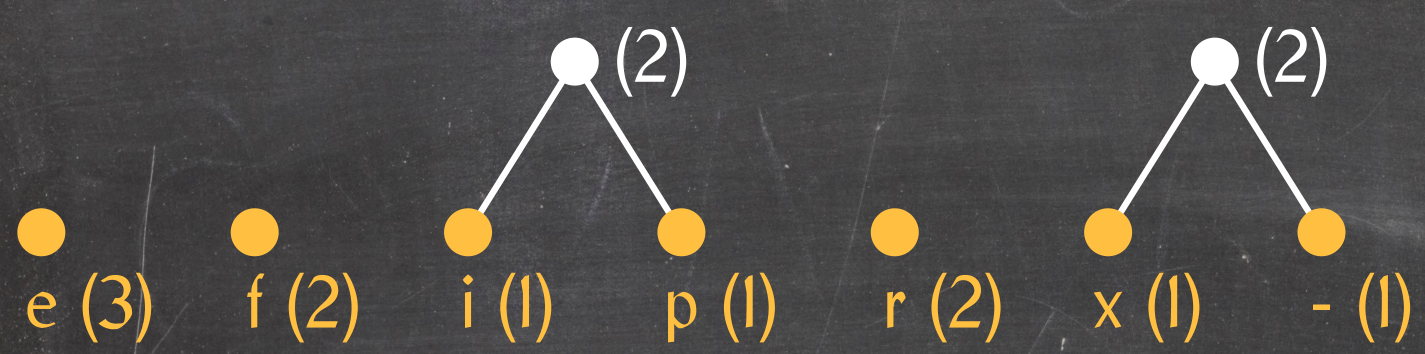

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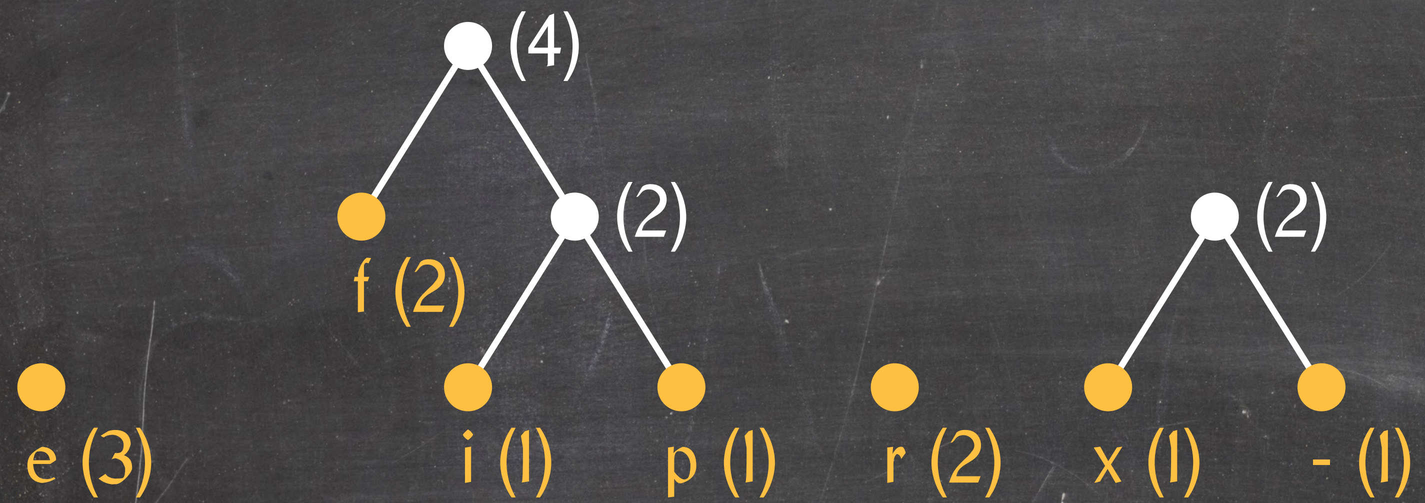
A Greedy Choice for Optimal Prefix Codes

We can build binary trees by starting with each leaf in its own tree, joining two trees under a common parent, and repeating this until only one tree is left.

"prefix-free"



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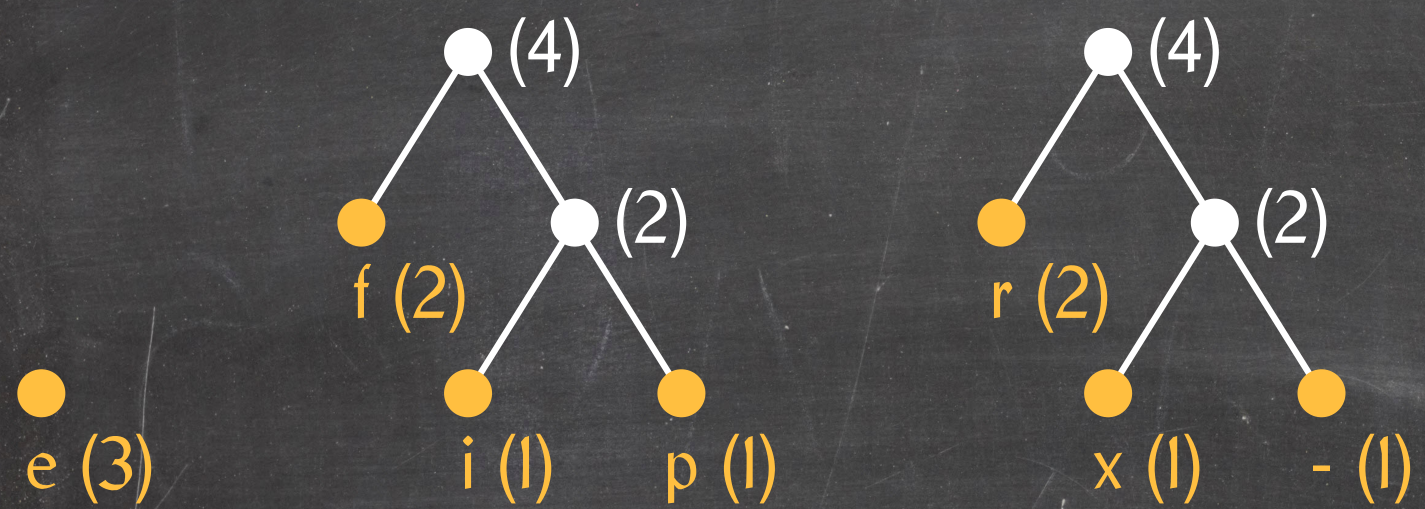

