第4讲 机器人动力学

谢 斌
xiebin@csu.edu.cn
中南大学人工智能与机器人实验室
http://airl.csu.edu.cn

Review

- Introduction to Kinematics of Robotics
- Link Description
- Frame Attachment
- Forward Kinematics
- Inverse Kinematics

学习目标

- 能够阐述机器人动力学的处理对象和目的；
- 能够应用牛顿欧拉法对单关节机器人进行动力学分析；
- 能够应用拉格朗日法进行动力学建模；
- 能够对比两类建模方法并根据情况进行选择。

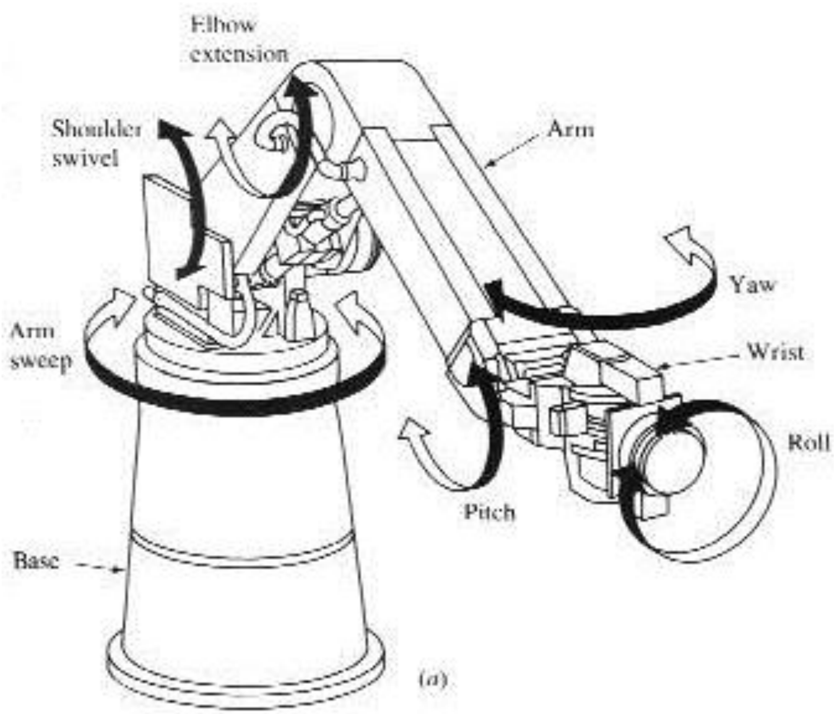
Man VS Machine ?

Man VS Machine ?

思考：如何控制守门员做出扑球动作？

Contents

- Introduction



Manipulator Dynamics considers the forces required to cause desired motion.

Considering the equations of motion arises from torques applied by the actuators, or from external forces applied to the manipulator.

4.1 Introduction to Dynamics

- There are two problems related to the dynamics that we wish to solve.

- **Forward Dynamics:** given a torque vector, $\boldsymbol{T}$, calculate the resulting motion of the manipulator, Θ, $\dot{\Theta}$, and $\ddot{\Theta}$. This is useful for simulating the manipulator.

- **Inverse Dynamics:** given a trajectory point, Θ, $\dot{\Theta}$, and $\ddot{\Theta}$, find the required vector of joint torques, $\boldsymbol{T}$.

  This formulation of dynamics is useful for the problem of controlling the manipulator.

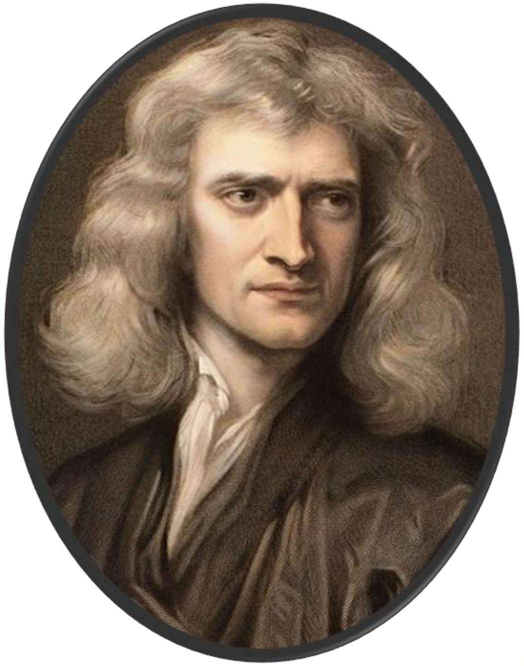
- Two methods for formulating dynamics model：
  - **Newton-Euler dynamic formulation**
    - Newton's equation along with its rotational analog, Euler's equation, describe how forces, inertias, and accelerations relate for rigid bodies, is a "**force balance**" approach to dynamics.
  - **Lagrangian dynamic formulation**
    - Lagrangian formulation is an "**energy-based**" approach to dynamics.

