More Efficient Sorting: Mergesort and Quicksort

Recursive Sorts

Recursive sorts Divide the data roughly in half and are called Again on the smaller data sets. This is called the Divide-and-Conquer paradigm. We will see 2 recursive sorts:

- Merge Sort
- QuickSort

Divide-and-Conquer

- Divide-and-conquer paradigm:
  - Divide: divide one large problem into 2 smaller problems of the same type.
  - Recur: solve the 2 subproblems.
  - Conquer: combine the 2 solutions into a solution to the larger problem.
- The base case for the recursion are subproblems of manageable size, usually 0 or 1.

Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm

Merge Sort

Merge-Sort

Merge-sort on an input sequence S with n elements consists of three steps:

- Divide: partition into 2 groups of about n/2 each
- Recur: recursively sort S_1 and S_2
- Conquer: merge S_1 and S_2 into a unique sorted sequence

Merging Two Sorted Sequences

- The conquer step merges the 2 sorted sequences A and B into one sorted sequence S
- ***How: Compare the lowest element of each of A and B and insert whichever is smaller.***
- Merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes $O(n)$ time

2 5 6 9 12 15 20 27 A

4 7 10 13 16 B

S 2 4 5 6

Merging Two Sorted Sequences

Algorithm *merge*(*A*, *B*)
    Input sorted sequences *A* and *B*
    Output sorted sequence of $A \cup B$

    $S \leftarrow$ empty sequence
    while !*A.isEmpty*() $\wedge$!*B.isEmpty*()
        if isLessThan(*A.first*().*element*(), *B.first*().*element*())
            S.insertLast(*A.remove*(*A.first*()))
        else
            S.insertLast(*B.remove*(*B.first*()))
    while !*A.isEmpty*()
            S.insertLast(*A.remove*(*A.first*()))
    while !*B.isEmpty*()
            S.insertLast(*B.remove*(*B.first*()))
    return *S*

Not In-Place

Merge-Sort

Algorithm ***mergeSort***(S)

Input sequence S with n elements,

Output sequence S sorted

if $S.size() > 1$

$(S_1, S_2) \leftarrow partition(S, n/2)$

$mergeSort(S_1)$

$mergeSort(S_2)$

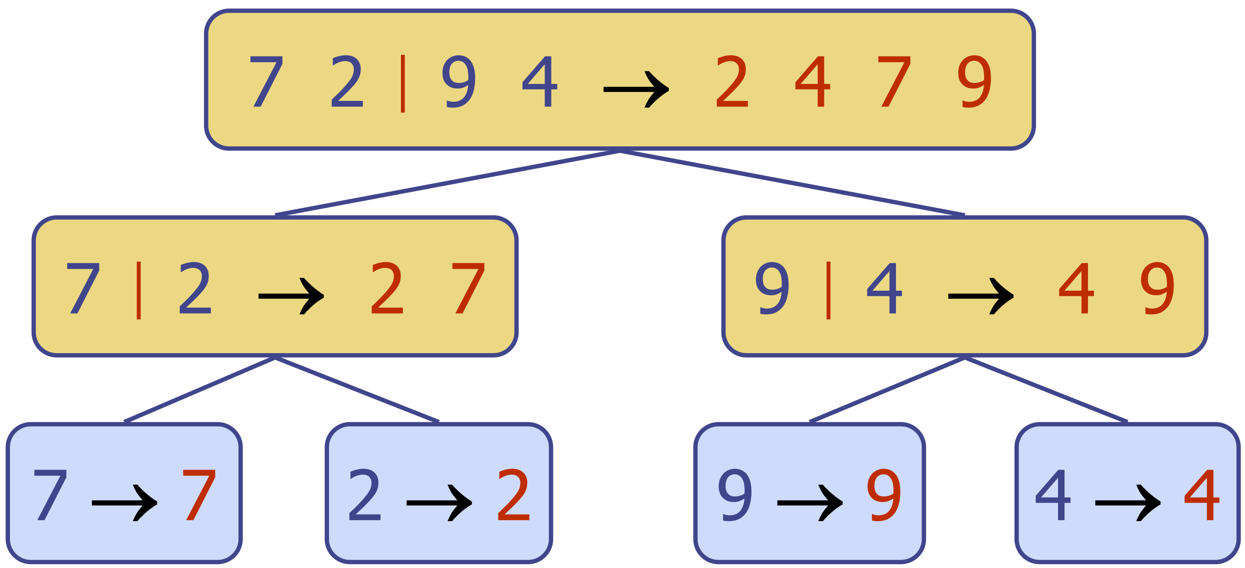
$S \leftarrow merge(S_1, S_2)$

Not In-Place

Merge-Sort Tree

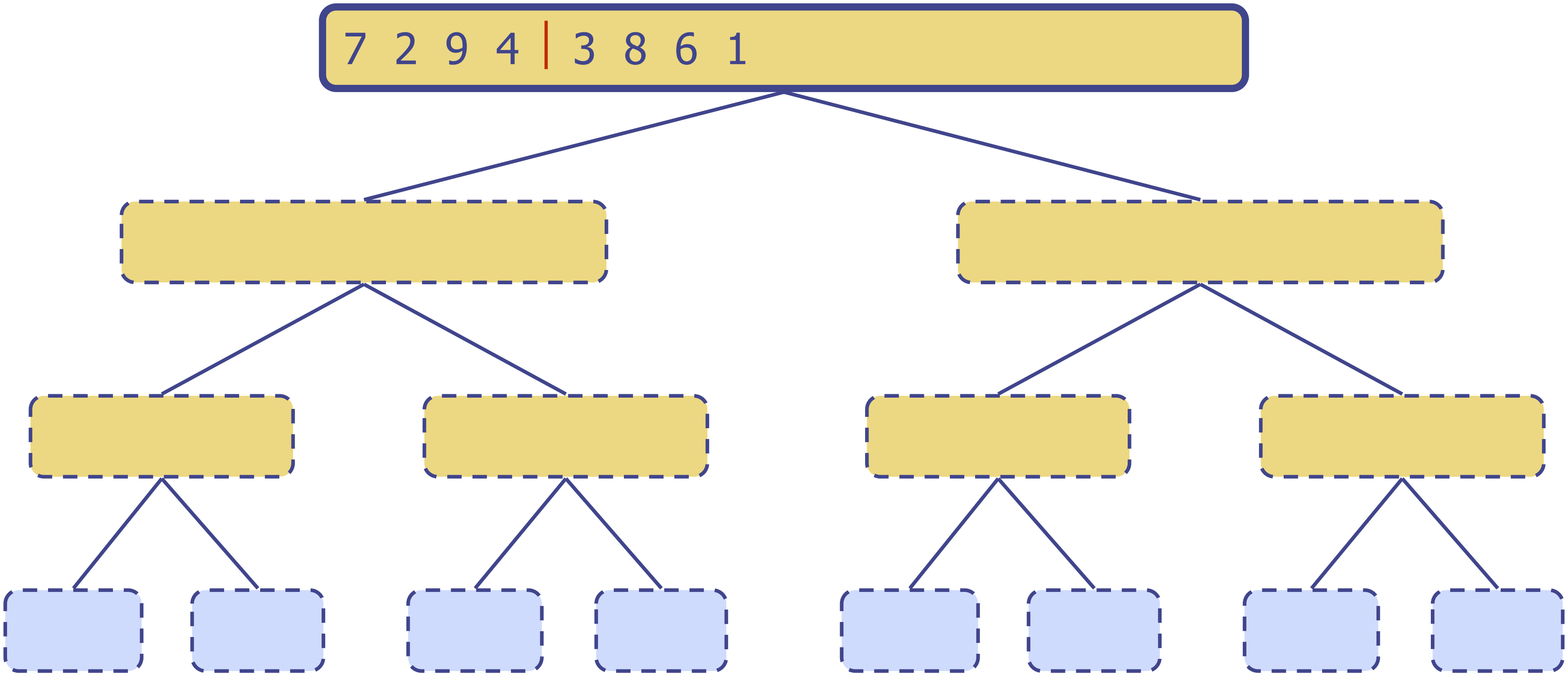
An execution of merge-sort is depicted by a binary tree

- each node represents a recursive call of merge-sort and stores
  - unsorted sequence before the execution and its partition
  - sorted sequence at the end of the execution
- the root is the initial call
- the children are calls on subsequences
- the leaves are calls on sequences of size 0 or 1



