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Problem 1

(a) $A = \begin{bmatrix} -(1+k^2)/R & k \\ 0 & -(2+k^2)/R \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda + (1+k^2)/R & -k \\ 0 & \lambda + (2+k^2)/R \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda + (1+k^2)/R)(\lambda + (2+k^2)/R) = 0$$

$$\Rightarrow \lambda_1 = -(1+k^2)/R, \quad \lambda_2 = -(2+k^2)/R$$

Because k, R are positive scalars, $\begin{cases} \lambda_1 < 0 \\ \lambda_2 < 0 \end{cases}$ can always be satisfied.

Which means the system is stable for any positive k, R.

(b) For $\lambda_1 = -(1+k^2)/R$:

$$(\lambda_1 I - A)v_1 = \begin{bmatrix} 0 & -k \\ 0 & \frac{-(1+k^2)+(2+k^2)}{R} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \Rightarrow v_{12} = 0, \ v_{11} \text{ is free.}$$

$$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = -(2+k^2)/R$:

$$(\lambda_2 I - A)v_2 = \begin{bmatrix} \frac{+(1+k^2)-(2+k^2)}{R} & -k \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \Rightarrow -\frac{1}{R}v_{21} - kv_{22} = 0$$

$$\Rightarrow v_2 = \begin{bmatrix} -kR \\ 1 \end{bmatrix}$$

(c) By (a), (b), we could diagonalize A as $A = V\Lambda V^{-1}$,
where $V = \begin{bmatrix} 1 & -kR \\ 0 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $V^{-1} = \begin{bmatrix} 1 & kR \\ 0 & 1 \end{bmatrix}$

Thus $\Phi(t) = e^{At} = V \cdot e^{\Lambda t} \cdot V^{-1} = \begin{bmatrix} 1 & -kR \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} 1 & kR \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} e^{-\frac{1+k^2}{R} \cdot t} & Rk\left(e^{-\frac{1+k^2}{R} \cdot t} - e^{-\frac{2+k^2}{R} t}\right) \\ 0 & e^{-\frac{2+k^2}{R} t} \end{bmatrix}$$

Basicly there will be terms: $e^{-\frac{1+k^2}{R}t}$ & $e^{-\frac{2+k^2}{R}t}$ in the STM.

Thus, the $x(t) \to 0$ when $t \to \infty$,

And the speed would be influenced by k, R:

the system goes faster when k is larger and R is smaller

(d) $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\frac{1+k^2}{R}\cdot t} & Rk\left(e^{-\frac{1+k^2}{R}\cdot t} - e^{-\frac{2+k^2}{R}t}\right) \\ 0 & e^{-\frac{2+k^2}{R}t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

suppose $x_1(0)=0$, $x_1(t) = \Phi_{12}\cdot x_2(0)$

$$= Rk\left(e^{-\frac{1+k^2}{R}\cdot t} - e^{-\frac{2+k^2}{R}t}\right) x_2(0)$$

$$= Rke^{-\frac{k^2}{R}t}\left(e^{-t} - e^{-2t}\right) x_2(0)$$

$$\frac{\partial \Phi_{12}}{\partial k} = (R-2k^2t)\left(e^{-\frac{1+k^2}{R}t} - e^{-\frac{2+k^2}{R}t}\right) = (R-2k^2t)e^{-\frac{k^2}{R}t}\left(e^{-\frac{t}{R}} - e^{-\frac{2t}{R}}\right)$$

$$\frac{\partial \Phi_{12}}{\partial t} = k\left[(k^2+2)e^{-\frac{2+k^2}{R}\cdot t} - (k^2+1)e^{-\frac{1+k^2}{R}t}\right] = ke^{-\frac{k^2}{R}t}\left[(k^2+2)e^{-\frac{2t}{R}} - (k^2+1)e^{-\frac{t}{R}}\right]$$

Let $\begin{cases} \partial\Phi_{12}/\partial k = 0 \\ \partial\Phi_{12}/\partial t = 0 \end{cases}$, $\Rightarrow \begin{cases} R-2k^2t = 0 & \cdots ① \\ (k^2+2)e^{-\frac{2t}{R}} - (k^2+1)e^{-\frac{t}{R}} = 0 & \cdots ② \end{cases}$

