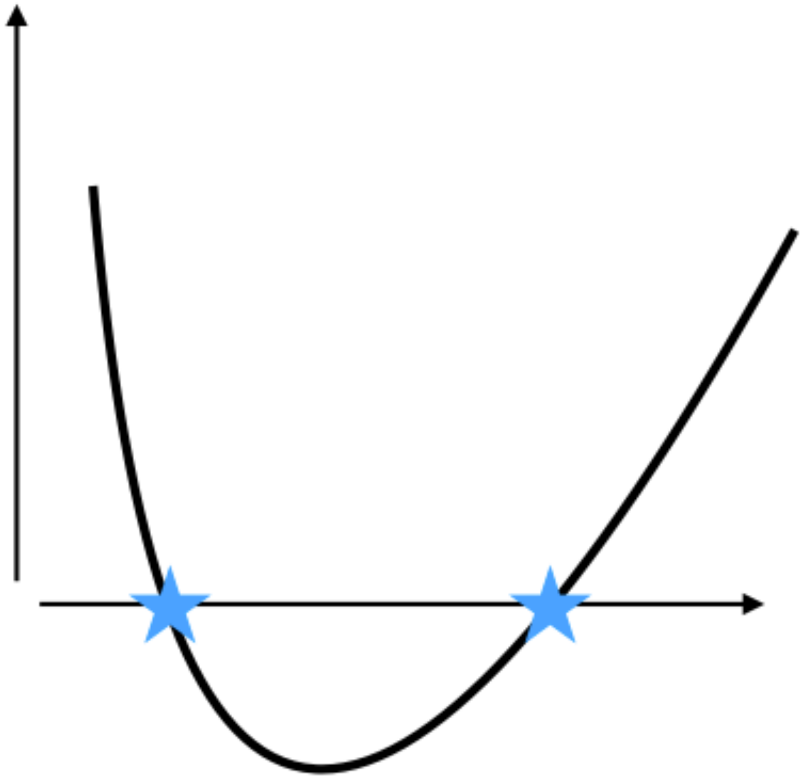
Root Finding



Trial and Error

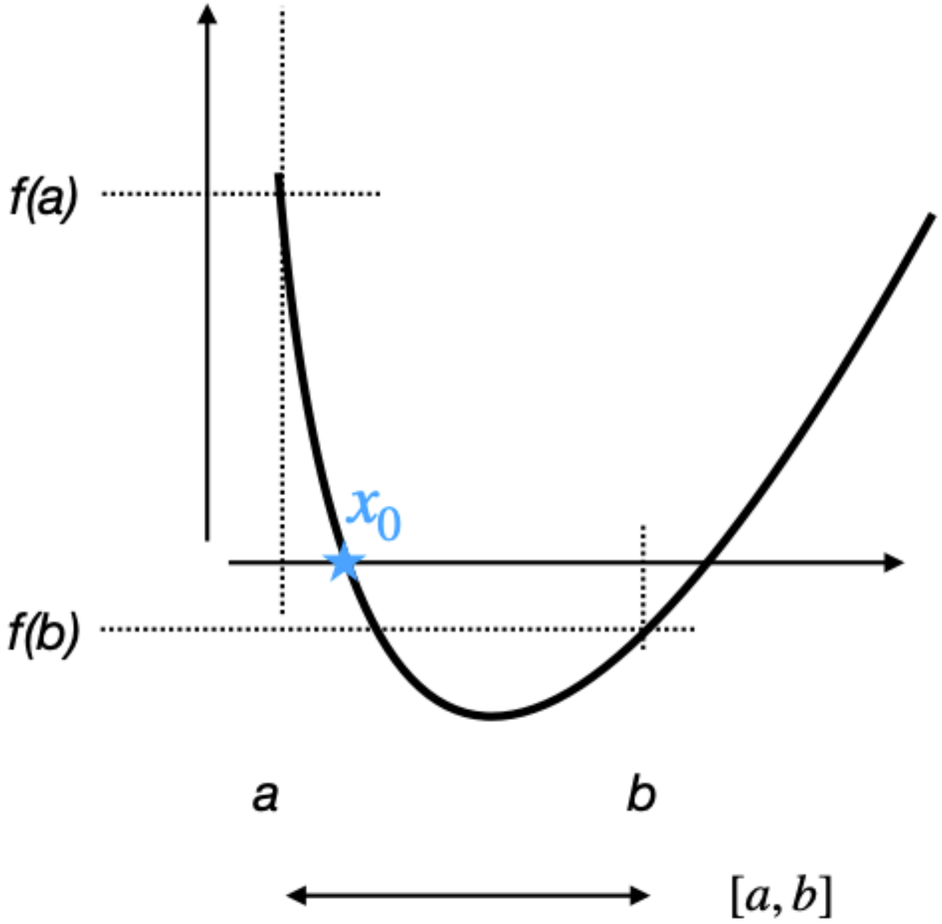
Start with x_0 (initial guess)