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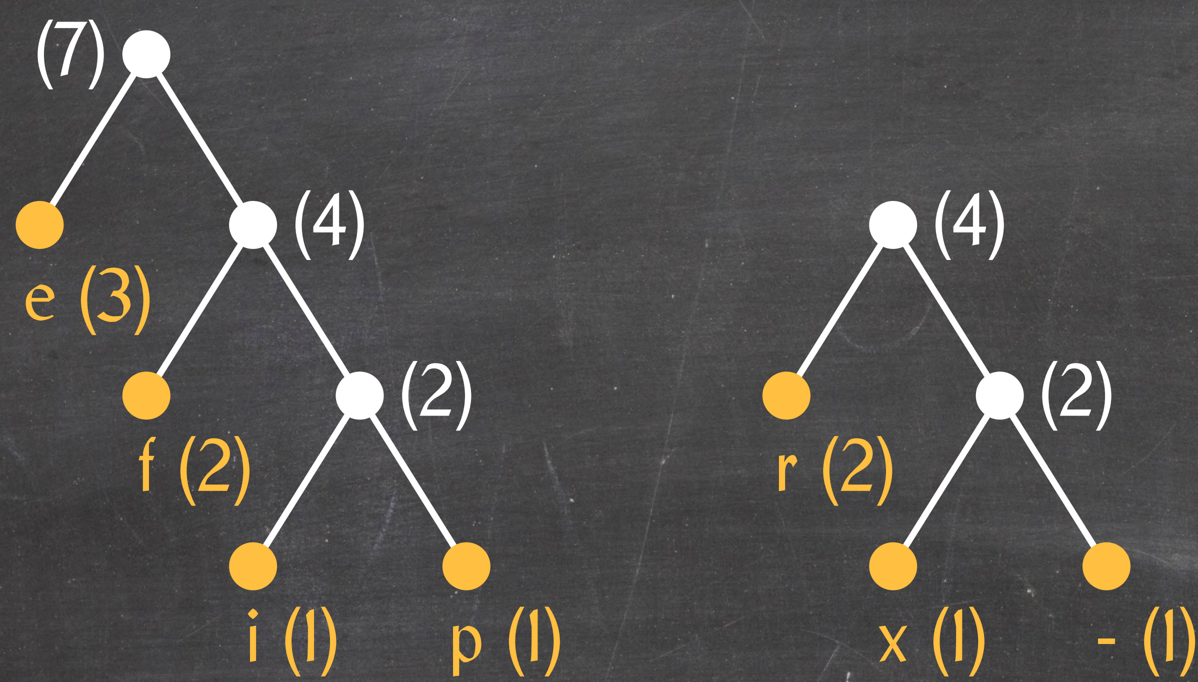

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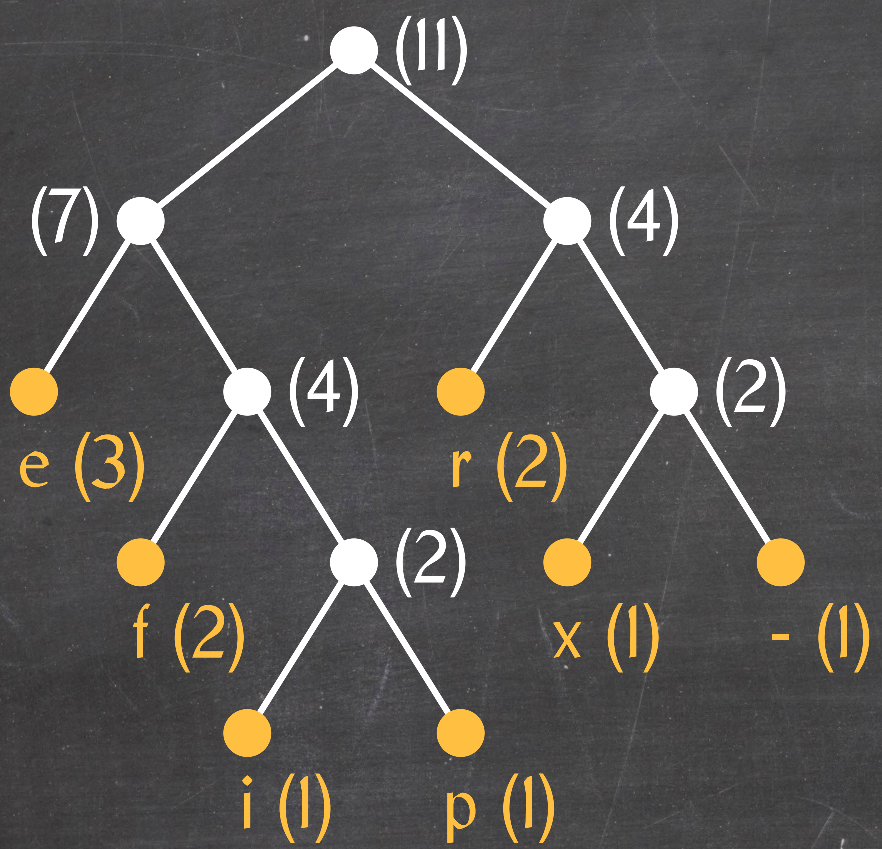

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Huffman's Algorithm

Huffman(T)

```
determine the set A of characters that occur in T and their frequencies
Q = an empty priority queue
for every character x ∈ A
    do create a node v associated with x and define f(v) = f(x)
       Q.insert(v, f(v))
while |Q| > 1
    do v = Q.deleteMin()
       w = Q.deleteMin()
       u = a new node with frequency f(u) = f(v) + f(w)
       make v and w children of u
       Q.insert(u, f(u))
return Q.deleteMin()
```

Lemma: Huffman's algorithm runs in $O(m \lg n)$ time, where $m = |T|$ and n is the size of the alphabet.

