1 Fields

F is a field.

$F \in F[t]$ inclusion as constant polys.

The 1 and 0 of F, $F[t]$ are the same.

$1F = 1Ft^0 + 0t^1 + 0t^2 + \cdots$

$0 = 0 + 0t^1 + \cdots$

cf:RCC same 0&1

Let $MnF := \{n \times n$ square matrices entries in $F\}$

$A, B \in MnF$, $A_{ij} = ij^{\text{th}}$ entry of A

then $A = B$ iff $A_{ij} = B_{ij}$ $\forall i_j$

Define:

$+$ on MnF by $(A+B)_{ij} = A_{ij} + B_{ij}$

$\bullet$ on MnF by $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$

then MnF satisfies all the axioms except:

(*)(M4), (M3) [&(M3')] (in general)

$n = 1$, M_1F is essentially F

$n > 1$, (*) holds

$$0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, 1 = I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

2 Vector Spaces

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a, b) | a, b \in \mathbb{R}\}$

coordinate addition & scalar multiplication

Define: let F be a field and V a set, we call V a **vector space over** F,

write:

V a VS/F under

$+$: $V \times V \to V$ (addition)

write $v + w := +(v, w)$

$\bullet$: $F \times V \to V$(scalar multiplication)

write $\alpha v := \bullet(\alpha, v)$

if the following axioms hold, $\forall v, w, z \in V$, $\forall \alpha, \beta \in F$

- $(v + w) + z = v + (w + z)$
- $\exists 0 \in V$, s.t. $v + 0 = v = 0 + v$
- (2) holds and $v + (-1) \cdot v = 0$
- $v + w = w + v$
- $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) \cdot v$
- $a \cdot v = v$
- $(\alpha + \beta)v = \alpha v + \beta v$
- $\alpha(v + w) = \alpha v + \alpha w$

Comments: $\forall a$ VS/F

- The zero of F is unique, write 0_F or 0. The zero of V is unique, write 0_V or 0. But almost always different.
- If $v, w \in V$, $\alpha \in F$, $\alpha v + w \in V$. But vw & $v\alpha$ does not make sense.
- Will write vectors using roman letters. Scalars, greek. Exception: $(x_1, \ldots, x_n) \in \mathbb{R}^n$.
- +: $V \times V \to V$ says, if $v, w \in V$, then $v + w \in V$. We say V is **closed** under +. Similarly, $\alpha \in F, v \in V \to \alpha v \in V$, we say V is closed under scalar multiplications.

3 Examples

F is a field

0.F

1. $F^n = \{(\alpha_1, \ldots, \alpha_n) | \alpha_i \in F, \forall i\}$

   under **component wise operations** i.e. addition or scalar multiplication, is in VS/F

2. let $I \subset \mathbb{R}$ be a subset, e.g. open on closed, internal. Then

   $Fcn(I) := \{f : I \to \mathbb{R} | fcn\}$

   set of **Real valued Functions** on I.

   Define $\forall f, g \in Fcn(I)$, $\forall \alpha \in R$.

+:=$f+g$ by $(f+g)(x) := f(x)+g(x), \forall x \in I$

•: αf by $(\alpha f)(x) := \alpha(f(x))$

0 is $0 \cdot x = 0$, $\forall x \in I$ is a VS/R.

3. $C(I) = \{f \in Fcn(I) | f continuous on I\}$

4. $Diff(I) = \{f \in Fcn(I) | f is diff on I\}$

5. $C^n(I) = \{f \in Fcn(I)|$ the n^{th} derivative $\mathrm{f}^{(\mathrm{n})}$ exists and is on $I\}$

6. $C^\infty(I) := \{f \in Fcn(I)|$ $\mathrm{fc}^{(\mathrm{n})}$ exists on I, $\forall n\}$

7. $C^w(I) := \{f \in Fcn(I)|$ f converges to its Taylor series in I in its neighborhood of each pt in I$\}$

8. $Int(I) := \{f \in Fcn(I)|$ f integrable on I$\}$

9. F[t] is a VS/F

10. let $n \geq 0$ in $\mathbb{Z}$

    let $F[t] = \{f \in F(I)|$ degree of $f \leq n\}$

    DEGREE of f in $f(I) = 1$, s.t. coefficient of $t^n \neq 0$.

11. let $F^{m \times n} = \{m \times n$ matrices entries in F$\}$

    $A = B$ iff $A_{ij} = Bij \forall i, j$

    with $(A+B)_{ij} = A_{ij} + Bij$

    $(\alpha A)_{ij} = \alpha A_{ij}$

12. let V be a VS/F, $X \neq \phi$ a set

    let $W := \{f : X \to V | f a fcn\}$

    Define: $\forall f, g \in X, \alpha \in F$

$(f+g)(x) := f(x) + g(x)$

$(\alpha f)(x) := \alpha f(x)$

$0(x) = 0_V, \forall x \in X$ is a VS/F

13. let $F \subset K$ be fields $+, \bullet$ from K, came 0&1. (e.g. RCC)

    Then K is a VS/F by $+$ on K

    And Restriction of $\bullet$ on K