

1 Fields

what properties do the REAL numbers have?

algebracally:

\exists two functions (maps):

$+: R \times R \mapsto R$ (addition)

$\cdot: R \times R \mapsto R$ (multiplication)

XXXX axioms:

Def: let F be a , then F is a field under maps:

addition: with $a + b := +(a, b)$

multiplication: with $ab := \cdot(a, b)$

satisfying all of the following:

$\forall a, b, c \in F$:

A1: $(a+b)+c = a+(b+c)$ (ASSOCIATIVITY)

A2: \exists an element $0 \in F$ such that $a + 0 = a = 0 + a$ (EXISTENCE OF ZERO)

A3: A2 hold and $\forall x \in F, \exists y \in F$ such that $x + y = 0 = y + x$ (EXISTENCE OF ADD)

A4: $x + y = y + x$ (COMMUTATIVITY)

M1: $a(bc) = (ab)c$

M2: $\exists 1 \in F$ such that $1 \neq 0$ and $a \cdot 1 = a = 1 \cdot a$

M3: M2 hold, $\forall x \neq 0 \in F \exists y \in F$ such that $xy = 1 = yx$

M4: $ab = ba$

D1: $a(b + c) = ab + ac$

D2: $(a + b)c = ac + bc$

comments: let F be a field, $a, b, c \in F$, then the followings are true

- $F \neq \emptyset$ and in fact at least there are 0, 1
- 0, 1 are unique
- If $a + b = 0$, then b is unique written $-a$
- If $a + b = 0 = a + c$, then $b = b + 0 = b + (a + c) = (b + a) + c = (a + b) + c = 0 + c = c$
- If $a + b = a + c$, then $b = c$
- If $a \neq 0$ and $ab = 1$, then b is unique written a^{-1}
- $0 \cdot a = 0$ TO BE FILLED
- If $a \cdot b = 0$, then $a = 0$ or $b = 0$
- If $ab = ac$, $a \neq 0$, then $b = c$
- $(-a)(-b) = ab$
- $-(-a) = a$
- If $a \neq 0$, $a^{-1} \neq 0$, then $(a^{-1})^{-1} = a$

2 example

1. \mathbb{R} is a field, \mathbb{Q} is a field, \mathbb{C} is a field
2. \mathbb{Z} is not a field
3. $F = \{0, 1\}$

4. \exists fields with n elements with $n = \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16\}$

5. Let F be a field

let $F[t] := \{\text{polynomials in indeterminate } t\}$:

$$(a_0 + a_1t + \dots + a_nt^n)(b_0 + b_1t + \dots + b_nt^n) = a_0b_0 + (a_0b_1 + a_1b_0)t + \dots$$

6. Let F be a field such that $F(t) := \left\{ \frac{f}{g} \mid f, g \in F[t], g \neq 0 \right\}$

with $\forall f, g, h, k \in F(t), g \neq 0, k \neq 0$

$$\frac{f}{g} + \frac{h}{k} = \frac{kf+hg}{gk}$$

$$\frac{f}{g} \cdot \frac{h}{k} = \frac{fh}{gk}$$

is a field with

$$0 = \frac{0}{1} = \frac{0}{g}$$

$$1 = \frac{f}{f}$$

field of rational plays