第2讲 数学基础

谢 斌

xiebin@csu.edu.cn

中南大学人工智能与机器人实验室

http://airl.csu.edu.cn

Review

1. 什么是机器人?
2. 机器人有哪些种类?
3. 为什么要使用机器人?
4. 当今全球的机器人市场情况如何?

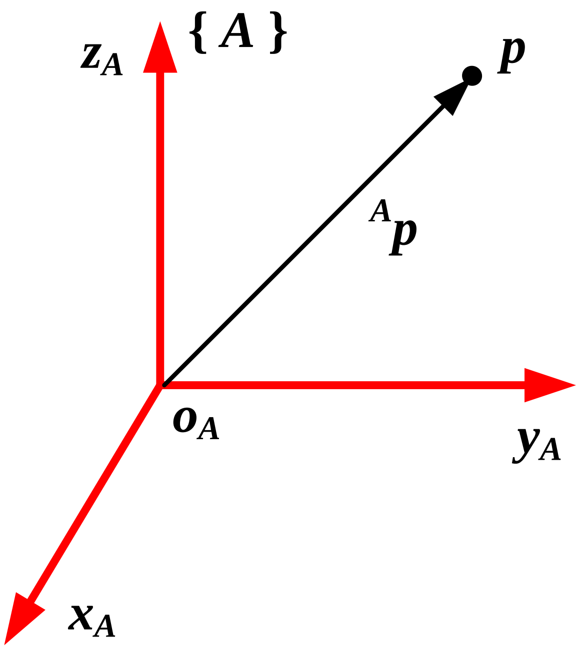
Questions！

1. 如何描述一个物体所处的位置、朝向？
2. 如何计算平移变换、旋转变换？
3. 什么是齐次坐标矩阵？
4. 如何求解不同坐标系之间的映射关系？

Contents

- 刚体位置 position

$$
{}^{A}\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}
$$



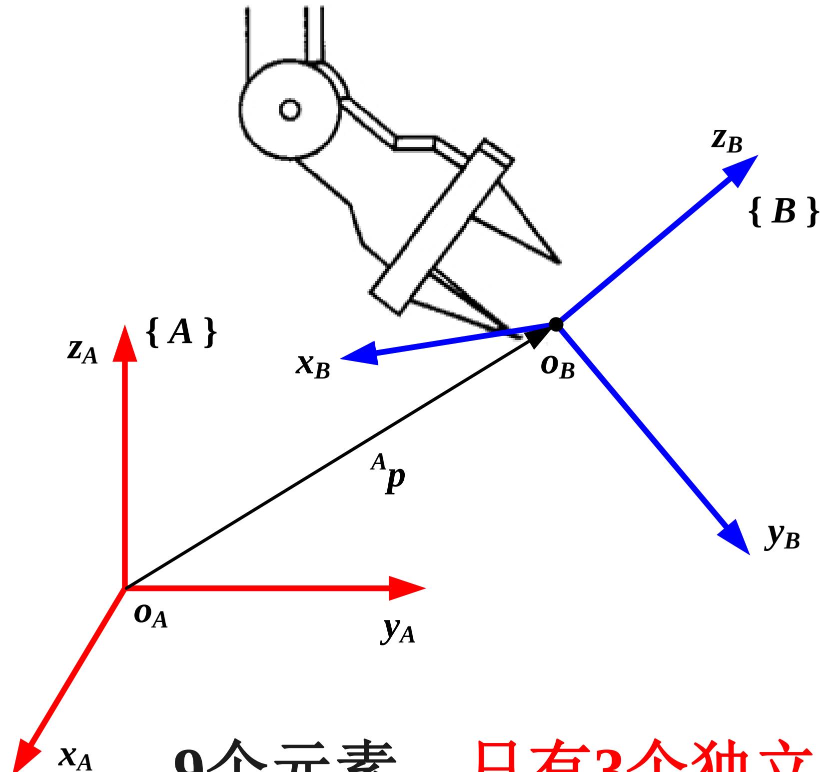
刚体姿态或方向 orientation

$$
{}^{A}_{B}R = \begin{bmatrix} {}^{A}x_B & {}^{A}y_B & {}^{A}z_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

Rotation Matrix

$$
{}^{A}_{B}R = \begin{bmatrix} x_B \cdot x_A & y_B \cdot x_A & z_B \cdot x_A \\ x_B \cdot y_A & y_B \cdot y_A & z_B \cdot y_A \\ x_B \cdot z_A & y_B \cdot z_A & z_B \cdot z_A \end{bmatrix}
$$

方向余弦direction cosines



9个元素，只有3个独立，满足6个约束条件。

- **Description of an orientation**

$$ {}^A_BR = \begin{bmatrix} {}^Ax_B & {}^Ay_B & {}^Az_B \end{bmatrix} = \begin{bmatrix} \boxed{x_B \cdot x_A \quad y_B \cdot x_A \quad z_B \cdot x_A} \\ x_B \cdot y_A \quad y_B \cdot y_A \quad z_B \cdot y_A \\ x_B \cdot z_A \quad y_B \cdot z_A \quad z_B \cdot z_A \end{bmatrix} $$

$$ {}^A_BR = \begin{bmatrix} {}^Ax_B & {}^Ay_B & {}^Az_B \end{bmatrix} = \begin{bmatrix} {}^Bx_A{}^T \\ {}^By_A{}^T \\ {}^Bz_A{}^T \end{bmatrix} $$

旋转矩阵R是正交矩阵，且满足

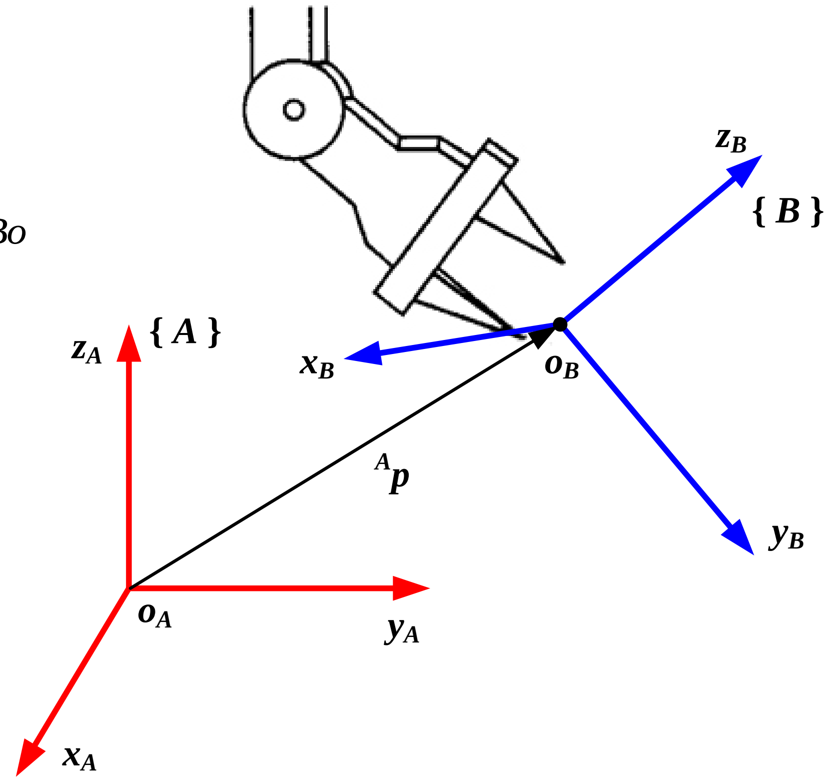
$$ {}^A_BR^{-1} = {}^A_BR^T; \ \left| {}^A_BR \right| = 1 $$

Positions, Orientations and Frames

Description of a frame

$$Frame\ \{B\} = \ {}^{A}x_{B},\ {}^{A}y_{B},\ {}^{A}z_{B},\ {}^{A}p_{Bo}$$

$$\{B\} = \left\{ {}^{A}_{B}R \quad {}^{A}\boldsymbol{p}_{Bo} \right\}$$

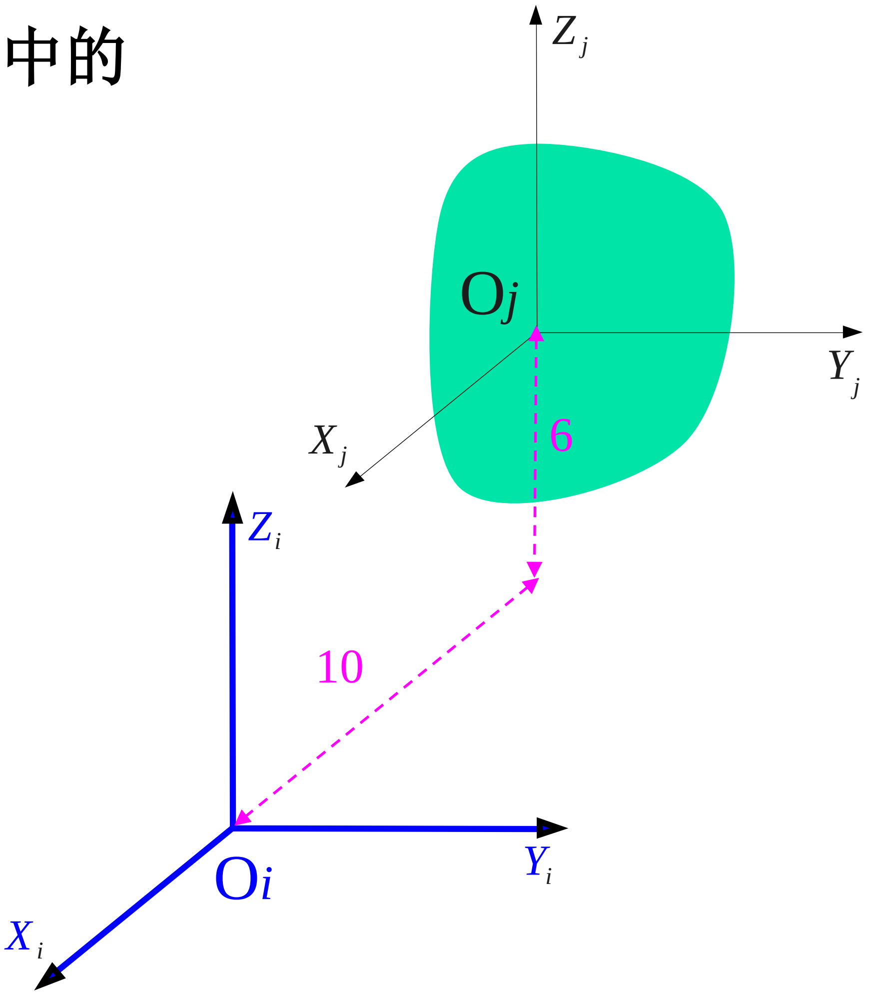


Frame is a set of four vectors giving position and orientation information.

