Limits and Continuity of Functions (Ch 6.4,6.5 & 6.7)

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Limits of a function

Consider $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}$ and a converging $\{x_n\} \in X$.

- $\{f(x_n)\} \in \mathbb{R}$ is the corresponding sequence of images of this function.

- If as $x_n \to a$, $f(x_n) \to L$, then $\lim_{x \to a} f(x) = L$
  - $\lim_{x \to a} f(x) = L$ is read as "L is the limit of $f(x)$ as x approaches a".

- When we say "x approaches a", x can approach a from either values lower than a or greater than a.

  - If x approaches a from values less than a (left hand side), then L is the *left-side limit of* $f(x)$.

  - If x approaches a from values higher than a (right hand side), then L is the *right-side limit of* $f(x)$.

Calculating Limits

Method 1 Graphically

Method 2 Numerically

Practice Problems

Find the left-side limits and the right-side limits of the following functions.

1. $\lim\limits_{x\to 2} \frac{x^2-4}{x-2}, x \neq 2$

2. $$\lim_{x\to 0} f(x) = \begin{cases} 1 & \text{for} \quad x \leq 0 \\ 2 & \text{for} \quad x > 0 \end{cases}$$

3. $$\lim_{x\to 4} f(x) = \begin{cases} 1 & \text{for} \quad x \neq 4 \\ -1 & \text{for} \quad x = 4 \end{cases}$$

2 Observations from last practice problems

- It is not necessary that $a \in X$ and $L \in f[X]$.

- It is not necessary that

$$\lim_{x \to a^-} f(x) = L_1$$

equals

$$\lim_{x \to a^+} f(x) = L_2$$

- However, if $L_1 = L_2 = L$ or

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

then we say that "$f(x)$ has a limit L as x approaches a and write

$$\lim_{x \to a} f(x) = L$$

Practice Problem

Find the limits of the following functions.

1. $\lim_{x \to a} c$

2. $\lim_{x \to a} x$

Two useful results about limits

1. $\lim_{x \to a} c = c$ at every point a.
2. $\lim_{x \to a} x = a$ at every point a.

Limit Theorems Involving Two Functions

Consider $f_1(x)$ and $f_2(x)$ such that $\lim_{x\to a} f_1(x) = L_1$ and $\lim_{x\to a} f_2(x) = L_2$ where L_1 and L_2 are two finite numbers then

1. Sum-difference limit theorem

$$\lim_{x\to a} (f_1(x) \pm f_2(x)) = L_1 \pm L_2$$

2. Product limit theorem

$$\lim_{x\to a} (f_1(x).f_2(x)) = L_1.L_2$$

3. Quotient limit theorem

$$\lim_{x\to a} \left(\frac{f_1(x)}{f_2(x)}\right) = \frac{L_1}{L_2} \quad (L_2 \neq 0)$$

Calculating limits through direct substitution

Find the limits of the following functions as $x \to 0$

1. $$f(x) = 7 - 9x + x^2$$

2. $$f(x) = (x+2)(x-3)$$

3. $$f(x) = \frac{3x+5}{x+2}$$

Calculating limits using factorization

Find the limits of the following functions:

1. $$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}, x \neq 2$$

2. $$\lim_{x \to 2} \frac{(x + 2)^3 - 8}{x}, x \neq 0$$

If the functions f and g are equal for all x close to a (but not necessarily at $x = a$) then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ whenever either limits exists.

Calculating limits using conjugates

Conjugates: Change the sign in the middle of two terms.

- conjugate of $a + b$ is $a - b$ and vice versa.
- $a \pm \sqrt{b}$ are conjugates.

Find the limit of

$$\lim_{x \to 4} \frac{2 - \sqrt{x}}{4 - x}$$

Finite limits of a function

- A function $f(x)$ has a **finite limit** L **as** $x \to a$**,** a **finite** if, any $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in (a - \delta, a + \delta)$, or equivalently $|x - a| < \delta$ then $|f(x) - L| < \epsilon$. Then we write,

$$\lim_{x \to a} f(x) = L$$

- Question: What does it mean to say that $f(x)$ does not tend to the number L as x tends to a?
  - If we can find an $\epsilon > 0$ such that for all $\delta > 0$, there exists a number x such that $|x - a| < \delta$ and $|f(x) - L| \geq \epsilon$

Practice Problem

Using $\epsilon\delta$ definition of limits, show that

$$\lim_{x \to 3} (3x - 2) = 7$$

- The function $f(x)$ has a **finite limit as** $|x|$ **becomes infinite** if, for every $\epsilon > 0, \exists\delta > 0$ such that if $|x| > \delta$ then $|f(x) - L| < \epsilon$. Then we write,

$$\lim_{x\to(\pm)\infty} f(x) = L$$

  - The horizontal line $y = L$ is a horizontal asymptote of f as x tends to $(\pm)\infty$.

- A function $f(x)$ has an **infinite limit as** $x \to a$**,** a **finite** if, any $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in (a - \delta, a + \delta)$, then $|f(x)| > \epsilon$. Then we write,

$$\lim_{x\to a} f(x) = (\pm)\infty$$

  - The vertical line $x = a$ is a vertical asymptote for the graph of f.

