Quantum Noise and error mitigation

Quantum Computing Minicourse ICTP-SAIFR

Stefano Carrazza[‡] and Matteo Robbiati[†]

8 April 2024

‡ Associate Professor & Researcher, University of Milan and INFN Milan, Italy.
† PhD candidate, University of Milan, Italy and CERN, Switzerland.

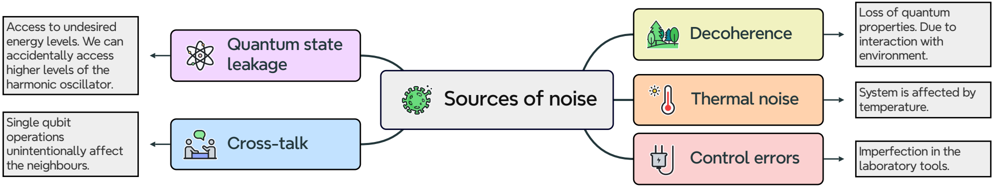
Various sources of noise

We call **quantum noise** any phenomenon that can corrupt the manipulation of the state of a system of qubits.

Various sources of noise

We call **quantum noise** any phenomenon that can corrupt the manipulation of the state of a system of qubits.

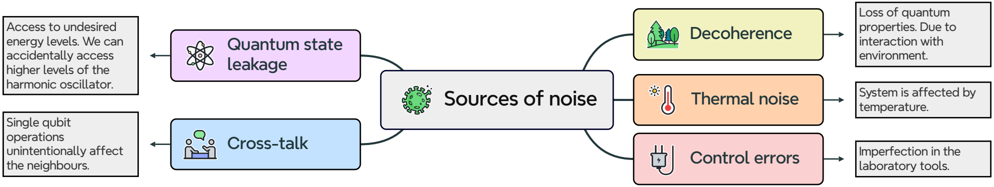
We can introduce various sources of noise.



Various sources of noise

We call **quantum noise** any phenomenon that can corrupt the manipulation of the state of a system of qubits.

We can introduce various sources of noise.

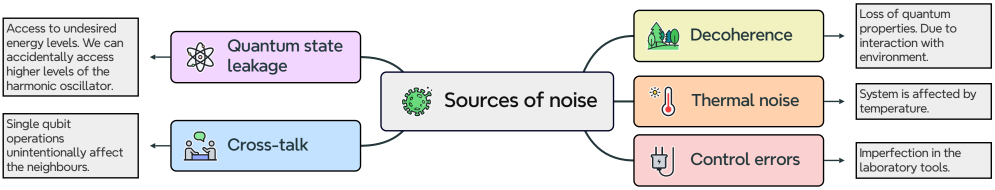


Each of them (even all together) can deteriorate a quantum state.

Various sources of noise

We call **quantum noise** any phenomenon that can corrupt the manipulation of the state of a system of qubits.

We can introduce various sources of noise.



Each of them (even all together) can deteriorate a quantum state.

What about the shot-noise? Is it similar to these noises?

Without noise we can use state vectors

For now we talked about state vector manipulation, where unitary gates $\mathcal{U}$ are used to evolve quantum states:

$$|\psi\rangle' = \mathcal{U} |\psi\rangle \,.$$

Without noise we can use state vectors

For now we talked about state vector manipulation, where unitary gates $\mathcal{U}$ are used to evolve quantum states:

$$|\psi\rangle' = \mathcal{U}\,|\psi\rangle \,.$$

This picture is fine until the quantum state is **pure**, namely, it is represented by a vector into the Hilbert space.

Without noise we can use state vectors

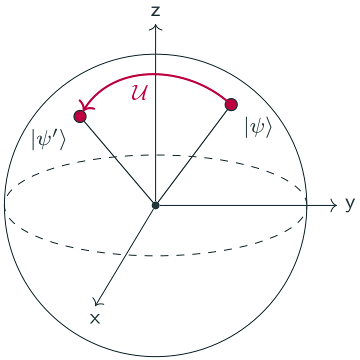
For now we talked about state vector manipulation, where unitary gates $\mathcal{U}$ are used to evolve quantum states:

$$|\psi\rangle' = \mathcal{U}\,|\psi\rangle\,.$$

This picture is fine until the quantum state is **pure**, namely, it is represented by a vector into the Hilbert space.

About pure states

- Pure states are preserved until the quantum system is isolated, thus doesn't interact with the environment.
- A pure state of a single qubit can be visualized as a point on the surface of the Bloch sphere.
- Using unitaries $\mathcal{U}$ we can move any point of the sphere into another point in a reversible way.