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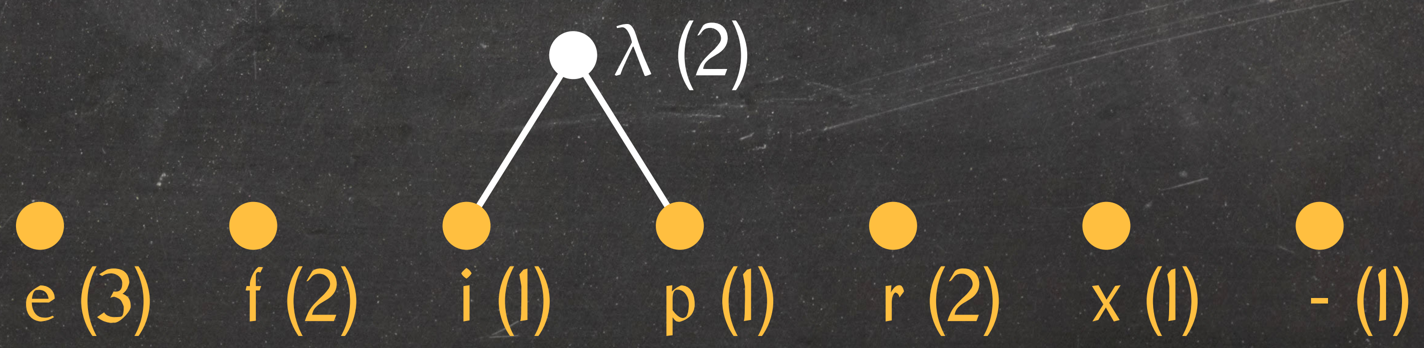

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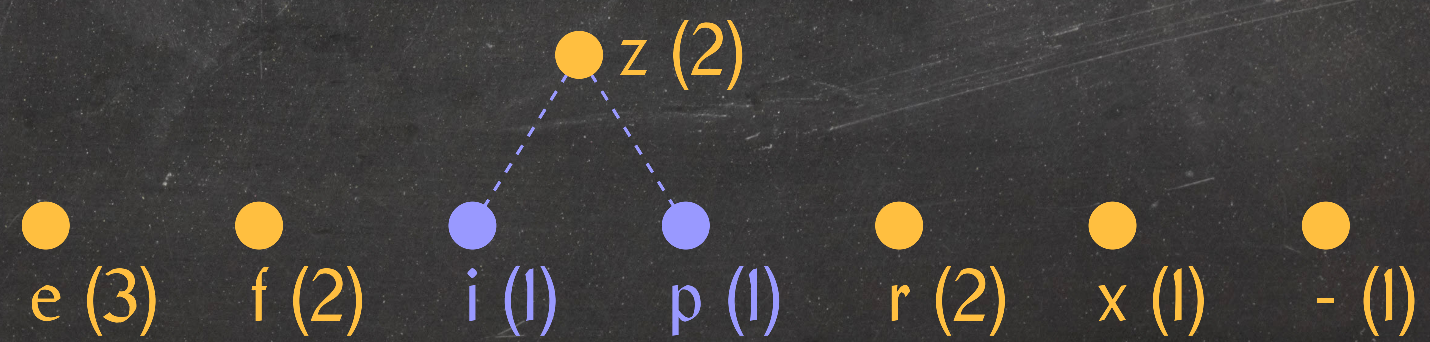
Replacing a and b with z in T produces a new text T' over an alphabet of size $n - 1$ where z has frequency $f(z)$.

“prefix-free”

$\Downarrow$

“zrefzx-free”



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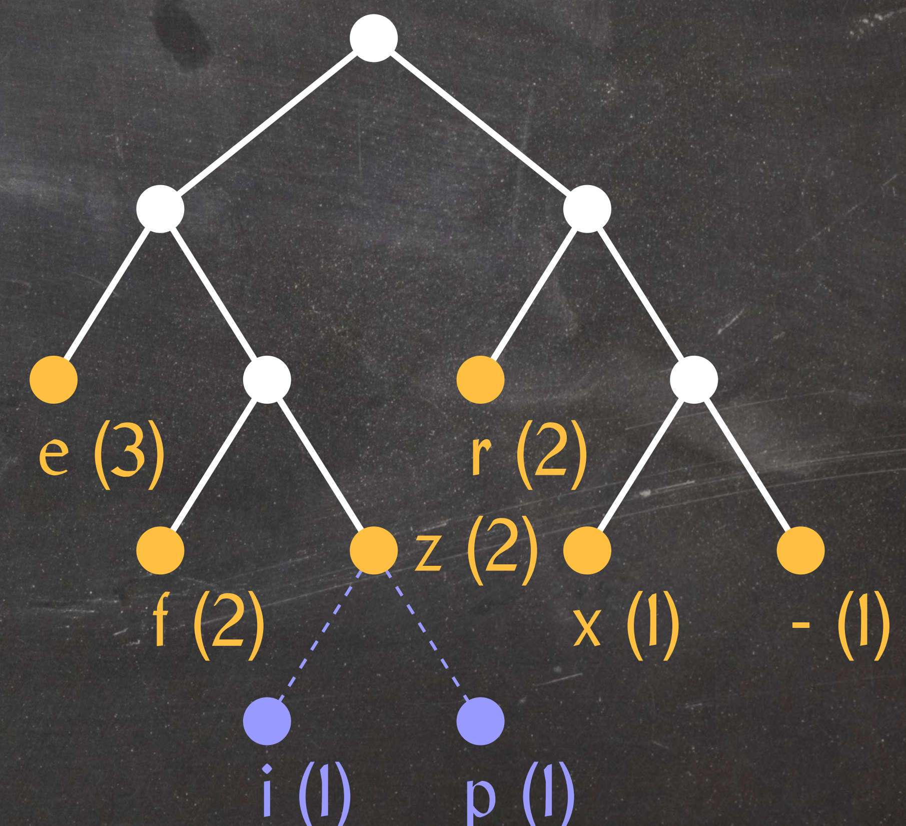

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By induction, it produces an optimal code $C'(\cdot)$ for T'.

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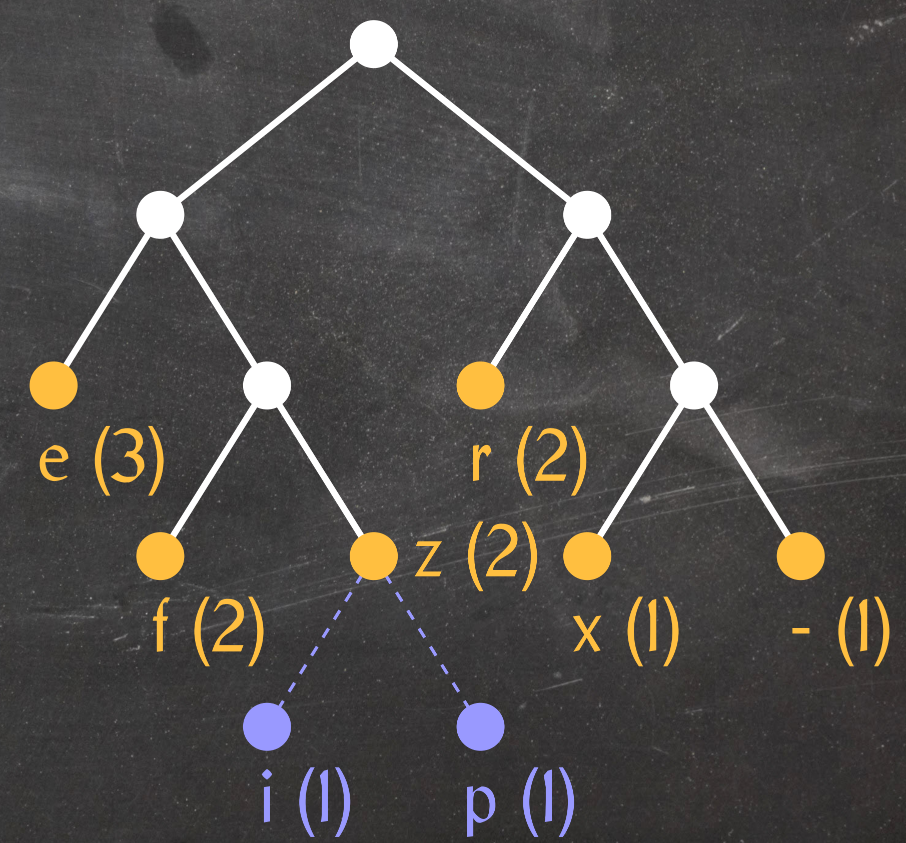

