For this problem set, you may find it useful to consult Ken Rosen's textbook *Discrete Math and Its Applications*.

1. Give the contrapositive of the following statement. "If every bird flies, then there is a hungry cat."

   Answer:

   - If there is not a hungry cat, then no bird flies.

2. A proposition is a statement that can be true or false but not both. Let A, B, and C be propositions. Let $\land$ denote logical AND, let $\lor$ denote logical OR, and let $\neg$ denote logical NOT. Argue that if $(A \lor B) \land (\neg B \lor C)$ is true, then $(A \lor C)$ must be true as well.

   Answer:
   - $(A \lor B) \land (\neg B \lor C) \Rightarrow (A \lor C)$ is true if all possible values in its domain map to "True" in its range.
   - The table below maps each possibility of A, B and C through each of the terms of the expression $(A \lor B) \land (\neg B \lor C) \Rightarrow (A \lor C)$
   - In each case the resulting value is true
   - Therefore $(A \lor B) \land (\neg B \lor C) \Rightarrow (A \lor C)$, or if $(A \lor B) \land (\neg B \lor C)$ is true, then $(A \lor C)$ must be true as well.

A	B	$\neg B$	C	$(A \lor B)$	$(\neg B \lor C)$	$(A \lor B) \land (\neg B \lor C)$	$(A \lor C)$	$(A \lor B) \land (\neg B \lor C) \Rightarrow (A \lor C)$
T	T	F	T	T	T	T	T	T
T	T	F	F	T	F	F	T	T
T	F	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	T	F	F	T	F	F	F	T
F	F	T	T	F	T	F	T	T
F	F	T	F	F	T	F	F	T

3. We use the notation $\mathrm{A} \Rightarrow \mathrm{B}$ to indicate that A implies B. This new proposition $\mathrm{A} \Rightarrow \mathrm{B}$ is true except when A is true and B is false. We write $\mathrm{A} \Leftrightarrow \mathrm{B}$ when either both A and B are true or both are false. Argue that $\mathrm{A} \Leftrightarrow \mathrm{B}$ if and only if $\mathrm{A} \Rightarrow \mathrm{B}$ and $\mathrm{B} \Rightarrow \mathrm{A}$.

   Answer:
   - Two expressions are equivalent if they map to the same values in a range for any values in their domain
   - All possible values of the domain are given in the truth table
   - If the values in the range of each expression match for every possible combination of values in the domain, they are equivalent
   - $\mathrm{A} \Rightarrow \mathrm{B}$ has the truth table shown below, demonstrating that $\mathrm{A} \Rightarrow \mathrm{B}$ is true except when A is true and B is false.
   - Similarly $\mathrm{B} \Rightarrow \mathrm{A}$ is true except when B is true and A is false (see table)
   - $\mathrm{A} \Leftrightarrow \mathrm{B}$ is defined to be true when either both A and B are true or both are false (see table).
   - $\mathrm{A} \Rightarrow \mathrm{B}$ and $\mathrm{B} \Rightarrow \mathrm{A}$ can be represented as $(\mathrm{A} \Rightarrow \mathrm{B}) \wedge (\mathrm{B} \Rightarrow \mathrm{A})$ and is also true only when either both A and B are true or both are false (see table).
   - Hence $\mathrm{A} \Leftrightarrow \mathrm{B}$ if and only if $\mathrm{A} \Rightarrow \mathrm{B}$ and $\mathrm{B} \Rightarrow \mathrm{A}$.

A	B	$\mathrm{A} \Rightarrow \mathrm{B}$	$\mathrm{B} \Rightarrow \mathrm{A}$	$(\mathrm{A} \Rightarrow \mathrm{B}) \wedge (\mathrm{B} \Rightarrow \mathrm{A})$	$\mathrm{A} \Leftrightarrow \mathrm{B}$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

4. We will use the notation | · | to indicate the number of elements in the set or its *cardinality*, e.g. $|A|$ is the number of elements in the set A. Consider four sets A,B,C,D such that the intersection of any three is empty. Use the inclusion-exclusion to give an expression for $|A \cup B \cup C \cup D|$ without using any union (∪) symbols.