The noise breaks the vector notation!

But what happens if we have noise?

The noise breaks the vector notation!

But what happens if we have noise? State vectors are no longer usable, because the system is not isolated.

The noise breaks the vector notation!

But what happens if we have noise? State vectors are no longer usable, because the system is not isolated.

Noise corrupts the quantum states, building **mixed states**, namely classical mixture of pure states.

The noise breaks the vector notation!

But what happens if we have noise? State vectors are no longer usable, because the system is not isolated.

Noise corrupts the quantum states, building **mixed states**, namely classical mixture of pure states. An useful tool to represent both pure and mixed states is the **density matrix**, which in general is defined as:

$$\rho = \sum_{i=0}^{n} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| ,$$

where p_i is the statistical weight of the pure state $|\psi_i\rangle$ in the mix. If $n = 1$, then $\rho = |\psi\rangle \langle\psi|$ and $|\psi\rangle$ is pure.

The noise breaks the vector notation!

But what happens if we have noise? State vectors are no longer usable, because the system is not isolated.

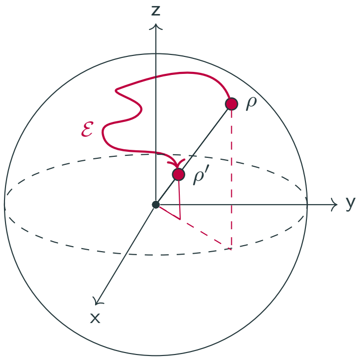
Noise corrupts the quantum states, building **mixed states**, namely classical mixture of pure states. An useful tool to represent both pure and mixed states is the **density matrix**, which in general is defined as:

$$\rho = \sum_{i=0}^{n} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| ,$$

where p_i is the statistical weight of the pure state $\left|\psi_i\right\rangle$ in the mix. If $n = 1$, then $\rho = \left|\psi\right\rangle \left\langle\psi\right|$ and $\left|\psi\right\rangle$ is pure.

About mixed states

- The evolution of a density matrix is described in terms of superoperators (or *channels*) $\rho' = \mathcal{E}(\rho)$, which, in the case of pure states act like unitaries $\rho' = \mathcal{E}^\dagger \rho \mathcal{E}$.
- A mixed state of a single qubit can be visualized as a point within the surface of the Bloch sphere.
- The effect of the noise can be represented by a superoperator which can map points on the surface into points located within the Bloch sphere.



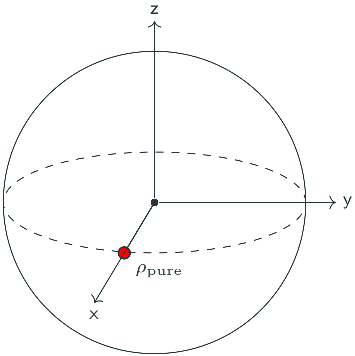
Pure and mixed states in the Bloch sphere

Pure and mixed states in the Bloch sphere

The maximally entangled state is a pure state

The density matrix of $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is

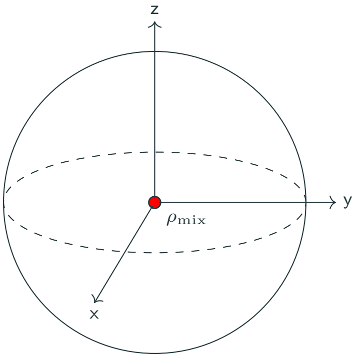
$$\rho_{\text{pure}} = |+\rangle\langle+| = \\ = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$



The state which maximally mix $|0\rangle$ and $|1\rangle$

The density matrix of $\frac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)$ is

$$\rho_{\text{mix}} = \frac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big) = \\ \frac{1}{2}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$



A dummy noise representation

We can now define a channel $\mathcal{N}$, which represents the action of the noise on a state: $\rho_{\text{noisy}} = \mathcal{N}(\rho)$.

A dummy noise representation

We can now define a channel $\mathcal{N}$, which represents the action of the noise on a state: $\rho_{\text{noisy}} = \mathcal{N}(\rho)$.

We can exploit the Pauli's operator to write a noise model:

- we use X to apply random bitflip $|0\rangle = X|1\rangle$ and $|1\rangle = X|0\rangle$ with probability p_x;
- we use Z to apply random phase flip $|0\rangle = Z|0\rangle$ and $-|1\rangle = Z|1\rangle$ with probability p_z;
- we use Y to apply more complex manipulations since $Y = iXZ$ with probability p_y.

