AME 541 HW1 Shuai Zhao uscid: 7722927131

Problem 1

(a) Since $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p_x\cos\theta + (p_y-1)\sin\theta \\ -p_x\sin\theta + (p_y-1)\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} p_x \\ p_y-1 \end{bmatrix}$,

$$\begin{bmatrix} p_x \\ p_y-1 \end{bmatrix} = \frac{1}{\cos^2\theta-(-\sin^2\theta)}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

i.e. $\begin{cases} p_x = x_1\cos x_3 - x_2\sin x_3 \\ p_y - 1 = x_1\sin x_3 + x_2\cos x_3 \end{cases}$

Since $x_1 = p_x\cos\theta + (p_y-1)\sin\theta$,

$$\begin{aligned}
\dot{x}_1 &= \dot{p}_x\cos\theta - p_x\sin\theta\,\dot\theta + \dot{p}_y\sin\theta + (p_y-1)\cos\theta\,\dot\theta \\
&= u\cos^2\theta - (x_1\cos x_3 - x_2\sin x_3)\omega\sin x_3 + u\sin^2\theta + (x_1\sin x_3 + x_2\cos x_3)\omega\cos x_3 \\
&= u - x_1\omega\sin x_3\cos x_3 + x_2\omega\sin^2 x_3 + x_1\omega\sin x_3\cos x_3 + x_2\omega\cos^2 x_3 \\
&= u + \omega x_2
\end{aligned}$$

Similarly, $x_2 = -p_x\sin\theta + (p_y-1)\cos\theta$

$$\begin{aligned}
\dot{x}_2 &= -\dot{p}_x\sin\theta - p_x\omega\cos\theta + \dot{p}_y\cos\theta - (p_y-1)\omega\sin\theta \\
&= -\omega x_1
\end{aligned}$$

Easy to notice that: $\dot{x}_3 = \dot\theta = \omega$

Therefore:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \omega x_2 + u \\ -\omega x_1 \\ +\omega \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}^{B}\begin{bmatrix} u \\ \omega \end{bmatrix}$$

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{C}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D}\begin{bmatrix} u \\ \omega \end{bmatrix}$$

(b) $\dot{x} = f(x,u,t) = \begin{bmatrix} \omega x_2 + U \\ -\omega x_1 \\ + \omega \end{bmatrix}$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 1 & x_2 \\ 0 & -x_1 \\ 0 & 1 \end{bmatrix}$$

$$y = g(x,u,t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial g}{\partial u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

At the eq point, $x^{eq} = 0$, $u^{eq} = 0$:

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tilde{u}$$

$$\tilde{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tilde{u}$$

(c) Since $p_x(t) = \sin t$, $p_y(t) = 1 - \cos t$, $\theta(t) = t$:

$$\begin{cases} x_1 = p_x \cos\theta + (p_y - 1)\sin\theta = \sin t \cos t - \cos t \sin t = 0 \\ x_2 = -\sin^2 t - \cos^2 t = -1 \\ x_3 = \theta = t \end{cases}$$

Thus, $\dot{x}_1 = 0$, $\dot{x}_2 = 0$, $\dot{x}_3 = t' = 1$

plug $\omega = U = 1$ into the state eqn :

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \omega x_2 + U \\ -\omega x_1 \\ + \omega \end{bmatrix} \Rightarrow \begin{cases} \dot{x}_1 = 1 \cdot (-1) + 1 = 0 \\ \dot{x}_2 = -1 \cdot 0 = 0 \\ \dot{x}_3 = 1 \end{cases},$$

which can match the conclusion above.

So,

$u = \omega = 1$, $p_x = \sin t$, $p_y = 1 - \cos t$, $\theta = t$ is a solution.

(d) $\dot{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ the state & output won't change.

So the system is a LTI system.