1. **Guess x_1 (trial)**
2. **Is $|x_1 - x_0| < \varepsilon$? (error)**
3. **Improve x_1**

iterate until

$|x_1 - x_0| < \varepsilon$

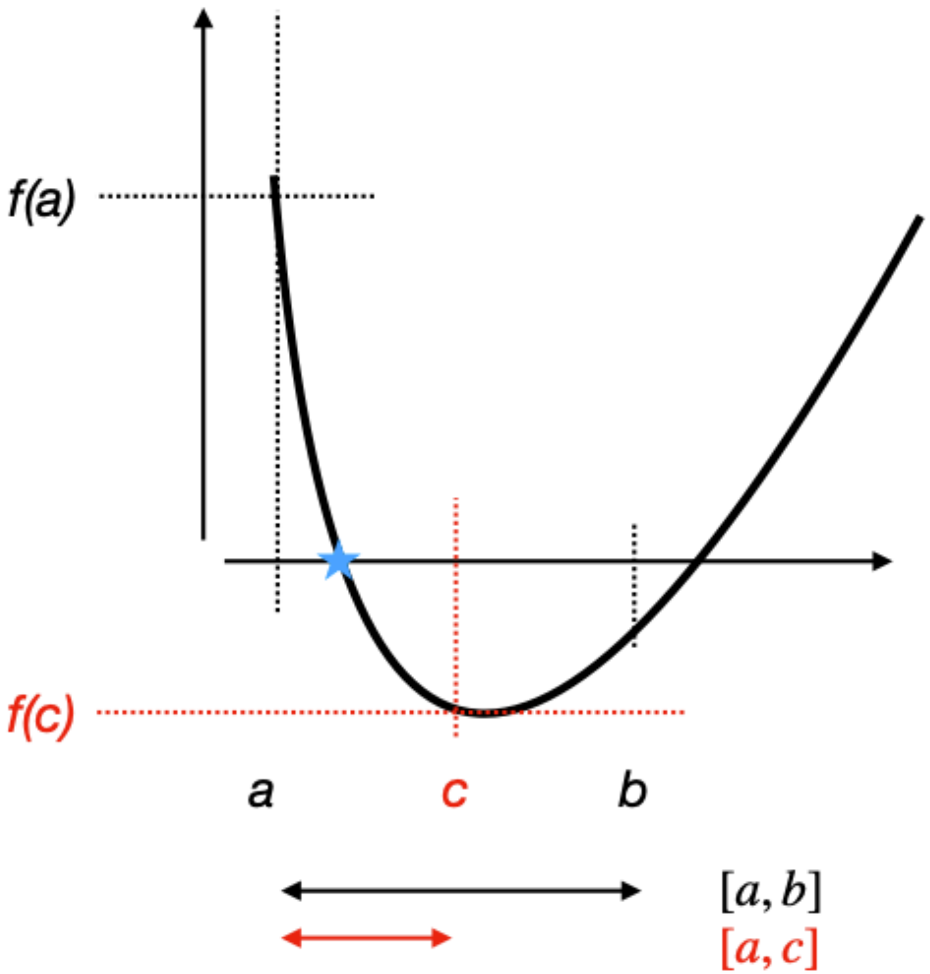
Bisection



$a < x_0 < b$

$f(a) > 0$ **and** $f(b) < 0$

Bisection



$f(a) > 0$ **and** $f(c) < 0$

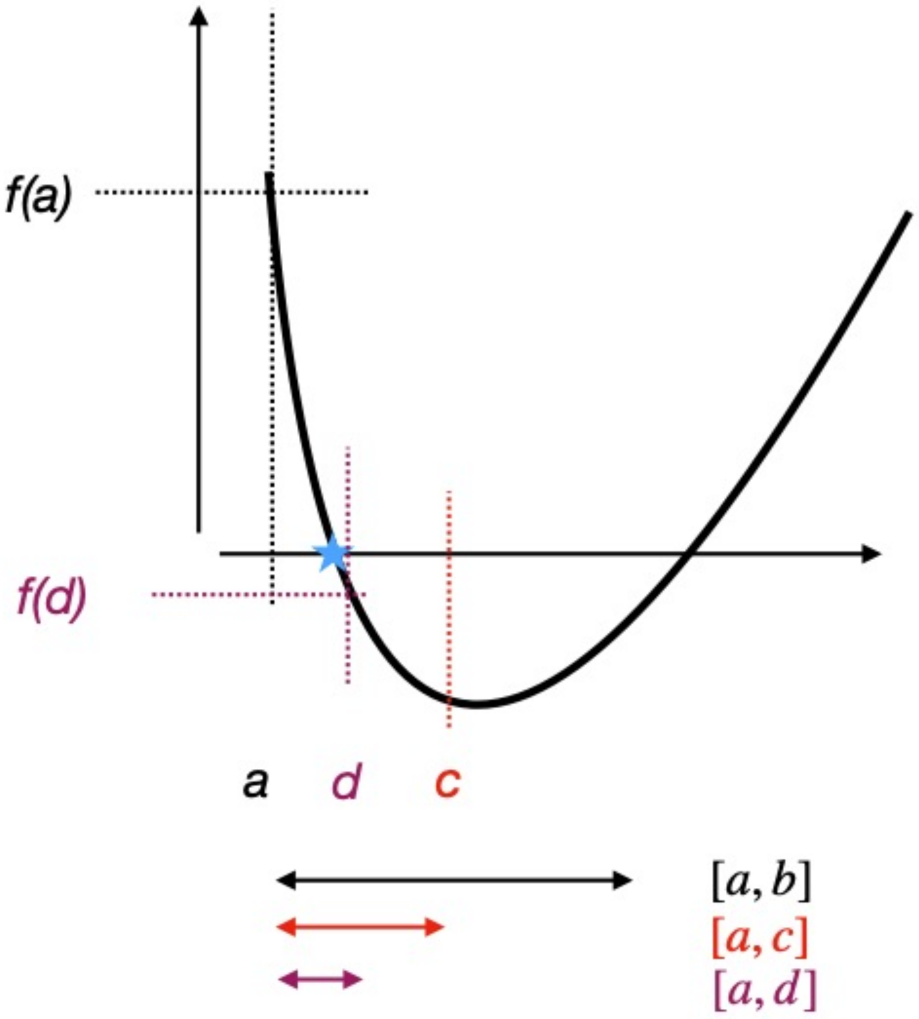
$a < x_0 < c$

Note:

$f(c) < 0$ **and** $f(b) < 0$

so root x_0 is *not* in [c, b]

