HPC PROJECT: GEOPHYSICAL FLOW ON THE SURFACE OF THE SPHERE

DI QI

The barotropic vorticity equation describes the evolution of a homogeneous (constant density), non-divergent, incompressible flow on the surface of the sphere

$$\frac{\partial \zeta}{\partial t} = -J\left(\psi, f+\zeta\right) - \nu\left(-1\right)^m \nabla^{2m}\zeta, \tag{0.1}$$

with the relative vorticity $\zeta = \nabla^2\psi = \nabla \times \mathbf{v}$. We follow the basic method-of-lines process to solve this equation. Specifically, use pseudo-spectral method to compute the spatial discretization, and use standard leapfrog scheme for time updating, then the numerical scheme is like

$$\frac{\zeta_{n+1} - \zeta_{n-1}}{2\Delta t} = -J\left(\psi_n, f+\zeta_n\right) - \nu\left(-1\right)^m \nabla^{2m}\zeta_n. \tag{0.2}$$

Variables on the sphere for spectral method is decomposed under the spherical harmonic basis, for example for the vorticity ζ

$$\zeta\left(\theta,\lambda\right) = \sum_{m=-M}^{M} \hat{a}_m\left(\sin\theta\right) e^{im\lambda}, \quad \hat{a}_m\left(\sin\theta\right) = \sum_{l=|m|}^{N(m)} a_{lm} P_{lm}\left(\sin\theta\right). \tag{0.3}$$

with $\lambda \in [0, 2\pi)$ the longitude, $\theta \in [-\pi/2, \pi/2]$ the latitude, and P_{lm} the associated Legendre functions. Derivative about the longitude λ becomes a multiplication of im from the Fourier decomposition, and derivative about the latitude θ can be achieved through the recursive formula for Legendre functions

$$\cos\theta \frac{dP_{lm}}{d\theta} = -l\epsilon_{l+1,m} P_{l+1,m} + \left(l+1\right)\epsilon_{l,m} P_{l-1,m}, \tag{0.4}$$

where $\epsilon_{l,m} = \left(\frac{l^2-m^2}{4l^2-1}\right)^{1/2}$. Following these conventions, the equations can be written for the time evolution of the spectral coefficients in terms of the spectral coefficients themselves. We summarize the important parts of the algorithm as follows:

- *Transforms between physical space and spectral space*: The state variables need to be frequently transformed between the physical domain and the spectral domain. Under the spherical harmonics, the transform (0.3) cannot be as easy and efficient as FFT. Still, at each fixed latitude θ, the coefficients $\hat{a}_m$ can be calculated individually with the standard 1 dimensional FFT algorithm (which can be efficiently paralleled also). For the second part of (0.3), use the fact that P_{lm} are orthonormal, then the final coefficients a_{lm} for the spherical harmonics can be calculated from projection to each Legendre function

  $$a_{lm} = \int_{-1}^{1} \hat{a}_m\left(\mu\right) P_{lm}\left(\mu\right) d\mu.$$

  Then we need to evaluate the latitude grid points $\mu = \sin\theta$ at the Gaussian quadrature points for reducing error.
- *Updating the spectral coefficients a_{lm} at each time step*: To compute the vorticity tendency $J\left(\psi_n, f+\zeta_n\right)$ at each time step, it is related to the derivatives about θ and λ. Under the spherical harmonics, derivatives about λ becomes a multiplication of im and derivatives about θ can be converted to the combination of two other modes $P_{l+1,m}$ and $P_{l-1,m}$ according to (0.4).

- *Dealiasing for high wavenumber truncation*: Due to the high wavenumber truncation and the nonlinearity in the equation, dealiasing techniques needs to be used to get rid of the aliasing error. Simply we apply the standard 2/3-rule introducing $3M+1$ grid points in the $2M+1$ discretization.

Parallel can be considered in the above processes to accelerate the code:

- For the transforms between physical and spectral domain, the entire grid on the sphere can be split into several bands with constant latitude θ. FFT can be carried out independently for each band on different processors. Then the Fourier coefficients $\hat{a}_m(\mu)$ need to be scattered and gathered to calculate the Gaussian quadrature (this part can be also decomposed into different independent bands with constant first wavenumber l).
- For updating the coefficients under the spherical harmonics, the process can be decomposed into several latitude bands for different processors. The recursive formula (0.4) indicates that we only need to communicate the data between neighboring (latitude) mode $l+1, l-1$ to update mode of wavenumber l.

I would like to try these further directions if time permitted:

- Solve the rotational shallow water equations on the sphere using the above pseudo-spectral strategy. Furthermore, it will be interesting to try the multi-level shallow water system with thermodynamics considered.
- Add passive tracers into the system, and use finite volume scheme to solve the advection equations for tracers. It is interesting to see how the tracers are advected by the flow on the sphere.
- Use GPU to accelerate the code further.