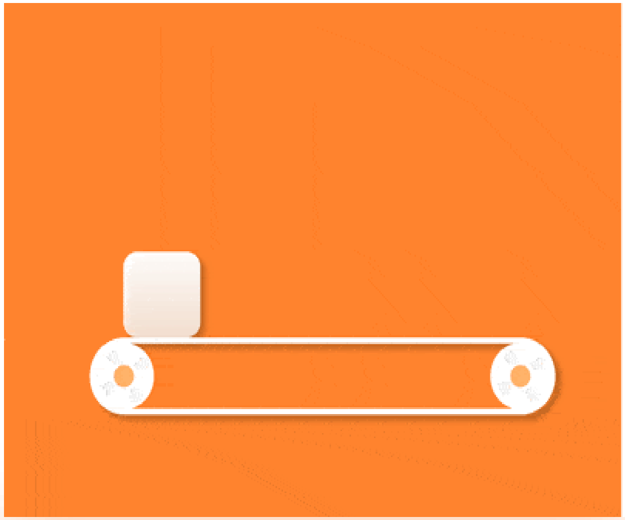
Contents

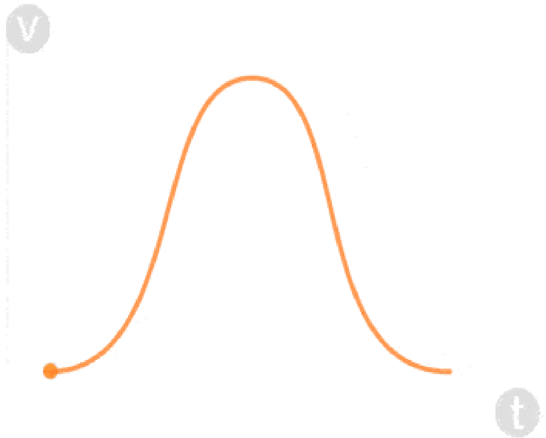
4.2 Newton-Euler Equations

- Newton's Law

$$F = m\,a$$





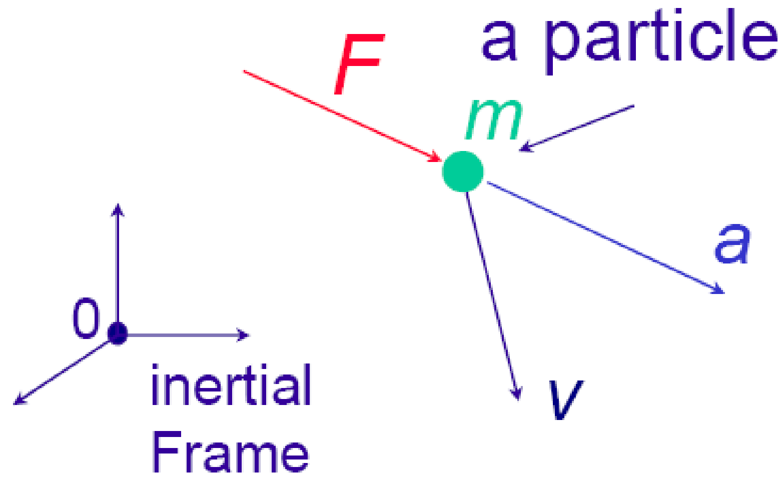


4.2 Newton-Euler Equations

- Newton-Euler Dynamic Equation

Newton's Law

$$F = m\,a$$



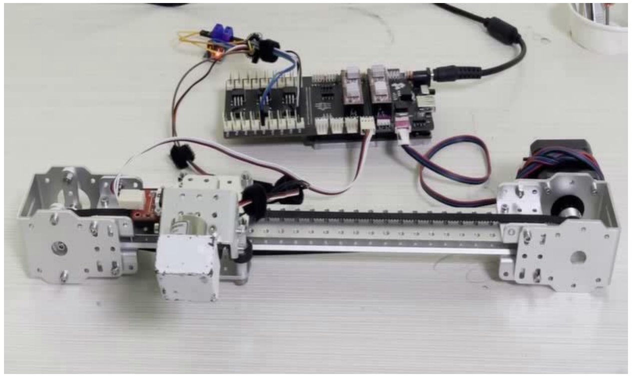
$$\frac{d}{dt}(mv) = F$$

rate of change of the linear momentum is equal to the applied force

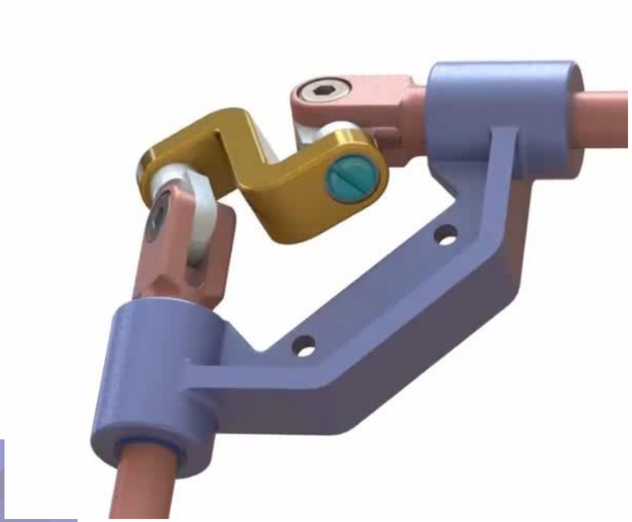
Linear Momentum

$$\varphi = mv$$

4.2 Newton-Euler Equations



平动关节 → 牛顿第二定律



转动关节 → 欧拉第二运动定律

莱昂哈德·欧拉

（1707年4月15日～1783年9月18日）

瑞士数学家、自然科学家，是18世纪数学界最杰出的人物之一。

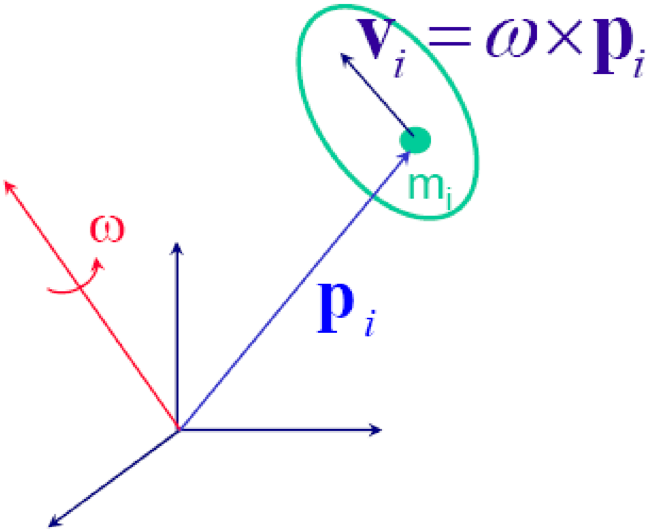
欧拉是18世纪数学界最杰出的人物之一，他不但为数学界作出贡献，更把整个数学推至物理的领域。

他是数学史上最多产的数学家，平均每年写出八百多页的论文，还写了大量的力学、分析学、几何学、变分法等的课本，《无穷小分析引论》、《微分学原理》、《积分学原理》等都成为数学界中的经典著作。

4.2 Newton-Euler Equations

- Newton-Euler Dynamic Equation

Rotational Motion



Angular Momentum

$$\sum_i \mathrm{p}_i \times m_i \mathrm{v}_i$$

$$\Rightarrow \phi = \sum_i m_i \mathrm{p}_i \times (\omega \times \mathrm{p}_i)$$

$$m_i \to \rho dv \qquad (\rho : \text{density})$$

4.2 Newton-Euler Equations

- Newton-Euler Dynamic Equation

Rotational Motion

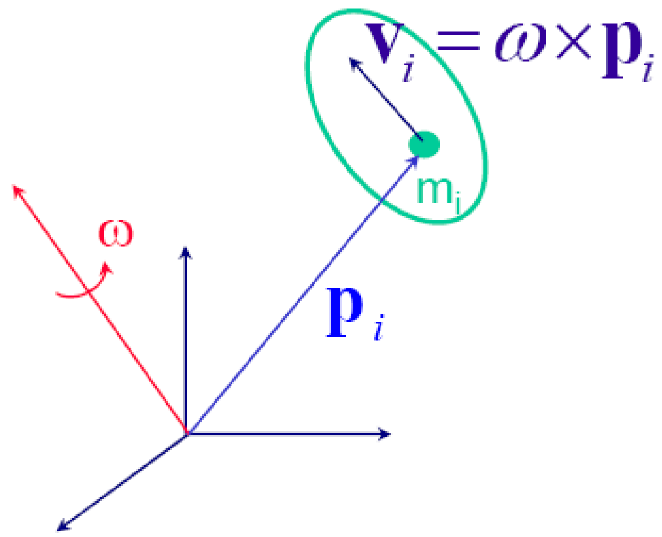
Angular Momentum

$$\phi = \int_V \mathrm{p} \times (\omega \times \mathrm{p}) \rho dv$$

$$\Rightarrow \phi = \left[\int_V -\hat{\mathrm{p}}\hat{\mathrm{p}} \rho dv \right] \omega$$

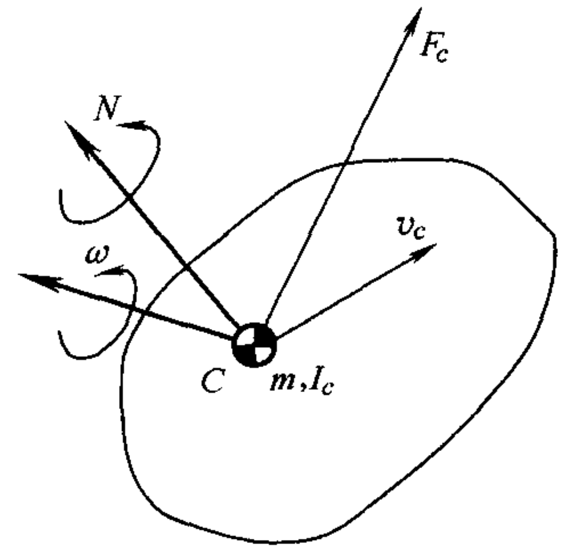
$$\phi = I \omega$$

Inertia Tensor



4.2 Newton-Euler Equations

- Newton-Euler Dynamic Equation



$$m\dot{v}_c = F_c$$ （**Newton Equation**）

$$I_c\dot{\omega} + \omega \times (I_c\omega) = N$$

（**Euler Equation**）

where m is the mass of a rigid body, $I_C \in R^{3\times 3}$ represent inertia tensor, F_C is the external force on the center of gravity, N is the torque on the rigid body, v_C represent the translational velocity, while ω is the angular velocity.

4.2 Newton-Euler Equations

- Newton-Euler Dynamic Equation

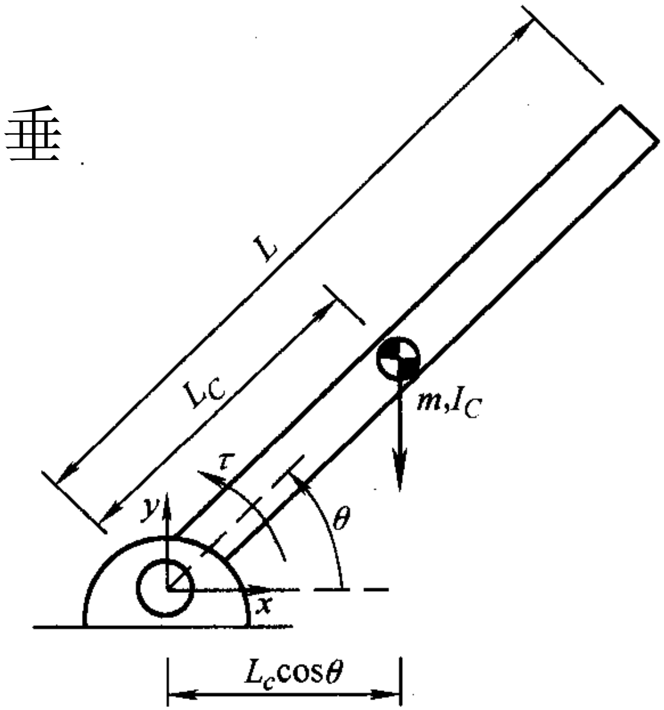
	linear	angular
惯性	质量m	张量I
动量	mv	Iω
外力	力F	力矩τ
加速度	线性加速度a	角加速度α
欧拉方程	F=ma	τ=Iα+ω×Iω

- 例1. 求解下图所示的1自由度机械手的运动方程式，在这里，由于关节轴制约连杆的运动，所以可以将运动方程式看作是绕固定轴的运动。

- 解：假设绕关节轴的惯性矩为 I，取垂直纸面的方向为 z 轴，则有

$$I\dot{\omega} = \begin{bmatrix} 0 \\ 0 \\ I\ddot{\theta} \end{bmatrix} \qquad \omega \times I\omega = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ I\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 \\ 0 \\ \tau - mgL_c \cos\theta \end{bmatrix}$$