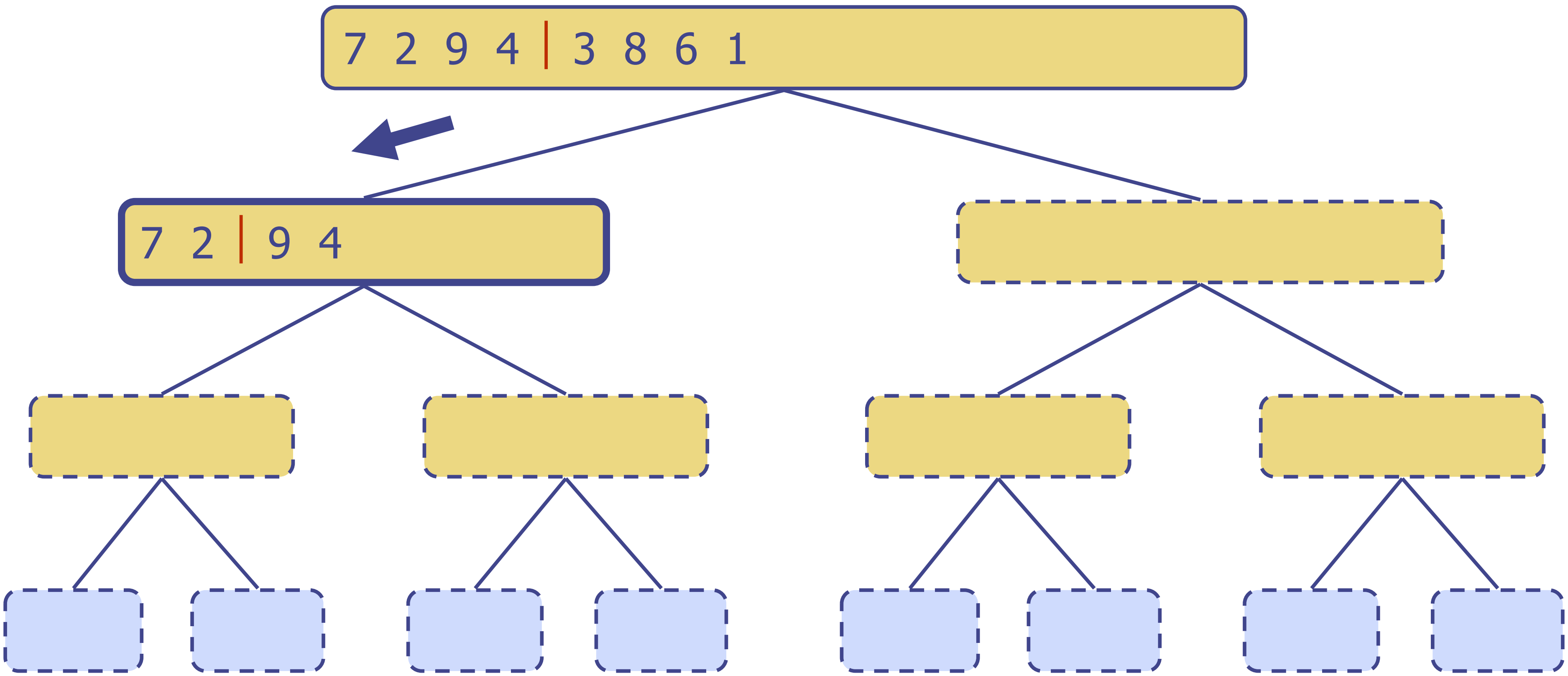
Execution Example

- Partition



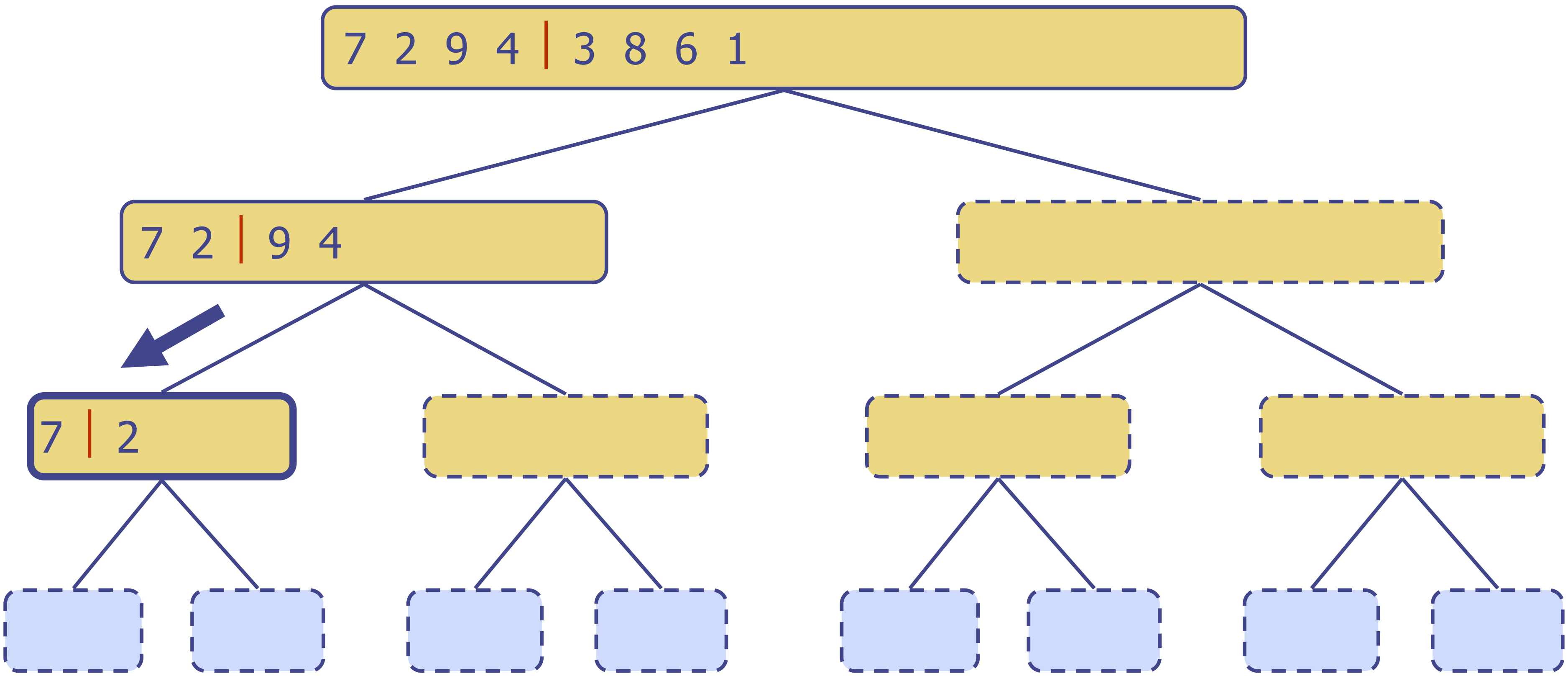
Execution Example (cont.)

- Recursive call, partition



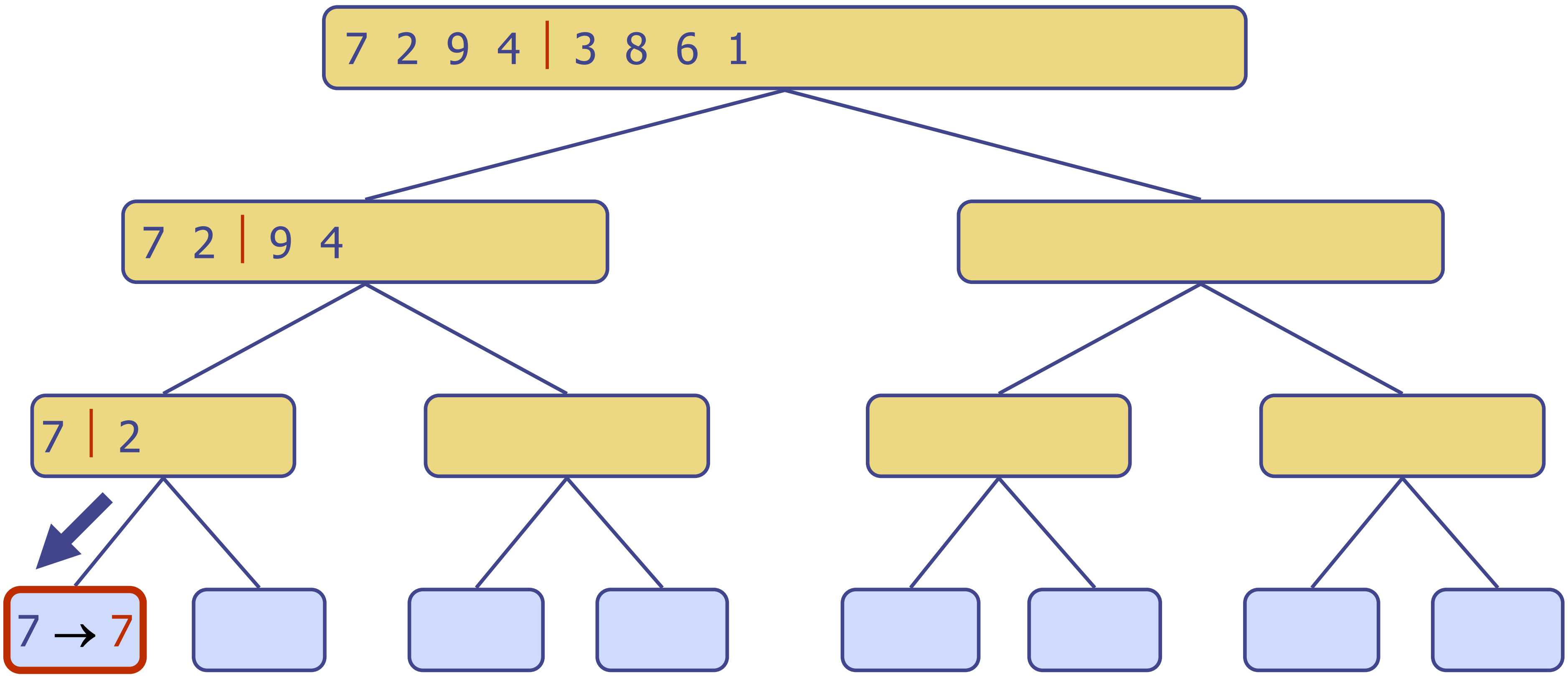
Execution Example (cont.)

- Recursive call, partition



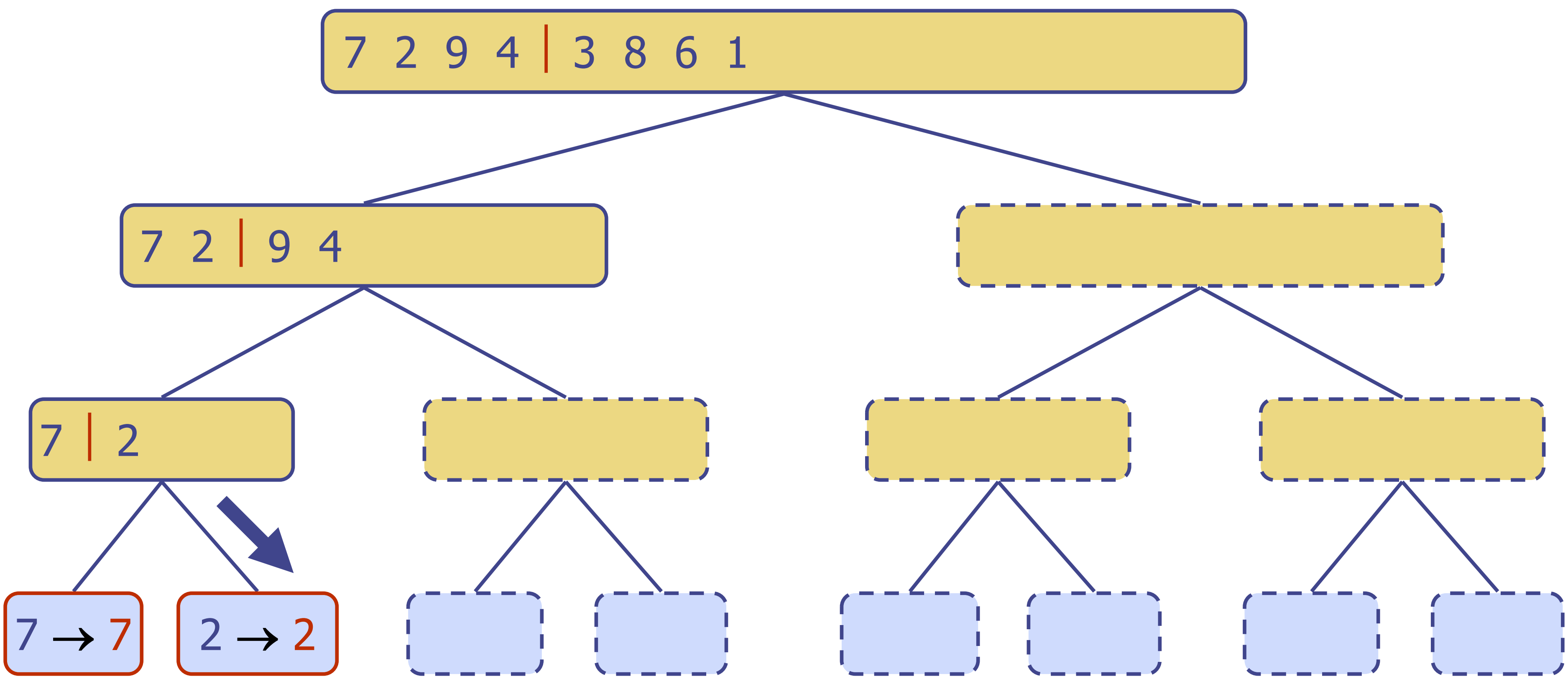
Execution Example (cont.)

