第3讲 机器人运动学

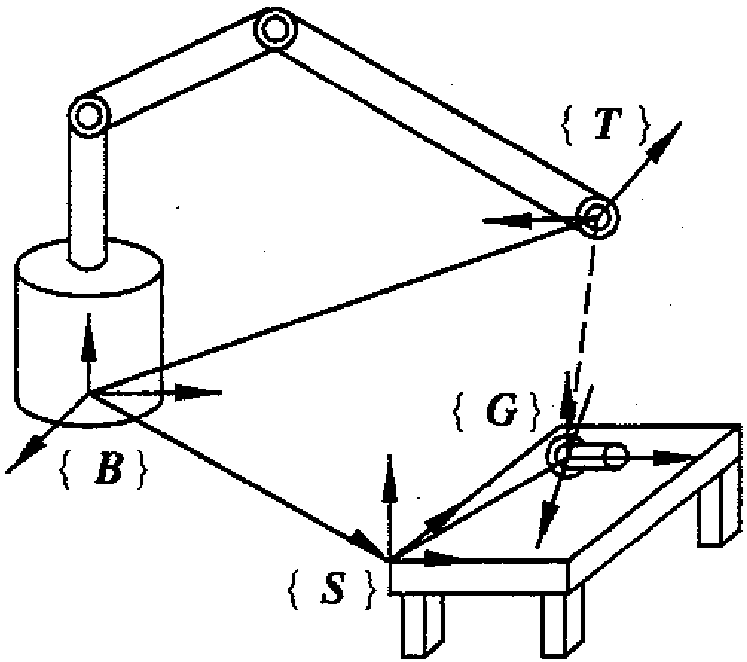
谢 斌
xiebin@csu.edu.cn
中南大学人工智能与机器人实验室
http://airl.csu.edu.cn

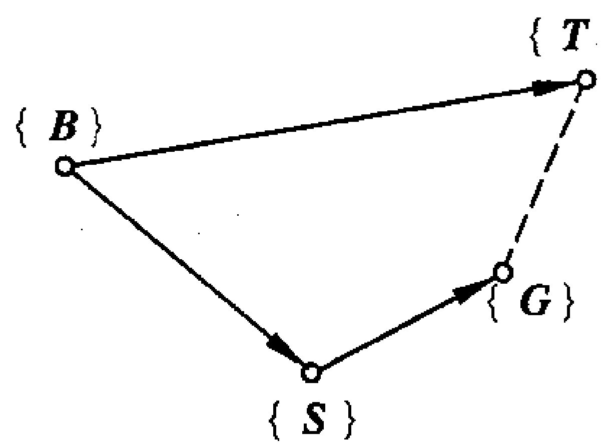
Review

- **Positions, Orientations, and Frames**
- **Mappings**
- **Operators**
- **Transform Equations**

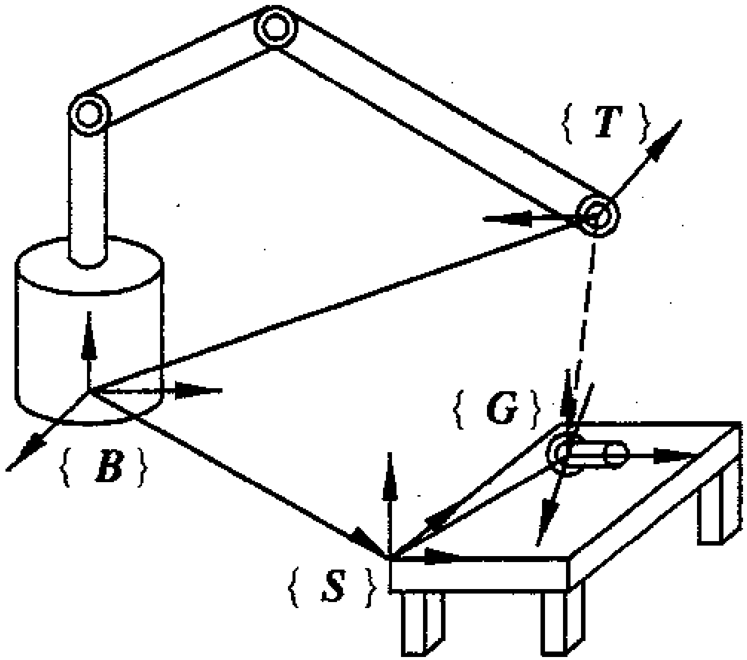
Example 2.8

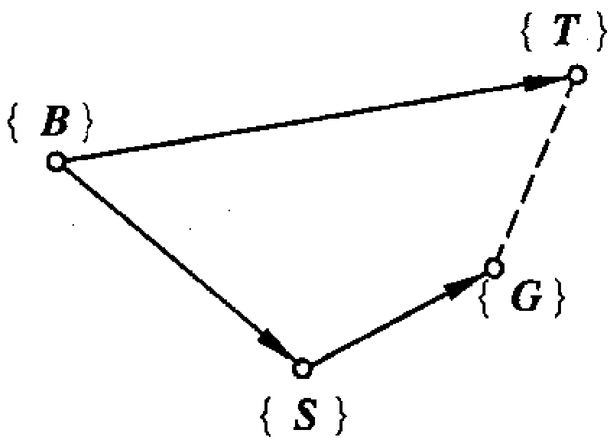
- Assume that we know the transform ${}^{B}_{T}T$, ${}^{B}_{S}T$, and ${}^{S}_{G}T$. Calculate the position and orientation of the bolt relative to the manipulator's hand, ${}^{T}_{G}T$.





Example 2.8





Solution:

$$ {}^{G}_{S}T \; {}^{S}_{B}T \; {}^{B}_{T}T \; {}^{T}_{G}T = \mathrm{I} $$

$$ \Rightarrow \; {}^{S}_{G}T^{-1} \; {}^{B}_{S}T^{-1} \; {}^{B}_{T}T \; {}^{T}_{G}T = \mathrm{I} $$

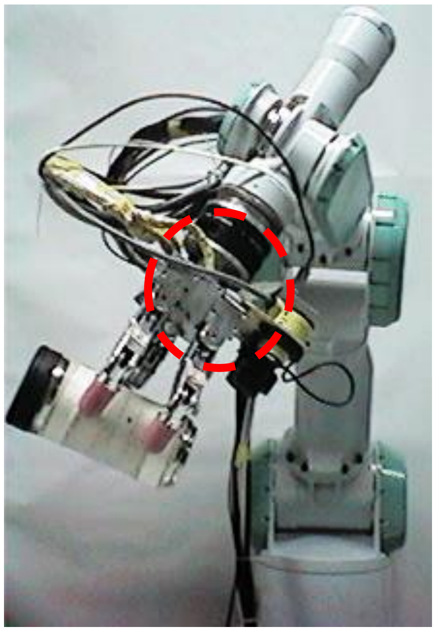
$$ \Rightarrow \; {}^{T}_{G}T = {}^{B}_{T}T^{-1} \; {}^{B}_{S}T \; {}^{S}_{G}T $$

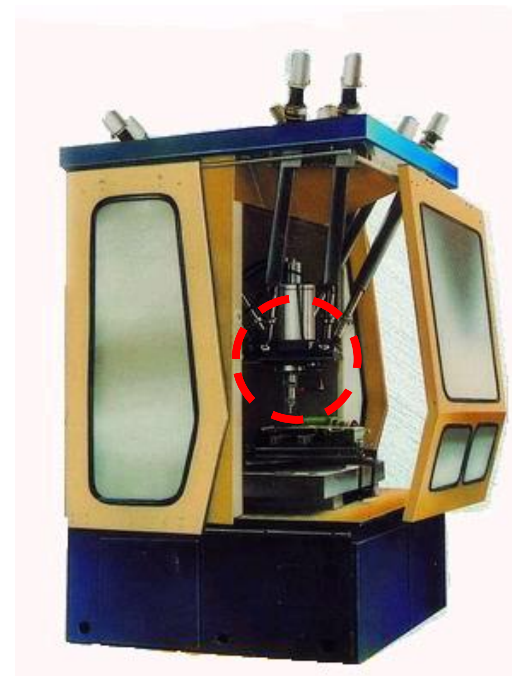
Questions

1. 如何描述机器人的连杆参数？
2. 已知机器人的连杆参数，如何求取末端执行器的姿态？
3. 想要机器人的末端执行器到达指定的位置和朝向，如何求解所有可能的连杆配置？

Contents

Introduction to Robot Kinematics

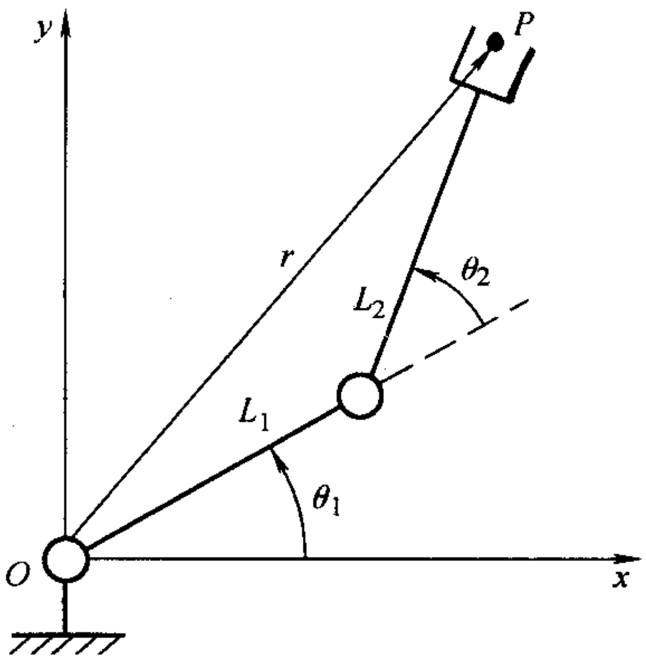




机器人运动学研究的是机器人的工作空间与关节空间之间的影射关系或机器人的运动学模型（**Model**），包括正（**Forward**）运动学和逆（**Inverse**）运动学两部分内容。

Introduction to Robot Kinematics

- **Kinematics** treats motion **without regard to the forces** that cause it. Within the science of kinematics one studies the **position, velocity, acceleration**, and all higher order derivatives of the position variables (with respect to time or any other variable).



从**几何学**的观点来处理**手指位置**P与**关节变量** L_1, L_2, θ_1 和 θ_2的关系称为**运动学**(Kinematics)。

Introduction to Robot Kinematics

- **Direct (also forward) kinematics**
  - **Given** are joint relations (rotations, translations) for the robot arm.
  - **Task:** What is the orientation and position of the end-effector?
- **Inverse kinematics**
  - **Given** is desired end-effector position and orientation.
  - **Task:** What are the joint rotations and translations to achieve this?

Example of Direct Kinematics

- Define position of end-effector and the joint variable，

$$r = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

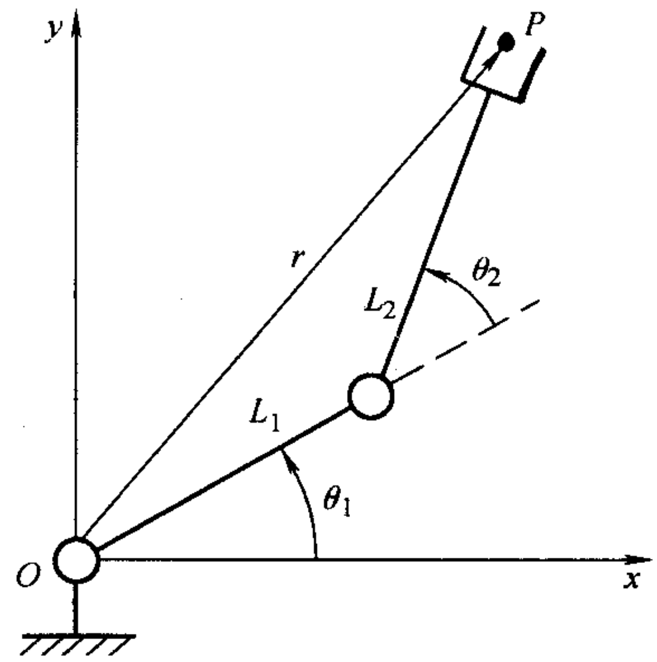
- According to geometry：

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

The general vector form

$$r = f(\theta)$$



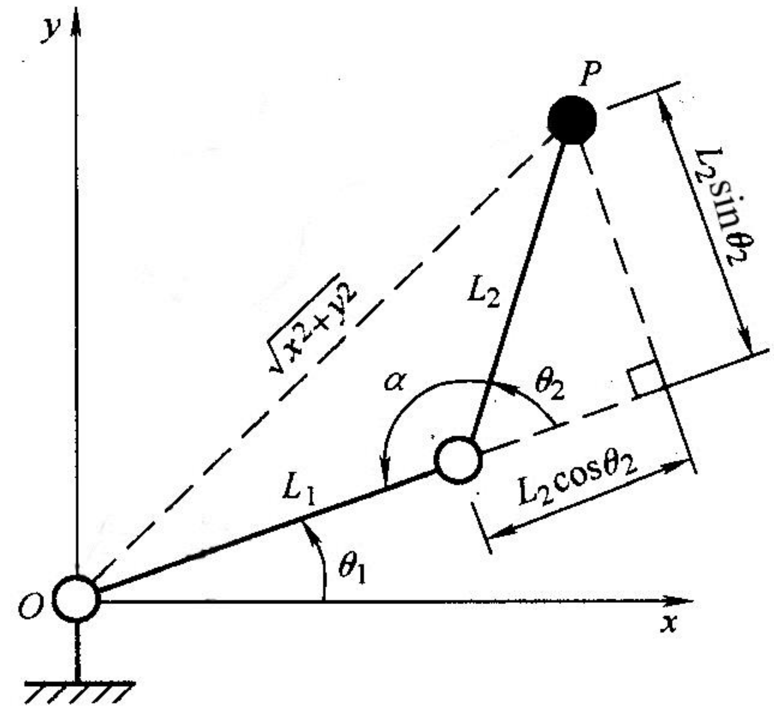
Example of Inverse Kinematics

$$\theta_2 = \pi - \alpha$$

$$\theta_1 = \arctan\left(\frac{y}{x}\right) - \arctan\left(\frac{L_2 \sin\theta_2}{L_1 + L_2\cos\theta_2}\right)$$