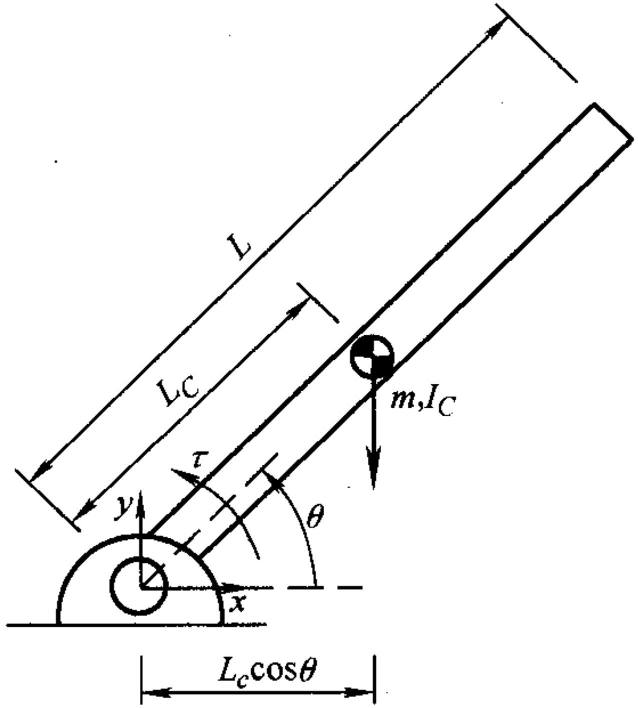


1自由度机械手

4.2 Newton-Euler Equations

$$I\dot{\omega}=\begin{bmatrix}0\\0\\I\ddot{\theta}\end{bmatrix}\qquad \omega\times I\omega=\begin{bmatrix}0\\0\\\dot{\theta}\end{bmatrix}\times\begin{bmatrix}0\\0\\I\dot{\theta}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$N=\begin{bmatrix}0\\0\\\tau-mgL_c\cos\theta\end{bmatrix}$$

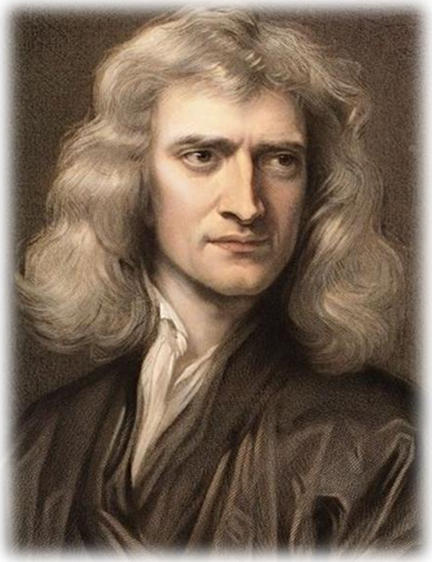


由欧拉运动方程式 $I_C\dot{\omega}+\omega\times(I_C\omega)=N$

$$I\ddot{\theta}+mgL_c\cos\theta=\tau$$

该式即为1自由度机械手的欧拉运动方程式。

除了牛顿欧拉法，还有什么动力学方法？



Contents

Kinetic and Potential Energy of a Rigid Body

$$K = \frac{1}{2} M_1 \dot{\boldsymbol{x}}_1^2 + \frac{1}{2} M_0 \dot{\boldsymbol{x}}_0^2$$

$$P = \frac{1}{2} k (\boldsymbol{x}_1 - \boldsymbol{x}_0)^2 - M_1 \boldsymbol{g} \boldsymbol{x}_1 - M_0 \boldsymbol{g} \boldsymbol{x}_0$$

$$D = \frac{1}{2} c (\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_0)^2$$

$$W = \boldsymbol{F} \boldsymbol{x}_1 - \boldsymbol{F} \boldsymbol{x}_0$$

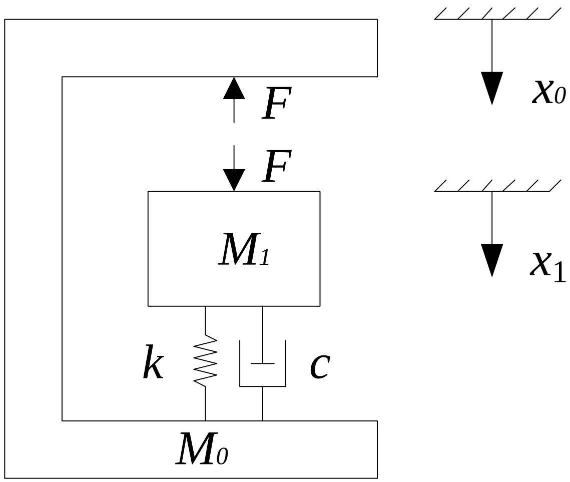
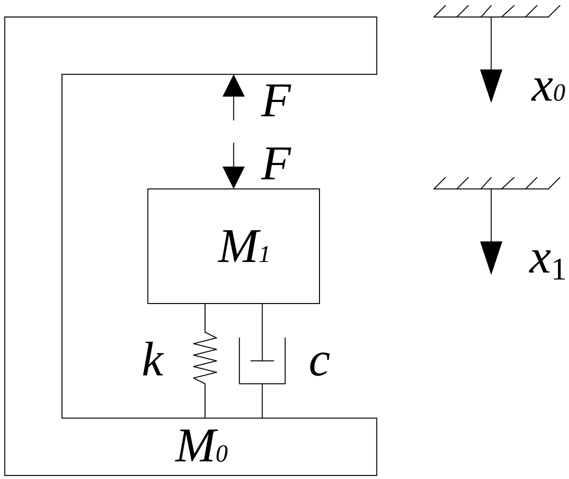


图4.1 一般物体的动能与位能

- $x_0 = 0, x_1$ is a generalized coordinate

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}_1}\right) - \frac{\partial K}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} + \frac{\partial P}{\partial x_1} = \frac{\partial W}{\partial x_1}$$

① ② ③ ④ ⑤



① Kinetic Energy due to (angular) velocity

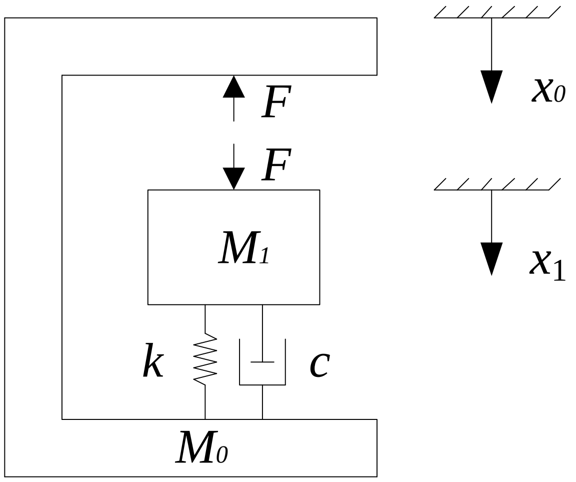
② Kinetic Energy due to position (or angle)

③ Dissipation Energy due to (angular) velocity

④ Potential Energy due to position

⑤ External Force or Torque

- x_0 and x_1 are both generalized coordinates



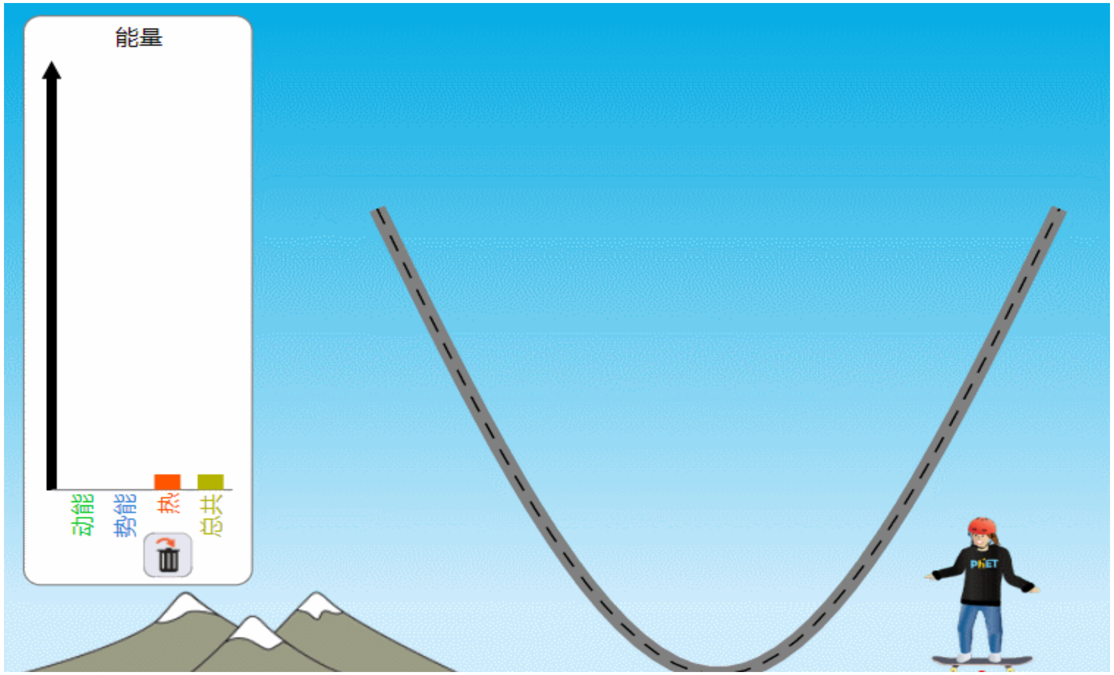
$$M_1\ddot{\boldsymbol{x}}_1 + c(\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_0) + k(\boldsymbol{x}_1 - \boldsymbol{x}_0) - M_1\boldsymbol{g} = \boldsymbol{F}$$

$$M_0\ddot{\boldsymbol{x}}_0 + c(\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_0) - k(\boldsymbol{x}_1 - \boldsymbol{x}_0) - M_0\boldsymbol{g} = -\boldsymbol{F}$$