Eg.2.1： 某刚体*j*在参考系*i*中的

位置 $^{o_i}_{o_j}P = ?$

方向 $^{O_i}_{O_j}R = ?$

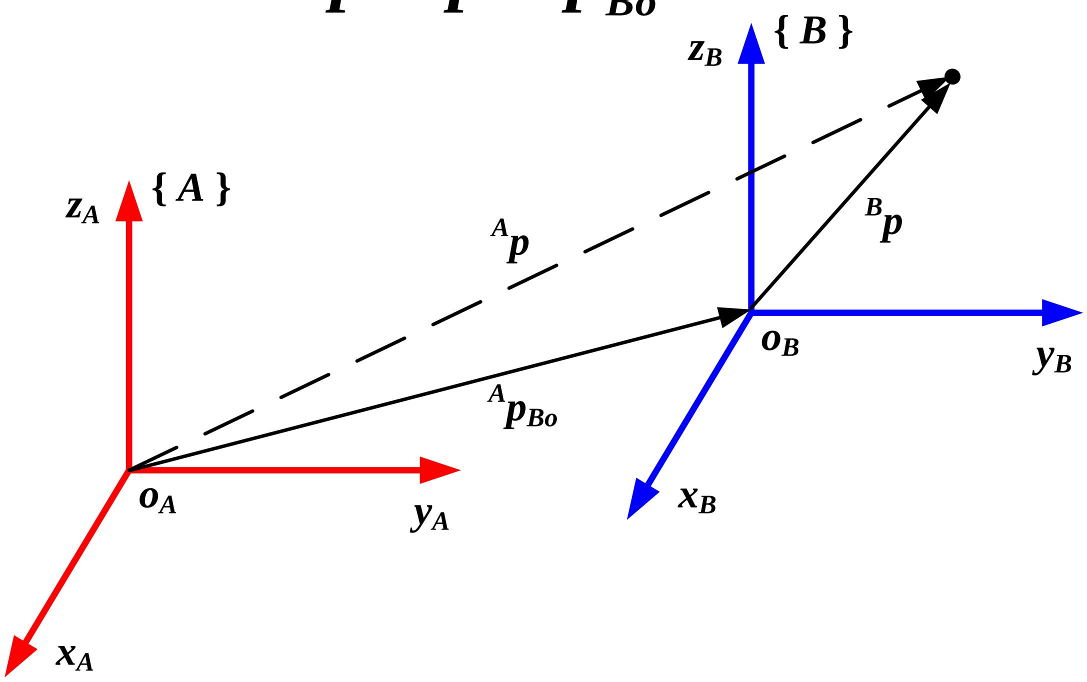


Contents

Mappings 映射

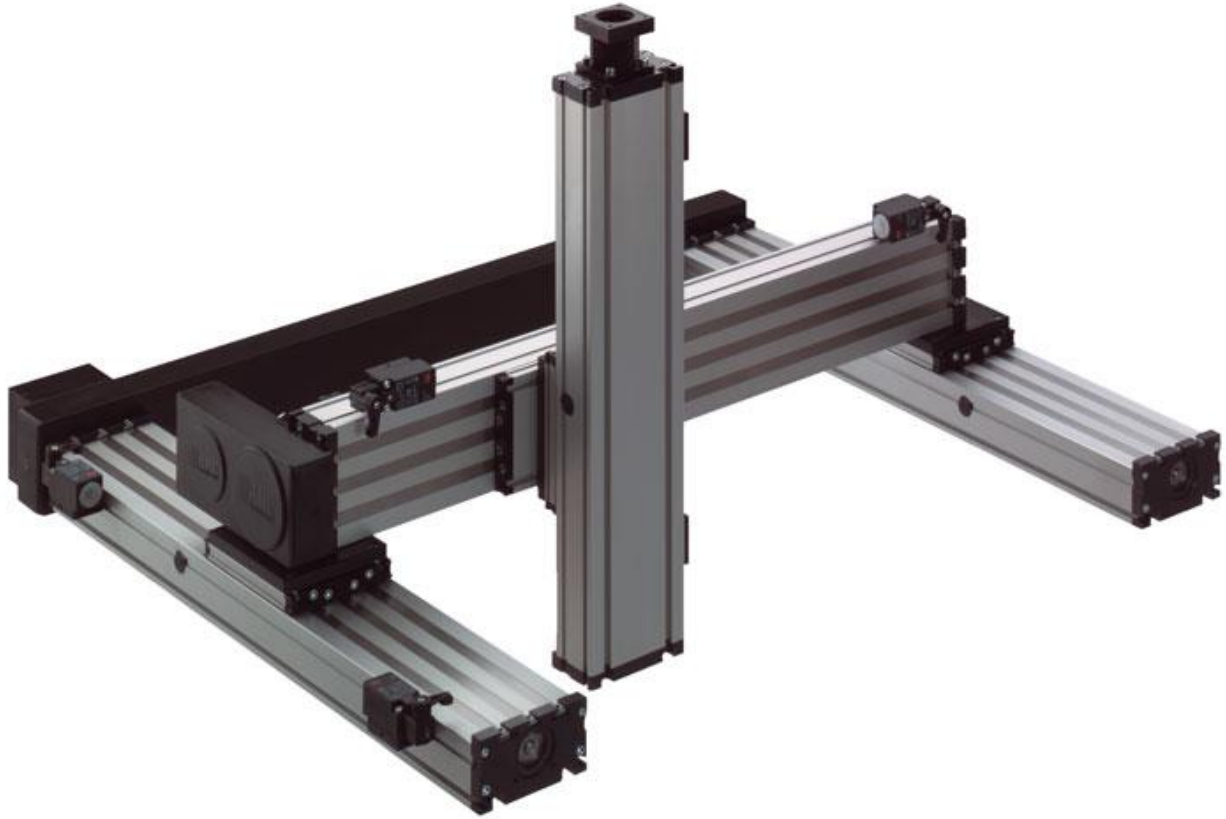
- Changing descriptions from frame to frame
- Translational Mapping (平移映射，坐标平移)

$$^{A}\boldsymbol{p} = {}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{p}_{Bo}$$



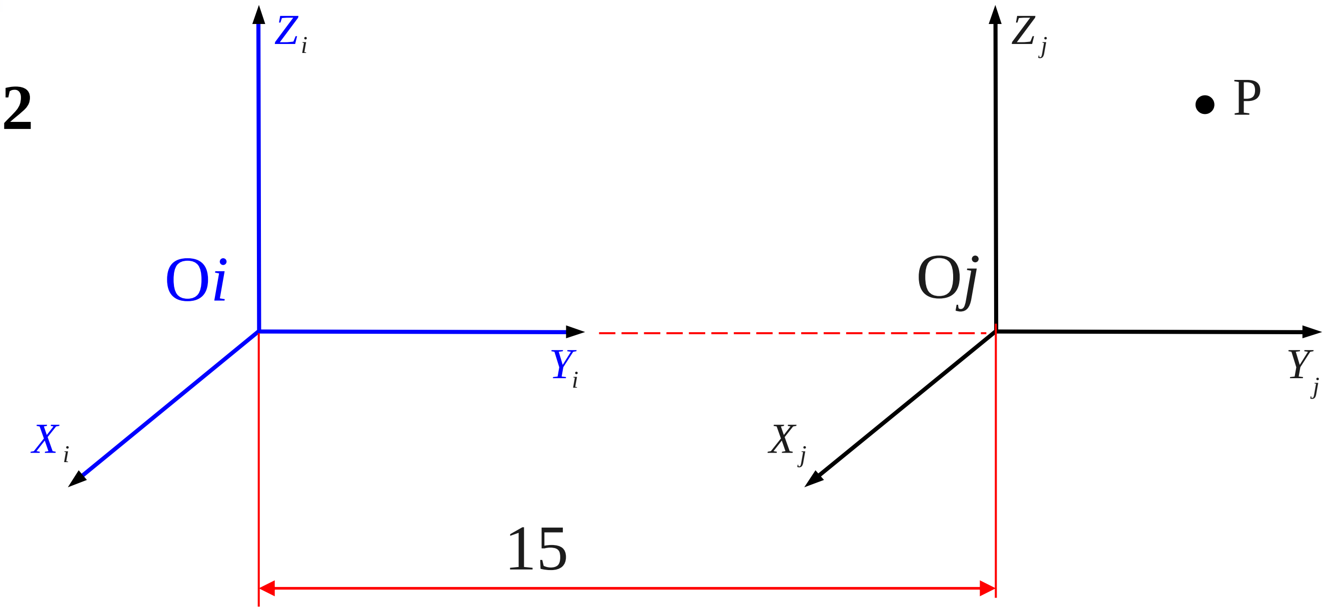
Translational Mappings 平移映射

适用的机器人类型举例（有平移关节）



Translational Mappings 平移映射

Eg.2.2



已知 ${}^{j}P = \begin{bmatrix} -5 & 6 & 7 \end{bmatrix}^T$ 求 P点在i坐标系中的坐标。

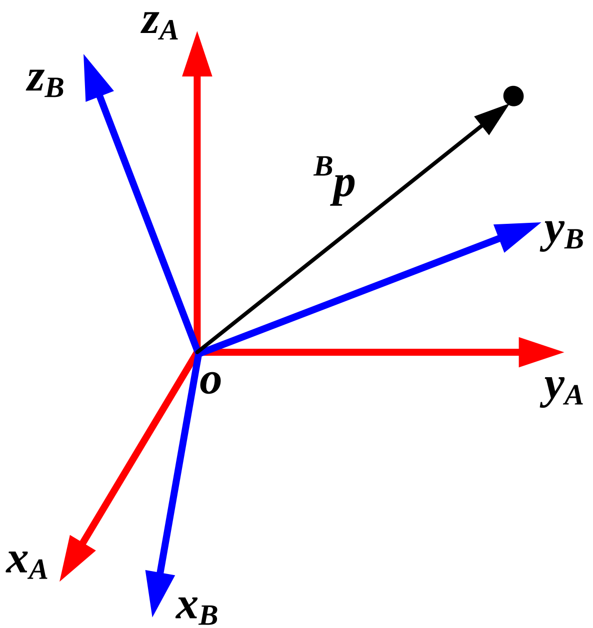
解答：${}^{i}P = {}^{j}P + {}^{Oj}_{i}P = \begin{bmatrix} -5 & 6 & 7 \end{bmatrix}^T + \begin{bmatrix} 0 & 15 & 0 \end{bmatrix}^T$

$= \begin{bmatrix} -5 & 21 & 7 \end{bmatrix}^T$

Mappings

- Changing descriptions from frame to frame
- Rotational Mapping (旋转映射)

$$^{A}\boldsymbol{p} = {}^{A}_{B}R\,{}^{B}\boldsymbol{p}$$

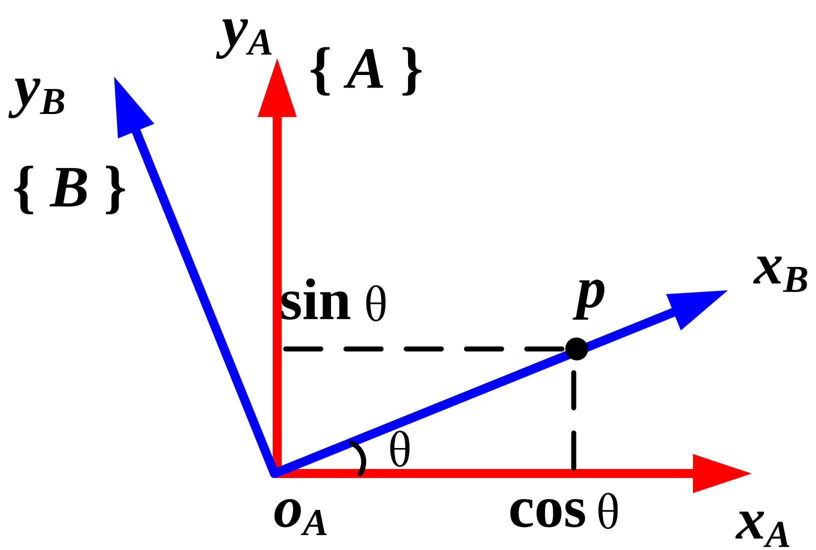


Rotational Mapping 旋转映射

- Rotation about an axis

$$
\begin{aligned}
{}^{A}x_p &= {}^{B}x_p \cdot \cos\theta \\
{}^{A}y_p &= {}^{B}x_p \cdot \sin\theta \\
{}^{A}z_p &= {}^{B}x_p \cdot 0 = 0
\end{aligned}
$$