where

$$\alpha = \arccos\left[\frac{-(x^2+y^2)+L_1^2+L_2^2}{2L_1L_2}\right]$$

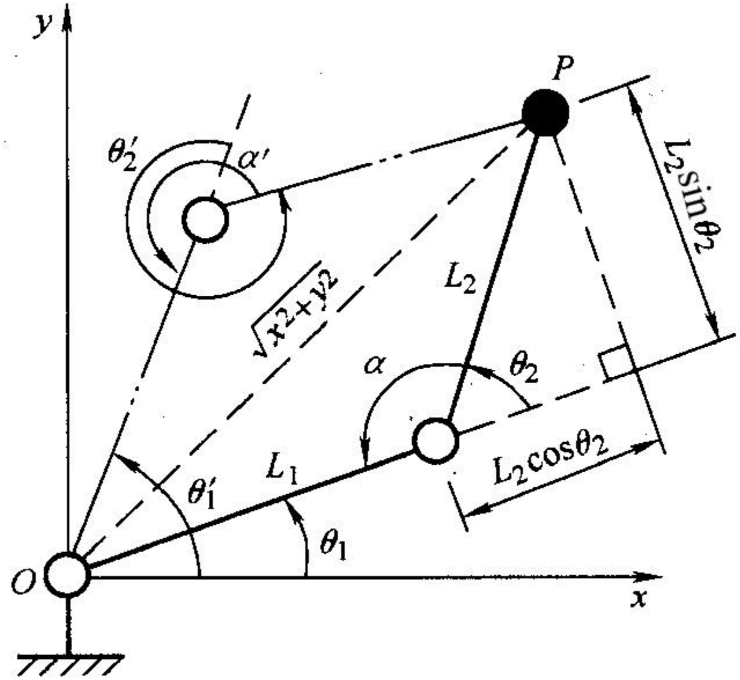


The general vector form

$$\theta = f^{-1}(r)$$

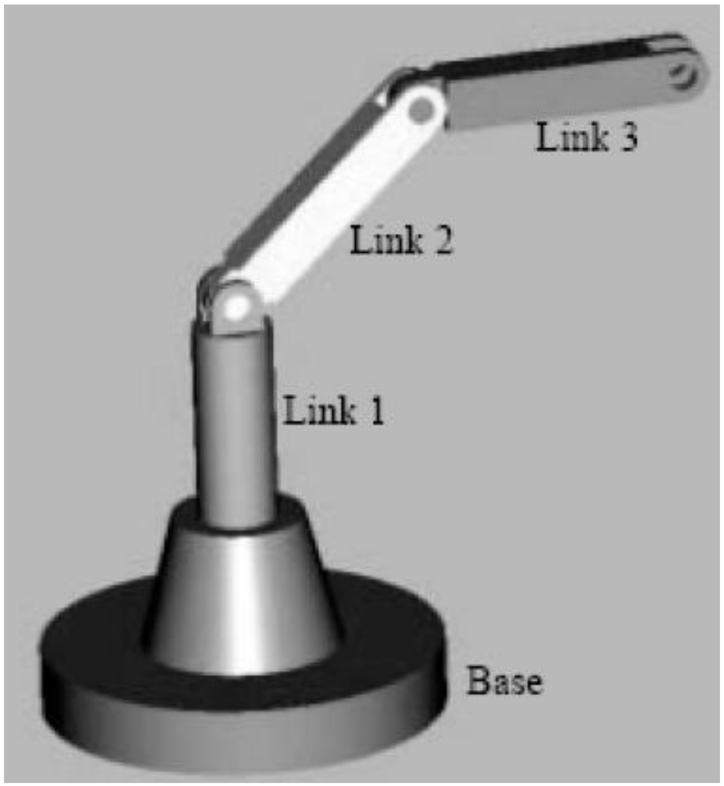
Multiple solutions of inverse kinematics

$$\alpha' = -\alpha$$



Contents

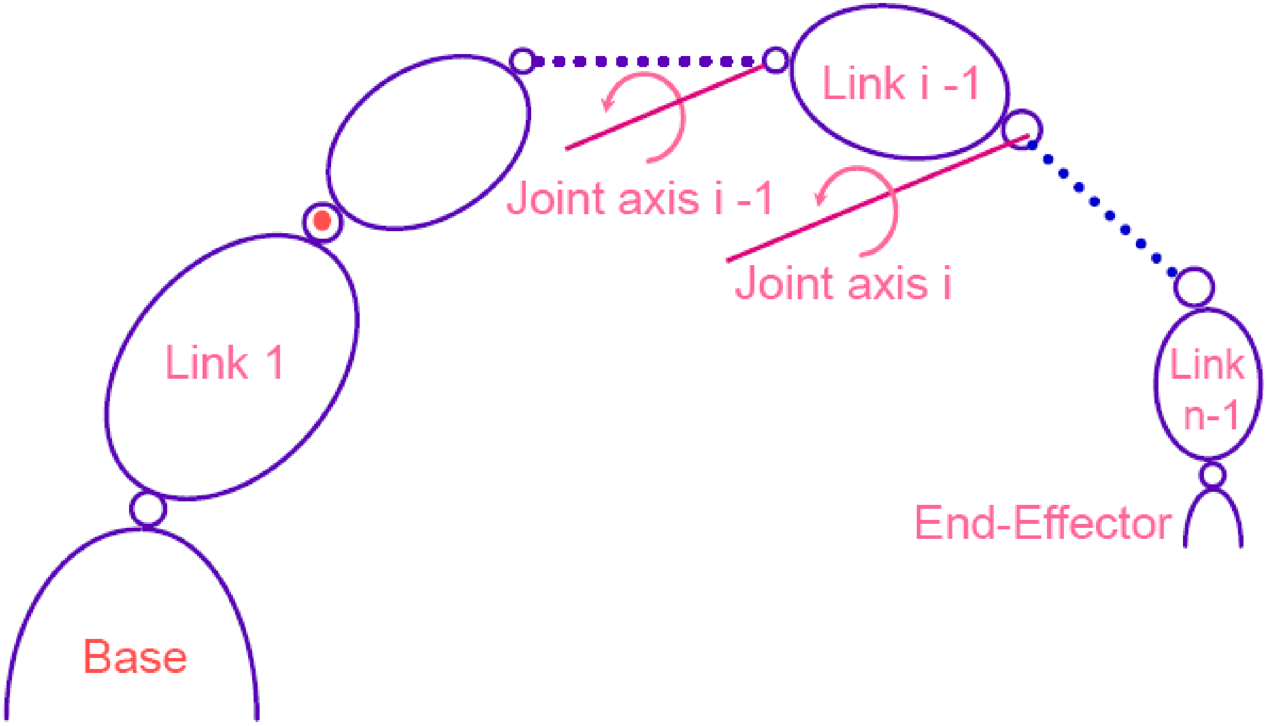
Link Description



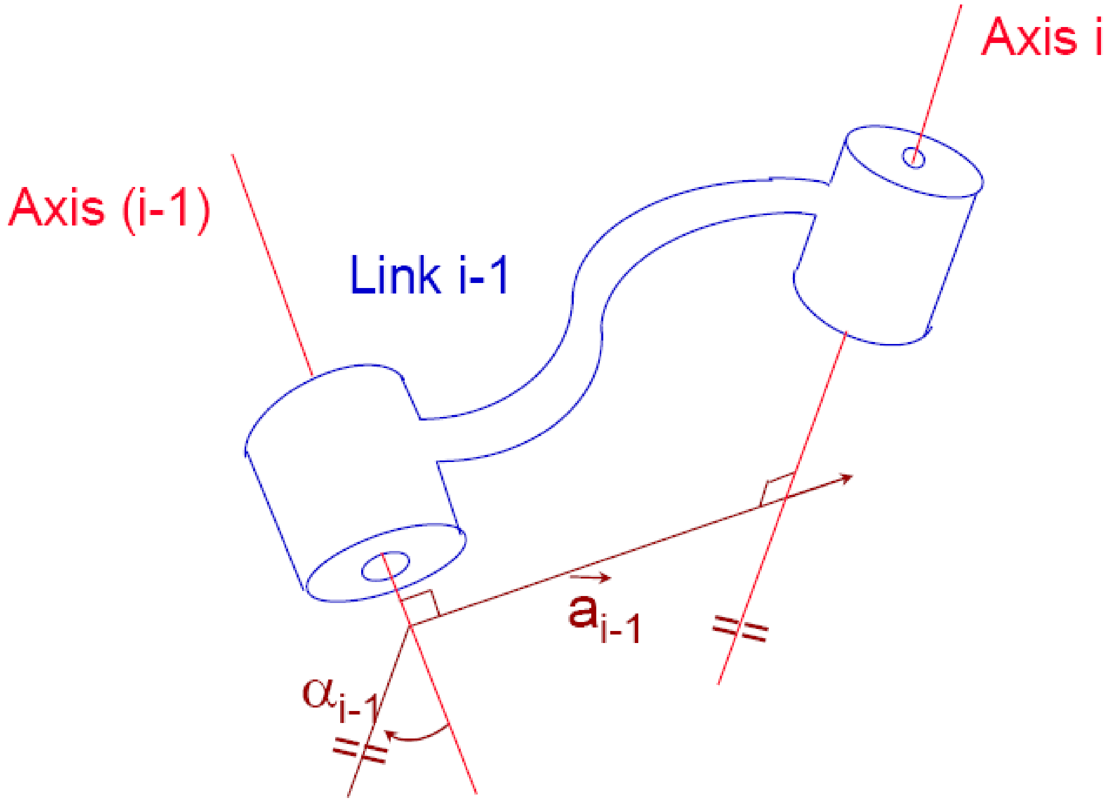
- Mechanics of a manipulator can be represented as a kinematics chain of rigid bodies (links) connected by revolute or prismatic joints.
- One end of the chain is constrained to a base, while an end-effector is mounted to the other end of the chain.
- The resulting motion is obtained by composition of the elementary motions of each link with respect to the previous one.

Link Description

Manipulator



Link Description



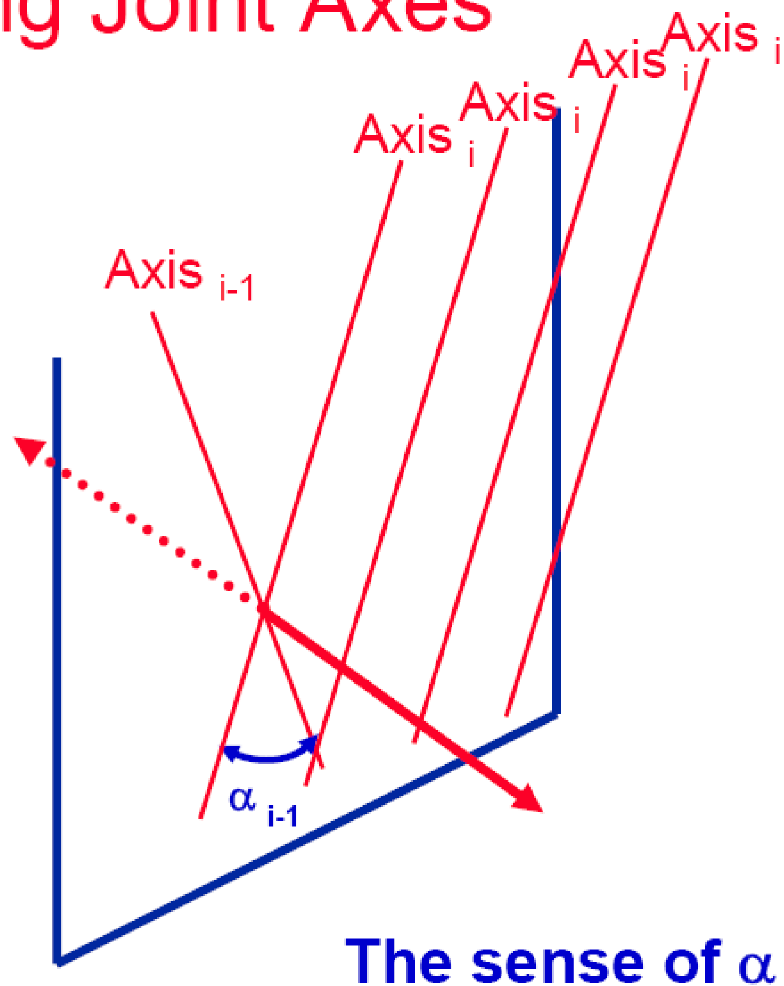
a_{i-1}: Link Length - mutual perpendicular

unique except for parallel axis

α_{i-1}: Link Twist - measured in the right-hand sense about $\vec{a}_{i-1}$

(连杆扭角)

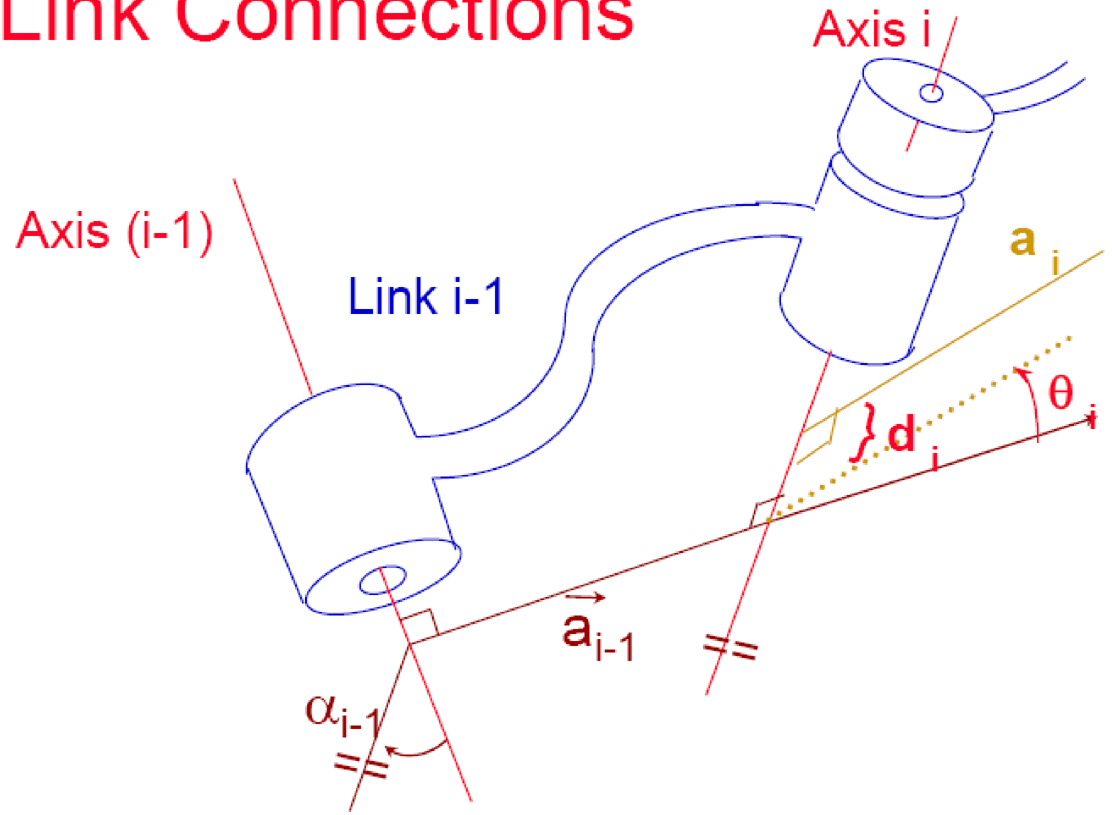
Intersecting Joint Axes



The sense of α_{i-1} is free

Link Description

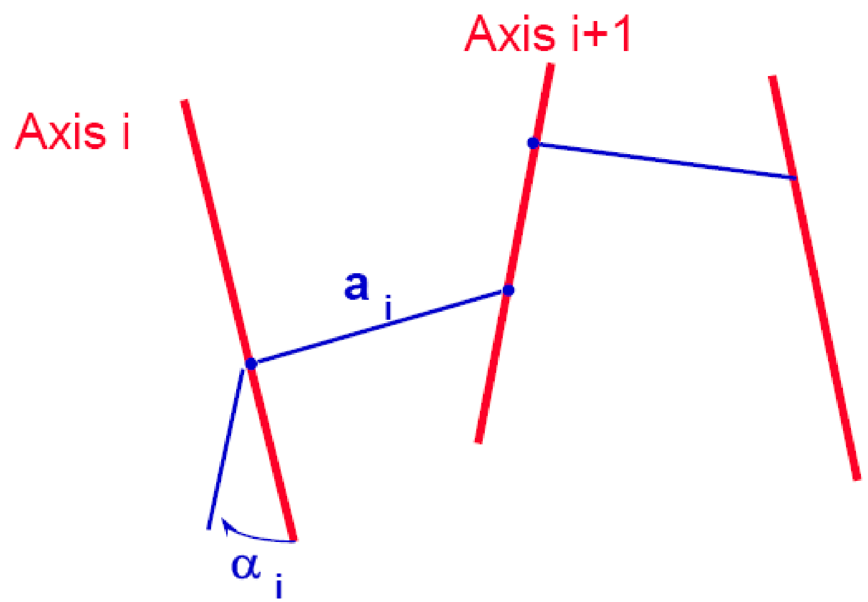
Link Connections



d_i: Link Offset -- variable if joint i is *prismatic* (平动关节)

θ_i : Joint Angle -- variable if joint i is *revolute* (转动关节)