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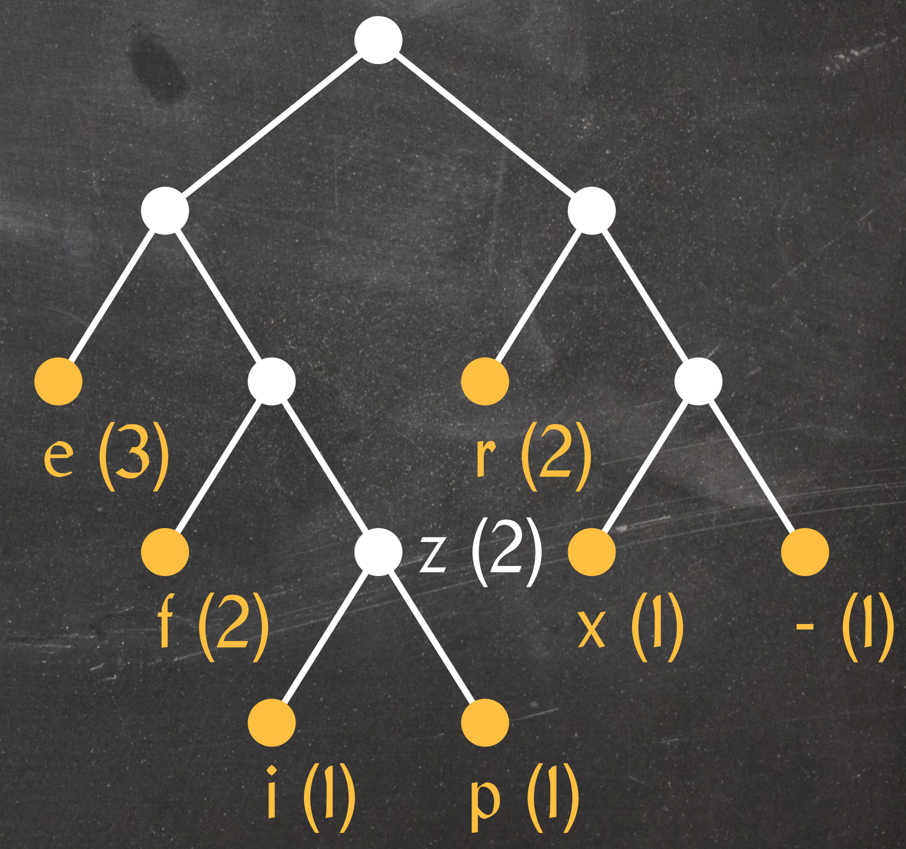

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Let $C''(\cdot)$ be the code for T' defined as

$$C''(x) = \begin{cases} C^*(x) & x \neq z \\ \sigma & x = z \text{ and } C^*(a) = \sigma 0 \end{cases}$$

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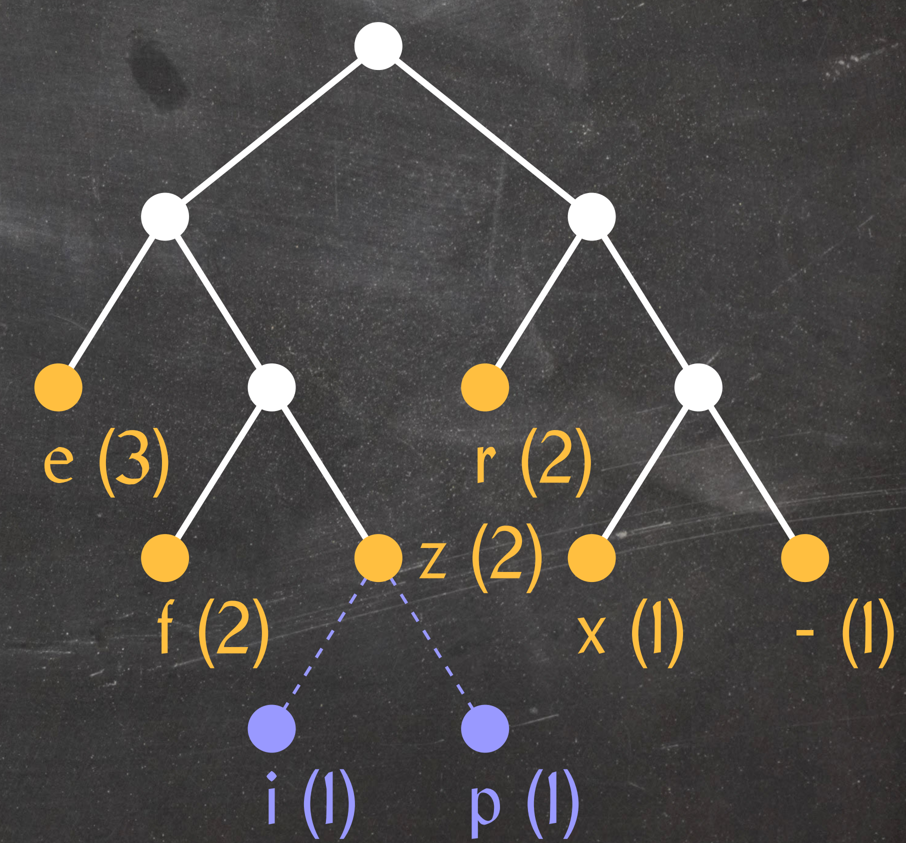