$$
R(z,\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$



Frame {B} rotated θ about z

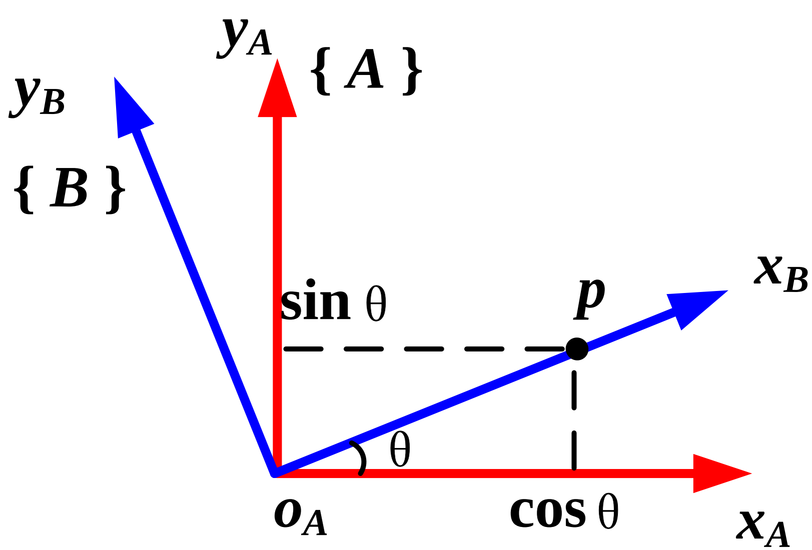
Rotational Mapping 旋转映射

- Rotation about an axis

$$R(x,\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & c\theta & -s\theta\\ 0 & s\theta & c\theta\end{bmatrix}$$

$$R(y,\theta)=\begin{bmatrix}c\theta & 0 & s\theta\\ 0 & 1 & 0\\ -s\theta & 0 & c\theta\end{bmatrix}$$

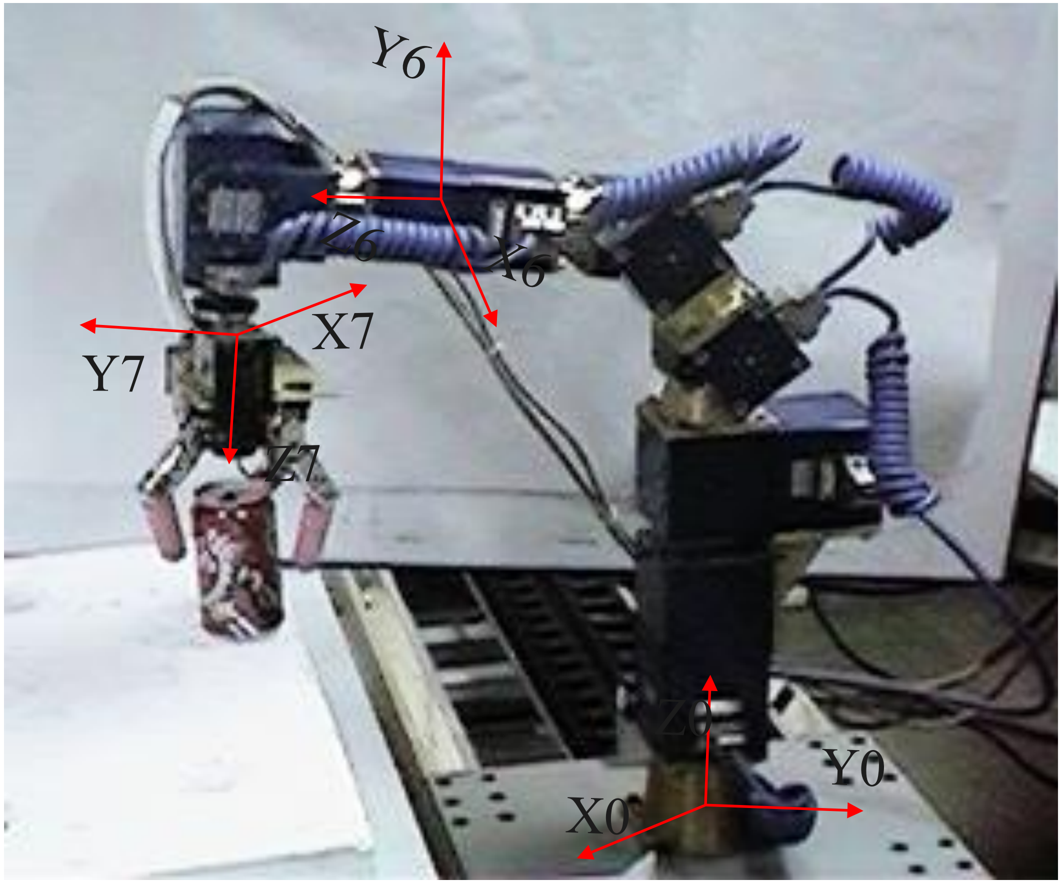
$$R(z,\theta)=\begin{bmatrix}c\theta & -s\theta & 0\\ s\theta & c\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$



Frame {B} rotated θ about z

Rotational Mapping 旋转映射

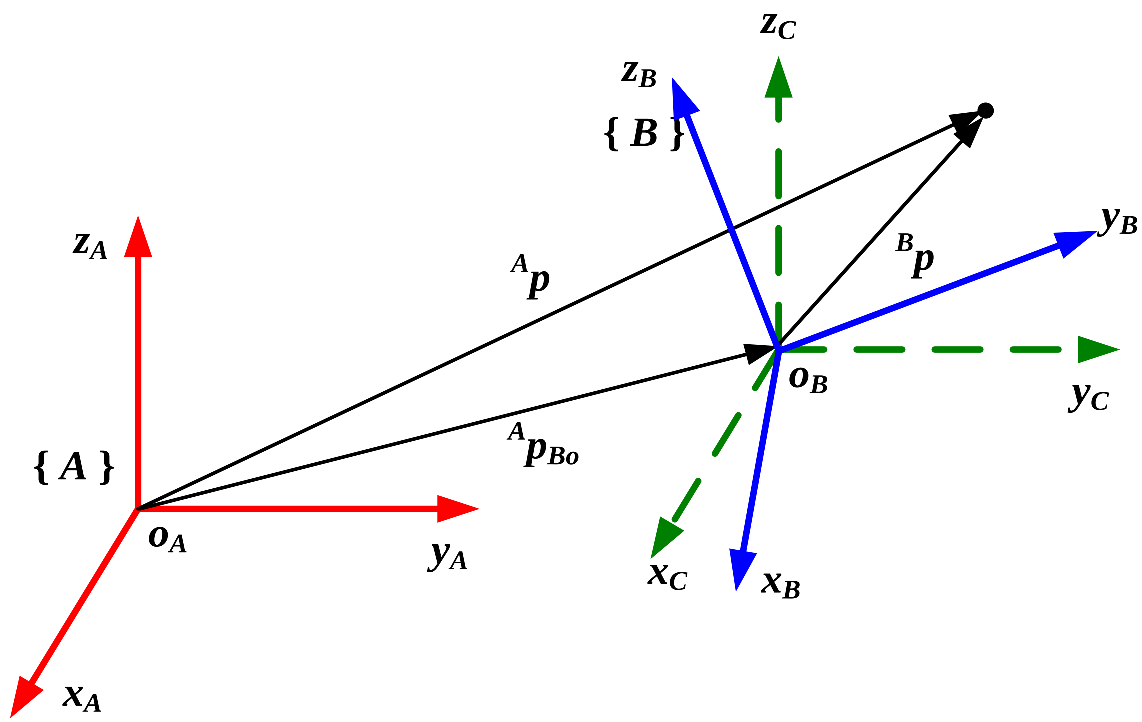
适用的机器人类型举例（有旋转关节）



Mappings 映射—复合映射

- General Mapping (复合映射)：平移+旋转

$$ {}^{A}\boldsymbol{p} = {}^{A}_{B}R\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{p}_{Bo} $$



Mappings 映射—复合映射

- General Mapping (复合映射)：平移+旋转

$$^{A}\boldsymbol{p}={}^{A}_{B}R\,{}^{B}\boldsymbol{p}+{}^{A}\boldsymbol{p}_{Bo}$$

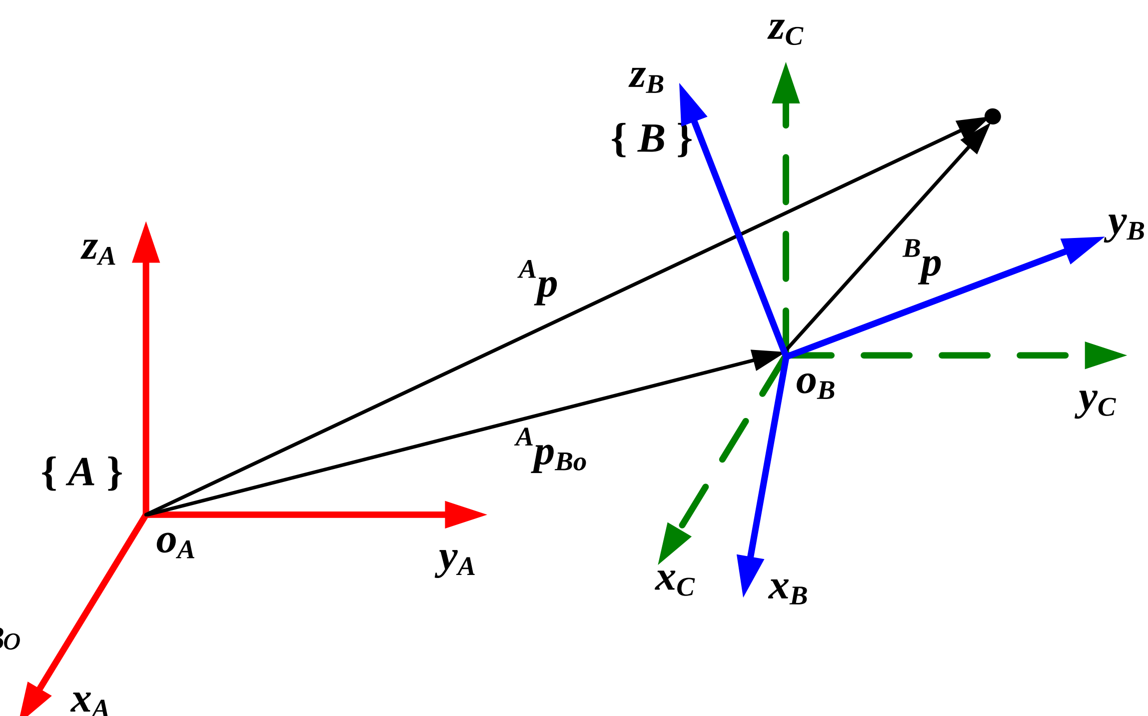
推导(中间坐标系C):

I（旋转）: C与B原点重合，
C与A姿态相同

$$^{C}\boldsymbol{p}={}^{C}_{B}R\,{}^{B}\boldsymbol{p}={}^{A}_{B}R\,{}^{B}\boldsymbol{p}$$

II（平移）: C与B原点重合

$$^{A}\boldsymbol{p}={}^{C}\boldsymbol{p}+{}^{A}\boldsymbol{p}_{Co}={}^{A}_{B}R\,{}^{B}\boldsymbol{p}+{}^{A}\boldsymbol{p}_{Bo}$$



