Lecture 3
GP, SDP, Relaxation

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- **Geometric Programming**
- Semi-definite Programming
- Non-convex and Approximate Algorithm

Geometric Programming

- What is This?
- Why We Need This?
- Standard Form and Transformation
- Use Cases
- Proof for Convexity

Geometric Programming

minimize $cx_1^{a_1}x_2^{a_2}x_3^{a_3}\ldots.x_n^{a_n}$

subject to $bx_1^{b_1}x_2^{b_2}x_3^{b_3}\ldots.x_n^{b_n} \leq 1$

$cx_1^{c_1}x_2^{c_2}x_3^{c_3}\ldots.x_n^{c_n} = 1$

minimize $x_1^{2}x_2^{-1}x_3^{3}$

subject to $x_1^{2}x_2^{3} \leq 1$

Why we need such optimization problem?

Geometric Programming

Geometric Programming

posynomial

minimize $\quad f_0(x)$

subject to $\quad f_i(x) \le 1, \qquad i = 1,2, \dots m$

$\qquad\qquad\quad g_i(x) = 1, \qquad i = 1,2, \dots p$

monomials

Monomials and Posynomial

posynomial

minimize $f_0(x)$

subject to $f_i(x) \leq 1, \quad i = 1,2, \dots m$

$g_i(x) = 1, \quad i = 1,2, \dots p$

monomials

Properties of Monomials

Monomials are closed under multiplication and division

if f and g are monomials:

- f*g is monomials
- f/g is monomials

Problem

Which of the followings are **monomials**?

A. $2xy$

$B.\ 2+3x+3y$

$C.\ sinx$

$D.\ {x_1}^3{x_2}^{-1/2}$

$E.\ 1$

$F.\ -xyz$

Problem

Which of the followings are **posynomial**?

A. $2xy + x + xy^{1/3}$

$B.$ $3xz - 2x$

$C.$ $2x$

$D.$ 1

$E.$ $2(1 + 2xy)^2$

$F.$ $3(1 + x)^{0.5}$

Standard Form

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 1, & i = 1,2, \dots m \\
 & g_i(x) = 1, & i = 1,2, \dots p
\end{array}
$$