First & Last Links



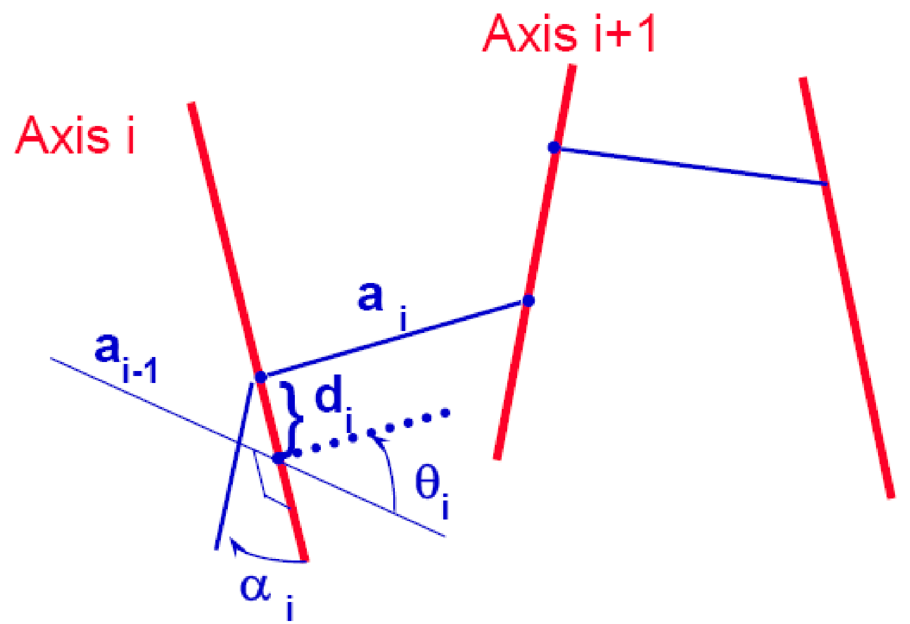
a_i and α_i depend on joint axes i and i+1

Axes 1 to n: determined

$\Rightarrow$ a_1 , a_2 … a_{n-1} and α_1, α_2 … α_{n-1}

Convention: $\boxed{a_0 = a_n = 0 \text{ and } \alpha_0 = \alpha_n = 0}$

First & Last Links



θ_i and d_i depend on links i-1 and i

$\Rightarrow$ $\theta_2, \theta_3, \ldots, \theta_{n-1}$ and $d_2, d_3, \ldots, d_{n-1}$

Convention: set the constant parameters to zero

Following joint type: d_1 or $\theta_1 = 0$ and d_n or $\theta_n = 0$

Denavit-Hartenberg Parameters

4 D-H parameters $(\alpha_i, a_i, d_i, \theta_i)$

3 fixed link parameters

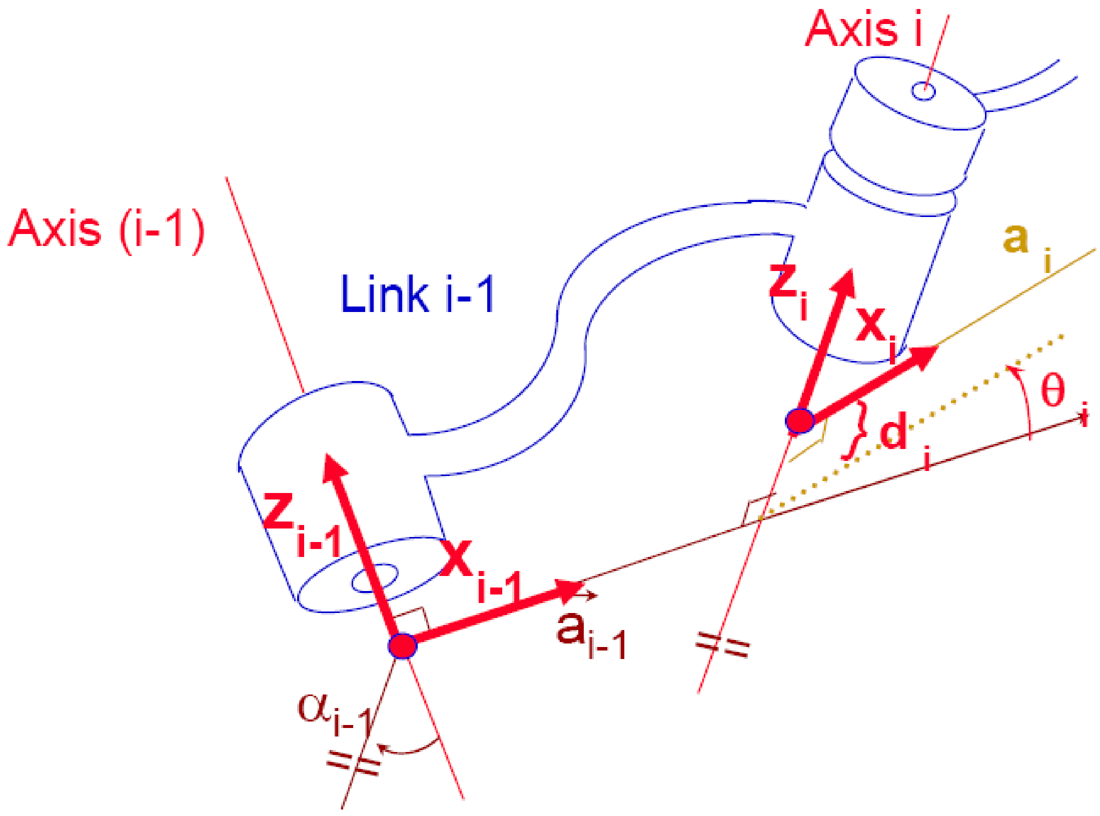
1 joint variable $\begin{cases} \theta_i & \text{revolute joint} \\ d_i & \text{prismatic joint} \end{cases}$