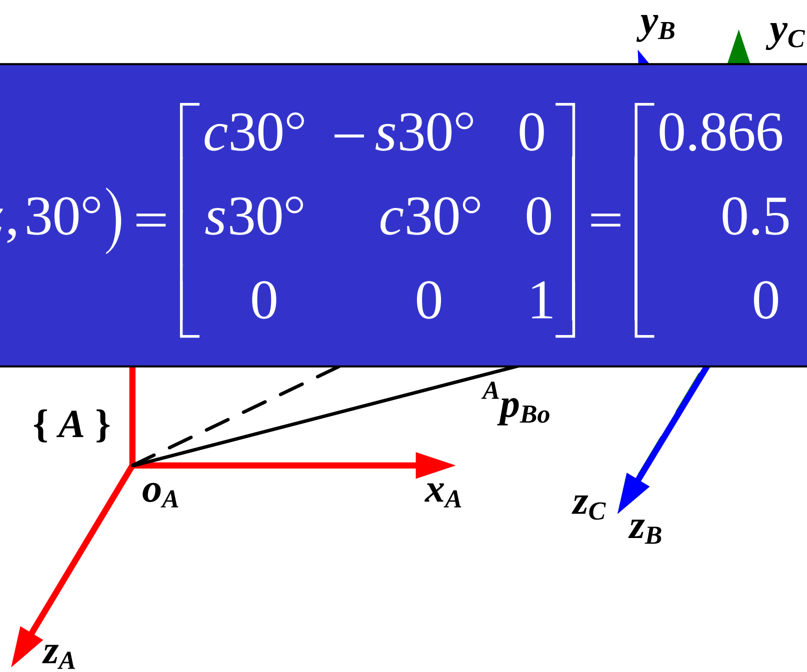
问题：是否可以先平移后旋转？

Mapping involving general frames

- **Eg.2.3** Figure below shows a frame $\{B\}$ which is rotated relative to frame $\{A\}$ about Z_A by 30 degrees, and translated 12 units in X_A, and 6 units in Y_A, find ${}^{A}\boldsymbol{p}$, where ${}^{B}\boldsymbol{p}=[3,7,0]^{T}$.

Solution:

$$ {}^{A}_{B}R = R(z,30^\circ) = \begin{bmatrix} c30^\circ & -s30^\circ & 0 \\ s30^\circ & c30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

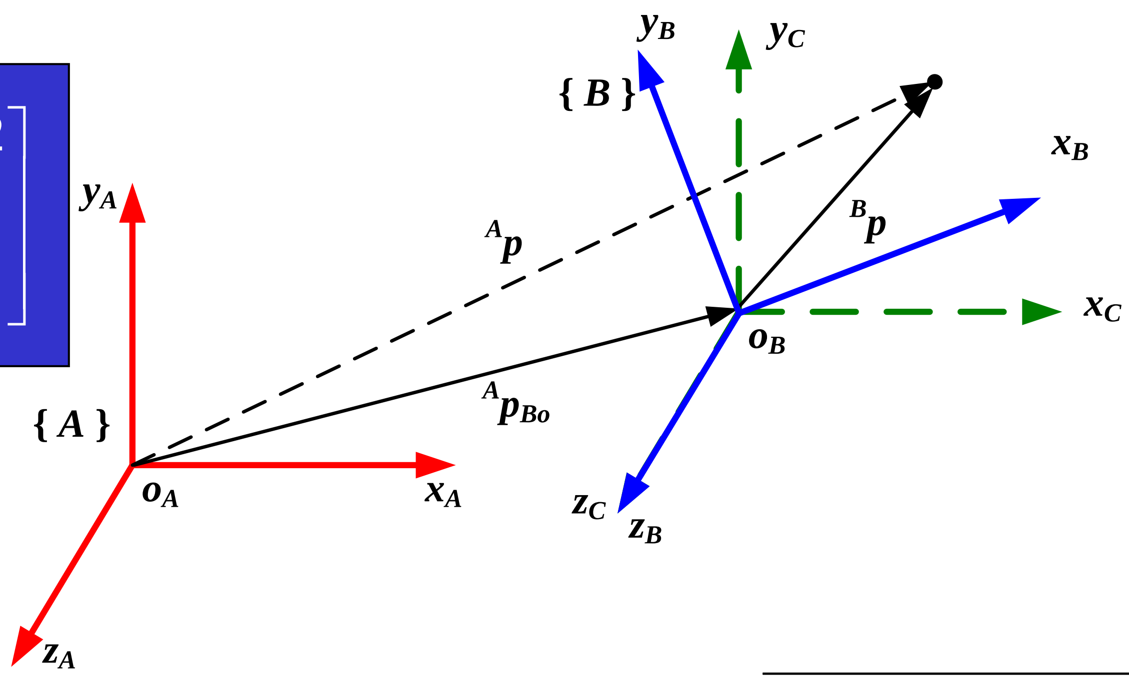


Mapping involving general frames

- **Eg.2.3** Figure below shows a frame $\{B\}$ which is rotated relative to frame $\{A\}$ about Z_A by 30 degrees, and translated 12 units in X_A, and 6 units in Y_A, find $^{A}\boldsymbol{p}$, where $^{B}\boldsymbol{p}=[3,7,0]^T$.

Solution:

$$^{A}p_{B_0} = \begin{bmatrix} 12 \\ 6 \\ 0 \end{bmatrix}$$

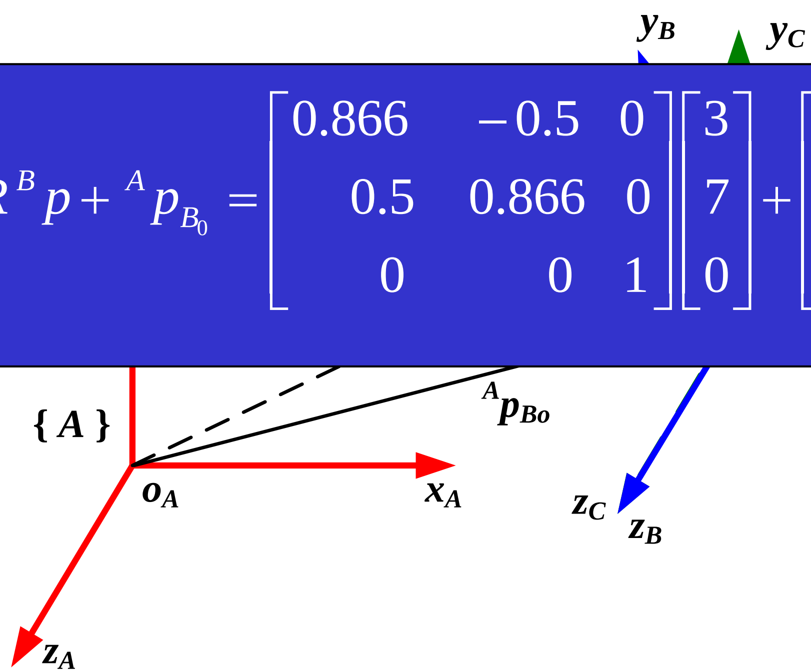


Mapping involving general frames

- **Eg.2.3** Figure below shows a frame $\{B\}$ which is rotated relative to frame $\{A\}$ about Z_A by 30 degrees, and translated 12 units in X_A, and 6 units in Y_A, find $^{A}\boldsymbol{p}$, where $^{B}\boldsymbol{p}=[3,7,0]^T$.

Solution:

$$^{A}p = {}^{A}_{B}R\,{}^{B}p + {}^{A}p_{B_0} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} + \begin{bmatrix} 12 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 11.098 \\ 13.562 \\ 0 \end{bmatrix}$$



Homogeneous Transform 齐次坐标变换

- 所谓**齐次坐标**就是将一个原本是 n 维的向量用一个 n+1 维向量来表示。一个向量的齐次表示是不唯一的，比如齐次坐标[8,4,2]、[4,2,1]表示的都是二维点[2,1]。
- 已知一直角坐标系中的某点坐标，则该点在另一直角坐标系中的坐标可通过**齐次坐标变换**求得。
- 齐次坐标提供了用**矩阵运算**把二维、三维甚至高维空间中的一个点集从一个坐标系变换到另一个坐标系的有效方法。

$$p=\begin{bmatrix}x\\y\\z\end{bmatrix} \Longrightarrow p=\begin{bmatrix}x\\y\\z\\1\end{bmatrix}=\begin{bmatrix}wx\\wy\\wz\\w\end{bmatrix}$$

非零的比例因子

直角坐标

齐次坐标

Homogeneous Transform 齐次坐标变换

1）点的齐次坐标： $P=\begin{bmatrix} x & y & z & \omega \end{bmatrix}^T$

$P=\begin{bmatrix} 2 & 3 & 4 & 1 \end{bmatrix}^T$， $P=\begin{bmatrix} 4 & 6 & 8 & 2 \end{bmatrix}^T$

无意义点 $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$， $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ 坐标原点

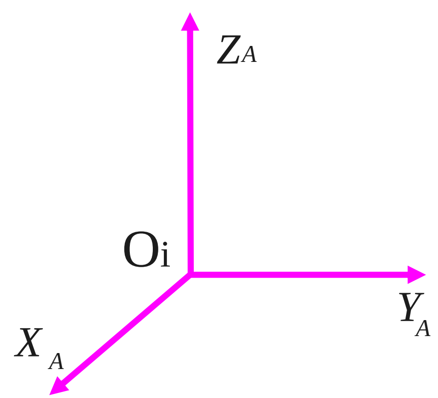
2）坐标轴方向的齐次坐标： a,b,c称为方向数

$$\begin{bmatrix} a & b & c & 0 \end{bmatrix}^T$$

X轴： $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$

Y轴： $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$

Z轴： $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$



Homogeneous Transform 齐次坐标变换

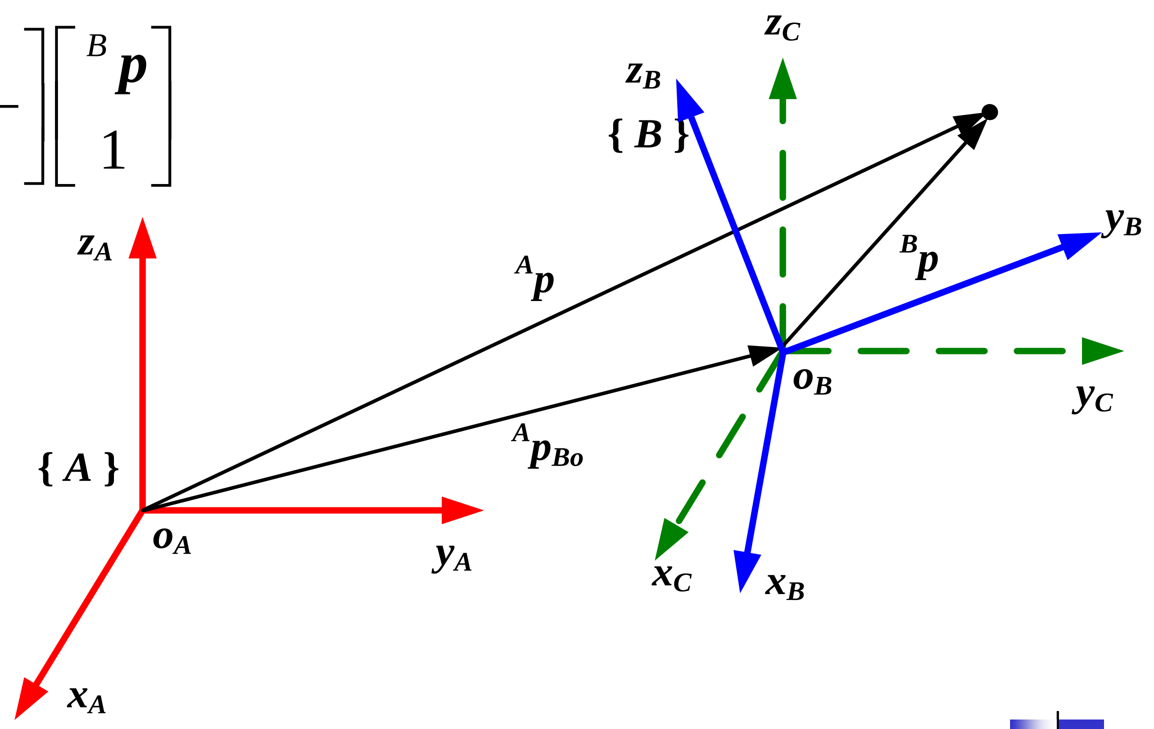
$$ {}^{A}\boldsymbol{p} = {}^{A}_{B}R\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{p}_{Bo} $$

