1 Fields

what properties do the REAL numbers have?

algebracally:

$\exists$ two functions (maps):

+: $R \times R \mapsto R$ (addition)

$\cdot : R \times R \mapsto R$ (multiplication)

XXXX axioms:

Def: let F be a , then F is a field under maps:

addition: with $a + b := +(a, b)$

multiplication: with $ab := \cdot(a, b)$

satisfying all of the following:

$\forall a, b, c \in F$:

A1: (a+b)+c = a+(b+c) (ASSOCIATIVITY)

A2: $\exists$ an element $0 \in F$ such that $a + 0 = a = 0 + a$ (EXISTENCE OF ZERO)

A3: A2 hold and $\forall x \in F, \exists y \in F$ such that $x + y = 0 = y + x$ (EXISTENCE OF ADD)

A4: $x + y = y + x$ (COMMUTATIVITY)

M1: $a(bc) = (ab)c$

M2: $\exists 1 \in F$ such that $1 \neq 0$ and $a \cdot 1 = a = 1 \cdot a$

M3: M2 hold, $\forall \alpha \neq x \in F \exists y \in F$ such that $xy = 1 = yx$

M4: $ab = ba$

D1: $a(b + c) = ab + ac$

D2: $(a + b)c = ac + bc$

comments: let F be a field, a, b, c $\in$ F, then the followings are true

- $F \neq \phi$ and in fact at least there are 0, 1
- 0, 1 are unique
- If $a + b = 0$, then b is unique written -a
- If $a + b = 0 = a + c$, then $b = b + 0 = b + (a + c) = (b + a) + c = (a + b) + c = 0 + c = c$
- If $a + b = a + c$, then $b = c$
- If $a \neq 0$ and $ab = 1$, then b is unique written a^{-1}
- $0 \cdot a = 0$ TO BE FILLED
- If $a \cdot b = 0$, then $a = 0$ or $b = 0$
- If $ab = ac$, $a \neq 0$, then $b = c$
- $(-a)(-b) = ab$
- $-(-a) = a$
- If $a \neq 0$, $a^{-1} \neq 0$, then $(a^{-1})^{-1} = a$

2 example

1. R is a field, Q is a field, C is a field

2. Z is not a field

3. F = {0, 1}

4. $\exists$ fields with n elements with n = {2, 3, 4, 5, 7, 8, 9, 11, 13, 16}

5. Let F be a field

let F[t] := {polynomials in inderterminant t}:

$(a_0 + a_1 t + ... + a_n t^n)(b_0 + b_1 t + ... + b_n t^n) = a_0 b_0 + (a_0 b_1 + a_1 b_0)t + ...$

6. Let F be a field such that $F(t) := \left\{ \frac{f}{g} | f, g \in F[t], g \neq 0 \right\}$

with $\forall f, g, h, k \in F(t), g \neq 0, k \neq 0$

$\frac{f}{g} + \frac{h}{k} = \frac{kf+hg}{gk}$

$\frac{f}{g} \cdot \frac{h}{k} = \frac{fh}{gk}$

is a field with

$0 = \frac{0}{1} = \frac{0}{g}$

$1 = fracff$

field of rational plays