Ingredients of a Mathematical Model

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Ch. 2.1 – 2.5(CW)

September 1, 2020

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What is a mathematical model?

What is a mathematical model?

A. The creation of mathematical formulas to represent a real world problem in mathematical models.
B. The creation of real world problems based solely on a theoretical formula already in existence.
C. Mathematical models are intelligent fashion stars.
D. Mathematical models are toys that can be purchased in hobby shops by mathematicians.
E. None of these are correct.

When the poll is active, you will be able to respond **http://www.PollEv.com/psharma024**.

Mathematical Model

Mathematical Model

A mathematical representation of the problem/question that is being studied.

Advantages of mathematical approach:

- "language" used is more concise and precise.
- allows us to borrow and utilize existing mathematical results.
- checks us for making unwanted assumptions.
- allows us to treat the general n-variable case.

Variables and Constants

- Variables: Something whose magnitude can change.
  - Endogenous Variables: Variables whose solution value we seek from the model.
    - Often denoted with y or Y
  - Exogenous Variables: Variables whose value is assumed to determined by forces external to the model.
    - Often denoted with x or X
- Constants: Does not change.
  - Logical constants: $7, 0.5, 2$.
  - Symbolic constants: a, b, c or A, B, C or α, β, γ
  - Co-efficient of a variable: A constant joined to a variable.

Equations

- Behavioral Equations: Specifies the manner in which a variables behaves in response to changes in other variables.

$$y = \alpha + \beta x$$

- Definitions Equations or Identity: Sets up an identity between two alternate expressions that have exactly the same meaning.

$$y \equiv \alpha + \beta x$$

  - $\equiv$ is read as "is identically equal to", is equivalent to" or "is defined to be equal to".

- Conditional Equations: Specify certain "conditions" which need to satisfied by the variables.

Real number system

Real line: Graphical representation of real numbers.

- Origin: "0"
- Positive Numbers: Numbers to the right of origin.
- Negative Numbers: Numbers to the left of origin.
- Rational Numbers: Numbers which can be written as ratio of two integers $\frac{W_1}{W_2}$ where $W_2 \neq 0$
  - Integers: Whole numbers
    - Natural Numbers: Positive integers
  - Fractions: Numbers which can be written as ratio of two integers $\frac{W_1}{W_2}$ where $W_2 \neq 0$ or 1
- Irrational Numbers: Numbers which can not be expressed as a ratio of two whole numbers.

Absolute Value

Absolute Value

The absolute value of any number x, denoted $|x|$ is defined as:

$$|x| \equiv \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

- By definition, $|x| \geq 0$ always.
- Geometrically, $|x|$ represents the distance between the point x and origin on the real line.
- Absolute value of x is also known as modulus of x or mod x.

Distance between two numbers

Distance between x and y

The distance between two numbers on the real line x and y, denoted $|x - y|$ is defined as:

$$|x - y| \equiv \begin{cases} x - y & \text{if} \quad x \geq y \\ y - x & \text{if} \quad x < y \end{cases}$$

- If x and y are two real numbers, then $|x - y| = |y - x|$ would represent the distance between x and y on the real line.
- $|x| + |y| \geq |x + y|$
- $|x|.|y| = |x.y|$

Practice Problem

What is the distance between

1. 7 and 2?

2. -3 and -5?

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Conditional and Bi-conditional Statements

Let X and Y be two statements.

1. Implication arrow: "$\implies$."
2. Equivalence arrow: "$\iff$."

- If X, then Y : A conditional statement which means the following:
  - $X \implies Y$.
  - X implies Y.
  - If X, then Y.
  - Y is a consequence of X.
  - X is a sufficient condition for Y.
  - Y is a necessary condition for X.

- $X \implies Y$ and $Y \implies X$ is a bi-conditional statement which means the following:
  - $X \iff Y$.
  - X if and only if Y.
  - X and Y are equivalent.
  - X is necessary and sufficient condition for Y.