Problem 2

(a) the system is $m\ddot{y} + g(y) = 0$

Let $x_1 = y$, $x_2 = \dot{x}_1 = \dot{y}$, then $\dot{x}_2 = \ddot{y}$

(1) For hardening spring:

$g(y) = k(1+y^2)y$

$$m\dot{x}_2 + k(1+x_1^2)x_1 = 0$$

$$\dot{x}_2 = -\frac{k}{m}(x_1^3 + x_1)$$

The state eqns are:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}(x_1^3 + x_1) \end{cases}$$

(2) For softening spring:

$g(y) = k(1-y^2)y$

$$m\dot{x}_2 + k(1-x_1^2)x_1 = 0$$

$$\dot{x}_2 = -\frac{k}{m}(-x_1^3 + x_1)$$

The state eqns are:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}(-x_1^3 + x_1) \end{cases}$$

(3) For linear spring:

$g(y) = ky$

$$m\dot{x}_2 + kx_1 = 0$$

$$\dot{x}_2 = -\frac{k}{m}x_1$$

The state eqns are:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 \end{cases}$$

(b) to find the eq point, Let $\dot{x} = 0$

(1) For hardening spring:

$$\dot{x}_1 = x_2 = 0$$

$$\dot{x}_2 = -\frac{k}{m}(x_1^3 + x_1) = 0$$

$$\Rightarrow x_1(1+x_1^2) = 0$$

$$\Rightarrow x_1 = 0$$

So the eq point is:

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

only 1 eq point

But not linear

hard to leave eq point

However a strong impact

will make it more instable

(2) For softening spring:

$$\dot{x}_1 = x_2 = 0$$

$$\dot{x}_2 = -\frac{k}{m}(-x_1^3 + x_1) = 0$$

$$\Rightarrow x_1(1-x_1^2) = 0$$

$$\Rightarrow x_1 = \{0, \pm 1\}$$

So the eq points are:

$$x = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

3 eq points,

which mean its motion

will be complex

(3) For linear spring:

$$\dot{x}_1 = x_2 = 0$$

$$\dot{x}_2 = -\frac{k}{m}x_1 = 0$$

$$\Rightarrow x_1 = 0$$

So the eq point is:

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Easiest model.

Only 1 eq point,

will be harmonic around eq point.

(c) • Causality : ALL 3 are casual.

depend only on present and past (force is determined by position)

• time-varing : ALL 3 are time-invariance

m, k won't change with time.

• linearity : ONLY the linear spring system is linear. state eqn

For the hardening and softening spring system, there is x_1^3 term in their

However, $(x_{1\alpha}+x_{1\beta})^3 \neq x_{1\alpha}^3 + x_{\beta}^3$, which means it's NOT linear.