A dummy noise representation

We can now define a channel $\mathcal{N}$, which represents the action of the noise on a state: $\rho_{\text{noisy}} = \mathcal{N}(\rho)$.

We can exploit the Pauli's operator to write a noise model:

- we use X to apply random bitflip $|0\rangle = X|1\rangle$ and $|1\rangle = X|0\rangle$ with probability p_x;
- we use Z to apply random phase flip $|0\rangle = Z|0\rangle$ and $-|1\rangle = Z|1\rangle$ with probability p_z;
- we use Y to apply more complex manipulations since $Y = iXZ$ with probability p_y.

Then the combined effect of these Pauli components is:

$$\mathcal{N}(\rho) = \left(1 - \sum_k p_k\right)\rho + \sum_k p_k P_k \rho P_k$$

where $\sum_k p_k \leq 1$, P_k are the Paulis $\{X, Y, Z\}$ and the first term describe when ρ remains unchanged.

A dummy noise representation

We can now define a channel $\mathcal{N}$, which represents the action of the noise on a state: $\rho_{\text{noisy}} = \mathcal{N}(\rho)$.

We can exploit the Pauli's operator to write a noise model:

- we use X to apply random bitflip $|0\rangle = X|1\rangle$ and $|1\rangle = X|0\rangle$ with probability p_x;
- we use Z to apply random phase flip $|0\rangle = Z|0\rangle$ and $-|1\rangle = Z|1\rangle$ with probability p_z;
- we use Y to apply more complex manipulations since $Y = iXZ$ with probability p_y.

Then the combined effect of these Pauli components is:

$$\mathcal{N}(\rho) = \left(1 - \sum_k p_k\right)\rho + \sum_k p_k P_k \rho P_k$$

where $\sum_k p_k \leq 1$, P_k are the Paulis $\{X, Y, Z\}$ and the first term describe when ρ remains unchanged.

Effect of the noise when computing expectation values

1. The effect of this noise model is to push any state ρ closer to the maximally mixed state $\rho_{\text{mix}} = \frac{1}{2^N} I$, where N is the number of qubits of the system and I is the identity.
2. The expectation value of a Pauli string (X, Y, Z or combinations P) over ρ_{mix} zero. In fact, since $\text{E}[P] = \text{Tr}[P] = 0$:

$$E_{\rho_{\text{mix}}}[P] = \text{Tr}[P\rho_{\text{mix}}] = \frac{1}{2^N}\text{Tr}[P\,I] = \frac{1}{2^N}\text{Tr}[P] = 0.$$

Reality is more complex

The Pauli noise representation is good enough as starting point, but it assumes simplifications:

Reality is more complex

The Pauli noise representation is good enough as starting point, but it assumes simplifications:

1. with this model we assume the noise affects single qubits independently. In practice, it acts like single-qubit gates. In real devices noise can be correlated across multiple qubits;

Reality is more complex

The Pauli noise representation is good enough as starting point, but it assumes simplifications:

1. with this model we assume the noise affects single qubits independently. In practice, it acts like single-qubit gates. In real devices noise can be correlated across multiple qubits;
2. we assume the noise is "Markovian", namely, the noise at a certain time t of the execution is independent on the state of the system at time $t - \delta t$. Many quantum systems exhibit non-Markovian dynamics;

Reality is more complex

The Pauli noise representation is good enough as starting point, but it assumes simplifications:

1. with this model we assume the noise affects single qubits independently. In practice, it acts like single-qubit gates. In real devices noise can be correlated across multiple qubits;
2. we assume the noise is "Markovian", namely, the noise at a certain time t of the execution is independent on the state of the system at time $t - \delta t$. Many quantum systems exhibit non-Markovian dynamics;
3. Pauli errors act on the system as discrete transformations. Actual quantum systems may experience continuous errors.

Reality is more complex

The Pauli noise representation is good enough as starting point, but it assumes simplifications:

1. with this model we assume the noise affects single qubits independently. In practice, it acts like single-qubit gates. In real devices noise can be correlated across multiple qubits;
2. we assume the noise is "Markovian", namely, the noise at a certain time t of the execution is independent on the state of the system at time $t - \delta t$. Many quantum systems exhibit non-Markovian dynamics;
3. Pauli errors act on the system as discrete transformations. Actual quantum systems may experience continuous errors.

