Sequences and Series

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September 28, 2020

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Sequences

- Sequence $\Longrightarrow$ a string of objects.
- The collection of objects is ordered in such a way such that it has an identified first member, second member, third member, and so on.
  - Amount of money deposited in a bank over a period of time.
- For our purposes,

Sequences

A function a from $\mathbb{N}$ to $\mathbb{R}$.

$$\{a_n\} = a_1, a_2, a_3, ...$$

- **Terms of a sequence**: $a_1, a_2, a_3, ...$
- a_n is the n^{th} term of the sequence $\{a_n\}$.
- **Finite Sequence**: A sequence with finite terms
- **Infinite Sequence**: A sequence which does not have finite terms.

Progression

Progressions $\implies$ sequences following specific pattern.

Arithmetic Progression (A.P.)

An arithmetic progression is a sequence $\{a_n\}$ with the property that

$$a_{n+1} = a_n + d$$

where $a_1 \equiv a$ is the *first term* and d is the common difference.

Then, an arithmetic progression is the sequence:

$$a, a+d, a+2d, a+3d, ...a+(n-1)d$$

Geometric Progression

Geometric Progression (G.P.)

A geometric progression is a sequence $\{a_n\}$ with the property that

$$\frac{a_{n+1}}{a_n} = r$$

where $a_1 \equiv a$ is the *first term* and r is the common ratio.

Then, the geometric progression is the sequence:

$$a, ar, ar^2, ar^3, \ldots\ldots ar^{n-1}$$

Harmonic Progression(H.P.)

A harmonic progression is a sequence $\{a_n\}$ with the property that

$$a_n = \frac{1}{a + (n-1)d}$$

where $a \neq 0$ is the first term, d is the common difference of a A.P and $\frac{-a}{d} \notin \mathbb{N}$.

Each term is the Harmonic Mean of the two neighboring terms.

- Harmonic mean is the reciprocal of the arithmetic mean of the reciprocals of the given set of observations.
- The harmonic mean of $2, 4$ is

$$\frac{1}{\left(\frac{\frac{1}{2}+\frac{1}{4}}{2}\right)} = \frac{2}{0.75} = 2.67$$

- The harmonic mean of $\frac{1}{a}$ and $\frac{1}{a+2d}$ is $\frac{1}{\left(\frac{a+a+2d}{2}\right)} = \frac{1}{a+d}$.

Practice Problems

1. Write the first five terms in the progression defined by:
   1. $$a_n = 2n + 5$$
   2. $$a_n = -\frac{3^n}{5}$$
2. What is the 20^{th} term of the progression defined by
$$a_n = (n-1)(2-n)(3+n)?$$
3. Write down the first three terms and a formula for the n^{th} term of each of the following sequence:
   1. the arithmetic progression with first term 2 and common difference 5.
   2. the geometric progression with first term 4 and common ratio 3.

Series

Let $\{a_n\}$ be a given sequence. Then, the series $\{s_n\}$ generated by the sequence is

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

- The symbol $\sum$ is the upper-case Greek-letter *sigma*.
- The symbol $\sum_{k=1}^{n} a_k$ is read as "sum of 1^{st} to n^{th} term of the sequence a_n".

Arithmetic Series

The series generated by summing the terms of an arithmetic progression is called Arithmetic Series.

$$s_n = a + (a + d) + (a + 2d) .... + (a + (n - 1)d)$$

Sum of a finite arithmetic series

The sum of first n terms of an arithmetic progression with first term a and common difference d is

$$s_n = n\left(a + \frac{n-1}{2} d\right) = \frac{n}{2}(a_1 + a_n)$$

- The sum of first n natural numbers is given by:

$$s_n = n\left(\frac{n+1}{2}\right)$$

Practice Problems

1. Find the sum of the first 15 terms of the sequence

$$48, 44, 40, 36, ...$$

2. Find the sum of the natural numbers from 1 to 100 inclusive.

Geometric Series

The series generated by summing the terms of a geometric progression is called Geometric Series.

$$s_n = a + ar + ar^2 + ar^3 .... + ar^{n-1}$$

Sum of a geometric progression

The sum of first n terms of a geometric progression with first term a and common ratio $r \neq 1$ is

$$s_n = a\left[\frac{1-r^n}{1-r}\right] = a\left[\frac{r^n-1}{r-1}\right]$$

Practice Problem

Find the sum of the first 8 terms of the geometric progression

$$10, 50, 250, 1250.......$$

Limit of a Sequence

Limit of a sequence

$\{a_n\} \to a, \iff \forall \epsilon > 0 \in \mathbb{R}, \exists\ n_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned} a - \epsilon &< a_n < a + \epsilon\ \forall n > n_\epsilon \\ \iff -\epsilon &< a_n - a < \epsilon\ \forall\ n > n_\epsilon \\ \iff & |a_n - a| < \epsilon\ \forall\ n > n_\epsilon \end{aligned}$$

- The symbol $\to$ is read as "tends to the limit".
- The notation "$\{a_n\} \to a$" reads as "the sequence a_n tends to the limit a".
- Other notation:
  - $\lim a_n = a$.
  - $\lim\limits_{n\to\infty} a_n = a$ which is read as "a_n tends to the limit a as n tends to ∞"
- The open interval $(a - \epsilon, a + \epsilon)$ is the "ϵ neighborhood of a".

$$N_\epsilon(a) = \{a_n : |a_n - a| = \epsilon\}$$

Practice Problems

In each of the following cases, state whether the sequence $\{a_n\}$ tends to a limit, and find the limit if it exists.

