AME 60614: Numerical Methods
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Problem Set 2

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All codes are submitted to Sakai, and only their file names will be mentioned in this report when they occurs.

1 Finite-Difference Schemes

1.1 a

Taking the determinant of $T_\alpha - \lambda I$ with $n \times n$ as D_n. Computing D_n by expanding the last row,

$$\begin{aligned}
&D_n = (\alpha - \lambda) D_{n-1} - (-1)^{2n-1} (-1)^{2n-2} D_{n-2} \\
&D_n = (\alpha - \lambda) D_{n-1} - D_{n-2} \\
&D_0 = 0 \\
&D_1 = (\alpha - \lambda)
\end{aligned}$$

Taking $(\alpha - \lambda)$ as $2x$, then the recurrence relation of D_n becomes,

$$\begin{aligned}
D_0 &= 0 \\
D_1 &= 2x \\
D_n &= 2xD_{n-1} - D_{n-2}
\end{aligned}$$

That is the famous Chebyshev polynomials of the second kind, and the roots are,

$$x_j = \cos\left(\frac{\pi j}{n+1}\right),\ j = 1, ..., n$$

Therefore, the eigenvalues can be solved,

$$\begin{aligned}
\alpha - \lambda_j = 2x_j &= 2\cos\left(\frac{\pi j}{n+1}\right) \\
\lambda_j &= \alpha - 2\cos\left(\frac{\pi j}{n+1}\right),\ j = 1, ..., n
\end{aligned}$$

For $k-th$ component of eigenvector associated with λ_j,

$$x_{k+2} - (\alpha - \lambda_j)\, x_{k+1} + x_k = 0$$

Such equation has different eigenroots. Therefore,

$$\begin{aligned}
x_0 &= x_{n+1} = 0 \\
x_k &= ar_1^k + br_2^k \\
a + b &= 0 \\
x_k &= a\left(e^{\frac{ik\pi j}{n+1}} - e^{-\frac{ik\pi j}{n+1}}\right) = 2ia\sin\left(\frac{k\pi j}{n+1}\right)
\end{aligned}$$

Since a is arbitrary, setting $a = \frac{1}{2i}$.

$$x_k = \sin\left(\frac{k\pi j}{n+1}\right)$$
$$\boldsymbol{q}_j = \left[\sin\left(\frac{\pi j}{n+1}\right), \sin\left(\frac{2\pi j}{n+1}\right), ..., \sin\left(\frac{n\pi j}{n+1}\right)\right]^T$$