09/30/20

Lec 10

Proof by Contradiction

Idea: If we want to prove that a statment A is true then we can start by assuming that $\neg A$ is true then produce a contradiction so our original assumption is false. So A must be true.

Example:

1. Let's prove that there is no smallest integer

$s \in \mathbb{Z}$ is "smallest" If $\forall x \in \mathbb{Z}, s \le x$

Proof: Assume to the contrary that there is a smalest integer $s \in \mathbb{Z}$

So that $\forall x \in \mathbb{Z}, s \le x$.

But consider $x = s-1 \in \mathbb{Z}$

Then $s \le x \equiv s \le s-1$
$\equiv 0 \le -1$

This is a contradiction

So there is no smallest integer $\square$

2. Lema: $\forall a \in \mathbb{Z}$, If a^2 is even then a is even

Proof: Assume to the contrary that $\exists a \in \mathbb{Z}$ Such that a^2 is even and a is odd

Then $\exists k \in \mathbb{Z}$ such that $a = 2k+1$

$$a^2 = (2k+1)^2$$
$$a^2 = 4k^2 + 4k + 1$$
$$= 2\underbrace{(2k^2+2k)}_{\in \mathbb{Z}}+1$$

so a^2 is odd. Contradiction

So the Lemma is true

Proposition: $\sqrt{2}$ is irrational

Proof: Assume that $\sqrt{2}$ is rational

Then $\exists\ a, b \in \mathbb{Z}, ab \neq 0$ such that $\sqrt{2} = \frac{a}{b}$

We can assume WLOG that a an b are both positive

We can also assume that a and b are not both even. For otherwise,
If they're both even then $\exists\ c, d \in \mathbb{Z}$ such that $a = 2c$, $b = 2d$ w/ $c < a$, $d < b$

But then $\frac{a}{b} = \frac{2c}{2d} = \frac{c}{d}$ and repeat

Now, $\sqrt{2} = \frac{a}{b} \Rightarrow \sqrt{2}\,b = a$

$\Rightarrow 2b^2 = a^2$

$\Rightarrow a^2$ is even

$\Rightarrow a$ is even (by previous Lemma)

but if a is even then $a = 2k$ for some $k \in \mathbb{Z}$

$2b^2 = a^2 = (2k)^2 = 4k^2$

$\Rightarrow b^2 = 2k^2$

$\Rightarrow b^2$ is even

$\Rightarrow b$ is even (by previous Lemma)

So both a and b must be even.
This is a contradiction

So $\sqrt{2} \notin \mathbb{Q}$ $\square$