- Recursive call, base case



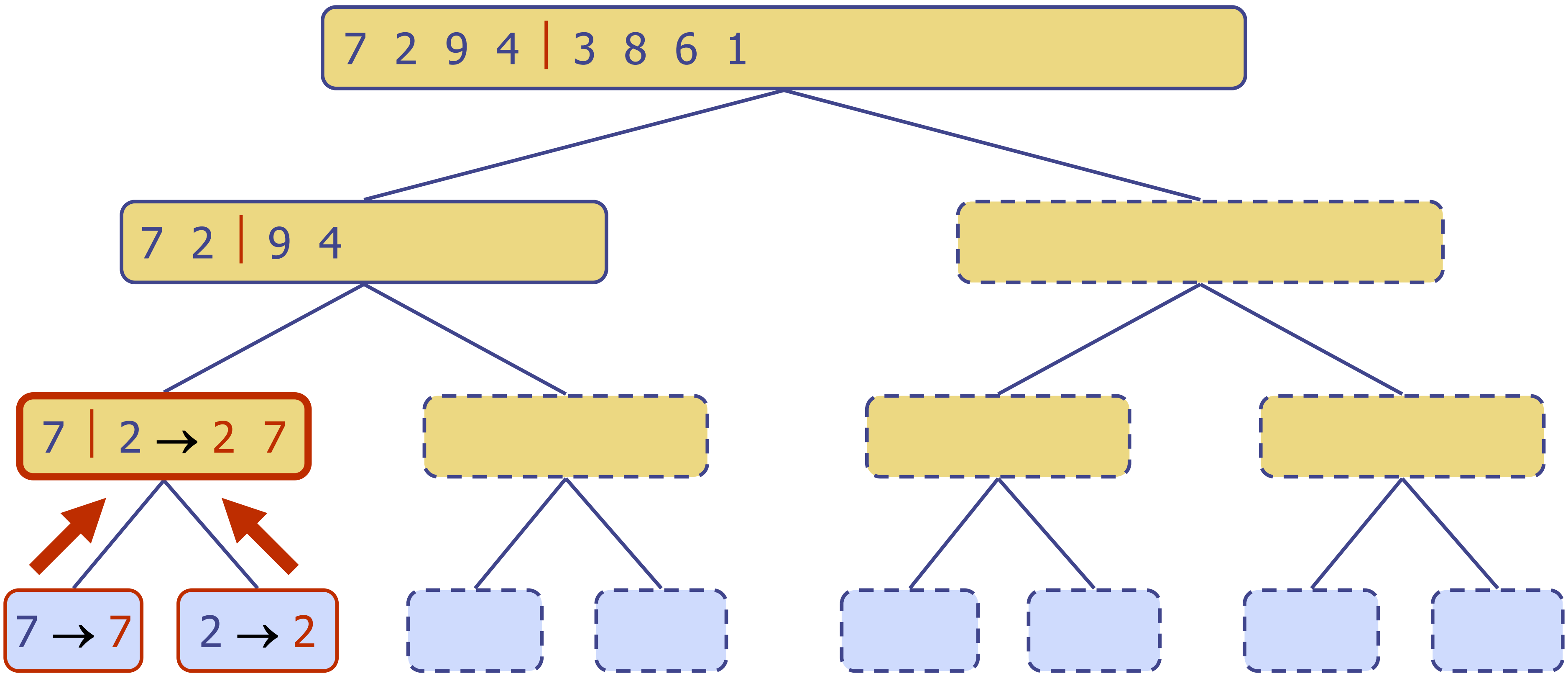
Execution Example (cont.)

- Recursive call, base case



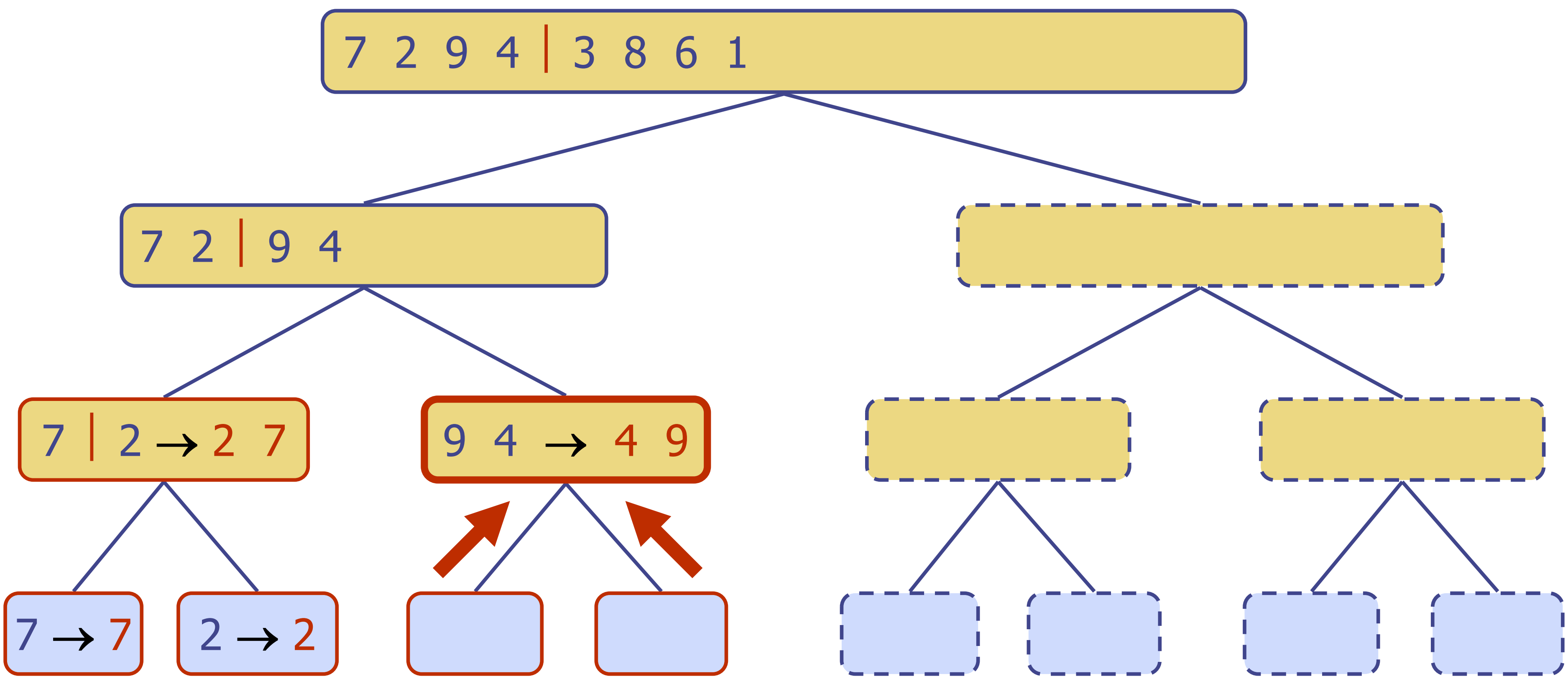
Execution Example (cont.)

- Merge



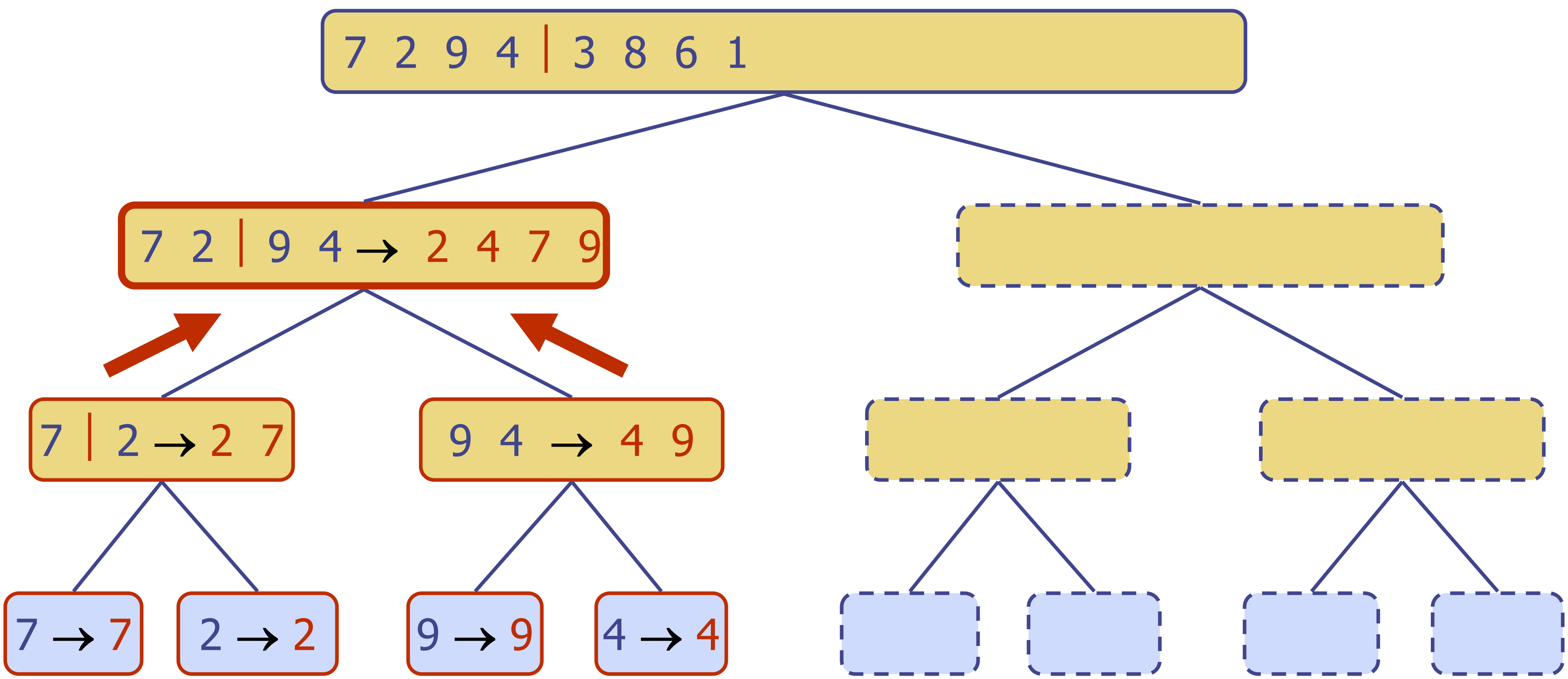
Execution Example (cont.)

