AMZ 541 HW2 Shuai Zhao ucid: 7722927131

Problem 1

(a) the system is $\dot{x}(t) = e^{-At} B e^{At} x(t) \quad \cdots (1)$

the structure " $e^{-At} B e^{At}$ " suggests we could use transformation

So, suppose a new system that $x'(t) = e^{At} x(t)$

Then, $\dot{x}'(t) = \frac{d}{dt}(e^{At} \cdot x(t)) = A e^{At} x(t) + e^{At} \cdot \dot{x}(t)$

$$= A e^{At} x(t) + e^{At} \cdot [e^{-At} \cdot B \cdot e^{At} \cdot x(t)]$$
$$= A e^{At} x(t) + B e^{At} x(t)$$
$$= (A+B) e^{At} x(t)$$
$$= (A+B) x'(t)$$

i.e. $\dot{x}'(t) = (A+B) x'(t)$

This is a LTI system, showing that: $x'(t) = e^{(A+B)(t-s)} \cdot x'(s)$

Take $x'(t) = e^{At} x(t)$ back: $e^{At} \cdot x(t) = e^{(A+B)(t-s)} \cdot e^{-As} x(s)$

Simplify: $x(t) = e^{-At} e^{(A+B)(t-s)} e^{-As} \cdot x(s)$

i.e. the S.T.M. is $\Phi(t,s) = e^{-At} e^{(A+B)(t-s)} e^{-As}$

(b) the definition of matrix exp : $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$

$$\int_0^t e^{A\sigma} d\sigma = \int_0^t \sum_{k=0}^{\infty} \frac{(A\sigma)^k}{k!} d\sigma$$
$$= \int_0^t [I + A\sigma + \frac{1}{2}(A\sigma)^2 + \frac{1}{3!}(A\sigma)^3 + \cdots + \frac{1}{k!}(A\sigma)^k + \cdots] d\sigma$$
$$= I \cdot t + A \cdot \frac{1}{2}t^2 + \frac{1}{2}A^2 \cdot \frac{1}{3}t^3 + \cdots + \frac{1}{k!}A^k \cdot \frac{1}{k+1}t^{k+1} + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{A^k}{(k+1)!} \cdot t^{k+1}$$

$$[e^{At} - I] \cdot A^{-1} = [\sum_{k=0}^{\infty} \frac{(At)^k}{k!} - I] \cdot A^{-1}$$
$$= [I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \cdots + \frac{1}{k!}(At)^k + \cdots - I] \cdot A^{-1}$$
$$= [At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{k!}A^k t^k + \cdots] \cdot A^{-1}$$
$$= I \cdot t + \frac{1}{2}At^2 + \frac{1}{3!}A^2t^3 + \cdots + \frac{1}{k!}A^{k-1}t^k + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{A^k}{(k+1)!} \cdot t^{k+1}$$

So, $\int_0^t e^{A\sigma} d\sigma = [e^{At} - I] \cdot A^{-1}$

For $\dot{x} = Ax + B\bar{u}$, the sol'n is :

$$x(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\tau)} B\bar{u}\, d\tau \qquad (\bar{u} \text{ is constant vector})$$
$$= e^{At} \cdot x(0) + e^{At} \int_0^t e^{-A\tau} d\tau \cdot B\bar{u}$$

According to the result above, $\int_0^t e^{A\sigma} d\sigma = [e^{At} - I] \cdot A^{-1}$

we could know : $\int_0^t e^{-A\tau} d\tau = (e^{-At} - I) \cdot (-A)^{-1} = (I - e^{-At}) \cdot A^{-1}$

Thus, $x(t) = e^{At} \cdot x(0) + e^{At}(I - e^{-At}) \cdot A^{-1} B\bar{u}$

$$= e^{At} \cdot x(0) + (e^{At} - I) A^{-1} B\bar{u}$$

Problem 2

(a) Linear:

Suppose we have two inputs : $u_1(k)$, $u_2(k)$

Then, $\begin{cases} \dot{x}_1(k+1) = Ax_1(k) + Bu_1(k) \\ y_1(k) = Cx_1(k) + Du_1(k) \\ x_1(k_0) = x_{10} \end{cases}$ $\begin{cases} \dot{x}_2(k+1) = Ax_2(k) + Bu_2(k) \\ y_2(k) = Cx_2(k) + Du_2(k) \\ x_2(k_0) = x_{20} \end{cases}$

For the state-space eqn :

$$\alpha x_1(k+1) + \beta x_2(k+1) = \alpha [Ax_1(k) + Bu_1(k)] + \beta [Ax_2(k) + Bu_2(k)]$$
$$= A[\alpha x_1(k) + \beta x_2(k)] + B[\alpha u_1(k) + \beta u_2(k)]$$

Similiarily for output eqn :

$$\alpha y_1(k) + \beta y_2(k) = \alpha [Cx_1(k) + Du_1(k)] + \beta [Cx_2(k) + Du_2(k)]$$
$$= C[\alpha x_1(k) + \beta x_2(k)] + D[\alpha u_1(k) + \beta u_2(k)]$$

Therefore, the system is linear

Time-ivariant :

Suppose we have a time switch on the input : $u(k) \rightarrow u(k+m)$

How to?

