Derivatives and Differentiation

Dr. Priyanka Sharma

October 12, 2020

Stuart School of Business
ILLINOIS INSTITUTE OF TECHNOLOGY

The Difference Quotient

- In our analysis, we are often interested in how one variable changes in response to changes in other variables.
  - For example, how change in consumer income changes equilibrium price and quantity?
  - More generally, if $y = f(x)$ then how does y response to change in x?

- Change is often denoted with "Δ" (the Greek capital delta, for "difference").
  - $\Delta x \equiv x^{new} - x^{old}$
  - $\Delta y \equiv y^{new} - y^{old} = f(x^{new}) - f(x^{old})$

Difference Quotient

The **difference quotient** measures the average rate of change in y, or the change in y per unit of change in x and is given by:

$$\frac{\Delta y}{\Delta x} = \frac{f(x^{new}) - f(x^{old})}{x^{new} - x^{old}} = \frac{f(x^{old} + \Delta x) - f(x^{old})}{\Delta x}$$

The difference quotient is a function of x^{old} and Δx.

The Derivative Function

- Often we are interested in the rate of change of y when Δx is very small.
- In other words, we are interested in

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Derivative Function

The derivative (or marginal) function measures the instantaneous rate of change in y and is given by:

$$\frac{dy}{dx} \equiv \frac{d}{dx} f(x) \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- $f(x)$ is a *primitive function*.
- $f'(x)$ is a derived *function*.
- The value of the derivative at x_0 is written as

$$\frac{dy}{dx}(x_0) \equiv \frac{df}{dx}(x_0) \equiv f'(x_0) \equiv \left.\frac{dy}{dx}\right|_{x=x_0}$$

Differentiability

- The process of finding the derivative function is called differentiation.
- We also use the phrase "**differentiate $f(x)$ with respect to x**" to mean "**find** $f'(x)$"
- A function is said to be differentiable at $x = x_0$ if

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists and is finite. In other words,

$$\lim_{\Delta x \to 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x)$$

Practice Problem

Given the function $y = \frac{1}{x^2}$

1. Find the difference quotient as a function of x and Δx.
2. Find the derivative function $\frac{dy}{dx}$.
3. Check whether the function is differentiable at $x = 0$.
4. Check whether the function is differentiable at $x = 3$ and $x = 4$.
5. If y is differentiable at $x = 3, 4$, then find $f'(3)$ and $f'(4)$.
6. Check whether the function is continuous at $x = 0, 3$ and 4.

Practice Problem

Consider the function $f(x) = |x-2| + 1$

1. Find the difference quotient as a function of x and Δx.
2. Check whether the function is differentiable at $x = 2$.
3. Check whether the function is continuous at $x = 2$.

Theorem

In general, if a function f is differentiable at a point x_0, then it is also continuous at that point.

- Continuity is a necessary condition for differentiability.
- Can you prove this result?

Derivative and Slope of a Straight line

- Consider a linear polynomial given by $y = ax + b$.
- Then the difference quotient between any two arbitrary points x_0, y_0 and x_1, y_1 on it is given by

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = a$$

- a is also called the **slope** of the line.
- If $a > 0$, then the line is said to be "upward sloping".
- If $a < 0$, then the line is said to be "downward sloping".
- For a straight line, the difference quotient coincides with the slope of the line.

Derivative of points along a curve

- In other words, the difference quotient between two points coincides with the slope of the line between two points.
- For a non-linear curve, the different quotient depends on both x_0 and Δx.
- The value of the derivative function at x_0 is given by the slope of the tangent to the function at point $(x_0, f(x_0))$

Rules of Differentiation

Find the derivative of

1. $f(x) = b$

Constant-Function Rule

The derivative of a constant-function $f(x) = b$ is 0.

$$\frac{d}{dx}f(x) = \frac{d}{dx}b = 0$$

2. $f(x) = x$
3. $f(x) = x^2$
4. $f(x) = x^n$

Generalized Power-function rule

The derivative of a power function $f(x) = cx^n$ is cnx^{n-1}.

$$\frac{d}{dx}f(x) = \frac{d}{dx}cx^n = cnx^{n-1}$$

Practice Problems

1. Find the derivative of each of the following functions:
    1. $f(x) = 12$
    2. $f(x) = \sqrt{x}$
    3. $f(x) = 12x^2$
2. Find $f'(1)$ and $f'(4)$.