• Memoryless : ALL 3 are NOT memoryless

The output position depends on the accumulation of velocity,

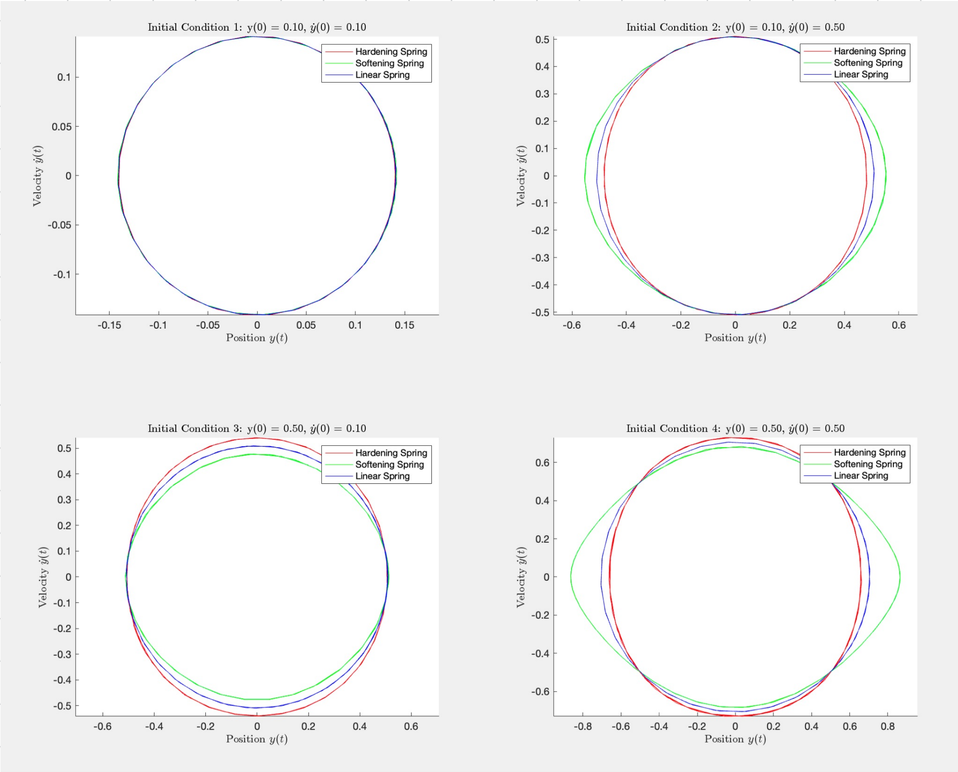
as well as the velocity does of acceleration.

So the outputs will be influenced by past state.

• finite-dim : ALL 3 are finite-dimentional.

Every output can be presented by x_1 & x_2.

(d)



Discussion:

-

	Amplitude	Maximum Velocity
Harden	Small	Large
linear	Medium	Medium
Soften	Large	Small

- The large the initial position,
  the greater difference in amplitude between springs.

- Similarly for the initial velocity & maximum velocity.

- It's hard for ode45 to find solution when the initial condition is about 1.
  That is because there're eq points near 1 for the softening spring system,
  which makes the output complex.

```matlab
%% Models
m = 1;
k = 1;

initial_conditions = [
    0.1, 0.1;
    0.1, 0.5;
    0.5, 0.1;
    0.5, 0.5
];

hd_spring = @(t, x) [x(2); -k/m * (1 + x(1)^2) * x(1)];
sf_spring = @(t, x) [x(2); -k/m * (1 - x(1)^2) * x(1)];
li_spring = @(t, x) [x(2); -k/m * x(1)];

%% plot
figure;
tspan = [0, 20];

for i = 1:4
    subplot(2, 2, i);
    hold on;

    % hd
    [t, x_hard] = ode45(hd_spring, tspan, initial_conditions(i, :));
    plot(x_hard(:, 1), x_hard(:, 2), 'r', 'DisplayName', 'Hardening Spring');

    % sf
    [t, x_soft] = ode45(sf_spring, tspan, initial_conditions(i, :));
    plot(x_soft(:, 1), x_soft(:, 2), 'g', 'DisplayName', 'Softening Spring');

    % li
    [t, x_linear] = ode45(li_spring, tspan, initial_conditions(i, :));
    plot(x_linear(:, 1), x_linear(:, 2), 'b', 'DisplayName', 'Linear Spring');

    xlabel('Position $y(t)$', 'Interpreter', 'latex');
    ylabel('Velocity $\dot{y}(t)$', 'Interpreter', 'latex');
    title(sprintf('Initial Condition %d: y(0) = %.2f, $\\dot{y}(0)$ = %.2f', i, initial_conditions(i, 1), initial_conditions(i, 2)), 'Interpreter', 'latex');
    legend;

    axis equal;
    hold off;
end
```

Problem 3.

(a) Let $\dot{x} = x(x^2-1) = 0$,

$x = \{-1, 0, 1\}$ are the eq points.