(b) Consider:

$$(I-\frac{A}{z})\cdot\sum_{k=0}^{\infty}(\frac{A}{z})^k$$
$$=(I-\frac{A}{z})\cdot(I+\frac{A}{z}+(\frac{A}{z})^2+(\frac{A}{z})^3+\cdots)$$
$$=(I+\frac{A}{z}+(\frac{A}{z})^2+(\frac{A}{z})^3+\cdots)-\frac{A}{z}(I+\frac{A}{z}+(\frac{A}{z})^2+(\frac{A}{z})^3+\cdots)$$
$$=(I+\frac{A}{z}+(\frac{A}{z})^2+(\frac{A}{z})^3+\cdots)-(\frac{A}{z}\ (\frac{A}{z})^2+(\frac{A}{z})^3+(\frac{A}{z})^4+\cdots)$$
$$=I$$

Thus:

$$\sum_{k=0}^{\infty}(\frac{A}{z})^k=(I-\frac{A}{z})^{-1}$$

By the definition of z-transform:

$$\mathcal{Z}(A^k)=\sum_{k=0}^{\infty}A^k\cdot z^{-k}=\sum_{k=0}^{\infty}(\frac{A}{z})^{-k}=(I-\frac{A}{z})^{-1}$$
$$=\left[\frac{1}{z}(zI-A)\right]^{-1}$$
$$=z(zI-A)^{-1}$$

i.e. $\mathcal{Z}(A^k)=zR(z)$, where $R(z)=(zI-A)^{-1}$

(c) $A=\begin{bmatrix}0 & 1\\ -0.5 & 0.3\end{bmatrix} \Rightarrow zI-A=\begin{bmatrix}z & -1\\ 0.5 & z-0.3\end{bmatrix}$

$\Rightarrow \det(zI-A)=z(z-0.3)+0.5=z^2-0.3z+0.5$

$$R(z)=(zI-A)^{-1}=\frac{1}{z^2-0.3z+0.5}\begin{bmatrix}z-0.3 & 1\\ -0.5 & z\end{bmatrix}$$

$$\mathcal{Z}(A^k)=z\cdot R(z)=\frac{z}{z^2-0.3z+0.5}\begin{bmatrix}z-0.3 & 1\\ -0.5 & z\end{bmatrix}$$

After the inverse z-transformation: (and $k=9$)

```
State transition matrix Phi(9):
   0.044785650000000  -0.022110890000000
   0.011055445000000   0.038152383000000
```

```
clc;
clear;

syms z k

A = [0 1; -0.5 0.3];
I = eye(2);
R_z = inv(z * I - A);
zR_z = z * R_z;

Phi_11 = iztrans(zR_z(1,1), z, k);
Phi_12 = iztrans(zR_z(1,2), z, k);
Phi_21 = iztrans(zR_z(2,1), z, k);
Phi_22 = iztrans(zR_z(2,2), z, k);

Phi_k = [Phi_11, Phi_12; Phi_21, Phi_22];

% k=9
Phi_k_at_9 = subs(Phi_k, k, 9);
Phi_k_at_9_num = double(Phi_k_at_9);

disp('State transition matrix Phi(9):');
disp(Phi_k_at_9_num);
```

```
State transition matrix Phi(9):
   0.044785650000000  -0.022110890000000
   0.011055445000000   0.038152383000000
```

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Problem 3

(a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

Notice that $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^3 = A^2 \cdot A = 0$

$\vdots$

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

eigenvalue decompose:

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix} \Rightarrow |\lambda I - A| = \lambda^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$$

then, solve $(\lambda I - A)\cdot\vec{u} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \Rightarrow u_2 = 0$

So the eigen vector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $\lambda_1 = \lambda_2 = 0$

(b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

Notice that $A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

$A^3 = A^2 \cdot A = -I \cdot A = -A$

$A^4 = A^2 \cdot A^2 = (-I)\cdot(-I) = I$

So,

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{1}{2}(-I)t^2 + \frac{1}{3!}(-A)t^3 + \frac{1}{4!}\cdot I \cdot t^4 + \cdots$$

$$= I\left(1 - \frac{1}{2}t^2 + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots\right) + A\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)$$

$$= \cos(t)\cdot I + \sin(t)\cdot A$$

$$= \begin{bmatrix} \cos(t) & 0 \\ 0 & \cos(t) \end{bmatrix} + \begin{bmatrix} 0 & \sin(t) \\ -\sin(t) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

eigenvalue decompose:

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} \Rightarrow |\lambda I - A| = \lambda^2 + 1 = 0 \Rightarrow \lambda_1 = i,\ \lambda_2 = -i$$

For $\lambda_1 = i$, solve:

$$(\lambda I - A)\vec{u} = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} iu_1 - u_2 = 0 \\ u_1 + iu_2 = 0 \end{cases} \Rightarrow iu_1 = u_2$$

