Optimization with more than one variable

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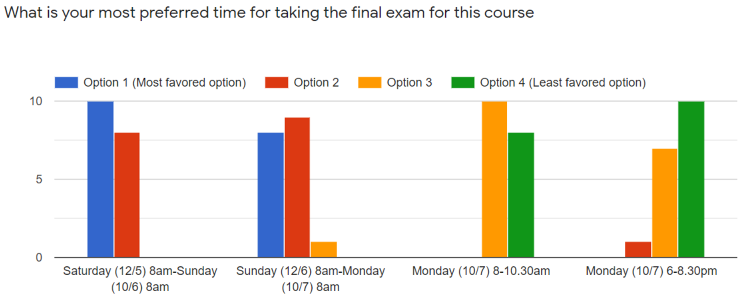
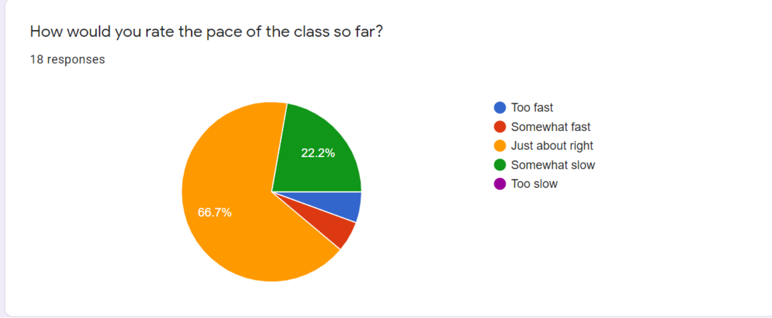
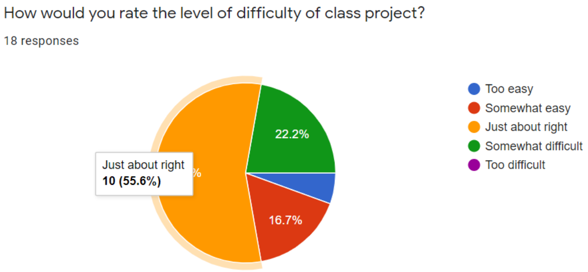
Exam Time



What is your most preferred time for taking the final exam for this course
Option 1 (Most favored option)
Option 2
Option 3
Option 4 (Least favored option)
10
5
0
Saturday (12/5) 8am-Sunday (10/6) 8am
Sunday (12/6) 8am-Monday (10/7) 8am
Monday (10/7) 8-10.30am
Monday (10/7) 6-8.30pm

Pace of the class




How would you rate the level of difficulty of class project?
18 responses
22.2%
16.7%
Just about right
10 (55.6%)
Too easy
Somewhat easy
Just about right
Somewhat difficult
Too difficult

Important Dates

- 2 December 2020: Project Evaluations Due
- 5 December 2020: Final Exam
- 9 December 2020: Final Project Due after incorporating peer feedback
- 14 December 2020: Bonus Homework Due

Winter Break Plans

- Thoroughly review materials covered in class.
- Additionally, read following chapters of your textbook:
  - Chapter 10 : Exponential and Logarithmic Functions
  - Chapter 14: Integral Calculus

Multivariable Functions

- So far, we focused on cases where dependent variable is a function of only one independent variable.
- But in reality dependent variables are functions of more than one independent variables.

$$y = f(x_1, x_2, x_3, ..., x_n)$$

  - where the variables $x_i (i = 1, 2, ..n)$ are all *independent* of one another $\Rightarrow$ changes in one do not affect the others.

- In analytical works, we are often interested in identifying the effects of change in *one* of the independent variables on the dependent variable (a.k.a *ceteris paribus*)
- The difference quotient can be expressed as:

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta, x_2, x_3, ..., x_n) - f(x_1, x_2, x_3, ..., x_n)}{\Delta x_1}$$

Partial Differentiation

- Taking limits of the difference quotient as $\Delta x_1 \to 0$, we get the **partial derivative of y with respect to x_1**.

$$f_1 \equiv \frac{\partial f}{\partial x_1} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

  - The symbol $\frac{\partial f}{\partial x_1}$ is read as "partial dee-eff-by-dee-x_1"

- The process of finding partial derivatives is known as **partial differentiation**.

- Mechanics of partial differentiation:
  - When differentiating with respect to x_i, all other independent variables($x_j, j \neq i$) are regarded as *constants*.
    - $\Rightarrow$ x_{-i}s will drop out if they are *additive constants* but retained if they are *multiplicative constants*.
  - Use the rules of simple differentiation for x_i.
- Question: What is the interpretation of a partial derivative?