posynomial

monomials

Example Problem

minimize $\quad x^{-1}y^{-1/2}z^{-1} + 2.3xz + 4xyz$

subject to $\quad (1/3)x^{-2}y^{-2} + (3/4)y^{1/2}z^{-1} \leq 1$

$x + 2y + 3z \leq 1$

$xy = 1$

Problem1: Chang to Standard Form

maximize x/y

subject to $2 \leq \mathrm{x} \leq 3$

$x^2 + 3y/z \leq \sqrt{y}$

$x/y = z^2$

Problem2: Chang to Standard Form

minimize $\sqrt{1+x^2}+(1+y/z)^{3.1}$

subject to $\dfrac{1}{x}+\dfrac{z}{y}\leq 1$

$(x/y+y/z)^{2.2}+x+y\leq 1$

Problem2: Chang to Standard Form

minimize $\sqrt{1+x^2} + (1+y/z)^{3.1}$

subject to $\frac{1}{x} + \frac{z}{y} \leq 1$

$(x/y + y/z)^{2.2} + x + y \leq 1$

Problem3: Chang to Standard Form

minimize $\max\{x + y, 1 + (y + z)^{1/2}\}$

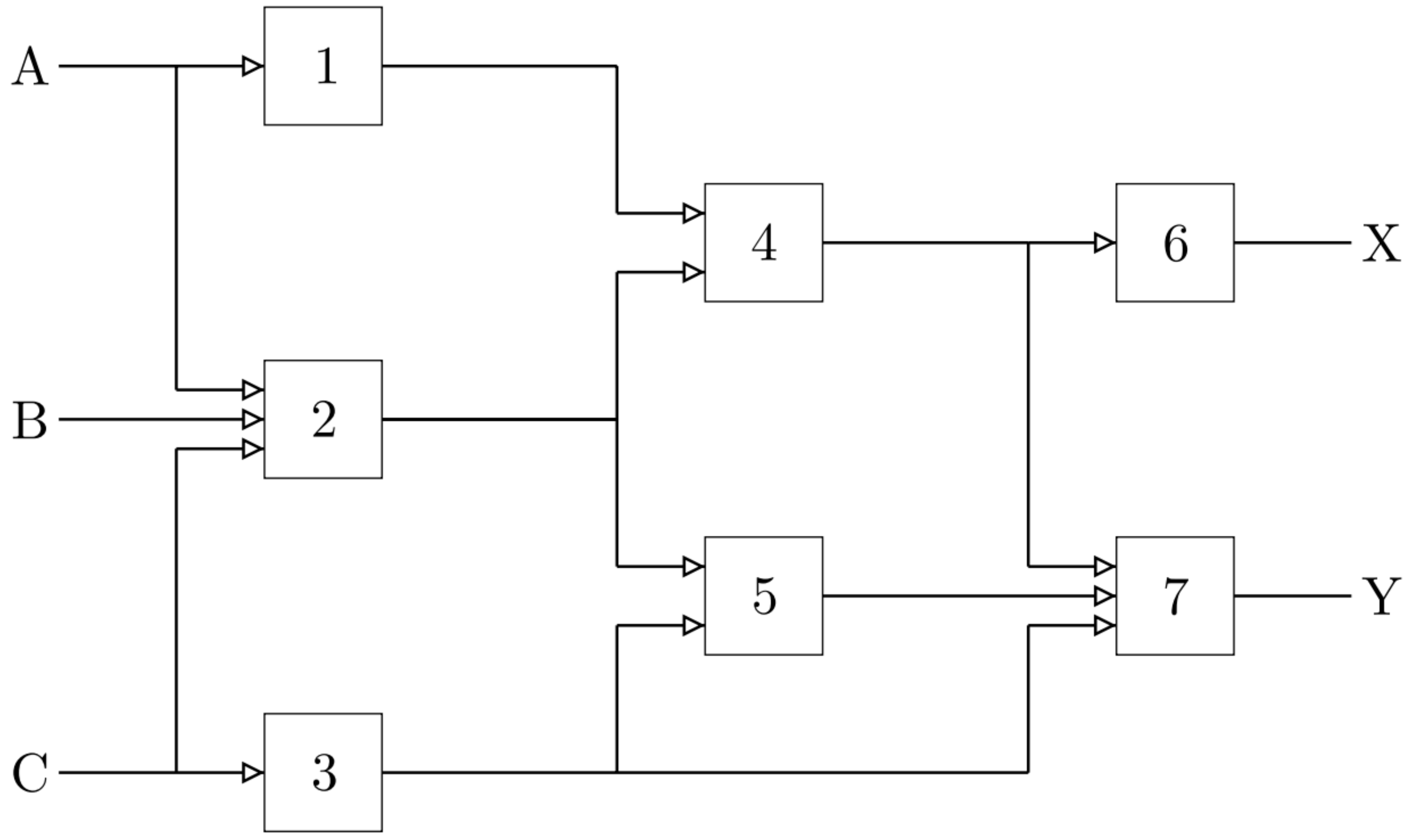
subject to $\max\{y, z^2\} + \max\{yz, 0.3\} \leq 1$