Mathematical Proofs

- Theorems: Most important results in mathematics.
- Every mathematical theorem can be written as an implication: $X \implies Y$.
  - X: premise
  - Y: conclusion
- 3 ways to prove a theorem.
  1. Direct Proof: Start with the premise X and work towards the conclusion Y.
  2. Indirect Proof: Start by assuming that Y is not true, and on that basis demonstrate that X is not true either.

     $$P \implies Q \text{ is equivalent to } \sim Q \implies \sim P$$

     - $\sim$ is read as "not."
  3. Proof by Contradiction: Start by assuming that X is true but Y is not and arrive at a contradiction.

Practice Problem

Use three different methods to prove that

$$-x^2 + 5x - 4 > 0 \implies x > 0$$

Sets Notation

- Set: Collection of distinct objects.
  - Objects may be numbers, persons, items, anything.

- Elements of the set: Objects in the set.

- Ways of writing a set:
  - Tabular form or enumeration form: Explicitly list all the elements

$$A = \{1, 2, 3\}$$

    - Order in which elements appear in a set is irrelevant.
  - Description form or Set-builder form

$$A = \{x : x \text{ is a positive integer between 1 and 3}\}$$

$$A = \{x | x \text{ is a positive integer between 1 and 3}\}$$

    - The notation "|" and ":" reads "such that".
    - The entire notation reads "A is the set of all positive integers between 1 & 3".

Types of Sets

- Infinite Set: Set with infinite number of elements
  - Denumerable or countable set

$$A = \{x|x \text{ is an integer}\}$$

  - Non-denumerable Set

$$A = \{x|x \text{ is a real number between 1 and 3}\}$$

- Finite Set: Set with a finite number of elements

$$A = \{1, 2, 3\}$$

  - All finite sets are countable.
- Null Set: Set with no elements

$$A = \{\}$$

$$A = \emptyset$$

- Singleton set: Set with exactly one element.

Subsets and Supersets

Subsets and Supersets

If every element of set A is also an element of set B, then A is a *subset* of B and B is said to be *superset* of A.

Symbolically, the above statement can be written as

Subsets and Supersets

$A \subset B \iff x \in A \implies x \in B$

- The symbol "$\subset$" is also read as "is contained in".
- Then $A \subset B$ is read as "A is contained in B" or "A is a subset of B".
- The symbol $\in$ reads "is an element of" or "belongs to the set".
- $x \in A$ is read as " x is an element of set A".
- $A \subset B$ is same as writing $B \supset A$.
- The symbol "$\supset$" is also read as "contains".
- $B \supset A$ is read as "B contains A" or "B is a superset of A".

Equal Sets

- Largest subset: the set itself.
- Smallest subset: null set $\emptyset$

Equal Sets

Let A and B be two sets. Then

$$A = B \iff A \subset B \text{ and } B \subset A$$

- Universal set: set containing a possible elements
  - often denoted with $\mathbb{U}$

Disjoint Sets

Disjoint Sets

Let A and B be two sets. Then $\nexists$ x such that $x \in A$ and $x \in B$.

- The notation "$\nexists$" is read as "there does not exist".
- The entire statement reads as "There does not exist an x such that x is an element of both A and B.

Operations on Sets

- Union

$$A \cup B \equiv \{x | x \in A \text{ or } x \in B\}$$

- Intersection

$$A \cap B \equiv \{x | x \in A \text{ and } x \in B\}$$

$$A \cap B = \emptyset \iff \text{A and B are disjoint sets.}$$

- Complement

$$A^c \equiv \{x | x \notin A \text{ and } x \in \mathbb{U}\}$$

- Difference/ Relative Complement

$$A - B \equiv \{x | x \in A \text{ and } x \notin B\}$$

Practice Problems

1. Find all possible subsets of $A = \{2, 3, 5\}$.

2. Given the sets $A = \{2, 3, 5\}$, $B = \{3, 5, 6\}$ and $C = \{7\}$, find:
    1. $A \cap C$
    2. $A \cup B$
    3. $A \cap A$
    4. $\mathbb{U}$
    5. $A \cap \mathbb{U}$
    6. $A \cap \emptyset$