Practice Problems

Find the partial derivatives of the following functions and calculate their values at the point $(x, y) = (1, 3)$.

1. $f(x, y) = x^2 y + y^5$
2. $f(x, y) = (x + 4)(3x + 2y)$
3. $f(x, y) = \frac{3x-2y}{x^2+3y}$

Gradient Vector

All the partial derivatives of a function $y = f(x_1, x_2, ..., x_n)$ can be collected under a single mathematical entity called the *gradient vector*, or simply the *gradient*, of function f.

Gradient Vector

$$gradf(x_1, x_2, ..x_n) = (f_1, f_2, ...f_n) = \nabla f(x_1, x_2, ..x_n)$$

where $f_i = \frac{\partial f}{\partial x_i}$.

- The mathematical symbol ∇ is the inverted version of the Greek letter Δ and is read as: "del".

Derivatives and Differentials

- Recall that, by definition, the derivative $\frac{dy}{dx} = f'(x)$ is the limit of the difference quotient:

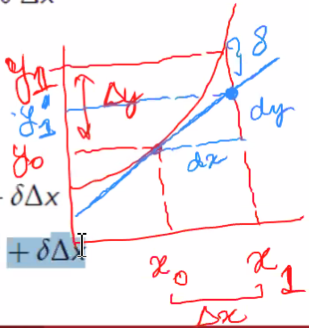
$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- What if $\Delta x \nrightarrow 0$? Then

$$\frac{dy}{dx} = f'(x) \neq \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- Define $\delta = \frac{\Delta y}{\Delta x} - \frac{dy}{dx}$. Then

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{dy}{dx} + \delta \\ \Rightarrow \Delta y &= \frac{dy}{dx}\Delta x + \delta \Delta x \\ \Rightarrow \Delta y &= f'(x)\Delta x + \delta \Delta x \end{aligned}$$



Differentials

- Replacing Δx and Δy by dx and dy, we get:

$$dy = f'(x)dx$$

- Written this way, the derivative $f'(x)$ can be used to approximate Δy.
- Given a specific value of $dx(\Delta x)$, we can get dy as an approximation of Δy, with the understanding that the smaller the Δx, the better the approximation.
- The quantities dx and dy are called the *differentials* of x and y, respectively.
- dy now depends of x as well as dx.
- Written this way, the derivative $f'(x)$ can be reinterpreted as the factor of proportionality between two finite changes dy and dx.

Total Differentials

- The concept of differentials when extended to functions of more than one variable gives rise to *total differentials* and *partial differentials*.

- For functions of more than one variable, the differential is given by:

$$dy = \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2 + ... + \frac{\partial y}{\partial x_n}dx_n$$

  where

  - dy is the total differential of y.
  - $\frac{\partial y}{\partial x_i}dx_i$ are the partial differentials of y.

Practice Problems

Find the total differential for each of the following functions:

1. $y = ax_1 + bx_2$.
2. $y = x_1^2 + x_2^3 + x_1x_2$.
3. $z = x^a y^b$.

$$dy = \frac{\partial y}{\partial x_1} \cdot dx_1 + \frac{\partial y}{\partial x_2} \cdot dx$$

$$\frac{\partial y}{\partial x_1} = [2x_1 + x_2]$$

$$\frac{\partial y}{\partial x_2} = [3x_2^2 + x_1]$$

Total Derivatives

- Dividing the total differentials with respect to dx_i gives total derivative of y with respect to x_i.

$$\frac{dy}{dx_i} = \frac{\partial y}{\partial x_1}\frac{dx_1}{dx_i} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dx_i} + ... + \frac{\partial y}{\partial x_n}\frac{dx_n}{dx_i}$$

- When $x_1, x_2, ..., x_n$ are all independent of one another, then

$$\frac{dy}{dx_i} = \frac{\partial y}{\partial x_i}$$

- Question:What if $x_1, x_2, ..., x_n$ are no longer independent of x_i?