   Answer:

   $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| - |A \cap B \cap C \cap D|$

5. State the formal definition of $O(n)$, and show that the function $f(n) = (n^4 + n^2 - 9)/(n^3 + 1)$ is $O(n)$.

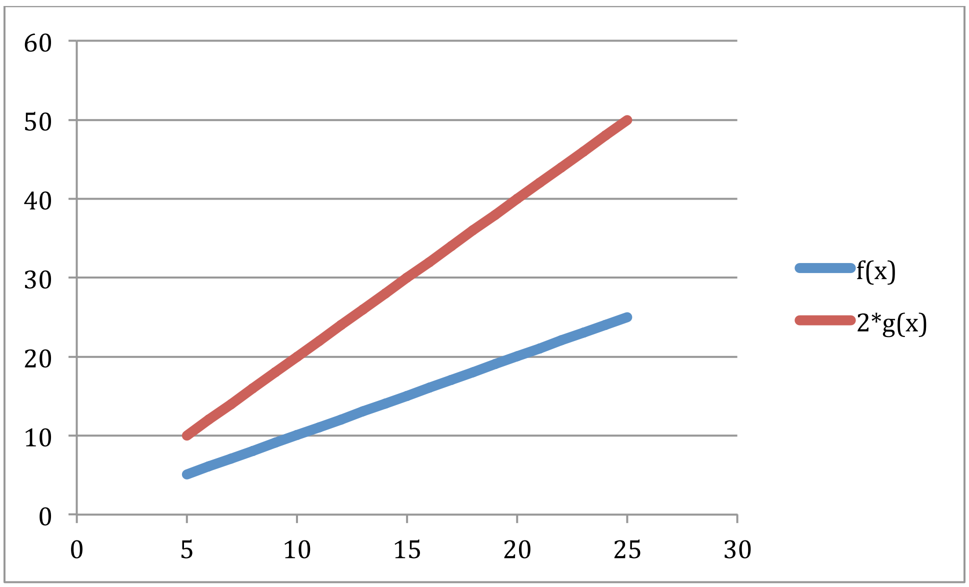
Answer:

Definition: Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f (x) is O(g(x)) if there are constants C and k such that

$|f(x)| \leq C|g(x)|$

whenever x > k. [Source: Rosen, *Discrete Mathematics and Its Applications, 7e*, pp. 205]

- Let C = 2, k = 5, and g(x) = n
- $|f(x)| \leq 2|g(x)|$ for x > 5 and diverging more as x grows (see graph)



- Further, the limit as n approaches infinity of f(x) = n

6. Let A be a set. We use the notation $\mathrm{P(A)}$ to indicate the power set of A, which consist of all subsets of A. For example, if $\mathrm{A} = \{0,1\}$, then $\mathrm{P(A)} = \{\{\},\{0\},\{1\},\{0,1\}\}$. Consider $\mathrm{Q(n)} = \mathrm{P}(\{1, \ldots \mathrm{n}\}) - \{\{\}\}$ and use an inductive argument to show that

$$\sum_{\{a_1,\ldots a_k\} \in Q(n)} \frac{1}{a_1 \ldots a_k} = n$$

for all positive integers n.

Answer:

- The following proof is by induction.
- **Basis**: First demonstrate P(1) = 1:

$$P(1) = \sum_{\{a_1\} \in Q(1)} \frac{1}{1} = 1$$

- **Induction step**: Then assume $P(k)$ and show that it is true for $P(k+1)$:

$$P(k) = \sum_{\{a_1,\ldots a_k\} \in Q(k)} \frac{1}{a_1 \ldots a_k} = k$$

$$P(k+1) = \sum_{\{a_1,\ldots a_{k+1}\} \in Q(k)} \frac{1}{a_1 \ldots a_{k+1}}$$

$$P(k+1) = \left(\sum_{\{a_1,\ldots a_k\} \in Q(k)} \frac{1}{a_1 \ldots a_k} \right) + \left(\frac{1}{\mathrm{k}+1} \right) + \left(\frac{1}{\mathrm{k}+1} \right) \left(\sum_{\{a_1,\ldots a_k\} \in Q(k)} \frac{1}{a_1 \ldots a_k} \right)$$

$$P(k+1) = P(k) + \left(\frac{1}{\mathrm{k}+1} \right) + \left(\frac{1}{\mathrm{k}+1} \right) P(k)$$

$$P(k+1) = P(k) + \left(\frac{P(k)+1}{\mathrm{k}+1} \right)$$

$$P(k+1) = k + \left(\frac{k+1}{\mathrm{k}+1} \right)$$

$$P(k+1) = k+1$$

- Thus the formula is correct for $n = k+1$ which proves the result.

7. Prove that the set of all languages that have a bounded maximum string length is countable.

Answer:

- Define language L, $L(w) = \{w \in \Sigma^k;\ where\ k\ is\ some\ positive\ integer\}$
- Let S_i be the set of all strings of length $i, i \leq k$
- Each S_i is countable, in fact the cardinality of the set S_i is the number of symbols in the alphabet raised to the kth power
- Let S^{Total} = the union of all S_i bounded by length k:
  - $\mathrm{S}^{total} = \bigcup_{i=1}^{k} S_i$
- The union of countable sets is countable (from class proof)
- Therefore the set of all languages that have a bounded maximum string length is countable