New subspaces from old:

If $V$ a vector space, $U, W$ are subspaces, $U \cap W$ is also a subspace.

We even showed this for any number of subspaces:

If $W_i, i \in I$ are all subspaces, the $\bigcap_{i \in I} W_i$ is a subspace.

Here $I$ is called an indexing set.

One next homework, we look at $U \cup W$

Hint: Try $V = \mathbb{R}^2$

$U$ =x-axis=span$((1,0)) = \{(x,0)|x \in \mathbb{R}\}$ $W$ =y-axis

sums of subspaces:

Have $V$ a vector space, $U, W$ subspaces.

Then $U + W = \{u + w | u \in U, w \in W\} = \{v \in V|$v can be written as $v = u + w, u \in U, w \in W\}$

Let's prove that $U + W$ is a subspace, need to show:

1. $0_v \in U + W$
2. If $v_1, v_2 \in U + W$, then $v_1 + v_2 \in U + W$
3. If $c \in F$ and $v \in U + W$ then $c \cdot v \in U + W$

- $0_v \in U + W$ because $0_v = 0_{v\, in\, U} + 0_{v\, in\, W}$
- Say $v_1, v_2 \in U + W$, want $v_1 + v_2 \in U + W$

  That means $v_1 = u_1 + w_1$ for some $u_1 \in U, w_1 \in W$

  $v_2 = u_2 + w_2$ for some $u_2 \in U, w_2 \in W$

  So $v_1 + v_2 = (u_1 + w_1)_(u_2 + w_2) = (u_1 + u_2)_{in\, U} + (w_1 + w_2)_{in\, W}$

- closed under scalar mult is an exercise

Fact: $U + W$ is the smallest subspace of $V$ that contains both $U$ and $W$.

More formally: $U \subset U + W$, $w \subset U + W$

If $X \in V$ is a subspace and $U \subset X, W \subset X$ then $U + W \subset X$

*Proof.* we won't do the whole thing, will show $U \subset U + W$

Let's show $U \subset U + W$

i.e. if $x \in U$, then $x \in U + W$ ==¿ $x = u + w$

$x = x + 0$, $x$ in $U$ and 0 in $W$ $\square$

Example 1: $V = \mathbb{R}^2$

Have $U = \text{span}\{(1, 0, 0)\}$, $W = \text{span}\{(0, 1, 1)\}$

$U + W$ is the plane that contains $U, W$

say $v \in U + W$ means $v = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Example 2: $V = \mathbb{R}^2, U = \mathbb{R}^2, W =$x-axis$=$ span$\{(1, 0)\}$

$U + W = \mathbb{R}^2$

Since $W \subset U$ already, adding in those vectors doesn't change $U$

Note: saying $v \in U + W$ means $v = u + w$ for some $u \in U, w \in W$. It is possible to have more than one choice for $u, w$

Fact: if $U \cap W = \{0\}$

then if $v \in U + W$, there's exactly one choice of $u, w$ to make $v = u + w$

*Proof.* suppose $v + u_1 + w_1$ and $v = u_2 + w_2$, we'll show $u_1 = u_2$, $w_1 = w_2$

$\therefore u_1 + w_2 = u_2 + w_2$

$\therefore u_1 - u_2 = w_1 - w_2$

$\therefore u_1 - u_2 \in U \cap W$

$\therefore u_1 - u_2 = 0$

$\therefore u_1 = u_2$.

similarly, $w_1 = w_2$ $\square$

In this case, your book says $U + W$ is a direct sum ("direct" means $U \cap W = \{0\}$)

Take $V, W$ any two vector spaces (over the same field F), define a new vector space $V \oplus W$ called the direct sum of $V, W$

vectors in $V \oplus W$ are ordered pairs $(v, w)$

$(x, y) + (a, b) = (x + a, y + b)$

$c \cdot (x, y) = (cx, cy)$

For $U + W$, $U, W$ were both subspaces of $V$

If $v \in U + W$, might be many ways to write $v = u + w$

For $U \oplus W$, $U, W$ don't need to be related ($\mathbb{R} \oplus [-1, 1]$ is ok), $(x, y) = (w, z) \leftrightarrow x = w, y = z$

Fact: If $U, W$ are subspaces of $V$, and $U \cap W = \{0\}$

Then there's an invertible linear map. $T : U \oplus W \to U + W$ (we call these isomorphic vector spaces)

We always have a linear map, $T : U \oplus W \to U + W$ defined by $T(u, w) = u + w$. (True even if $U \cap W \neq \{0\}$). This map is always surjective.

Need to show:

For any $y \in U + W$, $\exists\, x \in U \oplus W$ s.t. $T(x) = y$

since $y \in U + W$, then $y = u + w$ for some $u \in U, w \in W$

So $y = T(u, w)$ (the $x$ above is $(u, w)$)

To show injective, we'll use that $U \cap W = \{0\}$

Need: If $T(u_1, w_1) = T(u_2, w_2)$, then $(u_1, w_1) = (u_2, w_2)$

If $T(u_1, w_1) = T(u_2, w_2)$, that means $u_1 + w_1 = u_2 + w_2$

we just showed this means $u_1 = u_2, w_1 = w_2$

$\therefore (u_1, w_1) = (u_2, w_2)$