- Recursive call, …, base case, merge



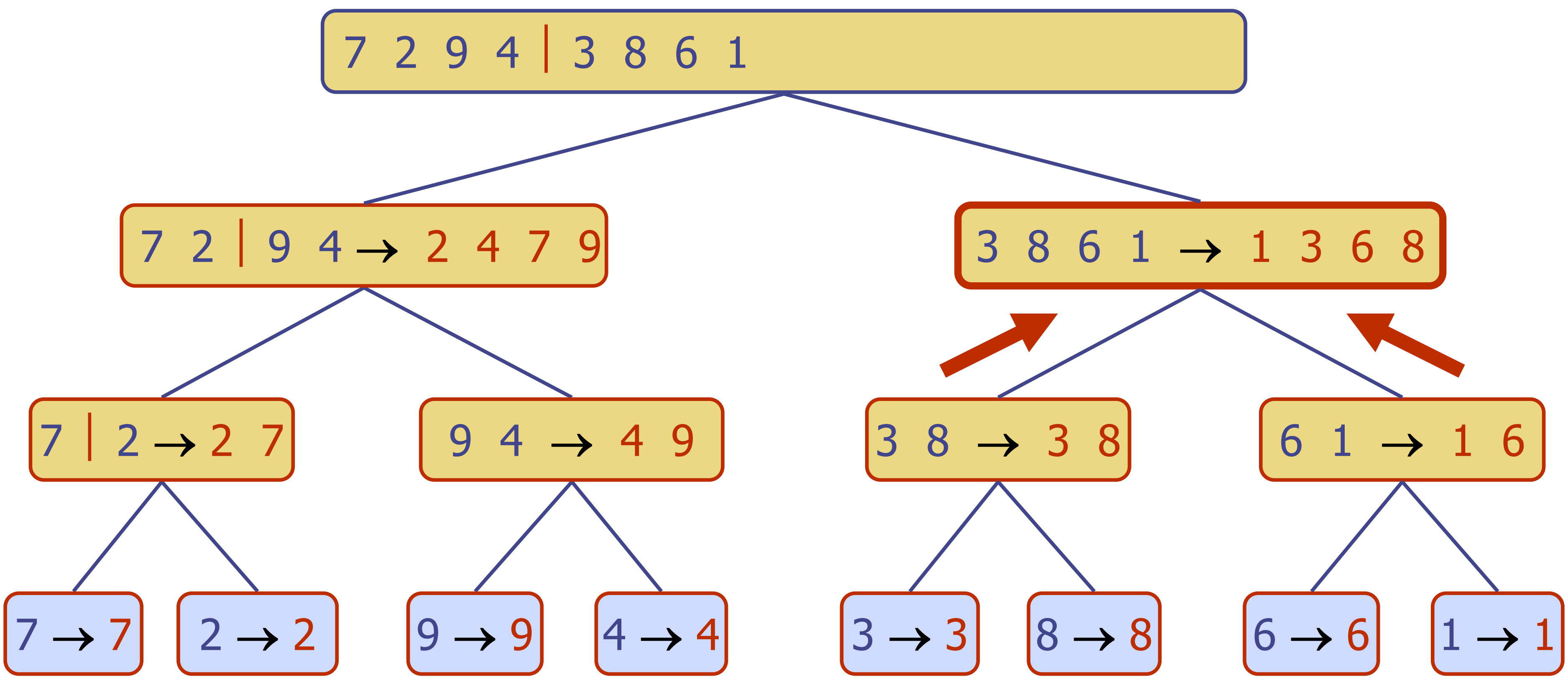
Execution Example (cont.)

- Merge



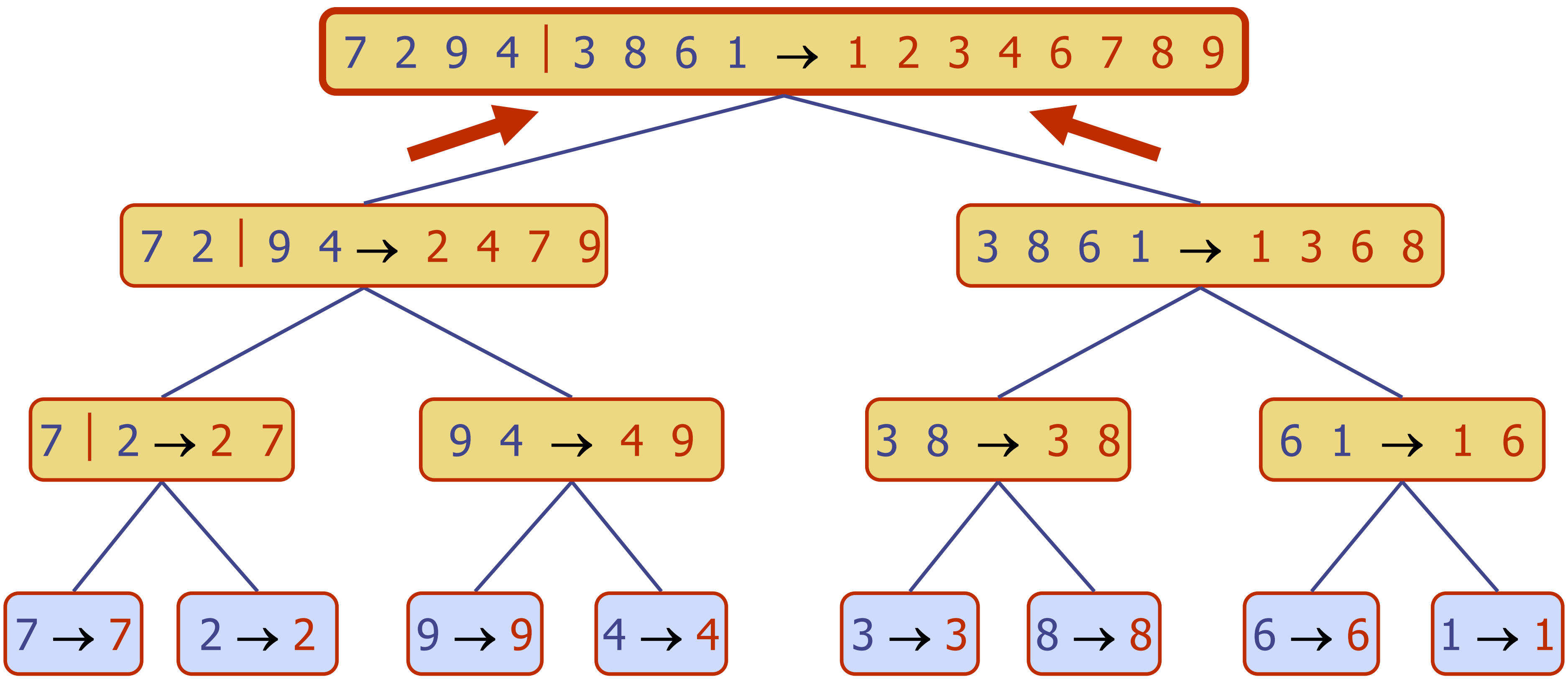
Execution Example (cont.)

- Recursive call, ..., merge, merge



Execution Example (cont.)

