Alternating Direction Method of Multipliers

Concept

- It was first introduced in the mid-1970s by Gabay, Mercier, Glowinski, and Marrocco, though similar ideas emerged as early as the mid-1950s. The algorithm was studied throughout the 1980s, and by the mid-1990s
- It takes the form of a *decomposition-coordination* procedure, in which the solutions to small local subproblems are coordinated to find a solution to a large global problem
- It can be viewed as an attempt to blend the benefits of **dual decomposition** and **augmented Lagrangian methods** (also called method of multipliers) for constrained optimization

https://en.wikipedia.org/wiki/Augmented_Lagrangian_method

Dual problem

- convex equality constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- Lagrangian: $L(x, y) = f(x) + y^T(Ax - b)$

y is dual variable or Lagrange multiplier

- dual function: $g(y) = \inf_x L(x, y)$
- dual problem: maximize $g(y)$
- recover $x^\star = \mathrm{argmin}_x L(x, y^\star)$

Assuming that strong duality holds, the optimal values of the primal and dual problems are the same

Dual ascent

- gradient method for dual problem: $y^{k+1} = y^k + \alpha^k \nabla g(y^k)$
- $\nabla g(y^k) = A\tilde{x} - b$, where $\tilde{x} = \operatorname{argmin}_x L(x, y^k)$
- dual ascent method is

$$
\begin{aligned}
x^{k+1} &:= \operatorname{argmin}_x L(x, y^k) && \text{// } x\text{-minimization} \\
y^{k+1} &:= y^k + \alpha^k (Ax^{k+1} - b) && \text{// dual update}
\end{aligned}
$$

$\alpha^k > 0$ is a step size   residual for the equality constraint

Dual decomposition

- suppose f is separable: $x_i \in \mathbf{R}^{n_i}$ are subvectors of x.

$$f(x) = f_1(x_1) + \cdots + f_N(x_N), \quad x = (x_1, \ldots, x_N)$$

- then L is separable in x: $L(x,y) = L_1(x_1,y) + \cdots + L_N(x_N,y) - y^T b$,

$$L_i(x_i, y) = f_i(x_i) + y^T A_i x_i \qquad A = [A_1 \ \cdots \ A_N],$$

- x-minimization in dual ascent splits into N separate minimizations

$$x_i^{k+1} \ := \ \operatorname*{argmin}_{x_i} L_i(x_i, y^k) \quad y^{k+1} \ := \ y^k + \alpha^k (\textstyle\sum_{i=1}^N A_i x_i^{k+1} - b)$$

which can be carried out in parallel

Each iteration requires a *broadcast and a gather* operation

Augmented Lagrangian methods (method of multipliers)

- Transform the primal problem

$$\begin{array}{ll} \text{minimize} & f(x) + (\rho/2)\|Ax - b\|_2^2 \\ \text{subject to} & Ax = b. \end{array}$$

$\rho > 0$ penalty parameter

- use **augmented Lagrangian** (Hestenes, Powell 1969), $\rho > 0$

$$L_\rho(x, y) = f(x) + y^T(Ax - b) + (\rho/2)\|Ax - b\|_2^2$$

- method of multipliers (Hestenes, Powell; analysis in Bertsekas 1982)

$$\begin{aligned} x^{k+1} &:= \underset{x}{\operatorname{argmin}}\, L_\rho(x, y^k) \\ y^{k+1} &:= y^k + \rho(Ax^{k+1} - b) \end{aligned}$$

(note specific dual update step length ρ)

ADMM

- ADMM problem form (with f, g convex)

$$\begin{array}{ll} \text{minimize} & f(x) + g(z) \\ \text{subject to} & Ax + Bz = c \end{array}$$

  - two sets of variables, with separable objective

variables $x \in \mathbf{R}^n$ and $z \in \mathbf{R}^m$,
$A \in \mathbf{R}^{p \times n}$, $B \in \mathbf{R}^{p \times m}$,
$c \in \mathbf{R}^p$.

