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Problem 5.1:

$$V_1(W_1) = \max_{W_2} u(W_1 - W_2)$$

Because the individual lives for one period, they don't save for the next period.

Therefore, $W_2 = \psi_1(W_1) = 0$, $V_1(W_1) = \mathrm{u}(W_1)$

Problem 5.2:

As the Problem 5.1 shows, $W_3 = 0$.

$$V_1(W_1) = \max_{W_2} u(W_1 - W_2) + \beta u(W_2).$$

Taking the derivative of both sides of the function with respect to W_2,

$$u^{'}(W_1 - W_2) = \beta u^{'}(W_2)$$

The function above can be written as $W_2 = \psi_1(W_1)$.

Problem 5.3:

Because $W_4 = 0$, therefore, $c_3 = W_3 - W_4 = W_3 - 0 = W_3$.

Taking the derivative with respect to W_3, we get:

$$u^{'}(W_2 - W_3) = \beta u^{'}(W_3)$$

To maximize the equation: $u(W_1 - W_2) + \beta u(W_2 - W_3) + \beta^2 u(W_3)$, we need to take the derivative with respect to W_2, we get:

$$u^{'}(W_1 - W_2) = \beta u^{'}(W_2 - W_3)$$

We plug in the number next to solve the problem. **(Pls see the Jupyter notebook)**

Problem 5.4:

The condition that characterizes the optimal choice in period T-1:

$$u^{'}(W_{T-1} - \psi_{T-1}(W_{T-1})) = \beta V_T^{'}(\psi_{T-1}(W_{T-1})) \text{ (with respect to } W_T\text{)}$$

The value function V_{T-1}:

$$V_{T-1}(W_{T-1}) \equiv \max_{W_T} u(W_{T-1} - \psi_{T-1}(W_{T-1})) + \beta V_T(\psi_{T-1}(W_{T-1}))$$

Problem 5.5:

$V_T(W_T) = \mathrm{u}(W_T)$. Because $u(c) = \ln(c)$, the optimal decision rule should satisfy the equation:

$$\frac{1}{W_{T-1}-\psi_{T-1}(W_{T-1})}=\beta\frac{1}{W_T}$$

And we have:

$$\psi_{T-1}(\overline{W})=\frac{\beta}{1+\beta}\overline{W}$$

$$V_{T-1}(\overline{W})=\ln(\frac{1}{1+\beta}\overline{W})+\beta\ln(\frac{\beta}{1+\beta}\overline{W})$$

When t=T, we get:

$$\psi_T(\overline{W})=0$$

$$V_T(\overline{W})=\ln(\overline{W})$$

Therefore, when $T<\infty$, $\psi_T(\overline{W})$ doesn't equal to $\psi_{T-1}(\overline{W})$ and $V_T(\overline{W})$ doesn't equal to $V_{T-1}(\overline{W})$.

Problem 5.6:

The condition:

$$u^{'}(W_{T-2}-W_{T-1})=\beta u^{'}(W_{T-1}-\psi_{T-1}(W_{T-1}))$$

$$W_{T-1}=\frac{\beta W_{T-2}+\psi_{T-1}(W_{T-1})}{1+\beta}$$

The value function:

$$V_{T-2}(W_{T-2})\equiv\max_{W_{T-1}}\ln(W_{T-2}-W_{T-1})+\beta V_{T-1}(W_{T-1})$$

Problem 5.7:

Claim:

$$\psi_{T\text{-}s}(W_{T-s})=\frac{\sum_{i=1}^{s}\beta^i}{\sum_{i=0}^{s}\beta^i}W_{T-s}$$

$$V_{T\text{-}s}(W_{T-s})=\sum_{i=0}^{s}\beta^i\ln(\frac{\beta^i}{\sum_{i=0}^{s}\beta^j}W_{T-s})$$

When s=1, these two equations are satisfied.

$$\psi_{T\text{-}1}(W_{T\text{-}1})=0=W_{T-1}$$

$$V_{T-1}(W_{T-1}) = \ln(W_{T-1})$$

Suppose the two equations of the claim hold when t=T, then at t=T+1, we have:

$$u^{'}(\psi_{T-1}(W_{T-1}) - \psi_T(W_T)) = \beta V^{'}(\psi_T(W_T))$$

$$V(\psi_{T\text{-}1}(W_{T\text{-}1})) = u(\psi_{T-1}(W_{T-1}) - \psi_T(W_T)) + \beta V(\psi_T(W_T))$$

When $s \to \infty$,

$$\lim_{s\to\infty} \psi_{T-s}(W_{T-s}) = \beta W_{T-s}$$

$$\lim_{s\to\infty} V_{T-s}(W_{T-s}) = \left(\frac{\ln(1-\beta)}{1-\beta} + \frac{\ln\beta}{(1-\beta)^2}\right) W_{T-s}$$

Problem 5.8:

We assume that $\mathbf{W_0}$ represent the cake left tomorrow. $W_0 \in [0, W]$

$$V(W) = \max_{W_0} u(W - W_0) + \beta V(W_0)$$

Problem 5.9—5.22:

Pls see the Jupyter notebook.