Linear Algebra Review
Part II

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Inverse

- The inverse of matrix $\boldsymbol{A}$ is denoted as $\boldsymbol{A}^{-1}$.
- If matrix $\boldsymbol{A}$ has an inverse, then it should be full rank or in other words, its determinant is non-zero.
- $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
- $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
- $(\boldsymbol{A}^{-1})^{T} = (\boldsymbol{A}^{T})^{-1}$

Trace

- The trace of a <u>square matrix</u> $\boldsymbol{A}$ is sum of the diagonal elements.
- $tr\boldsymbol{A} = tr\boldsymbol{A}^T$
- $tr(\boldsymbol{A} + \boldsymbol{B}) = tr\boldsymbol{A} + tr\boldsymbol{B}$
- $tr(t\boldsymbol{A}) = t\ tr\boldsymbol{A}$
- For matrices $\boldsymbol{A}, \boldsymbol{B}$ such that $\boldsymbol{AB}$ is a square matrix, then
$$tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$$
- For matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ such that $\boldsymbol{ABC}$ is a square matrix, then
$$tr(\boldsymbol{ABC}) = tr(\boldsymbol{BCA}) = tr(\boldsymbol{CAB})$$

Linearly Independent and Dependent

- For a set of vectors $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}\}$, they are said to be <u>linearly independent</u> if no vector can be represented as a linear combination of the remaining vectors.
- Conversely, if one vector belonging to the set can be represented as a linear combination of the remaining vectors, then the vectors are said to be <u>linearly dependent</u>. In this case,

$$\boldsymbol{x_n} = a_1\boldsymbol{x_1} + a_2\boldsymbol{x_2} + \cdots + a_{n-1}\boldsymbol{x_{n-1}}$$

Rank

- The column rank of matrix $\boldsymbol{A}$ is the size of the largest subset of columns of $\boldsymbol{A}$ that constitute a linearly independent set. Likewise, the row rank of matrix $\boldsymbol{A}$ is the size of the largest subset of columns of $\boldsymbol{A}$ that constitute a linearly independent set.
- The column rank of a matrix is equal to the row rank of a matrix, so their value is called the rank of a matrix.

Rank

- For an $n \times m$ matrix $\boldsymbol{A}$, $rank(\boldsymbol{A}) \leq \min(n, m)$ and if $rank(\boldsymbol{A}) = \min(n, m)$, then matrix $\boldsymbol{A}$ is said to be <u>full rank</u>.
- $rank(\boldsymbol{A}) = rank(\boldsymbol{A}^T)$
- $rank(\boldsymbol{A} + \boldsymbol{B}) \leq rank(\boldsymbol{A}) + rank(\boldsymbol{B})$
- For an $n \times m$ matrix $\boldsymbol{A}$ and a $m \times p$ matrix $\boldsymbol{B}$,

$$rank(\boldsymbol{AB}) \leq \min(rank(\boldsymbol{A}), rank(\boldsymbol{B}))$$

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Homogeneous Linear Equation

- $\boldsymbol{A}\,\boldsymbol{X} = \boldsymbol{0}$ is called homogeneous linear equation.
- Steps of determinate the structure of the solution:

  1. Calculate the determinant of $\boldsymbol{A}$

  2. If the result is non-zero, then there is only zero solution;

     otherwise, there is an infinite number of non-zero solutions.

Non-homogeneous Linear Equation

- $\boldsymbol{A}\,\boldsymbol{X} = \boldsymbol{b}$ (where $\boldsymbol{b}$ is not a zero-vector) is called <u>non-homogeneous linear equation</u>.
- Steps of determinate the structure of the solution:

  1. Calculate the rank of $\boldsymbol{A}$

  2. Calculate the rank of $\boldsymbol{A}$'s augmented matrix (putting vector $\boldsymbol{b}$ to the right of $\boldsymbol{A}$ to form this matrix), named $\boldsymbol{B}$

Non-homogeneous Linear Equation

- There are three cases:
- If $rank(\boldsymbol{A}) = rank(\boldsymbol{B}) = n$, then there is only one solution.
- If $rank(\boldsymbol{A}) = rank(\boldsymbol{B}) < n$, then there is an infinite number of solutions.
- If $rank(\boldsymbol{A}) < rank(\boldsymbol{B})$, then there is no solution.

Thanks for listening!