More rules of differentiation

Sum-Difference Rule

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$$

Product Rule

$$\frac{d}{dx}\left[f(x).g(x)\right] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) = f(x)g'(x) + g(x)f'(x)$$

Quotient Rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

Quotient Rule is a special case of product rule.

Practice Problems

Differentiate the following functions:

1. $7x^4 + 2x^3 - 3x + 37$
2. $(9x^2 - 2)(3x + 1)$
3. $\frac{x+9}{x}$

Chain Rule

If $z = f(y)$ is a differentiable function of y and $y = g(x)$ is a differentiable function of x, then the composite function $(fog)(x) or z(x) = f[g(x)]$ is a differentiable function of x and

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(y)g'(x)$$

- Accounts for the chain effect of a change in x

$$\Delta x \overset{via g}{\rightarrow} \Delta y \overset{via f}{\rightarrow} \Delta z$$

- In case of three functions, $z = f(y), y = g(x)$ and $x = h(w)$

$$\frac{dz}{dw} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dw}$$

Practice Problems

Find $\frac{dz}{dx}$ for the following functions:

1. $z = 3y^2$ where $y = 2x + 5$

2. $z = (x^2 + 3x - 2)^{17}$

- There is often more than one way of differentiating a particular function.
- It often saves times to simplify before differentiating.

L'Hôpital's Rule

L'Hôpital's Rule

If f and g are differentiable at a, with $f(a) = g(a) = 0$ and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Practice Problem

Compute the limit of the following function.

$$\lim_{x \to 2} \frac{3x^2 + 3x - 18}{x - 2}$$

Mean Value Theorem

If f is continuous in the closed bounded interval $[a, b]$ and differentiable in the open interval (a, b), then there exists at least one interior point ξ in (a, b) such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Test the mean-value theorem on $f(x) = x^3 - x$ in $[0, 2]$.

Monotonic Functions

- In the context of functions defined on real-space, one-to-one functions arise only in case of strictly monotone (or monotonic) functions.

Strictly Increasing functions

A function is said to be **strictly increasing function** if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$, for all x_1, x_2 in the domain of X.

- If $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$, then function is said to be **weakly increasing function** or **non-decreasing function**.

Strictly decreasing functions

A function is said to be **strictly decreasing function** if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$, for all x_1, x_2 in the domain of X.

- If $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$, then function is said to be **weakly decreasing function** or **non-increasing function**.

Practice Problems

Which of the following functions are monotonic?

1. $$y = x^2 (x > 0)$$

2. $$y = x^2 (x < 0)$$

3. $$y = x^2$$

Testing for monotonicity

Strictly increasing functions

If $f'(x) \geq 0$ for all x and $f'(x) = 0$ only for a finite number of values of x, then function is said to be **strictly increasing function**.

A function is said to be an increasing (or non-decreasing) function if $f'(x) \geq 0$ for all x.

Strictly decreasing functions

If $f'(x) \leq 0$ for all x and $f'(x) = 0$ only for a finite number of values of x, then function is said to be **strictly decreasing function**.

A function is said to be decreasing function (or non-increasing) if $f'(x) \leq 0$ for all x.

Practice Problems

Using calculus determine which of the following functions are monotonic?

