1 Functions

Notation: $f : A \to B$ means f is a function that inputs elements from the set A, and outputs elements from the set B.

ie. for each $a \in A$, there is some $f(a) \in B$

A is called the domain

B is called the codomain

e.g.: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$

non-examples:

$f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \sqrt{z}$

we say f is not well-defined since for each z there are two choices for $\sqrt{z}$

$f : \mathbb{Q} \to \mathbb{R}$ defined by $f(\frac{a}{b}) = a$

This is bad because $f(\frac{1}{2}) = 1$ but $f(\frac{2}{4}) = 2$

Definition: say $f : A \to B$, f is called injective (one to one) when if $f(a) = f(b)$ then $a = b$

i.e.: for each $y \in B$, there is at most one $x \in A$ so that $f(x) = y$

i.e.: if $a \neq b$, then $f(a) \neq f(b)$

e.g: $f(x) = x^3$

e.g.: $g : \mathbb{R} \to \mathbb{R}[x]$ defined by $g(c) = cx + c$

This is injective because if $g(c_1) = g(c_2)$, then $c_1 x + c_1 = c_2 x + c_2$ so $c_1 = c_2$

e.g.: $f(x) = x^2$ is not injective

Definition: say $f : A \to B$, f is surgective when for each $b \in B$, there is some $a \in A$ so that $f(a) = b$.

i.e.: for each $t \in B$, there is at least one $a \in A$ so that $f(a) = b$

e.g.: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not surgective because there is no $x \in \mathbb{R}$ so that $f(x) = -1$

e.g.: $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$ is surgective, because for each $y \in \mathbb{R}$, $g(\sqrt[3]{y}) = y$

Definition: say $f : A \to B$, f is called bijective when f is both injective and surgective $\leftarrow$ good for proofs

i.e.: For each $y \in B$, there is exactly one $x \in A$ so that $f(x) = y$

e.g.: $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$ is bijective.

e.g.: Have A be any set, and define $id_A : A \to A$, $id_A(x) = x$, no matter what A we pick, this is bijective.

(so if $A = \mathbb{R}$, then id_A is line $y = x$)

Proof:

Let's show id_A is injective

suppose $id_A(x) = id_A(y)$, want $x = y$.

$x = y$, done

Next show id_A is surjective

pick $y \in A$ we want to find $x \in A$ so that $id_A(x) = y$

Well $id_A(y) = y$, so we win.

Definition: Suppose $f : A \to B$, and $g : B \to C$, then $g \circ f : A \to C$ is called the composition of f and g, defined by $(g \circ f)(x) = g(f(x))$

e.g.: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$g : \mathbb{R} \to \mathbb{R}$, $g(x) = cos(x)$

$(g \circ f)(x) = \cos(x^2)$ and $(f \circ g)(x) = \cos^2(x)$

2 Inverse

Suppose $f : A \to B$, we say f is left-invertible if $\exists g : B \to A$ such that $(g \circ f)(x) = id_A$ and right-invertible if $(f \circ g)(x) = id_B$.

f is invertible if both left and right invertible.

Fact:

any left-inverse or right-inverse is actually a two-sided inverse.

f is injective $\leftrightarrow$ f is left-invertible

f is surjective $\leftrightarrow$ f is right-invertible

f is bijective $\leftrightarrow$ f is invertible

Proof (only for #3):

First assume f bijective, will prove f is invertible.

Since f is bijective, for each $b \in B$, there is exactly one $a \in A$ such that $f(a) = b$

Define: $g : B \to A$ by $g(b) =$ that one $a \in A$ such that $f(a) = b$

Look: $f(g(b)) = b$ so $f \circ g = id_B$

To be continued...