Grover search algorithm

Quantum Computing Minicourse ICTP-SAIFR

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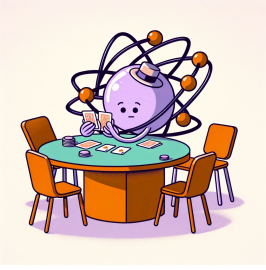
Motivation

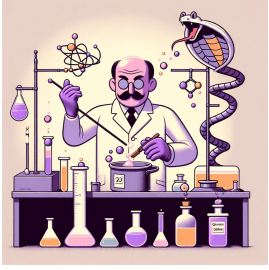
The Grover algorithm is powerful when searching an item among an unordered set of candidates.

Extract the jack of clubs from a Poker deck



Find a passcode composed of 10 numbers

Find an antidote to the Cobra poison, exploring 10^{20} molecules



? How many attempts could you need, in the worst scenario, to explore all the possibilities?

⚠ In the worst scenario, you will need to check 52 **cards**, 10^{10} **passcodes** and 10^{20} **molecules**.

Quadratic speedup

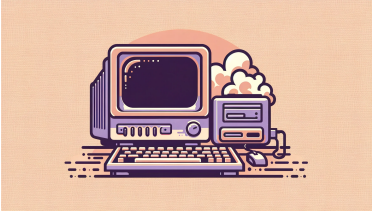
If we consider a time cost of $\delta = 10^{-8}$ seconds for any algorithmic call (quantum or classical) we would wait:

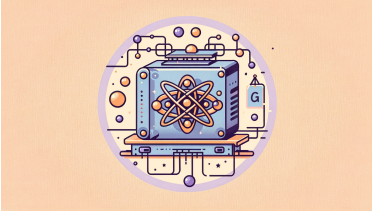
On a classical computer

- 0.52 μs to find the jack of clubs;
- 100 seconds to find the passcode;
- $\sim$ 31688 years to find the Cobra antidote.

On a quantum computer

- 0.0721 μs to find the jack of clubs;
- 0.001 seconds to find the passcode;
- 100 seconds to find the Cobra antidote.





The Grover algorithm solves this kind of search with a number of algorithmic calls proportional to $\sqrt{N}$, where N is the dimension of the search space.

The Grover algorithm

We consider a system of $N+1$ qubits: the system plus one ancilla. One of the $M = 2^N$ components of the system's state, which we call $|\omega\rangle$, will represent the item we are searching for.

1. prepare the system of N qubits into the maximally superposed state;
2. prepare the ancilla qubit into the $|-\rangle$ state;
3. apply the Grover operator, which is able to detect ω among the components and to amplify its amplitude;
4. repeat 3. for the optimal number of times;
5. perform the measurements on the system qubits: the target component $|\omega\rangle$ will come out with probability close to one.

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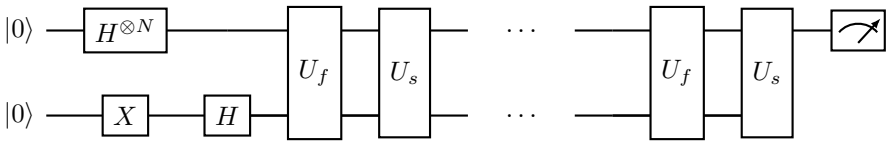
In terms of quantum circuit:



step 1 and 2: the state preparation

We consider a set of $M = 2^N$ unordered items and we encode them into the state of an N qubits system:

$$\begin{bmatrix} \text{item}_1 \\ \text{item}_2 \\ \dots \\ \text{item}_{2^N} \end{bmatrix} \quad \rightarrow \quad |\psi\rangle = \begin{bmatrix} \psi_{00\dots0} \\ \psi_{00\dots1} \\ \dots \\ \psi_{11\dots1} \end{bmatrix} \equiv \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_M \end{bmatrix}.$$

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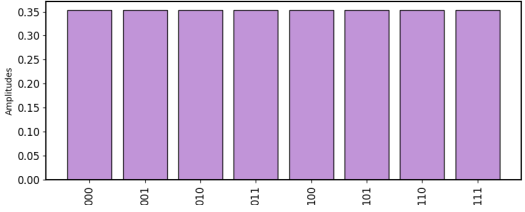
The first step of the algorithm is the state preparation into the following superposed state:

$$H^{\otimes N} |0\rangle_N \otimes HX |0\rangle = \left[\frac{1}{\sqrt{M}} \sum_{i=1}^{M} |x_i\rangle \right] \otimes |-\rangle = |s\rangle \otimes |-\rangle .$$

We move the system state from the computational zero to the maximally superposed state $|s\rangle$.