α_i and a_i : describe the Link i

d_i and θ_i : describe the Link's connection

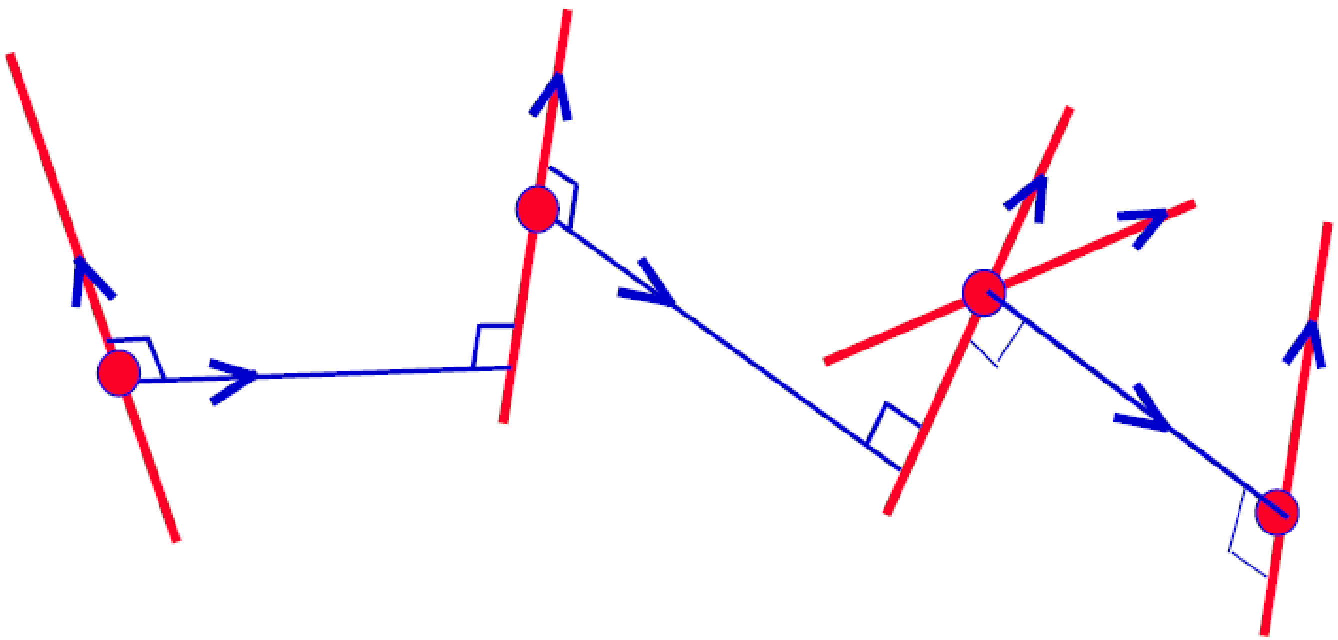
Contents

Frame Attachment



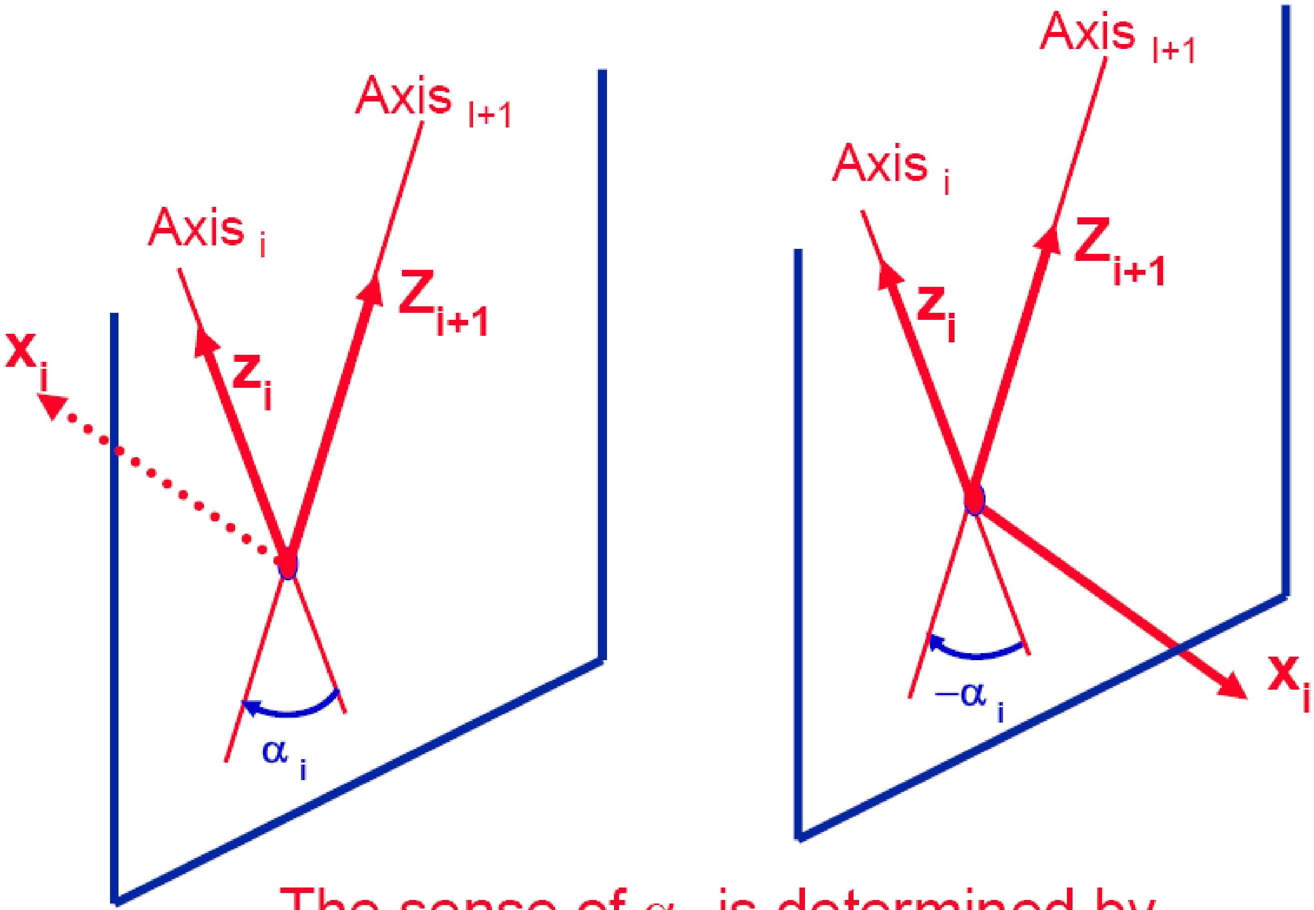
y-vectors: complete right-hand frames

Summary – Frame Attachment



1. Normals
2. Origins
3. Z-axes
4. X-axes

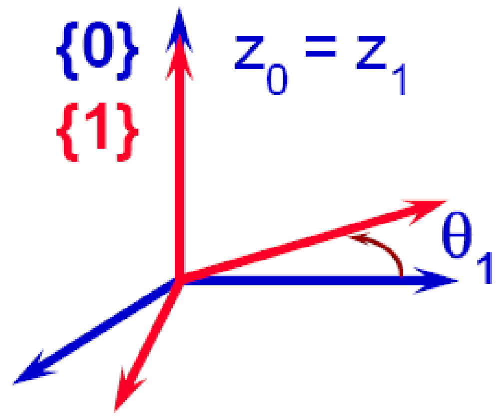
Intersecting Joint Axes



The sense of α_i is determined by the direction of x

First Link

Revolute



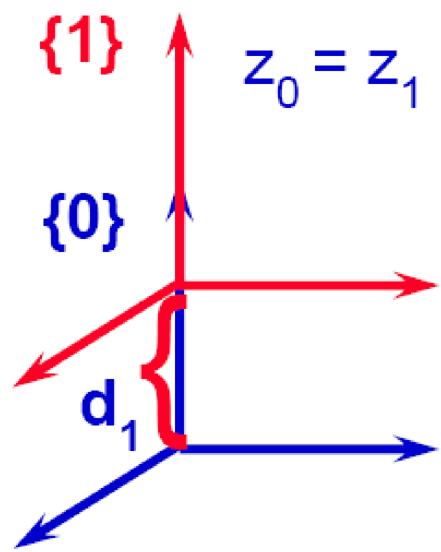
$a_0 = 0$

$\alpha_0 = 0$

$d_1 = 0$

$\theta_1 = 0 \longrightarrow \{0\} \equiv \{1\}$

Prismatic



$a_0 = 0$

$\alpha_0 = 0$

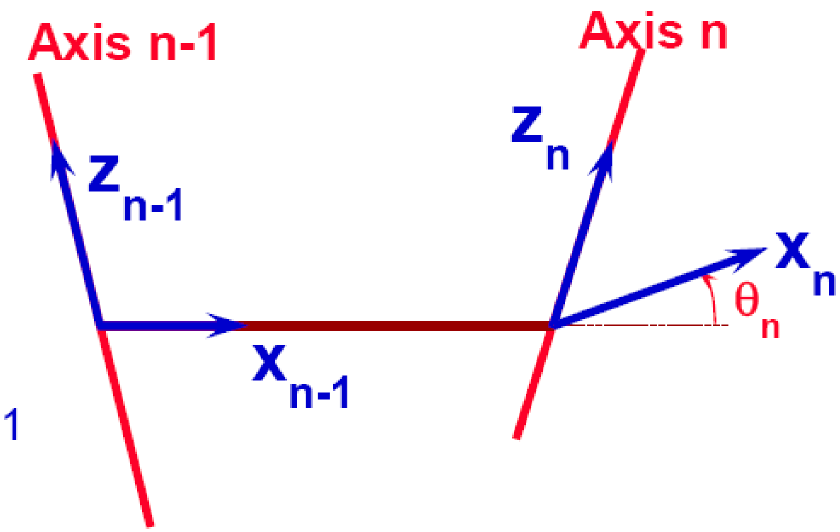
$\theta_1 = 0$

$d_1 = 0 \longrightarrow \{0\} \equiv \{1\}$

Last Link

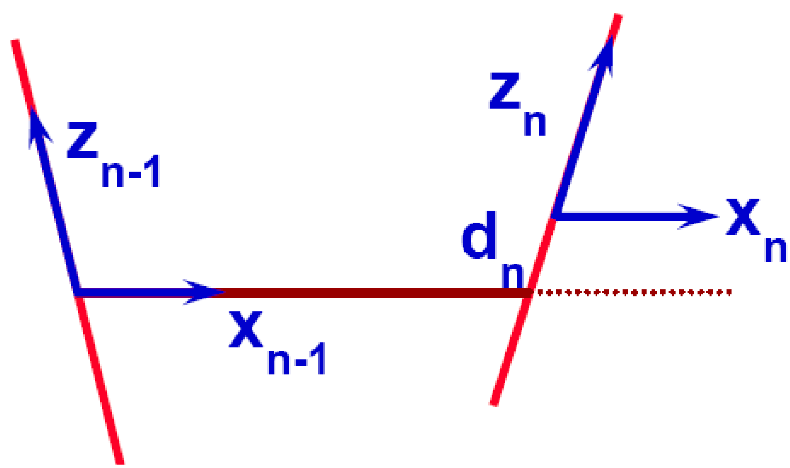
Revolute $\boxed{d_n = 0}$

$\theta_n = 0 \longrightarrow x_n = x_{n-1}$

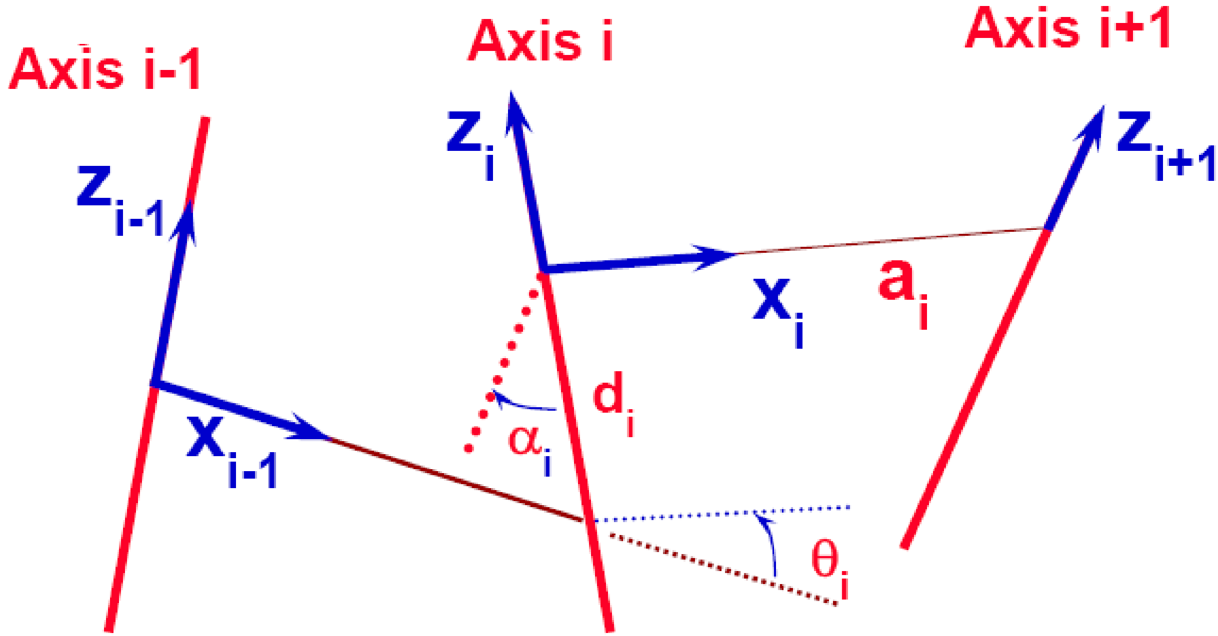


Prismatic $\boxed{\theta_n = 0}$

$d_n = 0 \longrightarrow x_n = x_{n-1}$



Summary of the link parameters



a_i : distance (z_i , z_{i+1}) along x_i

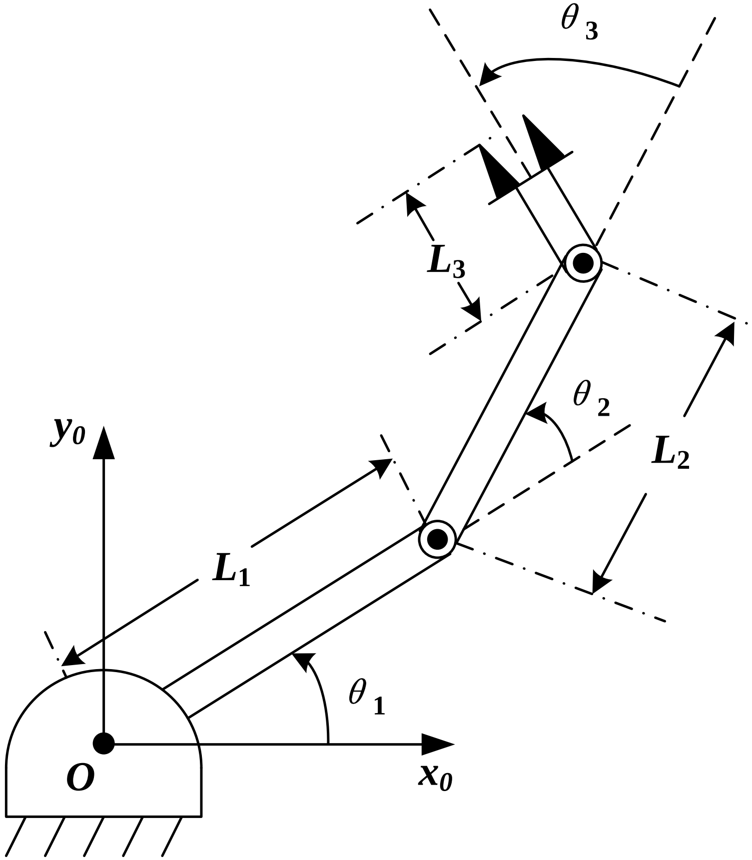
α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

θ_i : angle (x_{i-1} , x_i) about z_i

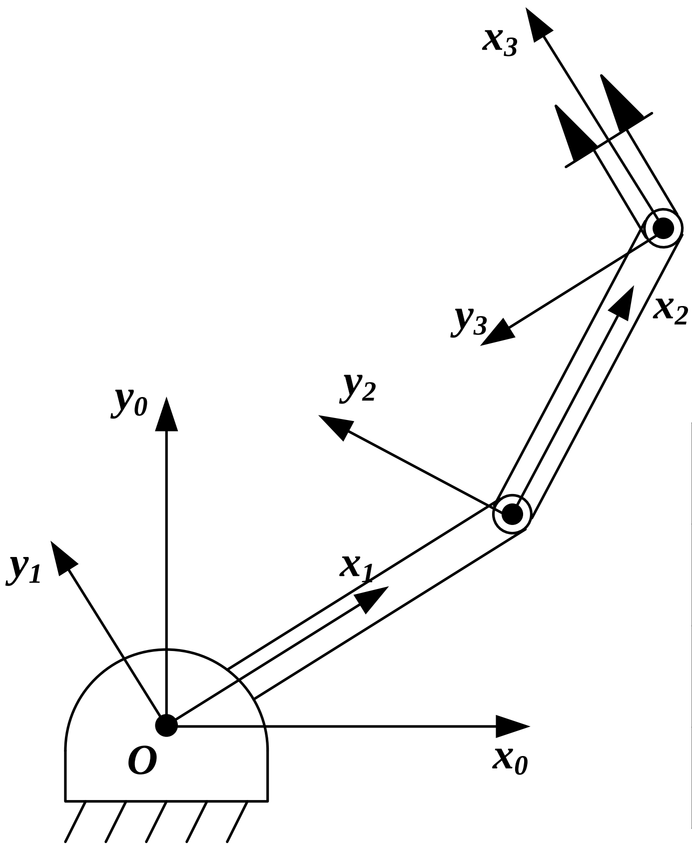
Denavit-Hartenberg Parameters

Example-RRR Arm



Denavit-Hartenberg Parameters

Example-RRR Arm



a_i = distance from z_i to z_{i+1} along x_i

α_i = angle from z_i to z_{i+1} about x_i

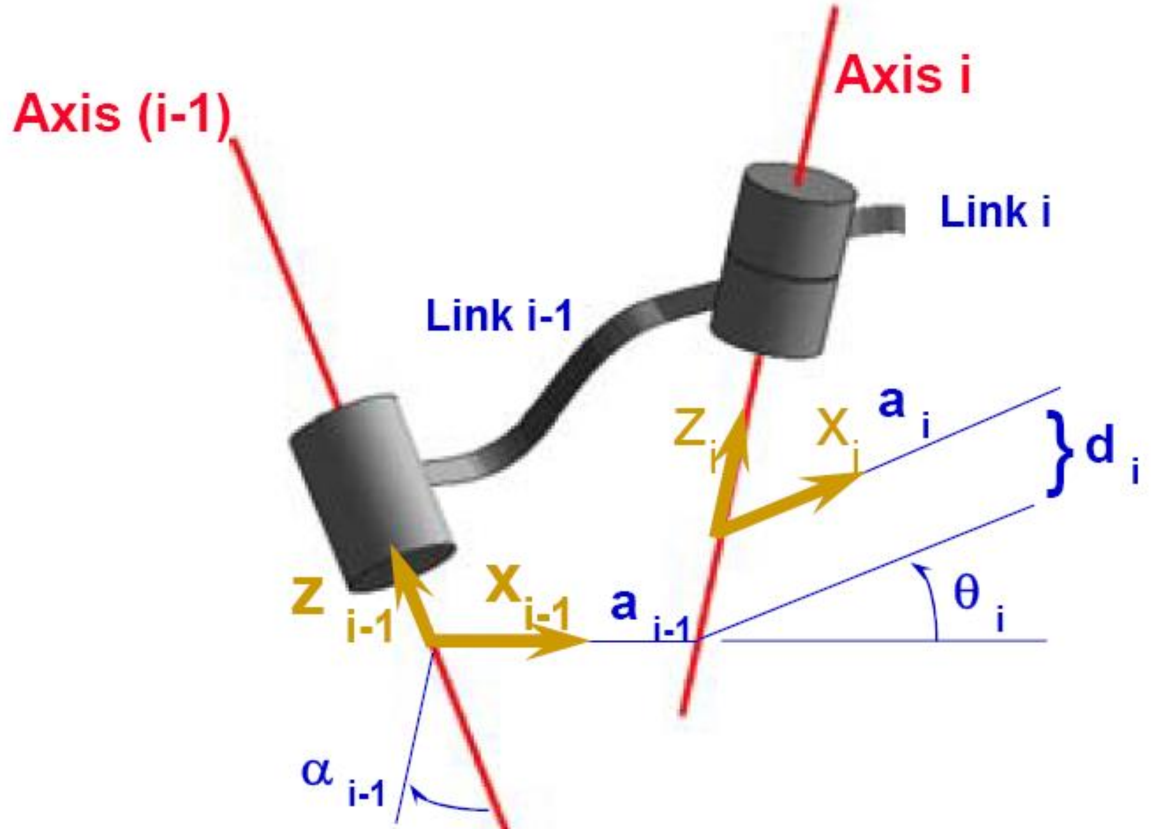
d_i = distance from x_{i-1} to x_i along z_i

θ_i = angle from x_{i-1} to x_i about z_i

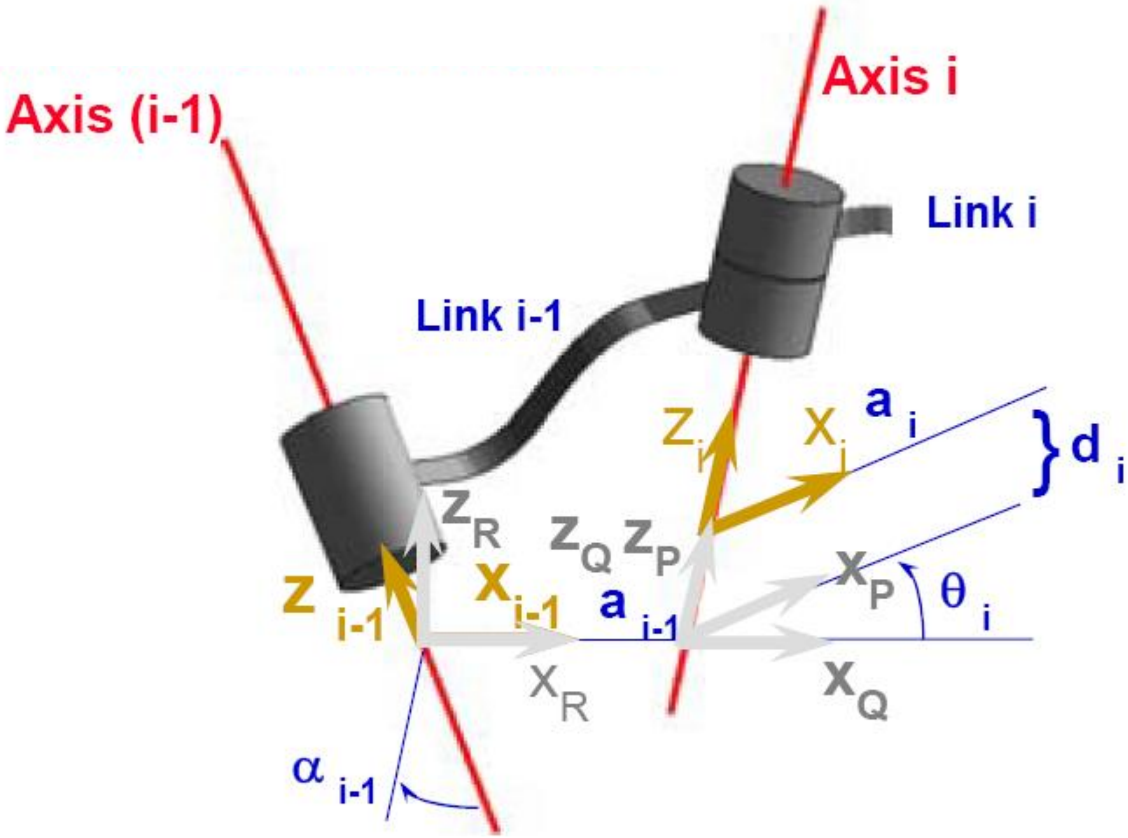
link	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