- Homogeneous Transformation

$$ \begin{bmatrix} {}^{A}\boldsymbol{p} \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}\boldsymbol{p}_{Bo} \\ \hline 0\ 0\ 0 & 1 \end{array} \right] \begin{bmatrix} {}^{B}\boldsymbol{p} \\ 1 \end{bmatrix} $$

Matrix Form:

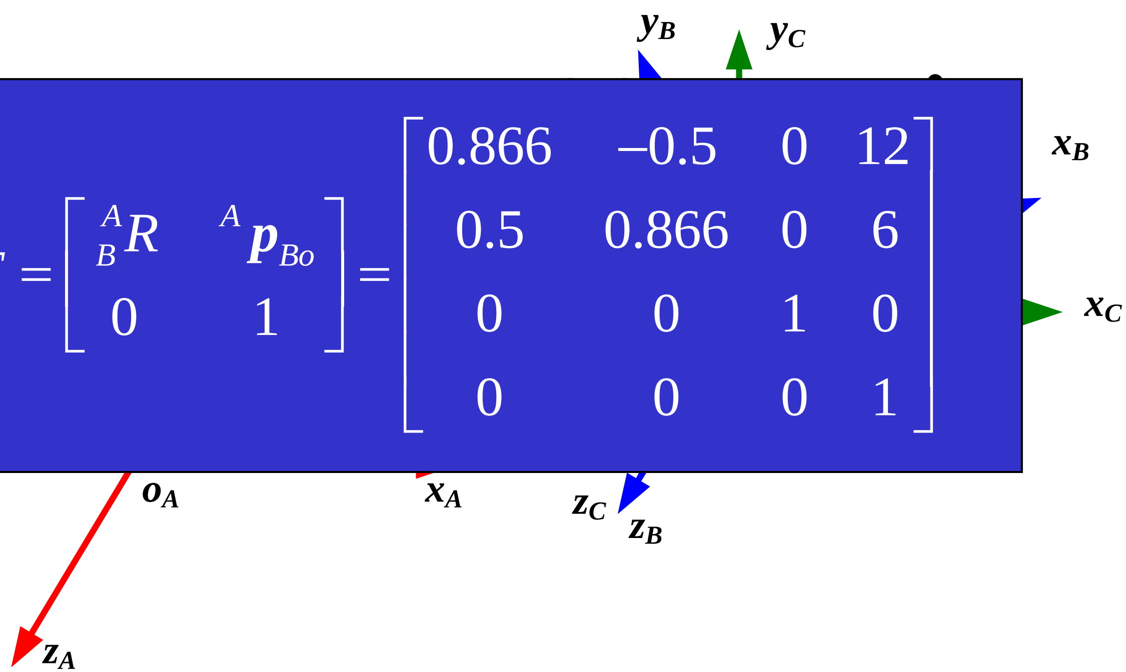
$$ {}^{A}\boldsymbol{p} = {}^{A}_{B}T\,{}^{B}\boldsymbol{p} $$



Homogeneous Transform 齐次坐标变换

- **Eg.2.4** Figure below shows a frame {*B*} which is rotated relative to frame {*A*} about Z_A by 30 degrees, and translated 12 units in X_A, and 6 units in Y_A, find ${}^{A}\boldsymbol{p}$ using homogeneous matrix, where ${}^{B}\boldsymbol{p}=[3,7,0]^T$.

Solution:



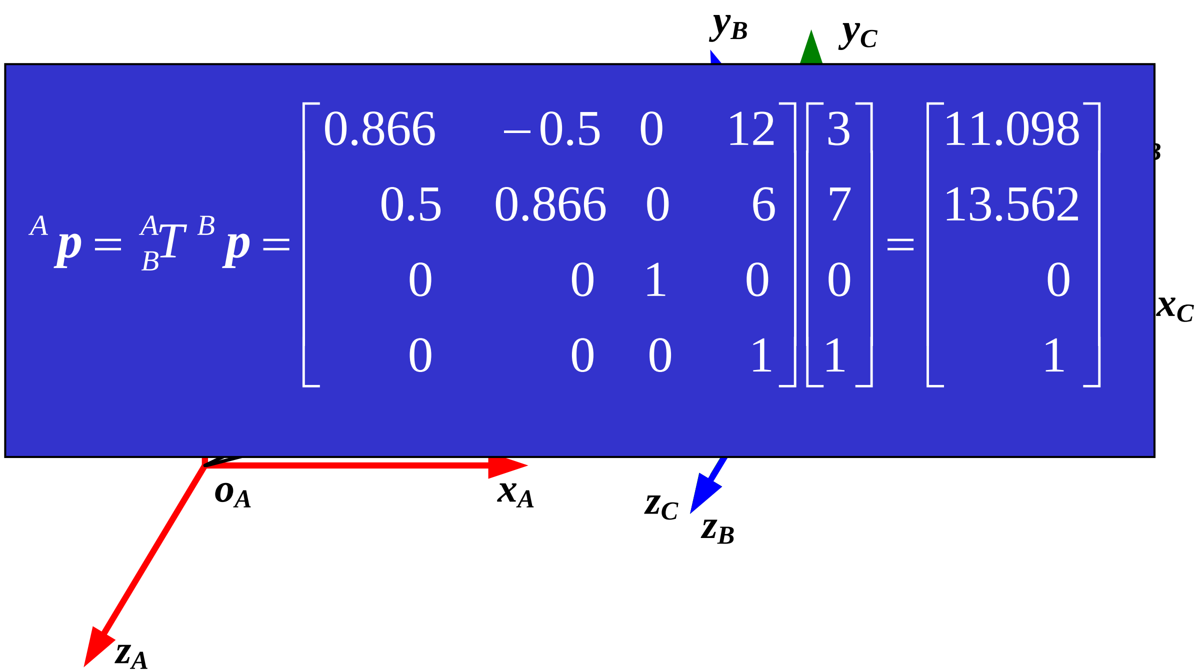
$$
{}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}\boldsymbol{p}_{Bo} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 12 \\ 0.5 & 0.866 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Homogeneous Transform

- **Eg.2.4** Figure below shows a frame $\{B\}$ which is rotated relative to frame $\{A\}$ about Z_A by 30 degrees, and translated 12 units in X_A, and 6 units in Y_A, find ${}^{A}\boldsymbol{p}$ using homogeneous matrix, where ${}^{B}\boldsymbol{p}=[3,7,0]^{T}$.

Solution:

$$ {}^{A}\boldsymbol{p} = {}^{A}_{B}T\,{}^{B}\boldsymbol{p} = \begin{bmatrix} 0.866 & -0.5 & 0 & 12 \\ 0.5 & 0.866 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 11.098 \\ 13.562 \\ 0 \\ 1 \end{bmatrix} $$



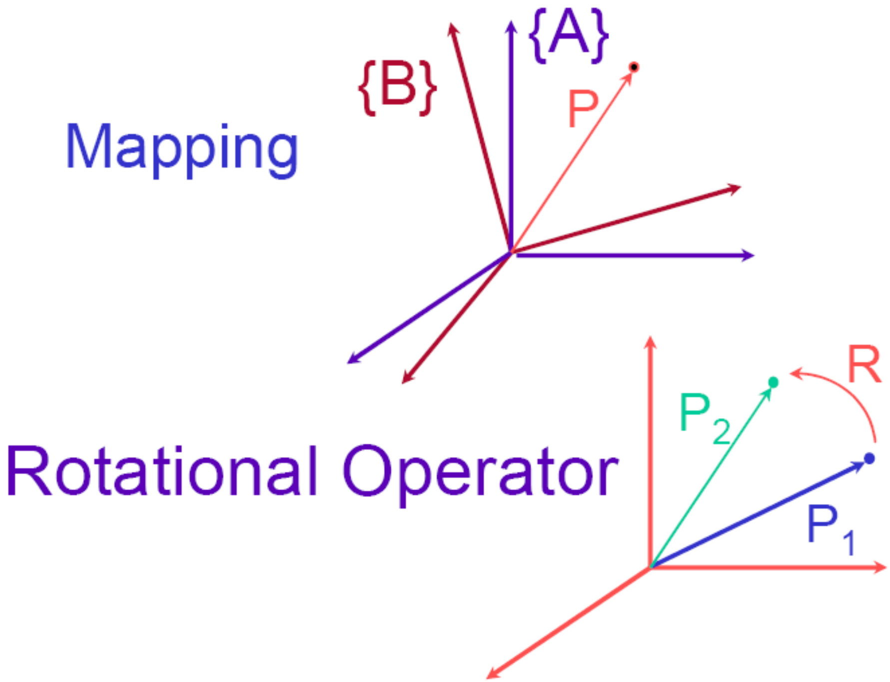
Contents

- 刚体位姿的数学描述

  Positions, Orientations, and Frames
- 坐标变换（点的映射）**Mappings**
- 算子**Operators**
- 变换等式**Transform Equations**

Operators

- Mapping: changing descriptions from frame to frame
- Operators: moving points (within the same frame)



$$^{A}P = {}^{A}_{B}R\, {}^{B}P$$

$$R:\ P_1 \longrightarrow P_2$$

$$\boxed{P_2 = R\, P_1}$$

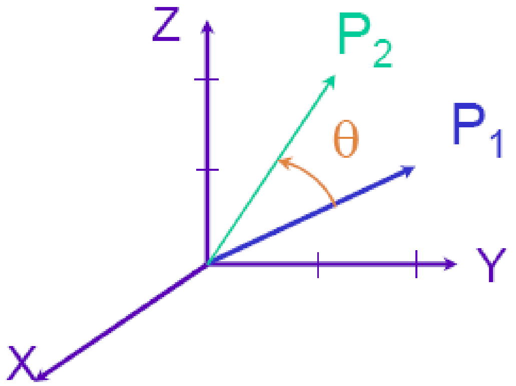
Rotational Operators

$$R_K(\theta): \quad P_1 \longrightarrow P_2$$

$$P_2 = R_K(\theta)\, P_1$$

Example

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



$$P_2 = R_X(\theta)P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Rotational Operators