(b) $f(x,u,t) = x(x^2-1)$,

$$\frac{\partial f}{\partial x} = (x^2-1) + x \cdot 2x = 3x^2-1 \quad , \quad \frac{\partial f}{\partial u} = 0$$

(1) For $x=-1$, $\left.\frac{\partial f}{\partial x}\right|_{x=-1} = 2$, $\dot{\tilde{x}} = 2(\tilde{x}+1)$

the solution is:

$$\tilde{x}(t) = (\tilde{x}(0)+1)e^{2t} - 1$$

Solving $\dot{\tilde{x}} = 2(\tilde{x}+1)$:

$$\frac{d\tilde{x}}{dt} = 2(\tilde{x}+1)$$

$$\int \frac{1}{2(\tilde{x}+1)} dx = \int dt$$

$$\ln(x+1) = 2t + c$$

$$x+1 = e^{2t+c}$$

$$x = C_1 e^{2t} - 1$$

Since $x(0) = C_1 e^0 - 1 \Rightarrow C_1 = x(0)+1$

$$x = (x(0)+1)e^{2t} - 1$$

(2) For $x=0$, $\left.\frac{\partial f}{\partial x}\right|_{x=0} = -1$, $\dot{\tilde{x}} = -x$

the sol'n is:

$$\tilde{x}(t) = x(0) \cdot e^{-t}$$

$$\frac{d\tilde{x}}{dt} = -\tilde{x}$$

$$\int \frac{d\tilde{x}}{\tilde{x}} = -\int dt$$

$$\ln(\tilde{x}) = -t + c$$

$$\tilde{x} = C_1 e^{-t}$$

$$\tilde{x}(0) = C_1 e^{-0} \Rightarrow C_1 = x(0)$$

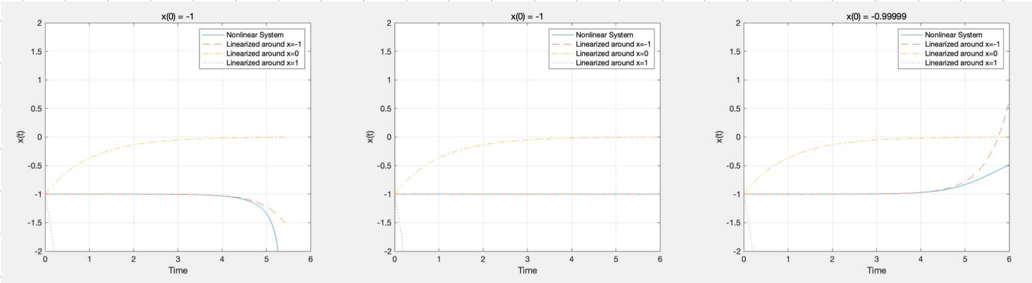
$$\tilde{x} = x(0)e^{-t}$$

(3) For $x=1$, $\left.\frac{\partial f}{\partial x}\right|_{x=1} = 2$, $\dot{\tilde{x}} = 2(\tilde{x}-1)$

Similarly as (1), the sol'n is:

$$\tilde{x}(t) = (x(0)-1)e^{2t} + 1$$

(c)



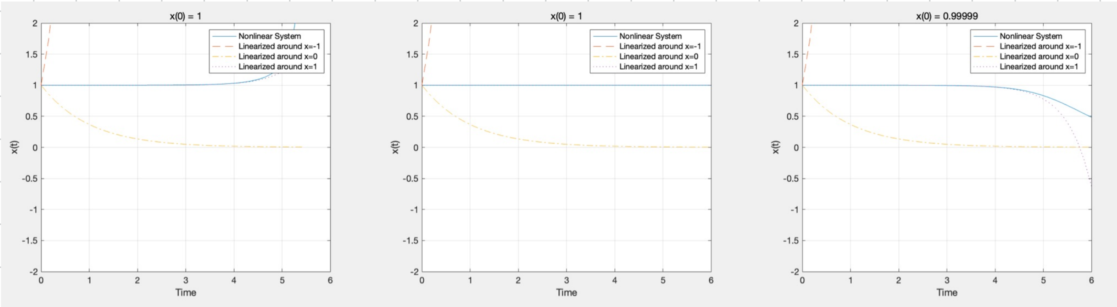
1° For initial condition @ -1: the system doesn't vibrate, the linearization works perfect.