$f(x, y(x))$

$x \to y \to f$

$$df = \underbrace{\frac{\partial f}{\partial x} \cdot dx}_{\text{Direct Effect}} + \underbrace{\frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \cdot dx}_{\text{Indirect effect of } x}$$

$$\frac{dy}{dx_i} = \underbrace{\frac{\partial y}{\partial x_i}}_{\text{Direct effect of } x_i} + \underbrace{\Sigma_{j \neq i} \frac{\partial y}{\partial x_j}\frac{dx_j}{dx_i}}_{\text{Indirect effect of } x_i}$$

Practice Problem

Find the total derivative $\frac{dy}{dw}$ given the function

$$y = f(x, w) = 3x - w^2$$

where $x = g(w) = 2w^2 + w + 4$.

$$\boxed{y = 3x - w^2}, \quad x = 2w^2 + w + 4$$

$$\frac{dy}{dw} = \frac{\partial y}{\partial w} + \frac{\partial y}{\partial x} \cdot \frac{dx}{dw}$$

$$= (-2w) + 3[4w + 1]$$

$$= 10w + 3$$

Generalization of the Chain Rule for Multivariate Functions

- For simplicity, assume that $y = f(x_1, x_2, ..., x_n)$ where $x_1, x_2, ...x_n$ are continuous and differentiable functions of a variable t.

The Chain Rule

If the first, second and mixed partial derivative of function f exist and are continuous functions of $x_1, x_2, .., x_n$, and $x_1, x_2, ...x_n$ are differentiable functions of a variable t, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + ... + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

Practice Problem

If

$$z = xy^4 + x^3y^2$$
$$x = 2 - 3t$$
$$y = 4 + 5t$$

use the chain rule to find $\frac{dz}{dt}$.

If

$$z = xy^4 + x^3y^2$$
$$x = 2 - 3t$$
$$y = 4 + 5t$$

use the chain rule to find $\frac{dz}{dt}$.

$= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$\boxed{y^4 + 3x^2y^2}$ $\quad -3 \quad$ $\boxed{4xy^3 + 2x^3y}$ $\quad 5$

$=$

Higher order partial derivatives

- **Second partial derivatives**

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right)$$

- **Mixed partial derivatives**

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

for $i \neq j$.

Practice Problems

Find the second and mixed partial derivatives of the following functions.

1. $f(x, y) = x^2y + y^5$

2. $f(x, y) = (x + 4)(3x + 2y)$

3. $f(x, y) = \frac{3x-2y}{x^2+3y}$

Mixed derivative theorem

Mixed Derivative Theorem aka Young's Theorem

If the first, second and mixed partial derivative of function f exist and are continuous functions of $x_1, x_2, .., x_n$, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

for all $i \neq j$.

Differential Version of Optimization Conditions

Consider a function $z = f(x)$. $\rightarrow dz = \boxed{f'(x) . dx}$

- Recall that, using derivatives, the optimization conditions are given by:
  1. First order condition: $f' = 0$.
  2. Second order condition for maximum: $f'' \leq 0$.
  3. Second order condition for minimum: $f'' \geq 0$.

0

$dz = 0$

- These conditions can equivalently be expressed as differentials
  1. First order condition: $dz = 0$.
  2. Second order condition for maximum: $d^2z \leq 0$ for arbitrary non-zero values of dx.
  3. Second order condition for minimum: $d^2z \geq 0$ for arbitrary non-zero values of dx.

$$d^2(z) = d(dz) = \boxed{d\left[f'(x)\, dx\right]}$$

$$= d[f'(x)]\, dx$$

$dx \rightarrow$ constants

$$d^2(z) = d[f'(x)]\, dx$$

$$= [f''(x) . dx]\, dx$$

$$= f''(x)\underbrace{(dx)^2}_{+}$$

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Finding extreme values of functions of two variables

Consider $z = f(x, y)$.

- The first order optimization condition requires

  $dz = f_x dx + f_y dy = 0$ for arbitrary value of dx and dy, not both zero

  Then the first order optimization condition can be expressed as:

  $$f_x = f_y = 0$$

- Through some additional work, it can be shown that
  - $d^2 z < 0$ iff $f_{xx} < 0$; $f_{yy} < 0$; and $f_{xx} f_{yy} > f_{xy}^2$.
  - $d^2 z > 0$ iff $f_{xx} > 0$; $f_{yy} > 0$; and $f_{xx} f_{yy} > f_{xy}^2$.

Practice Problem

Find and classify the extreme values of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$.

Optimization with Equality Constraints

- In most economic applications, the objective function $z = f(x, y)$ needs to be maximized subject to a constraint $g(x, y) = c$.

- The Lagrange-Multiplier method is to convert a constrained-extremum problem into an "unconstrained" extremum problem.
- The *Lagrangian Function* incorporates the constraint by writing the objection function as follows:

λ: choice

$$Z(x, y, \lambda) = f(x, y) + \lambda[c - g(x, y)]$$

- The stationary values can be identified by simultaneously solving

$$Z_x = 0; Z_y = 0; Z_\lambda = 0$$

x^*, y^*, λ^*

Practice Problem

Find the extremum of $z = xy$ subject to $x + y = 6$.