- Augment the objective

$$\begin{array}{ll} \min\limits_{x} & f(x) + g(z) + \dfrac{\rho}{2}\|Ax + Bz - c\|_2^2 \\ \text{subject to} & Ax + Bz = c \end{array}$$

- Augmented Lagrangian

$$L_\rho(x, z, y) = f(x) + g(z) + y^T(Ax + Bz - c) + (\rho/2)\|Ax + Bz - c\|_2^2$$

ADMM

- ADMM repeats the steps, for k=1, 2, 3

$$
\begin{aligned}
x^{k+1} &:= \operatorname{argmin}_x L_\rho(x, z^k, y^k) && \text{// } x\text{-minimization} \\
z^{k+1} &:= \operatorname{argmin}_z L_\rho(x^{k+1}, z, y^k) && \text{// } z\text{-minimization} \\
y^{k+1} &:= y^k + \rho(Ax^{k+1} + Bz^{k+1} - c) && \text{// dual update}
\end{aligned}
$$

- Note that if we minimized over x and z jointly, reduces to method of multipliers
- Else in an alternating fashion, which accounts for the term *alternating direction*

Example: Lasso regression

- Problem

$$\text{minimize} \quad (1/2)\|Ax - b\|_2^2 + \lambda\|x\|_1$$

- ADMM form

$$\begin{array}{ll} \text{minimize} & (1/2)\|Ax - b\|_2^2 + \lambda\|z\|_1 \\ \text{subject to} & x - z = 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & f(x) + g(z) \\ \text{subject to} & x - z = 0, \end{array}$$

$$f(x) = (1/2)\|Ax - b\|_2^2$$

$$g(z) = \lambda\|z\|_1.$$

- Update

$$\begin{aligned} x^{k+1} &:= (A^TA + \rho I)^{-1}(A^Tb + \rho(z^k - u^k)) \\ z^{k+1} &:= S_{\lambda/\rho}(x^{k+1} + u^k) \\ u^{k+1} &:= u^k + x^{k+1} - z^{k+1}. \end{aligned}$$

Example: Lasso regression

➢ Soft thresholding operator S

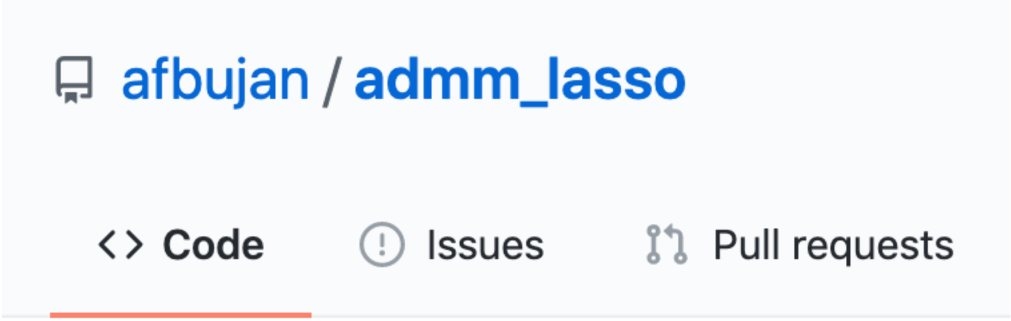
$$S_\kappa(a) = \begin{cases} a-\kappa & a>\kappa \\ 0 & |a|\le\kappa \\ a+\kappa & a<-\kappa, \end{cases}$$

$$S_\kappa(a) = (a-\kappa)_+ - (-a-\kappa)_+.$$

Code

ADMM

```
lambda = 1;
rho = 1/lambda;

tic;

x = zeros(n,1);
z = zeros(n,1);
u = zeros(n,1);

[L U] = factor(A, rho);

for k = 1:MAX_ITER

    % x-update
    q = Atb + rho*(z - u);
    if m >= n
       x = U \ (L \ q);
    else
       x = lambda*(q - lambda*(A'*(U \ ( L \ (A*q) ))));
    end
```



https://web.stanford.edu/~boyd/papers/prox_algs/lasso.html#8
https://github.com/afbujan/admm_lasso/blob/master/lasso_admm.py

Thanks

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交替方向乘子法（ADMM）算法的流程和原理是怎样的?

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Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers

S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein

Foundations and Trends in Machine Learning, 3(1):1–122, 2011. (Original draft posted November 2010.)

- Paper
- Matlab examples
- MPI example
- ADMM links and resources

https://stanford.edu/~boyd/papers/admm_distr_stats.html