Grover search algorithm

Quantum Computing Minicourse ICTP-SAIFR

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8 April 2024

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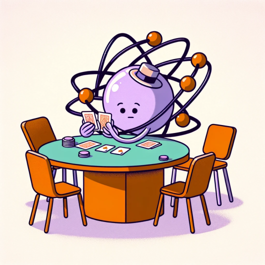
Motivation

The Grover algorithm is powerful when searching an item among an unordered set of candidates.

Extract the jack of clubs from a Poker deck



Find a passcode composed of 10 numbers

Find an antidote to the Cobra poison, exploring 10^{20} molecules

? How many attempts could you need, in the worst scenario, to explore all the possibilities?

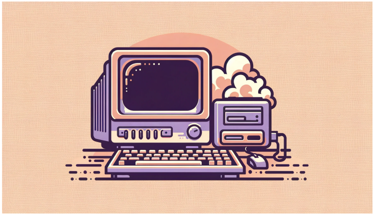
⚠ In the worst scenario, you will need to check 52 **cards**, 10^{10} **passcodes** and 10^{20} **molecules**.

Quadratic speedup

If we consider a time cost of $\delta = 10^{-8}$ seconds for any algorithmic call (quantum or classical) we would wait:

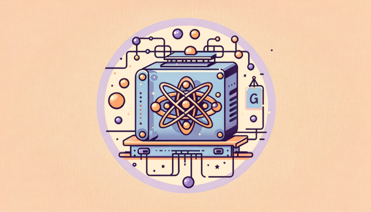
On a classical computer

- 0.52 μs to find the jack of clubs;
- 100 seconds to find the passcode;
- $\sim$ 31688 years to find the Cobra antidote.



On a quantum computer

- 0.0721 μs to find the jack of clubs;
- 0.001 seconds to find the passcode;
- 100 seconds to find the Cobra antidote.



The Grover algorithm solves this kind of search with a number of algorithmic calls proportional to $\sqrt{N}$, where N is the dimension of the search space.

The Grover algorithm

The key steps of the Grover algorithm:

1. prepare a system of N qubits into a maximally superposed state;
2. prepare an ancilla qubit into the $|-\rangle$ state;
3. apply an oracle operator U_f which can mark the correct solution;
4. apply a diffusion operator U_s which amplifies the correct solution;
5. repeat 3. and 4. for the optimal number of times.

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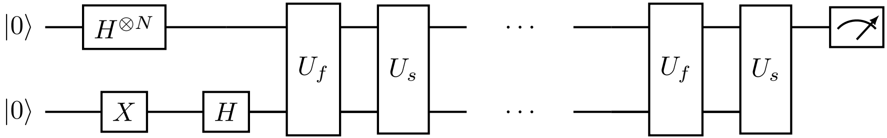
In terms of quantum circuit:



step 1 and 2: the state preparation

We consider a set of 2^N unordered items and we encode them into the state of an N qubits system:

$$\begin{bmatrix} \text{item}_1 \\ \text{item}_2 \\ \dots \\ \text{item}_{2^N} \end{bmatrix} \qquad \rightarrow \qquad |\psi\rangle = \begin{bmatrix} \psi_{00\dots0} \\ \psi_{00\dots1} \\ \dots \\ \psi_{11\dots1} \end{bmatrix} \equiv \begin{bmatrix} x_0 \\ x_1 \\ \dots \\ x_{2^N-1} \end{bmatrix}.$$

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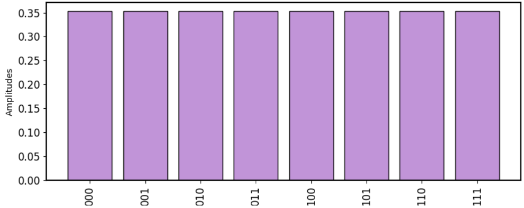
The first step of the algorithm is the state preparation into the following superposed state:

$$H^{\otimes N+1} |0\rangle_N |1\rangle = \left[\frac{1}{2^{N/2}} \sum_{i=0}^{2^N-1} |x_i\rangle \right] \otimes |-\rangle .$$

We move the system state from the computational zero to the maximally superposed state $|s\rangle$.



