MATH130165h: Homework 1

Due Mar 14, 2025

Problem 1. [10 pt] For $x \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$, verify the following inequalities and give examples of a nonzero vector or matrix for which equality is achieved.

(a) $\|x\|_\infty \le \|x\|_2 \le \sqrt{m}\,\|x\|_\infty$;

(b) $\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\,\|A\|_\infty$;

(c) Show the equivalence between induced matrix norms $\|\cdot\|_p$ and $\|\cdot\|_q$ for $1 \le p < q \le \infty$.

Problem 2. [10 pt] For each of the following statement, find an example matrix.

(a) a matrix whose singular values are the same as its eigenvalues;

(b) a matrix whose singular values are the same as the absolute values of its eigenvalues (the matrix has at least one negative eigenvalue);

(c) a matrix whose singular values are not the same as the absolute values of its eigenvalues;

Theorem I. Given a matrix $A \in \mathbb{R}^{m\times n}$. The singular values of A are denoted as $\{\sigma_i\}_{i=1}^{\min(m,n)}$ with their corresponding left and right singular vectors being $\{u_i\}_{i=1}^{\min(m,n)}$ and $\{v_i\}_{i=1}^{\min(m,n)}$. For any r with $0 \le r \le \min(m,n)$, the matrix A_r as defined,

$$A_r = \sum_{i=1}^{r} u_i \sigma_i v_i^*,$$

satisfies

$$\|A - A_r\|_{\mathrm{F}} = \inf_{\substack{B\in\mathbb{C}^{m\times n}\\ \mathrm{rank}(B)\le r}} \|A - B\|_{\mathrm{F}}.$$

Problem 3. [20 pt] Prove the Theorem I.

Problem 4. [15 pt] Let $P \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|P\|_2 \ge 1$, with equality if and only if P is an orthogonal projector.

Problem 5. [20 pt] Implement a matrix-matrix multiplication algorithm in C++.

Problem 6. [10 pt] Call "dgemm" function in LAPACK for a matrix-matrix multiplication in C++.

Problem 7. [10 pt] Apply your code in Problem 5 and Problem 6 to two random matrices of size 5000×5000 and compare their runtime.