3.1 代入法

代入法共分为两步：

(1) 猜测解的形式；

(2) 用数学归纳法求出解中的常数，并证明解是正确的。

例如：T(n) = 2T(n/2)+n

(1) 猜测T(n) ≤ cnlgn；

(2) 证明：令m=n/2，有T(n/2) ≤ cn/2lg(n/2)；

则：

$$
\begin{aligned}
T(n) &\le 2(cn/2\lg(n/2))+n \\
&\le cn\lg(n/2) + n \\
&= cn\lg n - cn\lg 2 + n \\
&= cn\lg n - cn + n \\
&\le cn\lg n
\end{aligned}
$$

3.2 递归树

例T(n) = 3T(n/4) + O(n^2)，其递归树如图3-1所示。图中结点中的数字表示合并问题解的代价，因此该递归式的解为图中所有结点中数字之和。易得该递归树最多有$\log_4 n$+1层，如图中左侧所示；另外图右侧表明了每层的数字之和，则递归式T(n) = 3T(n/4) + O(n^2)的解为T(n) = O(n^2)，详细结算过程如下：

$$
\begin{aligned}
T(n) &= cn^2 + (3/16)\,cn^2 + (3/16)^2\,cn^2 + \ldots + (3/16)^{\log_4 n-1}\,cn^2 + \Theta(n^{\log_4 3}) \\
&= \sum_{i=0}^{\log_4 n-1}\left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&\le \sum_{i=0}^{\infty}\left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{1}{1-(3/16)}cn^2 + \Theta(n^{\log_4 3}) \qquad \boxed{\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}} \\
&= (16/13)\,cn^2 + \Theta(n^{\log_4 3}) = O(n^2)
\end{aligned}
$$

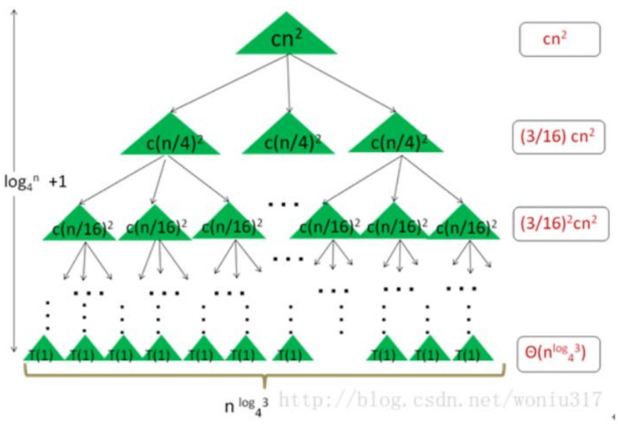

图3-1 T(n) =3T(n/4) + O(n^2)的递归树

3.2 主方法（The master method）

若 a≥1, b≥1, f(n)为一个函数，递归式 $T(n) = aT(n/b) + f(n)$。另 $\varepsilon > 0$，则有：

(1) 若 $f(n) = O(n^{\log_b a-\varepsilon})$，则 $T(n) = \Theta(n^{\log_b a})$;

(2) 若 $f(n) = \Theta(n^{\log_b a})$，则 $T(n) = \Theta(n^{\log_b a}\lg n)$;

(3) 若 $f(n) = \Omega(n^{\log_b a+\varepsilon})$ 且 $af(n/b) \leqslant cf(n)$，其中 c<1，则 $T(n) = \Theta(f(n))$。

例如：

a) 递归式 $T(n) = 9T(n/3)+n$ 中 a=9, b=3, $f(n)=n= O(n^{\log_3 9-\varepsilon})$，满足主方法(1)所示的条件，所以其解 $T(n) = \Theta(n^2)$。

b) 递归式 $T(n)=T(2n/3)+1$ 中 a=1, b=3/2, $f(n)=1=\Theta(n^{\log_{3/2} 1})$，满足主方法(2)所示的条件，所以其解 $T(n) = \Theta(\lg n)$。

Akra–Bazzi method

Formulation [edit]

The Akra–Bazzi method applies to recurrence formulas of the form[1]

$$T(x) = g(x) + \sum_{i=1}^{k} a_i T(b_i x + h_i(x)) \qquad \text{for } x \geq x_0.$$

The conditions for usage are:

- sufficient base cases are provided
- a_i and b_i are constants for all i
- $a_i > 0$ for all i
- $0 < b_i < 1$ for all i
- $|g(x)| \in O(x^c)$, where c is a constant and O notates Big O notation
- $|h_i(x)| \in O\left(\frac{x}{(\log x)^2}\right)$ for all i
- x_0 is a constant

The asymptotic behavior of $T(x)$ is found by determining the value of p for which $\sum_{i=1}^{k} a_i b_i^p = 1$ and plugging that value into the equation[2]

$$T(x) \in \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right)\right)$$

(see Θ). Intuitively, $h_i(x)$ represents a small perturbation in the index of T. By noting that $\lfloor b_i x \rfloor = b_i x + (\lfloor b_i x \rfloor - b_i x)$ and that the absolute value of $\lfloor b_i x \rfloor - b_i x$ is always between 0 and 1, $h_i(x)$ can be used to ignore the floor function in the index. Similarly, one can also ignore the ceiling function. For example, $T(n) = n + T\left(\frac{1}{2}n\right)$ and $T(n) = n + T\left(\left\lfloor \frac{1}{2}n \right\rfloor\right)$ will, as per the Akra–Bazzi theorem, have the same asymptotic behavior.

Example [edit]

Suppose $T(n)$ is defined as 1 for integers $0 \leq n \leq 3$ and $n^2 + \frac{7}{4}T\left(\left\lfloor \frac{1}{2}n \right\rfloor\right) + T\left(\left\lceil \frac{3}{4}n \right\rceil\right)$ for integers $n > 3$. In applying the Akra–Bazzi method, the first step is to find the value of p for which $\frac{7}{4}\left(\frac{1}{2}\right)^p + \left(\frac{3}{4}\right)^p = 1$. In this example, $p = 2$. Then, using the formula, the asymptotic behavior can be determined as follows[3]:

$$\begin{aligned} T(x) &\in \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)\right) \\ &= \Theta\left(x^2 \left(1 + \int_1^x \frac{u^2}{u^3}\, du\right)\right) \\ &= \Theta(x^2(1 + \ln x)) \\ &= \Theta(x^2 \log x). \end{aligned}$$