Contents

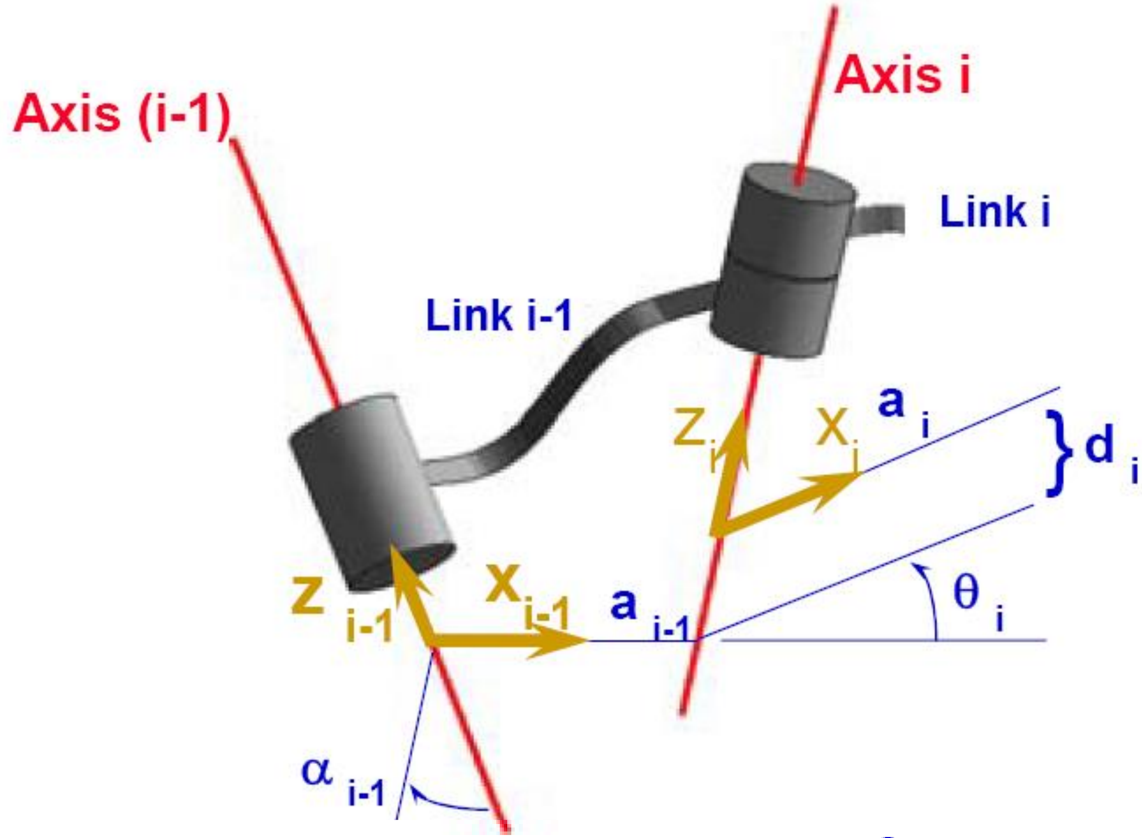
Forward Kinematics



Forward Kinematics

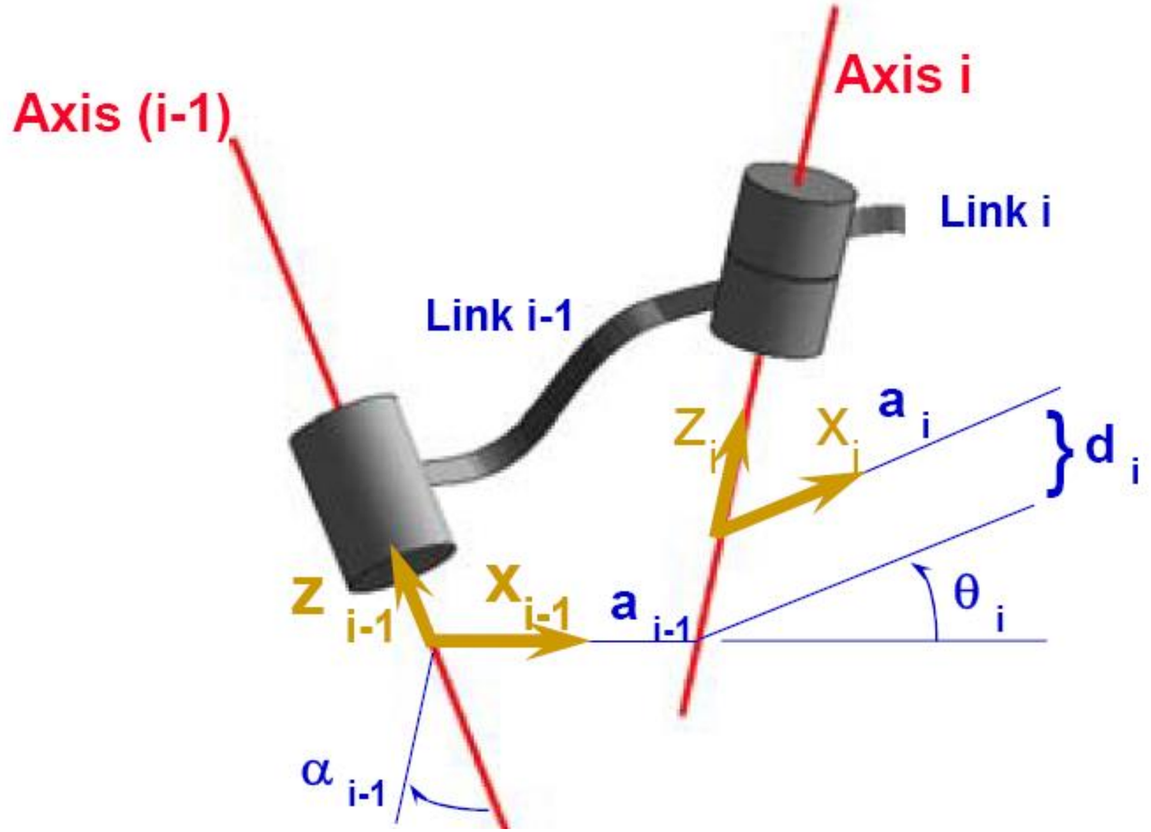


Forward Kinematics



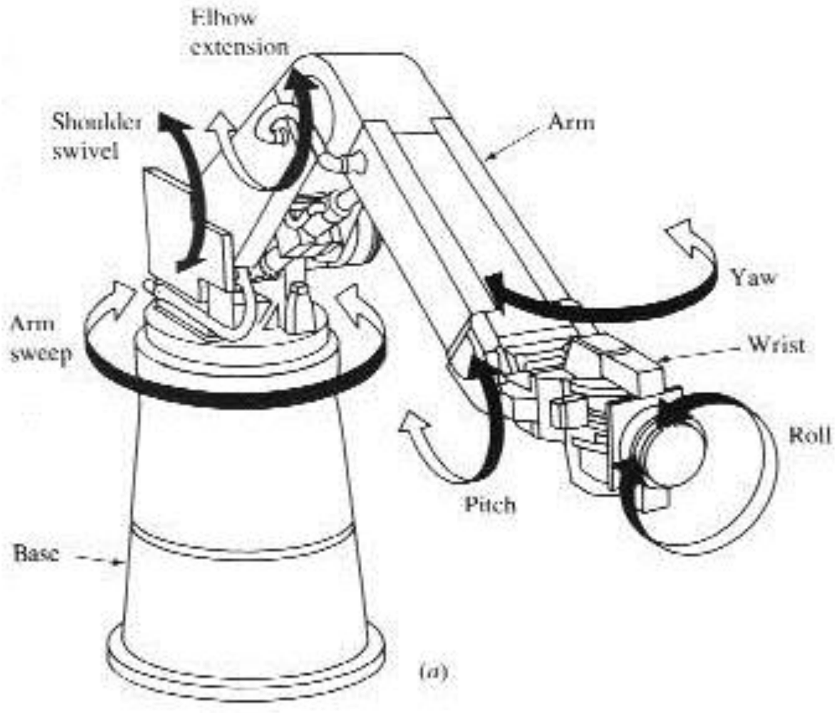
$$
{}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Forward Kinematics

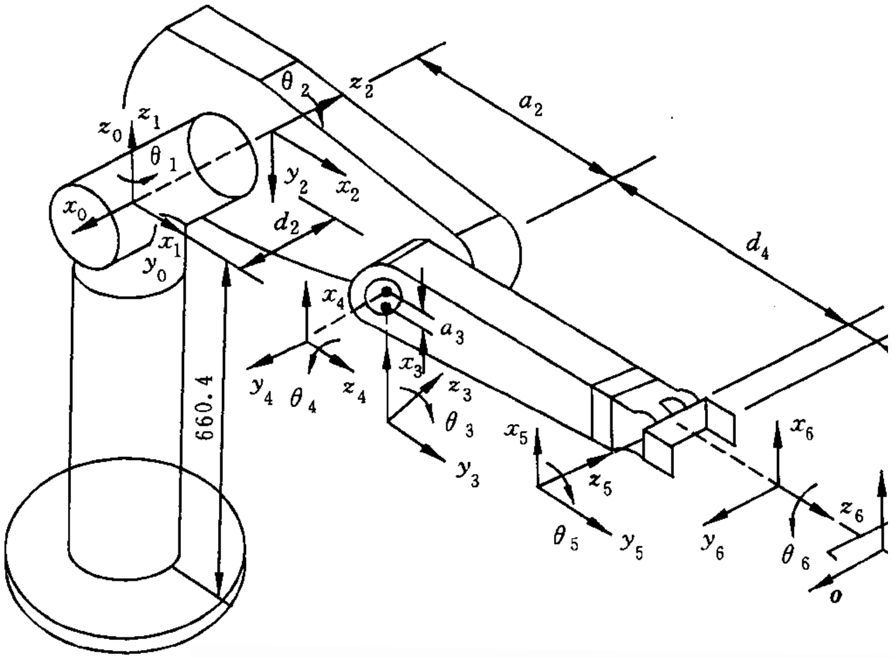


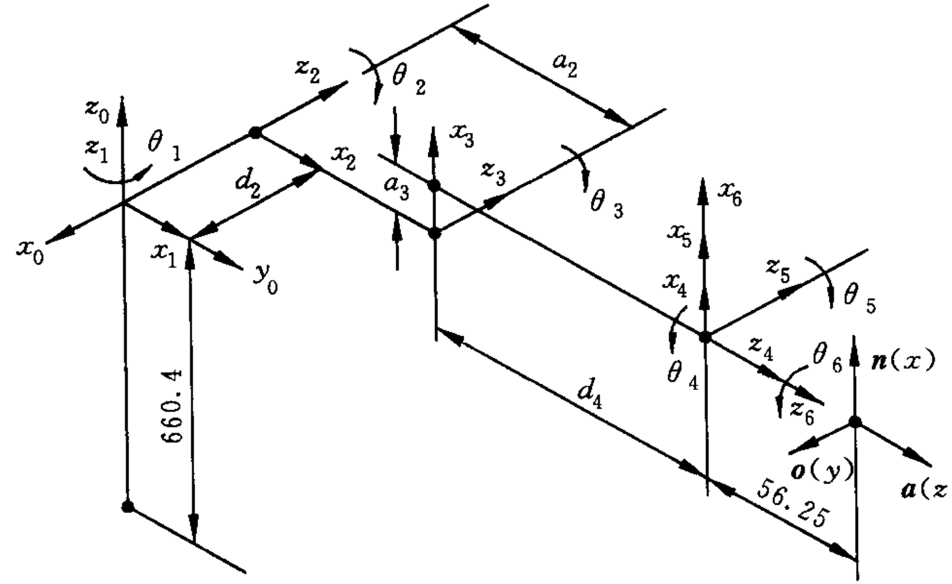
Forward Kinematics: ${}^{0}_{N}T = {}^{0}_{1}T \; {}^{1}_{2}T \; \ldots \; {}^{N-1}_{N}T$

Forward Kinematics of PUMA 560



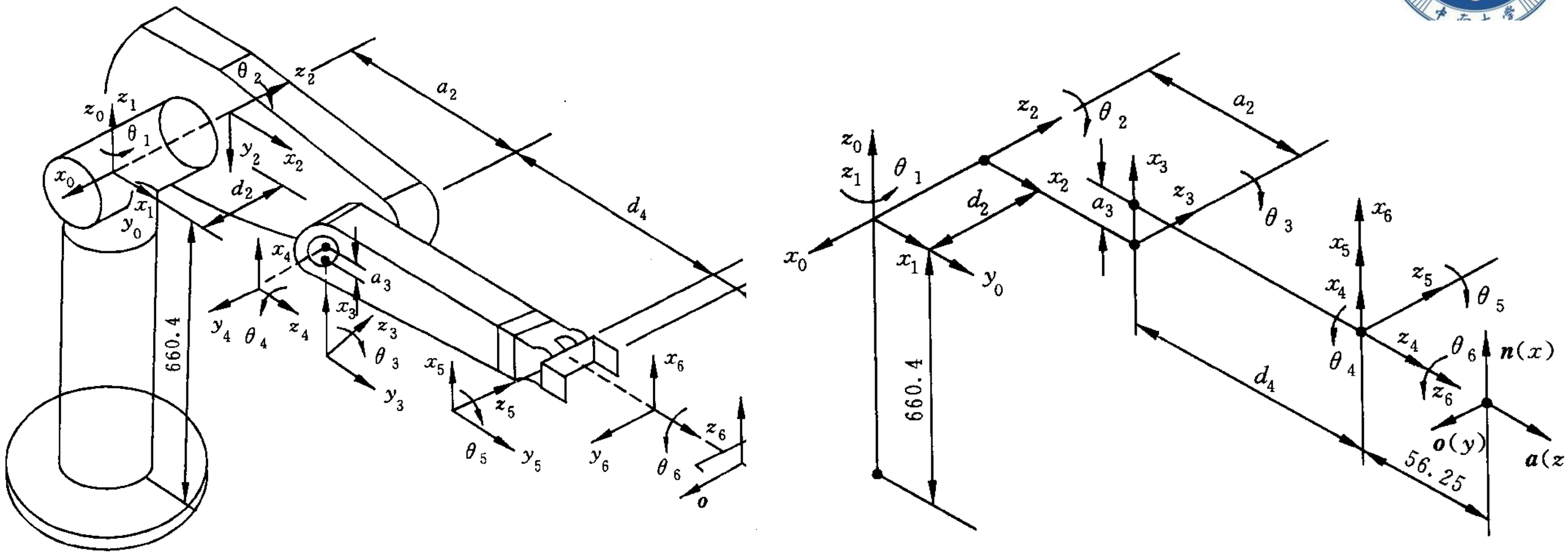
Forward Kinematics of PUMA 560





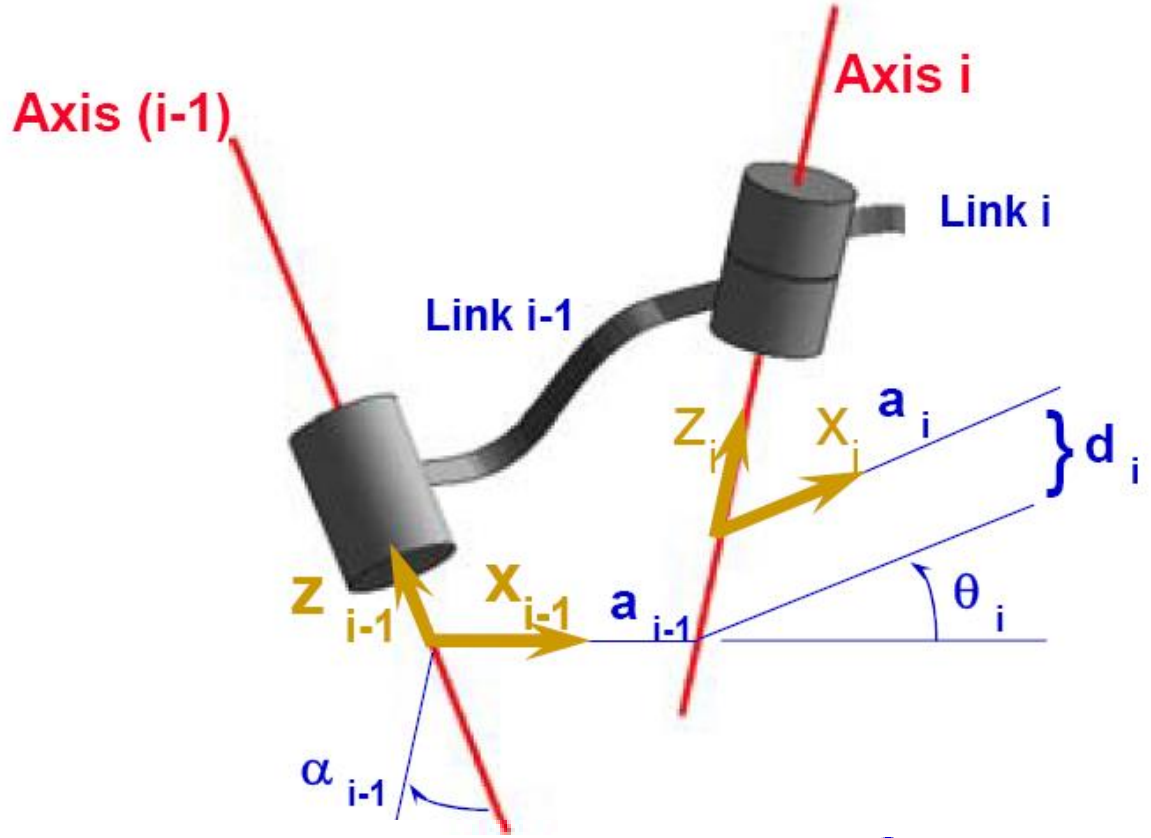
连杆 i	变量 θ_i	α_{i-1}	a_{i-1}	d_i	变量范围
1	$\theta_1(90^\circ)$	0°	0	0	$-160^\circ \sim 160^\circ$
2	$\theta_2(0^\circ)$	-90°	0	d_2	$-225^\circ \sim 45^\circ$
3	$\theta_3(-90^\circ)$	0°	a_2	0	$-45^\circ \sim 225^\circ$
4	$\theta_4(0^\circ)$	-90°	a_3	d_4	$-110^\circ \sim 170^\circ$
5	$\theta_5(0^\circ)$	90°	0	0	$-100^\circ \sim 100^\circ$
6	$\theta_6(0^\circ)$	-90°	0	0	$-266^\circ \sim 266^\circ$