- Homogeneous Transformation of Rotation

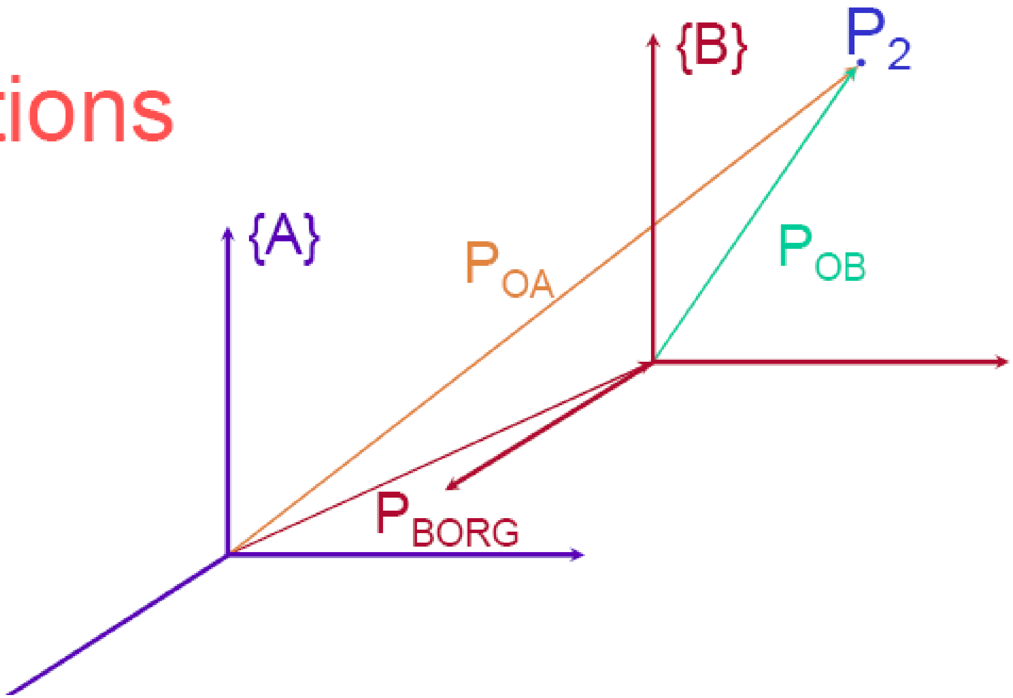
$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & c\theta & -s\theta & 0\\ 0 & s\theta & c\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$Rot(y,\theta)=\begin{bmatrix}c\theta & 0 & s\theta & 0\\ 0 & 1 & 0 & 0\\ -s\theta & 0 & c\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$Rot(z,\theta)=\begin{bmatrix}c\theta & -s\theta & 0 & 0\\ s\theta & c\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

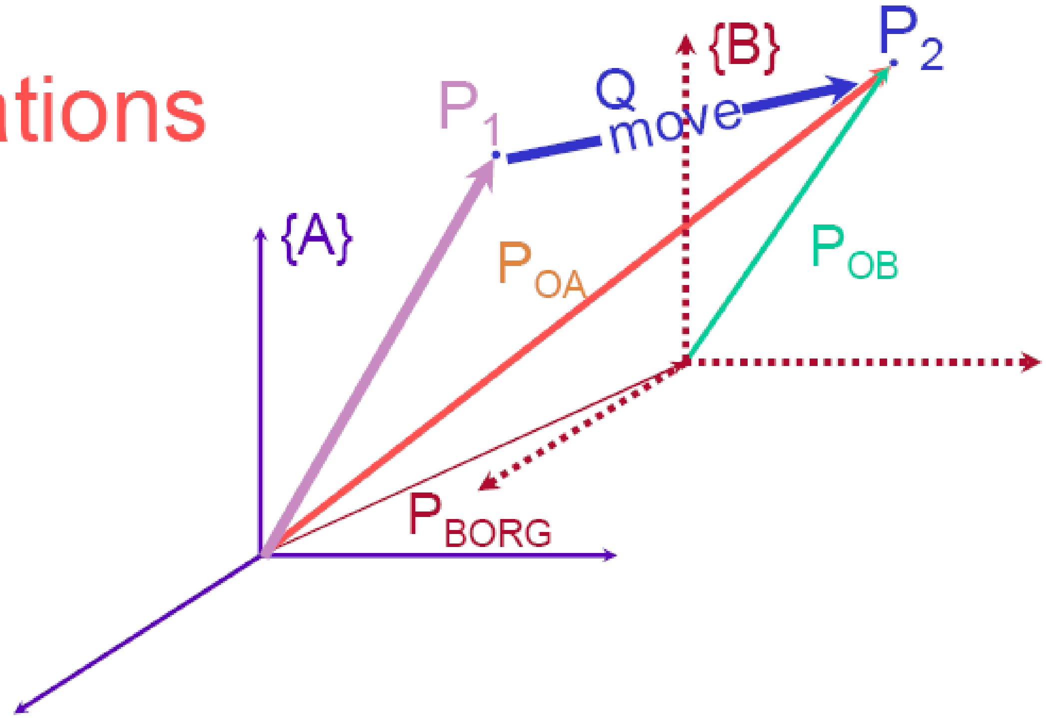
Translational Operators

Translations



Translational Operators

Translations



Mapping: $P_{BORG}: \quad P_{OB} \longrightarrow P_{OA}$ (same point)

2 diff. vectors

$$P_{OA} = P_{OB} + P_{BORG}$$

Translational Operator:

$Q: \quad P_1 \longrightarrow P_2$ (2 points, 2 diff vectors)

$$P_2 = P_1 + Q$$

Translational Operators

- Homogeneous Transformation of Translation
- 空间中的某点用矢量***ai***+***bj***+***ck***描述，该点也可表示为：

$$Trans(a,b,c)=\begin{bmatrix}1&0&0&a\\0&1&0&b\\0&0&1&c\\0&0&0&1\end{bmatrix}$$

- 对已知矢量 ***u***=[*x*,*y*,*z*,*w*]^T 进行平移变换所得的矢量 ***v*** 为：

$$v=Trans(a,b,c)\cdot u=\begin{bmatrix}1&0&0&a\\0&1&0&b\\0&0&1&c\\0&0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}x+aw\\y+bw\\z+cw\\w\end{bmatrix}=\begin{bmatrix}x/w+a\\y/w+b\\z/w+c\\1\end{bmatrix}$$

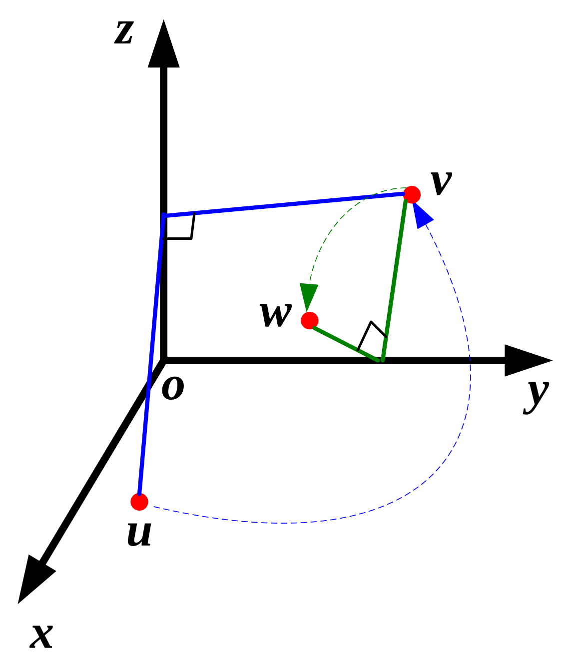
Compound Operator

- **Eg.2.5** Figure below shows a vector $\boldsymbol{u}=7\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}$. We wish to rotate it about z axis by 90 degrees to get $\boldsymbol{v}$, and then rotate $\boldsymbol{v}$ about y axis by 90 degrees to get $\boldsymbol{w}$. Find the homogeneous coordinate of $\boldsymbol{v}$, $\boldsymbol{w}$.

Solution:

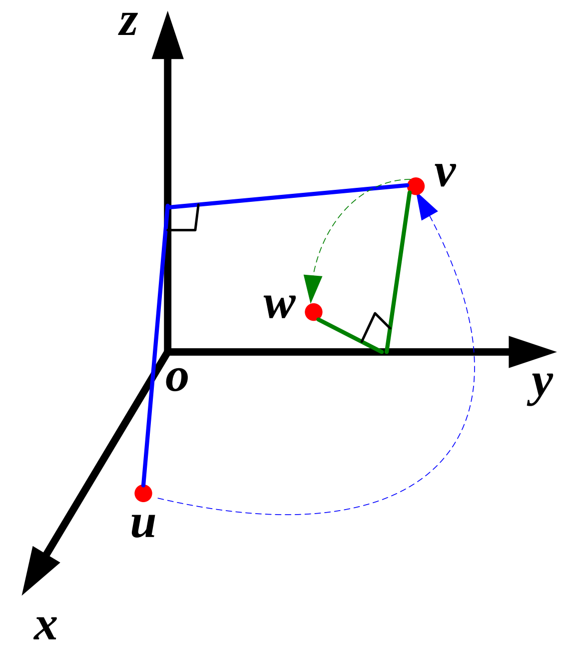
$$v = Rot(z,90°)\cdot u = \begin{bmatrix} c90° & -s90° & 0 & 0 \\ s90° & c90° & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 2 \\ 1 \end{bmatrix}$$

$$w = Rot(y,90°)\cdot v = \begin{bmatrix} c90° & 0 & s90° & 0 \\ 0 & 1 & 0 & 0 \\ -s90° & 0 & c90° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 7 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 1 \end{bmatrix}$$



Compound Operator

- **Eg.2.5** Figure below shows a vector $\boldsymbol{u}=7\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}$. We wish to rotate it about z axis by 90 degrees to get $\boldsymbol{v}$, and then rotate $\boldsymbol{v}$ about y axis by 90 degrees to get $\boldsymbol{w}$. Find the homogeneous coordinate of $\boldsymbol{v}$, $\boldsymbol{w}$.



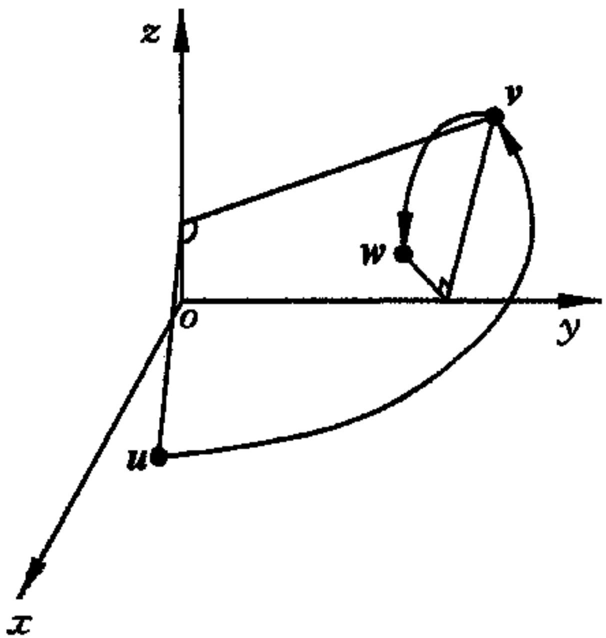
Solution： Combined these two operators

$$w = Rot\left(y,90°\right)Rot\left(z,90°\right)u$$

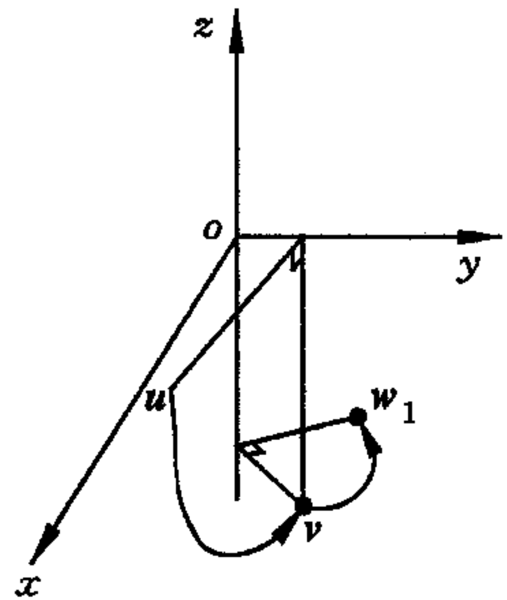
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 1 \end{bmatrix}$$

Order of Rotation

- If we change the order of rotation, rotate ***u*** about y axis by 90 degrees first, then rotate about z axis by 90 degrees, we can get a different result of ***w***.