Laws of Set Operations

Similar to laws of algebra, set operations obey certain following rules:

- Idempotent Laws
  1. $A \cup A = A$
  2. $A \cap A = A$

- Associative Laws
  1. $(A \cup B) \cup C = A \cup (B \cup C)$
  2. $(A \cap B) \cap C = A \cap (B \cap C)$

- Commutative laws
  1. $A \cup B = B \cup A$
  2. $A \cap B = B \cap A$

- Distributive laws
  1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Laws of Set Operations

- Identity laws
  1. $A \cup \varnothing = A$
  2. $A \cup \mathbb{U} = \mathbb{U}$
  3. $A \cap \mathbb{U} = A$
  4. $A \cap \varnothing = \varnothing$

- Complement laws
  1. $A \cup A^c = \mathbb{U}$
  2. $(A^c)^c) = A$
  3. $A \cap A^c = \varnothing$
  4. $\mathbb{U}^c = \varnothing$
  5. $\varnothing^c = \mathbb{U}$

De Morgan's laws

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

Practice Problems

1. Let A, B, C be sets such that $A \subset B$ and $A \cap C = \emptyset$. Which of the following expressions can be simplified and how?
   1. $A \cap (B \cup C)$
   2. $(A \cap B) \cup C$
   3. $A \cup (B \cap C)$
   4. $(A \cup B) \cap C$

2. Given the sets $A = \{2, 3, 5\}$, $B = \{3, 5, 6\}$ and $C = \{3, 4, 6, 7\}$, verify the distributive law.

Sets and Real Numbers

- $\mathbb{R}$ is used to denote the set of all real numbers

$$\mathbb{R} \equiv \{x|x \text{ is a real number on the real line}\} \subset \mathbb{R}$$

- $\mathbb{N}$ is used to denote set of all positive integers

$$\mathbb{N} \equiv \{x|x \text{ is a positive integer in } \mathbb{R}\} \subset \mathbb{R}$$

- $\mathbb{I}$ is used to denote set of all integers

$$\mathbb{I} \equiv \{x|x \text{ is a integer in } \mathbb{R}\} \subset \mathbb{R}$$

- $\mathbb{Q}$ is used to denote the set of all rational numbers

$$\mathbb{Q} \equiv \{x|x = \frac{p}{q} \text{ where } p, q \in \mathbb{I} \text{ and } p \neq 0\} \subset \mathbb{R}$$

  - The notation $\in$ reads "is an element of" or "belongs to the set". Then $p \in \mathbb{I}$ is read as " p is an element of set $\mathbb{I}$".

Sets and Intervals on $\mathbb{R}$

An interval I is a subset of $\mathbb{R}$ with the following two properties:

1. I has more than one element in it.
2. $a, b \in I \implies c \in I \forall a < c < b$.

- The notation "$\forall$" is read as "for all".
- The entire statement is read as "if a and b are elements of an interval I, then c is also an element of I, for all c between a and b.

Let $a, b \in \mathbb{R}$ where $a < b$, then we have the following terminology:

- Closed interval on $\mathbb{R}$

$$A = \{x | a \leq x \leq b\} \subset \mathbb{R}$$

$$A = [a, b]$$

Note that $a \in A$ and $b \in A$

Intervals on $\mathbb{R}$

- Open interval on $\mathbb{R}$

$$B = \{x|a < x < b\} \subset \mathbb{R}$$

$$B = (a, b)$$

Note that $a \notin B$ and $b \notin B$

- Closed-open interval on $\mathbb{R}$

$$C = \{x|a \leq x < b\} \subset \mathbb{R}$$

$$C = [a, b)$$

Note that $a \in C$ and $b \notin C$

- Open-closed interval on $\mathbb{R}$

$$D = \{x|a < x \leq b\} \subset \mathbb{R}$$

$$D = (a, b]$$

Note that $a \notin D$ and $b \in D$

Practice Problems

Write the following in set notation and in interval notation.