Bisection



$f(a) > 0$ **and** $f(d) < 0$

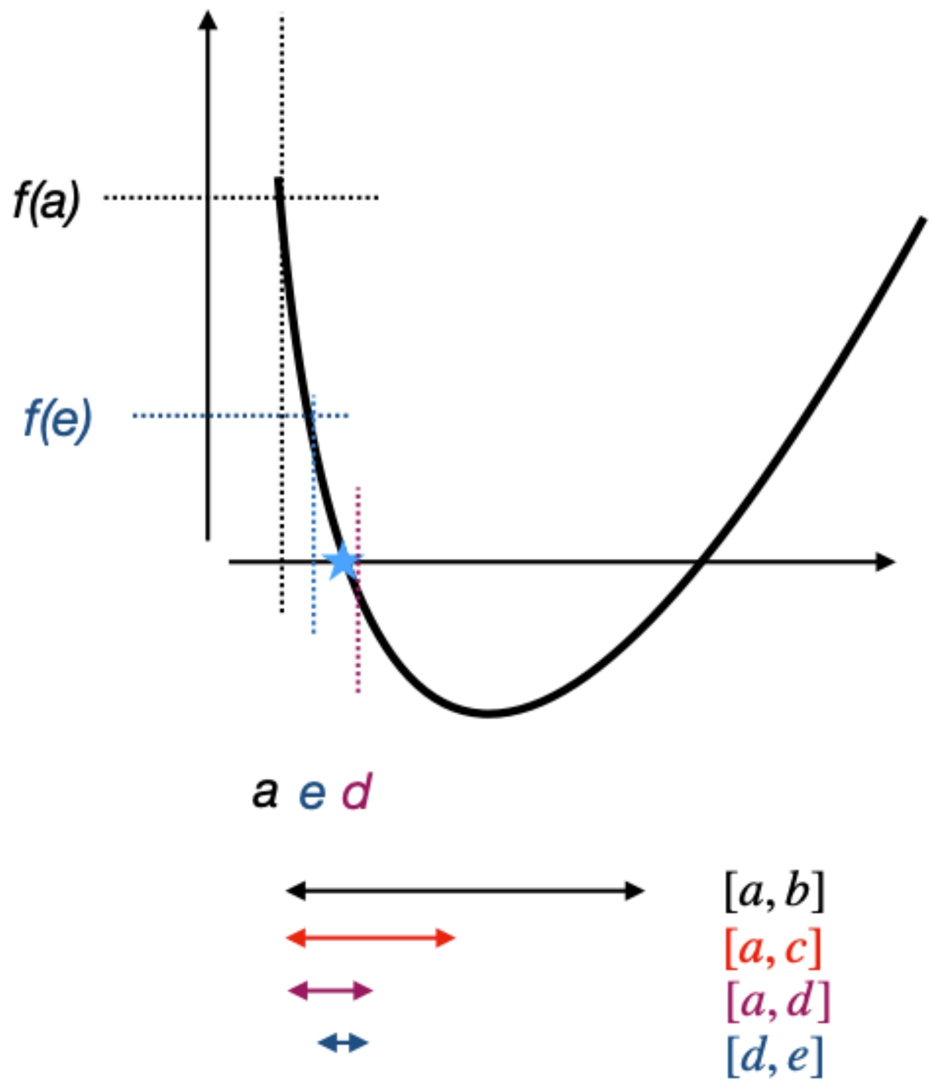
$a < x_0 < d$

Note:

$f(c) < 0$ **and** $f(d) < 0$

so root x_0 is *not* in [d, c]

Bisection



$f(e) > 0$ **and** $f(d) < 0$

$e < x_0 < d$

Note:

$f(a) > 0$ **and** $f(e) > 0$

so root x_0 is *not* in [a, e]

Bisection Algorithm

Start with x_0 (initial guess)

1. **bisect x_1**
2. **pick half with sign change**
3. **is $|x_1 - x_0| < \varepsilon$?**

