New subspaces from old:

If V a vector space, u, w are subspaces, $u \cap w$ is also a subspace.

We even showed this for any number of subspaces:

If $w_i, i \in I$ are all subspaces, the $\bigcap_{i \in I} w_i$ is a subspace.

Here I is called an indexing set.

One next homework, we look at $u \cup w$

Hint: Try $V = \mathbb{R}^2$

u =x-axis=span$((1,0)) = \{(x,0)|x \in \mathbb{R}\}$ w =y-axis

sums of subspaces:

Have V a vector space, u, w subspaces.

Then $u + w = \{u + w | u \in U, w \in W\} = \{v \in V|$v can be written as $v = u + w, u \in U, w \in W\}$

Let's prove that $u + w$ is a subspace, need to show:

1. $0_v \in u + w$
2. If $v_1, v_2 \in u + w$, then $v_1 + v_2 \in u + w$
3. If $c \in F$ and $v \in u + w$ then $c \cdot v \in u + w$

- $0_v \in u + w$ because $0_v = 0_{v\, in\, u} + 0_{v\, in\, w}$
- Say $v_1, v_2 \in u + w$, want $v_1 + v_2 \in u + w$

  That means $v_1 = u_1 + w_1$ for some $u_1 \in u, w_1 \in w$

  $v_2 = u_2 + w_2$ for some $u_2 \in u, w_2 \in w$

  So $v_1 + v_2 = (u_1 + w_1)_(u_2 + w_2) = (u_1 + u_2)_{in\, u} + (w_1 + w_2)_{in\, w}$

- closed under scalar mult is an exercise

Fact: $U + W$ is the smallest subspace of V that contains both u and w.

More formally: $u \in u + w$, $w \in u + w$

If $X \in V$ is a subspace and $U \in X, W \in X$ then $U + W \in X$

pf: we won't do the whole thing, will show $U \subset U + W$

Let's show $U \subset U + W$

i.e. if $x \in U$, then $x \in U + W$ ==¿ $x = u + w$

$x = x + 0$, x in U and 0 in W

Note: saying $v \in U + W$ means $v = u + w$ for some $u \in U, w \in W$

Fact: if $U \cap W = \{0\}$

then if $v \in U + W$, there's exactly one choice of u, w to make $v = u + w$

pf: suppose $v + u_1 + w_1$ and $v = u_2 + w_2$, we'll show $u_1 = u_2$, $w_1 = w_2$

So $u_1 + w_2 = u_2 + w_2$

$u_1 - u_2 = w_1 - w_2$

So $u_1 - u_2 \in U \cap W$, so $u_1 - u_2 = 0$ so $u_1 = u_2$. similarly, $w_1 = w_2$

In this case, your book says $U + W$ is a direct sum ("direct" means $U \cap W = \{0\}$)

Take V, W any two vector spaces (over the same field F), define a new vector space $V \oplus W$ called the direct sum of V, W

vectors in $V \oplus W$ are ordered pairs (v, w)

$(x, y) + (a, b) = (x + a, y + b)$

$c \cdot (x, y) = (cx, cy)$

For $U + W$, U, W were both subspaces of V

If $v \in U + W$, might be many ways to write $v + u + w$

For $U \oplus W$, u, w don't need to be related ($\mathbb{R} \oplus [-1, 1]$ is ok), $(x, y) = (w, z) \leftrightarrow x = w, y = z$

Fact: If U, W are subspaces of V, and $U \cap W = \{0\}$

Then there's an invertible linear map. $T : U \oplus W \to U + W$ (we call these isomorphic vector spaces)

We always have a linear map, $T : U \oplus W \to U + W$ defined by $T(u, w) = u + w$. (True even if $U \cap W \neq \{0\}$). This map is always surjective.

Need to show:

For any $y \in U + W$, $\exists x \in U \oplus W$ s.t. $T(x) = y$

since $y \in U + W$, then $y = U + W$ for some $u \in U, w \in W$

So $y = T(u, w)$ (the x above is (u, w))

To show injective, we'll use that $U \cap W = \{0\}$

Need: If $T(u_1, w_1) = T(u_2, w_2)$, then $(u_1, w_1) = (u_2, w_2)$

If $T(u_1, w_1) = T(u_2, w_2)$, that means $u_1 + w_1 = u_2 + w_2$

we just showed this means $u_1 = u_2, w_1 = w_2$