For initial condition around -1:

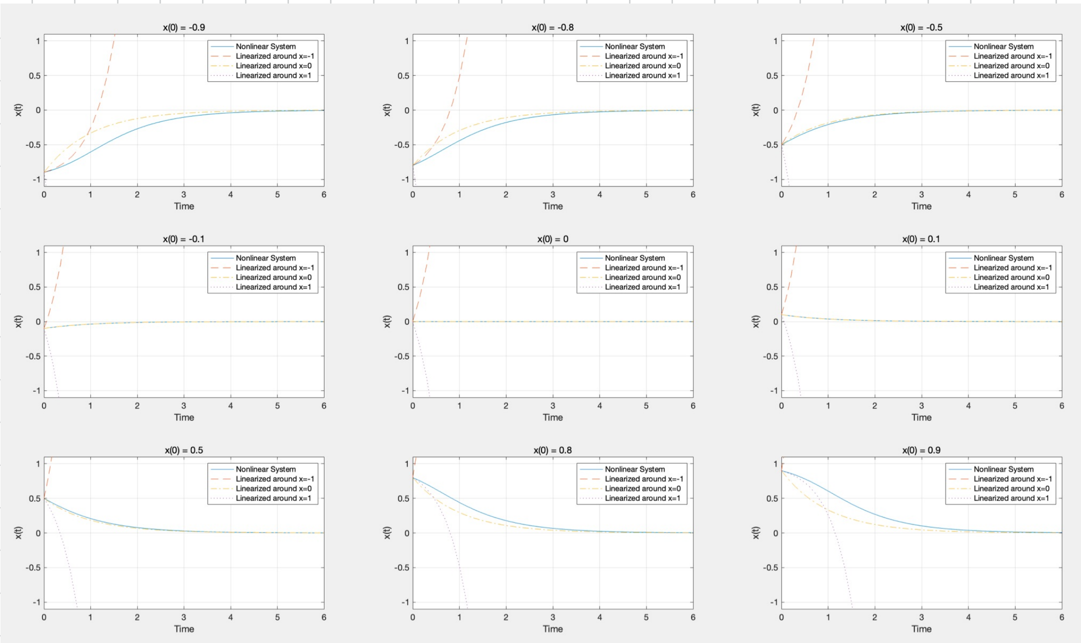
<1> the linearization around $x=-1$ works well at the begining,

<2> but the error increase as time.

<3> linearization around $x=0, +1$ can't match.



2° Similarly for $x=1$.



3° When the $x(0) \in (-1, 1)$, the linearization around $x=0$ works better.
The closer $x(0)$ is to 0, the better the linearization at $x=0$ fits

(d) $g(x) = \frac{x^2}{2} - \frac{x^4}{4} \Rightarrow g'(x) = x - x^3 = x(1-x^2)$

So $\dot{x} = -g'(x) = x(x^2-1)$

According to (c),

(1) When $x(0) \approx \pm 1$

although $\dot{x}(0) = 0$, the system still won't converge to ± 1.

which means $x = \pm 1$ are saddle points (unstable)

That will let x go away from ± 1.

(2) When $x(0) \approx 0$, the system will soon converge to $x = 0$.