$$x = \frac{1}{2}(a + b)$$

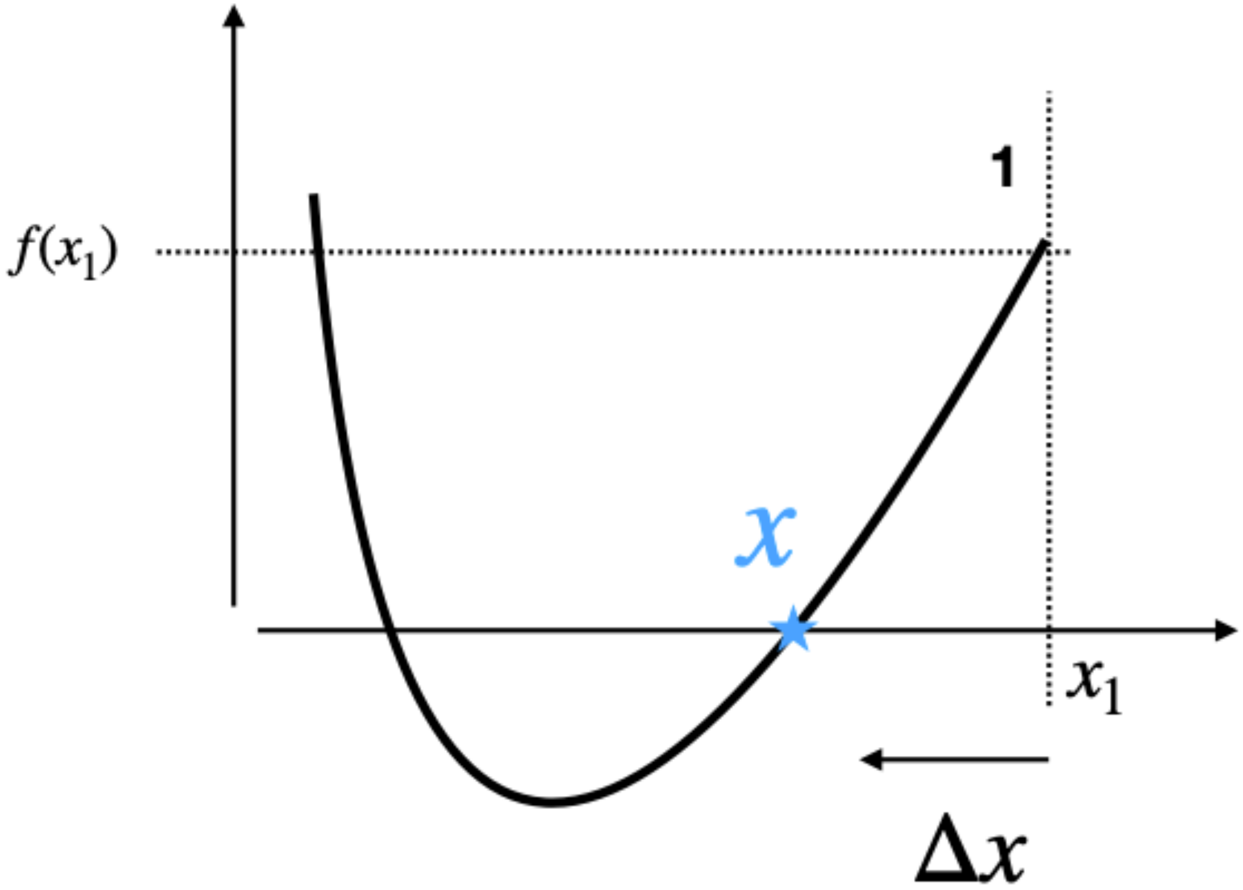
if $f(a)f(x) < 0$

$$x_0 \in [a, x]$$
$$b \leftarrow x$$

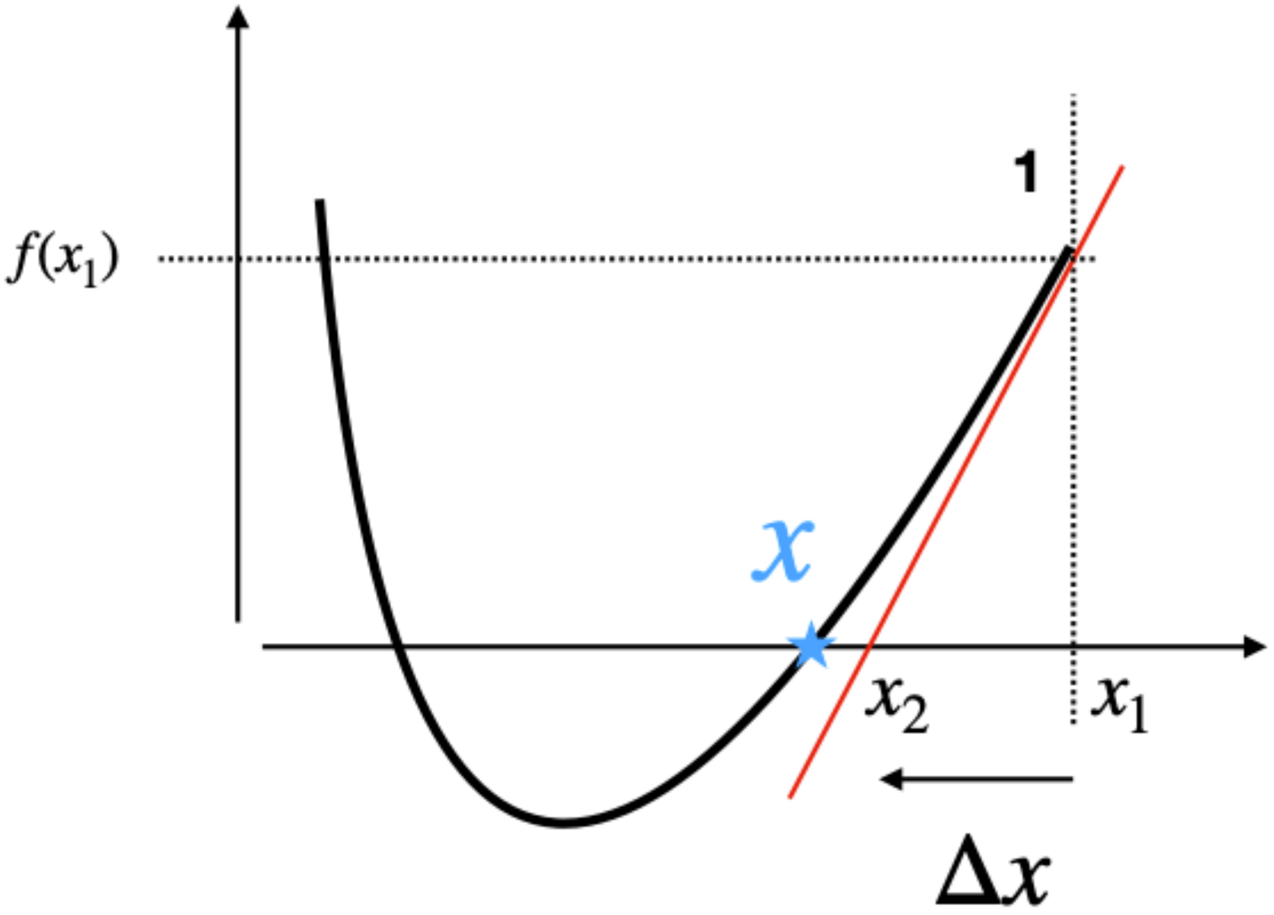
else

$$x_0 \in [x, b]$$
