For fun problem:

suppose $f : A \to B$, $g : B \to C$

$g \circ f$ is injective

what can you say about g and f

similar question with $g \circ f$ surjective

Linear Independence:

Suppose V a VS/F, and $v_1, \ldots, v_n \in V$

These vectors are called linearly independent when the only solution to $c_1 v_1 + \ldots + c_n v_n = 0, c_i \in F$ is the trivial solution $c_1 = \ldots = c_n = 0$

If $S \subset V$ is a subset of V, we call S linearly independent when if $c_1 v_1 + \ldots + c_n v_n = 0$ for some $c_i \in F, v_i \in S$ then $c_1 = \ldots = c_n = 0$

i.e. all finite collections of elements in S are linearly independent.

If $v_1, \ldots, v_n \in V$ then we define $\text{span}\{v_1, \ldots, v_n\} = \{c_1 v_1 + \ldots + c_n v_n \,|\, c_i \in F\} = \{\text{linear combinations of } v_1 \ldots v_n\}$

$\text{span}\{v_1 \ldots v_n\}$ is the smallest subspace of V that contains $v_1 \ldots v_n$

If $S \subset V$ is a subset

$\text{span}\{S\} = \{c_1 v_1 + \ldots + c_n v_n \,|\, c_i \in F, v_i \in S\} = \{$finite linear combinations of elements in $S\}$

Example

Have $V \in \mathbb{R}^2$, let's show that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is linearly independent and span of it is V

Proof. To prove $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_1, \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_2$ is linearly independent

Assume $c_1v_1 + c_2v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for some $c_1, c_2 \in \mathbb{R}$

Need to show $c_1 = c_2 = 0$

Yes this is true

To prove $\text{span}\{v_1, v_2\} = V$, we need to show that only vector in V can be represented as a linear combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Pick $(x, y) \in V$, $xv_1 + yv_2 = (x, y)$ $\square$

Still have $V \in \mathbb{R}^2$

$V = \text{span}\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 24 \\ -15 \end{pmatrix}\}$

To prove this we would need to take $(x, y) \in V$, find $c_1, c_2 \in R$ so that

$c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 24 \\ -15 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
can do this by row reduction

$$\begin{bmatrix} 3 & 24 & | & x \\ 1 & -15 & | & y \end{bmatrix}$$

If we row reduce and get on solution, that means your set doesn't span all of V

We could also say $V = \text{span}\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 15 \end{pmatrix}, \begin{pmatrix} \pi \\ -e \end{pmatrix} \}$

turns out $V = \text{span}\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \}$, the extra vectors are "redundant"

Linear Independence Facts

If $S \subset V$ contains exactly one element, then S is linearly dependent iff $S = \{0_v\}$

Proof.

$\rightarrow$ direction

Say $S = \{x\}$, we know $c \cdot x = 0_v$, for some $c \neq 0$ (because S is independent), divided by c, $x = 0_v$

$\leftarrow$ direction

If $S = \{0_v\}$

$1 \cdot 0_v = 0_v$ and $1 \neq 0$

So this is a nontrivial linear combination that gives 0_v

So S is dependent $\square$

Exercise

Find 3 vectors in $\mathbb{R}^3$ so that any pair of vectors is independent, but the set of all 3 is linearly dependent

Moral of the story:

If I want to prove S is linearly independent, I cannot split S into parts. Need to use the definition of linear independence on the entire S.

Fact:

If S_1 and S_2 are two subsets of V, and both S_1 and S_2 are linearly independent.

Then $S_1 \cap S_2$ is linearly independent $\rightleftarrows$ $\text{span}\{S_1\} \cap \text{span}\{S_2\} = \{0_v\}$.

Proof.

Trick: If $c_1x_1 + \ldots + c_nx_n \in \text{span}\{S_1\}$ and $d_1y_1 + \ldots + d_ny_n \in \text{span}\{S_2\}$ and they're equal, then $c_1x_1 + \ldots + c_nx_n - d_1y_1 - \ldots - d_ny_n = 0$

Example: Working with $V = C[0, \pi]$

prove that $\{sin, cos\}$ is linearly independent

To do that, assume $c_1 sin + c_2 cos = 0$

This means: For every $x \in [0, \pi]$, $c_1 sin(x) + c_2 cos(x) = 0_v(x) = 0$

Pick $x = 0$, $c_1 0 + c_2 1 = 0$ so $c_2 = 0$

Pick $x = \frac{\pi}{2}, c_1 1 + x_2 0 = 0$, so $c_1 = 0$ $\square$

Fact: If $S \subset V$ and span$\{S\} = V$

If $S \subset U \subset V$ then span$\{U\} = V$

Fact: If $S \subset V$ and S is linearly independent

If $U \subset S$ then U is linearly independent

If $c_1 x_1 + \ldots + c_n x_n = 0$, $(x_1 \ldots x_n \in U)$

Since $x_1 \ldots x_n \in S$ and S is linearly independent

must have $c_1 = \ldots = c_n = 0$