- Merge



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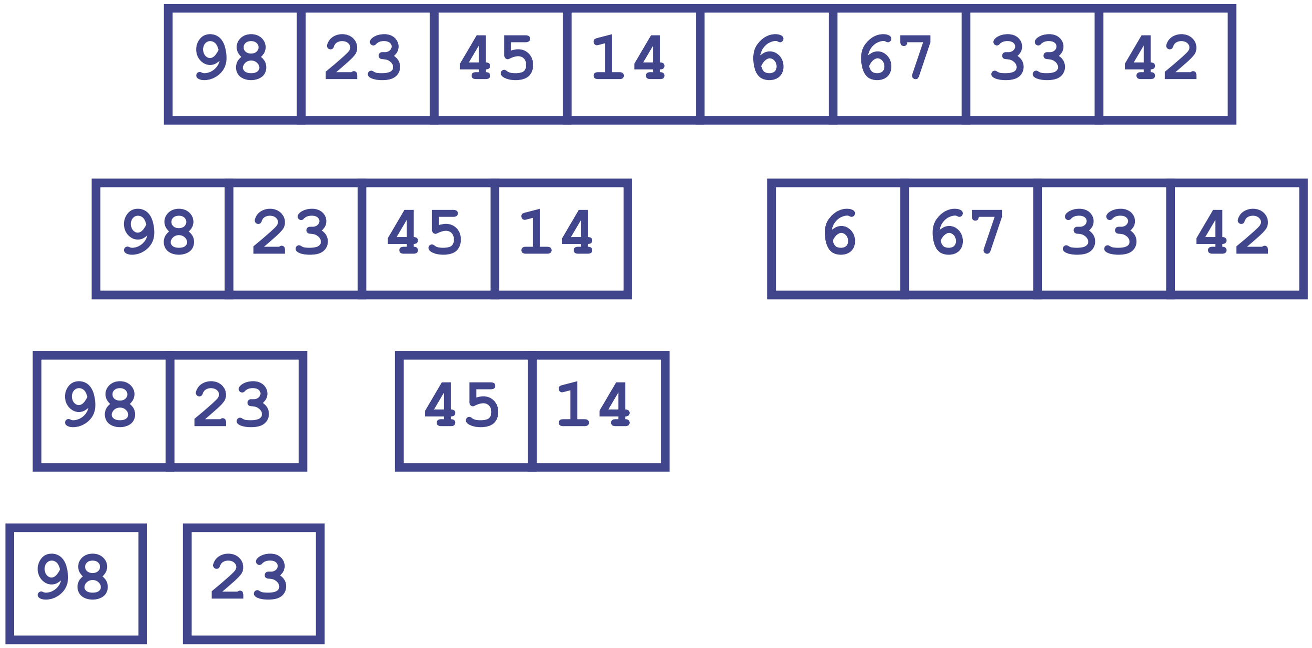
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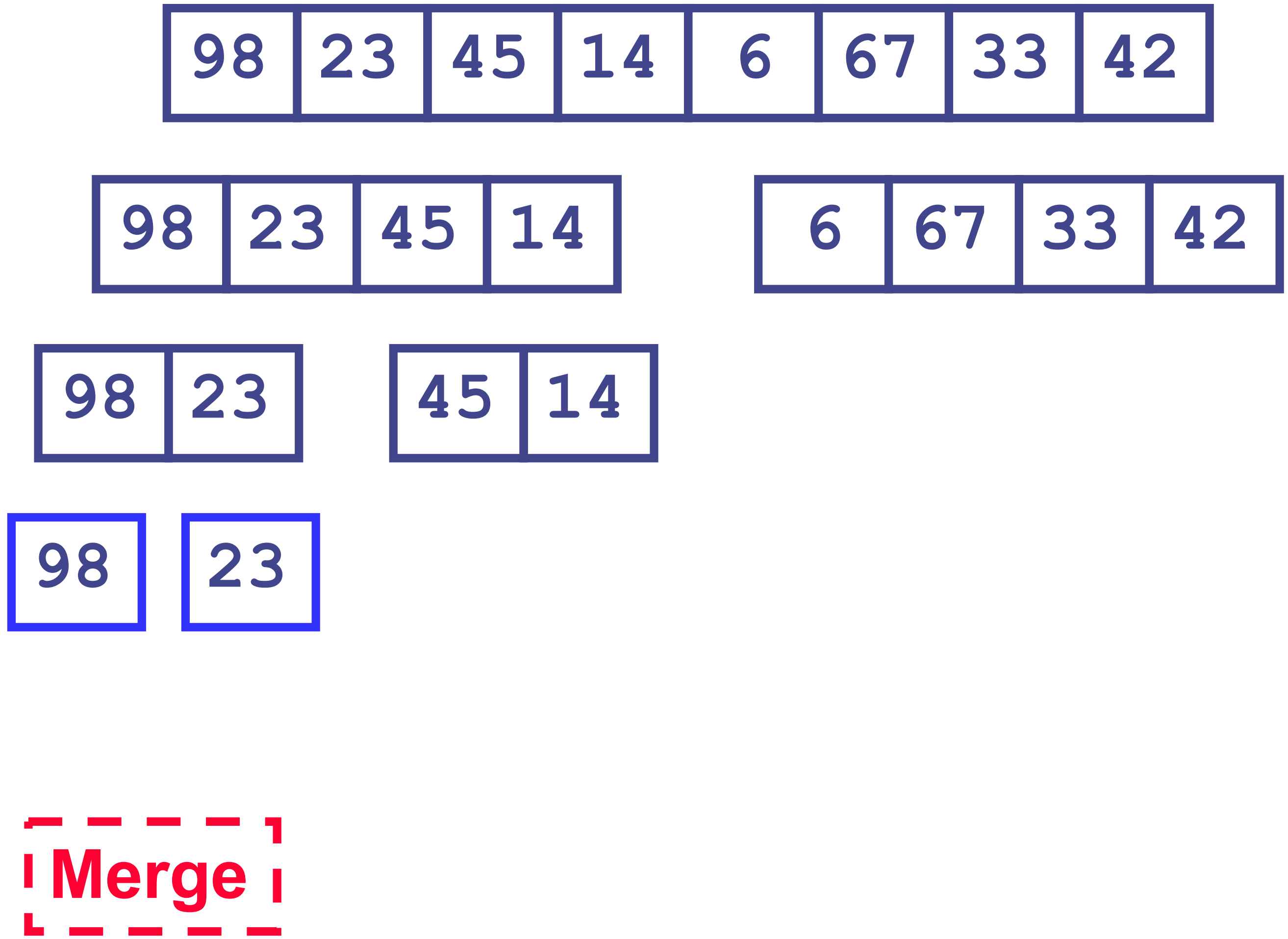
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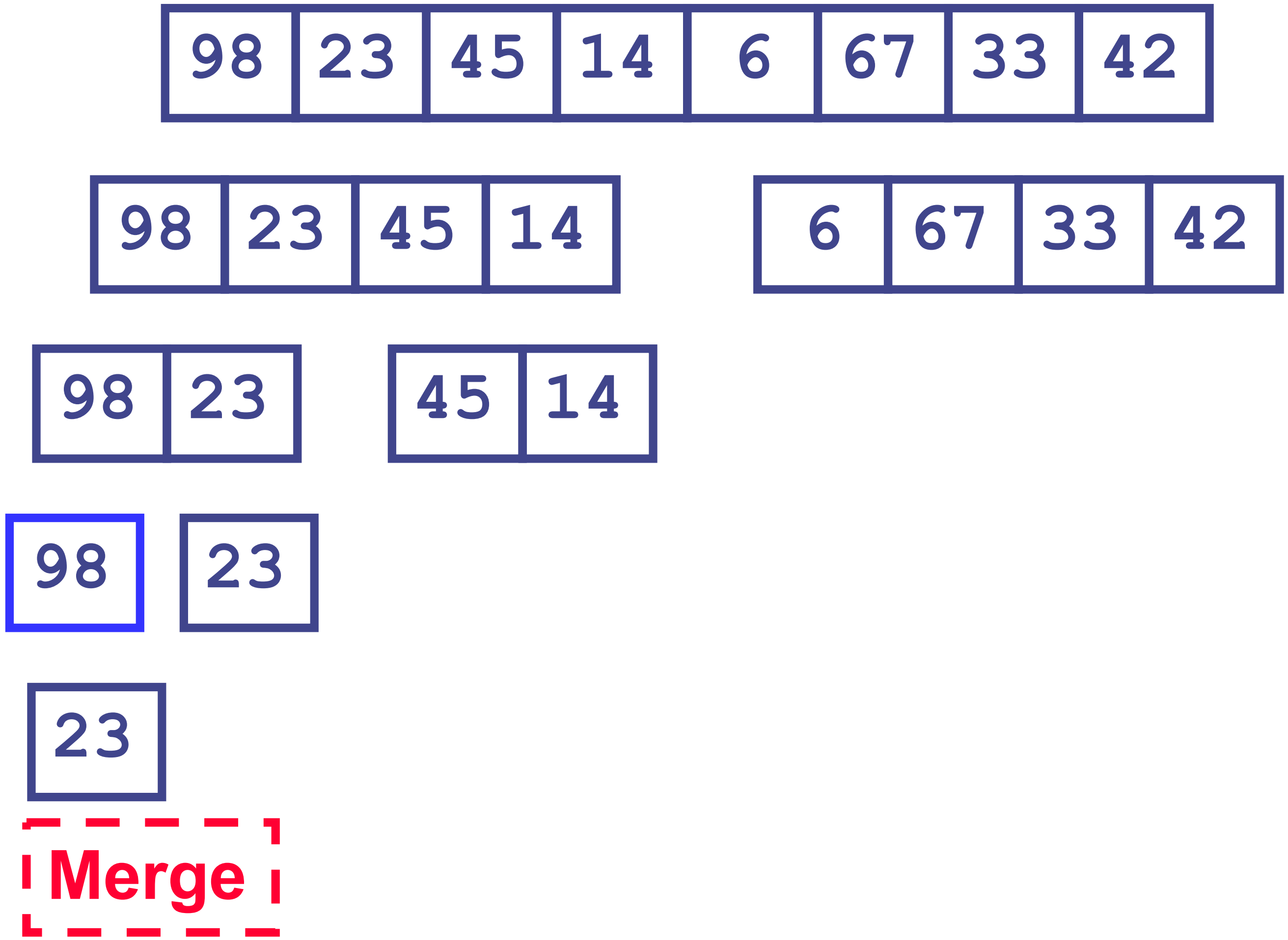