$$\frac{3xy}{z} = 1$$

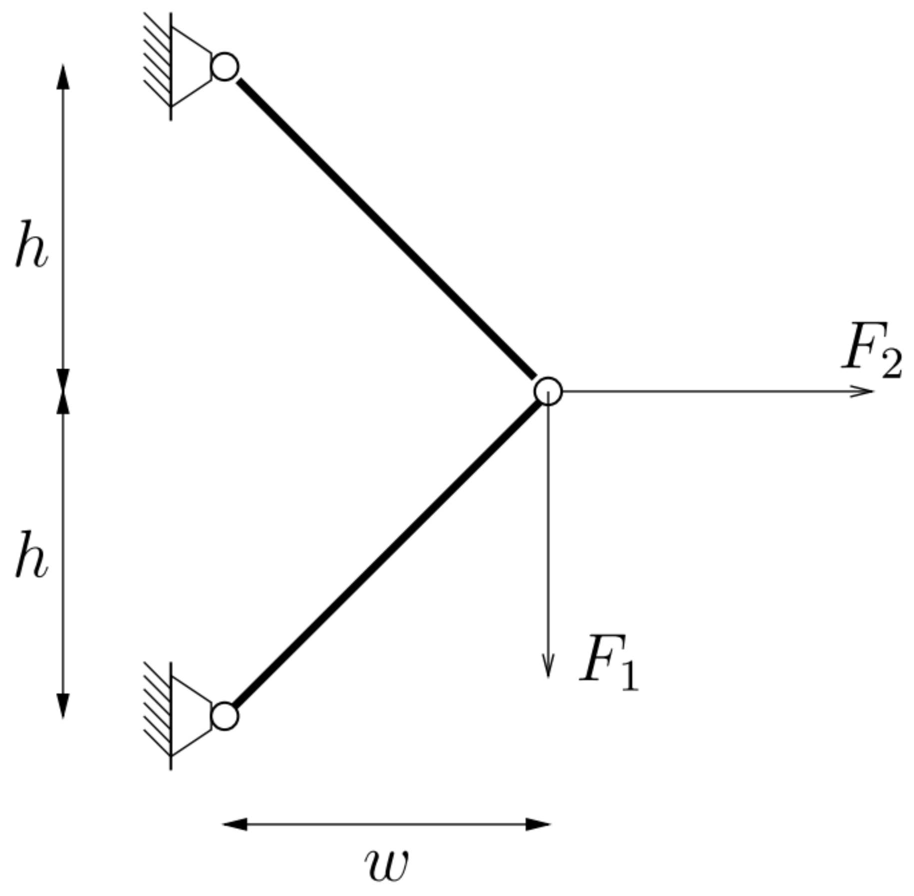
Problem

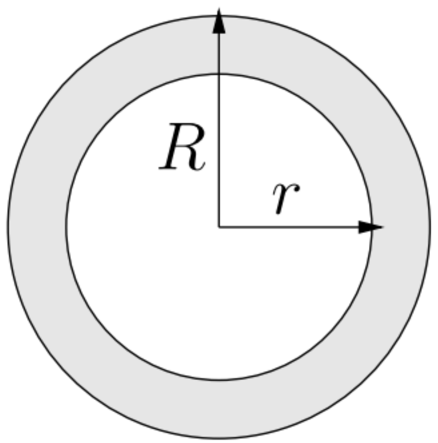
We optimize the shape of a box-shaped structure with height h, width w, and depth d. We have a limit on the total wall area $2(hw + hd)$, and the floor area wd, as well as lower and upper bounds on the aspect ratios h/w and w/d. Subject to these constraints, we wish to maximize the volume of the structure, hwd.