Amplitudes of $|x\rangle_N$ with $N = 3$ and $|\omega\rangle = |001\rangle$

Step 3: the oracle U_f

We consider now a function $f : \{0,1\}^N \to \{0,1\}$ which can detect the correct solution $|\omega\rangle$:

$$f(x) = \begin{cases} 1 & \text{if } x = \omega, \\ 0 & \text{otherwise.} \end{cases}$$

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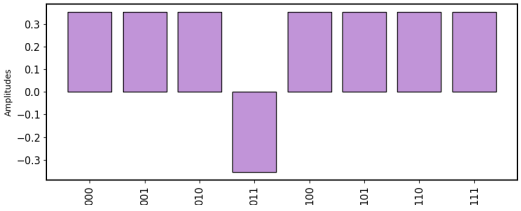
In practice, this can be done by setting up a multi-controlled operation which applies a phase kickback only if the control state is $|\omega\rangle$.

$$U_f \left|x\right\rangle \left|-\right\rangle = (-1)^{f(x)} \left|x\right\rangle \left|-\right\rangle .$$

Its action on $|\omega\rangle$ is:

$$U_f \left|\omega\right\rangle \left|-\right\rangle = - \left|\omega\right\rangle \left|-\right\rangle .$$

where $f(x)$ follows the rule exposed before.



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Step 4: the diffusion operator U_s

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whose action is a reflection of the system with respect to the state:

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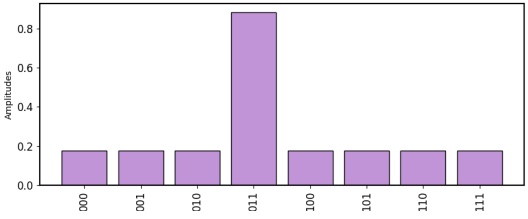
U_s is also known as "inversion by the mean", in fact, it can be shown it implements an inversion w.r.t. the mean value of the amplitudes of $|x\rangle$.



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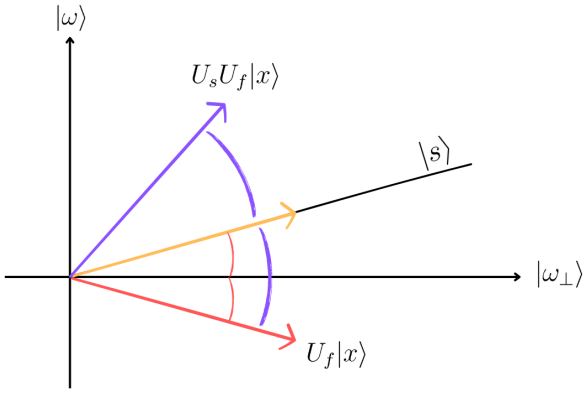
Graphical intuition

We can visualize the Grover's action using vectors.



We are iteratively moving the system state through the target $|\omega\rangle$.

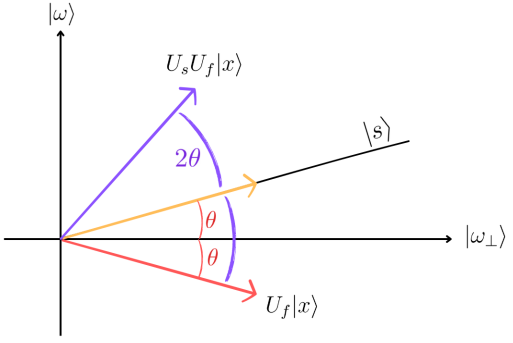
How many times do we need to iterate Grover?

As we can deduce from the previous slide, there exist an **optimal number** of Grover iterations fixed by geometry.

1. we can decompose $|x\rangle$ into the *winning* and the *losing* components $|s\rangle = \sqrt{\frac{1}{M}}\,|\omega\rangle + \sqrt{\frac{M-1}{M}}\,|\omega_\perp\rangle$.
2. The same vector can be defined in terms of the angle in the plane: $|s\rangle = \sin\theta\,|\omega\rangle + cos\theta\,|\omega_\perp\rangle$.
3. from 1. and 2. we can write $\theta = \arcsin\left(1/\sqrt{M}\right)$ and, if M is large, $\theta \approx 1/\sqrt{M}$.
4. the action of $U_s U_f$ on $|x\rangle$ is equal to a rotation of 2θ of the vector.
5. after k iteration of Grover, the angle has become: $\alpha = (2k+1)\,\theta$, and, to maximize $\sin\alpha$:

$$\alpha = \frac{\pi}{2} \quad \rightarrow \quad k = \frac{\pi}{4\theta} - \frac{1}{2} = \frac{\pi}{4}\sqrt{M} - \frac{1}{2}.$$

6. from 5. we need to get an integer, since we are talking about iterations. Commonly $\theta \approx \frac{\pi}{4}\sqrt{M}$.



Let's code!