So the eigen vector for $\lambda_1 = i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}$

For $\lambda_1 = -i$, solve:

$$(\lambda I - A)\vec{u} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} -iu_1 - u_2 = 0 \\ u_1 - iu_2 = 0 \end{cases} \Rightarrow -iu_1 = u_2$$

So the eigen vector for $\lambda_2 = -i$ is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

Problem 4

A S.T.M should satisfy : $\frac{\partial \Phi(t,t_0)}{\partial t} = A(t)\cdot \Phi(t,t_0)$

If $\Phi(t,t_0) = e^{\int_{t_0}^{t} A(\sigma)d\sigma}$ is a S.T.M ,

$$\frac{\partial \Phi}{\partial t} = e^{\int_{t_0}^{t} A(\sigma)d\sigma} \cdot \frac{\partial}{\partial t}\left[\int_{t_0}^{t} A(\sigma)d\sigma\right] =$$

$$= e^{\int_{t_0}^{t} A(\sigma)d\sigma} \cdot A(t)$$

$$= \sum_{k=0}^{\infty}\left[\frac{1}{k!}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^k\right]\cdot A(t)$$

Since $\int_{t_0}^{t} A(\sigma)d\sigma \cdot A(t) = A(t)\cdot \int_{t_0}^{t} A(\sigma)d\sigma$,

$$\Rightarrow \left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^2 A(t) = \int_{t_0}^{t} A(\sigma)d\sigma \cdot A(t)\cdot \int_{t_0}^{t} A(\sigma)d\sigma$$

$$= A(t)\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^2$$

Similarly, $\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^k \cdot A(t) = A(t)\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^k$, $\forall k = 0, 1, 2, \cdots$

Thus, $\frac{\partial \Phi}{\partial t} = \sum_{k=0}^{\infty}\left[\frac{1}{k!}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^k\right]\cdot A(t)$

$$= \left(1 + \int_{t_0}^{t} A(\sigma)d\sigma + \frac{1}{2}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^2 + \frac{1}{3!}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^3 + \cdots\right)\cdot A(t)$$

$$= A(t) + \int_{t_0}^{t} A(\sigma)d\sigma \cdot A(t) + \frac{1}{2}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^2\cdot A(t) + \frac{1}{3!}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^3\cdot A(t) + \cdots$$

$$= A(t) + A(t)\cdot\int_{t_0}^{t} A(\sigma)d\sigma + A(t)\cdot\frac{1}{2}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^2 + A(t)\cdot\frac{1}{3!}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^3 + \cdots$$

$$= A(t)\cdot\left(1 + \int_{t_0}^{t} A(\sigma)d\sigma + \frac{1}{2}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^2 + \frac{1}{3!}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^3 + \cdots\right)$$

$$= A(t)\cdot\sum_{k=0}^{\infty}\left[\frac{1}{k!}\cdot\left(\int_{t_0}^{t} A(\sigma)d\sigma\right)^k\right]$$

$$= A(t)\cdot e^{\int_{t_0}^{t} A(\sigma)d\sigma}$$

i.e. $\frac{\partial \Phi}{\partial t} = A(t)\cdot \Phi$

Besides, $\Phi(t_0,t_0) = \Phi(t,t_0)\big|_{t=t_0} = e^{\int_{t_0}^{t_0} A(\sigma)d\sigma} = e^0 = I.$

So, Φ is an S.T.M

Problem 5.

The equation is: $\ddot{y}(t) + (2-\cos(2t))y(t) = 0$

Let $x_1 = y$, $x_2 = \dot{y}$, then $\begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = (\cos(2t)-2)\cdot y(t) = (\cos(2t)-2)\cdot x_1 \end{cases}$

so the state eqn is:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (\cos(2t)-2)x_1 \end{cases} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \cos(2t)-2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

the output eqn is:

$$y = x_1 \Rightarrow y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Answer by Matlab:

```
on the time interval of length equal to three periods of oscillations for t0 = 0
   0.942135130380487   0.378508906373619
  -0.297404112355399   0.942171885496012

on the time interval of length equal to three periods of oscillations for t0 = 1
   1.086870981799333   0.244552932093098
  -0.546381236772937   0.797533945490601
```