1. $$y = x^2$$

2. $$y = x^5 - 10x^3 + 45x$$

3. $$y = x - x^{-1}, \quad x > 0$$

4. $$y = x + |x|$$

Monotonicity and Inverse Functions

Inverse Function

If f is a monotonic function and the solution of the equation $f(x) = y$ is $x = g(y)$, then g is called the inverse function of f.

- In general, $f(g(y)) = x$ and $g(f(x)) = y$.
- The inverse function $g(y)$ must also be monotonic.
- If g is the inverse function of f, then f is the inverse function of g.
- In general, $g(x) \neq \frac{1}{f(x)}$

Inverse Function Rule

Inverse Function Rule

Let f be a monotonic function with inverse function g; if, for given x, f is differentiable at x and $f'(x) \neq 0$, then g is differentiable at $y = f(x)$, and

$$g'(y) = \frac{1}{f'(x)}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

- Another notation used for g is f^{-1}.
- The graph of $f(x)$ and $g(x)$ are mirror images of each other with reference to the 45° through the origin.

Practice Problem

Consider $f(x) = 7x + 21$.

1. Find its inverse function $g(x)$.
2. Check whether $g(x) = \frac{1}{f(x)}$
3. Find $f(g(x))$ and $g(f(x))$.
4. Find $\frac{df}{dx}$, $\frac{dg}{dx}$ and verify the inverse function rule.
5. Verify that the graphs of two function bear a mirror image relationship to each other.

Critical points

Critical Points

Let f be a differentiable function. The points on the curve $y = f(x)$ where $f'(x) = 0$ are called *critical points*.

- The value taken by the function at such a point is called a *critical value*.
- At critical points, the tangent to the curve is a straight line parallel to the x-axis.
- There are three kinds of critical points:
  1. **Maximum points** $\Rightarrow f'(x) > 0$ for points to the left of this point and $f'(x) < 0$ for points to the right of this point.
  2. **Minimum points** $\Rightarrow f'(x) < 0$ for points to the left of this point and $f'(x) > 0$ for points to the right of this point.
  3. **Points of inflection** $\Rightarrow f'(x)$ does not change sign passing through the point.
- Maximum and Minimum points of a curve are **turning points.**
- Points of inflection are not turning points.

Practice Problems

Find and classify the critical points of the curve

$$y = x^3 - 9x^2 + 24x + 10$$

Sketch the curve.

Total and Marginal Functions

- In many business applications, the primitive function $f(x)$ represents *total* function.
- Then the derived function $f'(x)$ represents its *marginal* function.
- You should be able to visualize the marginal function from the total function.
- This can be done by identifying how the slope of the tangent to the function changes as x changes.

Practice Problems

Sketch the graph of the following functions and their corresponding marginal functions.

1. $f(x) = 1 + \frac{1}{2}x$
2. $f(x) = 20 - x^2$
3. $f(x) = 20 + x^2$
4. $f(x) = \begin{cases} 5 - x & \text{for} \quad x \leq 3 \\ x - 1 & \text{for} \quad x > 3 \end{cases}$

Continuously differentiable function

If the derivative of a function is continuous everywhere, then the function is said to be *continuously differentiable*.

The Second Derivative

- If f is a differentiable function, then $f'(x)$ is a function, which may itself be differentiable.

Second Derivative of the Function

The derivative of the derivative function is called the second derivative of the function and is denoted by

$$f''(x) \equiv \frac{d}{dx}\left(\frac{dy}{dx}\right) \equiv \frac{d^2y}{dx^2}$$

- $f''(x)$ is read as "eff double prime x".
- $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is read as "derivative of $\frac{dy}{dx}$ with respect to x".
- $\frac{d^2y}{dx^2}$ is read as "d-two y over d x-squared".
- f is said to **twice differentiable**.
- $f''(x)$ is continuous, then $f(x)$ is said to be **twice continuously differentiable**.