Forward Kinematics of PUMA 560



$$^{0}T_{6}={}^{0}T_{1}(\theta_{1})\,{}^{1}T_{2}(\theta_{2})\,{}^{2}T_{3}(\theta_{3})\,{}^{3}T_{4}(\theta_{4})\,{}^{4}T_{5}(\theta_{5})\,{}^{5}T_{6}(\theta_{6})$$

Forward Kinematics



$$
{}^{i-1}_{\ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Forward Kinematics of PUMA 560

$$ {}^{0}T_{1} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}T_{2} = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}T_{3} = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}T_{4} = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{4}T_{5} = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{5}T_{6} = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

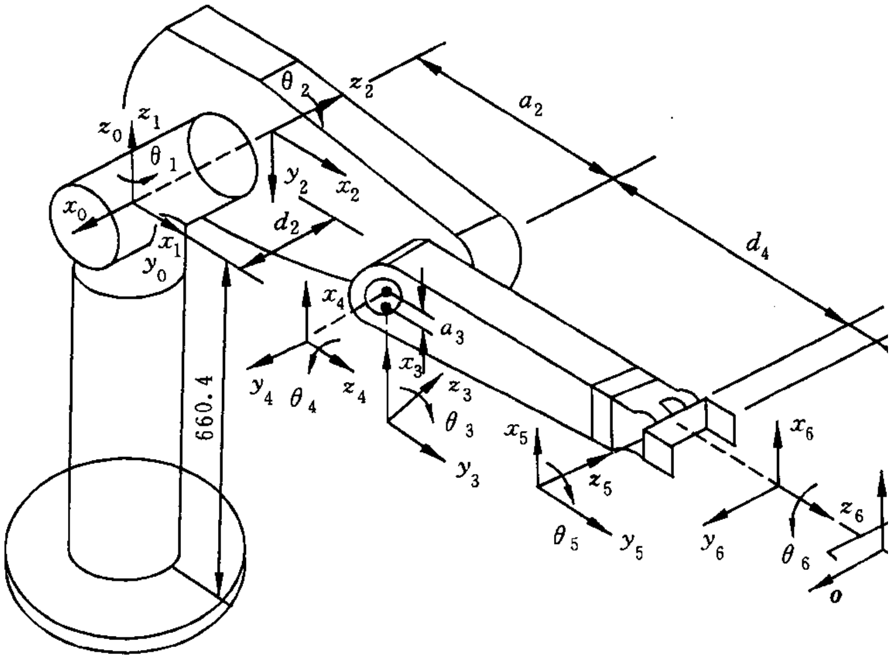
$$ {}^{0}T_{6} = {}^{0}T_{1}(\theta_1)\,{}^{1}T_{2}(\theta_2)\,{}^{2}T_{3}(\theta_3)\,{}^{3}T_{4}(\theta_4)\,{}^{4}T_{5}(\theta_5)\,{}^{5}T_{6}(\theta_6) $$

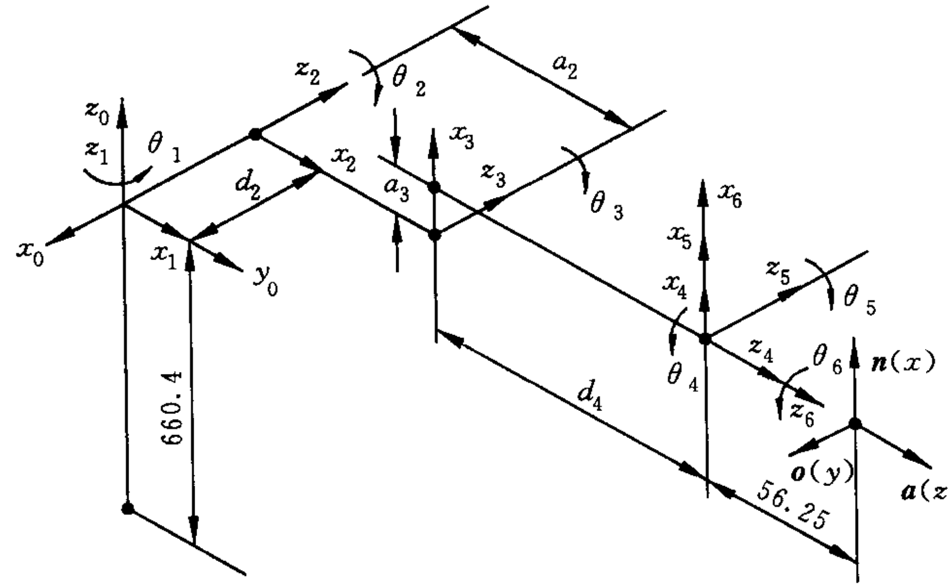
Forward Kinematics of PUMA 560

$$
{}^{0}T_{6} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
\left.
\begin{aligned}
n_x &= c_1\left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] + s_1(s_4c_5c_6 + c_4s_6),\\
n_y &= s_1\left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] - c_1(s_4c_5c_6 + c_4s_6),\\
n_z &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6;\\
o_x &= c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6),\\
o_y &= s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5c_6),\\
o_z &= -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6,\\
a_x &= -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5,\\
a_y &= -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5,\\
a_z &= s_{23}c_4s_5 - c_{23}c_5;\\
p_x &= c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_2s_1,\\
p_y &= s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_2c_1\\
p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
\right\}
$$

Contents

Definition of Inverse Kinematics





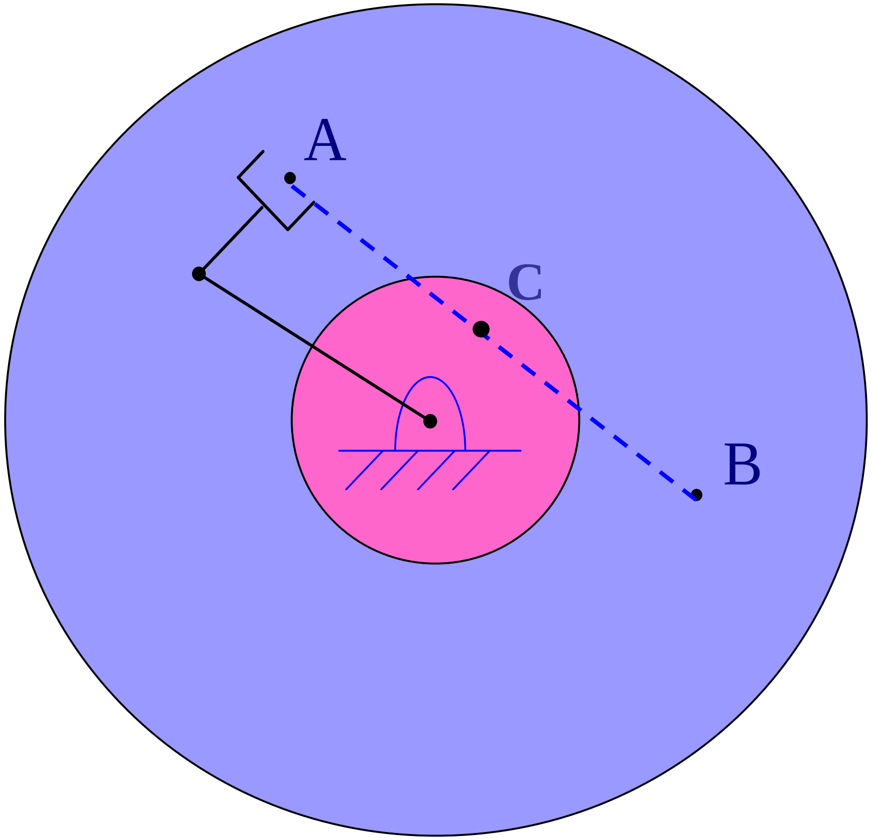
$$ {}^{0}T_6 = {}^{0}T_1(\theta_1)\,{}^{1}T_2(\theta_2)\,{}^{2}T_3(\theta_3)\,{}^{3}T_4(\theta_4)\,{}^{4}T_5(\theta_5)\,{}^{5}T_6(\theta_6) $$

Given ${}^{0}_{N}T$ find $\theta_1, \theta_2, \ldots, \theta_n$

Existence of Solutions

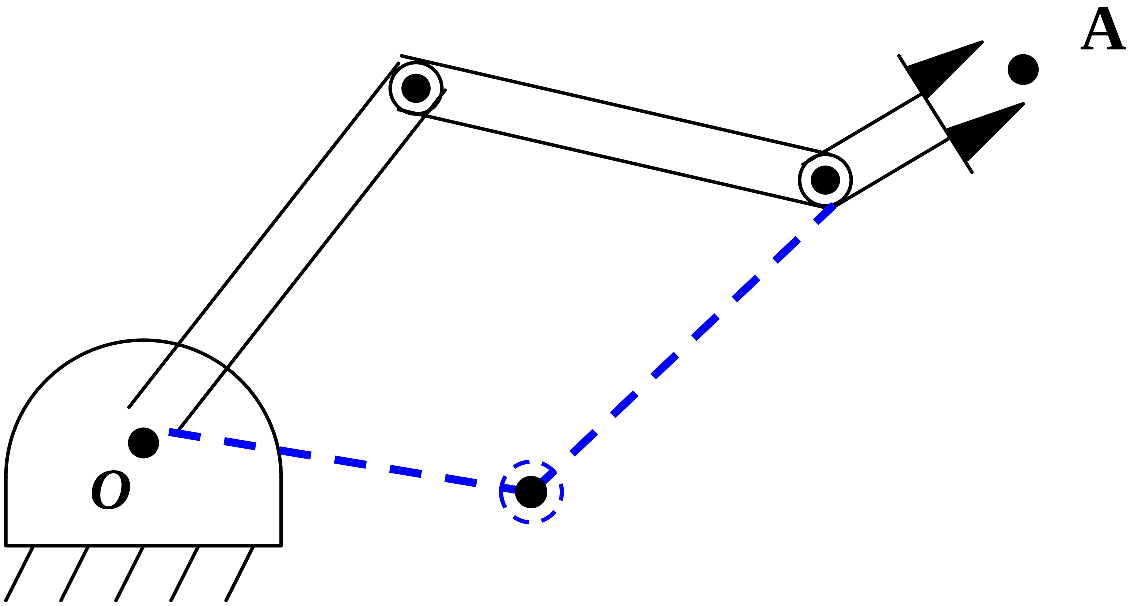
- Reachable workspace of a two- link manipulator.

For a solution to exist, the specified goal point must lie within the workspace.



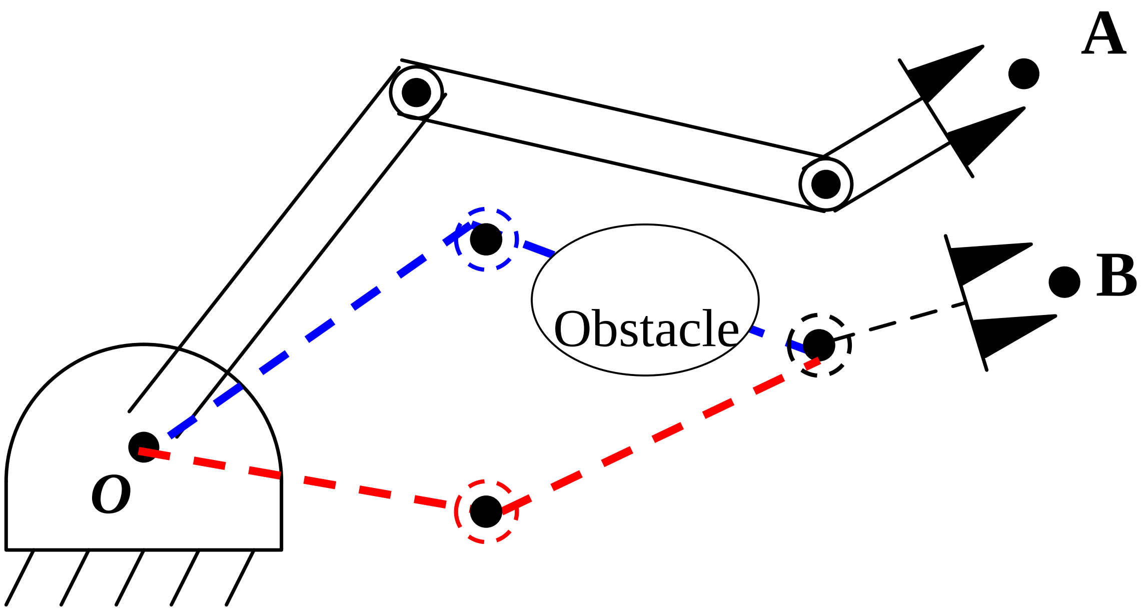
Multiple Solutions

- Three-link manipulator have multiple solutions.



Multiple Solutions

- Choose the *closest possible* solution.

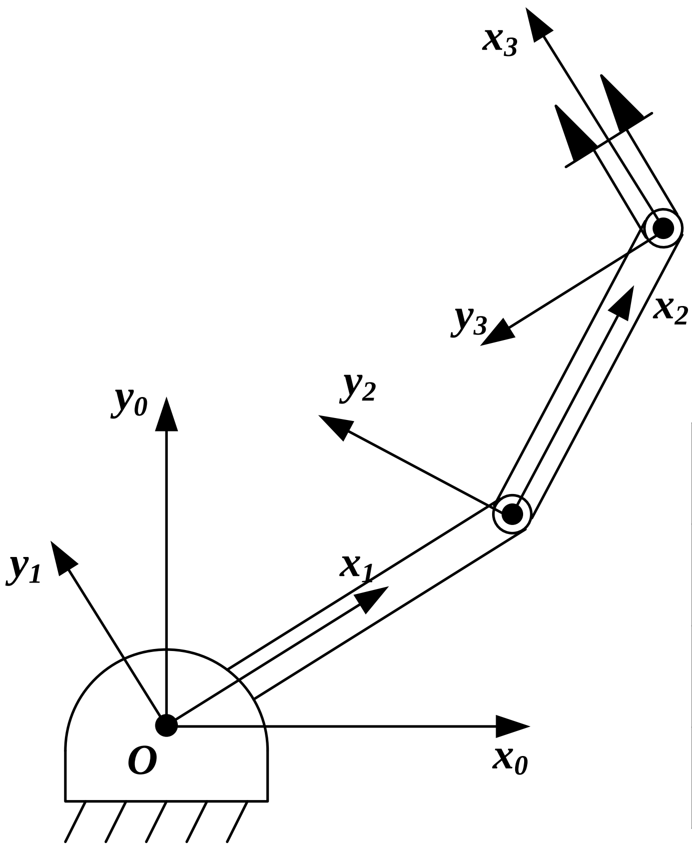


Method of Solution

- Closed form solutions.
  - algebraic
  - geometric
- Numerical solutions.
  - much slower due to iterative nature.
  - can’t guarantee to find all possible solutions.

