AME 60614: Numerical Methods
Fall 2021

Problem Set 2
Due: October 14, 2021

Prof. Jonathan F. MacArt

Submission guidelines: A hard copy of written work is due in class. Additionally, submit through Sakai an archive (`tar` or similar) of all files requested in this assignment. Place files from individual problems into separate folders within your archive, *i.e.*, `p1/`, `p2/`, etc. Please give this archive a name of the form `lastname_firstname_ps2.tar`.

Contact `jmacart@nd.edu` with any questions.

1 Iterative Methods for Linear Systems

This problem is partially taken from Chapter 4 of Yousef Saad, Iterative Methods for Sparse Linear Systems*, which is available at* `http://www-users.cs.umn.edu/~saad/IterMethBook_2ndEd.pdf`. *Theorems from this chapter might be particularly useful in solving this problem.*

Consider an $n \times n$ tridiagonal matrix of the form

$$\mathrm{T}_\alpha = \begin{bmatrix} \alpha & -1 & & & & \\ -1 & \alpha & -1 & & & \\ & -1 & \alpha & -1 & & \\ & & -1 & \alpha & -1 & \\ & & & -1 & \alpha & -1 \\ & & & & -1 & \alpha \end{bmatrix},$$

where α is a real parameter.

a. Verify (analytically) that the eigenvalues of T_α are given by

$$\lambda_j = \alpha - 2\cos\left(\frac{\pi j}{n+1}\right) \qquad j = 1, \ldots, n, \tag{1}$$

and that an eigenvector associated with each λ_j is

$$\mathbf{q}_j = \left[\sin\left(\frac{\pi j}{n+1}\right), \sin\left(\frac{2\pi j}{n+1}\right), \ldots, \sin\left(\frac{n\pi j}{n+1}\right)\right]^T, \tag{2}$$

where $(\cdot)^T$ is the vector transpose. Under what condition on α does this matrix become positive definite?

b. With $\alpha = 2$, the matrix T_2 corresponds to the negative of the second-order central difference stencil for a second derivative. In general, iterative schemes solve the linear system

$$\mathrm{A}x = b.$$

Decomposing the matrix A as $\mathrm{A} = \mathrm{M} - \mathrm{N}$, the update from iteration k to iteration $k+1$ is

$$\mathrm{M}x^{(k+1)} = \mathrm{N}x^{(k)} + b.$$

Defining the *iteration matrix* as $\mathrm{G} \equiv \mathrm{M}^{-1}\mathrm{N} = \mathrm{I} - \mathrm{M}^{-1}\mathrm{A}$, the iterative update may be rewritten:

$$x^{(k+1)} = \mathrm{G}x^{(k)} + f,$$

where $f = \mathrm{M}^{-1}b$ is the right-hand side. The matrix M can be considered a "preconditioner" and depends on the specific solution scheme to be employed.

(i) Will Jacobi iteration converge for $\mathrm{A} = \mathrm{T}_2$? If so, compute its convergence factor. (Take the convergence factor to be the spectral radius of the iteration matrix, *i.e.*, $\rho(\mathrm{G})$.)

(ii) Will Gauss–Seidel iteration converge for $\mathrm{A} = \mathrm{T}_2$? If so, compute its convergence factor. (*Hint: Refer to Theorem 4.16 in Saad.*)

(iii) For which values of ω will SOR iteration converge? What is the optimum value of ω for this problem?

c. Letting $\alpha = 2$ as before, compute the ratio of the number of Jacobi iterations to the number of Gauss–Seidel iterations required to converge the solution to the same error.

d. Letting $\alpha = 2$ again, consider the linear system

$$\mathrm{T}_2\mathbf{x} = \mathbf{b}.$$

With $n = 100$, let

$$\begin{aligned} \mathbf{x} &= \mathbf{q}_j, \\ \mathbf{b} &= \lambda_j \mathbf{q}_j, \end{aligned}$$

such that this is the eigenvalue problem for T_2. The eigenvalues λ_j are given by Eq. (1), and the eigenvectors $\mathbf{q}_j$ are given by Eq. (2).

- Solve the eigenvalue problem for $j = 1$ using Jacobi iteration, using an initial guess of $\mathbf{x}^0 = \mathbf{0}$ and 1000 iterations. Define the error at the k^{th} iteration as

$$\epsilon^k = \max_{i \le n} \left| x_i - x_i^k \right|.$$

  Plot the error as a function of iteration number on a semi-log-y plot (x-axis linear). What is the convergence rate in this case? Is it the same as the j^{th} eigenvalue of the Jacobi iteration matrix? If not, why?
- Solve the eigenvalue problem for $j = 2$. What is the convergence rate in this case?
- Based on these results, how would you expect the error to behave as a function of the number of iterations for an arbitrary $\mathbf{x}$ that is a *linear combination* of the eigenvectors $\mathbf{q}_j$? (Test it for, e.g., $\mathbf{b} = \lambda_2\mathbf{q}_2 + \lambda_4\mathbf{q}_4$.) Is the j^{th} eigenvalue of the Jacobi iteration matrix an accurate estimate of the convergence factor when k is small? What about in the limit as $k \to \infty$? Be sure to explain your reasoning.

Include in your `.tar` archive your Jacobi solver code (in any programming language).

2 Multigrid Methods

(Multigrid example taken from LeVeque). Consider the one-dimensional Poisson equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x)$$

with Dirichlet boundary conditions $u(0) = 1$ and $u(1) = 3$. Let

$$f(x) = -20 + a\phi'' \cos\phi - a\left(\phi'\right)^2 \sin\phi,$$

where $a = 0.5$ and $\phi(x) = 20\pi x^3$. As we will see in our discussion of ODEs, this is a prototypical *boundary-value problem.* The true solution is

$$u(x) = 1 + 12x - 10x^2 + a\sin\phi.$$

a. Discretize the second derivative using the second-order central-difference scheme on 255 interior grid points, and use Jacobi iteration to solve the resulting linear system for u. Use $u^{(0)} = 1 + 2x$ for your initial guess. Plot the numerical solution after 20, 100, and 1000 iterations overtop the analytic solution. Discuss your findings.

b. Now solve this problem using a multigrid method with 7 levels of grids, that is, using $2^j - 1$ interior grid points, where $j = 8, 7, 6, 5, 4, 3, 2$. Plot the solution after one V-cycle, starting from the same initial guess, and using a total of 20 Jacobi iterations on each level (10 when descending the V and 10 when ascending). How does the convergence of the multigrid approach compare to the convergence of plain Jacobi iteration?

c. In a reasonable amount of computational time, what is the smallest error (use the L_2 norm) that you can achieve with each method above? How does the computational cost of the two methods compare for a fixed error tolerance?

Be sure to include your codes in your `.tar` archive.