Interpretation of Second derivative

- $f''(x)$ measures the rate of change of $f'(x)$.
- $f''(x)$ measures the rate of change of the slope of tangent to the point $(x, f(x))$.
- $f''(x)$ measures the rate of change of rate of change.
- $f''(x)$ measures how the slope of the tangent to the curve changes as x changes.
    - $f''(x) > 0$ implies that $f'(x)$ is increasing.
    - $f''(x) < 0$ implies that $f'(x)$ is decreasing.
- The second derivative relates to the curvature of the function.

Using derivatives to identify the shape of a function

- $f'(x) > 0$ and $f''(x) > 0$
  - Slope of the function is positive and increasing.
  - **Function is increasing at an increasing rate.**
- $f'(x) < 0$ and $f''(x) > 0$
  - Slope of the function is negative and increasing.
  - **Function is decreasing at an increasing rate.**
- $f'(x) > 0$ and $f''(x) < 0$
  - Slope of the function is positive and decreasing.
  - **Function is increasing at an decreasing rate.**
- $f'(x) < 0$ and $f''(x) < 0$
  - Slope of the function is negative and decreasing.
  - **Function is decreasing at an decreasing rate.**

Second Derivative Test for classifying critical points

Second Derivative Test

1. **Maximum points** $\Rightarrow f''(x) < 0$.
2. **Minimum points** $\Rightarrow f''(x) > 0$.

- The second derivative test is silent for the case where $f''(x) = 0$. In that case, revert to the first derivative test.

Practice Problems

Using the first and second derivative test, find and classify the critical points of the following function:

1. $$y = 2x^3 - 3x^2 - 12x + 9$$

2. $$y = x^4$$

3. $$y = -x^4$$

4. $$y = x^3$$

Optimization

Optimization

Finding the maximum or minimum value of a function, sometimes subject to one or more constraints on the independent variable.

- Maximize profits or other desirable variables.
- Minimize costs or other undesirable variables.

- Local optimum: optimum value within the "neighborhood" of the point.
- Global optimum: optimum value for all x in the domain of the function.

Local Maximum

Necessary conditions for local maximum

If the function f has a local maximum at $x = x^*$, then $f'(x^*) = 0$ and $f''(x^*) \leq 0$.

Sufficient conditions for local maximum

If $f'(x^*) = 0$ and $f''(x^*) < 0$, then the function f has a local maximum at $x = x^*$.

- **First order conditions** for a local maximum: $\Rightarrow f'(x^*) = 0$.
- **Second order conditions** for a local maximum: $\Rightarrow f''(x^*) \leq 0$ and $f''(x^*) < 0$.

Local Minimum

Necessary conditions for local minimum

If the function f has a local minimum at $x = x^*$, then $f'(x^*) = 0$ and $f''(x^*) \geq 0$.

Sufficient conditions for local minimum

If $f'(x^*) = 0$ and $f''(x^*) > 0$, then the function f has a local minimum at $x = x^*$.

- **First order conditions** for a local minimum: $\Rightarrow f'(x^*) = 0$.
- **Second order conditions** for a local minimum: $\Rightarrow f''(x^*) \geq 0$ and $f''(x^*) > 0$.

Global Maximum

Global Maximum Point

A local maximum point (x^*, y^*) of the curve $y = f(x)$, with the additional property that

$$y^* \geq f(x) \text{ for all } x \text{ in the domain of } f$$

How to find global maximum?

1. Find all the critical points and identify the maximum points.
2. Identify how function behaves as $x \to \pm\infty$.
3. Compare the value of $f(x)$ at critical points and at $x \to \pm\infty$.

Global Minimum Point

A local minimum point (x^*, y^*) of the curve $y = f(x)$, with the additional property that

$$y^* \leq f(x) \text{ for all } x \text{ in the domain of } f$$

Practice Problems

Find the global maxima and minima for

1. $$y = 2x^3 - 3x^2 - 12x + 9$$

2. $$y = -2x^3 + 3x^2 + 12x - 9$$

3. $$y = 3 + 2x - x^2$$

4. $$y = x^4 - 4x^3 + 5$$

5. $$y = \frac{1}{8}(x^4 - 4x^3 + 5)^3$$

6. $$y = |x|$$

Takeaways from Practice Problems

1. The function f has a local minimum at $x = x^*$ if and only if the function $-f$ has a local maximum at $x = x^*$, and the same is true for global minima and maxima.

