09/25/20

Lec 08

Disproving Existentially Quantified Statments

Examples:

1. Claim: $\exists x \in \mathbb{R} \quad \sin(x+1) + \cos(x^3 - x) + \sin(2x) = 5$

   Disproof: $\forall y \in \mathbb{R} \quad \sin y \leq 1$ and $\cos y \leq 1$

   So in praticular

   $$\sin(x+1) + \cos(x^3 - x) + \sin(2x) \leq 1+1+1 = 3$$

   $\therefore$ LHS $\neq 5 \quad \forall x \in \mathbb{R}$

2. Claim: $\exists n \in \mathbb{Z} \quad n^2 + 1 = 7$

   Disproof: Suppose that there is an $n \in \mathbb{Z}$ such that $n^2 + 1 = 7$

   Then $n^2 = 6$

   This is imposemble

   Since $2^2 = 4$ and $3^2 = 9$ then $2 < n < 3$ $\square$

Proving / Disproving Nested Statments

Examples:

1. Claim: $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2+y^2=1$

   Disproof: Take $x=5$ and $y=5$
   Then $5^2+5^2=50 \neq 1$ $\square$

2. Claim: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2+y^2=1$

   Disproof: Take $x=4$
   Then $x^2+y^2=16+y^2$ so if this is 1
   then $y^2=-15$. Which is imposible $\square$

3. Claim: $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2+y^2=1$

   Proof: Take $x=0$, $y=1$
   Then $0^2+1^2=1$ $\square$

4. Claim: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2+y^2=1$

   Disproof:

   Suppose $\exists x_0 \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$

   $x_0^2+y^2=0$

   In Praticular if $y=0$ then
   $x_0^2+0^2=1 \Leftrightarrow x_0^2=1$

   contradiction

   While if $y=1$ then
   $x_0^2+1^2=1 \Leftrightarrow x_0^2=0$

   $\square$

Proving Implications

" $A \Rightarrow B$ "

Strategy:

1. Assume A (the hypothoses)
2. < Argue >
3. Deduce B (the conclusion)

warning: DO NOT assume B (the conclusion)

Examples:

1. Claim: $\forall k \in \mathbb{Z}$, If k^3 is a perfect square then $9k^{11}$ is a perfect square

an integer a is a perfect square if $\exists b \in \mathbb{Z}$ such that $b^2 = a$

Proof: Let $k \in \mathbb{Z}$, Assume k^3 is a perfect square, So $\exists b \in \mathbb{Z}$ such that $b^2 = k^3$

Then
$$\begin{aligned} 9k^{11} &= 3^2 k^8 k^3 \\ &= 3^2 (k^4)^2 b^2 \\ &= (3k^4 b)^2 \end{aligned}$$
$(3k^4 b) \in \mathbb{Z}$

Since $9k^{11} = (\text{integer})^2$, $9k^{11}$ is a perfect square $\square$

2. Claim: If $n \in \mathbb{N}$ is even then so is n^2+n

an integer $a \in \mathbb{Z}$ is even if $\exists k \in \mathbb{Z}$ such that $a = 2k$

Proof: Let $n \in \mathbb{N}$, Assume n is even, so $\exists k \in \mathbb{Z}$ such that $n = 2k$

$$n^2 + n = (2k^2 + 2k) = 4k^2 + 2k = 2(2k^2 + k)$$

Since $2k^2 + k \in \mathbb{Z}$, $n^2 + n$ is even $\square$

3 Claim: If $n \in \mathbb{N}$ is odd then so is $2n^3 - n$

an integer $a \in \mathbb{Z}$ is odd if $\exists k \in \mathbb{Z}$ such that $a = 2k+1$

Proof. Let $n \in \mathbb{N}$ assume n is odd, so $\exists k \in \mathbb{Z}$ such that $n = 2k+1$

$$
\begin{aligned}
2n^3 - n &= 2(2k+1)^3 - (2k+1) \\
&= 2(8k^3 + 12k^2 + 6k + 1) - 2k - 1 \\
&= 2(8k^3 + 12k^2 + 5k + 1) - 1 \\
&= 2(8k^3 + 12k^2 + 5k + 1) - 1 + (2 - 2) \\
&= 2(8k^3 + 12k^2 + 5k \quad) + 1
\end{aligned}
$$

Since $8k^3 + 12k + 5k - 1 \in \mathbb{Z}$, $2n^3 - n$ is odd.