Other Problem – Circuit design

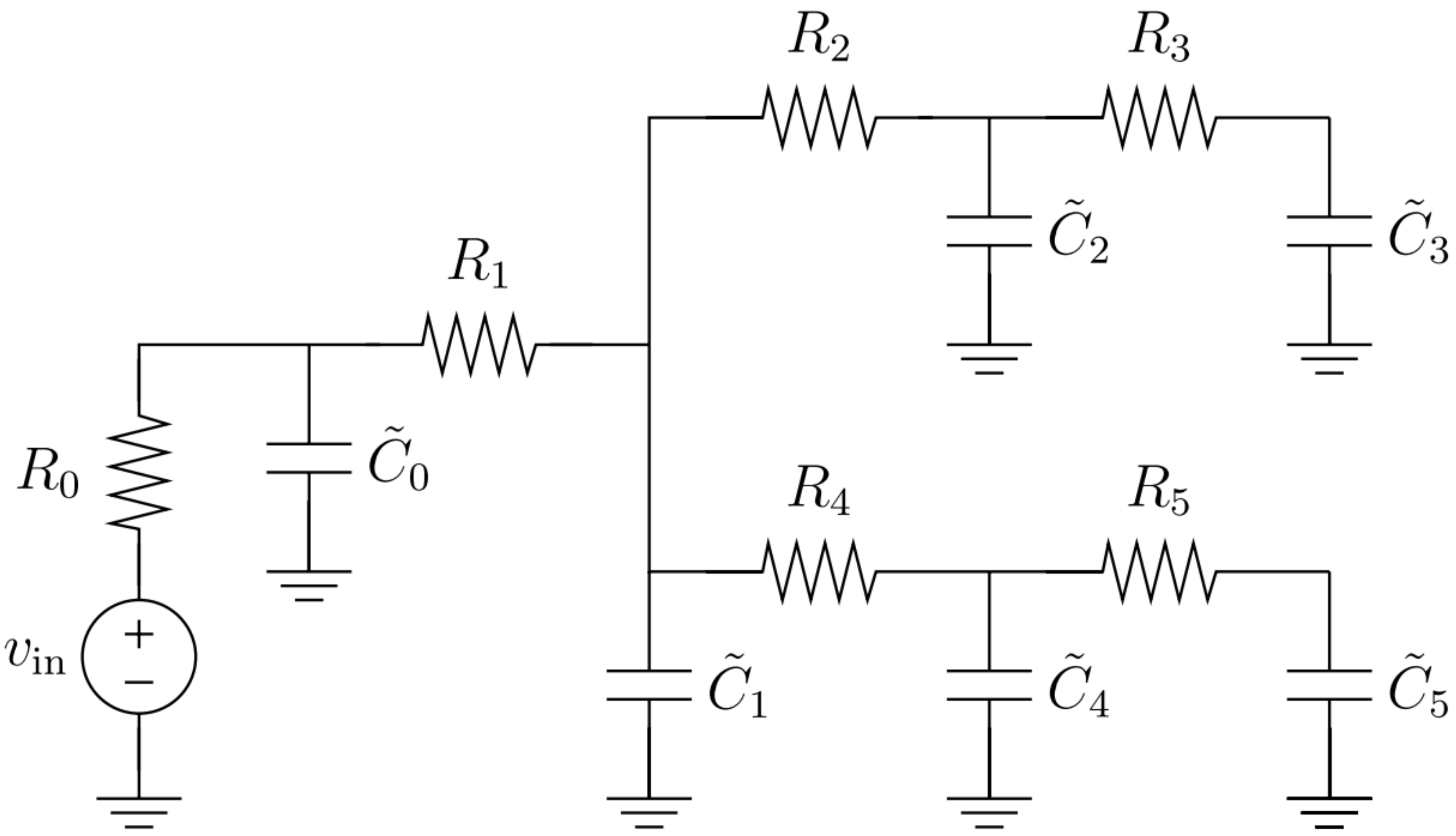


Other Problem – Truss Design





Other Problem – Wire Sizing



Convexity of Geometric Programing

$$\begin{aligned} \text{minimize} \quad & \underbrace{f_0(x)}_{\text{posynomial}} \\ \text{subject to} \quad & \underbrace{f_i(x)}_{\text{posynomial}} \le 1, \quad i = 1,2, \dots m \\ & \underbrace{g_i(x)}_{\text{monomials}} = 1, \quad i = 1,2, \dots p \end{aligned}$$

Q: Is it convex?

Convex Optimization

Two Conditions

- Feasible region: Convex set
- Objective: Convex function

Convexity of Geometric Programing

Convexity of Geometric Programing

Convexity of Geometric Programing

Feasibility Analysis

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \le 1, & i = 1,2, \ldots m \\ & g_i(x) = 1, & i = 1,2, \ldots p \end{array}$$

Feasibility Analysis

minimize $f_0(x)$

subject to $f_i(x) \leq 1, \qquad i = 1,2,\ldots m$

$g_i(x) = 1, \qquad i = 1,2,\ldots p$

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Recall: Linear Programming

minimize $\quad c^T x$

subject to $\quad {a_i}^T x = b_i \quad i = 1,2, \ldots p$

$x \geq 0$

Basics for Positive Semi-definite Matrix

If matrix M is PSD, then

For symmetric matrix M$\in S^n$, we have M = QDQ^T

Semi-definite Programming

minimize $C * X$

subject to $A_i * X = b_i \quad i = 1,2, \ldots p$

$X \succcurlyeq 0$

Semi-definite Programming

minimize $\quad C * X$

subject to $\quad A_i * X = b_i \qquad i = 1,2,\ldots p$

$X \succcurlyeq 0$

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix}$$

$b_1 = 11 \qquad b_2 = 19$

SDP is Convex?