2. Suppose $g(x) = H(f(x))$, where H is a *strictly increasing function*. If the function f has a global maximum point at (x^*, y^*), then the function g has a global maximum point at $(x^*, H(y^*))$

3. Calculus methods can be applied to finding maxima and minima only if the relevant function is differentiable.

Boundary maxima and minima

- Often times, we are interested in finding the maximum or minimum value of a function, subject to the constraint that $x \geq 0$.

- Then there are two ways in which a local maximum subject to the constraint that $x \geq 0$ can occur:

  1. **Interior Local Maximum** $x^* > 0$ such that $f'(x^*) = 0$, and $f'(x) > 0$ for points to the left of this point and $f'(x) < 0$ for points to the right of this point.

  2. **Boundary Local Maximum** $f'(0) \leq 0$, and $f'(x) < 0$ for all $x > 0$.

- Global maximum subject to the constraint that $x \geq 0$ can be found by comparing the local maxima and consider the behavior of $f(x)$ as $x \to \infty$.

Convex Set

Some definitions:

- **line segment**: part pf a straight line that lies between two given points.
- Straight line is something that extends infinitely far in both directions.

Convex Set

A set X is a convex set if the line segment joining any two points of X is entirely contained in X.

Convex Functions

Convex Function

The function f is convex if and only if

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

for all x_1, x_2, α such that $x_1 \neq x_2$ and $0 < \alpha < 1$.

- Graphically, a function with the property that the set of all points which are on or above its graph is a convex set.
- The sum of two convex functions is also a convex function.

Practice Problems

Check if the following functions are convex:

1. x^a for $a \geq 1$

2. $3x^2 + 4|x|$

Differentiable Convex Functions

Let f be a differentiable convex function, and suppose that $x_1 < x_2$ and $0 < \alpha < 1$. Then the following two conditions hold:

1. $$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

2. $$f'(x_2) \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Combining the two, we get

Property 1

A differentiable function f is convex if and only if

$$f'(x_1) \leq f'(x_2) \text{ for all } x_1 < x_2$$

Differentiable Convex Function

Based on the two conditions, we can also say the following:

Property 2

A twice-differentiable function f is convex if and only if

$$f''(x) \geq 0 \text{ for all } x$$

Property 3

If f is a differentiable convex function, then

$$f(x_0 + h) \geq f(x_0) + hf'(x_0)$$

for all x_0 and h.

Property 4

If f is a differentiable convex function and $f'(x_0) = 0$, then $(x_0, f(x_0))$ is a global minimum point.

Strictly Convex Functions

- A function f is strictly convex if and only if

$$f(\alpha x_1 + (1-\alpha)x_2) < \alpha f(x_1) + (1-\alpha)f(x_2)$$

  for all x_1, x_2, α such that $x_1 \neq x_2$ and $0 < \alpha < 1$.

- A differentiable function f is strictly convex if and only if

$$f'(x_1) < f'(x_2) \text{ for all } x_1 < x_2$$

- A twice-differentiable function f is strictly convex if and only if

$$f''(x) > 0 \text{ for all } x$$

Concave Functions

- The function f is concave if and only if

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2)$$

  for all x_1, x_2, α such that $x_1 \neq x_2$ and $0 < \alpha < 1$.

- A differentiable function f is concave if and only if

$$f'(x_1) > f'(x_2) \text{ for all } x_1 < x_2$$

- A twice-differentiable function f is concave if and only if

$$f''(x) \leq 0 \text{ for all } x$$

- If f is a differentiable concave function, then

$$f(x_0 + h) \leq f(x_0) + hf'(x_0)$$

  for all x_0 and h.