(a) $Rot(y,90°)Rot(z,90°)$



(b) $Rot(z,90°)Rot(y,90°)$

Compound Operator

- We are given a single frame {A} and a position vector ${}^{A}\mathbf{P}$ described in this frame. We then transform ${}^{A}\mathbf{P}$ by first rotating it about $\hat{Z}_A$ by an angle ϕ, then rotating about $\hat{Y}_A$ by an angle θ. Determine the 3 × 3 rotation matrix operator, $R(\phi,\theta)$, which describes this transformation.

Solution: Suppose the first rotation converts ${}^{A}P \rightarrow {}^{A}P'$, and the second rotation converts ${}^{A}P' \rightarrow {}^{A}P''$. Then we have:

$$^{A}P' = R_z(\phi)\,{}^{A}P$$

$$^{A}P'' = R_y(\theta)\,{}^{A}P'$$

$$\Rightarrow {}^{A}P'' = R_y(\theta) R_z(\phi)\,{}^{A}P$$

Compound Operator

- We are given a single frame {A} and a position vector ${}^{A}\mathbf{P}$ described in this frame. We then transform ${}^{A}\mathbf{P}$ by first rotating it about $\hat{Z}_A$ by an angle ϕ, then rotating about $\hat{Y}_A$ by an angle θ. Determine the 3 × 3 rotation matrix operator,

$R(\phi,\theta)$, which describes this transformation.

Solution: $${}^{A}P'' = R_y(\theta) R_z(\phi)\, {}^{A}P$$

$$\Rightarrow R(\phi,\theta) = R_y(\theta) R_z(\phi)$$

$$\Rightarrow R(\phi,\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta c\phi & -c\theta s\phi & s\theta \\ s\phi & c\phi & 0 \\ -s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix}$$

Compound Operator

- **Eg.2.6** 固连在坐标系$\{A\}$上的点${}^A\boldsymbol{p} = [7, 3, 1]^T$，绕z轴旋转90度，接着平移$[4, -3, 7]^T$，然后再绕y轴旋转90度，求出齐次变换矩阵T及该点变换后的齐次坐标。

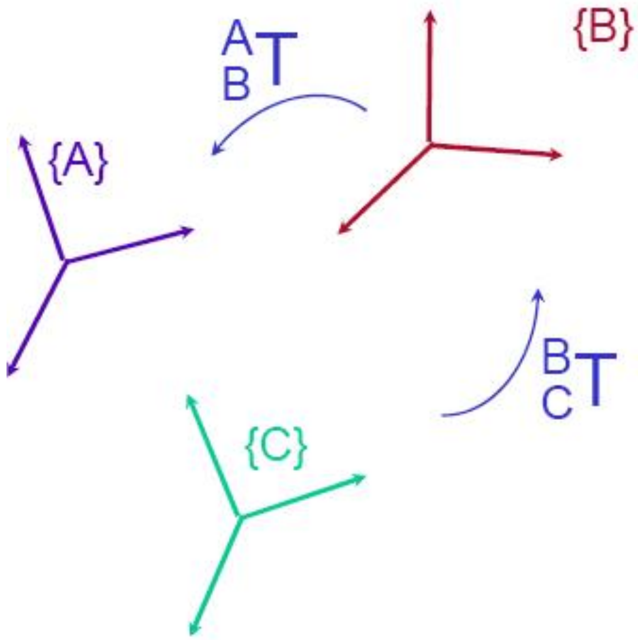
$$T = Rot(y, 90)\,Trans(4, -3, 7)\,Rot(z, 90)$$

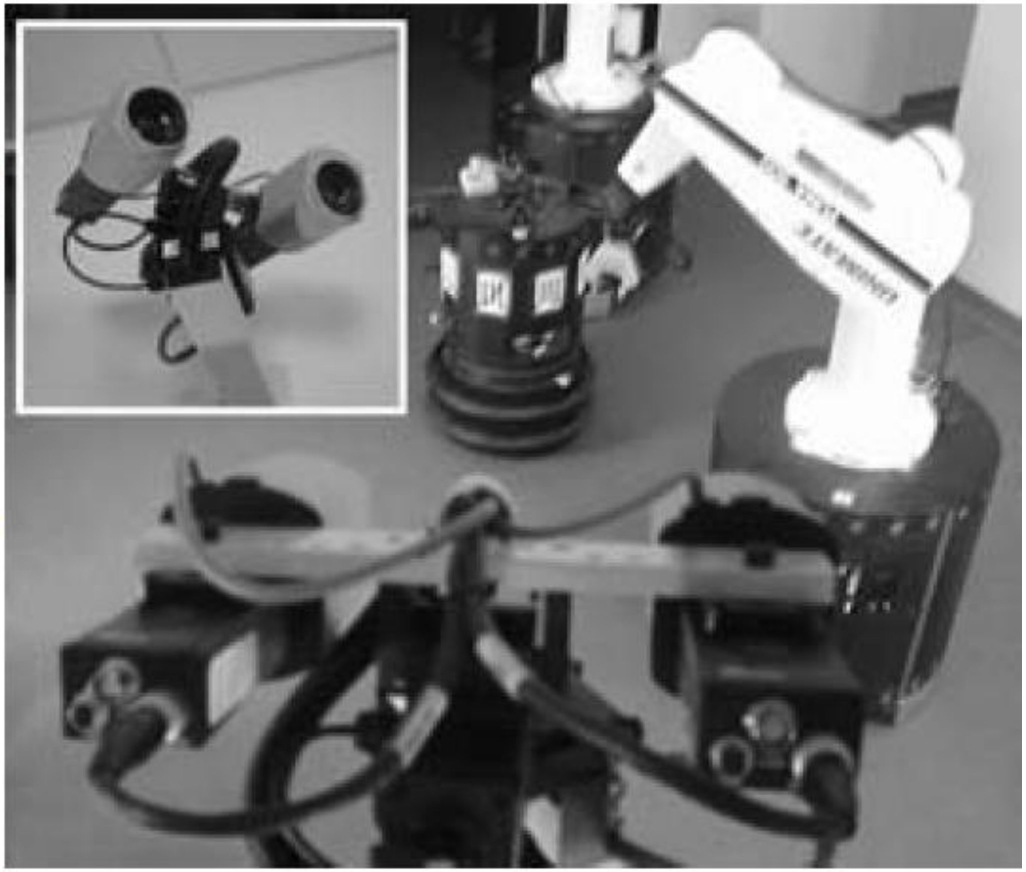
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p' = Tp = \begin{bmatrix} 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

Contents

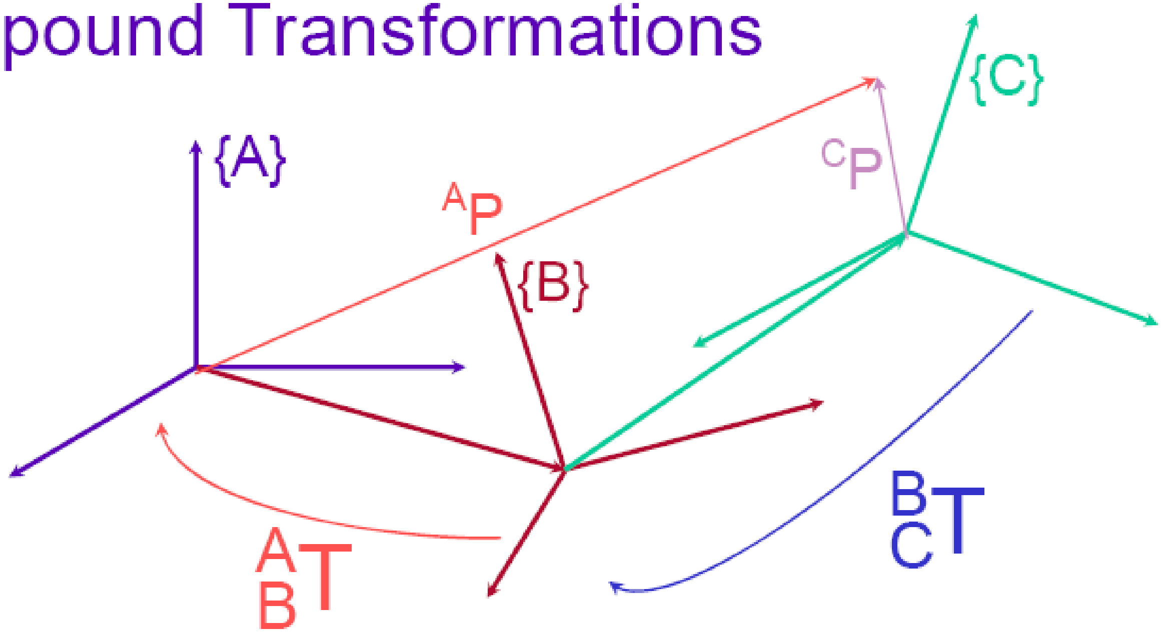
Transform Equations





Transform Equations

Compound Transformations



Transform Equations

- Inverse Transformation

  从坐标系{B}相对{A}的描述 ${}^{A}_{B}T$，求得坐标系{A}相对{B}的描述 ${}^{B}_{A}T$，是**齐次变换求逆问题**。

- 对于给定的 ${}^{A}_{B}T$，求解 ${}^{B}_{A}T$，等价于给定 ${}^{A}_{B}R$ 和 ${}^{A}\boldsymbol{p}_{Bo}$ 计算 ${}^{B}_{A}R$ 和 ${}^{B}\boldsymbol{p}_{Ao}$。

$$
{}^{A}_{B}T = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}\boldsymbol{p}_{Bo} \\ \hline 0\;0\;0 & 1 \end{array}\right] \quad \Rightarrow {}^{B}_{A}T = \left[\begin{array}{c|c} {}^{B}_{A}R & {}^{B}\boldsymbol{p}_{Ao} \\ \hline 0\;0\;0 & 1 \end{array}\right] = ?
$$

Transform Equations

- Inverse Transformation

对于给定的 ${}^{A}_{B}T$ ， 求解 ${}^{B}_{A}T$ ， 等价于给定 ${}^{A}_{B}R$ 和 ${}^{A}\boldsymbol{p}_{Bo}$ 计算 ${}^{B}_{A}R$ 和 ${}^{B}\boldsymbol{p}_{Ao}$ 。

$$^{B}\left({}^{\boldsymbol{A}}\boldsymbol{p}_{Bo}\right) = {}^{B}_{A}R\,{}^{\boldsymbol{A}}\boldsymbol{p}_{Bo} + {}^{\boldsymbol{B}}\boldsymbol{p}_{Ao}$$

$$\Rightarrow \quad {}^{\boldsymbol{B}}\boldsymbol{p}_{Ao} = -{}^{B}_{A}R\,{}^{\boldsymbol{A}}\boldsymbol{p}_{Bo} = -{}^{A}_{B}R^{T}\,{}^{\boldsymbol{A}}\boldsymbol{p}_{Bo}$$

$$\Rightarrow \quad {}^{B}_{A}T = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{\boldsymbol{A}}\boldsymbol{p}_{Bo} \\ \hline 0 & 1 \end{array}\right]$$