minimize $C \ * X$

subject to $A_i * X = b_i \quad i = 1,2,...p$

$X \succcurlyeq 0$

LP is special case of SDP

minimize $c^T x$

subject to $a_i{}^T x = b_i \quad i = 1,2, \ldots p$

$x \geq 0$

minimize $C * X$

subject to $A_i * X = b_i \quad i = 1,2, \ldots p$

$X \succcurlyeq 0$

Major applications of SDP

- Non-convex problem relaxes to SDP
- Directly modeling using SDP

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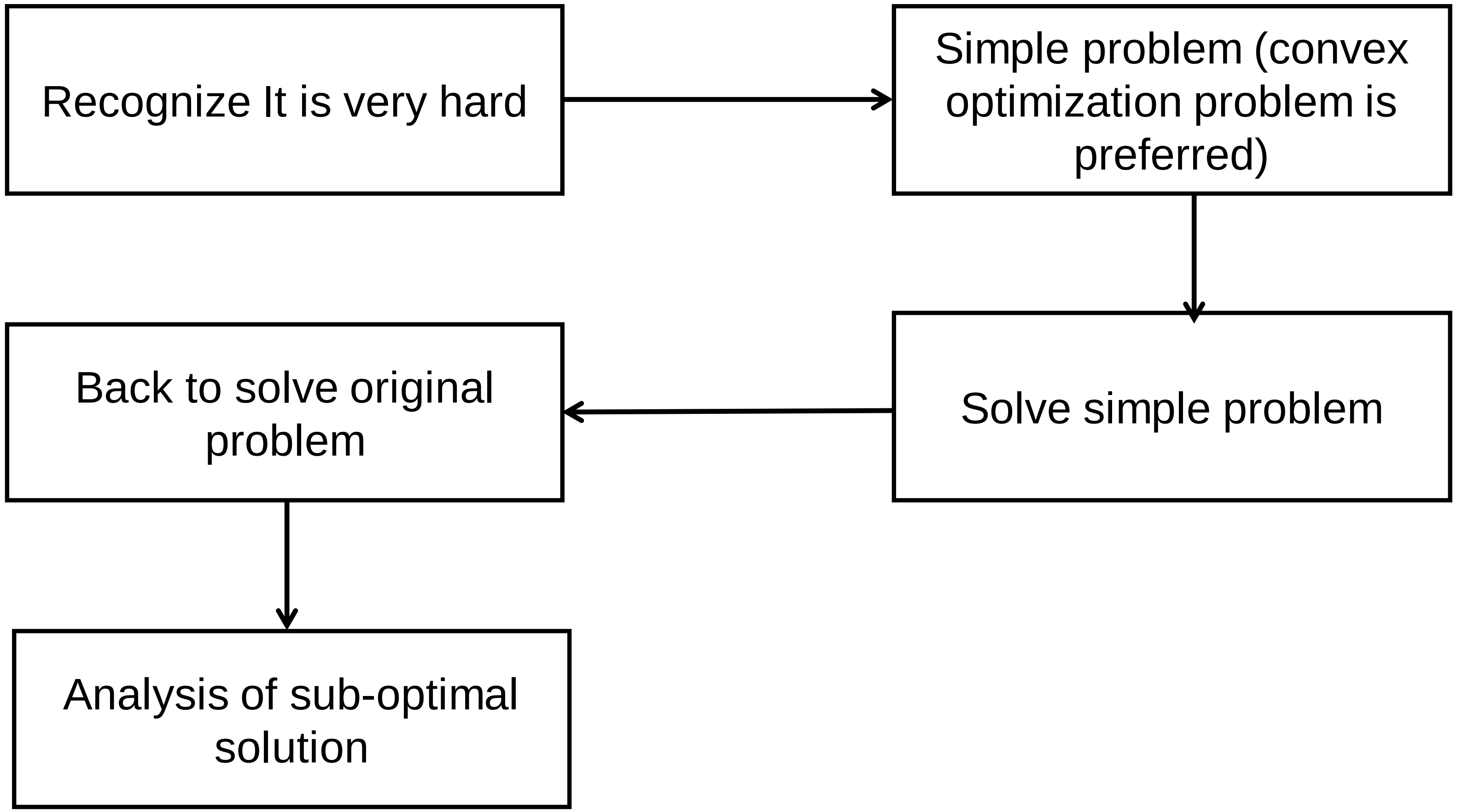
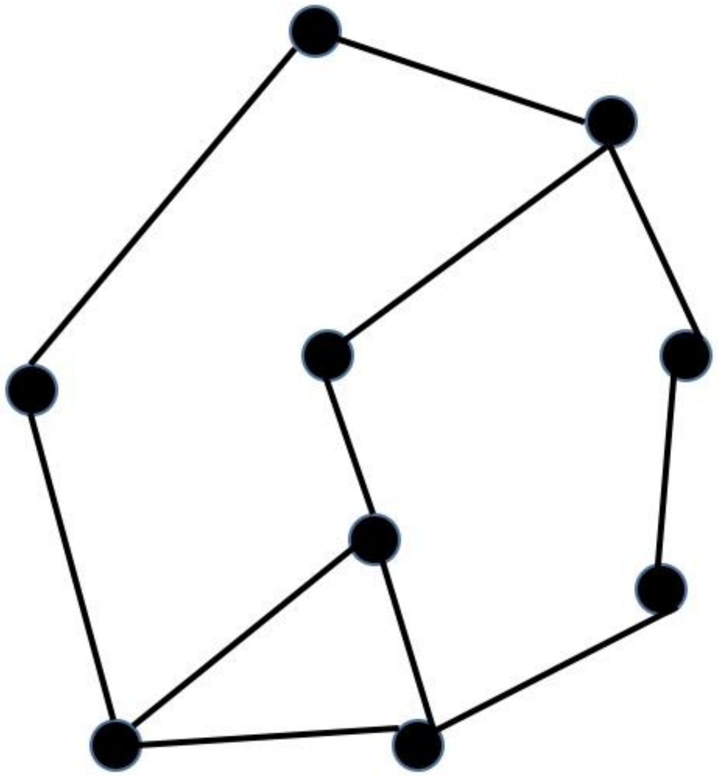
Non-convex problems

