AME 60614: Numerical Methods
Fall 2021

Problem Set 3

Qihao Zhuo

1 Modified Wavenumber Analysis

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= \alpha \frac{\partial^2 \phi}{\partial x^2} \\
\phi_j &= \psi(t) e^{ikx_j} \\
\frac{d\psi}{dt} &= -\alpha k^2 \phi
\end{aligned}
$$

Considering the second-order one-sided scheme,

$$
\begin{aligned}
\frac{d\phi_j}{dt} &= \frac{\alpha}{\Delta x^2}\left(-\phi_{j+3} + 4\phi_{j+2} - 5\phi_{j+1} + 2\phi_j\right) \\
&= \frac{\alpha}{\Delta x^2}\left(-\psi e^{ikx_j} e^{ik3\Delta x} + 4\psi e^{ikx_j} e^{ik2\Delta x} - 5\psi e^{ikx_j} e^{ik\Delta x} + 2\psi e^{ikx_j}\right) \\
&= \frac{\alpha\phi}{\Delta x^2}\left(-e^{ik3\Delta x} + 4e^{ik2\Delta x} - 5e^{ik\Delta x} + 2\right) \\
&= \frac{\alpha\phi}{\Delta x^2}\left(-\cos 3\Delta x - i\sin 3\Delta x + 4\cos 2\Delta x + 4i\sin 2\Delta x - 5\cos\Delta x - 5i\sin\Delta x + 2\right) \\
&= \frac{\alpha}{\Delta x^2}\left[\left(2 - \cos 3\Delta x + 4\cos 2\Delta x - 5\cos\Delta x\right) - i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right]\phi \\
&= -\frac{\alpha}{\Delta x^2}\left[\left(-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x\right) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right]\phi \\
-\alpha k'^2\phi &= -\frac{\alpha}{\Delta x^2}\left[\left(-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x\right) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right]\phi \\
-\alpha k'^2 &= -\frac{\alpha}{\Delta x^2}\left[\left(-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x\right) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right] \\
k'^2 &= \frac{1}{\Delta x^2}\left[\left(-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x\right) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right] \\
k'^2\Delta x^2 &= \left(-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x\right) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)
\end{aligned}
$$

$k'\Delta x$ is a complex number and $|k'\Delta x|_{max} > 2$. Thus it will lead to numerical instablity.

2 One-Dimensional Diffusion Equation

The time step could still be uniform, so the forward-time scheme,

$$
\frac{\partial u}{\partial t} = \frac{u_j^{n+1} - u_j^n}{\Delta t}
$$

For non-uniform grid,

$$
\begin{aligned}
u_{j+1} &= u_j + \frac{\partial u}{x}(x_{j+1}-x_j) + \frac{\partial^2 u}{\partial x^2}(x_{j+1}-x_j)^2 \\
u_{j-1} &= u_j + \frac{\partial u}{x}(x_{j-1}-x_j) + \frac{\partial^2 u}{\partial x^2}(x_{j-1}-x_j)^2 \\
\frac{u_{j+1}}{x_{j+1}-x_j} &= \frac{u_j}{x_{j+1}-x_j} + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}(x_{j+1}-x_j) \\
\frac{u_{j-1}}{x_j-x_{j-1}} &= \frac{u_j}{x_j-x_{j-1}} - \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}(x_j-x_{j-1}) \\
\frac{u_{j+1}}{x_{j+1}-x_j} + \frac{u_{j-1}}{x_j-x_{j-1}} &= \frac{u_j}{x_{j+1}-x_j} + \frac{u_j}{x_j-x_{j-1}} + \frac{\partial^2 u}{\partial x^2}(x_{j+1}-x_{j-1}) \\
\Rightarrow \frac{\partial^2 u}{\partial x^2} &= \frac{1}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}u_{j+1} \\
&\quad - \left(\frac{1}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)} - \frac{1}{(x_{j+1}-x_{j-1})(x_j-x_{j-1})}\right)u_j + \frac{1}{(x_{j+1}-x_{j-1})(x_j-x_{j-1})}u_{j-1} \\
&= \frac{1}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}u_{j+1} - \frac{1}{(x_{j+1}-x_j)(x_j-x_{j-1})}u_j + \frac{1}{(x_{j+1}-x_{j-1})(x_j-x_{j-1})}u_{j-1}
\end{aligned}
$$