Contents


```
clear;
clc;
close all;
```

define parameters

```
omega = 2;
alpha = 1;
```

A(t)

```
A = @(t) [0, 1; -(omega - alpha * cos(2*t)), 0];
```

ODE: Phi_dot = A * Phi

```
odefun = @(t, X) reshape(A(t) * reshape(X, 2, 2), 4, 1);
X0 = reshape(eye(2), 4, 1);

% for initial condition t0=0 and t0=1
tspan_0 = [0 3*pi];
tspan_1 = [1 1 + 3*pi];

% solve ODE
[t_0, X_0] = ode45(odefun, tspan_0, X0);
[t_1, X_1] = ode45(odefun, tspan_1, X0);

% solution(4x1) to Phi(2x2)
Phi_t_t0_0 = zeros(length(t_0), 2, 2);
Phi_t_t0_1 = zeros(length(t_1), 2, 2);

for i = 1:length(t_0)
    Phi_t_t0_0(i, :, :) = reshape(X_0(i, :), 2, 2);
end
for i = 1:length(t_1)
    Phi_t_t0_1(i, :, :) = reshape(X_1(i, :), 2, 2);
end
```

print answer

```
disp('on the time interval of length equal to three periods of oscillations for t0 = 0');
Phi_t0_0_end = reshape(Phi_t_t0_0(end, :, :), 2, 2);
disp(Phi_t0_0_end);

disp('on the time interval of length equal to three periods of oscillations for t0 = 1');
Phi_t0_1_end = reshape(Phi_t_t0_1(end, :, :), 2, 2);
disp(Phi_t0_1_end);
```

```
on the time interval of length equal to three periods of oscillations for t0 = 0
   0.942135130380487   0.378508906373619
  -0.297404112355399   0.942171885496012

on the time interval of length equal to three periods of oscillations for t0 = 1
   1.086870981799333   0.244552932093098
  -0.546381236772937   0.797533945490601
```

plot just for checking

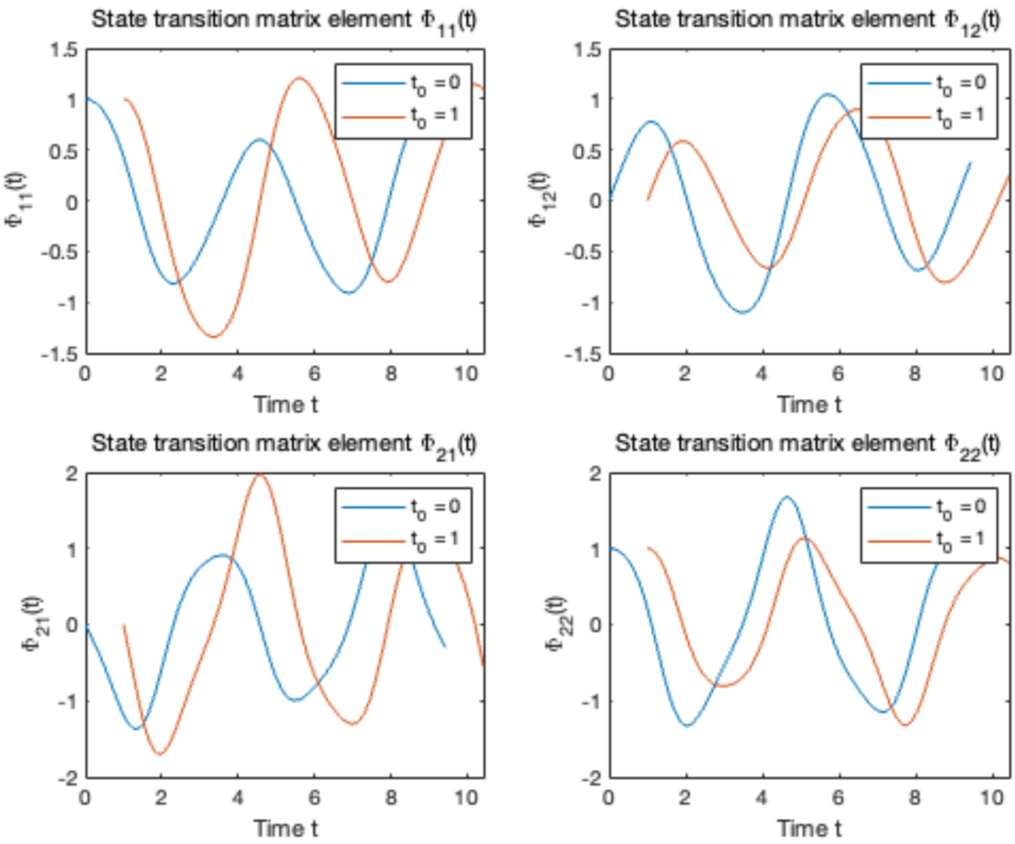
```
figure;
subplot(2,2,1);
plot(t_0, squeeze(Phi_t_t0_0(:,1,1)), 'DisplayName', 't_0 = 0');
hold on;
plot(t_1, squeeze(Phi_t_t0_1(:,1,1)), 'DisplayName', 't_0 = 1');
xlabel('Time t');
ylabel('\Phi_{11}(t)');
legend;
title('State transition matrix element \Phi_{11}(t)');

subplot(2,2,2);
plot(t_0, squeeze(Phi_t_t0_0(:,1,2)), 'DisplayName', 't_0 = 0');
hold on;
plot(t_1, squeeze(Phi_t_t0_1(:,1,2)), 'DisplayName', 't_0 = 1');
xlabel('Time t');
ylabel('\Phi_{12}(t)');
legend;
title('State transition matrix element \Phi_{12}(t)');

subplot(2,2,3);
plot(t_0, squeeze(Phi_t_t0_0(:,2,1)), 'DisplayName', 't_0 = 0');
hold on;
plot(t_1, squeeze(Phi_t_t0_1(:,2,1)), 'DisplayName', 't_0 = 1');
xlabel('Time t');
ylabel('\Phi_{21}(t)');
legend;
title('State transition matrix element \Phi_{21}(t)');

subplot(2,2,4);
plot(t_0, squeeze(Phi_t_t0_0(:,2,2)), 'DisplayName', 't_0 = 0');
hold on;
plot(t_1, squeeze(Phi_t_t0_1(:,2,2)), 'DisplayName', 't_0 = 1');
xlabel('Time t');
ylabel('\Phi_{22}(t)');
```

```
legend;
title('State transition matrix element \Phi_{22}(t)');
```



