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Lec 09

Divisability

" $(m \mid n)$ "

Definition: for $m, n \in \mathbb{Z}$, we say m devides n, if $\exists k \in \mathbb{Z}$ such that $n = km$

Examples:

① Claim: $\forall a \in \mathbb{Z}$, $1 \mid a$

Proof: Let $a \in \mathbb{Z}$

Then $a = \underbrace{a}_{\in \mathbb{Z}} \cdot 1$ $\square$

② Claim: $\forall a \in \mathbb{Z}$, $-1 \mid a$

Proof: Let $a \in \mathbb{Z}$

Then $a = \underbrace{(-a)}_{\in \mathbb{Z}}(-1)$ $\square$

③ Claim: $0 \mid 0$

Proof: $0 = \underbrace{1}_{\in \mathbb{Z}} \cdot 0$ $\square$

Warning: " $a \mid b$ " is NOT the same as " $\frac{b}{a}$ "

It is OK to write $0 \mid 0$ but NOT OK to write $\frac{0}{0}$

④ Claim: $\forall a \in \mathbb{Z}$, if $0 \mid a$ then $a = 0$

Proof: Let $a \in \mathbb{Z}$ and assume $0 \mid a$
Then $\exists k \in \mathbb{Z}$ such that $a = k \cdot 0 = 0$ $\square$

5 Proposition: Transitivity of Divisability (TD)
$\forall a,b,c \in \mathbb{Z}$ If $a|b$ and $b|c$ then $a|c$

Proof: Let $a,b,c \in \mathbb{Z}$ and assume that $a|b$ and $b|c$

Then $\exists k \in \mathbb{Z}$ such that $b = ka$ and $\exists l = \mathbb{Z}$ such that $c = lb$

Since,
$$c = lb$$
$$c = l(ka)$$
$$c = \underbrace{(lk)}_{\in \mathbb{Z}} a \quad \square$$

6 Claim 1: $\forall a,b,c \in \mathbb{Z}$, If $a|b$ and $a|c$ then $a|bc$

Proof: Let $a,b,c \in \mathbb{Z}$ and assume $a|b$ and $a|c$

Then $\exists k, l \in \mathbb{Z}$ such that $b = ka$ and $c = la$

So,
$$bc = ka(la)$$
$$bc = \underbrace{(kla)}_{\in \mathbb{Z}} a \quad \square$$

Claim 2: The Converse, $\forall a,b,c \in \mathbb{Z}$ If $a|bc$ then $a|b$ and $a|c$

Disproof: Take $a=2$, $b=3$, $c=4$

Then $a|bc$ (because $2|12$) but $a \nmid b$

7 Proposition: **Divisibility of Integer Combinations**

$\forall a,b,c \in \mathbb{Z}$ If $a|b$ and $a|c$ then ($\forall x,y \in \mathbb{Z}$ $a | bx + cy$

Proof: Let $a,b,c \in \mathbb{Z}$ and assume $a|b$ and $a|c$

Then $\exists k, l \in \mathbb{Z}$ such that $b = ka$ and $c = la$

Let $x, y \in \mathbb{Z}$, Consider $bx + cy$

$$\begin{aligned} bx + cy &= kax + lay \\ &= \underbrace{(kx + ly)}_{\in \mathbb{Z}} a \end{aligned} \quad \square$$

Proposition: **The Converse**

$\forall a,b,c \in \mathbb{Z}$, if $a | bx + cy$ $\forall x,y \in \mathbb{Z}$ then $a|b$ and $a|c$

Proof: Let $a,b,c \in \mathbb{Z}$, assume $a | bx + cy$ $\forall x,y \in \mathbb{Z}$

In praticular, If $x = 1$ and $y = 0$,

then $a | b\cdot 1 + c \cdot 0 \Leftrightarrow a|b$

and similarly, If $x = 0$ and $y = 1$,

then $a | b \cdot 0 + c \cdot 1 \Leftrightarrow a|c$

The following modification is False

$\forall a,b,c,x,y \in \mathbb{Z}$, If $a | bx + cy$, then $a|b$ and $a|c$

Disproof: Take $a = 2$, $x = y = 0$, $b = 3$, $c = 2$

Then $a | bx + cy$ but $a \nmid b$

Proof by Contropositive

Since an implication is logicaly equivilant to its contropositive, to prove the implication we can instead prove the contropositive

$A \Rightarrow B$ contropositive $\neg B \Rightarrow \neg A$

Examples:

1. Claim: $\forall x \in \mathbb{Z}$ If x^2-6x+8 is odd then x is odd

$\equiv \forall x \in \mathbb{Z}$, If NOT ($x$ is odd) then NOT (x^2-6x+8)

$\equiv \forall x \in \mathbb{Z}$, If x is even then x^2-6x+8 is even

Proof: Let $x \in \mathbb{Z}$ and assume that x is even

Then $\exists K \in \mathbb{Z}$ $x = 2K$

So
$$x^2-6x+8 = (2K)^2 - 6(2K) + 8 = 4K^2 - 12K + 8 = 2\underbrace{(2K^2-6K+4)}_{\in \mathbb{Z}}$$

So x^2-6x+8 is even $\square$

(2) Claim: $\forall x \in \mathbb{R}$ If $x^2 - x - 2 > 0$ then $x > 2$ or $x < -1$

$\equiv \forall x \in \mathbb{R}$, If $x \leq 2$ and $x \geq -1$ then $x^2 - x - 2 \leq 0$

Proof: Let $x \in \mathbb{R}$ assume $-1 \leq x \leq 2$

Then $x^2 - x + 2 = (x-2)(x+1)$
Now $-1 \leq x \Leftrightarrow x + 1 \geq 0$
and $x \leq 2 \Leftrightarrow x - 2 \leq 0$

So $x^2 - x + 2 = \underbrace{(x-2)}_{\leq 0}\underbrace{(x+1)}_{\geq 0} \leq 0$ $\square$

Alternative Proof: We can prove the statment

$\forall x \in \mathbb{R}$, If $x^2 - x - 2 > 0$ then $x > 2$ or $x < -1$

Let $x \in \mathbb{R}$ and assume $x^2 - x - 2 > 0$ consider two cases: If $x > 2$ or if $x \leq 2$

In the first case we're done!

In the second case we must prove that $x < -1$

So let's assume $x^2 - x - 2 > 0$ and $x \leq 2$

Then $\underbrace{x^2 - x - 2}_{<0} = \underbrace{(x-2)}_{\leq 0}\underbrace{(x+1)}_{\text{must be } < 0}$

$\equiv x + 1 < 0 \equiv x < -1$ $\square$

Observation: The previous proof Used the equivalence

$$A \Rightarrow B \vee C \equiv (A \wedge \neg B) \Rightarrow C$$

This is sometimes called "proof by elimination"