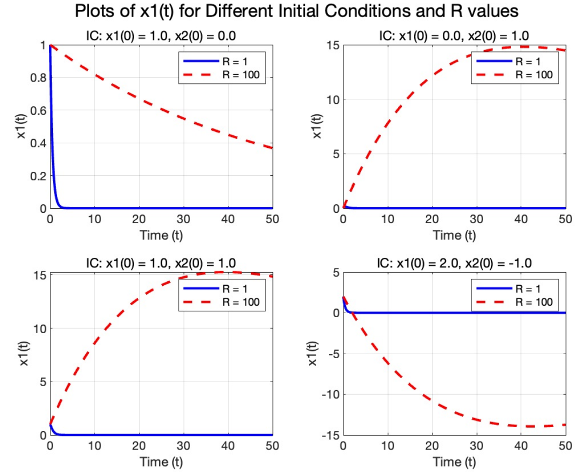
By ①. we could know: $R = 2k^2t \Rightarrow k^2 = \frac{R}{2t}$

Thus, $\left(\frac{R}{2t}+2\right) e^{-\frac{2t}{R}} - \left(\frac{R}{2t}+1\right)e^{-\frac{t}{R}} = 0$

Then we could get $t = \cdots ?$

$$k = \sqrt{\frac{R}{2t}} = \cdots ?$$

(e)



$x_1(t)$ varies with differento I.C.s,
but all of them will eventually converge to 0,
while for $R=1$, it goes faster than for $R=100$.

(f) Since $\tau = t/R$, $\frac{d\tau}{dt} = \frac{1}{R}$

then $\dot{x}(t) = \frac{dx}{dt} = \frac{dx}{d\tau} \cdot \frac{d\tau}{dt} = \frac{1}{R}\frac{dx}{d\tau} = Ax + Bu$

So, $\frac{dx}{d\tau} = RAx + RBu$

i.e. if using compressed time scale $\tau = t/R$,
the state eqn becomes:

$$\dot{x} = \hat{A}x + \hat{B}u,$$

where $\hat{A} = RA = \begin{bmatrix} -(1+K^2) & kR \\ 0 & -(2+k^2) \end{bmatrix}$, $\hat{B} = RB = \begin{bmatrix} 0 \\ R \end{bmatrix}$

Therefore, the eigenvalues of A change to $\lambda_1 = -(1+K^2)$, $\lambda_2 = -(2+K^2)$
which means the STM doesn't relate to R anymore.
i.e. the system speed can't be changed by R.

Problem 2.

to solve the $A^TP+PA=-Q=-\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$$\begin{bmatrix}-(1+k^2)/R & 0\\ k & -(2+k^2)/R\end{bmatrix}\begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}+PA+\begin{bmatrix}1&0\\0&1\end{bmatrix}=0$$

$$\Rightarrow\begin{cases}1-\frac{2}{R}(1+k^2)P_{11}=0\\ kP_{11}-(\frac{1+k^2}{R}+\frac{2+k^2}{R})P_{12}=0\\ 2kP_{12}-\frac{2}{R}(2+k^2)P_{22}+1=0\end{cases}\Rightarrow\begin{cases}P_{11}=\frac{R}{2(k^2+1)}\\ P_{12}=P_{21}=\frac{R^2k}{2(k^2+1)(2k^2+3)}\\ P_{22}=\frac{R(R^2k^2+2k^4+5k^2+3)}{2(k^2+1)(k^2+2)(2k^2+3)}\end{cases}$$

i.e. $$P=\begin{bmatrix}\frac{R}{2(k^2+1)} & \frac{R^2k}{2(k^2+1)(2k^2+3)}\\ \frac{R^2k}{2(k^2+1)(2k^2+3)} & \frac{R(R^2k^2+2k^4+5k^2+3)}{2(k^2+1)(k^2+2)(2k^2+3)}\end{bmatrix}$$

the eigenvalues of P are:

$$\lambda_1=\frac{R}{4(2k^6+9k^4+13k^2+6}\left(12k^2+4k^2+R^2k^2+9+\sqrt{(R^2k^2+1)(12k^2+4k^2+R^2k^2+9)}\right)$$
$$\lambda_2=\frac{R}{4(2k^6+9k^4+13k^2+6}\left(12k^2+4k^2+R^2k^2+9-\sqrt{(R^2k^2+1)(12k^2+4k^2+R^2k^2+9)}\right)$$

Easy to find $\lambda_1>0$ for $k,R>0$;

For λ_2, since $R^2k^2+1<12k^2+4k^2+R^2k^2+9$,

$\sqrt{(R^2k^2+1)(12k^2+4k^2+R^2k^2+9)}<12k^2+4k^2+R^2k^2+9$

Thus, $\lambda_2>0$

So, $\lambda_1,\lambda_2>0$, which means P is positive definite.