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Problem 6

(a) $$A = \begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix}, \quad A^* = \begin{bmatrix} A_{11}^T & \alpha A_{21}^T \\ 0 & A_{22}^T \end{bmatrix}$$

$$A \cdot A^* = \begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} A_{11}^T & \alpha A_{21}^T \\ 0 & A_{22}^T \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^T & \alpha A_{11}A_{21}^T \\ \alpha A_{21}A_{11}^T & \alpha^2 A_{21}A_{21}^T + A_{22}A_{22}^T \end{bmatrix}$$

$$A^* \cdot A = \begin{bmatrix} A_{11}^T & \alpha A_{21}^T \\ 0 & A_{22}^T \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^T A_{11} + \alpha^2 A_{21}^T A_{21} & \alpha A_{21}^T A_{22} \\ \alpha A_{22}^T A_{21} & A_{22}^T A_{22} \end{bmatrix}$$

$A \cdot A^* \neq A^* \cdot A \Rightarrow A$ is NOT normal

(b) the S.T.M. would be :

$$\Phi(t) = e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}(At)^2$$

$$A = \begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^2 & 0 \\ \alpha A_{21}A_{11} + \alpha A_{22}A_{21} & A_{22}^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} A_{11}^2 & 0 \\ \alpha A_{21}A_{11} + \alpha A_{22}A_{21} & A_{22}^2 \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}^3 & 0 \\ \alpha A_{21}A_{11}^2 + \alpha A_{22}A_{21}A_{11} + \alpha A_{22}^2 A_{21} & A_{22}^2 \end{bmatrix}$$

$\vdots$

$$A^k = \begin{bmatrix} A_{11}^k & 0 \\ \alpha \sum_{m=0}^{k-1} A_{22}^m A_{21} A_{11}^{k-1-m} & A_{22}^k \end{bmatrix}$$

$$\text{So, } \Phi = e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} A_{11}^k t^k & 0 \\ \alpha \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{m=0}^{k-1} A_{22}^m A_{21} A_{11}^{k-1-m} & \sum_{k=0}^{\infty} \frac{1}{k!} A_{22}^k t^k \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} e^{A_{11}t} & 0 \\ \alpha \int_0^t e^{A_{22}(t-\tau)} A_{21} e^{A_{11}\tau} d\tau & e^{A_{22}t} \end{bmatrix}}$$

the resolvent is: $R = (zI - A)^{-1}$

$$A = \begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix}$$

$$zI - A = \begin{bmatrix} zI - A_{11} & 0 \\ -\alpha A_{21} & zI - A_{22} \end{bmatrix}$$

$$(zI - A)^{-1} = \begin{bmatrix} (zI - A_{11})^{-1} & 0 \\ -(zI - A_{22})^{-1}(-\alpha A_{21})(zI - A_{11})^{-1} & (zI - A_{22})^{-1} \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} (zI - A_{11})^{-1} & 0 \\ \alpha (zI - A_{22})^{-1} A_{21} (zI - A_{11})^{-1} & (zI - A_{22})^{-1} \end{bmatrix}}$$

(c) Now, with $A_{11} = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$, $A_{22} = -1$, $A_{21} = [1 \quad 2]$

$$A = \begin{bmatrix} -1 & -2 & 0 \\ -2 & -5 & 0 \\ \alpha & 2\alpha & -1 \end{bmatrix}$$

(1) Right eigenvector:

$$A - \lambda I = \begin{bmatrix} -1-\lambda & -2 & 0 \\ -2 & -5-\lambda & 0 \\ \alpha & 2\alpha & -1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (-1-\lambda) \det \begin{bmatrix} -1-\lambda & -2 \\ -2 & -5-\lambda \end{bmatrix} = (-1-\lambda)[(-1-\lambda)(-5-\lambda) - 4]$$
$$= (-1-\lambda)(\lambda^2 + 6\lambda + 1)$$

eigen value: $(1+\lambda)(\lambda^2 + 6\lambda + 1) = 0$

for $(1+\lambda) = 0 \Rightarrow \lambda_1 = -1$

for $(\lambda^2 + 6\lambda + 1) = (\lambda + 3)^2 - 8 = 0 \Rightarrow \lambda_2 = -2\sqrt{2} - 3$, $\lambda_3 = 2\sqrt{2} - 3$

eigen vector:

$\lambda_1 = -1$, $(A - \lambda I)\vec{w} = \begin{bmatrix} 0 & -2 & 0 \\ -2 & -4 & 0 \\ \alpha & 2\alpha & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$$\begin{cases} -2w_2 = 0 \\ -2w_1 - 4w_2 = 0 \\ \alpha w_1 + 2w_2 = 0 \end{cases} \Rightarrow w_1 = w_2 = 0$$

Thus, for $\lambda_1 = -1$, $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\lambda_2=-2\sqrt{2}-3,\ (A-\lambda I)w=\begin{bmatrix} 2+2\sqrt{2} & -2 & 0 \\ -2 & -2+2\sqrt{2} & 0 \\ \alpha & 2\alpha & 2+2\sqrt{2} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\begin{cases} (1+\sqrt{2})w_1 - w_2 = 0 \\ w_1 + (1-\sqrt{2})w_2 = 0 \\ \alpha w_1 + 2\alpha w_2 + (2+2\sqrt{2})w_3 = 0 \end{cases} \Rightarrow \begin{cases} w_2 = (1+\sqrt{2})w_1 \\ w_3 = \frac{\alpha}{2}(-\sqrt{2}-1)\cdot w_1 \end{cases}$$

Thus, for $\lambda_2=-2\sqrt{2}-3,\ \vec{w}=\begin{bmatrix} 1 \\ 1+\sqrt{2} \\ -\frac{\alpha}{2}(\sqrt{2}+1) \end{bmatrix}$

$$\lambda_3= 2\sqrt{2}-3,\ (A-\lambda I)w=\begin{bmatrix} 2-2\sqrt{2} & -2 & 0 \\ -2 & -2-2\sqrt{2} & 0 \\ \alpha & 2\alpha & 2-2\sqrt{2} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\begin{cases} (1-\sqrt{2})w_1 - w_2 = 0 \\ w_1 + (1+\sqrt{2})w_2 = 0 \\ \alpha w_1 + 2\alpha w_2 + (2-2\sqrt{2})w_3 = 0 \end{cases} \Rightarrow \begin{cases} w_2 = (1-\sqrt{2})w_1 \\ w_3 = \frac{\alpha(w_1+2w_2)}{2(\sqrt{2}-1)} = \frac{\alpha}{2}\cdot\frac{3-\sqrt{2}}{\sqrt{2}-1}\cdot w_1 \end{cases}$$

$$= \frac{\alpha}{2}(\sqrt{2}-1)\cdot w_1$$

Thus, for $\lambda_3= 2\sqrt{2}-3,\ \vec{w}=\begin{bmatrix} 1 \\ 1-\sqrt{2} \\ \frac{\alpha}{2}(\sqrt{2}-1) \end{bmatrix}$

(2) Left eigen vector : $\vec{u}^T A = \lambda\vec{u} \Rightarrow A^T\vec{u} = \lambda\vec{u}$

$$\lambda_1=-1,\ (A^T-\lambda I)u=\begin{bmatrix} 0 & -2 & \alpha \\ -2 & -4 & 2\alpha \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{cases} -2u_2 + \alpha u_3 = 0 \\ -2u_1 - 4u_2 + 2\alpha u_3 = 0 \end{cases} \Rightarrow \begin{cases} u_1 = 0 \\ u_2 = \frac{\alpha}{2}u_3 \end{cases}$$

Thus, for $\lambda_1=-1,\ \vec{u}=\begin{bmatrix} 0 \\ \alpha/2 \\ 1 \end{bmatrix}$

$$\lambda_2=-2\sqrt{2}-3,\ (A^T-\lambda I)u=\begin{bmatrix} 2-2\sqrt{2} & -2 & \alpha \\ -2 & -2+2\sqrt{2} & 2\alpha \\ 0 & 0 & 2-2\sqrt{2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{cases} (2-2\sqrt{2})u_1 - 2u_2 + \alpha u_3 = 0 \\ -2u_1 + (-2+2\sqrt{2})u_2 + 2\alpha u_3 = 0 \\ (2-2\sqrt{3})u_3 = 0 \end{cases} \Rightarrow \begin{cases} u_1 = (\sqrt{2}-1)\ u_2 \\ u_3 = 0 \end{cases}$$

Thus, for $\lambda_2=-2\sqrt{2}-3,\ \vec{u}=\begin{bmatrix} \sqrt{2}-1 \\ 1 \\ 0 \end{bmatrix}$

$$\lambda_3 = 2\sqrt{2}-3, (A^T - \lambda I)v = \begin{bmatrix} 2+2\sqrt{2} & -2 & \alpha \\ -2 & -2-2\sqrt{2} & 2\alpha \\ 0 & 0 & 2+2\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{cases} (2+2\sqrt{2})v_1 - 2v_2 + \alpha v_3 = 0 \\ -2v_1 + (-2-2\sqrt{2})v_2 + 2\alpha v_3 = 0 \\ (2+2\sqrt{2})v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_1 = -(\sqrt{2}+1)v_2 \\ v_3 = 0 \end{cases}$$

