

For this problem set, you may find it useful to consult Ken Rosen's textbook *Discrete Math and Its Applications*.

1. Give the contrapositive of the following statement. “If every bird flies, then there is a hungry cat.”

Answer:

- If there is not a hungry cat, then not every bird flies.
2. A proposition is a statement that can be true or false but not both. Let A, B, and C be propositions. Let \wedge denote logical AND, let \vee denote logical OR, and let \neg denote logical NOT. Argue that if $(A \vee B) \wedge (\neg B \vee C)$ is true, then $(A \vee C)$ must be true as well.

Answer:

- $(A \vee B) \wedge (\neg B \vee C) \Rightarrow (A \vee C)$ is true if all possible values in its domain map to “True” in its range.
- The table below maps each possibility of A, B and C through each of the terms of the expression $(A \vee B) \wedge (\neg B \vee C) \Rightarrow (A \vee C)$
- In each case the resulting value is true
- Therefore $(A \vee B) \wedge (\neg B \vee C) \Rightarrow (A \vee C)$, or if $(A \vee B) \wedge (\neg B \vee C)$ is true, then $(A \vee C)$ must be true as well.

A	B	$\neg B$	C	$(A \vee B)$	$(\neg B \vee C)$	$(A \vee B) \wedge (\neg B \vee C)$	$(A \vee C)$	$(A \vee B) \wedge (\neg B \vee C) \Rightarrow (A \vee C)$
T	T	F	T	T	T	T	T	T
T	T	F	F	T	F	F	T	T
T	F	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	T	F	F	T	F	F	F	T
F	F	T	T	F	T	F	T	T
F	F	T	F	F	T	F	F	T

3. We use the notation $A \Rightarrow B$ to indicate that A implies B . This new proposition $A \Rightarrow B$ is true except when A is true and B is false. We write $A \Leftrightarrow B$ when either both A and B are true or both are false. Argue that $A \Leftrightarrow B$ if and only if $A \Rightarrow B$ and $B \Rightarrow A$.

Answer:

- Two expressions are equivalent if they map to the same values in a range for any values in their domain
- All possible values of the domain are given in the truth table
- If the values in the range of each expression match for every possible combination of values in the domain, they are equivalent
- $A \Rightarrow B$ has the truth table shown below, demonstrating that $A \Rightarrow B$ is true except when A is true and B is false.
- Similarly $B \Rightarrow A$ is true except when B is true and A is false (see table)
- $A \Leftrightarrow B$ is defined to be true when either both A and B are true or both are false (see table).
- $A \Rightarrow B$ and $B \Rightarrow A$ can be represented as $(A \Rightarrow B) \wedge (B \Rightarrow A)$ and is also true only when either both A and B are true or both are false (see table).
- Hence $A \Leftrightarrow B$ if and only if $A \Rightarrow B$ and $B \Rightarrow A$.

A	B	$A \Rightarrow B$	$B \Rightarrow A$	$(A \Rightarrow B) \wedge (B \Rightarrow A)$	$A \Leftrightarrow B$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

4. We will use the notation $|\cdot|$ to indicate the number of elements in the set or its *cardinality*, e.g. $|A|$ is the number of elements in the set A . Consider four sets A, B, C, D such that the intersection of any three is empty. Use the inclusion-exclusion to give an expression for $|A \cup B \cup C \cup D|$ without using any union (\cup) symbols.

Answer:

$$\begin{aligned}
 |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| \\
 &\quad - |C \cap D|
 \end{aligned}$$

- Since the intersection of any three sets is empty, all terms with greater than three elements may be dropped as empty

5. State the formal definition of $O(n)$, and show that the function $f(n) = \frac{n^4 + n^2 - 9}{n^3 + 1}$ is $O(n)$.

Answer:

Definition: Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$. [Source: Rosen, *Discrete Mathematics and Its Applications*, 7e, pp. 205]

- Assuming $n > 1$, we can drop the absolute values to find a C such that:

$$\frac{f(n)}{g(n)} \leq C \frac{f(n)}{g(n)} = C$$

- First:

$$f(n) = \frac{n^4 + n^2 - 9}{n^3 + 1} = \frac{n^4}{(n^3 + 1)} + \frac{n^2}{(n^3 + 1)} - \frac{9}{(n^3 + 1)}$$