Written in Matrices form:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_0 \end{bmatrix}\begin{bmatrix} \ddot{\boldsymbol{x}}_1 \\ \ddot{\boldsymbol{x}}_0 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_0 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ -\boldsymbol{F} \end{bmatrix}$$

Contents



$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

动能　　势能

约瑟夫•拉格朗日

（Joseph-Louis Lagrange，1736~1813）

法国著名数学家、物理学家。他在数学、力学和天文学三个学科领域中都有历史性的贡献，其中尤以数学方面的成就最为突出。

拉格朗日是分析力学的创立者。他在所著《分析力学》(1788)中，吸收并发展了欧拉、达朗贝尔等人的研究成果，应用数学分析解决质点和质点系(包括刚体、流体)的力学问题。

此外，拉格朗日还是数学分析仅次于欧拉的最大开拓者，在18世纪创立的主要分支中都有开拓性贡献。包括变分法、微分方程、方程论、数论、函数和无穷级数以及拉格朗日内插公式等。

4.4 Lagrange's Equation

- **Langrangian Function** L is defined as:

$$L = K - P \tag{4.1}$$

Kinetic Energy Potential Energy

- Dynamic Equation of the system (Langrangian Equation):

$$\boldsymbol{F}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}, i = 1, 2, \cdots n \tag{4.2}$$

where q_i is the generalized coordinates, $\dot{q}_i$ represent corresponding velocity, F_i stand for corresponding torque or force on the ith coordinate.

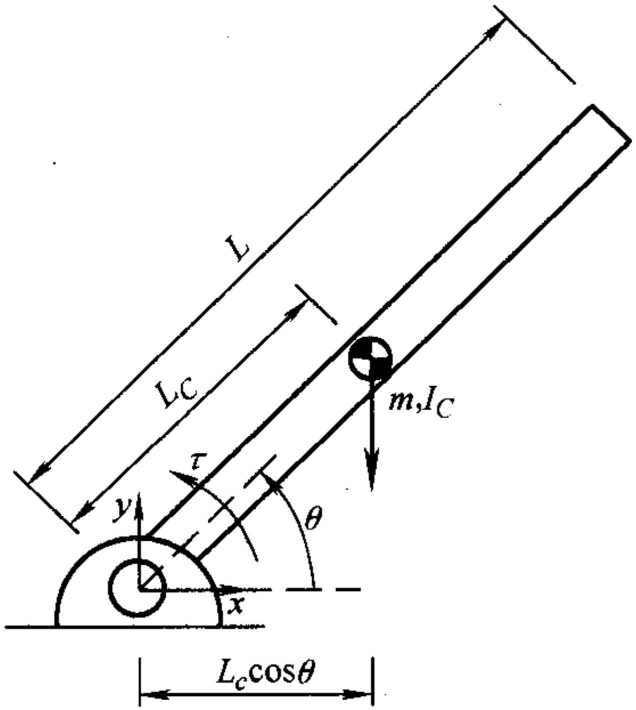
例2.通过拉格朗日运动方程式求解之前推导的1自由度机械手。

解：假设θ为广义坐标，则有

$$K=\frac{1}{2}I\dot{\theta}^2 \qquad P=mgL_c\sin\theta$$

$$L=K-P=\frac{1}{2}I\dot{\theta}^2-mgL_c\sin\theta$$

$$\frac{\partial L}{\partial\dot{\theta}}=I\dot{\theta} \qquad \frac{\partial L}{\partial\theta}=-mgL_c\cos\theta$$



由拉格朗日运动方程 $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}}\right)-\frac{\partial L}{\partial q}=\tau$

$$=> \quad I\ddot{\theta}+mgL_c\cos\theta=\tau$$

4.4 Lagrange’s Equation

- Kinetic and Potential Energy of a 2-links manipulator

$$K_1=\frac{1}{2}m_1v_1^2,\ v_1=d_1\dot{\theta}_1,\ P_1=m_1gh_1,\ h_1=-d_1\cos\theta_1$$

- Kinetic Energy K_1 and Potential Energy P_1 of link 1

$$K_1=\frac{1}{2}m_1d_1^2\dot{\theta}_1^2,\ P_1=-m_1gd_1\cos\theta_1$$

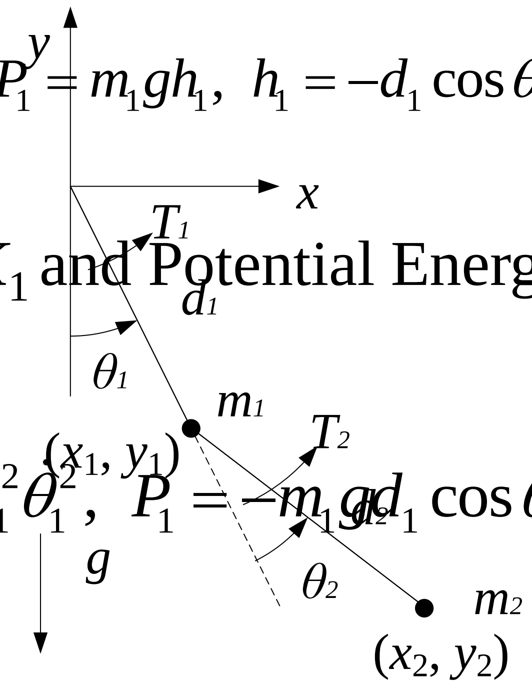


图4.2 二连杆机器手（1）

4.4 Lagrange's Equation

- Kinetic Energy K_2 and Potential Energy P_2 of link 2

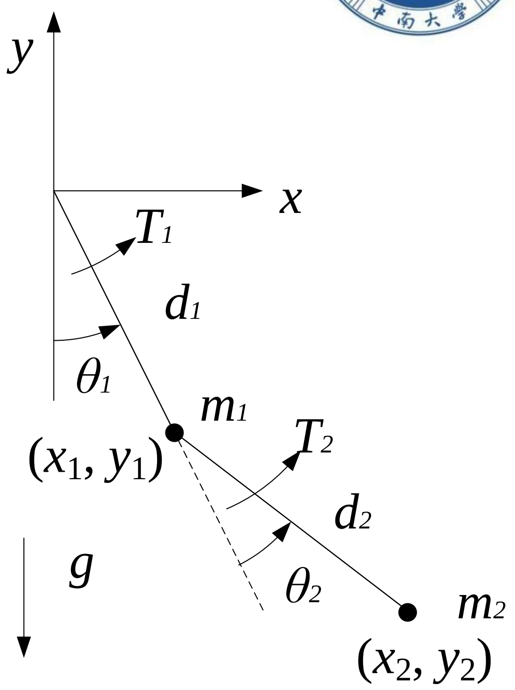
$$K_2 = \frac{1}{2} m_2 v_2^2, \quad P_2 = mgy_2$$

where

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2$$

$$x_2 = d_1 \sin\theta_1 + d_2 \sin(\theta_1 + \theta_2)$$

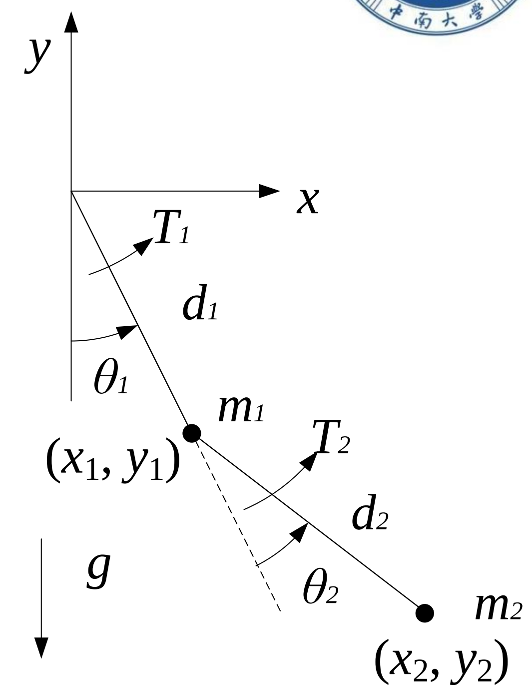
$$y_2 = -d_1 \cos\theta_1 - d_2 \cos(\theta_1 + \theta_2)$$



$$\Rightarrow \begin{cases} K_2 = \frac{1}{2} m_2 d_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 d_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2 d_1 d_2 \cos\theta_2 \left(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right) \\ P_2 = -m_2 g d_1 \cos\theta_1 - m_2 g d_2 \cos(\theta_1 + \theta_2) \end{cases}$$