Thus, for $\lambda_3 = 2\sqrt{2}-3$, $\vec{v} = \begin{bmatrix} -(\sqrt{2}+1) \\ 1 \\ 0 \end{bmatrix}$

(3) in order to fit the bi-orthogonal condition:

consider $V^T W = \begin{bmatrix} 1 & & \\ & 2\sqrt{2} & \\ & & -2\sqrt{2} \end{bmatrix}$

so $\vec{v_2}$ & $\vec{v_3}$ should be devided by $2\sqrt{2}$ & $-2\sqrt{2}$

therefore,

Left vector: $V = \begin{bmatrix} 0 & \frac{\sqrt{2}-1}{2\sqrt{2}} & -\frac{\sqrt{2}+1}{-2\sqrt{2}} \\ \frac{\alpha}{2} & 1/2\sqrt{2} & 1/(-2\sqrt{2}) \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}-\frac{\sqrt{2}}{4} & -\frac{1}{2}-\frac{\sqrt{2}}{4} \\ \frac{\alpha}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 1 & 0 & 0 \end{bmatrix}$

Right vector: $W = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & -\frac{\alpha}{2}(\sqrt{2}+1) & \frac{\alpha}{2}(\sqrt{2}-1) \end{bmatrix}$

eigen values $\Lambda = \begin{bmatrix} -1 & & \\ & -2\sqrt{2}-3 & \\ & & 2\sqrt{2}-3 \end{bmatrix}$

such V, W can satisfy $V^T W = I$,

i.e. satisfy bi-othogonal condition.

(e) the S.T.M is:

$$\Phi(t) = e^{At} = W \cdot e^{\Lambda t} \cdot V$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & -\frac{\alpha}{2}(\sqrt{2}+1) & \frac{\alpha}{2}(\sqrt{2}-1) \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & e^{(-2\sqrt{2}-3)t} & \\ & & e^{(2\sqrt{2}-3)t} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2}-\frac{\sqrt{2}}{4} & -\frac{1}{2}-\frac{\sqrt{2}}{4} \\ \frac{\alpha}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 1 & 0 & 0 \end{bmatrix}$$

So,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Phi(t) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Since $Y = [0 \ \ 1] \, x$,

we only care about $x_2(t) \Rightarrow$ we only need to derive the 3rd row of Φ

$$y(t) = x_2(t) = \begin{bmatrix} \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\Phi(t) = e^{At} = W \cdot e^{\Lambda t} \cdot V$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & -\frac{\alpha}{2}(\sqrt{2}+1) & \frac{\alpha}{2}(\sqrt{2}-1) \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & e^{(-2\sqrt{2}-3)t} & \\ & & e^{(2\sqrt{2}-3)t} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2}-\frac{\sqrt{2}}{4} & -\frac{1}{2}-\frac{\sqrt{2}}{4} \\ \frac{\alpha}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\alpha}{2}(\sqrt{2}+1) & \frac{\alpha}{2}(\sqrt{2}-1) \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & e^{(-2\sqrt{2}-3)t} & \\ & & e^{(2\sqrt{2}-3)t} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2}-\frac{\sqrt{2}}{4} & -\frac{1}{2}-\frac{\sqrt{2}}{4} \\ \frac{\alpha}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & -\frac{\alpha}{2}(\sqrt{2}+1)e^{(-2\sqrt{2}-3)t} & \frac{\alpha}{2}(\sqrt{2}-1)e^{(2\sqrt{2}-3)t} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2}-\frac{\sqrt{2}}{4} & -\frac{1}{2}-\frac{\sqrt{2}}{4} \\ \frac{\alpha}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 1 & 0 & 0 \end{bmatrix}$$

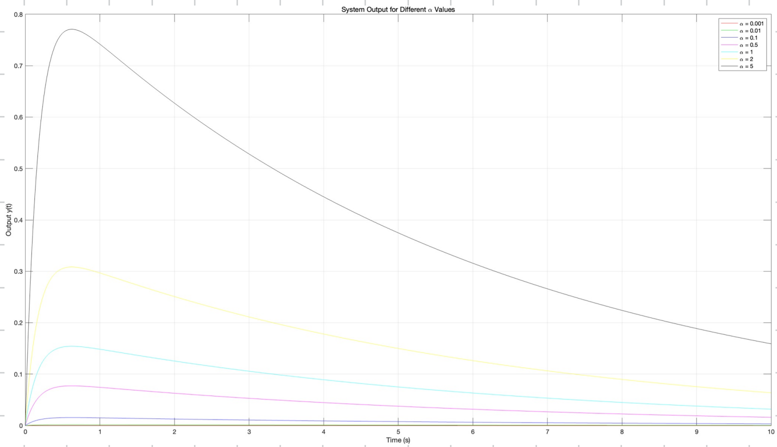
$$\Rightarrow \quad \Phi_{31} = -\frac{1}{4}(\sqrt{2}-1)\alpha e^{-(3+2\sqrt{2})t}\left[(3+2\sqrt{2})\alpha - 2e^{4\sqrt{2}t}\right]$$

$$\Phi_{32} = -\frac{1}{8}(2+\sqrt{2})\alpha e^{-(3+2\sqrt{2})t} - \frac{1}{4}(\sqrt{2}-2)e^{-t}$$

$$\Phi_{33} = \frac{1}{8}(2+\sqrt{2})\alpha e^{-(3+2\sqrt{2})t} + \frac{1}{4}(\sqrt{2}+2)e^{-t}$$