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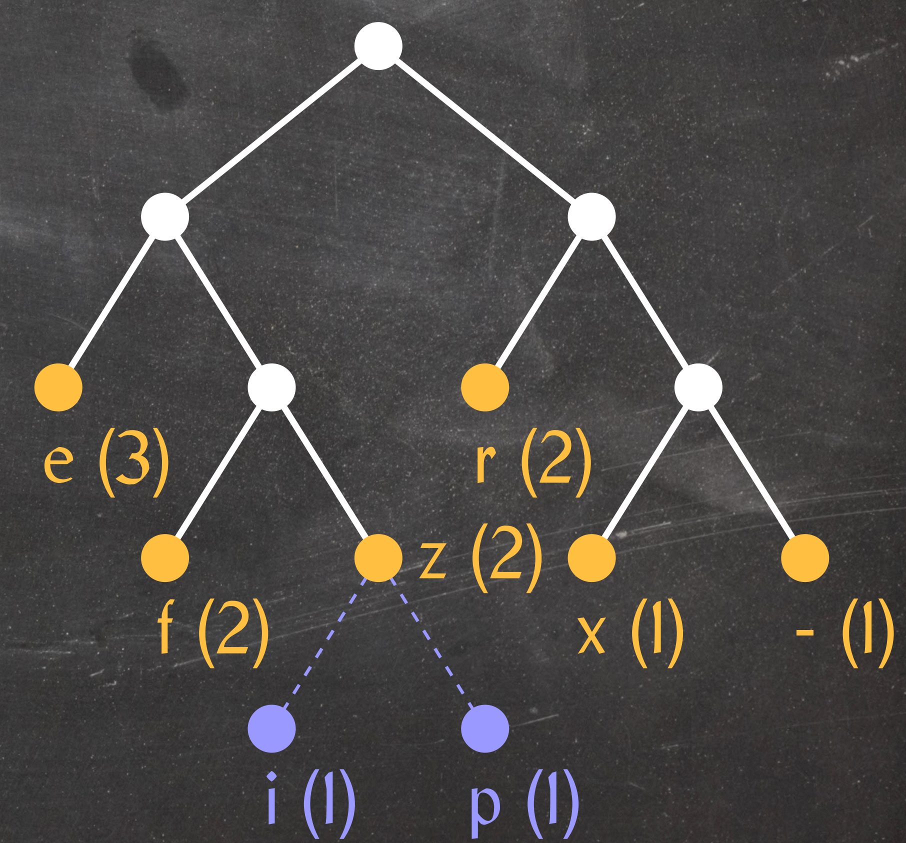

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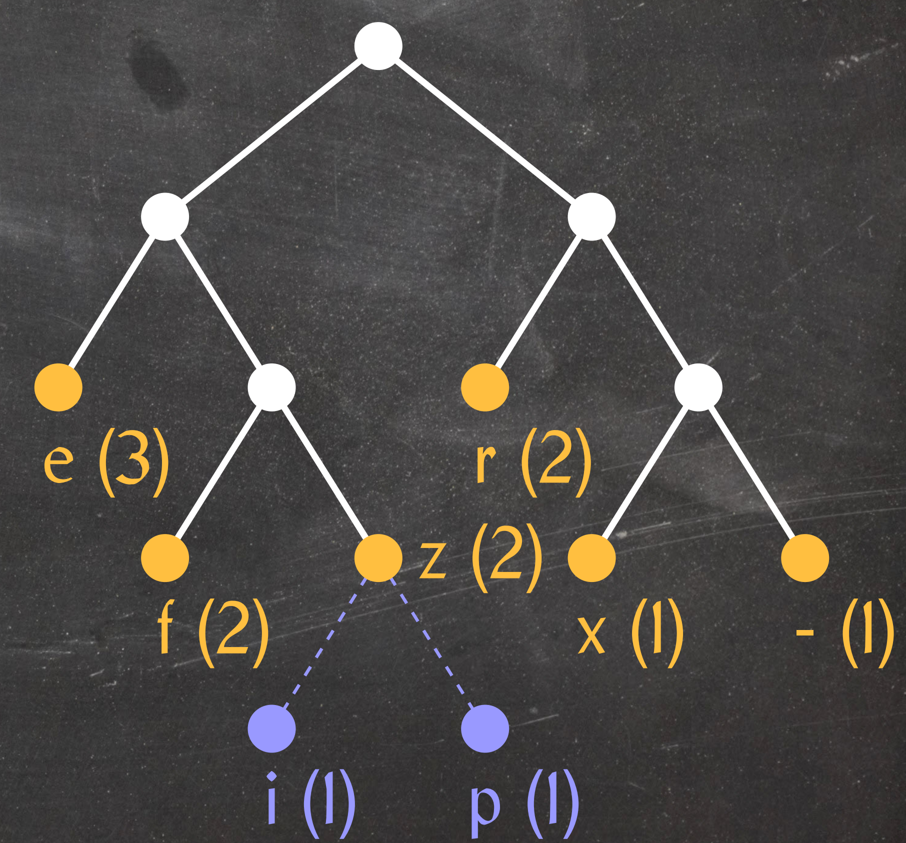

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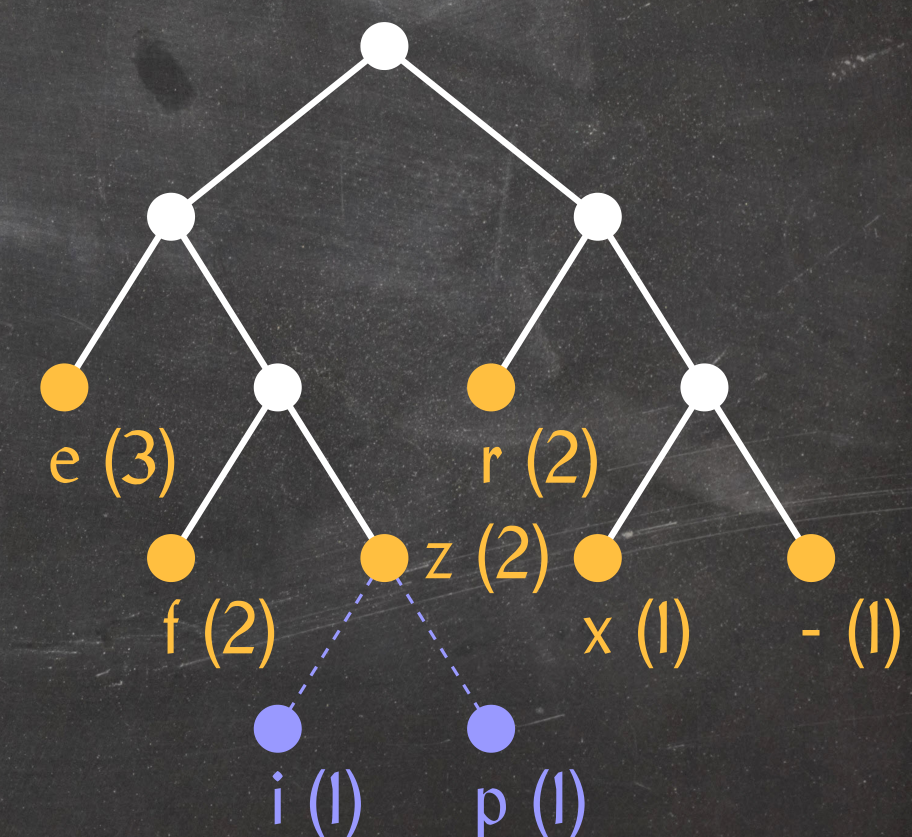

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$\Rightarrow$ $|C''(T')| < |C'(T')|$, a contradiction because $C'(\cdot)$ is optimal for T'.

