AME 541 HW3 uscid : 7722927131

Problem 1

(1) $A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, $e^{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}t} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$

$\boxed{A_1^t = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$ $e^{A_1 t} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}$

(2) $A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

eigenvalue :

$\det(\lambda I - A_2) = \det\left(\begin{bmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda-1 \end{bmatrix}\right) = (\lambda-1)\cdot\lambda\cdot(\lambda-1) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 0$

eigenvectors :

a. for $\lambda_1 = 1$, $(\lambda_1 I - A_2) v = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \cdot v = 0 \Rightarrow v_{12} = v_{13} = 0$

so we only have eigenvector $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

b. then consider : $(\lambda_1 I - A_2)\vec{v_2} = \vec{v_1}$

$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -v_{22} = 1 \\ v_{22} - v_{23} = 0 \end{cases}$

Thus, $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

c. for $\lambda_2 = 0$, $(\lambda_2 I - A_2) v = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} v = 0 \Rightarrow \begin{cases} -v_{31} - v_{32} = 0 \\ -v_{33} = 0 \end{cases}$

so $\vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

So, Jordan form of A_2 :

$J_2 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$P = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

such J_2, P can satisfy : $A_2 = P J_2 P^{-1}$

Therefore, $A_2^t = (PJ_2P^{-1})^t = PJ_2^tP^{-1}$, $e^{A_2t} = Pe^{J_2t}P^{-1}$

$$J_2^t = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A_2^t = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 1 & 1 & t-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$e^{A_2t} = \sum_{k=0}^{\infty} \frac{A_2^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} 1 & 1 & k-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & e^t & te^t - e^t \\ 0 & 0 & e^t \\ 0 & 0 & e^t \end{bmatrix}$$

$$e^{J_2t} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow e^{A_2t} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} e^t & e^t & (t-1)e^t \\ 0 & 0 & e^t \\ 0 & 0 & e^t \end{bmatrix}}$$

(3) $A_3 = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{array}\right] = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$

$$B = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$$

$$B^t = \begin{bmatrix} 2^t & 0 \\ t\cdot 2^t & 2^t \end{bmatrix}, \quad e^{Bt} = \begin{bmatrix} e^{2t} & 0 \\ 2te^{2t} & e^{2t} \end{bmatrix}$$

$$C^t = \begin{bmatrix} 3^t & t\cdot 3^t \\ 0 & 3^t \end{bmatrix}, \quad e^{Ct} = \begin{bmatrix} e^{3t} & 3te^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

So,

$$\boxed{A_3^t = \begin{bmatrix} 2^t & 0 & 0 & 0 \\ t\cdot 2^t & 2^t & 0 & 0 \\ 0 & 0 & 3^t & 0 \\ 0 & 0 & t\cdot 3^t & 3^t \end{bmatrix}, \quad e^{A_3 t} = \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 2te^{2t} & e^{2t} & 0 & 0 \\ 0 & 0 & e^{3t} & 3te^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}}$$

Problem 2

(a) $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, whose eigenvalues are $-2, 1, -1$

Since $\lambda = 1 > 0$, the system is UNSTABLE.

(b) $R = (sI - A)^{-1}$

$$= \begin{bmatrix} \frac{1}{s+2} & & \\ & \frac{1}{s-1} & \\ & & \frac{1}{s+1} \end{bmatrix}$$

$H(s) = C(sI - A)^{-1}B + D$

$$= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+2} & & \\ & \frac{1}{s-1} & \\ & & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 1$$

$$= \frac{s+3}{s+2}$$

Problem 3

(a) Plug $x_1 = x_2 = 1$ into the state-space :

$$f_1 : \dot{x}_1 = -x_1^2 + x_1x_2 = -1 + 1 = 0$$
$$f_2 : \dot{x}_2 = -2x_2^2 + x_2 - x_1x_2 + 2 = -2 + 1 - 1 + 2 = 0$$

Thus, $\bar{x} = [1 \ \ 1]^T$ is an eq-point.

(b) Let
$$\begin{cases} \dot{x}_1 = -x_1^2 + x_1x_2 = 0 & \cdots (1) \\ \dot{x}_2 = -2x_2^2 + x_2 - x_1x_2 + 2 = 0 & \cdots (2) \end{cases}$$

$$(1) : \quad x_1^2 = x_1x_2$$
$$(2) : \quad -2x_2^2 + x_2 - x_1^2 + 2 = 0 \Rightarrow x_1^2 = -2x_2^2 + x_2 + 2$$

So, for any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that can satisfy $x_1^2 = -2x_2^2 + x_2 + 2$ is an eq-point which means $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is NOT the only one.

(c) Linearization around $\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{\substack{x_1 = 1 \\ x_2 = 1}} = \begin{bmatrix} -2x_1 + x_2 & x_1 \\ -x_2 & -4x_2 + 1 - x_1 \end{bmatrix}\Bigg|_{\substack{x_1 = 1 \\ x_2 = 1}} = \begin{bmatrix} -1 & 1 \\ -1 & -4 \end{bmatrix}$$

$$\dot{\hat{x}} = \begin{bmatrix} -1 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} \hat{x}_1 - 1 \\ \hat{x}_2 - 1 \end{bmatrix}$$

Resolvent :

$$sI - A = \begin{bmatrix} s+1 & -1 \\ 1 & s+4 \end{bmatrix}$$

$$\det(sI - A) = (s+1)(s-4) + 1 = s^2 + 5s + 5$$

$$R = (sI - A)^{-1} = \frac{1}{s^2 + 5s^2 + 5} \begin{bmatrix} s+4 & 1 \\ -1 & s+1 \end{bmatrix}$$

State-transition Matrix :

$$\Phi = \mathcal{L}^{-1}(R) = e^{-\frac{5}{2}t} \begin{bmatrix} \cosh(\frac{\sqrt{5}}{2}t) + \frac{3\sqrt{5}}{5}\sinh(\frac{\sqrt{5}}{2}t) & \frac{2\sqrt{5}}{5}\sinh(\frac{\sqrt{5}}{2}t) \\ -\frac{2\sqrt{5}}{5}\sinh(\frac{\sqrt{5}}{2}t) & \cosh(\frac{\sqrt{5}}{2}t) - \frac{3\sqrt{5}}{5}\sinh(\frac{\sqrt{5}}{2}t) \end{bmatrix}$$

(d) If the initial condition is near the eq-point, since the STM has a term $e^{-\frac{5}{2}t}$, $x(t)$ will converage to 0, which leads to a stable responce.

However, if I.C. is far away from eq, then we can not predict the response.

And it can be easily seen with Matlab ploo below.