4.4 Lagrange's Equation

- Total Kinetic and Potential Energy of a 2-links manipulator are



$$K = K_1 + K_2$$

$$= \frac{1}{2}(m_1 + m_2)d_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 d_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$+ m_2 d_1 d_2 \cos\theta_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \quad (4.3)$$

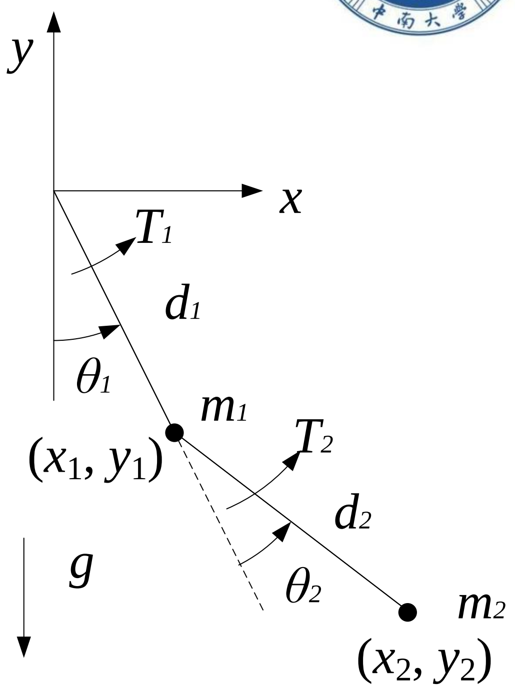
$$P = P_1 + P_2$$

$$= -(m_1 + m_2)gd_1\cos\theta_1 - m_2 g d_2\cos(\theta_1 + \theta_2) \quad (4.4)$$

4.4 Lagrange's Equation

- Lagrangian Formulation

  Lagrangian Function L of a 2-links manipulator:



$$L = K - P$$

$$= \frac{1}{2}(m_1 + m_2)d_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 d_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)$$

$$+ m_2 d_1 d_2 \cos\theta_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) + (m_1 + m_2)gd_1\cos\theta_1 + m_2 g d_2 \cos(\theta_1 + \theta_2) \qquad (4.5)$$

$$\boldsymbol{F}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}, i = 1,2,\cdots n$$

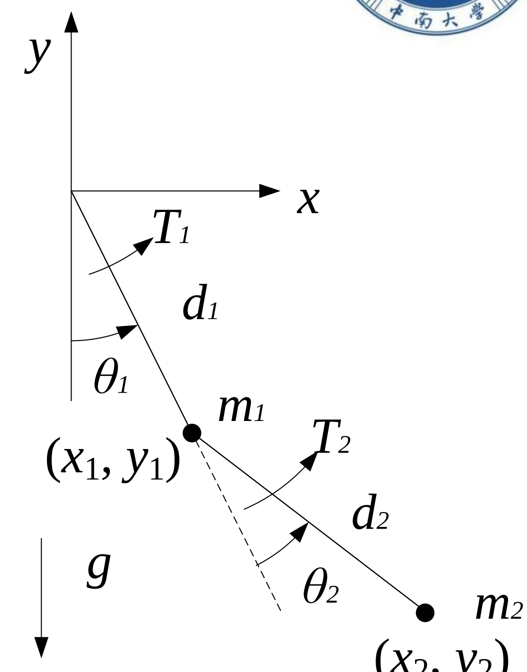
4.4 Lagrange's Equation

- Lagrangian Formulation

Dynamic Equations:

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$



有效惯量(effective inertial)：关节i的加速度在关节i上产生的惯性力

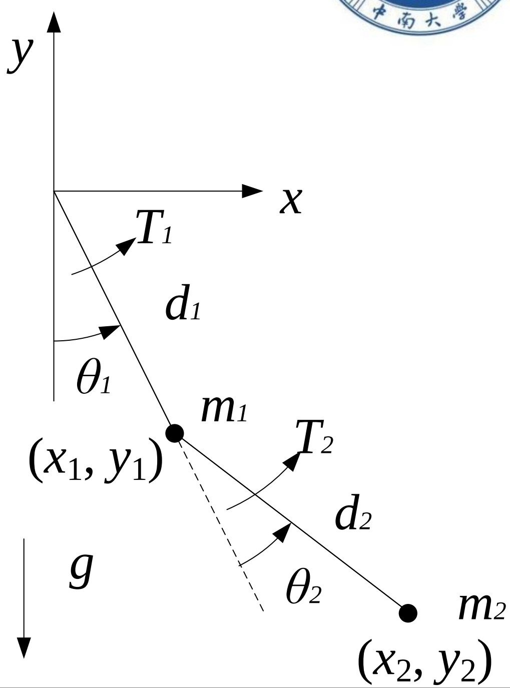
$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} \boxed{D_{11}} & D_{12} \\ D_{21} & \boxed{D_{22}} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2 \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \qquad (4.10)$$

- Lagrangian Formulation

Dynamic Equations:

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$

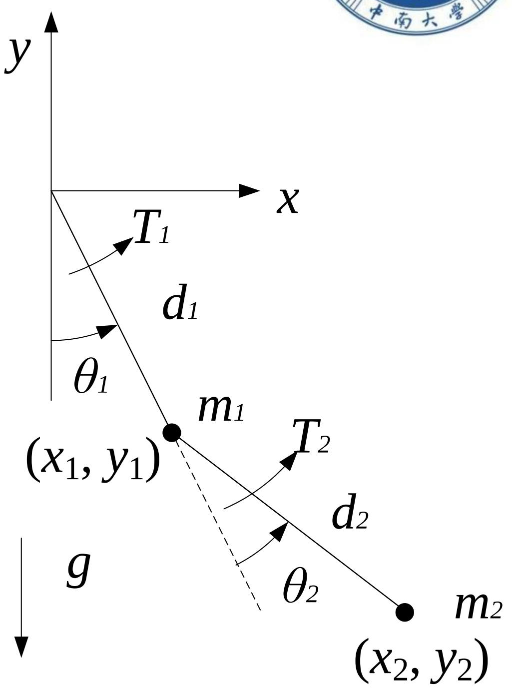


耦合惯量(coupled inertial)：关节i,j的加速度在关节j,i上产生的惯性力

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & \boxed{D_{12}} \\ \boxed{D_{21}} & D_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \qquad (4.10)$$

- Lagrangian Formulation

Dynamic Equations:



$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$

向心加速度(acceleration centripetal)系数：关节*i,j*的速度在关节*j,i*上产生的向心力

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \qquad (4.10)$$

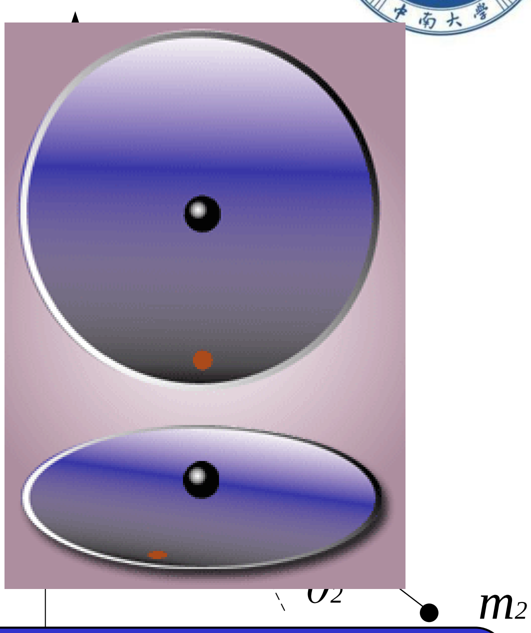
4.4 Lagrange's Equation

- Lagrangian Formulation

Dynamic Equations:

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$



哥氏加速度(Coriolis accelaration)系数：
关节*j*,*k*的速度引起的在关节*i*上产生的哥氏力(Coriolis force)

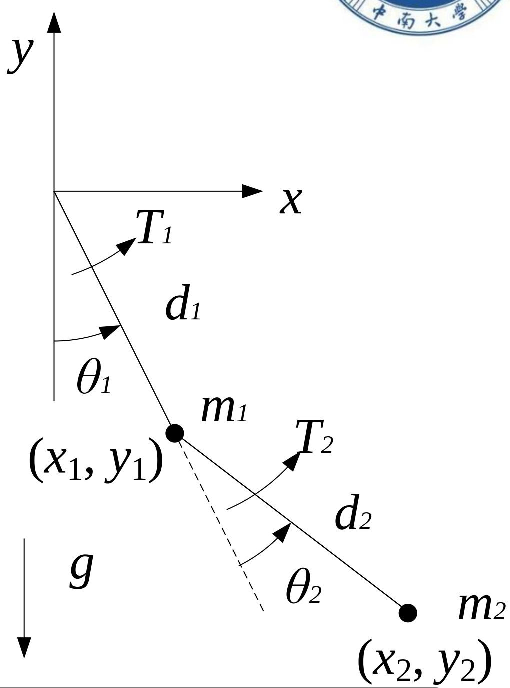
$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \qquad (4.10)$$