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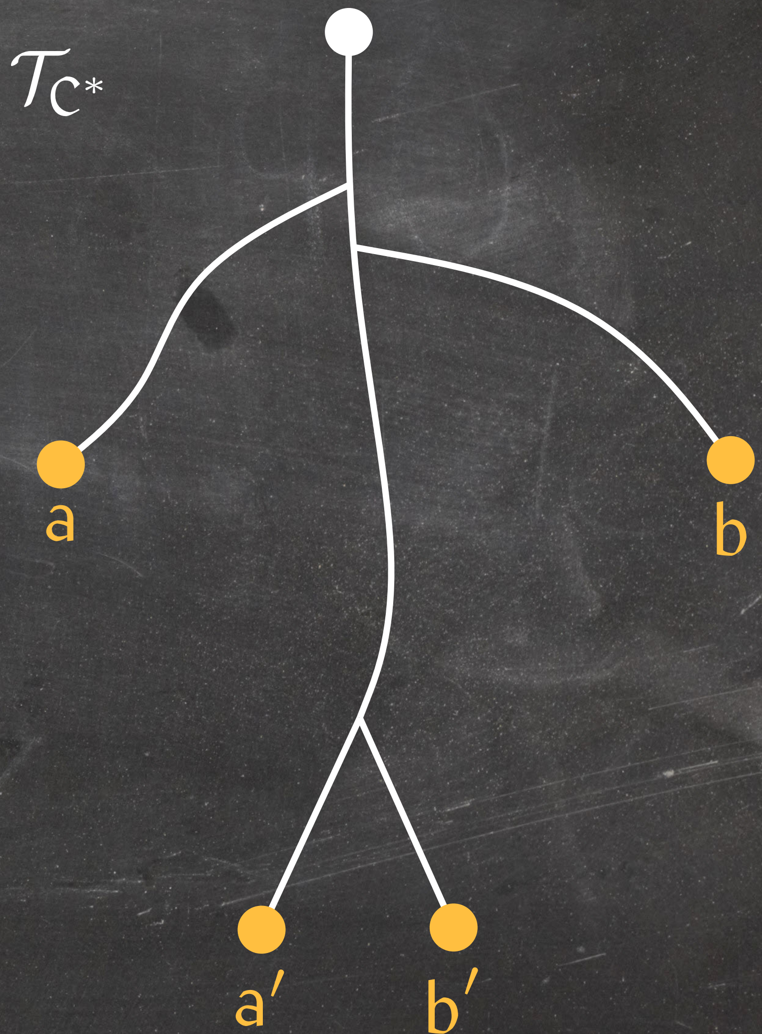

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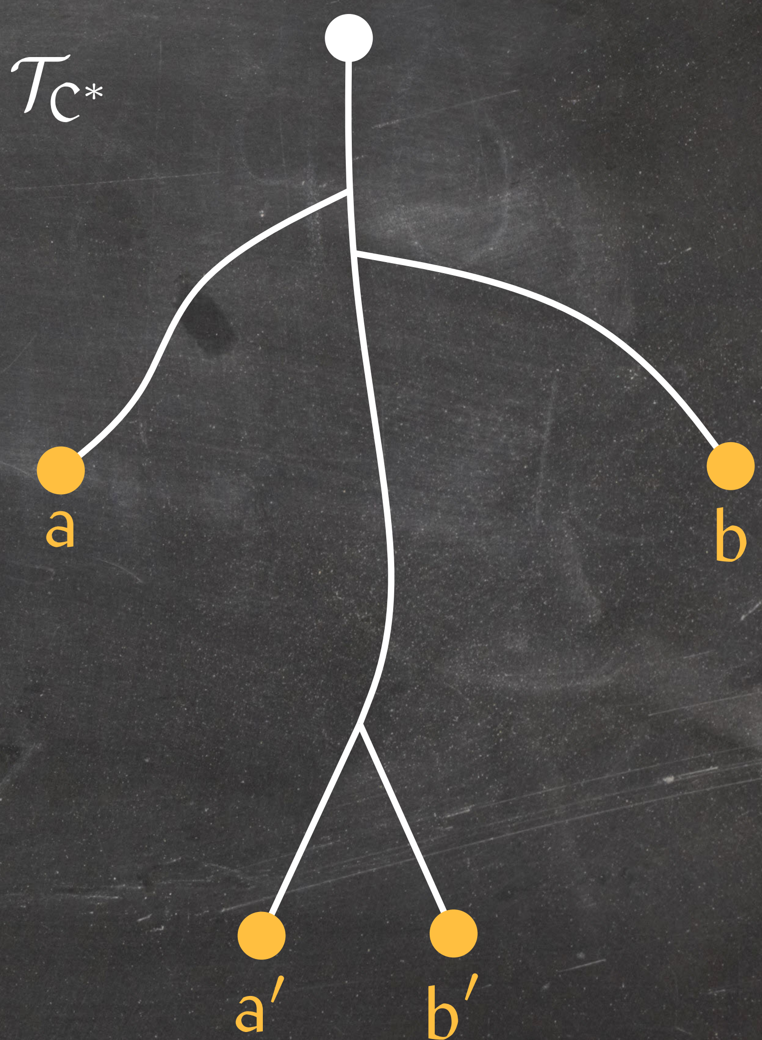

