Grover search algorithm

Quantum Computing Minicourse ICTP-SAIFR

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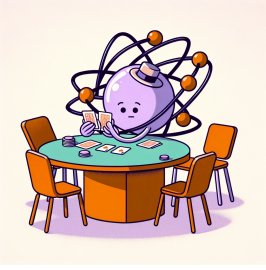
Motivation

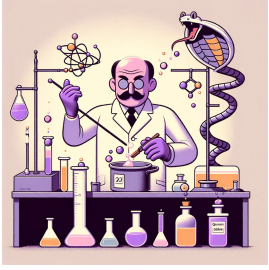
The Grover algorithm is powerful when searching an item among an unordered set of candidates.

Extract the jack of clubs from a Poker deck



Find a passcode composed of 10 numbers

Find an antidote to the Cobra poison, exploring 10^{20} molecules



? How many attempts could you need, in the worst scenario, to explore all the possibilities?

⚠ In the worst scenario, you will need to check 52 **cards**, 10^{10} **passcodes** and 10^{20} **molecules**.

Quadratic speedup

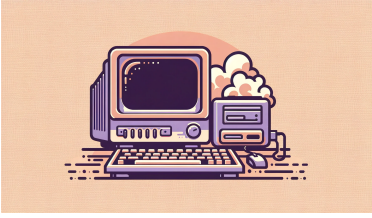
If we consider a time cost of $\delta = 10^{-8}$ seconds for any algorithmic call (quantum or classical) we would wait:

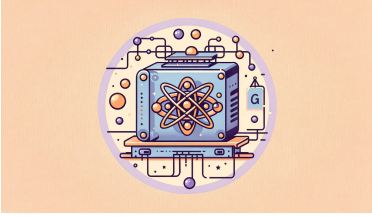
On a classical computer

- 0.52 μs to find the jack of clubs;
- 100 seconds to find the passcode;
- $\sim$ 31688 years to find the Cobra antidote.

On a quantum computer

- 0.0721 μs to find the jack of clubs;
- 0.001 seconds to find the passcode;
- 100 seconds to find the Cobra antidote.





The Grover algorithm solves this kind of search with a number of algorithmic calls proportional to $\sqrt{N}$, where N is the dimension of the search space.

The Grover algorithm

The key steps of the Grover algorithm:

1. prepare a system of N qubits into a maximally superposed state;
2. prepare an ancilla qubit into the $|-\rangle$ state;
3. apply an oracle operator U_f which can mark the correct solution;
4. apply a diffusion operator U_s which amplifies the correct solution;
5. repeat 3. and 4. for the optimal number of times.

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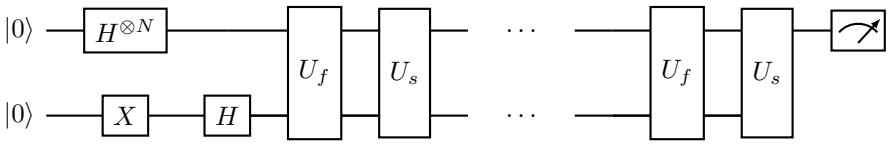
In terms of quantum circuit:



step 1 and 2: the state preparation

Lets consider 2^N elements composing our search space.

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We can encode them into the amplitudes of a quantum state of N qubits:

$$\begin{bmatrix} \text{item}_1 \\ \text{item}_2 \\ \dots \\ \text{item}_{2^N} \end{bmatrix} \quad \rightarrow \quad |\psi\rangle = \begin{bmatrix} \psi_{00\dots0} \\ \psi_{00\dots1} \\ \dots \\ \psi_{11\dots1} \end{bmatrix} \equiv \begin{bmatrix} x_0 \\ x_1 \\ \dots \\ x_{2^N-1} \end{bmatrix}.$$

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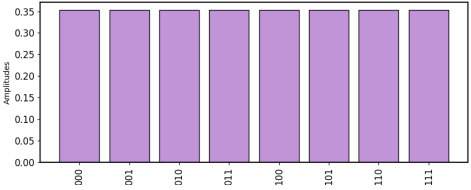
The first step of the algorithm is the state preparation into the following superposed state:

$$H^{\otimes N+1} |0\rangle_N |1\rangle = \left[\frac{1}{2^{N/2}} \sum_{i=0}^{2^N-1} |x_i\rangle \right] \otimes |-\rangle .$$



Step 3: the oracle U_f

We consider now a function $f : \{0,1\}^N \to \{0,1\}$. whose goal is to mark the correct solution:

$$f(x) = \begin{cases} 1 & \text{if correct,} \\ 0 & \text{otherwise.} \end{cases}$$

This function is typically embedded into a quantum oracle U_f which is able to recognize the solution for us. It's important to underline that even if the oracle can detect the solution, may don't know it's exact value.

The oracle marks the solution (one of the amplitudes) by flipping its sign thanks to a phase-kickback procedure.

$$U_f |x\rangle |-\rangle = (-1)^{f(x)} |x\rangle |-\rangle .$$

This can be done using a multi-controlled operation which triggers the kickback only when the system state corresponds to the target one.

Step 4: the diffusion operator U_s

Let's code!