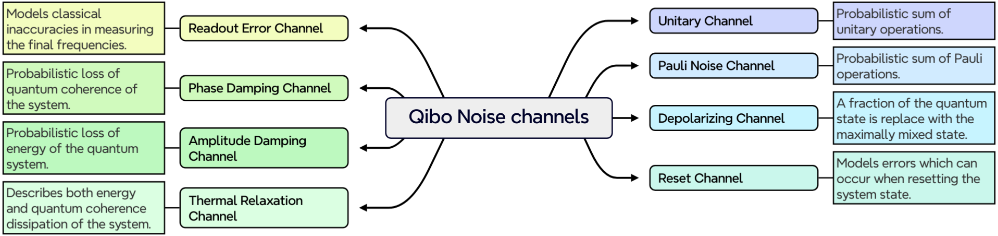
Several more complex noise models exist and many of them are implemented in `Qibo`:

Reality is more complex

The Pauli noise representation is good enough as starting point, but it assumes simplifications:

1. with this model we assume the noise affects single qubits independently. In practice, it acts like single-qubit gates. In real devices noise can be correlated across multiple qubits;
2. we assume the noise is "Markovian", namely, the noise at a certain time t of the execution is independent on the state of the system at time $t - \delta t$. Many quantum systems exhibit non-Markovian dynamics;
3. Pauli errors act on the system as discrete transformations. Actual quantum systems may experience continuous errors.

Several more complex noise models exist and many of them are implemented in `Qibo`:



How can we face this problem?

- The noise problem
  - Quantum error correction
    - The errors are removed completely
    - Requires a huge amount of qubits
  - Quantum error mitigation
    - Statistical analysis to mitigate the errors (not removed)
    - Few qubits systems are enough!

Clifford Data Regression

The goal of quantum error mitigation is to use our knowledge about the noise to try to mitigate its effect.

Clifford Data Regression

The goal of quantum error mitigation is to use our knowledge about the noise to try to mitigate its effect.

Suppose we are interested in computing the expectation value $\langle \mathcal{O} \rangle^0$ of an observable $\mathcal{O}$ over the state we get applying the quantum circuit $\mathcal{C}^0$ to a state ρ.

Clifford Data Regression

The goal of quantum error mitigation is to use our knowledge about the noise to try to mitigate its effect.

Suppose we are interested in computing the expectation value $\langle \mathcal{O} \rangle^0$ of an observable $\mathcal{O}$ over the state we get applying the quantum circuit $\mathcal{C}^0$ to a state ρ.

But our system is noisy! Thus what we get is a noisy expectation value $\langle \mathcal{O} \rangle^0_{\mathrm{noisy}}$.

Clifford Data Regression

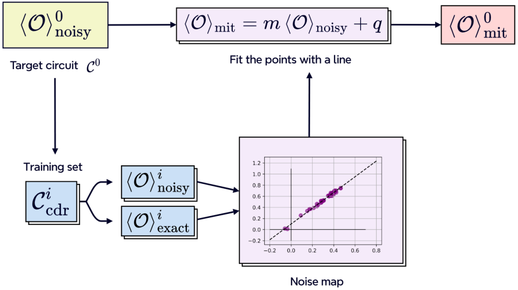
The goal of quantum error mitigation is to use our knowledge about the noise to try to mitigate its effect.

Suppose we are interested in computing the expectation value $\langle\mathcal{O}\rangle^0$ of an observable $\mathcal{O}$ over the state we get applying the quantum circuit $\mathcal{C}^0$ to a state ρ.

But our system is noisy! Thus what we get is a noisy expectation value $\langle\mathcal{O}\rangle^0_{\rm noisy}$.

Clifford Data Regression algorithm

1. Sample a set of circuits $\mathcal{C}^i_{\rm cdr}$, of the same size of the target $\mathcal{C}^0$, but fast to simulate[1].
2. For each $C^i_{\rm cdr}$ calculate exact and noisy exp value.
3. Make a scatter plot of exact versus noisy expectation values and fit the points with a line.
4. This line is actually a noise map, which can be used to map our $\mathcal{C}^0_{\rm noisy}$ into a mitigated $\mathcal{C}^0_{\rm mit}$.
5. The obtained map can be used for any new circuit of the same size of $\mathcal{C}^0$.



This method assumes the noise is affecting the gates!

[1] We generate these circuits by randomly replacing some gates of the circuit with Clifford gates. When dealing with rotations, one can replace the angles with multiples of $\pi/2$. For more info on Clifford simulators: arXiv:quant-ph/0406196.

Let's code!