Therefore the system is stable.

Problem 3

Consider a new system $\dot{x} = \hat{A}x$, where $\hat{A} = A + \mu I$

then its lyapunov function:

$$\hat{A}^T P + P\hat{A} + Q = (A+\mu I)^T P + P(A+\mu I) + Q$$
$$= (A^T + \mu I)P + P(A+\mu I) + Q$$
$$= A^T P + PA + 2\mu P + Q = 0$$

i.e. $\hat{A}^T P + P\hat{A} + Q = 0$ has positive definite P & Q solutions.

So, the eigenvalues of $\hat{A}$, $\hat{\lambda}_i < 0$
which means the sol'n of $\det(\hat{\lambda} I - \hat{A}) = 0$ is $\hat{\lambda}_i < 0$.

Since $\hat{A} = A + \mu I$,
the eigenvalues of A λ satisfies: $\hat{\lambda} = \lambda + \mu$,
then, $\hat{\lambda}_i = \lambda_i + \mu < 0 \Rightarrow \lambda_i < -\mu$

Problem 4

(a) Because $H(s)$ has a pair of poles $\sigma \pm j\omega$,

$$d(s) = (s - (\sigma + j\omega))(s - (\sigma - j\omega))$$
$$= s^2 - (\sigma + j\omega)s - (\sigma - j\omega)s + (\sigma + j\omega)(\sigma - j\omega)$$
$$= s^2 - 2\sigma s + \sigma^2 + \omega^2$$

Substitute $d(s)$ to $Y(s) \cdot \frac{1}{H(s)} = Y(s)d(s) = U(s)$

$$s^2 Y(s) - 2\sigma s Y(s) + (\sigma^2 + \omega^2) Y(s) = U(s)$$

Then, apply $\mathcal{L}^{-1}$ to it:

$$y''(t) - 2\sigma y'(t) + (\sigma^2 + \omega^2) y(t) = u(t)$$

(b) Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, then $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$

$$\Rightarrow \begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = 2\sigma\dot{y} - (\sigma^2 + \omega^2)y + u = 2\sigma x_2 - (\sigma^2 + \omega^2)x_1 + u \end{cases}$$

So the state eqn of the system is:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -(\sigma^2+\omega^2) & 2\sigma \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

Suppose we have $r = T^{-1}x$, $x = Tr$,

then: $\dot{r}(t) = T^{-1}AT\, r(t) + T^{-1}Bu$

So $\bar{A} = T^{-1}AT$

$$\Rightarrow T\bar{A} - AT = \begin{bmatrix} \sigma a - \omega b - c & \omega a + \sigma b - d \\ (\sigma^2+\omega^2)a - \sigma c - \omega d & (\sigma^2+\omega^2)b + \omega c - \sigma d \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} \sigma a - \omega b - c = 0 \\ \omega a + \sigma b - d = 0 \\ (\sigma^2+\omega^2)a - \sigma c - \omega d = 0 \\ (\sigma^2+\omega^2)b + \omega c - \sigma d = 0 \end{cases}$$

It's null space is: $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = c_1 \begin{bmatrix} \sigma/(\sigma^2+\omega^2) \\ -\omega/(\sigma^2+\omega^2) \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \omega/(\sigma^2+\omega^2) \\ \sigma/(\sigma^2+\omega^2) \\ 0 \\ 1 \end{bmatrix}$

Choose $c_1 = \sigma^2+\omega^2$, $c_2 = 0$.

then $T = \begin{bmatrix} \sigma & -\omega \\ \sigma^2+\omega^2 & 0 \end{bmatrix}$,

which can make $\bar{A} = T^{-1}AT = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$