RESEARCH STATEMENT

QIHANG SUN

1. Introduction

My research primarily focuses on analytic number theory and modular forms. Much of my work involves applications of estimating sums of various types of Kloosterman sums. To date, I have 1 paper on an explicit bound for traces of singular moduli (accepted by *Res. Number Theory*) and have submitted 2 papers on uniform estimates for sums of Kloosterman sums with their applications to integer partitions.

Kloosterman sums are special exponential sums and appear in many problems in number theory. For integers $c > 0$, m and n, the standard Kloosterman sum is defined as

$$S(m,n,c) := \sum_{d \pmod{c}^*} e\left(\frac{m\overline{d} + nd}{c}\right), \quad e(z) := e^{2\pi i z}, \quad \overline{d}d \equiv 1 \pmod{c}. \tag{1.1}$$

Kloosterman [Klo26] first introduced this definition in to investigate whether the quadratic form $a_1n_1^2 + a_2n_2^2 + a_3n_3^2 + a_4n_4^2$ with fixed $a_i \in \mathbb{N}$ represents all sufficiently large natural numbers. Another application is to estimate the shifted sum of divisor functions. Let $\tau(n)$ be the number of divisors of the positive integer n and

$$D(N,f) := \sum_{n=1}^{N} \tau(n)\tau(n+h), \quad \text{for some fixed integer } h \geq 1.$$

Heath-Brown [HB79] applied the Weil bound (2.1) of Kloosterman sums to prove

$$D(N,f) = \text{explicit main terms} + O(N^{\frac{5}{6}+\varepsilon}), \quad \text{uniformly for } 1 \leq h \leq N^{\frac{5}{6}}.$$

Using Kuznetsov's trace formula, Deshouillers and Iwaniec [DI82] obtained a much better error bound $O(N^{\frac{2}{3}+\varepsilon})$ for all $h \geq 1$.

The application of Kloosterman sums in my main projects arises from problems related to the integer partition function $p(n)$, which is the number of ways to write n as a sum of positive integers. The generating function of $p(n)$ is $q^{\frac{1}{24}}/\eta(z)$, where $\eta(z)$ is Dedekind's eta function with $q = e^{2\pi i z}$ and $\operatorname{Im} z > 0$. Since $\eta(z)$ is a weight $\frac{1}{2}$ modular form, we are able to write the approximations of $p(n)$ and related functions (e.g. rank of partitions) as sums of generalized Kloosterman sums. The bounds on Kloosterman sums gives the growth rate of errors for such approximations.

The research statement is structured as follows: Sections 2 through 4 describe the projects I have completed, while the final section offers a brief overview of my future research directions and ongoing challenges.

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, United States

E-mail address: qihangs2@illinois.edu.

2. Uniform bounds on sums of Kloosterman sums

The standard Kloosterman sums (1.1) have a well-known Weil bound

$$|S(m,n,c)| \le \sigma_0(c)\sqrt{(m,n,c)}\sqrt{c}, \quad \text{where } \sigma_k(\ell) = \sum\nolimits_{d|\ell} d^k \tag{2.1}$$

is the divisor function. The Weil bound implies a square-root cancellation for estimating

$$\mathrm{KS}(x) := \sum_{c\le x} \frac{S(m,n,c)}{c} \ll \sigma_0((m,n)) x^{\frac12}\log x. \tag{2.2}$$

Kuznetsov's trace formula [Kuz80] resulted in $\mathrm{KS}(x) \ll_{m,n} x^{\frac16}(\log x)^{\frac13}$. Sarnak and Tsimerman [ST09] obtained a bound which is uniform in m and n for $mn>0$:

$$\mathrm{KS}(x) \ll_\varepsilon \left(x^{\frac16} + (mn)^{\frac16} + (m+n)^{\frac18}(mn)^{\frac{\theta}{2}}\right)(mnx)^\varepsilon. \tag{2.3}$$

For general $k\in\mathbb{R}$ and weight k multiplier system ν, we define Kloosterman sums by

$$S(m,n,c,\nu) := \sum_{0\le a,d<c,\ \gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma} \overline{\nu}(\gamma) e\left(\frac{\tilde m a + \tilde n d}{c}\right)$$

where Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with $\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)\in\Gamma$ and $\tilde n := n-\alpha_\nu$ where $\alpha_\nu\in[0,1)$ is defined by $e(-\alpha_\nu) = \nu(\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right))$. Goldfeld and Sarnak [GS83] proved the asymptotic formula

$$\sum_{c\le x}\frac{S(m,n,c,\nu)}{c} = \sum_{\frac12<s_j<1} \tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1} + O_{\Gamma,\nu,m,n,k,\varepsilon}\left(x^{\frac{\beta}{3}+\varepsilon}\right), \tag{2.4}$$

where the sum runs over exceptional eigenvalues $\lambda_j = s_j(1-s_j)$ of the hyperbolic Laplacian Δ_k and $\tau_j(m,n)$ involves Maass forms with eigenvalue λ_j.