Amplitudes of $|x\rangle_N$ with $N = 3$ and $|\omega\rangle = |001\rangle$

Step 3: the oracle U_f

We consider now a function $f : \{0,1\}^N \to \{0,1\}$ which can detect the correct solution $|\omega\rangle$:

$$f(x) = \begin{cases} 1 & \text{if } x = \omega, \\ 0 & \text{otherwise.} \end{cases}$$

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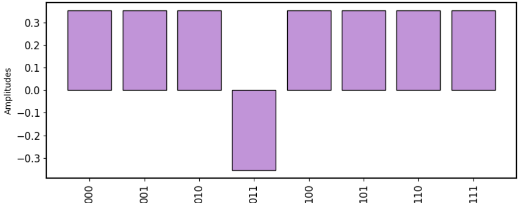
In practice, this can be done by setting up a multi-controlled operation which applies a phase kickback only if the control state is $|\omega\rangle$.

$$U_f |x\rangle |-\rangle = (-1)^{f(x)} |x\rangle |-\rangle ,$$

where $f(x)$ follows the rule exposed before.



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To this purpose, we introduce the diffusion operator

$$U_s = 2\,|s\rangle\,\langle s| - I,$$

whose action is a reflection of the system with respect to the state:

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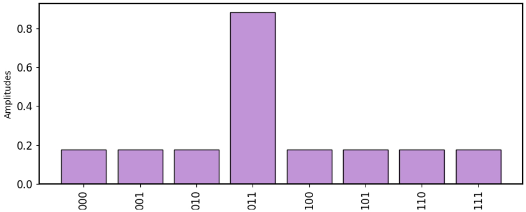
U_s is also known as "inversion by the mean", in fact, it can be shown it implements an inversion w.r.t. the mean value of the amplitudes of $|x\rangle$.



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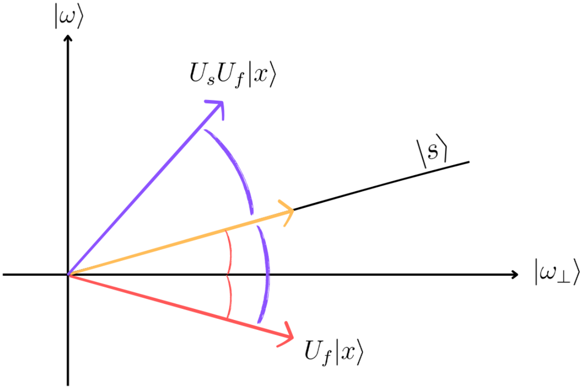
Graphical intuition

We can visualize the Grover's action using vectors.



We are iteratively moving the system state through the target $|\omega\rangle$.

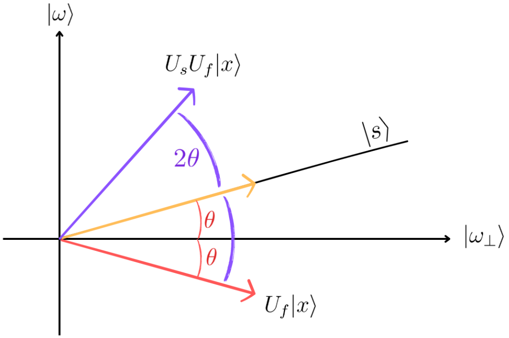
How many times do we need to iterate Grover?

As we can deduce from the previous slide, there exist an **optimal number** of Grover iterations fixed by geometry.

1. we can decompose $|x\rangle$ into the *winning* and the *losing* components $|s\rangle = \sqrt{\frac{1}{N}}\,|\omega\rangle + \sqrt{\frac{N-1}{N}}\,|\omega_\perp\rangle$.
2. The same vector can be defined in terms of the angle in the plane: $|s\rangle = \sin\theta\,|\omega\rangle + cos\theta\,|\omega_\perp\rangle$.
3. from 1. and 2. we can write $\theta = \arcsin\left(1/\sqrt{N}\right)$ and, if N is large, $\theta \approx 1/\sqrt{N}$.
4. the action of $U_s U_f$ on $|x\rangle$ is equal to a rotation of 2θ of the vector.
5. after k iteration of Grover, the angle has become: $\alpha = (2k+1)\,\theta$, and, to maximize $\sin\alpha$:

$$\alpha = \frac{\pi}{2} \quad \rightarrow \quad k = \frac{\pi}{4\theta} - \frac{1}{2} = \frac{\pi}{4}\sqrt{N} - \frac{1}{2}.$$

6. from 5. we need to get an integer, since we are talking about iterations. Commonly $\theta \approx \frac{\pi}{4}\sqrt{N}$.



Let's code!