Then: $y(t) = \Phi_{31} x_{11}(0) + \Phi_{32} x_{12}(0) + \Phi_{33} x_2(0)$

(f)



Let $x[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

we can get y in fig.

the overshoot of y increases as α rises.

Contents


```
clear;
clc;
close all;
```

eigenvalues eigenvectors

```
syms alpha t

% A
A = [-1, -2, 0;
     -2, -5, 0;
      alpha, 2*alpha, -1];

% right vectors
[W, D] = eig(A);
W(:,2) = W(:,2) / W(1,2);
W(:,3) = W(:,3) / W(1,3);
W = simplify(W);

% left vectors
[V, D_left] = eig(A');
V(:,2) = V(:,2) / (2*2^(1/2));
V(:,3) = V(:,3) / -(2*2^(1/2));
V = simplify(V);
```

Phi

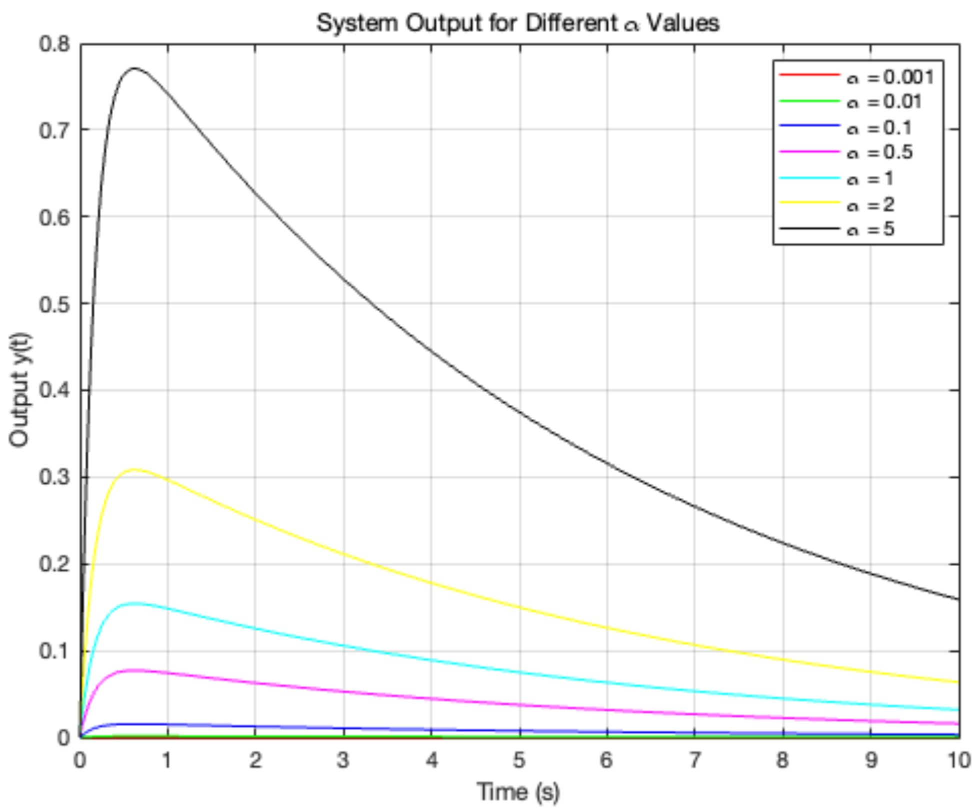
```
third_row = W(3, :);
Phi_last_row = third_row * expm(D * t) * V';
```

Output

initial condition

```
x0 = [1; 0; 0];

t_vals = linspace(0, 10, 1000);
alpha_values = [0.001, 0.01, 0.1, 0.5, 1, 2, 5];
colors = {'r', 'g', 'b', 'm', 'c', 'y', 'k'};

figure;
for i = 1:length(alpha_values)
    alpha_val = alpha_values(i);

    % y
    y_t = Phi_last_row * x0;
```

```
    y_numeric = subs(y_t, {alpha, t}, {alpha_val, t_vals});
    y_numeric = double(y_numeric);
    plot(t_vals, y_numeric, 'Color', colors{i}, 'DisplayName', ['\alpha = ', num2str(alpha_val)]);
    hold on;
end

xlabel('Time (s)');
ylabel('Output y(t)');
title('System Output for Different \alpha Values');
legend('show');
grid on;
hold off;
```



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