1. The set of real numbers between 2 and 10, inclusive.
2. The set of real numbers less than 15.
3. The set of real numbers greater than 20, inclusive.

Ordered Pairs

- In sets, we do not care about the order in which elements appear.

$$\{x, y\} = \{y, x\}$$

  - We could call $\{x, y\}$ an *un-ordered pair*

- If we designate the element "x" as the "first listing" of the set and the element "y" as the second listing of the set, then we have *ordered pair*, denoted by (x, y).
  - Let x=25 depict the age of a student in this class.
  - Let y=55 depict the weight of a student in this class.
  - Then *ordered pair* $(25, 55)$ depicts the (age,weight) of a student in this class.
  - $(25, 55) \neq (55, 25)$

Cartesian Product of sets X and Y

Cartesian Product of sets X and Y

The cartesian product of two sets X and Y is defined as follows:

$$X \times Y \equiv \{(x, y) |\ x \in X, y \in Y\}$$

- The notation "$X \times Y$" is read as "X cross Y".
- The set "$X \times Y$" is the set of all possible ordered pairs (x, y), where $x \in X$ and $y \in Y$.
- Also, known as "Product Set of X and Y"
- If $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, then

$$X \times Y = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$$

Practice Problems

1. Given $A = \{1, 3, 4\}$, $B = \{x, y\}$ and $C = \{m, n\}$, find:
   1. $A \times B$
   2. $B \times C$
   3. $A \times C$
   4. $B \times A$
   5. $A \times B \times C$

- Ordered Triple: (x_1, x_2, x_3)
- Ordered n-tuple: $(x_1, x_2, x_3, ...x_n)$

2. Is $A \times B = B \times A$? Why or why not?
3. Under what conditions is it true that $A \times B = B \times A$?

Cartesian plane

Cartesian plane or Euclidean two-space $\mathbb{R}^2$

$$\mathbb{R}^2 = \{(x, y) |\ x \in \ \mathbb{R}, y \in \mathbb{R}\}$$

- Cartesian plane is the product set $\mathbb{R} \times \mathbb{R}$.
- Graphically, represented using a rectangular co-ordinate plane or xy-plane.
    - Shown as two straight lines intersecting at right angles to each other.
        - Point of intersection $\implies$ Origin $(0, 0)$
        - Horizontal line $\implies$ x-axis.
        - Vertical line $\implies$ y-axis.
    - Each ordered pair in $\mathbb{R}^2$ is reflected by a point on this plane.
        - first number $\implies$ x-coordinate, measures the horizontal distance from the point to the y-axis.
        - second number $\implies$ y-coordinate, measures the vertical distance from the point to the x-axis.
    - All points along x-axis have a y-coordinate of 0 and all points along y-axis have a x-coordinate of 0.

Function

A function from a set X into a set Y is a rule f which assigns every element of set X to a member of set Y and is written as

$$f : X \to Y$$

$$X \xrightarrow{f} Y$$

$$y = f(x) \text{ where } y \in Y \ x \in X$$

- The notation "$f : X \to Y$" is read as "f is a function from X into Y" or "f maps from X into Y".
- Set X is called the *domain* of the function f.
- Set Y is called the *co-domain* of the function f.
- If $y \in Y$ is the element in Y assigned by f to an $x \in X$, then y *is the value of* f *at* x, or, y *is the image of* x *under* f
- The notation "$y = f(x)$" is read as "y is a function of x".

