AME 60614: Numerical Methods
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Problem Set 3

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1 Modified Wavenumber Analysis

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}$$
$$\phi_j = \psi(t) e^{ikx_j}$$
$$\frac{d\psi}{dt} = -\alpha k^2 \phi$$

Considering the second-order one-sided scheme,

$$\begin{aligned}
\frac{d\phi_j}{dt} &= \frac{\alpha}{\Delta x^2}\left(-\phi_{j+3} + 4\phi_{j+2} - 5\phi_{j+1} + 2\phi_j\right) \\
&= \frac{\alpha}{\Delta x^2}\left(-\psi e^{ikx_j} e^{ik3\Delta x} + 4\psi e^{ikx_j} e^{ik2\Delta x} - 5\psi e^{ikx_j} e^{ik\Delta x} + 2\psi e^{ikx_j}\right) \\
&= \frac{\alpha\phi}{\Delta x^2}\left(-e^{ik3\Delta x} + 4e^{ik2\Delta x} - 5e^{ik\Delta x} + 2\right) \\
&= \frac{\alpha\phi}{\Delta x^2}\left(-\cos 3\Delta x - i\sin 3\Delta x + 4\cos 2\Delta x + 4i\sin 2\Delta x - 5\cos\Delta x - 5i\sin\Delta x + 2\right) \\
&= \frac{\alpha}{\Delta x^2}\left[(2 - \cos 3\Delta x + 4\cos 2\Delta x - 5\cos\Delta x) - i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right]\phi \\
&= -\frac{\alpha}{\Delta x^2}\left[(-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right]\phi \\
-\alpha k^{'2}\phi &= -\frac{\alpha}{\Delta x^2}\left[(-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right]\phi \\
-\alpha k^{'2} &= -\frac{\alpha}{\Delta x^2}\left[(-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right] \\
k^{'2} &= \frac{1}{\Delta x^2}\left[(-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)\right] \\
k^{'2}\Delta x^2 &= (-2 + \cos 3\Delta x - 4\cos 2\Delta x + 5\cos\Delta x) + i\left(\sin 3\Delta x - 4\sin 2\Delta x - 5\sin\Delta x\right)
\end{aligned}$$

$k^{'}\Delta x$ is a complex number and $|k^{'}\Delta x|_{max} > 2$. Thus it will lead to numerical instablity.

2 One-Dimensional Diffusion Equation