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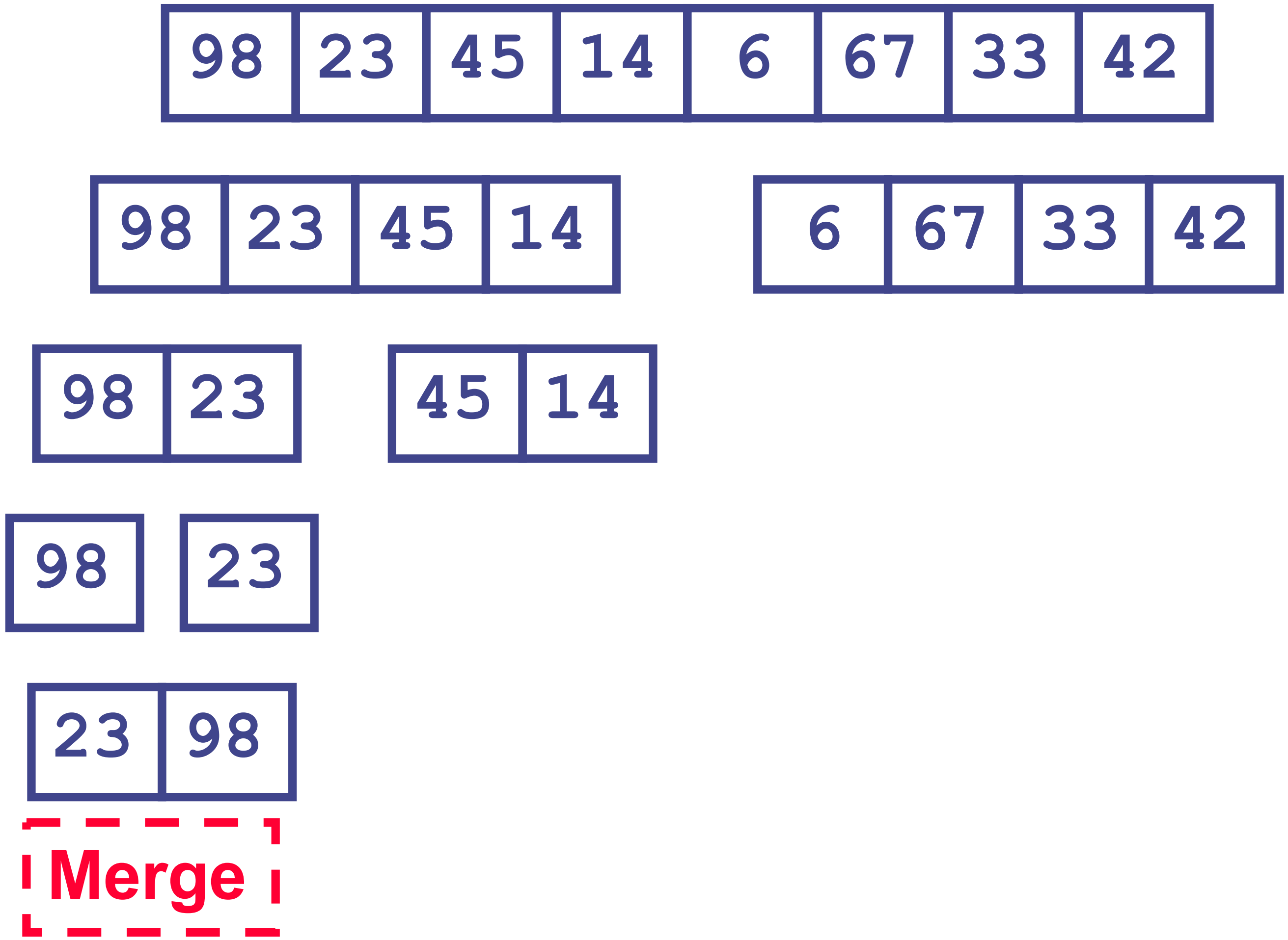
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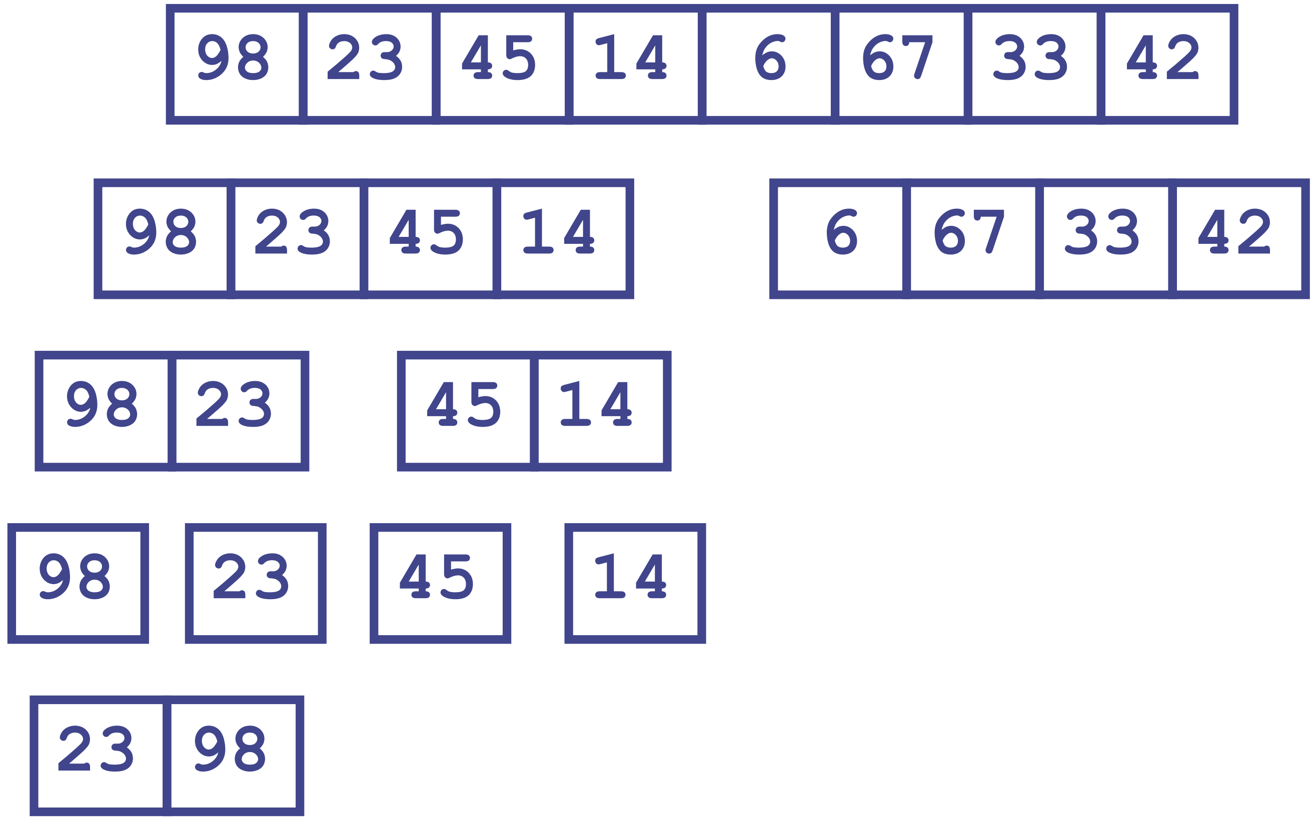
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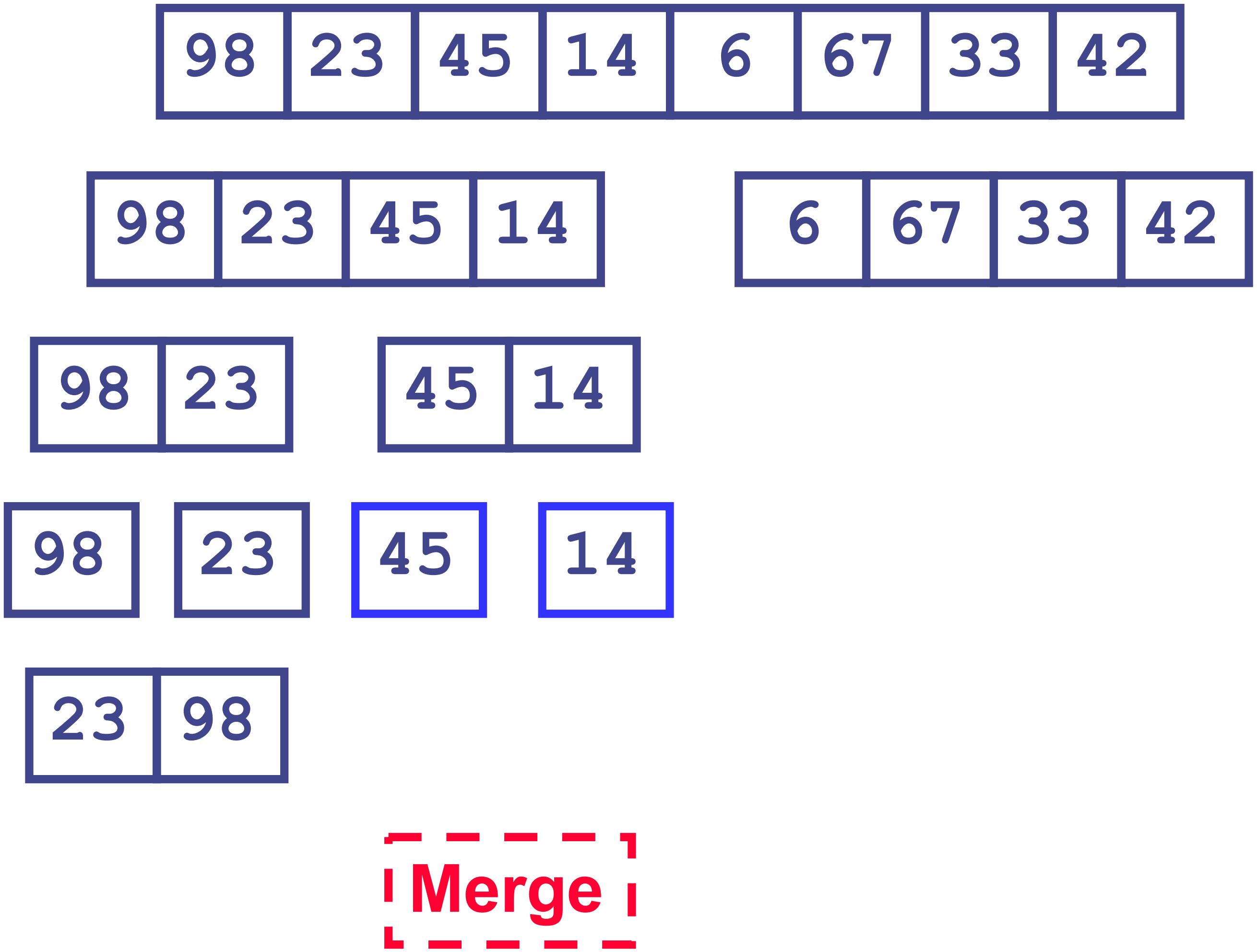