- Lagrangian Formulation

Dynamic Equations:

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$



Written in Matrices Fo

重力项(gravity)：关节*i,j*处的重力

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_1 \end{bmatrix} + \boxed{\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}} \qquad (4.10)$$

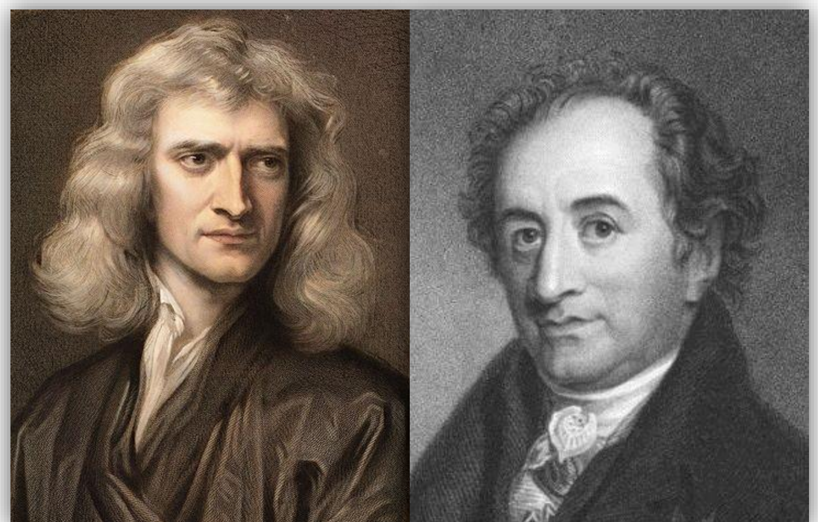
- 对上例指定一些数字，以估计此二连杆机械手在静止和固定重力负载下的 T_1 和 T_2 的数值。
- 取 $d_1=d_2=1$，$m_1=2$，计算$m_2=1,4$和100（分别表示机械手在**地面空载**、**地面满载**和**在外空间负载**的三种不同情况；在外空间由于失重而允许有较大的负载）三个不同数值下各系数的数值。

注意：有效惯量的变化将对机械手的控制产生显著影响！

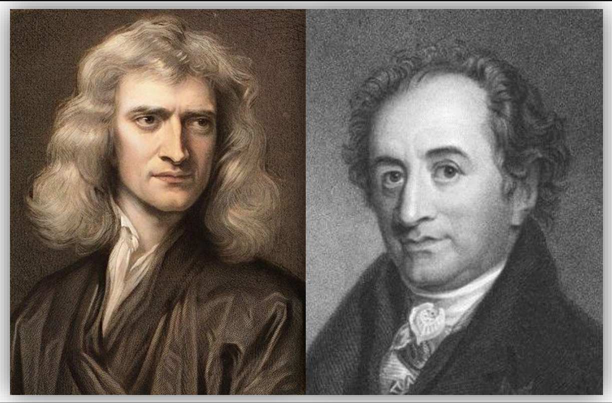
表4.1

负载	θ_2	$\cos\theta_2$	D_{11}	D_{12}	D_{22}	I_1	I_2
地面空载	0° 90° 180° 270°	1 0 -1 0	6 4 2 4	2 1 0 1	1 1 1 1	6 4 2 4	2 3 2 3
地面满载	0° 90° 180° 270°	1 0 -1 0	18 10 2 10	8 4 0 4	4 4 4 4	18 10 2 10	2 6 2 6
外空间负载	0° 90° 180° 270°	1 0 -1 0	402 202 2 202	200 100 0 100	100 100 100 100	402 202 2 202	2 102 2 102

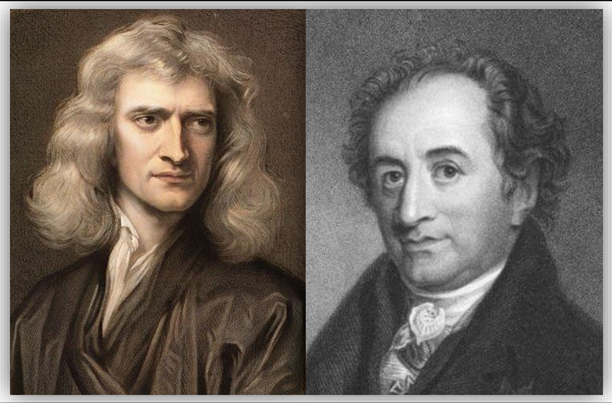
牛顿欧拉法 & 拉格朗日法



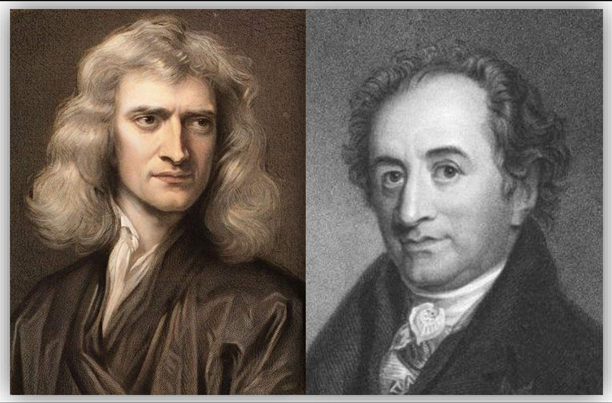
牛顿欧拉法 & 拉格朗日法

		
公式	$$F = m a$$ $$\vec{\tau} = I\vec{\alpha} + \vec{\omega} \times I\vec{\omega}$$ 牛顿欧拉法的表达方式在解决实际问题时会显得十分复杂；力学方程组包含大量的微分方程，在处理约束问题时，虽然独立变量减少了，可相关约束方程又增加了，加大了解决问题的难度。 败	$$F_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_i} - \frac{\partial L}{\partial y_i}, i = 1, 2, \cdots, n$$ 比如:对于有n个质点所组成的受到K个约束条件限制的力学体系，牛顿力学求解需 3N+K个方程；拉格朗日方程则只需3N-K个，但约束越多，则拉格朗日越显其锋芒。 胜

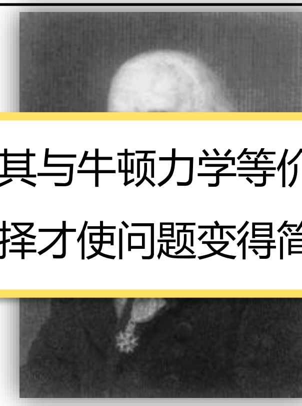
牛顿欧拉法 & 拉格朗日法

		
问题着眼点	牛顿欧拉法着眼点放在作用在物体上的受力情况。 处理问题需要考虑各个质点的受力，是矢量问题，解决问题是既要注意其大小再要注意其方向。 败	拉格朗日力学着眼于对整个系统的能量概念。 采用能量（标量）来解决问题，降低了问题的难度。 胜

牛顿欧拉法 & 拉格朗日法

		
计算量	牛顿－欧拉法着眼于每一个连杆的运动，即便对于多自由度的机械手其计算量也不增加，因此算法易于编程。 胜	对多自由度的机械手，拉格朗日法可以直接推导运动方程式，但随着自由度的增多演算量将大量增加。 败

牛顿欧拉法 & 拉格朗日法

		
公式	相关约束较多，加大了解决问题的难度。 败	相关约束较少，拉格朗日占优。 胜
问题着眼点	着眼于作用在物体上的受力情况。是矢量问题。 败	着眼于对整个系统的能量概念。是标量问题。 胜
计算量	即便对于多自由度的机械手其计算量也不增加。 胜	随着自由度的增多演算量将大量增加。 败

拉格朗日力学只是选择从另外角度来研究力学，其与牛顿力学等价，在处理问题时各有优缺，只有在适当的地方合适选择才使问题变得简单!

Contents

4.5 Dynamic Equation of a Manipulator

- Forming dynamic equation of any manipulator described by a series of A-matrices:

  (1) Computing the **Velocity** of any given point;

  (2) Computing total **Kinetic Energy**;

  (3) Computing total **Potential Energy**;

  (4) Forming **Lagrangian Function** of the system;

  (5) Forming Dynamic Equation through **Lagrangian Equation**.

