Solving System of Linear Equations

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Question

What is the difference between functions and equations?

When the poll is active, you will be able to respond **http://www.PollEv.com/psharma024**.

Types of equations and their roots

Roots of an equations

Only those values of x that satisfy the equation $f(x) = 0$. Also known as the zeroes of the equation.

- A (univariate) linear equation is the equation of the form

$$ax + b = 0$$

- The unique root of linear equation is

$$x = -\frac{b}{a}$$

Solving a quadratic equation

- A quadratic equation is the equation of the form

$$ax^2 + bx + c = 0$$

- A general approach to solving a quadratic equation involves completing the square and then using that expression to find the values of x.

- Employing this method to above equation, we obtain the two "roots" of the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  where $b^2 - 4ac \geq 0$

  - When $b^2 - 4ac > 0$, the roots are real and distinct.
  - When $b^2 - 4ac = 0$, the two roots are identical.
  - When $b^2 - 4ac < 0$, no real roots exist.

Practice Problems

Solve the following equations:

1. $a - bx = -c + dx$

2. $-3x^2 + 30x - 27 = 0$

3. $x^2 - 6x + 8 = 0$

Polynomial Equations

Consider the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0$$

- Difficult to find roots, but there are some results to guide us.

Fundamental Theorem of Algebra

Every polynomial of degree n can be written as the product of the polynomials of first or second degree.

- In general, a polynomial of degree-n has n roots.

Polynomial Equations

Theorem 1

Given the polynomial equation

$$x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$$

where all co-efficient are integers, and the coefficient of x^n is unity, <u>if</u> there exist integer roots, then each must be a divisor of a_0.

Find the integer roots of the cubic equation

$$x^3 - x^2 - 4x + 4 = 0$$

Polynomial Equations

Theorem 2

Given the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0$$

<u>if</u> there exists a rational root $\frac{r}{s}$, where r and s are integers without a common divisor except unity, then r is a divisor of a_0, and s is a divisor of a_n.

Find the rational roots of the quartic equation

$$2x^4 + 5x^3 - 11x^2 - 20x + 12 = 0$$

Polynomial Equations

Theorem 3

Given the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0$$

if the coefficients $a_n, a_{n-1}, ..., a_0$ add up to zero, then $x = 1$ is a root of the equation.

Determine if $x = 1$ is a root of the following polynomial equation:

$$x^3 - 2x^2 + 3x - 2 = 0$$

Practice Problems

Sketch the graph of the following equations:

- $3x + 4y = 12$

Inequalities

Rules of manipulating Inequalities

1. $x < y \iff x + a < y + a$
2. $x < y \iff a.x < a.y$ where $a > 0$
3. $x < y \iff a.x > a.y$ where $a < 0$

Practice Problems

1. Use the rules for manipulating inequalities to determine the values of x which satisfy the following:
    1. $2x + 1 > 0$
    2. $2x + 7 \geq x - 5$

2. Which of the following sets of real numbers are intervals?
    1. $$\{x \in \mathbb{R} | x + 1 > 0 \text{ and } 5x \leq 4 + 3x\}$$
    2. $$\{x \in \mathbb{R} | x + 1 < 0 \text{ or } 5x \geq 4 + 3x\}$$

3. Sketch the graph of the following equations:
    - $x + 2y \leq 3$
    - $2x - 3y \geq 13$

System of Linear Equations

- A **linear equation** in the variables $x_1, x_2, ..., x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

where b is a constant and $a_1, a_2, ..., a_n$ are co-efficients.

- A **system of linear equations** is a collection of one or more linear equations involving the same variables — say, $x_1, x_2, ..., x_n$.

- A **solution** of the system is a list $(s_1, s_2, ..., s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, ..., s_n$ are substituted for $x_1, x_2, ..., x_n$ respectively.
  - The solution is an ordered pair of numbers that satisfies all equations simultaneously.

Solving a system of equations

- A system of linear equations has
  1. no solution
  2. exactly one solution
  3. infinitely many solutions
- A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions.
- A system of linear equation is said to be **inconsistent** if it has no solution.
- One way in which solution can be identified is: Graphical approach.

Possibility 1 All lines intersect at a single point $\implies$ Solution is unique and is given by the ordered pair that represents this point.

Possibility 2 All the lines coincide $\implies$ Infinitely many solutions exist.

Possibility 3 Two or more lines run parallel to each other $\implies$ No solution exists.

Possibility 4 Lines intersect at more than one point, but there exists atleast one that does not pass through any given intersection point.

Practice Problems

Consider the following system of equations. Sketch their graphs and identify their solution graphically. Also, state whether each system of equations is consistent or inconsistent.

1. $$3x + 2y = 8$$
   $$2x + 5y = 9$$

2. $$3x + 2y = 8$$
   $$6x + 4y = 9$$

3. $$3x + 2y = 8$$
   $$6x + 4y = 16$$

Elimination Method for Solving Equations

1. Step 1: Leave the first equation alone and multiply rest of the equations with a suitable multiple such that the co-efficient of x_1 is same across all equations.

2. Step 2: Subtract first equation from all the equations so that equations (2)–(n) do not contain x_1 any more.

3. Step 3: Leave the first and second equations alone and multiply rest of the equations with a suitable multiple such that the co-efficient of x_2 for equations (3)-(n) is same as co-efficient of equation 2.

4. Step 4: Subtract second equation from equations (3)-(n) so that equations (3)–(n) do not contain x_1 any more.

5. Step 5: Keep repeating the process until equation (n) contains only x_n.

6. Step 6: Solve equation in Step 5 for x_n and plug it back in modified equation (n-1) to solve for x_{n-1}.

Practice Problems

Consider the following system of equations. Identify their solution using elimination method.

1. $$3x + 2y = 8$$
$$2x + 5y = 9$$

2. $$3x + 2y = 8$$
$$6x + 4y = 9$$

3. $$3x + 2y = 8$$
$$6x + 4y = 16$$

4. $$2x + 4y + z = 5$$
$$x + y + z = 6$$
$$2x + 3y + 2z = 6$$

Substitution Method for Solving Equations

- Step 1: Re-arrange the terms of equation (n) to specify x_n as a function of $x_1, x_2, ..x_{n-1}$.
- Step 2: Plug x_n from Step 1 above in equation (n-1) and re-arrange its terms to specify x_{n-1} as a function of $x_1, x_2, ..x_{n-2}$.
- Step 3: Iteratively follow the process laid out above to derive at the solution of x_1 and use its value to identify solutions for rest of the variable.

Practice Problem

Solve the following system of equations using method of substitution.

$$3x + 2y = 8$$
$$2x + 5y = 9$$