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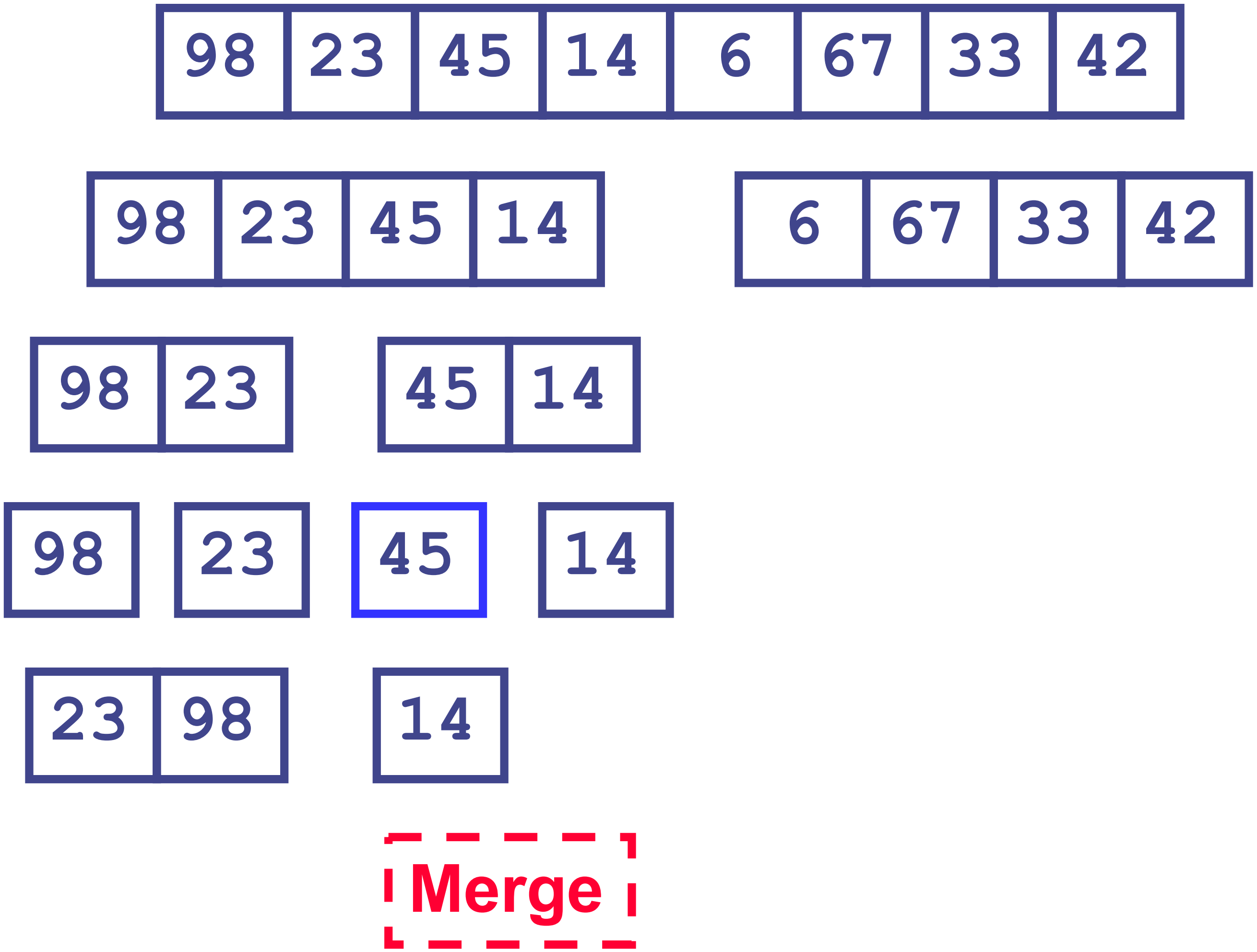
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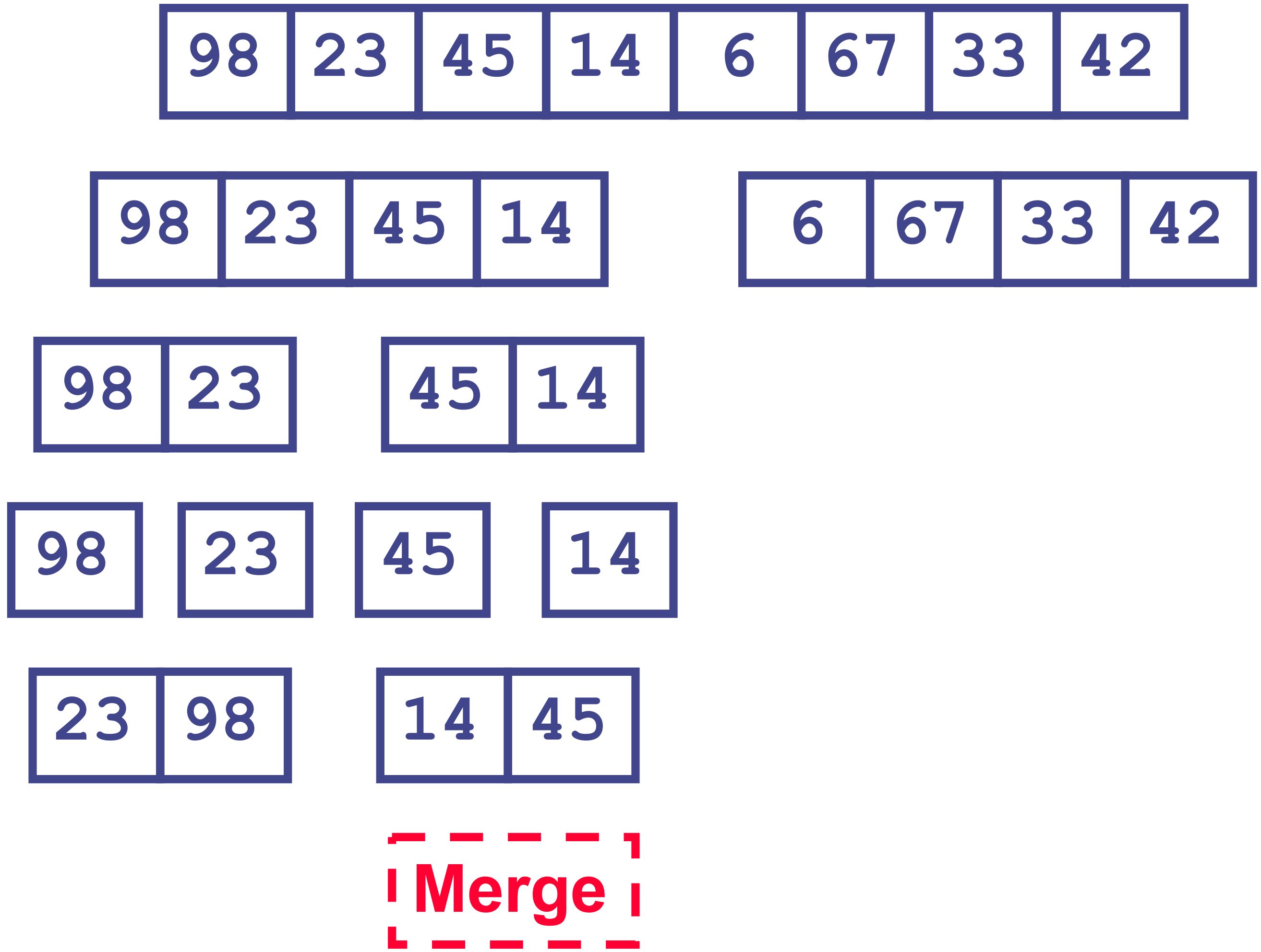
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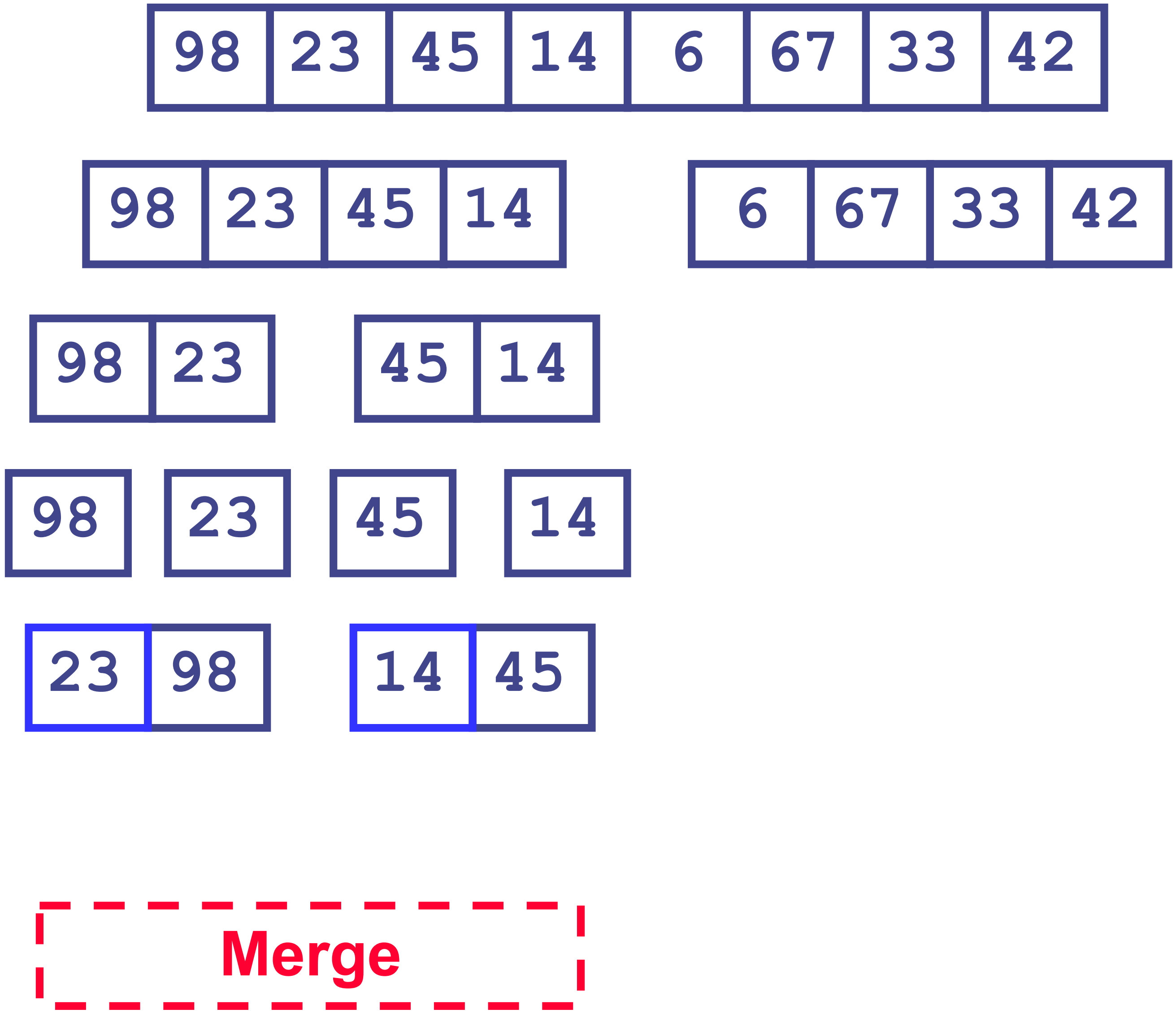