- and $n > 1$ implies that:

$$n > \frac{n^4}{(n^3 + 1)}, n > \frac{n^2}{(n^3 + 1)}, \text{ and } n > -\frac{9}{(n^3 + 1)}$$

- Choose $k = 1$ and “increase” the elements of $f(n)$ to simplify:

$$\frac{f(n)}{g(n)} = \frac{\frac{n^4}{(n^3 + 1)} + \frac{n^2}{(n^3 + 1)} - \frac{9}{(n^3 + 1)}}{n} \leq \frac{n + n + n}{n} = 3$$

- Thus the function $f(n) = (n^4 + n^2 - 9)/(n^3 + 1)$ is $O(n)$ because $\frac{n^4 + n^2 - 9}{n^3 + 1} \leq 3n$, whenever $n > 1$

6. Let A be a set. We use the notation $P(A)$ to indicate the power set of A, which consist of all subsets of A. For example, if $A = \{0,1\}$, then $P(A) = \{\{\}, \{0\}, \{1\}, \{0,1\}\}$. Consider $Q(n) = P(\{1, \dots, n\}) - \{\{\}\}$ and use an inductive argument to show that

$$\sum_{\{a_1, \dots, a_k\} \in Q(n)} \frac{1}{a_1 \dots a_k} = n$$

for all positive integers n.

Answer:

- The following proof is by induction.
- **Basis:** First demonstrate $P(1) = 1$:

$$P(1) = \sum_{\{a_1\} \in Q(1)} \frac{1}{1} = 1$$

- **Induction step:** Then assume $P(k)$ and show that it is true for $P(k + 1)$:

$$P(k) = \sum_{\{a_1, \dots, a_k\} \in Q(k)} \frac{1}{a_1 \dots a_k} = k$$

$$P(k + 1) = \sum_{\{a_1, \dots, a_{k+1}\} \in Q(k)} \frac{1}{a_1 \dots a_{k+1}}$$

$$P(k + 1) = \left(\sum_{\{a_1, \dots, a_k\} \in Q(k)} \frac{1}{a_1 \dots a_k} \right) + \left(\frac{1}{k+1} \right) + \left(\frac{1}{k+1} \right) \left(\sum_{\{a_1, \dots, a_k\} \in Q(k)} \frac{1}{a_1 \dots a_k} \right)$$

$$P(k + 1) = P(k) + \left(\frac{1}{k+1} \right) + \left(\frac{1}{k+1} \right) P(k)$$

$$P(k + 1) = P(k) + \left(\frac{P(k) + 1}{k+1} \right)$$

$$P(k + 1) = k + \left(\frac{k + 1}{k+1} \right)$$

$$P(k + 1) = k + 1$$

- Thus the formula is correct for $n = k + 1$ which proves the result.

7. Prove that the set of all languages over $\{0,1\}$ that have a bounded maximum string length is countable.

Answer:

- Let $\Sigma = \{0,1\}$
- Define language L , $L(w) = \{w \in \Sigma^k; \text{ where } k \text{ is some positive integer}\}$
- Let S_i be the set of all strings of length i , $i \leq k$
- Any of these S_i is countable because there exists a bijection of the positive integers onto the elements of the set. There are, however, infinitely many of these sets.
- To address this we use the following procedure: By putting each S_i in size order across the top of a grid, and successive natural numbers down the left side of the grid, we view a mapping/bijection from 1, to S_0 , 2 to S_1 , and so on: anti-diagonal mappings starting at each element in the left of the array, going up-right.
- Each diagonal set has a finite number of elements (and is thus countable), so each element in each set will be covered by the anti-diagonalization.
- Thus, there will exist a bijection of positive integers onto these diagonals (and in turn their elements), meaning that the countable (yet not finite) number of countable sets is also countable as a whole.
- Therefore the set of all languages over $\{0,1\}$ that have a bounded maximum string length is countable