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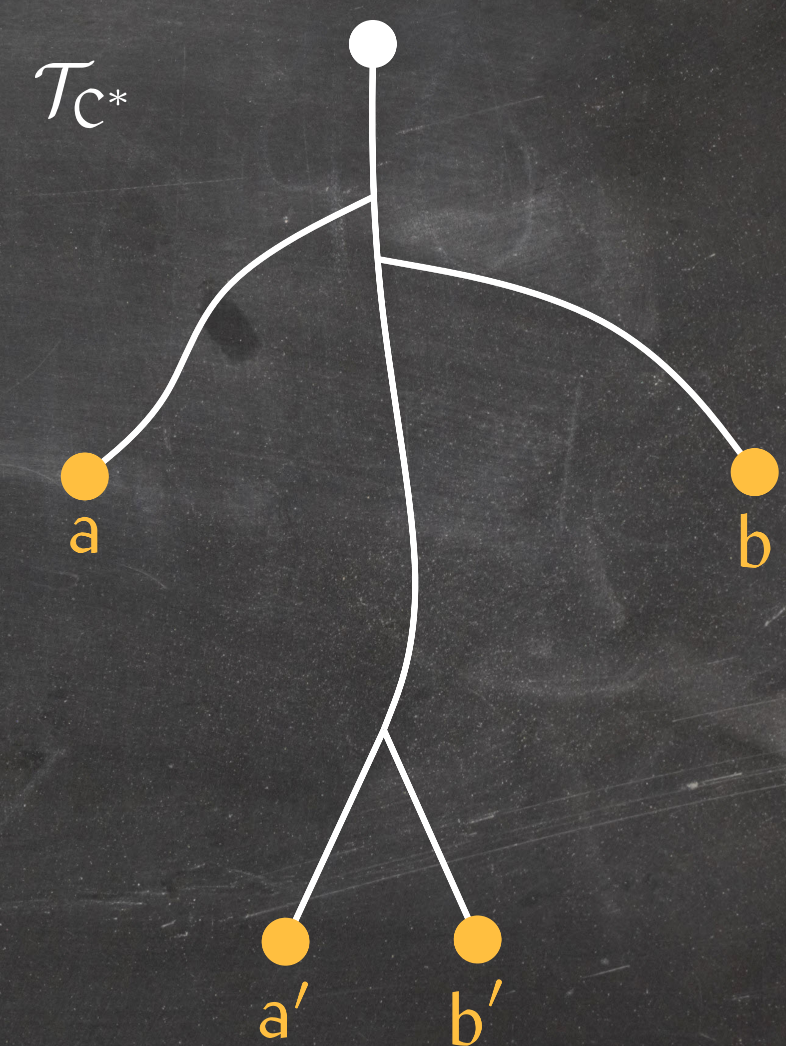

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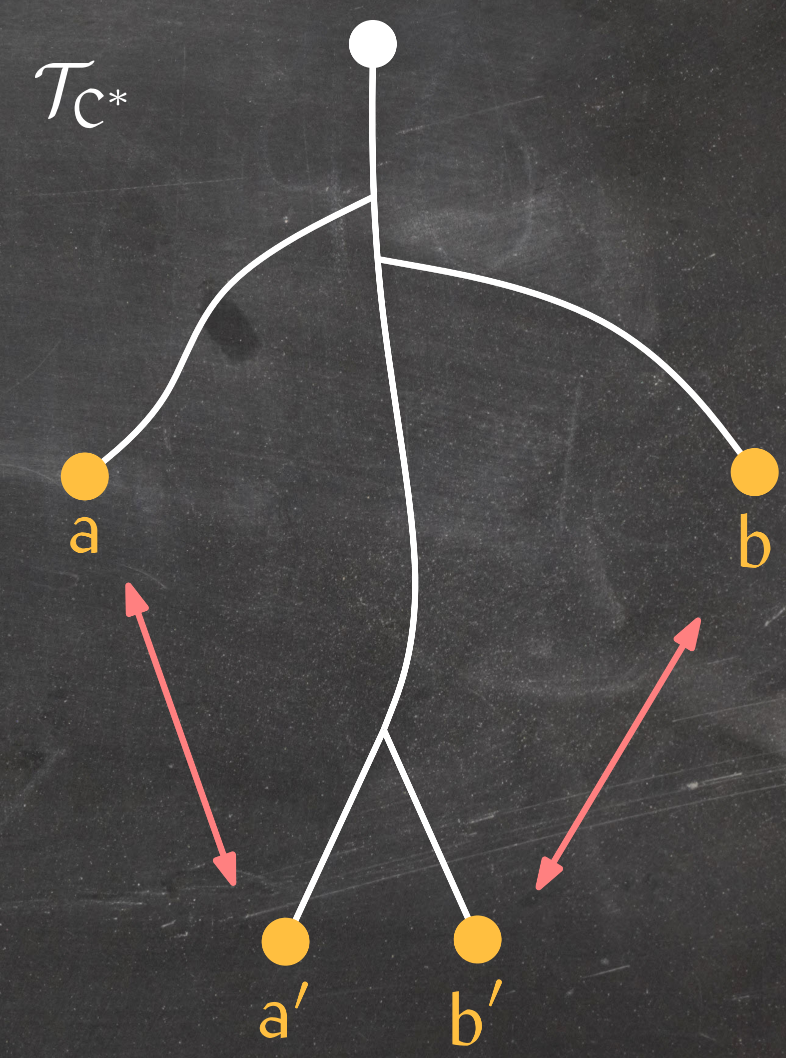

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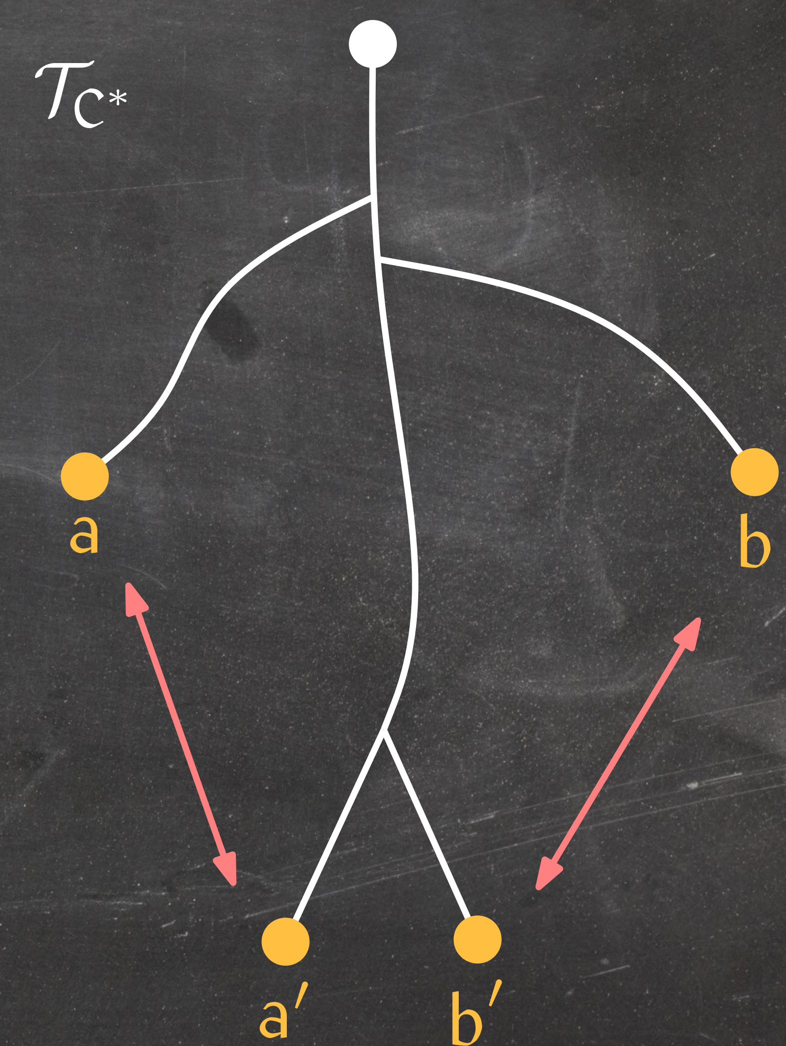

