Ancillary qubits

Quantum Computing Minicourse ICTP-SAIFR

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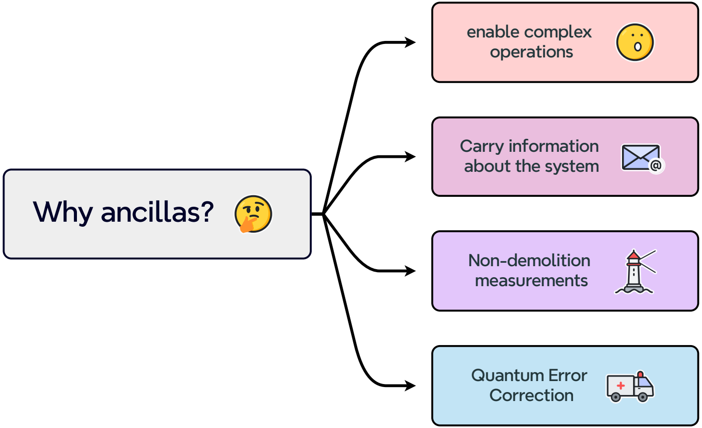
Ancillary qubits

Ancilla qubits are extra qubits, which help a qubit system in some computations. For example:



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Important

This happens if the ancilla is prepared into a state which is eigenvector of the controlled operation.

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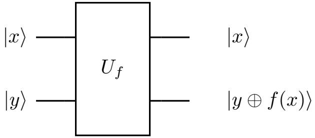

where the symbol $\oplus$ represents the "addition modulo 2" operation, equivalent to the logical operation XOR:

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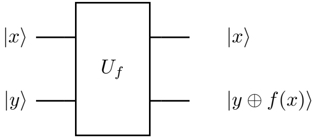

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In other terms, the action of this oracle can be expressed as $U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$.

A remarkable case: the phase flip

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The previous formula can be extended to the case of using multi-controlled operations once the policy is rewritten so that:

$$f(x) = \begin{cases} 1 & \text{if} \quad x = |11\dots 1\rangle \\ 0 & \text{otherwise.} \end{cases}$$

Is the phase kickback useful?

Some of the most powerful and famous quantum computing algorithms make use of the phase kickback

Figure 1: Deutsch-Josza

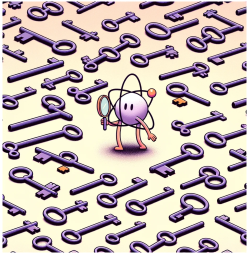

Figure 2: Grover

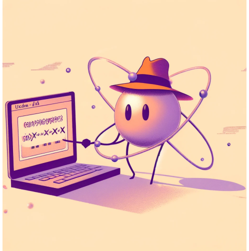

Figure 3: Shor

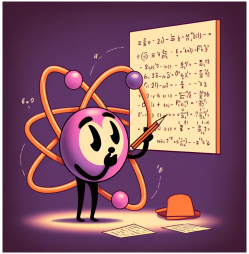

Figure 4: HHL

All of them are proved to outperform any classical algorithm, leading to a theoretical **quantum advantage**!

Let's code!