Algebraic Solution

Example-RRR Arm



a_i = distance from z_i to z_{i+1} along x_i

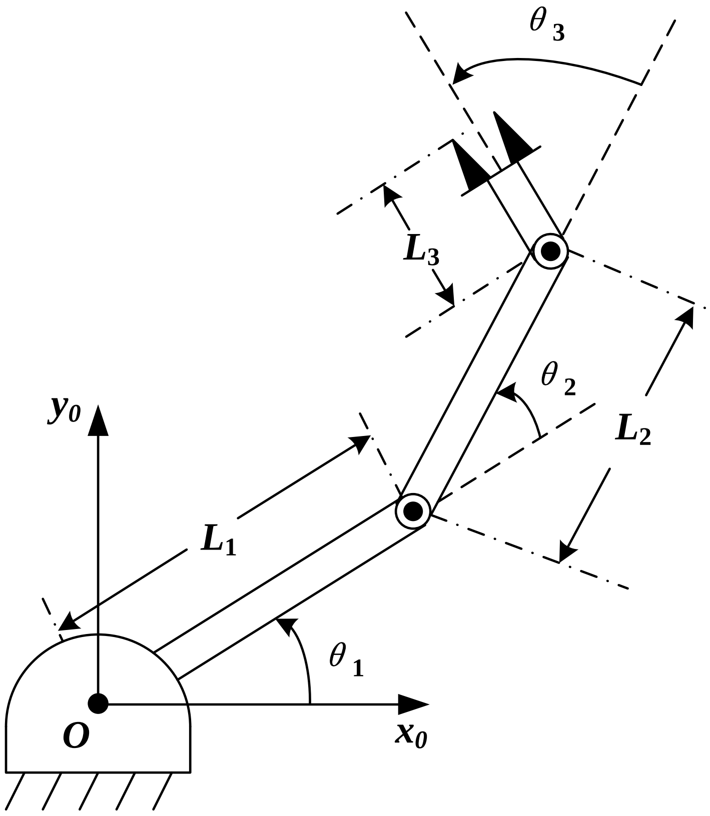
α_i = angle from z_i to z_{i+1} about x_i

d_i = distance from x_{i-1} to x_i along z_i

θ_i = angle from x_{i-1} to x_i about z_i

link	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

Algebraic Solution



$$
{}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{B}_{W}T = \begin{bmatrix} c_{\phi} & -s_{\phi} & 0 & x \\ s_{\phi} & c_{\phi} & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Algebraic Solution

$$\begin{bmatrix} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{B}_{W}T = \begin{bmatrix} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x = l_1c_1 + l_2c_{12} \\ y = l_1s_1 + l_2s_{12} \end{cases}$$

$$\Rightarrow \; x^2 + y^2 = l_1^2 + l_2^2 + 2l_1l_2c_2$$

$$\Rightarrow \; c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$

Algebraic Solution

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\Rightarrow \quad s_2 = \pm\sqrt{1 - c_2^2}$$

$$\Rightarrow \quad \theta_2 = \text{Atan2}\left(s_2, c_2\right)$$ (two-argument arctangent routine)

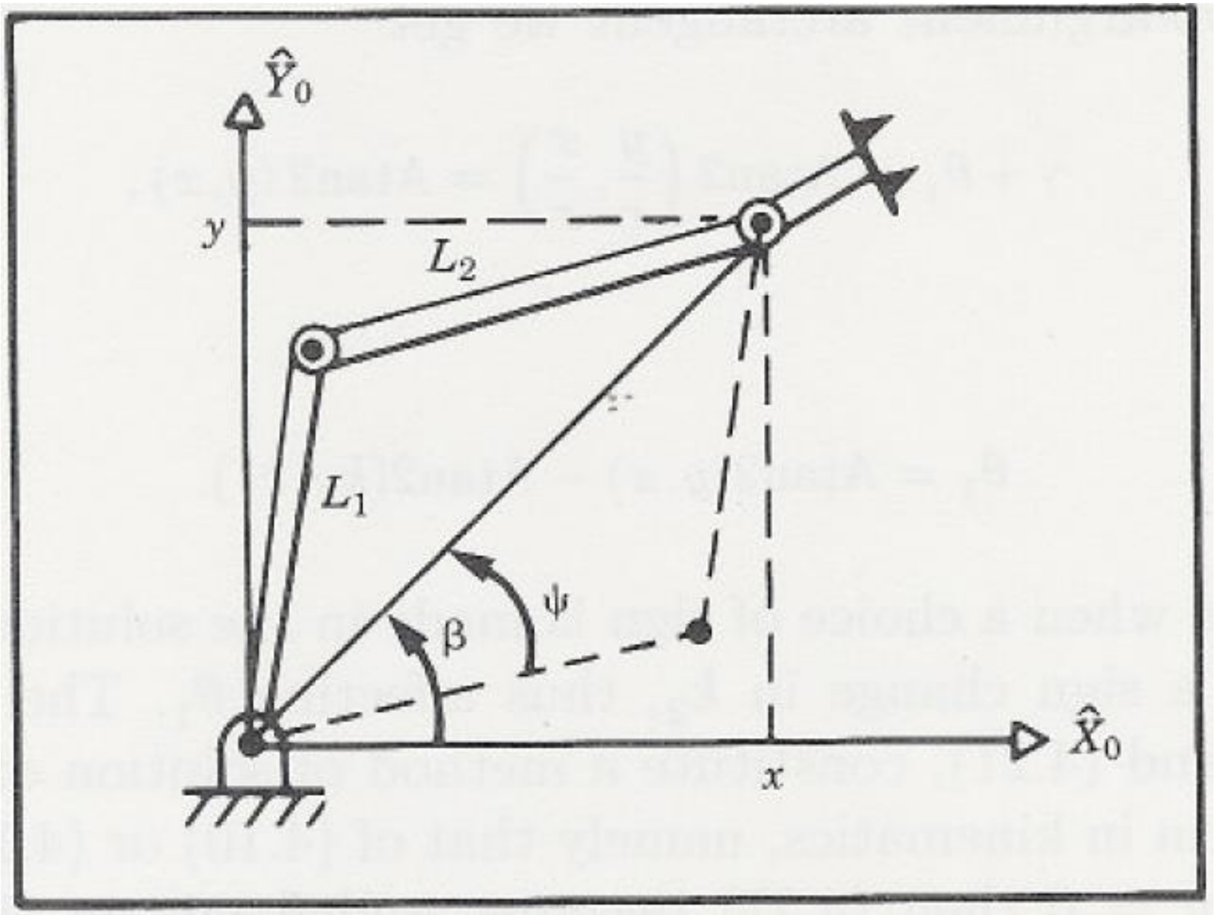
$$\theta_1 = \text{Atan2}\left(y, x\right) - \text{Atan2}\left(k_2, k_1\right)$$

where $k_1 = l_1 + l_2 c_2$

$k_2 = l_2 s_2$

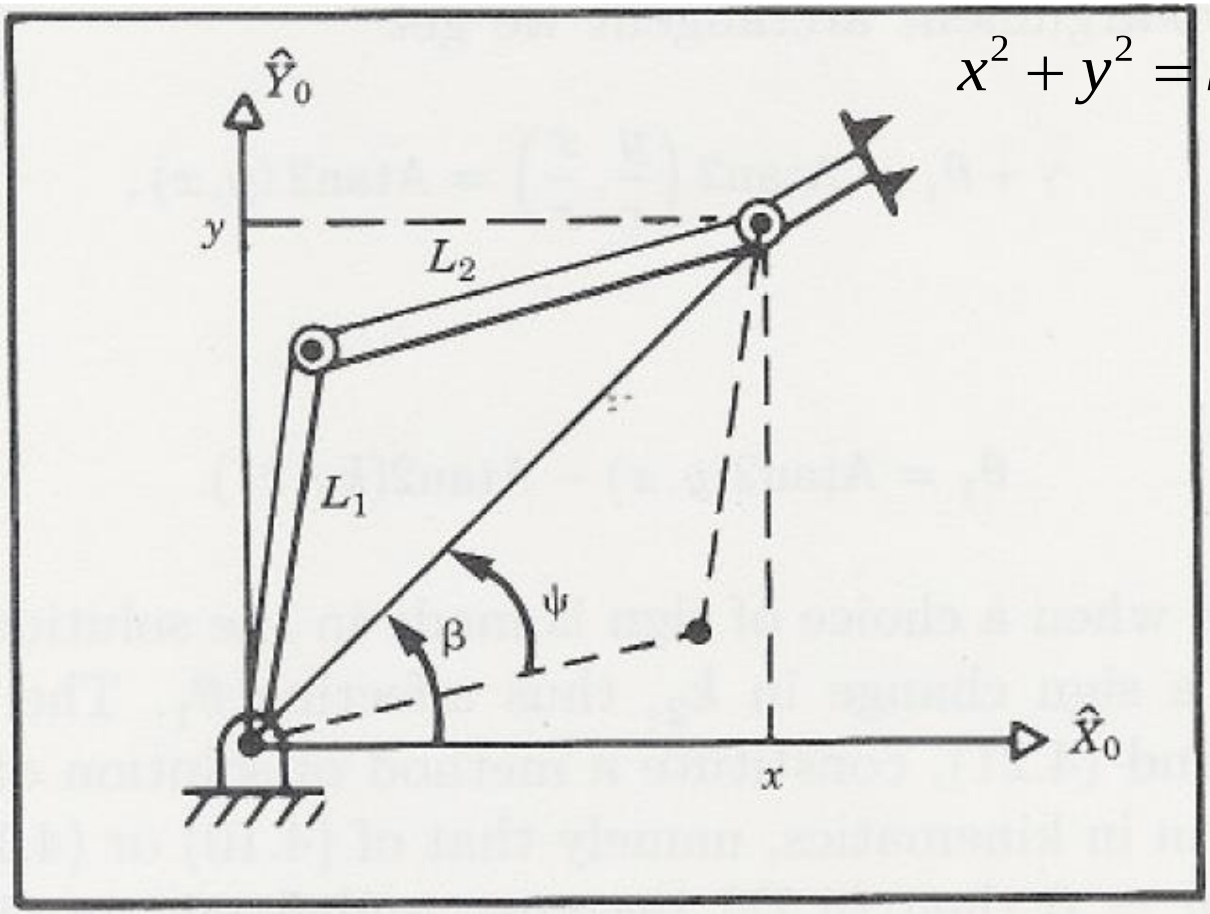
finally $\theta_1 + \theta_2 + \theta_3 = \text{Atan2}\left(s_\phi, c_\phi\right) = \phi$

Geometric Solution



Geometric Solution

apply the "law of cosines" to solve for θ_2 :



$$x^2 + y^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos(180 + \theta_2)$$

$$=> \quad c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

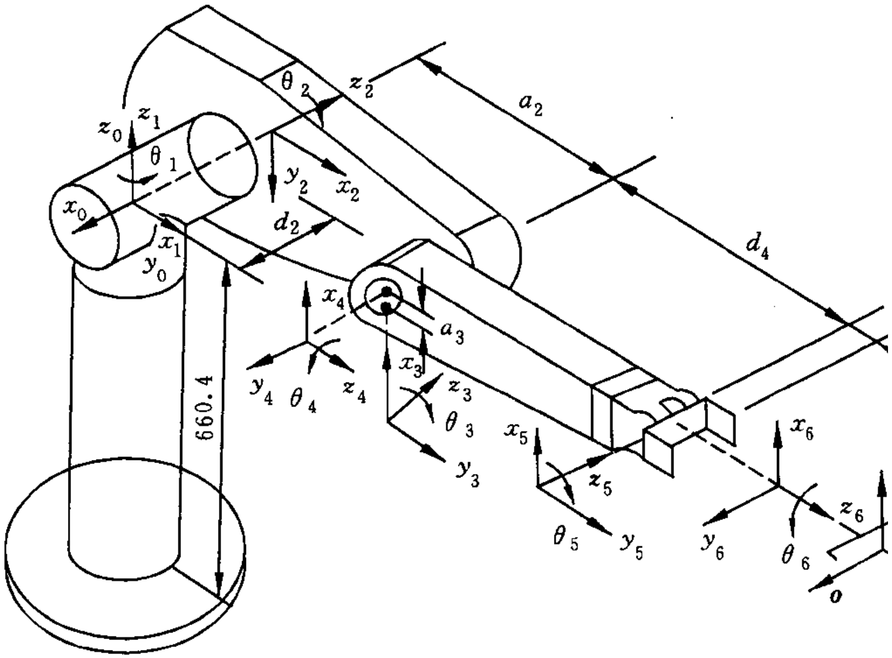
$$\beta = \mathrm{Atan2}(y, x)$$

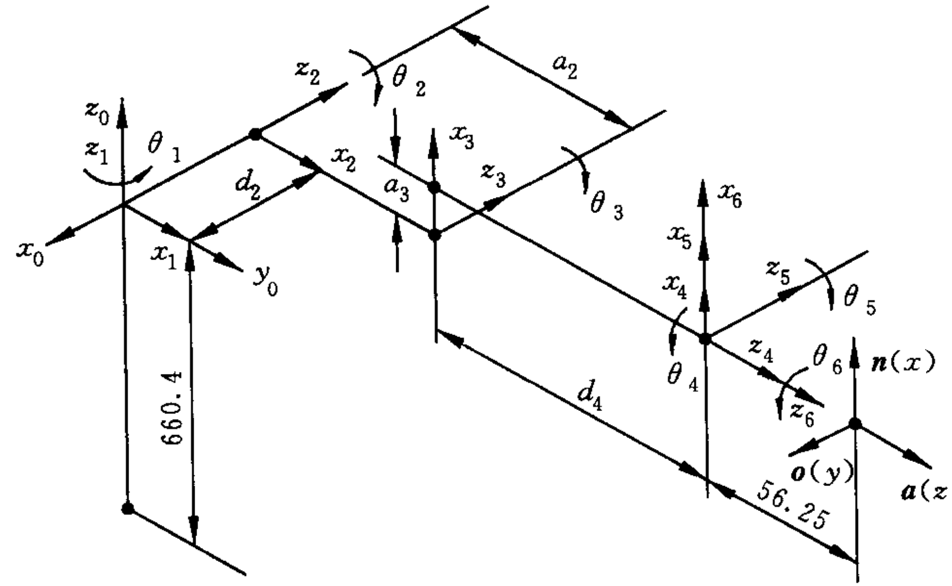
$$\cos\psi = \frac{x^2 + y^2 + l_1^2 - l_2^2}{2l_1\sqrt{x^2 + y^2}}$$

$$=> \quad \theta_1 = \beta \pm \psi$$

$$\theta_1 + \theta_2 + \theta_3 = \phi$$

Inverse Kinematics of PUMA 560





$$^{0}T_6 = {}^{0}T_1(\theta_1)\,{}^{1}T_2(\theta_2)\,{}^{2}T_3(\theta_3)\,{}^{3}T_4(\theta_4)\,{}^{4}T_5(\theta_5)\,{}^{5}T_6(\theta_6)$$

Given $^{0}_{6}T$ find $\theta_1, \theta_2, \ldots, \theta_6$

Inverse Kinematics of PUMA 560

$$
{}^0T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0T_1(\theta_1)\,{}^1T_2(\theta_2)\,{}^2T_3(\theta_3)\,{}^3T_4(\theta_4)\,{}^4T_5(\theta_5)\,{}^5T_6(\theta_6)
$$

- 1.solve θ_1

Left multiply ${}^0T_1^{-1}(\theta_1)$ on both sides

$$
{}^0T_1^{-1}(\theta_1)\,{}^0T_6 = {}^1T_2(\theta_2)\,{}^2T_3(\theta_3)\,{}^3T_4(\theta_4)\,{}^4T_5(\theta_5)\,{}^5T_6(\theta_6)
$$

$$
\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1T_6 = \begin{bmatrix} {}^1n_x & {}^1o_x & {}^1a_x & {}^1p_x \\ {}^1n_y & {}^1o_y & {}^1a_y & {}^1p_y \\ {}^1n_z & {}^1o_z & {}^1a_z & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Inverse Kinematics of PUMA 560