$$a \leftarrow x$$

Newton-Raphson



Newton-Raphson

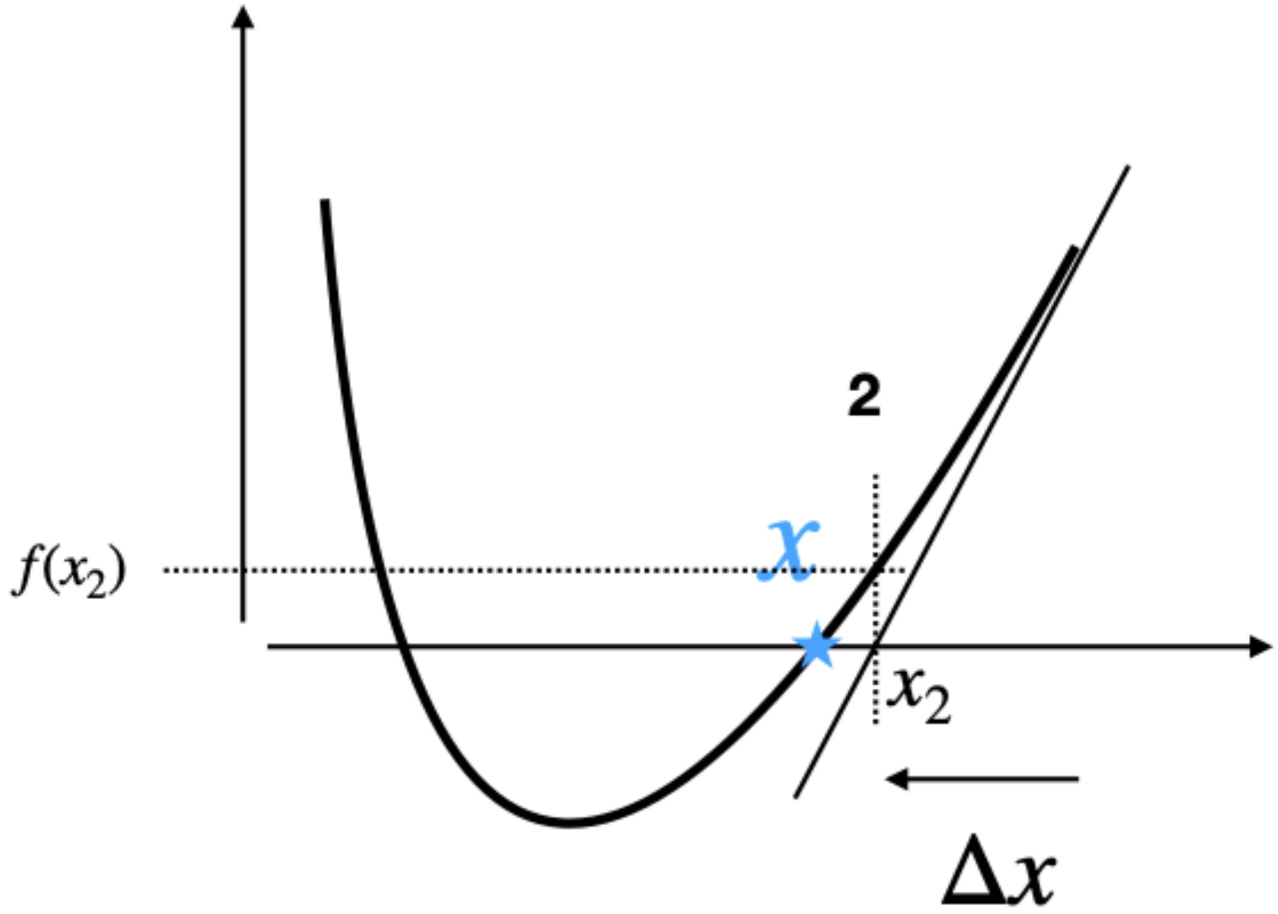


$$y(x) = mx + b$$

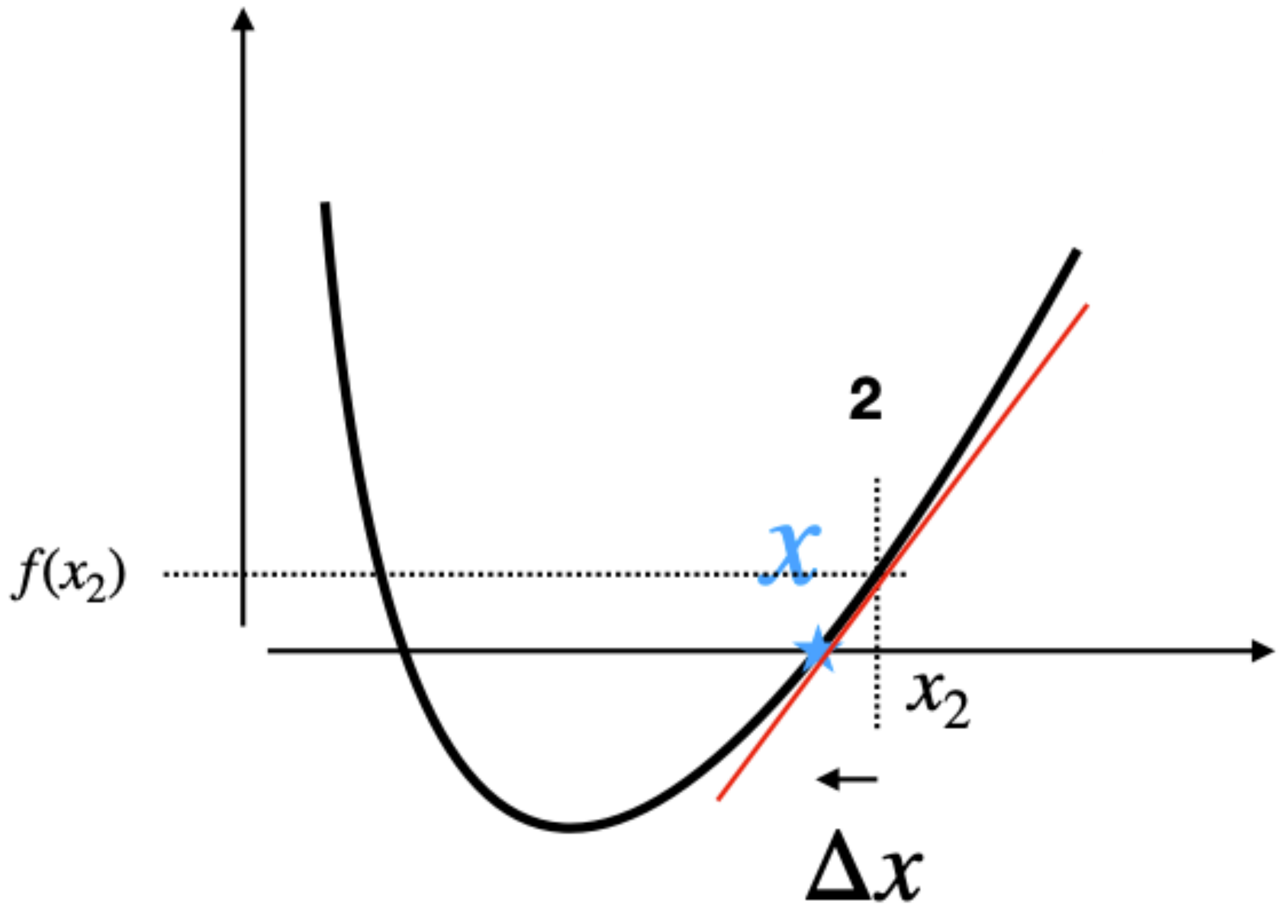
$$y(x) = f(x_1) + f'(x_1)x = 0$$

$$x_2 = x_1 - \underbrace{\frac{f(x_1)}{f'(x_1)}}_{\Delta x}$$

Newton-Raphson



Newton-Raphson



$$y(x) = mx + b$$

$$y(x) = f(x_2) + f'(x_2)x = 0$$

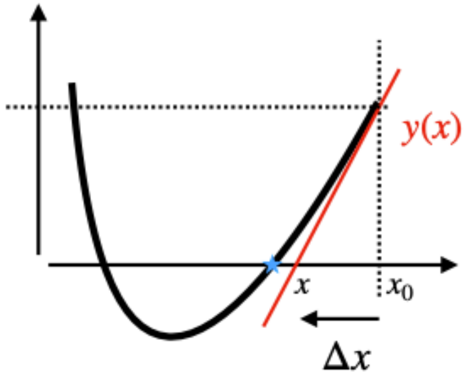
$$x_3 = x_2 - \underbrace{\frac{f(x_2)}{f'(x_2)}}_{\Delta x}$$

Newton-Raphson Algorithm

x_0 initial guess for root

x updated guess

$x = x_0 + \Delta x$ correction?