- Vertex cover problem
- Set cover problem

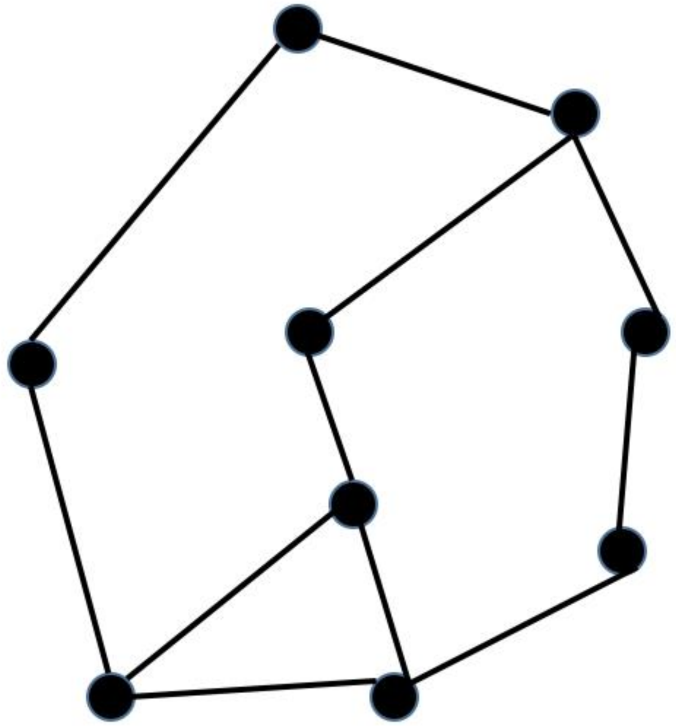
General Approach for very Hard Problem



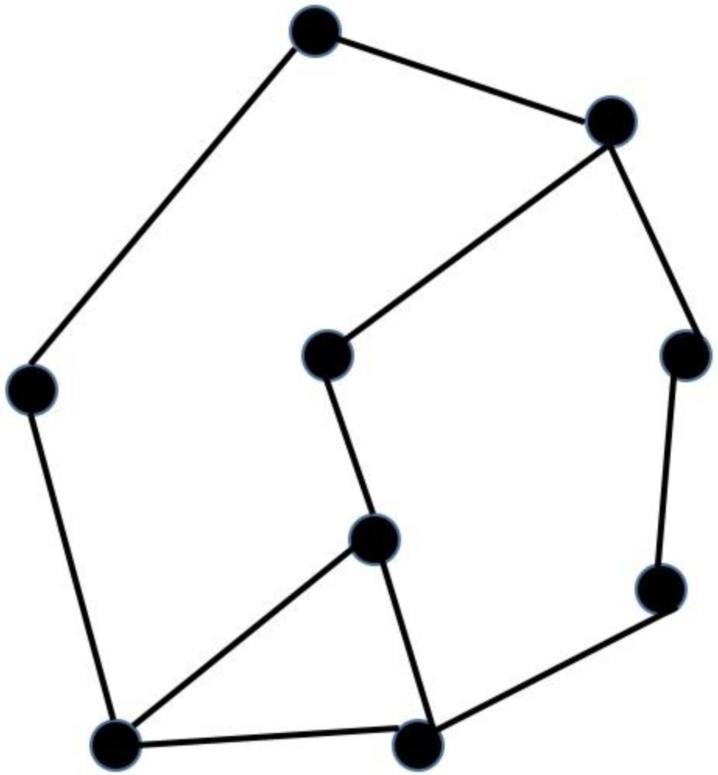
Vertex cover problem



Mathematical Programming

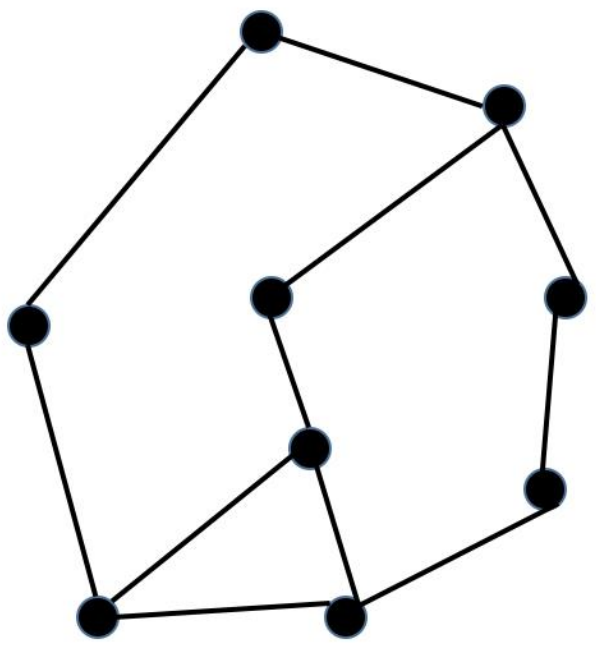


Integer Linear Programming Formulation



It is a hard problem!

Linear Programming Relaxation



Relax to simple problem and solve it

Rounding to get integer solution

Back to solve original problem

Analysis of Solution – Correctness?

Analysis of Solution – How far from true optimal?

True optimal solution: OPT

The lower and upper bound of our solution ???

Set Cover Problem

假设我们有个全集U (Universal Set), 以及m个子集合 $S_1, S_2 \dots, S_m$, 目标是要寻找最少的集合，使得集合的union等于U.

例子： U = {1,2,3,4,5}，S: {S_1={1,2,3}, S_2={2,4}, S_3={1,3}, S_4={4}, S_5={3,4}, S_6={4,5}}，最少的集合为：{1,2,3}, {4,5}，集合个数为2.

Set Cover Problem

例子： U = {1,2,3,4,5}，S: {S_1={1,2,3}, S_2={2,4}, S_3={1,3}, S_4={4}, S_5={3,4}, S_6={4,5}}，最少的集合为：{1,2,3}, {4,5}，集合个数为2.

Mathematical Programming

$$minimize \sum_{i=1}^{m} x_i$$

$$s.t. \sum_{i:e \in s_i} x_i \geq 1$$

$$x_i \in \{0,1\} \quad i = 1, \dots, m$$

It is a hard problem!

Convert to Linear Programming

$$minimize \sum_{i=1}^{m} x_i$$

$$s.t. \sum_{i:e\in s_i} x_i \geq 1$$

$$x_i \in [0,1] \quad i = 1, \ldots, m$$

Relax to simple problem and solve it

Randomize Rounding

The LP solution will give solutions for x_i, i.e., $x_1 = 0.6$

We view this as the probability of this set being selected

Back to solve original problem

Analysis of Solution – Correctness?

Analysis of the solution

Analysis of Solution – How far from true OPT

Analysis of the solution