4.5.1 Computing the Velocity

- Velocity of point P on link-3:

$$^{0}\boldsymbol{v}_{p} = \frac{d}{dt}(^{0}\boldsymbol{r}_{p}) = \frac{d}{dt}(T_{3}\,{}^{3}\boldsymbol{r}_{p}) = \dot{T}_{3}\,{}^{3}\boldsymbol{r}_{p}$$

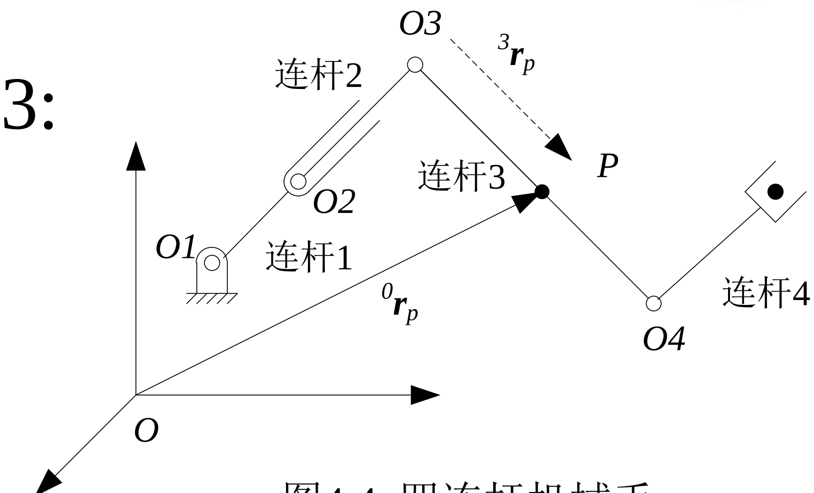


图4.4 四连杆机械手

- Velocity of any given point on link-i:

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \left(\sum_{j=1}^{i}\frac{\partial T_{i}}{\partial q_{j}}\dot{q}_{j}\right){}^{i}\boldsymbol{r} \qquad (4.15)$$

4.5.1 Computing the Velocity

- Acceleration of point P:

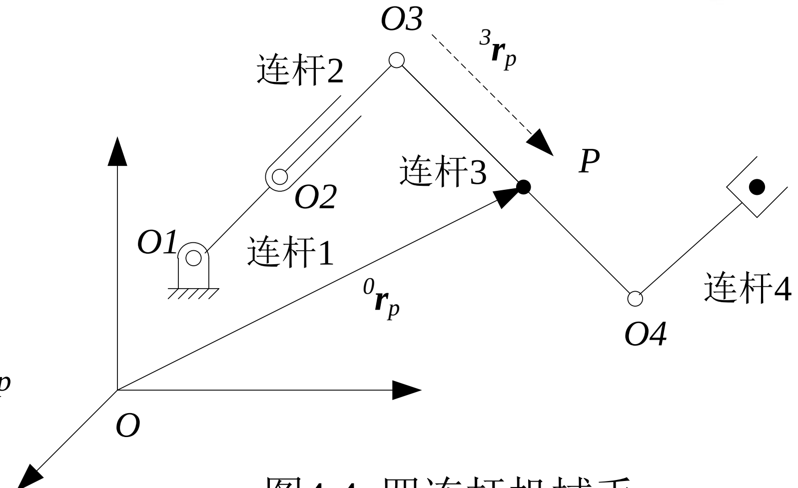


图4.4 四连杆机械手

$$^0\boldsymbol{a}_p = \frac{d}{dt}({}^0\boldsymbol{v}_p) = \frac{d}{dt}(\dot{T}_3\,{}^3\boldsymbol{r}_p) = \frac{d}{dt}\left(\sum_{j=1}^{3}\frac{\partial T_3}{\partial q_i}\dot{q}_i\right){}^3\boldsymbol{r}_p$$

$$= \left(\sum_{j=1}^{3}\frac{\partial T_3}{\partial q_i}\frac{d}{dt}\dot{q}_i\right)\left({}^3\boldsymbol{r}_p\right) + \left(\sum_{k=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 T_3}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j\right)\left({}^3\boldsymbol{r}_p\right)$$

$$= \left(\sum_{j=1}^{3}\frac{\partial T_3}{\partial q_i}\ddot{q}_i\right)\left({}^3\boldsymbol{r}_p\right) + \left(\sum_{k=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 T_3}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j\right)\left({}^3\boldsymbol{r}_p\right)$$

4.5.1 Computing the Velocity

- Square of velocity

$$({}^0\boldsymbol{v}_p)^2 = ({}^0\boldsymbol{v}_p)\cdot({}^0\boldsymbol{v}_p) = Trace[({}^0\boldsymbol{v}_p)\cdot({}^0\boldsymbol{v}_p)^T]$$

$$= Trace\left[\sum_{j=1}^{3}\frac{\partial T_3}{\partial q_j}\dot{q}_j({}^3\boldsymbol{r}_p)\cdot\sum_{k=1}^{3}\left(\frac{\partial T_3}{\partial q_k}\dot{q}_k\right)({}^3\boldsymbol{r}_p)^T\right]$$

$$= Trace\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial T_3}{\partial q_j}({}^3\boldsymbol{r}_p)({}^3\boldsymbol{r}_p)^T\frac{\partial T_3}{\partial q_k}^T\dot{q}_j\dot{q}_k\right]$$

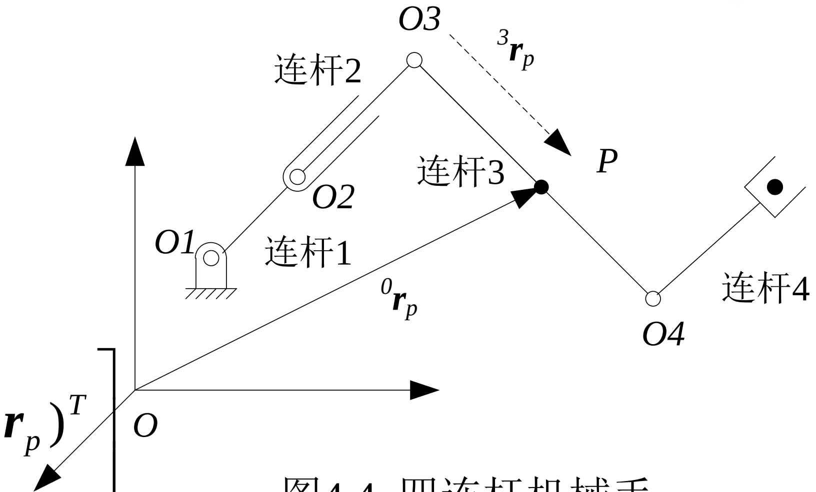


图4.4 四连杆机械手

The trace of an square matrix is defined to be the sum of the diagonal elements.

4.5.1 Computing the Velocity

- Square of velocity of any given point:

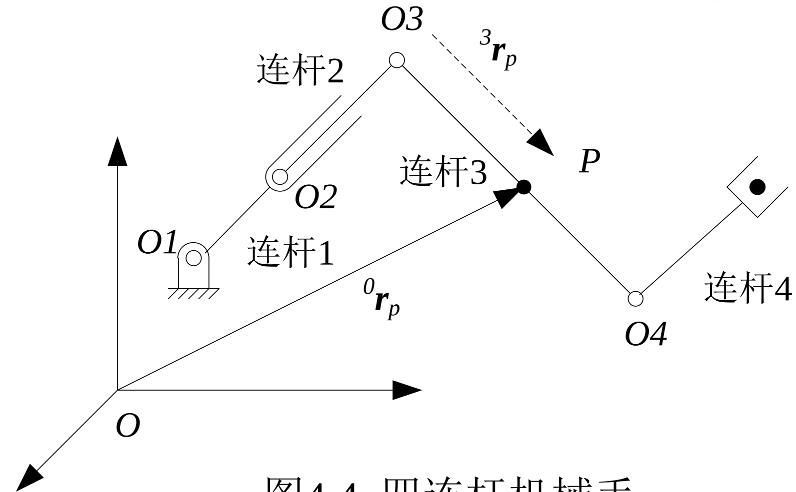


图4.4 四连杆机械手

$$\boldsymbol{v}^2 = \left(\frac{dr}{dt}\right)^2 = Trace\left[\sum_{j=1}^{i}\frac{\partial T_i}{\partial q_j}\dot{q}_j\,{}^{i}\boldsymbol{r}\sum_{k=1}^{i}\left(\frac{\partial T_i}{\partial q_k}\dot{q}_k\,{}^{i}\boldsymbol{r}\right)^T\right]$$

$$= Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial T_i}{\partial q_k}{}^{i}\boldsymbol{r}\,{}^{i}\boldsymbol{r}^T\left(\frac{\partial T_i}{\partial q_k}\right)^T\dot{q}_k\dot{q}_k\right] \qquad (4.16)$$

- Computing the Kinetic Energy

令连杆3上任一质点*P*的质量为dm，则其动能为：

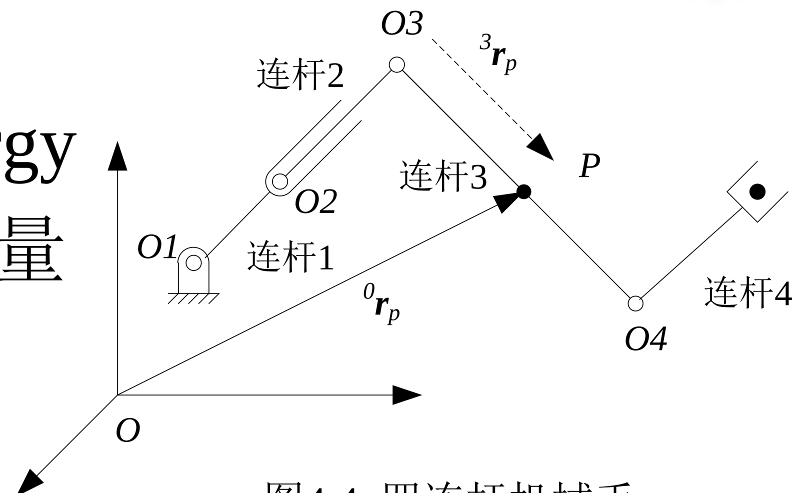


图4.4 四连杆机械手

$$dK_3 = \frac{1}{2} v_p^2 dm$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial T_3}{\partial q_i} {}^{3}r_p ({}^{3}r_p)^T \left(\frac{\partial T_3}{\partial q_k} \right)^T \dot{q}_i \dot{q}_k \right] dm$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial T_3}{\partial q_i} ({}^{3}r_p dm {}^{3}r_p{}^{T})^T \left(\frac{\partial T_3}{\partial q_k} \right)^T \dot{q}_i \dot{q}_k \right]$$

