Linear Algebra Review
Part I

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Notation

$$3\,x_1 + x_2 = 4$$
$$x_1 + 2\,x_2 = 3$$

- We can write it more compactly as:

$$\boldsymbol{A}\,\boldsymbol{X} = \boldsymbol{b}$$

where $\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Notation

- Vector: $\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
- Matrix: $\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_m} \end{bmatrix}$

Identity Matrix and Diagonal Matrix

- $\boldsymbol{I} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ (3 * 3 Example)
- $\boldsymbol{AI} = \boldsymbol{A} = \boldsymbol{IA}$

- $\boldsymbol{D} = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}$

Transpose

- Vector: $\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \boldsymbol{a}^T = [a_1 \quad \dots \quad a_n]$
- Matrix: $\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_m} \end{bmatrix},$

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{mn} \end{bmatrix} = [\boldsymbol{a}_1^T \quad \dots \quad \boldsymbol{a}_m^T]$$

Vector Multiplication

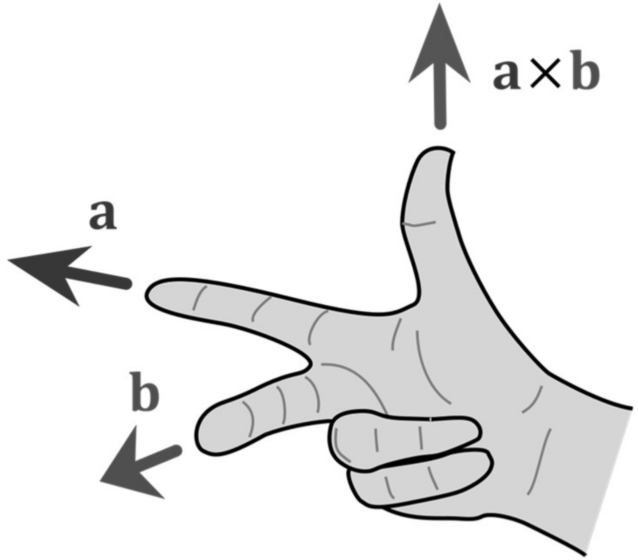
- Inner Product (Dot Product):

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \boldsymbol{a}^T\boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

- Outer Product:

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \boldsymbol{a}\boldsymbol{b}^T = \begin{bmatrix} a_1 b_1 & \cdots & a_1 b_n \\ \vdots & \ddots & \vdots \\ a_n b_1 & \cdots & a_n b_n \end{bmatrix}$$

Vector Multiplication



- Cross Product: (Example: 3D coordinate)

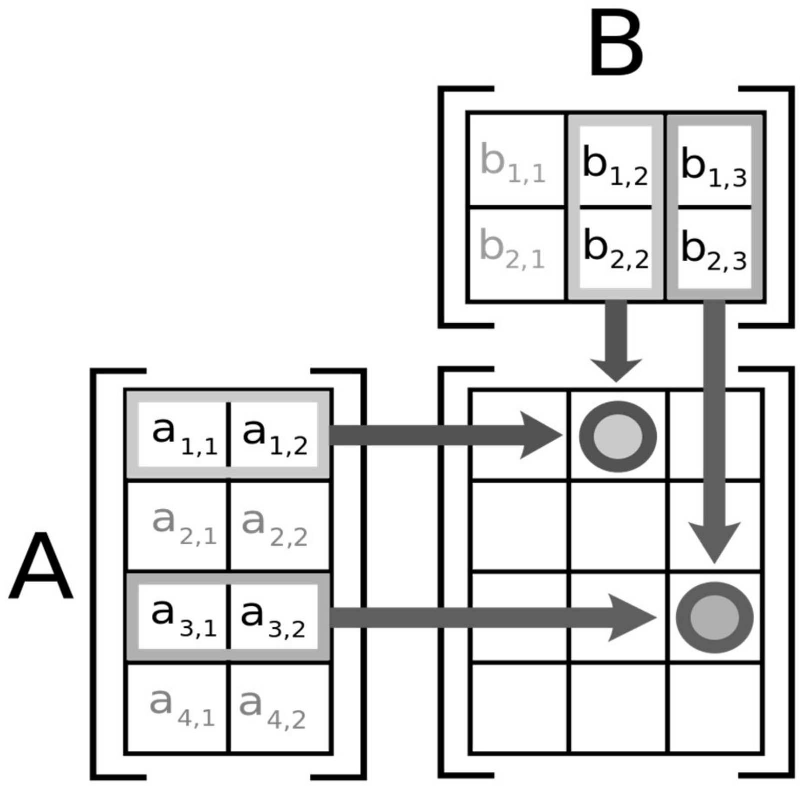
$$\boldsymbol{a} = (a_1, a_2, a_3), \boldsymbol{b} = (b_1, b_2, b_3),$$

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 \ - a_3 b_2, a_3 b_1 \ - a_1 b_3, a_1 b_2 \ - b_1 a_2)$$

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||b| \ \sin < \boldsymbol{a}, \boldsymbol{b} >$$

Matrix Multiplication

- If the size of matrix $\boldsymbol{A}$ is $n \times m$, and the size of matrix $\boldsymbol{B}$ is $m \times p$, then the size of the result of $\boldsymbol{AB}$ is $n \times p$. (If the width of $\boldsymbol{A}$ is not matched with the length of $\boldsymbol{B}$, then they can't be multiplied together.)



Matrix Multiplication

- Associative: $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
- Distributive: $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$
- BUT not commutive: $\boldsymbol{AB} \neq \boldsymbol{BA}$

Matrix Transpose (Ext.)

- $(\boldsymbol{A}^T)^T = \boldsymbol{A}$
- $(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$
- $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$
- If $\boldsymbol{A}^T = \boldsymbol{A}$, then we will say that matrix $\boldsymbol{A}$ is a <u>symmetric matrix</u>.

Determinant

$$\begin{aligned}
|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b\begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c\begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix} \\
&= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= aei + bfg + cdh - ceg - bdi - afh.
\end{aligned}$$

Determinant

$$\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & 2 & 8 \end{vmatrix} = 2\begin{vmatrix} 5 & 7 \\ 2 & 8 \end{vmatrix} - 3\begin{vmatrix} 6 & 7 \\ 1 & 8 \end{vmatrix} + 4\begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix}$$

$$= 2(40-14) - 3(48-7) + 4(12-5)$$

$$= 52 - 123 + 28$$

$$= -43$$

Determinant

- $\det(\boldsymbol{A}^T) = \det(\boldsymbol{A})$, $\det(\boldsymbol{I}) = 1$
- $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B})$ if $\boldsymbol{A}, \boldsymbol{B}$ are square matrix with the equal size
- $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$ if $\boldsymbol{A}$ is an $n \times n$ matrix
- If $\det(\boldsymbol{A}) = 0$, then we say that $\boldsymbol{A}$ is a singular matrix.

Thanks for listening!