Combing them for the FTCS scheme for non-uniform grid.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha\frac{\partial^2 u}{\partial x^2} \\
\frac{u_j^{n+1}-u_j^n}{\Delta t} &= \frac{\alpha}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}u_{j+1} - \frac{\alpha}{(x_{j+1}-x_j)(x_j-x_{j-1})}u_j + \frac{\alpha}{(x_{j+1}-x_{j-1})(x_j-x_{j-1})}u_{j-1} \\
u_j^{n+1} &= u_j \\
&\quad + \alpha\Delta t\left(\frac{1}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}u_{j+1} - \frac{1}{(x_{j+1}-x_j)(x_j-x_{j-1})}u_j + \frac{1}{(x_{j+1}-x_{j-1})(x_j-x_{j-1})}u_{j-1}\right)
\end{aligned}
$$

Now considering periodic boundary conditions, $u_0 = u_N$. So in practice the point x_0 could be seen as x_N. That is, the left point of x_0 could be x_{N-1} and the right point of x_N could be x_1. And to keep $u_0 = u_N$, only one of u_0 and u_N will be used in simulation. For convenience of notation, in matlab, u_N is used.

$$
\begin{aligned}
\frac{\partial^2 u_1}{\partial x^2} &= \frac{1}{(x_2-x_0)(x_2-x_1)}u_2 - \frac{1}{(x_2-x_1)(x_1-x_0)}u_1 + \frac{1}{(x_2-x_0)(x_1-x_0)}u_0 \\
&= \frac{1}{(x_2-x_0)(x_2-x_1)}u_2 - \frac{1}{(x_2-x_1)(x_1-x_0)}u_1 + \frac{1}{(x_2-x_0)(x_1-x_0)}u_N \\
\frac{\partial^2 u_N}{\partial x^2} &= \frac{1}{(x_{N+1}-x_{N-1})(x_{N+1}-x_N)}u_{N+1} - \frac{1}{(x_{N+1}-x_N)(x_N-x_{N-1})}u_N + \frac{1}{(x_{N+1}-x_{N-1})(x_N-x_{N-1})}u_{N-1} \\
&= \frac{1}{(x_{N+1}-x_{N-1})(x_{N+1}-x_N)}u_1 - \frac{1}{(x_{N+1}-x_N)(x_N-x_{N-1})}u_N + \frac{1}{(x_{N+1}-x_{N-1})(x_N-x_{N-1})}u_{N-1}
\end{aligned}
$$

So the matrix form of the scheme is,

$$
\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ ... \\ u_{N-1}^{n+1} \\ u_N^{n+1} \end{bmatrix} = I \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_{N-1} \\ u_N \end{bmatrix} + \alpha\Delta t A
$$

Matrix A is shown below,

$$\begin{bmatrix} -\frac{1}{(x_2-x_1)(x_1-x_0)} & \frac{1}{(x_2-x_0)(x_2-x_1)} & \cdots & \cdots & \frac{1}{(x_2-x_0)(x_1-x_0)} \\ \frac{1}{(x_3-x_1)(x_2-x_1)} & -\frac{1}{(x_3-x_2)(x_2-x_1)} & \frac{1}{(x_3-x_1)(x_3-x_2)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \frac{1}{(x_N-x_{N-2})(x_{N-1}-x_{N-2})} & -\frac{1}{(x_N-x_{N-1})(x_{N-1}-x_{N-2})} & \frac{1}{(x_N-x_{N-2})(x_N-x_{N-1})} \\ \frac{1}{(x_{N+1}-x_{N-1})(x_{N+1}-x_N)} & \cdots & \cdots & \frac{1}{(x_{N+1}-x_{N-1})(x_N-x_{N-1})} & -\frac{1}{(x_{N+1}-x_N)(x_N-x_{N-1})} \end{bmatrix}$$