Practice Problems

Find the limits of the following functions

1. $$\lim_{x \to 0} \frac{1}{x}$$

2. $$\lim_{x \to 2} \frac{1}{\sqrt{2-x}}$$

3. $$\lim_{x \to \infty} \frac{1}{x}$$

Infinite Limits of a function

- The function $f(x)$ has an **infinite limit as** $|x|$ **becomes infinite** if, for every $\epsilon > 0, \exists \delta > 0$ such that if $|x| > \delta$ then $|f(x)| > \epsilon$. Then we write,

$$\lim_{x \to (\pm)\infty} f(x) = (\pm)\infty$$

One-sided Limits

Left-hand limit

The left hand limit of a function $f(x)$ as x approaches a is a number L_1 such that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in (a-\delta, a)$, then $|L_1 - f(x)| < \epsilon$. Then we write,

$$\lim_{x \to a^-} f(x_n) = L_1$$

Right-hand limit

The left hand limit of a function $f(x)$ as x approaches a is a number L_2 such that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in (a, a+\delta)$, then $|L_2 - f(x)| < \epsilon$. Then we write,

$$\lim_{x \to a^+} f(x_n) = L_2$$

Continuity at a point

Suppose a function is defined on a domain that includes an open interval around a. Then f is said to be continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

A function $f : X \to \mathbb{R}$ is said to be continuous at $x = a$ if the following three conditions are met:

1. $a \in X$.
2. $\lim_{x \to a} f(x) = L$ exists and is finite.
3. $\lim_{x \to a} f(x) = f(a)$

For a function to be continuous at a, it has to be the case that L is finite and $(a, L) \in Gr(f)$

Practice Problems

Check whether each of the following functions is continuous at $x = a$

1. $$f(x) = \frac{x^2 - 4}{x - 2}, x \neq 2, a = 2$$

2. $$f(x) = x + 2, a = 2$$

3. $$f(x) = \begin{cases} 1 & \text{for} \quad x \leq 0 \\ 2 & \text{for} \quad x > 0 \end{cases}, a = 0$$

4. $$f(x) = \begin{cases} 1 & \text{for} \quad x \neq 4 \\ -1 & \text{for} \quad x = 4 \end{cases}, a = 4$$

5. $$f(x) = \frac{1}{x}, a = 0$$

Continuous Function

Continuous Function

A $f : X \to \mathbb{R}$ is continuous if it is continuous at all $x \in X$.

A function is said to be **continuous on the interval (a,b)** if it is continuous at all points $x \in (a, b)$.

Practice Problems

Check if the following functions are continuous at all x in their domains.

- $$f(x) = \frac{x^2 - 4}{x - 2}, x \neq 2$$

- $$f(x) = x + 2$$

All rational functions are continuous, wherever they are defined.

Properties of continuous functions

Consider functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ which are continuous at $x = a$. Then

1. $f + g : X \to \mathbb{R}$ is continuous at $x = a$.
2. $f - g : X \to \mathbb{R}$ is continuous at $x = a$.
3. $f.g : X \to \mathbb{R}$ is continuous at $x = a$.
4. $\frac{f}{g} : X \to \mathbb{R}$ is continuous at $x = a$, if $g(a) \neq 0$.

Intermediate Value Theorem

Intermediate Value Theorem

Let f be a function that is continuous for all x in the closed interval $[a, b]$, and assume that $f(a) \neq f(b)$. As x varies between a and b, so $f(x)$ takes on every value between $f(a)$ and $f(b)$.

An immediate and useful consequence of the intermediate-value theorem is the following:

Corollary to IVT

Let f be a function that is continuous for all x in the closed interval $[a, b]$, and assume that $f(a)$ and $f(b)$ have different signs (either $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$). Then, $\exists$ at least $c \in (a, b)$ such that $f(c) = 0$.

Practice Problems

1. Prove that the following equation has at least one solution between 0 and 1: $x^6 + 3x^2 - 2x - 1 = 0$.

2. Prove that for any positive number a, the equation $x^3 = a$ has a unique positive solution $x = c$.

Extreme Value Theorem

- If $f(x)$ has domain D, then
  - $c \in D$ is a **maximum point** for $f \iff f(x) \leq f(c)$ for all $x \in D$.
  - $d \in D$ is a **minimum point** for $f \iff f(x) \geq f(d)$ for all $x \in D$.
- The maximum and minimum points can also be referred as **extreme points**.
- $f(c)$ is called the maximum value.
- $f(d)$ is called the minimum value.

The Extreme Value Theorem

If a function is continuous in a closed, bounded interval $[a, b]$, then f attains both a maximum value and a minimum value in $[a, b]$.

Practice Problem

Explain why function f defined for all $x \in [0, 5]$ by

$$f(x) = \frac{x^6 + 5x^3 - 2x + 8}{x^4 + 10}$$

has both a maximum and a minimum value. (Do not try to find these values.)

One-sided continuity

Left-continuous at a

Suppose f is defined on a domain including the half-open interval $(c, a]$. f is left-continuous at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Right-continuous at a

Suppose f is defined on a domain including the half-open interval $[a, d)$. f is right-continuous at a if

$$\lim_{x \to a^+} f(x) = f(a)$$

A function is continuous at a if and only if it is both left- and right-continuous at a.

Practice Problem

Check which of the following functions are left-continuous at $x = a$, and which functions are right-continuous at $x = a$.

1. $$f(x) = \frac{x^2 - 4}{x - 2}, x \neq 2, a = 2$$

2. $$f(x) = x + 2, a = 2$$

3. $$f(x) = \begin{cases} 1 & \text{for} \quad x \leq 0 \\ 2 & \text{for} \quad x > 0 \end{cases}, a = 0$$

4. $$f(x) = \begin{cases} 1 & \text{for} \quad x \neq 4 \\ -1 & \text{for} \quad x = 4 \end{cases}, a = 4$$

5. $$f(x) = \frac{1}{x}, a = 0$$