- Kinetic Energy of any particle on link-i with position vector ${}^{i}\boldsymbol{r}$:

$$dK_i = \frac{1}{2} Trace\left[\sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial T_i}{\partial q_j} {}^{j}\boldsymbol{r}\, {}^{i}\boldsymbol{r}^{T} \frac{\partial T_i^{\,T}}{\partial q_k} \dot{q}_j \dot{q}_k \right] dm$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial T_i}{\partial q_j} ({}^{i}\boldsymbol{r} dm\, {}^{i}\boldsymbol{r}^{T})^{T} \frac{\partial T_i^{\,T}}{\partial q_k} \dot{q}_j \dot{q}_k \right]$$

- Kinetic Energy of link-3:

$$K_3 = \int_{\text{link}3} dK_3 = \frac{1}{2} Trace\left[\sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial T_3}{\partial q_j} \left(\int_{\text{link}3} {}^{3}\boldsymbol{r}_p\, {}^{3}\boldsymbol{r}_p^{\,T} dm \right) \left(\frac{\partial T_3}{\partial q_k} \right)^{T} \dot{q}_j \dot{q}_k \right]$$

4.5.2 Computing the Kinetic and Potential Energy

- Kinetic Energy of any given link-i:

$$K_i = \int_{\text{link } i} dK_i$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial T_i}{\partial q_j} I_i \left(\frac{\partial T_i}{\partial q_k} \right) \dot{q}_j \dot{q}_k \right] \tag{4.17}$$

- Total Kinetic Energy of the manipulator:

$$K = \sum_{i=1}^{n} K_i = \frac{1}{2} \sum_{i=1}^{n} Trace\left[\sum_{j=1}^{n} \sum_{k=1}^{i} \frac{\partial T_i}{\partial q_j} I_i \frac{\partial T_i^{T}}{\partial q_k} \dot{q}_i \dot{q}_k \right] \tag{4.19}$$

- Computing the Potential Energy

  Potential Energy of a object (mass *m*) at *h* height:

$$P = mgh$$

so the Potential Energy of any particle on link-i with position vector ${}^{i}\boldsymbol{r}$:

$$\boldsymbol{g}^T = [g_x, g_y, g_z, 1] \qquad dP_i = -dm\boldsymbol{g}^{T_0} r = -\boldsymbol{g}^T T_i {}^i r dm$$

where

$$P_i = \int_{\text{link } i} dP_i = -\int_{\text{link } i} \boldsymbol{g}^T T_i {}^i r dm = -\boldsymbol{g}^T T_i \int_{\text{link } i} {}^i r dm$$

$$= -\boldsymbol{g}^T T_i m_i {}^i r_i = -m_i \boldsymbol{g}^T T_i {}^i r_i$$

- Potential Energy of any particle on link-i with position vector ${}^{i}\boldsymbol{r}$:

$$dP_i = -dm\boldsymbol{g}^{T_0} r = -\boldsymbol{g}^T T_i {}^{i} r dm$$

- Total Potential Energy of the manipulator:

$$P = \sum_{i=1}^{n}(P_i - P_{ai}) \approx \sum_{i=1}^{n} P_i = -\sum_{i=1}^{n} m_i \boldsymbol{g}^T T_i {}^{i} r_i \qquad (4.21)$$

4.5.3 Forming the Dynamic Equation

- Lagrangian Function

$$L = K_t - P$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_i} I_i \frac{\partial T_i^{T}}{\partial q_k}\right)\dot{q}_j \dot{q}_k + \frac{1}{2}\sum_{i=1}^{n} I_{ai}\dot{q}_i^2 + \sum_{i=1}^{n} m_i \boldsymbol{g}^T T_i \, {}^i r_i ,$$

$$n = 1,2\cdots \qquad (4.22)$$

4.5.3 Forming the Dynamic Equation

- Derivative of Lagrangian function

$$\frac{\partial L}{\partial \dot{q}_p} = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_p} I_i \frac{\partial T_i^{\,T}}{\partial q_k}\right)\dot{q}_k$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_i} I_i \frac{\partial T_i^{T}}{\partial q_p}\right)\dot{q}_j + I_{ap}\dot{q}_p$$

$$p = 1, 2, \cdots n$$

4.5.3 Forming the Dynamic Equation

- According to Eq.(4.18), I_i is a symmetric matrix, lead to

$$Trace\left(\frac{\partial T_i}{\partial q_j} I_i \frac{\partial T_i^T}{\partial q_k}\right) = Trace\left(\frac{\partial T_i}{\partial q_k} I_i^T \frac{\partial T_i^T}{\partial q_j}\right) = Trace\left(\frac{\partial T_i}{\partial q_k} I_i \frac{\partial T_i^T}{\partial q_j}\right)$$

$$\frac{\partial L}{\partial \dot{q}_p} = \sum_{i=1}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_k} I_i \frac{\partial T_i^T}{\partial q_p}\right)\dot{q}_k + I_{ap}\dot{q}_p$$

4.5.3 Forming the Dynamic Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_p} = \sum_{i=p}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_k} I_i \frac{\partial T_i^{\,T}}{\partial q_p}\right)\ddot{q}_k + I_{ap}\ddot{q}_p$$

$$+\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^{\,T}}{\partial q_i}\right)\dot{q}_j \dot{q}_k$$

$$+\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_p \partial q_k} I_i \frac{\partial T_i^{\,T}}{\partial q_i}\right)\dot{q}_j \dot{q}_k$$

$$= \sum_{i=p}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_k} I_i \frac{\partial T_i^{\,T}}{\partial q_p}\right)\ddot{q}_k + I_{ap}\ddot{q}_p$$

$$+2\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^{\,T}}{\partial q_k}\right)\dot{q}_j \dot{q}_k$$

4.5.3 Forming the Dynamic Equation

$$\frac{\partial L}{\partial q_p} = \frac{1}{2}\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^{\,T}}{\partial q_k}\right)\dot{q}_j \dot{q}_k$$

$$+\frac{1}{2}\sum_{i=p}^{n}\sum_{i=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_k \partial q_p} I_i \frac{\partial T_i^{\,t}}{\partial q_j}\right)\dot{q}_j \dot{q}_k + \sum_{i=p}^{n} m_i \boldsymbol{g}^T \frac{\partial T_i}{\partial q_p}{}^i r_i$$

$$= \sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_p \partial q_j} I_i \frac{\partial T_i^{\,T}}{\partial q_k}\right)\dot{q}_j \dot{q}_k + \sum_{i=p}^{n} m_i \boldsymbol{g}^T \frac{\partial T_i}{\partial q_p}{}^i r_i$$

4.5.3 Forming the Dynamic Equation

- Dynamic Equation of a n-link manipulator:

$$T_i = \sum_{j=i}^{n}\sum_{k=1}^{j} Trace\left(\frac{\partial T_j}{\partial q_k} I_j \frac{\partial T_j}{\partial q_i}^T\right)\ddot{q}_k + I_{ai}\ddot{q}_i$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} Trace\left(\frac{\partial^2 T_i}{\partial q_k \partial q_m} I_j \frac{\partial T_j}{\partial q_i}^T\right)\dot{q}_k\dot{q}_m - \sum_{j=1}^{n} m_j \boldsymbol{g}^T \frac{\partial T_i}{\partial q_i}{}^i r_i \qquad (4.23)$$

$$T_i = \sum_{j=1}^{n} D_{ij}\ddot{q}_j + I_{ai}\ddot{q}_i + \sum_{j=1}^{6}\sum_{k=1}^{6} D_{ijk}\dot{q}_j\dot{q}_k + D_i \qquad (4.24)$$

注意：上述**惯量项**与**重力项**在机械手的控制中特别重要，它们将直接影响到机械手系统的**稳定性**和**定位精度**。只有当机械手高速运动时，向心力和哥氏力才变得重要。

4.6 Summary

- Two methods to form dynamic equation of a rigid body:
  - Newton-Euler Equation (Force-balance)
  - Lagrange’s Equation (Energy-based)
- Summarize steps to form Lagrange’s Equation of n-link manipulators:
  - Computing the **Velocity** of any given point;
  - Computing total **Kinetic Energy**;
  - Computing total **Potential Energy**;
  - Forming **Lagrangian Function** of the system;
  - Forming Dynamic Equation through **Lagrangian Equation**.

谢 谢！