For a wide class of multiplier systems, I proved a uniform version of [GS83] in analogy to the uniform bound (2.3) for standard Kloosterman sums:

Theorem 2.1 ([Sun23b, Theorem 1.3] combined with [Sun23a, Theorem 1.3])**.** *Suppose ν is an admissible weight $k=\pm\frac12$ multiplier on $\Gamma_0(N)$. Factor $|B\tilde\ell| = t_\ell u_\ell^2 w_\ell^2$ where t_ℓ is square-free, $u_\ell | M^\infty$ and $(w_\ell, M)=1$ for $\ell\in\{m,n\}$. Then*

$$\sum_{N|c\le X}\frac{S(m,n,c,\nu)}{c} = \sum_{r_j\in i(0,\frac14]} \tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1} + O_{\nu,\varepsilon}\left(\left(A_u(m,n) + X^{\frac16}\right)|\tilde m\tilde n X|^\varepsilon\right), \tag{2.5}$$

where the sum runs over the eigenvalues $\lambda_j<\frac14$ of the hyperbolic Laplacian Δ_k, $\tau_j(m,n)$ are as in (2.4), and

$$A_u(m,n) := \left(|\tilde m|^{\frac{131}{294}} + u_m\right)^{\frac18}\left(|\tilde n|^{\frac{131}{294}} + u_n\right)^{\frac18}|\tilde m\tilde n|^{\frac{3}{16}}.$$

When u_m and u_n are $O_\nu(1)$, e.g. $B\tilde m$ and $B\tilde n$ are square-free or coprime to M, then

$$A_u(m,n) \ll (\tilde m\tilde n)^{\frac14-\frac{1}{147}}.$$

Here the integers B and M are parameters associated to the multiplier systems. For the application in the next section, we have $24|B$.

3. Applications to integer partitions

For the integer partition function $p(n)$, Hardy and Ramanujan proved the asymptotics

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}n}}.$$

In 1938, Rademacher [Rad38] proved an exact formula for $p(n)$, in terms of Kloosterman sums:

$$p(n) = \frac{2\pi e(-\frac{1}{8})}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^{\infty} \frac{S(1, 1-n, c, \nu_\eta)}{c} I_{\frac{3}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right). \tag{3.1}$$

Suppose $\Lambda = \{\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_\kappa\}$ is a partition of n, i.e. $\sum_{j=1}^{\kappa} \Lambda_j = n$. Dyson [Dys44] introduced the rank of a partition as $\operatorname{rank}(\Lambda) := \Lambda_1 - \kappa$ to interpret Ramanujan's congruences $p(\ell n + \beta) \equiv 0 \pmod{\ell}$ for $(\ell, \beta) = (5, 4)$ or $(7, 5)$. Let $N(m, n)$ denote the number of partitions of n with rank m and $N(a, b; n)$ denote the number of partitions of n with rank $\equiv a \pmod{b}$. Let $q = \exp(2\pi i z) = e(z)$ for $z \in \mathbb{H}$ (the upper-half complex plane) and w be a root of unity. It is known that the generating function of $N(m, n)$ is

$$\mathcal{R}(w; q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n}, \tag{3.2}$$

where $(a; q)_n := \prod_{j=0}^{n-1}(1 - aq^j)$. For example, $\mathcal{R}(1; q) = 1 + \sum p(n) q^n$ is the generating function for partitions. For integers $b > a > 0$, denote $\zeta_b := \exp(\frac{2\pi i}{b})$ and define $A(\frac{a}{b}; n)$ by

$$\mathcal{R}(\zeta_b^a; q) =: 1 + \sum_{n=1}^{\infty} A\left(\frac{a}{b}; n\right) q^n.$$

The function $\mathcal{R}(w; q)$ has many beautiful connections and properties. When $w = -1$, it is known that $N(0, 2; n) - N(1, 2; n) = A(\frac{1}{2}; n)$ is the Fourier coefficient of Ramanujan's third order mock theta function $f(q)$. Ramanujan conjectured an asymptotic formula for $A(\frac{1}{2}; n)$, which was proved by Dragonette [Dra52] and improved by Andrews [And66]. The exact formula for $A(\frac{1}{2}; n)$ was finally proved by Bringmann and Ono [BO06, Theorem 1.1]:

$$A\left(\frac{1}{2}; n\right) = \frac{2\pi\, e(-\frac{1}{8})}{(24n-1)^{\frac{1}{4}}} \sum_{2|c>0} \frac{S(0, n, c, \overline{\psi})}{c} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{6c}\right), \tag{3.3}$$

where ψ is the weight $\frac{3}{2}$ multiplier system for $\frac{\eta(z)^5}{\eta(2z)^2}$ on $\Gamma_0(2)$.