- 1.solve θ_1

$$ {}^0T_1^{-1}(\theta_1)\,{}^0T_6 = {}^1T_2(\theta_2)\,{}^2T_3(\theta_3)\,{}^3T_4(\theta_4)\,{}^4T_5(\theta_5)\,{}^5T_6(\theta_6) $$

$$ \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ \boxed{-s_1 \quad c_1 \quad 0 \quad 0} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1T_6 = \begin{bmatrix} {}^1n_x & {}^1o_x & {}^1a_x & {}^1p_x \\ {}^1n_y & {}^1o_y & {}^1a_y & \boxed{{}^1p_y} \\ {}^1n_z & {}^1o_z & {}^1a_z & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Equating (2,4) elements from both sides

$$ => \qquad -s_1 p_x + c_1 p_y = {}^1p_y = d_2 $$

Using trigonometric substitution:

$$ p_x = \rho\cos\phi; \qquad p_y = \rho\sin\phi $$

where

$$ \rho = \sqrt{p_x^2 + p_y^2}; \quad \phi = \mathrm{atan2}\left(p_y, p_x\right) $$

Inverse Kinematics of PUMA 560

$$-s_1 p_x + c_1 p_y = {}^1p_y = d_2$$

- solve θ_1

$$p_x = \rho\cos\phi; \quad p_y = \rho\sin\phi$$

$$\rho = \sqrt{p_x^2 + p_y^2}; \quad \phi = \text{atan2}\left(p_y, p_x\right)$$

$$=> \begin{cases} \sin(\phi-\theta_1) = d_2/\rho; \quad \cos(\phi-\theta_1) = \pm\sqrt{1-(d_2/\rho)^2} \\ \phi-\theta_1 = \text{atan2}\left[\dfrac{d_2}{\rho}, \pm\sqrt{1-\left(\dfrac{d_2}{\rho}\right)^2}\right] \\ \theta_1 = \text{atan2}(p_y, p_x) - \text{atan2}(d_2, \pm\sqrt{p_x^2+p_y^2-d_2^2}) \end{cases}$$

Where plus-or-minus sign leads to two possible solutions of θ_1

Inverse Kinematics of PUMA 560

- 2. solve θ_3

$$\begin{bmatrix} \boxed{c_1 \quad s_1 \quad 0 \quad 0} \\ -s_1 \quad c_1 \quad 0 \quad 0 \\ \boxed{0 \quad 0 \quad 1 \quad 0} \\ 0 \quad 0 \quad 0 \quad 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1T_6 = \begin{bmatrix} {}^1n_x & {}^1o_x & {}^1a_x & \boxed{{}^1p_x} \\ {}^1n_y & {}^1o_y & {}^1a_y & {}^1p_y \\ {}^1n_z & {}^1o_z & {}^1a_z & \boxed{{}^1p_z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Equating (1,4) and (3,4) elements from both sides

$$=> \quad \begin{cases} c_1 p_x + s_1 p_y = {}^1p_x = a_3 c_{23} - d_4 s_{23} + a_2 c_2 \\ p_z = {}^1p_z = -a_3 s_{23} - d_4 c_{23} - a_2 s_2 \end{cases}$$

$$=> \quad a_3 c_3 - d_4 s_3 = k$$

where

$$k = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_2^2 - d_4^2}{2a_2}$$

Inverse Kinematics of PUMA 560

- solve θ_1

$$-s_1 p_x + c_1 p_y = {}^1p_y = d_2$$

$$=> \theta_1 = \mathrm{atan2}(p_y, p_x) - \mathrm{atan2}(d_2, \pm\sqrt{p_x^2 + p_y^2 - d_2^2})$$

- solve θ_3

$$a_3 c_3 - d_4 s_3 = k$$

$$\theta_3 = \mathrm{atan2}(a_3, d_4) - \mathrm{atan2}(k, \pm\sqrt{a_3^2 + d_4^2 - k^2})$$

where

$$k = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_2^2 - d_4^2}{2a_2}$$

plus-or-minus sign leads to two possible solutions of θ_3

Inverse Kinematics of PUMA 560

- 3.solve θ_2

$${}^{0}T_3^{-1}(\theta_1,\theta_2,\theta_3)\,{}^{0}T_6 = {}^{3}T_4(\theta_4)\,{}^{4}T_5(\theta_5)\,{}^{5}T_6(\theta_6)$$

$$\begin{bmatrix} \boxed{c_1c_{23} \quad s_1c_{23} \quad -s_{23} \quad -a_2c_3} \\ \boxed{-c_1s_{23} \quad -s_1s_{23} \quad -c_{23} \quad a_2s_3} \\ -s_1 \quad c_1 \quad 0 \quad -d_2 \\ 0 \quad 0 \quad 0 \quad 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{3}T_6$$

Equating (1,4) and (2,4) elements from both sides

$$=> \begin{cases} c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 = a_3 \\ -c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 = d_4 \end{cases}$$

Inverse Kinematics of PUMA 560

$$\Rightarrow \begin{cases} s_{23} = \dfrac{\left(-a_3 - a_2 c_3\right) p_z + \left(c_1 p_x + s_1 p_y\right)\left(a_2 s_3 - d_4\right)}{p_z^2 + \left(c_1 p_x + s_1 p_y\right)^2} \\ c_{23} = \dfrac{\left(-d_4 + a_2 c_3\right) p_z - \left(c_1 p_x + s_1 p_y\right)\left(-a_2 c_3 - a_3\right)}{p_z^2 + \left(c_1 p_x + s_1 p_y\right)^2} \end{cases}$$

$$\Rightarrow \theta_{23} = \theta_2 + \theta_3 = \operatorname{atan2}\Big[\left(-a_3 - a_2 c_3\right) p_z + \left(c_1 p_x + s_1 p_y\right)\left(a_2 s_3 - d_4\right), \left(-d_4 + a_2 c_3\right) p_z - \left(c_1 p_x + s_1 p_y\right)\left(-a_2 c_3 - a_3\right)\Big]$$

- solve θ_2

  There are four values of θ_{23} according to the four possible combinations of solutions for θ_1 and θ_3 .

  Then, four possible solutions for θ_2 are computed as

$$\theta_2 = \theta_{23} - \theta_3$$

Inverse Kinematics of PUMA 560

- 4.solve θ_4

$$ {}^{0}T_3^{-1}(\theta_1,\theta_2,\theta_3)\,{}^{0}T_6 = {}^{3}T_4(\theta_4)\,{}^{4}T_5(\theta_5)\,{}^{5}T_6(\theta_6) $$

$$ \begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{3}T_6 $$

Equating (1,3) and (2,3) elements from both sides

$$ \Rightarrow \begin{cases} c_1c_{23}a_x + s_1c_{23}a_y - s_{23}a_z = -c_4s_5 \\ -s_1a_x + c_1a_y = s_4s_5 \end{cases} $$

$$ \Rightarrow \theta_4 = \operatorname{atan2}(-a_xs_1 + a_yc_1, -a_xc_1c_{23} - a_ys_1c_{23} + a_zs_{23}) $$

Inverse Kinematics of PUMA 560

- 5.solve θ_5

$$ {}^0T_4^{-1}(\theta_1,\theta_2,\theta_3,\theta_4)\,{}^0T_6 = {}^4T_5(\theta_5)\,{}^5T_6(\theta_6) $$

$$ {}^0T_4^{-1}(\theta_1,\theta_2,\theta_3,\theta_4) = \begin{bmatrix} c_1c_{23}c_4+s_1s_4 & s_1c_{23}c_4-c_1s_4 & -s_{23}c_4 & -a_2c_3c_4+d_2s_4-a_3c_4 \\ -c_1c_{23}s_4+s_1c_4 & -s_1c_{23}s_4-c_1c_4 & s_{23}s_4 & a_2c_3s_4+d_2c_4+a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3-d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Equating (1,3) and (3,3) elements from both sides

$$ \Rightarrow \begin{cases} a_x(c_1c_{23}c_4+s_1s_4)+a_y(s_1c_{23}c_4-c_1s_4)-a_z(s_{23}c_4) = -s_5 \\ a_x(-c_1s_{23}) + a_y(-s_1s_{23}) + a_z(-c_{23}) = c_5 \end{cases} $$

$$ \Rightarrow \theta_5 = \mathrm{atan2}(s_5, c_5) $$

Inverse Kinematics of PUMA 560

- 6.solve θ_6

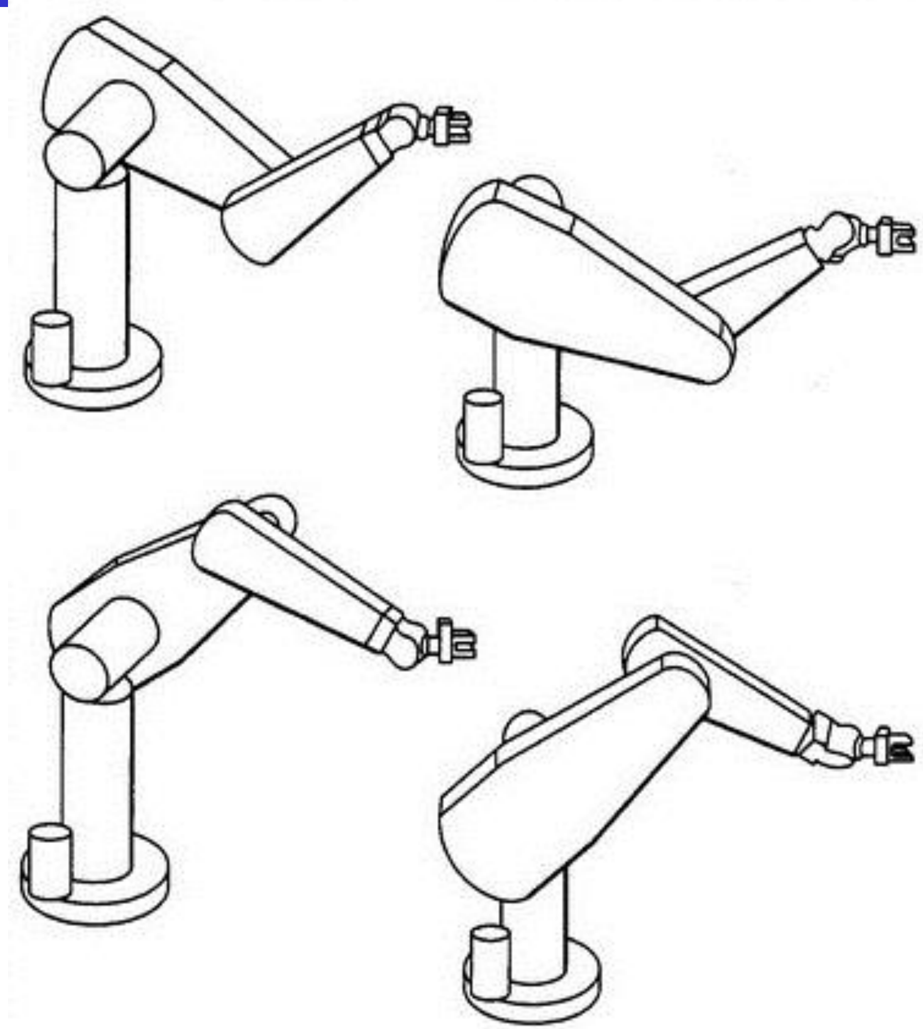
$$ {}^{0}T_5^{-1}(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)\,{}^{0}T_6 = {}^{5}T_6(\theta_6) $$

Equating (3,1) and (1,1) elements from both sides

$$ \Rightarrow \begin{cases} -n_x(c_1c_{23}s_4 - s_1c_4) - n_y(s_1c_{23}s_4 + c_1c_4) + n_z(s_{23}s_4) = s_6 \\ n_x\left[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5\right] + n_y\left[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5\right] \\ \qquad - n_z(s_{23}c_4c_5 + c_{23}s_5) = c_6 \end{cases} $$

$$ \Rightarrow \theta_6 = \mathrm{atan2}(s_6, c_6) $$

Inverse Kinematics of PUMA 560



Possible solutions of the PUMA560

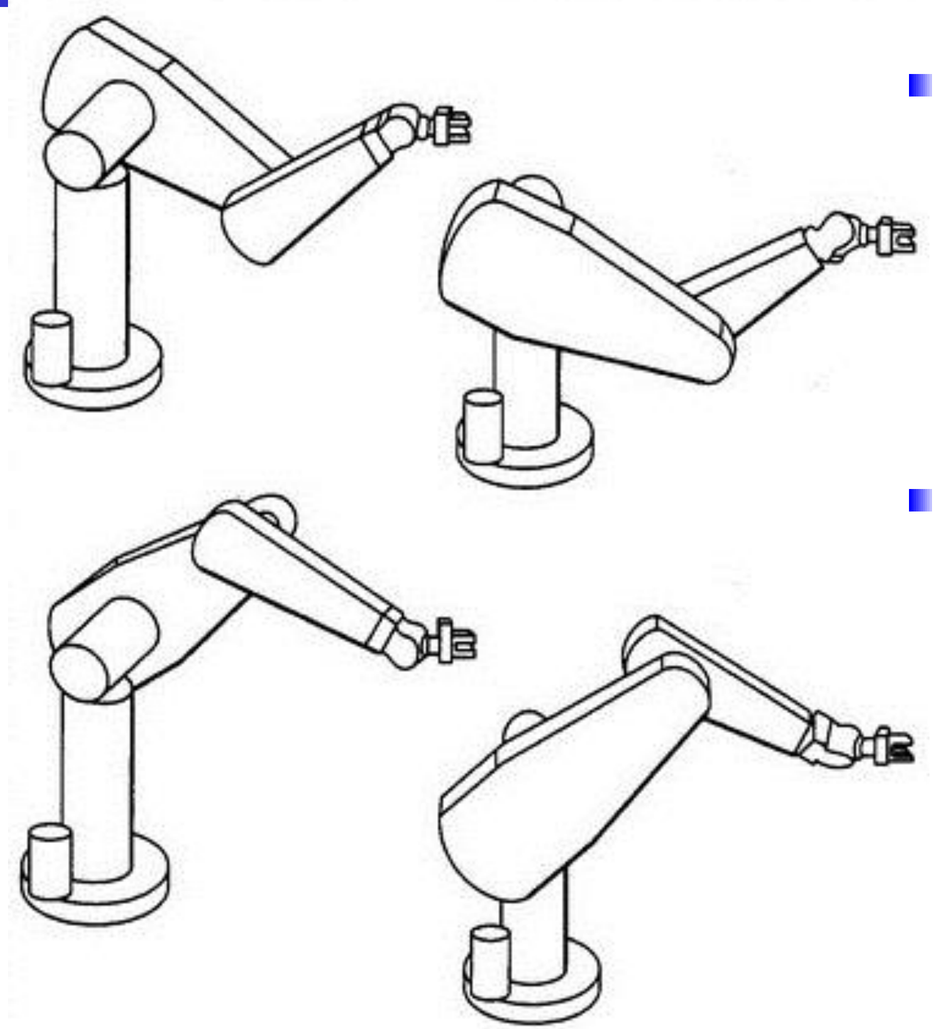
For each solution pictured, there is another solution in which the last three joints "flip " to an alternate configuration according to the formulas:

$$\theta_4' = \theta_4 + 180^\circ$$

$$\theta_5' = -\theta_5$$

$$\theta_6' = \theta_6 + 180^\circ$$

Inverse Kinematics of PUMA 560



Possible solutions of the PUMA560

- After all eight solutions have been computed, some or all of them may have to be discarded because of joint limit violations.
- Of the remaining valid solutions, usually the one closest to the present manipulator configuration is chosen.

Summary

- Introduction of Kinematics
  - Forward / Inverse Kinematics
- Link Description
  - Denavit-Hartenberg Parameters
  - Frame Attachment
- Forward Kinematics
  - Transform Equations
  - Example – forward kinematics of PUMA560
- Inverse Kinematics
  - Algebraic / Geometric
  - Example – inverse kinematics of PUMA560

谢 谢！