(3) For any x in the increasing interval, at which point, $\dot{g}(x) > 0$

So, $-\dot{g}(x) < 0$

Then,

add this $-\dot{g}(x)$ to x will let the function value go down along the function slope.

until reach the bottom (minimum)

similiarly as x is on the decreasing interval.

```matlab
clear;
clc;
close all;

%% Model
dxdt = @(t, x) x * (x^2 - 1);
tspan = [0 6];
initial_conditions_group1 = [-1.00001, -1, -0.99999, 1.00001, 1, 0.99999];
initial_conditions_group2 = [-0.9, -0.8, -0.5, -0.1, 0, 0.1, 0.5, 0.8, 0.9];

%% around -1 & 1
figure;
for i = 1:length(initial_conditions_group1)
    x0 = initial_conditions_group1(i);

    [t, x] = ode45(dxdt, tspan, x0);

    subplot(2, 3, i);
    plot(t, x, 'DisplayName', 'Nonlinear System');
    hold on;

    x_linear_minus_1 = -1 + (x0 + 1) * exp(2 * t);
    plot(t, x_linear_minus_1, '--', 'DisplayName', 'Linearized around x=-1');

    x_linear_0 = x0 * exp(-t);
    plot(t, x_linear_0, '-.', 'DisplayName', 'Linearized around x=0');

    x_linear_1 = 1 + (x0 - 1) * exp(2 * t);
    plot(t, x_linear_1, ':', 'DisplayName', 'Linearized around x=1');

    title(['x(0) = ', num2str(x0)]);
    xlabel('Time');
    ylabel('x(t)');
    ylim([-2, 2]);
    legend;
    grid on;
    hold off;
end

%% around 0
figure;
for i = 1:length(initial_conditions_group2)
    x0 = initial_conditions_group2(i);

    [t, x] = ode45(dxdt, tspan, x0);

    subplot(3, 3, i);
    plot(t, x, 'DisplayName', 'Nonlinear System');
    hold on;

    x_linear_minus_1 = -1 + (x0 + 1) * exp(2 * t);
    plot(t, x_linear_minus_1, '--', 'DisplayName', 'Linearized around x=-1');

    x_linear_0 = x0 * exp(-t);
    plot(t, x_linear_0, '-.', 'DisplayName', 'Linearized around x=0');

    x_linear_1 = 1 + (x0 - 1) * exp(2 * t);
    plot(t, x_linear_1, ':', 'DisplayName', 'Linearized around x=1');

    title(['x(0) = ', num2str(x0)]);
    xlabel('Time');
    ylabel('x(t)');
    ylim([-1.1, 1.1]);
    legend;
    grid on;
    hold off;
end
```

Problem 4

- Form the gradient flow dynamics:

suppose $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x^T = [x_1 \; x_2]$,

then:

$$xx^T - \Lambda = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}[x_1 x_2] - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} x_1^2-\lambda_1 & x_1x_2 \\ x_1x_2 & x_2^2-\lambda_2 \end{bmatrix}$$

Thus,

$$\| xx^T - \Lambda \|_F^2 = \text{trace}(M^T M) = \sum_{i,j} M_{ij}^2 = (x_1^2-\lambda_1)^2 + 2x_1^2x_2^2 + (x_2^2-\lambda_2)^2$$

So,

$$g(x) = \frac{1}{4}\| xx^T - \Lambda \|_F^2 = \frac{1}{4}[(x_1^2-\lambda_1)^2 + 2x_1^2x_2^2 + (x_2^2-\lambda_2)^2]$$

to derive the gradient:

$$\frac{\partial g(x)}{\partial x_1} = \frac{1}{4}[2(x_1^2-\lambda_1)\cdot 2x_1 + 4x_1x_2^2] = x_1^3 - \lambda_1 x_1 + x_1x_2^2 = (x_1^2-\lambda_1+x_2^2)\cdot x_1$$

similiarly, $\frac{\partial g(x)}{\partial x_2} = (x_1^2 - \lambda_2 + x_2^2)\cdot x_2$

Therefore, the model is:

$$\boxed{\dot{x} = -\nabla g(x) = \begin{bmatrix} -(x_1^2-\lambda_1+x_2^2)\cdot x_1 \\ -(x_1^2-\lambda_2+x_2^2)\cdot x_2 \end{bmatrix}}$$