Strictly Concave Functions

- A function f is strictly concave if and only if

$$f(\alpha x_1 + (1-\alpha)x_2) > \alpha f(x_1) + (1-\alpha) f(x_2)$$

  for all x_1, x_2, α such that $x_1 \neq x_2$ and $0 < \alpha < 1$.

- A differentiable function f is strictly concave if and only if

$$f'(x_1) > f'(x_2) \text{ for all } x_1 < x_2$$

- A twice-differentiable function f is strictly concave if and only if

$$f''(x) < 0 \text{ for all } x$$

Practice Problems

Which of the following functions are convex? Which are concave?

1. $x^2 - 2x + 2$

2. $x^3 - x^2$

- As a function of x the second derivative can be differentiated with respect to x again to produce a third derivative....
- ...which in turn can be the source of a fourth derivative,....
- ...which in turn can be the source of a fifth derivative,...
- ...and so on, as long as the differentiability condition is met.
- The higher order derivatives are symbolized as

$$f^{(3)}(x), f^{(4)}(x), f^{(5)}(x), ..., f^{(n)}(x)$$

$$\frac{d^3y}{dx}, \frac{d^4y}{dx}, \frac{d^5y}{dx}, ..., \frac{d^ny}{dx}$$

Partial Derivatives

- So far, we focused on cases where dependent variable is a function of only one independent variable.
- But in reality dependent variables are functions of more than one independent variables.

$$y = f(x_1, x_2, x_3, ..., x_n)$$

- In analytical works, we are often interested in identifying the effects of change in *one* of the independent variables on the dependent variable (a.k.a *ceteris paribus*)
- The difference quotient can be expressed as:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta, x_2, x_3, ..., x_n) - f(x_1, x_2, x_3, ..., x_n)}{\Delta x_1}$$

Partial Differentiation

- Taking limits of the difference quotient as $\Delta x \Rightarrow 0$, we get the **partial derivative of y with respect to x_1**.

$$f_1 \equiv \frac{\partial f}{\partial x_1} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

  - The symbol $\frac{\partial f}{\partial x_1}$ is read as "partial dee-eff-by-dee-x_1"

- The process of finding partial derivatives is known as **partial differentiation**.

- Mechanics of partial differentiation:
  - When differentiating with respect to x_i, all other independent variables are regarded as *constants*.
  - Use the rules of simple differentiation for x_i.

Higher order partial derivatives

- **Second partial derivatives**

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right)$$

- **Mixed partial derivatives**

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

for $i \neq j$.

Practice Problems

Find the first, second and mixed partial derivatives of the following functions.

1. $f(x, y) = x^2y + y^5$

2. $f(x, y) = (x + 4)(3x + 2y)$

Mixed derivative theorem

Mixed Derivative Theorem

If the first, second and mixed partial derivative of function f exist and are continuous functions of $x_1, x_2, .., x_n$, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

for all $i \neq j$.

Total Differentials

- The actual change in y when x changes by Δx is given by the differential:

$$dy = f'(x)dx$$

- For functions of more than one variable, the differential is given by:

$$dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_n}dx_n$$

The Chain Rule for Functions of more than one variable

The Chain Rule

If the first, second and mixed partial derivative of function f exist and are continuous functions of $x_1, x_2, .., x_n$, and $x_1, x_2, ...x_n$ are differentiable functions of a variable t, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + ... + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

- For the special case where $t = x_1$ (or any one x_i), the **total derivative of f with respect to x_1** is:

$$\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx_1} + ... + \frac{\partial f}{\partial x_n}\frac{dx_n}{dx_1}$$

Practice Problem

If

$$z = xy^4 + x^3y^2$$
$$x = 2 - 3t$$
$$y = 4 + 5t$$

use the chain rule to find $\frac{dz}{dt}$.