When w is a third root of unity, we find that $A(\frac{1}{3}; n) = A(\frac{2}{3}; n)$ is the Fourier coefficient of a Ramanujan's sixth order mock theta function denoted by $\gamma(q)$ in [AH91]. Bringmann deduced a formula [Bri09, Theorem 1.1] for $A(\frac{1}{3}; n)$:

$$A\left(\frac{1}{3}; n\right) = \frac{4\sqrt{3}\, i}{\sqrt{24n-1}} \sum_{3|c\leq\sqrt{n}} \frac{B_{1,3,c}(-n, 0)}{\sqrt{c}} \sinh\left(\frac{\pi\sqrt{24n-1}}{6c}\right) + O(n^{\varepsilon}).$$

Bringmann and Ono [BO12, Theorem 1.3] stated that the above sum, when extended to infinity, gives an exact formula for $A(\frac{1}{3}; n)$. I provide a detailed proof for the exact formula in [Sun23b]:

Theorem 3.1 ([Sun23b, Theorem 2.2]). *The following equation is true:*

$$A\left(\frac{1}{3};n\right)=\frac{2\pi\, e(-\frac{1}{8})}{(24n-1)^{\frac{1}{4}}}\sum_{3|c>0}\frac{S(0,n,c,\ (\frac{\cdot}{3})\overline{\nu_\eta})}{c}I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{6c}\right). \tag{3.4}$$

I also proved the following theorem as a corollary of Theorem 2.1:

Theorem 3.2 ([Sun23a, Theorem 1.5]). *With the same setting as Theorem 2.1, for $\beta=\frac{1}{2}$ or $\frac{3}{2}$ and $\alpha>0$, we have*

$$\sum_{N|c>\alpha\sqrt{|\tilde{m}\tilde{n}|}}\frac{S(m,n,c,\nu)}{c}\mathcal{M}_\beta\left(\frac{4\pi\sqrt{|\tilde{m}\tilde{n}|}}{c}\right)\ll_{\alpha,\nu,\varepsilon}|\tilde{m}\tilde{n}|^{\frac{143}{588}+\varepsilon}, \tag{3.5}$$

unless $\sum_{r_j=\frac{i}{4}}\tau_j(m,n)\neq 0$ when all $\tilde{m}$, $\tilde{n}$ and $k=\pm\frac{1}{2}$ have the same sign. Here $\mathcal{M}_\beta$ is the Bessel function I_β or J_β.

If $R_j(n,x)$ is the tail sum on $c: j|c>x$ for $j=1,2,3$ corresponding to (3.1), (3.3) and (3.4), respectively, I proved

$$R_j(n,\alpha\sqrt{n})\ll_{\alpha,\varepsilon} n^{-\frac{1}{147}+\varepsilon},\quad \text{for all } n \text{ and } j=1,2,3.$$

This improves the result of [AD19] when $j=1,2$ by removing hypothesis that $24n-1$ is square-free and the result of [AA18] when $j=1$ by removing the requirement $35^m\nmid(24n-1)$.

4. Effective estimates for traces of singular moduli

This is joint work with González [GS23] and has been accepted by *Res. Number Theory.* Let $j(z)=q^{-1}+744+O(q)$ for $q=e(z)$ be the modular j-invariant and for a discriminant $d<0$, let K be the corresponding quadratic field over $\mathbb{Q}$ and let $h(d)$ be the class number. Define the positive definite quadratic form $Q(x,y):=ax^2+bxy+cy^2=[a,b,c]$ for $a,b,c\in\mathbb{Z}$ and define z_Q as the root of $Q(x,1)$ in $\mathbb{H}$. Also define $z_d:=\frac{i}{2}\sqrt{|d|}$ if $d\equiv 0 \pmod 4$ and $z_d:=-\frac{1}{2}+\frac{i}{2}\sqrt{|d|}$ if $d\equiv 1 \pmod 4$.

Let $j_m\in\mathbb{C}[j]$ be the Faber polynomials of j; i.e. $j_m(z)=q^{-m}+O(q)$. Duke [Duk06] confirmed the following conjecture of Bruinier, Jenkins and Ono [BJO06]:

$$\lim_{d\to-\infty}\frac{1}{h(d)}\left(\operatorname{Tr} j_1(z_d)-\sum_{z_Q\in\mathcal{R}(1)}e(-z_Q)\right)=-24,$$

where $\mathcal{R}(Y)$ is the rectangle $\mathcal{R}(Y)=\left\{z=x+iy\in\mathbb{H}:-\frac{1}{2}\le x<\frac{1}{2}\text{ and } y>Y\right\}$. Andersen and Duke [AD22] generalized this result to give an asymptotic formula for $\operatorname{Tr}_d j_m(z_D)$ with error bound.

