Coordinate Theorem

let V be a VS/F with basis $\mathbb{B} = \{v_1 \dots v_n\}$, and $v \in V$.

Then $\exists!\, \alpha_1 \dots \alpha_n \in F$ s.t. $v = \alpha_1 v_1 + \dots \alpha_n v_n$.

($\exists!$ $\rightarrow$exists unique)

Proof. span$\{\mathbb{B}\} = V$;

$\mathbb{B}$ satisfies span$\{\mathbb{B}\} = V$, $u, v \in$Span($\mathbb{B}$)

$\alpha_1 \dots \alpha_n$ unique

Suppose $\alpha_1 v_1 \dots \alpha_n v_n = v = \beta_1 v_1 + \dots + \beta_n v_n,\ \beta_1 \dots \beta_n \in F$

So, $(\alpha_1 - \beta_i)v_1 + \dots + (\alpha_n - \beta_n)v_n = 0$

Hence $\alpha_i = \beta_i$, $\forall i$ as $\mathbb{B}$ is linearly independent.

$\square$

Question: is theorem true if V is a VS/F with basis $\mathbb{B}$?

Important Exercise: let V be a VS/F, $v_1, \dots, v_n \in V$.

Then span$\{v_1 \dots v_n\}$=span$\{v_2 \dots v_n\}$ iff $v_1 \in$ span$\{v_2 \dots v_n\}$

Induction

To prove a statement by induction, let P(n) be statement for any $n \in \mathbb{Z}^+$

that is either true of false for each n

To show P(n) is true $\forall n$

To do that you first prove P(1) is true

Now suppose P(n) is true (the induction hypothesis)

Then, show P(n) $\rightarrow$ P(n+1)

[Note: n must be arbitrary]

[Note: can stand at any n e.g. $n = 0$]]

Toss Out: let V be a VS/F and suppose that V can be spanned by a finite number of vectors, then V is a fdvs/F

More precisely, if $v_1 \ldots v_n \in V$ s.t. $\{v_1 \ldots v_n\}$ spans V, then a subset of $\{v_1 \ldots v_n\}$ is a basis for V

Proof. If $V = 0$, done by definition

So we may assume $V \neq 0$ and $V = \text{span}\{v_1 \ldots v_n\}$

We prove the theorem by induction on n

$n = 1$ as $V \neq 0, v_1 \neq 0$

Hence $\{v_1\}$ is a basis for V since $\alpha v_1 = 0 \rightarrow \alpha = 0$

hence $n = 1$ is true

Induction hypothesis: suppose span($w_1 \ldots w_n$)=v then a subset of $\{w_1 \ldots w_n\}$ is a basis

To show the result holds if $V = \text{span}\{v_1 \ldots v_{n+1}\}$

if $\{v_1 \ldots v_{n+1}\}$ is linearly independent, then a basis by definition and done

Suppose $\{v_1 \ldots v_{n+1}\}$ is linearly dependent

Then $\exists \alpha_1 \ldots \alpha_{n+1} \in F$ not all zero

s.t. $\alpha_1 v_1 + \ldots + \alpha_{n+1} v_{n+1} = 0$

Changing notation, we may assume that $\alpha_{n+1} \neq 0$ so ${\alpha_{n+1}}^{-1}$ exists Then

$v_{n+1} = -{\alpha_{n+1}}^{-1} \alpha_1 v_1 - \ldots {\alpha_{n+1}}^{-1} \alpha_n v_n \in \text{span}\{v_1 \ldots v_n\}$

By important exercise

$V = \text{span}\{v_1 \ldots v_{n+1}\} = \text{span}\{v_1 \ldots v_n\}$

By the case n, $\{v_1 \ldots v_n\}$. Hence $\{v_1 \ldots v_{n+1}\}$ contains a basis for V. $\square$

Example

1. let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in F^n$ & $S_n := \{e_1 \ldots e_n\} \subset F^n$

   let $V \in F^n$ then $\exists \alpha_1 \ldots \alpha_n \in F$ s.t. $v = (\alpha_1 \ldots \alpha_n) = \alpha_1 e_1 \ldots \alpha_n e_n$

   So $V = \text{span}\{S\}$ if $0 = \alpha_1 e_1 + \ldots + \alpha_n e_n = (\alpha_1 \ldots \alpha_n) = (0, \ldots, 0)$

   $\rightrightarrows \alpha_i = 0 \, \forall \, i$

   $\therefore S_n$ is a basis called Standard Basis for F^n

   More generally, let $e_{ij} \in F^{m \times n}$ be the matrix with 0's in all entries, except the ij$^{\text{th}}$ place where it is 1.

   Then $S = S_{m,n} = \{e_{ij} | 1 \leq i \leq m, \, 1 \leq j \leq n\}$ is a basis for $F^{m \times n}$

   Called standard basis for $F^{m \times n}$

2. let $V = F[t]$ if $f \in V$, then $\exists \alpha_0 \ldots \alpha_n \in F$ s.t. $f = \alpha_0 + \alpha_1 t + \ldots + \alpha_n t^n$

   So, $\text{span}\{1, t, t^2, \ldots\} = F[t]$ & $\{1, t, t^2, \ldots\}$ is linearly independent hence a basis for $F[t]$

   $\alpha_0 + \alpha_1 t + \ldots + \alpha_n t^n = 0$, then $\alpha_i = 0 \; \forall i$

3. $F[t]_n = \{f \in F[i] \,|\, \text{degree } f \leq n \text{ or } f = 0\} \subset F[t]$

   $F[t]_n = \text{span}\{1, t, \ldots, t^n\}$ & $\{1, t, \ldots, t^n\}$ is linearly independent on $\{1, t, t^2, \ldots\}$

4. $V = \mathbb{C}$ as a VS/R then $\{1, \sqrt{-1}\}$ is a basis for $\mathbb{C}$ as a VS/R

   $\alpha 1 + \beta\sqrt{-1}$, $\alpha, \beta \in \mathbb{R}$ uniquely

   $V = \mathbb{C}$ as a VS/$\mathbb{C}$ then $\{1\}$ is a basis of $\mathbb{C}$ as a VS/$\mathbb{C}$

More generally, F is a field F is a VS/F with basis $\{V\}$ if $V \neq 0$

if $\alpha \in F$ then $\alpha = \alpha v^{-1} v$ as v has a mult inverse

e.g. $\{\pi\}$ is a basis for $\mathbb{R}$ as a VS/R

5. $\{e^{-x}, e^{3x}\}$ is a basis for $V = \{f \in \mathbb{C}^2 \mid f'' - 2f' - 3f = 0\}$

6. Given $v_1, \ldots, v_m \in F^n$, know how to find span$\{v_1, \ldots, v_m\}$ and a basis for the span

1 Replacement Theorem

Let V be a VS/F & $\{v_1 \ldots v_n\}$ is a basis for V

Suppose $v \in V$ and $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$, $\alpha_1 \ldots \alpha_n \in F$ with $\alpha_i \neq 0$, then $\{v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n\}$ is a basis for V