More on functions

Graph of a function

The *graph Gr(f)* of the function $f : X \rightarrow \ Y$ is:

$$\text{Gr(f)} \ \equiv \{(x, f(x))|x \in X\}$$

$\text{Gr(f)} \subset X \times Y$

Range of a function

The *range* $f[X]$ of the function $f : X \rightarrow \ Y$ is:

$$f[X] \equiv \{f(x)|x \in X \subset \ Y\}$$

Types of functions

Onto functions

The function $f : X \rightarrow Y$ is *surjective or onto* if:

$$\forall y \in Y, \exists x \in X \text{ such that } y = f(x)$$

A function is onto $\iff f[X] = Y$

One-to-one functions

The function $f : X \rightarrow Y$ is injective or one-to-one if:

$$\forall x, x' \in X, f(x) \neq f(x') \iff x \neq x'$$

A one-to-one function is invertible. That is, there exists

$$f^{-1} : f[X] \rightarrow X$$

Constant Functions

$$y = b \text{ where b is a constant}$$

- The range of a constant function consists only of one element.
- In the coordinate plane, such a function will appear as a horizontal line.

Linear Functions

$$y = f(x) = ax + b$$

- f is a linear function.
- Graph of f is always a straight line.
- b: intercept term or y-intercept term.
    - $(0, b)$ always lies on the line.
- a is the slope of the line.
    - When $a > 0$, the line slants upward to the right.
    - When $a < 0$, the line slants downward to the right.
    - Higher is $|a|$, more steep is the line.

Practice Problems

Consider a function $f : X \to Y$ such that $f(x) = 5 + 3x$ where $x \in \mathbb{R}$.

1. Find the range of this function and express it as a set.

2. Express the graph of this function as a set.

3. Sketch the graph of this function.

4. Consider $X = \{x | x \in [1, 4]\} \subset \mathbb{R}$ and $Y = \mathbb{R}$.
    - Is f a surjective function?
    - Is f an injective function?
    - Does f have an inverse? If so, what is it?

$$\begin{aligned} y = f(x) &= ax^2 + bx + c \\ &= a\left[x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right] + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

- If $a > 0$, then $ax^2 + bx + c$ has a minimum at

$$-\frac{b}{2a}, c - \frac{b^2}{4a}$$

- If $a < 0$, then $ax^2 + bx + c$ has a maximum at

$$-\frac{b}{2a}, c - \frac{b^2}{4a}$$

Polynomial Function

1. Polynomial Function: A function of the general form

$$y = a_0x^0 + a_1x^1 + a_2x^2 + ... + a_nx^n$$

where $n \geq 0$

- Exponents: superscript indicators of the power of x
- Degree of the polynomial: the value of n, the highest power involved
- y is said to be a polynomial of degree n

1. Polynomial of degree 0 $\Longleftrightarrow$ Constant Function
2. Polynomial of degree 1 $\Longleftrightarrow$ Linear Function
3. Polynomial of degree 2 $\Longleftrightarrow$ Quadratic Function
4. Polynomial of degree 3 $\Longleftrightarrow$ Cubic Function

Rational Function

2. Rational function: A function which can be expressed as the ratio of two polynomials

$$y = \frac{a_0x^0 + a_1x^1 + a_2x^2 + ... + a_nx_a^n}{b_0x^0 + b_1x^1 + b_2x^2 + ... + b_nx_b^n}$$

where $n_a, n_b \geq 0$

Examples include:

- $$y = \frac{x+1}{x^2+4x}$$
- Rectangular Hyperbole
$$y = \frac{a}{x}$$

Practice Problems

Sketch the graph of the following functions:

1. $y = 5$
2. $y = 16 + 2x$
3. $y = -x^2 + 5x - 2$
4. $y = \frac{36}{x}$
5. $y = x$

Rules of Exponents

1. $x^0 = 1$
2. $x^1 = x$
3. $x^m \times x^n = x^{m+n}$
4. $\frac{x^m}{x^n} = x^{m-n}$
5. $x^{-n} = \frac{1}{x^n}$
6. $x^{\frac{1}{n}} = \sqrt[n]{x}$
7. $(x^m)^n = x^{mn}$
8. $x^m \times y^m = (xy)^m$

Practice Problems

1. Condense the following expressions:
   1. $x^6 \times x^4$
   2. $\frac{x^3}{x^{-2}}$
   3. $\frac{x^{\frac{1}{2}} \times x^{\frac{1}{3}}}{x^{\frac{2}{3}}}$