- to derive the eq point, let $\dot{x} = 0$

$$\begin{cases} -(x_1^2-\lambda_1+x_2^2)\cdot x_1 = 0 & \Rightarrow x_1 = 0 \text{ or } x_1^2+x_2^2=\lambda_1 \\ -(x_1^2-\lambda_2+x_2^2)\cdot x_2 = 0 & \Rightarrow x_2 = 0 \text{ or } x_1^2+x_2^2=\lambda_2 \end{cases}$$

(a) $x_1 = 0$, $x_2 = 0$

(b) $x_1 = 0$, $x_1^2+x_2^2 = x_2^2 = \lambda_2 \Rightarrow x_1 = 0$, $x_2 = \pm\sqrt{\lambda_2}$

(c) $x_2 = 0$, $x_1^2+x_2^2 = x_1^2 = \lambda_1 \Rightarrow x_1 = \pm\sqrt{\lambda_1}$, $x_2 = 0$

(d) $x_1^2+x_2^2 = \lambda_1 > \lambda_2 \Rightarrow$ NO sol'n

Above all, there are $\boxed{5 \text{ eq points}: \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{\lambda_2} \end{bmatrix}, \begin{bmatrix} 0 \\ -\sqrt{\lambda_2} \end{bmatrix}, \begin{bmatrix} \sqrt{\lambda_1} \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{\lambda_1} \\ 0 \end{bmatrix} \right\}}$

- linearize :

$$g_1(x_1,x_2) = -(x_1^3 - \lambda_1 x_1 + x_1 x_2^2) \quad , \quad g_2(x_1,x_2) = x_1^2 x_2 - \lambda_2 x_2 + x_2^3$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -3x_1^2+\lambda_1-x_2^2 & -2x_1x_2 \\ -2x_1x_2 & -x_1^2+\lambda_2-3x_2^2 \end{bmatrix}$$

(1) For $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} , \quad \dot{\tilde{x}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tilde{x}$$

(2) For $\begin{bmatrix} 0 \\ \sqrt{\lambda_2} \end{bmatrix}$:

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \lambda_1-\lambda_2 & 0 \\ 0 & -2\lambda_2 \end{bmatrix} , \quad \dot{\tilde{x}} = \begin{bmatrix} \lambda_1-\lambda_2 & 0 \\ 0 & -2\lambda_2 \end{bmatrix} \tilde{x}$$

(3) For $\begin{bmatrix} 0 \\ -\sqrt{\lambda_2} \end{bmatrix}$:

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \lambda_1-\lambda_2 & 0 \\ 0 & -2\lambda_2 \end{bmatrix} , \quad \dot{\tilde{x}} = \begin{bmatrix} \lambda_1-\lambda_2 & 0 \\ 0 & -2\lambda_2 \end{bmatrix} \tilde{x}$$

(4) For $\begin{bmatrix} \sqrt{\lambda_1} \\ 0 \end{bmatrix}$:

$$\frac{\partial g}{\partial x} = \begin{bmatrix} -2\lambda_1 & 0 \\ 0 & -\lambda_1+\lambda_2 \end{bmatrix} , \quad \dot{\tilde{x}} = \begin{bmatrix} -2\lambda_1 & 0 \\ 0 & -\lambda_1+\lambda_2 \end{bmatrix} \tilde{x}$$

(5) For $\begin{bmatrix} -\sqrt{\lambda_1} \\ 0 \end{bmatrix}$:

$$\frac{\partial g}{\partial x} = \begin{bmatrix} -2\lambda_1 & 0 \\ 0 & -\lambda_1+\lambda_2 \end{bmatrix} , \quad \dot{\tilde{x}} = \begin{bmatrix} -2\lambda_1 & 0 \\ 0 & -\lambda_1+\lambda_2 \end{bmatrix} \tilde{x}$$