Comparing with [AD22], we proved a bound which is completely explicit (but weaker in the exponent of some parameters). Let $\tau(n)$ be the number of divisors of n.

Theorem 4.1 ([GS23]). *Let D, d and m be as in* [AD22, Theorem 2.2], *and let $\ell(\delta)$ be a constant satisfying $\tau(c)\leq \ell(\delta)c^\delta$ for all c. Then for any $\delta\in(0,\frac14]$ and $0<Y\leq\frac{1}{2\pi m}$,*

$$\left|\operatorname{Tr}_d j_m(z_D)+24\delta_d\sigma_1(m)h(D)-\sum_{z_Q\in\mathcal{R}(Y)}\chi_d(Q)(e(-mz_Q)-e(-m\overline{z_Q}))\right|$$
$$\leq |D|^{\frac{13}{12}+\frac{\delta}{2}}m^{\frac32}\tau(D)\tau(m)Y^{\frac13-\delta}\zeta^2(1+\delta)\ell(\delta)\log\frac{2|D|^{\frac12}}{Y}|\log\delta|\cdot\begin{cases}106954 & d<0,\\ 24957m & d>0.\end{cases}\tag{4.1}$$

We also obtained an explicit bound for Y which ensures that left hand side of (4.1) is less than $\frac12$. Since $\operatorname{Tr}_d j_m(z_D)\in\mathbb{Z}$, this means that we can approximate the trace with the sum of an explicit number of terms. A main tool in our work is an explicit bound for sums of plus-space Kloosterman sums, which is of independent interest:

Theorem 4.2 ([GS23]). *For $m>0$, $n<0$, $\delta\in(0,\frac14]$ and $\ell(\delta)$ defined as in Theorem 4.1, we have*

$$\left|\sum_{4|c\leq x}\frac{S^+_{\frac12}(m,n,c)}{c}\right|\leq 26\,x^{\frac16+\delta}m^{\frac34}|n|^{\frac14}\tau(m)^{\frac12}\tau(n)^{\frac12}\zeta^2(1+\delta)\ell(\delta)|\log\delta|\log x.\tag{4.2}$$

5. Ongoing projects

5.1. **Rademacher-type exact formulas for ranks of partitions.** Motivated by the results of Bringmann and Ono [Bri09], [BO10], [BO12] and Garvan [Gar17], I would like to explain the sum

$$A\left(\frac{a}{c};n\right)=\frac{4\sqrt3 i}{\sqrt{24n-1}}\sum_{c|k\leq\sqrt n}\frac{B_{a,c,k}(-n,0)}{\sqrt k}\sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$
$$+\frac{8\sqrt3\sin(\frac{\pi a}{c})}{\sqrt{24n-1}}\sum_{\substack{1\leq k\leq\sqrt n,\ c\nmid k\\ r\geq0,\ \delta_{c,k,r}>0}}\frac{D_{a,c,k}(-n,m_{a,c,k,r})}{\sqrt k}\sinh\left(\frac{\pi\sqrt{2\delta_{c,k,r}(24n-1)}}{k\sqrt3}\right)+O_c(n^\varepsilon)$$

in terms of sums of Kloosterman sums as in the case of $A(\frac12;n)$ and $A(\frac13;n)$ mentioned in Section 3. I also want to prove that the formula actually converges to $A(\frac ab;n)$ when summed to infinity. However, when $c\geq5$ is prime, $B_{a,c,k}$ is no longer a Kloosterman sum with a multiplier system, but in current works I find a good explanation in vector-valued sense.

5.2. **Better uniform bounds on sums of Kloosterman sums.** Andersen and Wu [AW22] obtained a better uniform bound in the case of the eta-multiplier. Writing $24n-1=tw^2$ where t is square-free, they obtained

$$R_1(n,\alpha\sqrt n)\ll_{\alpha,\varepsilon} n^{-\frac12+\varepsilon}t^{-\frac1{36}}w^{-\frac16}.$$

Here $R_1(n,x)$ is the tail sum for $c>x$ in (3.1).

Their proof uses the hybrid subconvexity bound for the central value of L-functions. I wish to generalize their result to get uniform bounds of Kloosterman sums with respect to a family of multiplier systems and on $\Gamma_0(N)$ for all N, which will improve my results in Theorem 2.1 and Theorem 3.2.

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