Example 2.7

- Given a transformation matrix:

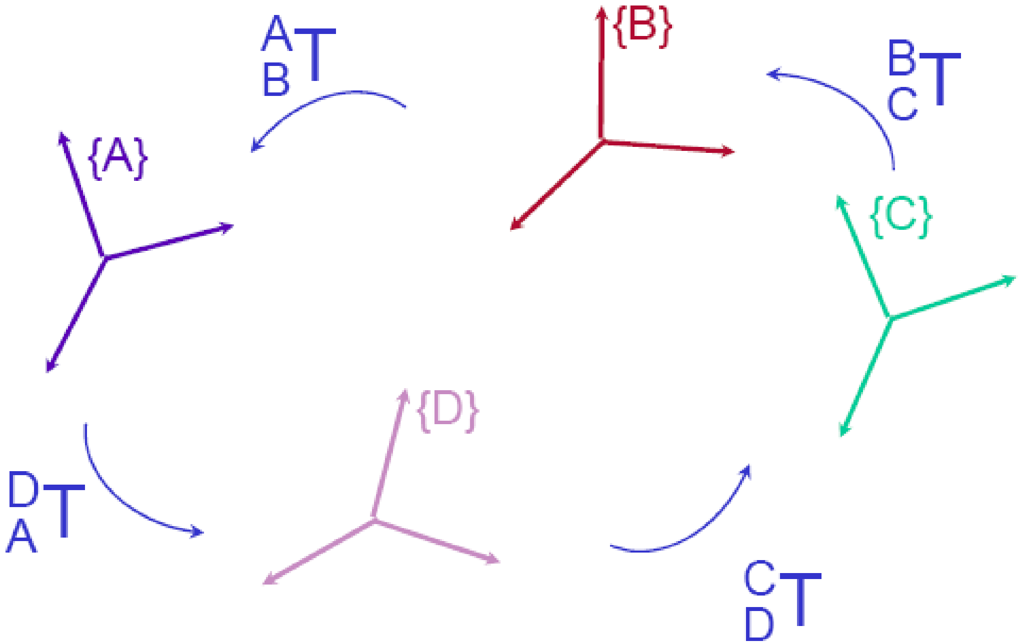
$$ {}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & c\theta & -s\theta & 2 \\ 0 & s\theta & c\theta & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- Find ${}^{B}_{A}T$

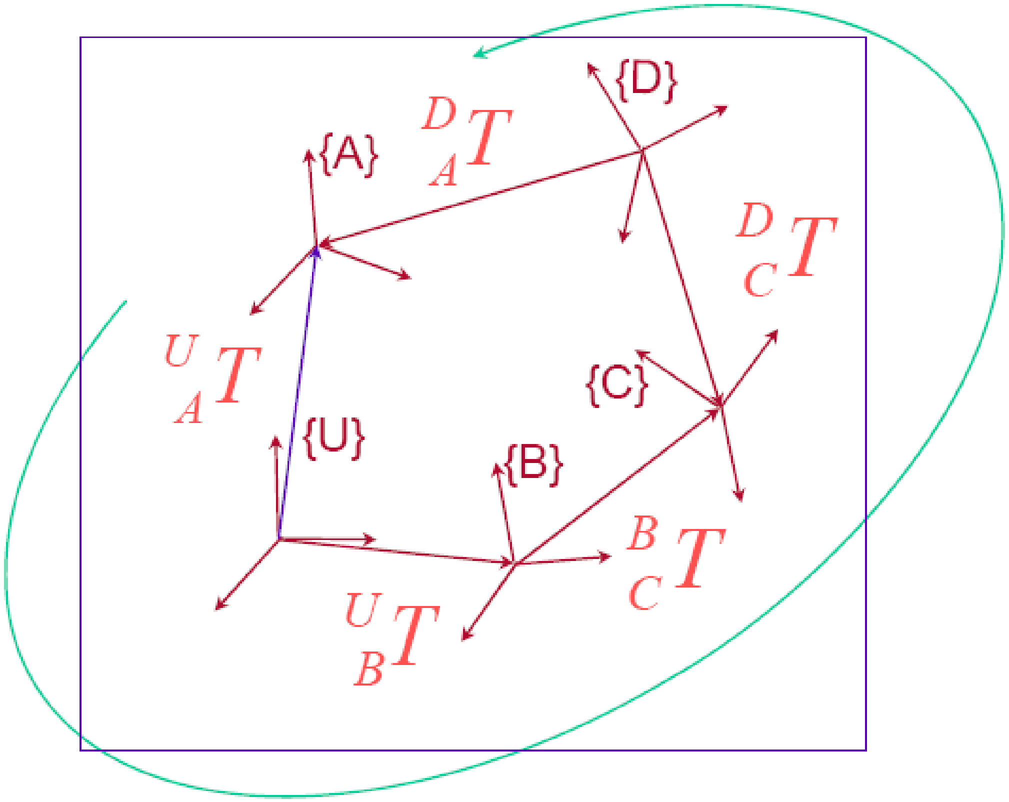
Solution:

$$ {}^{B}_{A}T = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\, {}^{A}\boldsymbol{p}_{Bo} \\ \hline 0 & 1 \end{array} \right] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & c\theta & s\theta & -2c\theta - 3s\theta \\ 0 & -s\theta & c\theta & 2s\theta - 3c\theta \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Transform Equations

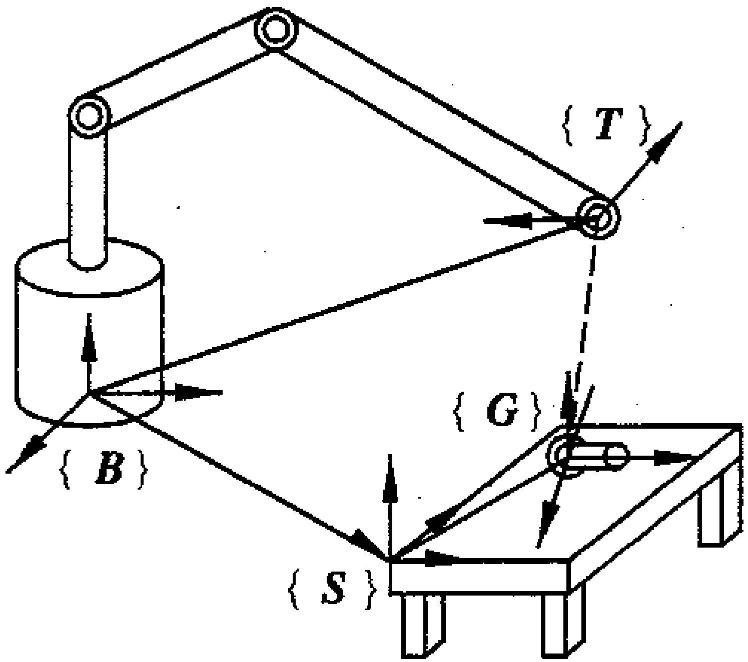


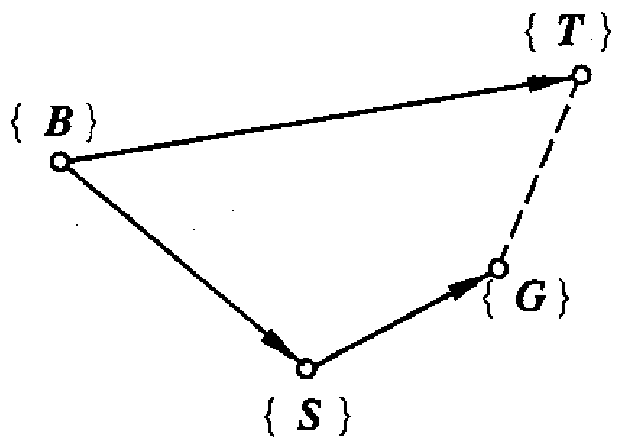
Transform Equations



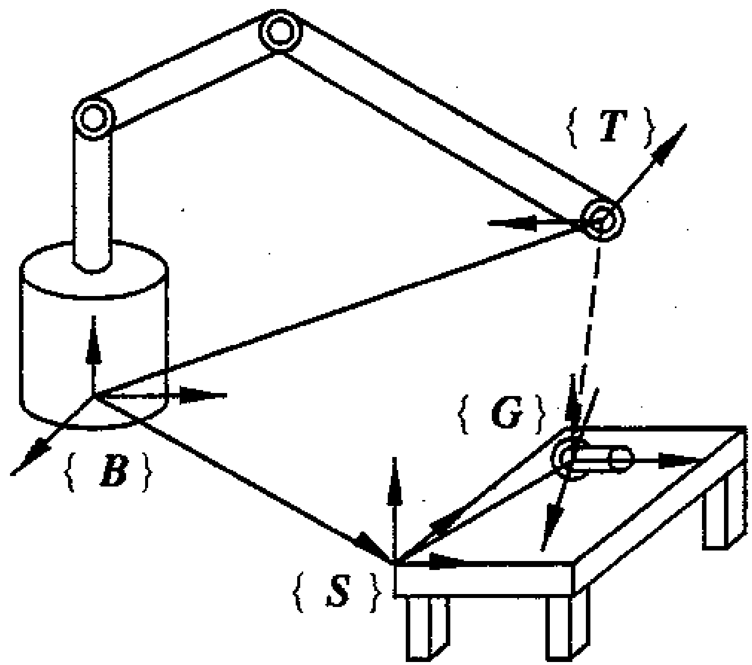
Example 2.8

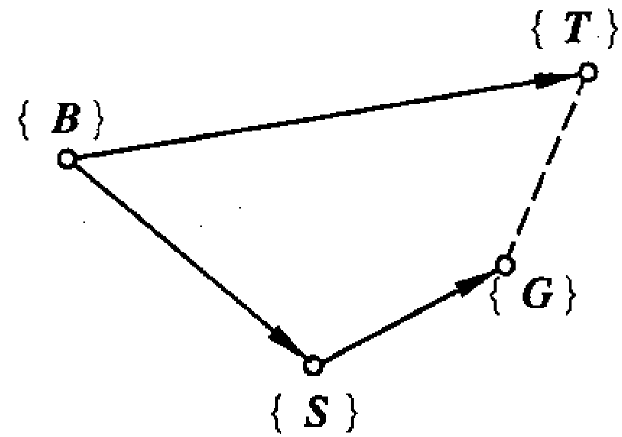
- Assume that we know the transform ${}^{B}_{T}T$, ${}^{B}_{S}T$, and ${}^{S}_{G}T$. Calculate the position and orientation of the bolt relative to the manipulator's hand, ${}^{T}_{G}T$.





Example 2.8





Solution:

$$ {}^{G}_{S}T \; {}^{S}_{B}T \; {}^{B}_{T}T \; {}^{T}_{G}T = \mathrm{I} $$

$$ \Rightarrow \; {}^{S}_{G}T^{-1} \; {}^{B}_{S}T^{-1} \; {}^{B}_{T}T \; {}^{T}_{G}T = \mathrm{I} $$

$$ \Rightarrow \; {}^{T}_{G}T = {}^{B}_{T}T^{-1} \; {}^{B}_{S}T \; {}^{S}_{G}T $$

Summary

- 位置、方向和坐标系描述
- 映射和运算符
  - 平移变换
  - 旋转变换
  - 一般变换
- 齐次变换
- 变换方程
  - 复合变换
  - 逆变换
- 机器人运动学、动力学和控制的数学基础。

谢 谢！