AME 60614: Numerical Methods
Fall 2021

Problem Set 4
Due: December 2, 2021

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Submission guidelines: A hard copy of written work is due in class. Additionally, submit through Sakai an archive (`tar` or similar) of all files requested in this assignment. Place files from individual problems into separate folders within your archive, *i.e.*, `p1/`, `p2/`, etc. Please give this archive a name of the form `lastname_firstname_ps4.tar`.

Contact `jmacart@nd.edu` with any questions.

1 Modified Wavenumber Analysis

This problem is partially taken from Chapter 5 of Parviz Moin, Fundamentals of Engineering Numerical Analysis. *Sections of this chapter could prove helpful in its completion.*

Use modified wavenumber analysis to show that the application of a second-order, one-sided spatial differencing scheme given by

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_j = \frac{-\phi_{j+3} + 4\phi_{j+2} - 5\phi_{j+1} + 2\phi_j}{\Delta x^2}$$

to the diffusion equation would lead to numerical instability.

2 One-Dimensional Diffusion Equation

Consider the one-dimensional linear diffusion equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

which is to be solved on the domain $x \in [-1, 1]$ using a non-uniform grid. The grid spacing for $j = 0, 1, \ldots, N$ ($N + 1$ points) is given by

$$x_j = \frac{\tanh\left(5\frac{j-N/2}{N/2}\right)}{\tanh(5)}.$$

Consider the solution of Eq. (1) on this non-uniform grid using the forward-time, centered-space (FTCS) scheme with periodic boundary conditions, that is, $u(x = -1) = u(x = 1)$.

a. Compute numerically the eigenvalues of the semi-discretized system. What is the ratio $\lambda_{\max}/\lambda_{\min}$ for $N = 128$? (*Hint: Be careful to ensure a full-rank matrix in implementing the boundary conditions.*)

b. Using matrix stability analysis, plot the maximum allowable product $\alpha\Delta t$ for stable solutions versus $N = \{32, 64, 128, 256, 512, 1024\}$.

c. Estimate the maximum stable time step size using von Neumann stability analysis. (*Hint: What is an appropriate approximation to the nonuniform grid for stability-analysis purposes?*) How does this estimate compare to that from matrix stability analysis?

3 Multi-Dimensional Convection Equation

Consider the multi-dimensional convection equation,

$$\frac{\partial u}{\partial t} + c_x \frac{\partial u}{\partial x} + c_y \frac{\partial u}{\partial y} = 0,$$

on the domain $x \in [0, 2\pi]$ and $y \in [0, 2\pi]$. All boundaries are periodic, and the initial condition is given by

$$u(x, y, 0) = \exp\left(\frac{1}{(x-\pi)^2 - 1}\right) \exp\left(\frac{1}{(y-\pi)^2 - 1}\right); \ |x - \pi| \leq 1; \ |y - \pi| \leq 1$$

and zero elsewhere. The convective velocities are $c_x = 1$ and $c_y = 2$ in the x- and y-directions, respectively.

a. Design an algorithm to compute an acceptably accurate solution to this equation in the shortest computational time. This is subject to two requirements:

   - Your solution must retain at least 99 % of the initial energy, that is, the domain integral (i.e., discrete sum) of the square of the solution.
   - Your solution must be performed on a grid of at least size 512×512 but may need to be larger to satisfy the previous criterion. Is it faster to use "good" numerical methods on a coarse grid or "bad" numerical methods on a fine grid?

   Justify your choices for spatial discretization, temporal discretization, and maximum time step (in terms of the grid spacing, which should be uniform and the same in both directions).

b. Implement your algorithm and determine the solution after ten cycles (i.e., the tenth time the exact solution would match the initial condition). How does your solution compare to the initial condition? In your report, include the energy content of your solution (as a percentage of the initial energy) and the runtime required.

Include your code in your `.tar` archive.