1. $a_n = \frac{n+1}{2n}$
2. $a_n = 1 + \frac{1}{2}n$
3. $a_n = 5(-1)^n$
4. $a_n = x^n$
5. $a_n = n^b$

Two useful facts about limits

1. Given $x \in \mathbb{R}$, $\lim_{n\to\infty} x^n = 0 \iff -1 < x < 1$.
2. Given $x \in \mathbb{R}$, $\lim_{n\to\infty} n^b = 0 \iff b < 0$.

Sum of the Series

Sum of the series: The constant associated with the $s_n = \sum_{k=1}^{n} a_k$.

Convergent Series

A series is said to be convergent if

$$\lim_{n\to\infty} s_n = s$$
$$\iff \quad \sum_{k=1}^{\infty} a_k = s$$

A series is said to be divergent if it is not convergent.

Is arithmetic series convergent or divergent?

Sum of a geometric progression (infinite version)

- Recall that, for a geometric progression,

$$s_n = a\left[\frac{1-r^n}{1-r}\right]$$

- $\lim\limits_{n\to\infty} r^n = 0 \iff -1 < r < 1$

Given $a \neq 0$, the series

$$a + ar + ar^2 + ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

is convergent if and only if $-1 < r < 1$ in which case

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

Practice Problems

Obtain the series corresponding to the following sequences and say whether the series is convergent or not. If it is convergent, find its limit.

1. $$a_n = 2n + 5$$

2. $$a_n = -\frac{3^n}{5}$$

3. $$a_n = a\left(\frac{1}{2}\right)^{n-1}$$

Practice Problems

Find the sum to the infinity of

1. the geometric progression with first term 4 and common ratio $\frac{1}{4}$

2. the geometric progression with first term 4 and common ratio 4

The Power of Compounding

Suppose you invest \$100 at an annual interest rate of 10% today. Suppose that interest is paid on yearly basis. How much money will you have

- one year from now?
- two years from now?
- three years from now?
- four years from now?

Future Value of P

If a sum P is invested at an annual rate of interest of r, compounded annually, the value of investment after n years is

$$M = P\left(1 + r\right)^n$$

- Future value of a principal P is the amount M to which P will compound to over a period of n years.
- The process of finding future value from present values is known as **compounding**.

Practice Problems

1. Suppose \$50 is invested at an annual interest rate of 4%. If interest is compounded annually, what is the value of the investment at the beginning of 7^{th} year?

2. Suppose \$50 is invested at an annual interest rate of 4% for first three years, 5% for the fourth year and 6% for the subsequent years. If interest is compounded annually, what is the value of the investment at the beginning of 7^{th} year?

Practice Problems

Suppose \$50 is invested at an annual interest rate of 4%. If interest is compounded quarterly, what is the value of the investment at the beginning of 7^{th} year?

If a sum P is invested at an annual rate of interest of r, compounded m times per year, the value of investment after n years is

$$M = P\left(1 + \frac{r}{m}\right)^{mn}$$

In this case, the **Annual Percentage Rate** is

$$r' = \left(1 + \frac{r}{m}\right)^{m} - 1$$

Practice Problems

1. If \$50 is invested at an annual interest rate of r, what value of r is required for the investment to be worth \$70 at the end of six years?

2. If \$50 is placed in a account at the beginning of each year, and the rate of interest is 4% per annum, how much money is in account just after the 7^{th} investment has been made?

If a sum P is *repeatedly* invested at a rate of interest of r per year, compounded annually, then amount of money in the account at the beginning of year n (just after n^{th} investment is)

$$\frac{P}{r}([1+r]^n - 1)$$

"Rule of 72"

- Tell you how long it takes (roughly!) to double your money.

$$T = \frac{72}{r}$$

- Tells you what interest rate you need (roughly!) to double your money.

$$r = \frac{72}{T}$$

- Can be used to quickly answer questions like:
    - How long will it take for your money to double using compounding interest?
    - How long will it take for your debt to double?
    - How many times money will double within a specified period?
    - What is the interest rate that an investment should earn to double with in a specified time period?

Comparing the math

Interest rate	Actual Years	Rule of 72
1%	69.66	72
2%	35	36
3%	23.45	24
4%	17.67	18
5%	14.21	14.4
6%	11.9	12
7%	10.24	10.29

Interest rate and Doubling Time

Interest Rate	Time to doubling
1.5%	48
2%	36
3%	24
4%	18
6%	12
12%	6
24%	3

Things to keep in mind!

- It is only an approximation.
- Assumes that interest rate remains unchanged.
- Does not allow for additional payments to the original amount.

Present Value and Discounting

Suppose you would like to have \$100 in your account 3 years from now. The ongoing rate of return is 10%. How much money should you put in your account

- 2 years from now?
- 1 year from now?
- today?

Present Value of P

In order to have a sum M available in n years from now, at an annual rate of interest of r compounded annually, the value of investment to be made today is

$$P = \frac{M}{(1+r)^n}$$

Present Value and Discounting

- Present value of a amount M is the principal amount P which will compound M to over a period of n years.
- The process of finding present value from future values is known as discounting.
- Discounting is the opposite of compounding.
- The ratio $\frac{1}{1+r}$ is called the **discount factor**.

Practice Problems

1. Suppose that a \$200,000 mortgage is to repaid over 20 years. Suppose that the rate of interest of 4.5% per annum, compounded monthly. The repayments are made monthly as well. Assuming that each month's repayment is a constant amount Y, what must Y be?

Present value of a finitely recurring income stream

The present value of a finite stream of income Y for n years starting one year from now

$$P_n = \frac{Y}{r}\left(1 - [1 + r]^{-n}\right)$$

2. What is the value of an infinite annual income stream of \$1200 when rate of interest is 4% per annum?

Present value of an infinite income stream

The present value of an infinite stream of income Y for a rate of return r is

$$\frac{Y}{r}$$