$x = x_0 + \Delta x$

$$f(x = x_0 + \Delta x) \approx f(x_0) + \Delta x \left.\frac{df}{dx}\right|_{x_0}$$

$$f(x_0) + f'(x_0)\Delta x = 0$$

$$\Delta x = -\frac{f(x_0)}{f'(x_0)}$$

while $|f(x)| > \epsilon$ or $|x_n - x_{n-1}| > \epsilon$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_1)}$$

Newton-Raphson Advantages

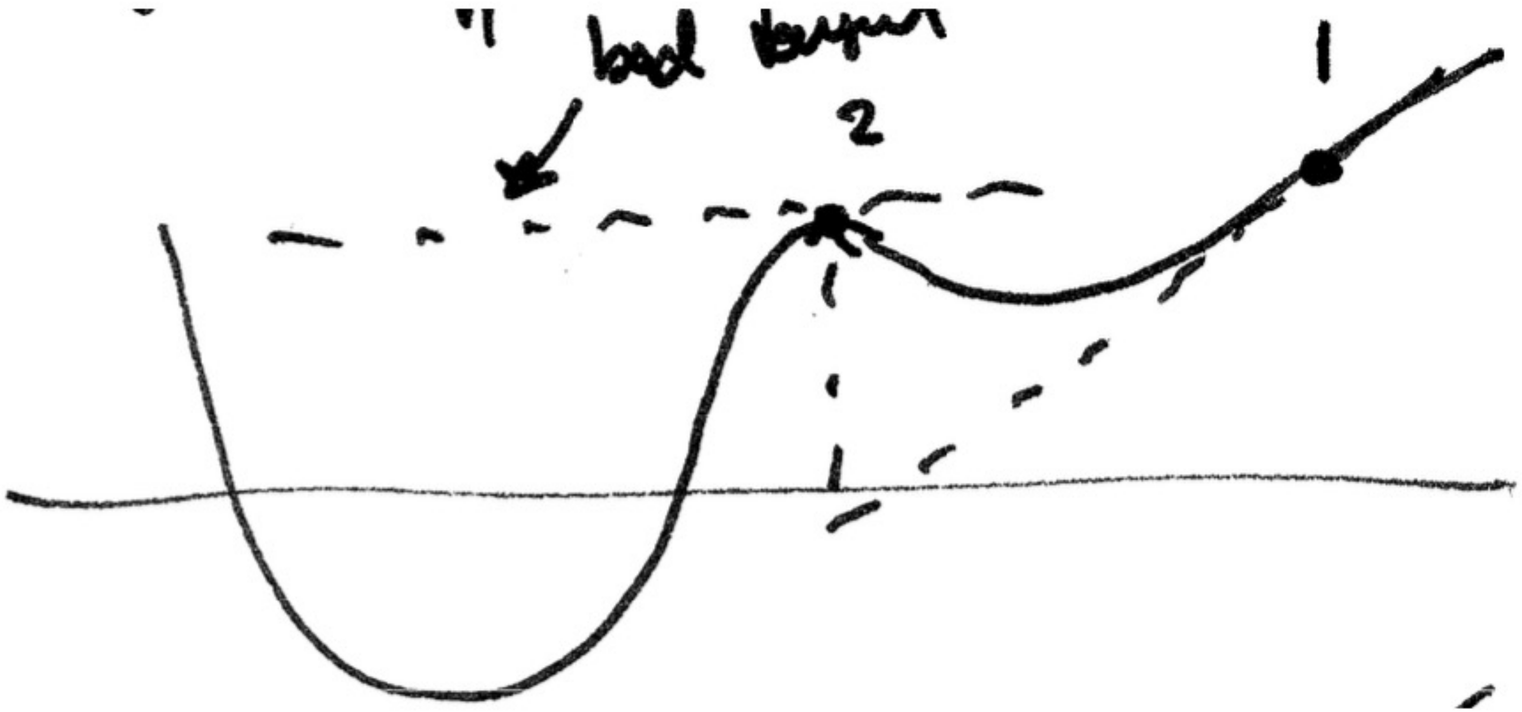
Pros:

- Converges quickly (quadratic), hence fast
- Works best with analytical derivative (but you can use numerical ones

Cons:

- starting guess must be close to root
- can fail to converge in certain situations

Newton-Raphson Failure Mode



Newton-Raphson Failure Mode