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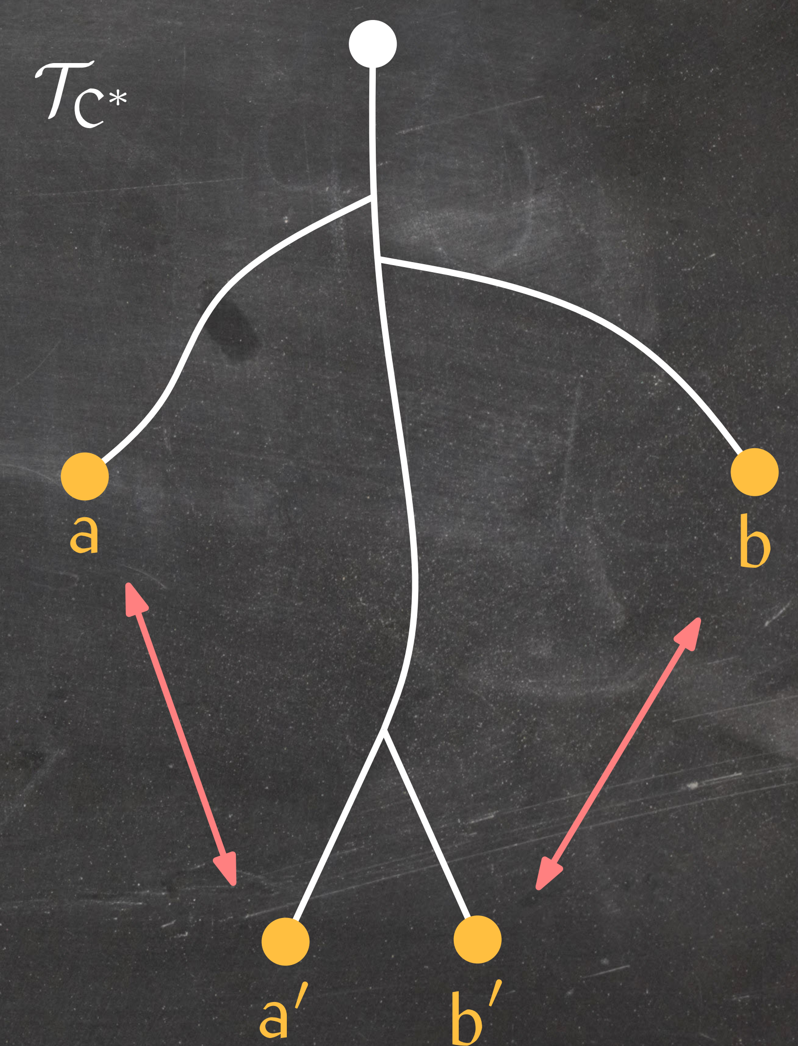
Since a and b are siblings in $\mathcal{T}_C$, this proves the claim.



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$$\begin{aligned} |C(T)| - |C^*(T)| = {} & f(a)|C(a)| + f(b)|C(b)| + f(a')|C(a')| + f(b')|C(b')| - \\ & f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \end{aligned}$$

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&= f(a)|C^*(a')| + f(b)|C^*(b')| + f(a')|C^*(a)| + f(b')|C^*(b)| - \\
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&= (f(a) - f(a'))\,(|C^*(a')| - |C^*(a)|) + (f(b) - f(b'))\,(|C^*(b')| - |C^*(b)|)
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&= f(a)|C^*(a')| + f(b)|C^*(b')| + f(a')|C^*(a)| + f(b')|C^*(b)| - \\
&\quad\; f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= \underbrace{(f(a) - f(a'))}_{\le 0}\underbrace{(|C^*(a')| - |C^*(a)|)}_{\ge 0} + \underbrace{(f(b) - f(b'))}_{\le 0}\underbrace{(|C^*(b')| - |C^*(b)|)}_{\ge 0}
\end{aligned}
$$

Correctness of Huffman's Algorithm

Claim: There exists an optimal prefix-free code $C(\cdot)$ for T such that the two least frequent characters a and b in T are siblings in $\mathcal{T}_C$.

Given: $|C^*(a)| \le |C^*(a')|$, $|C^*(b)| \le |C^*(b')|$, $f(a) \le f(b)$, and $f(a') \le f(b')$.

$\Rightarrow$ $f(a) \le f(a')$ and $f(b) \le f(b')$.

$$\begin{aligned}
|C(T)| - |C^*(T)| &= f(a)|C(a)| + f(b)|C(b)| + f(a')|C(a')| + f(b')|C(b')| - \\
&\quad\ f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= f(a)|C^*(a')| + f(b)|C^*(b')| + f(a')|C^*(a)| + f(b')|C^*(b)| - \\
&\quad\ f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= \underbrace{(f(a) - f(a'))}_{\le 0}\underbrace{(|C^*(a')| - |C^*(a)|)}_{\ge 0} + \underbrace{(f(b) - f(b'))}_{\le 0}\underbrace{(|C^*(b')| - |C^*(b)|)}_{\ge 0} \\
&\le 0
\end{aligned}$$

Summary

Greedy algorithms make natural local choices in their search for a globally optimal solution.

Many good heuristics are greedy:

- Simple
- Work well in practice

Proof that a greedy algorithm finds an optimal solution:

- Induction
- Exchange argument

Useful data structures:

- Union-find data structure
- Thin Heap

Analysis of a sequence of data structure operations:

- Amortized analysis
- Potential functions