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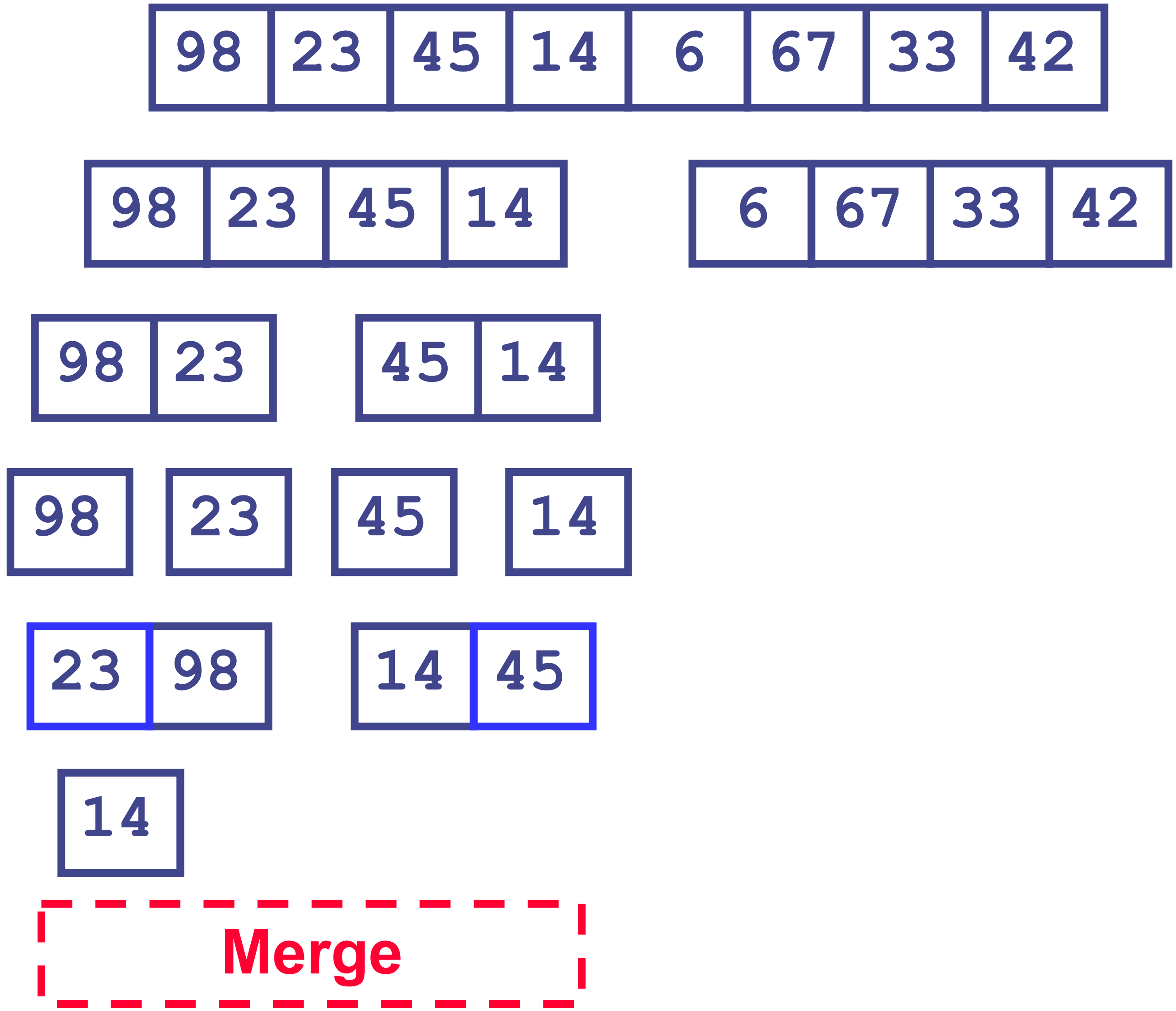
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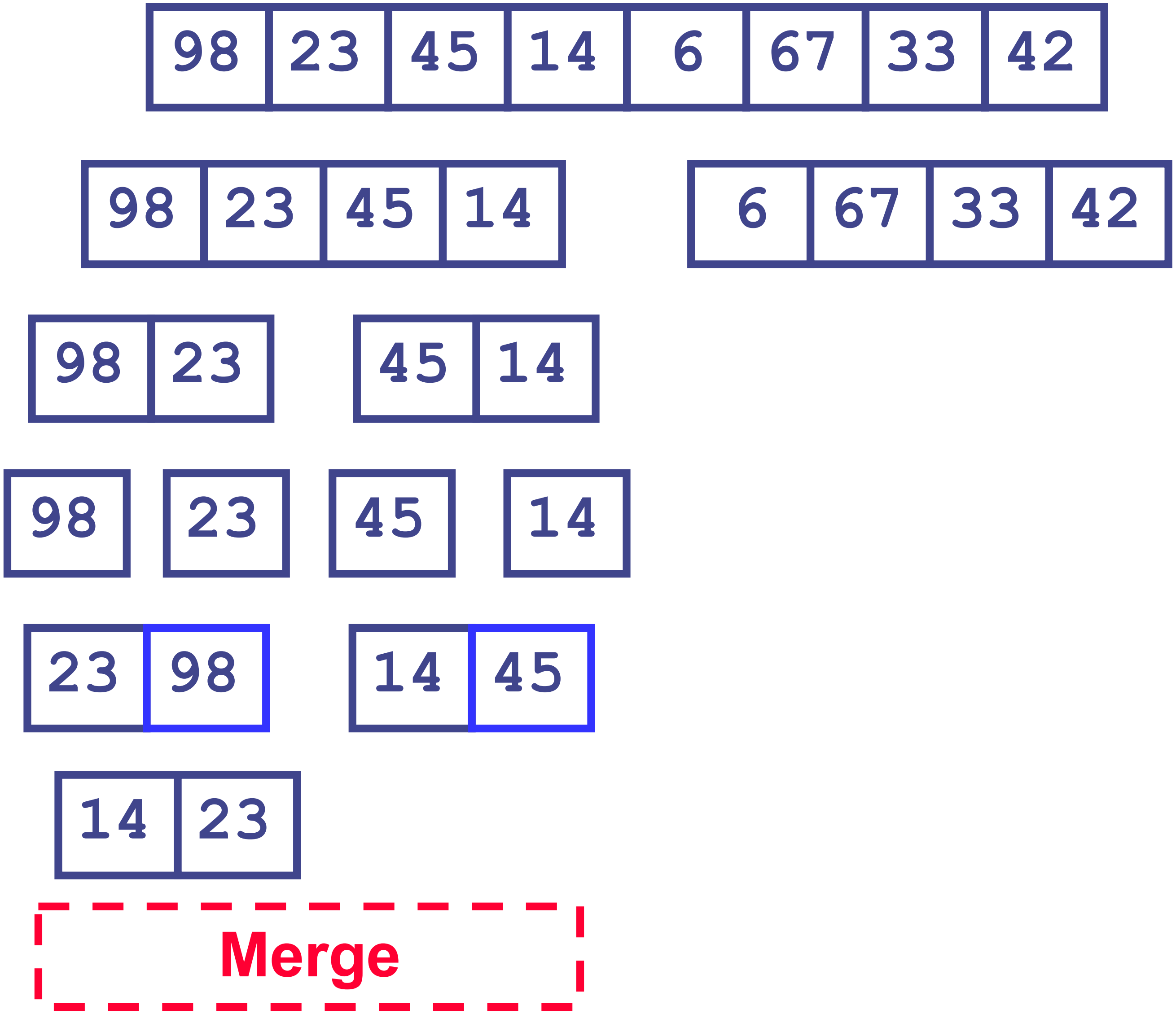
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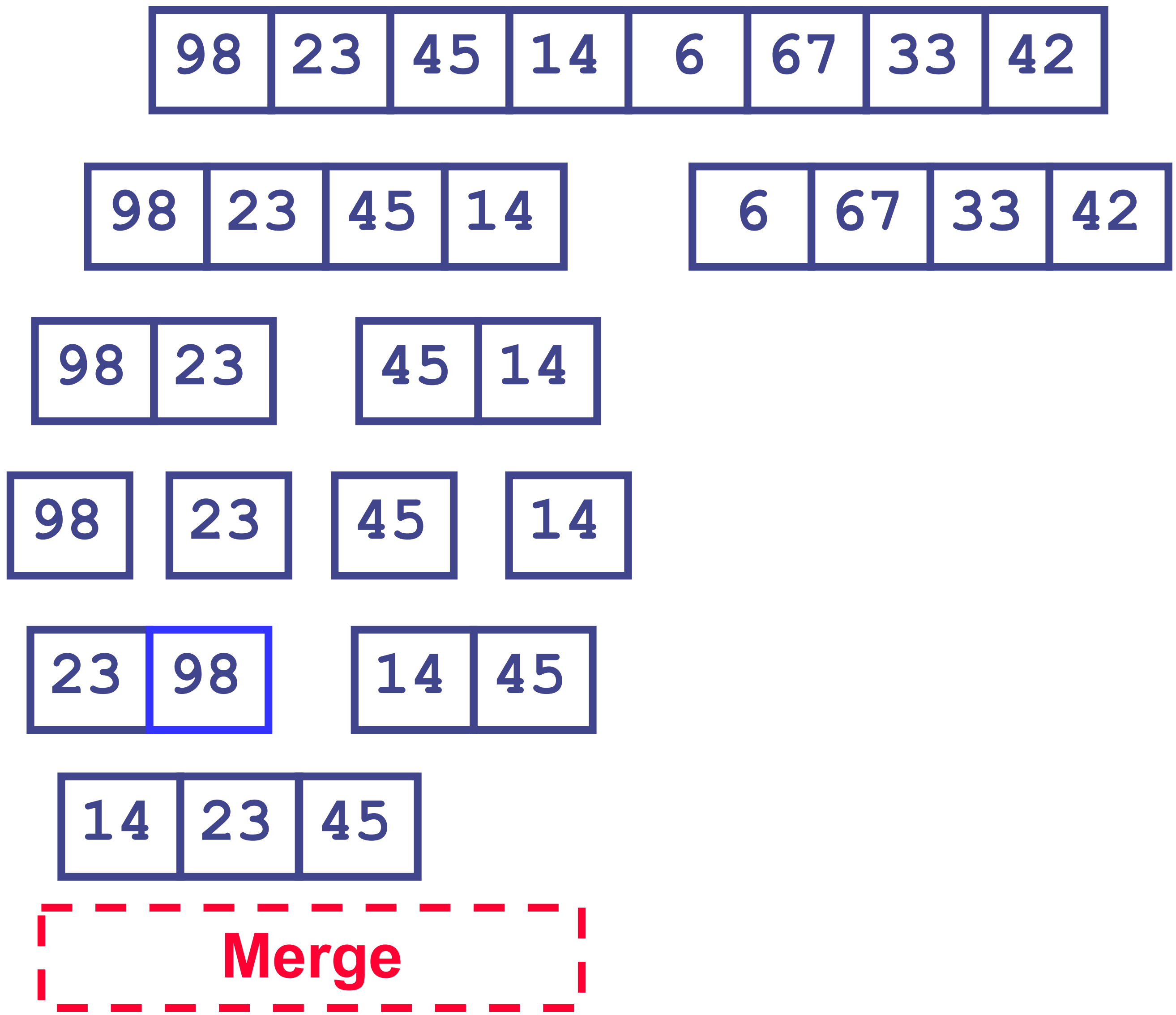
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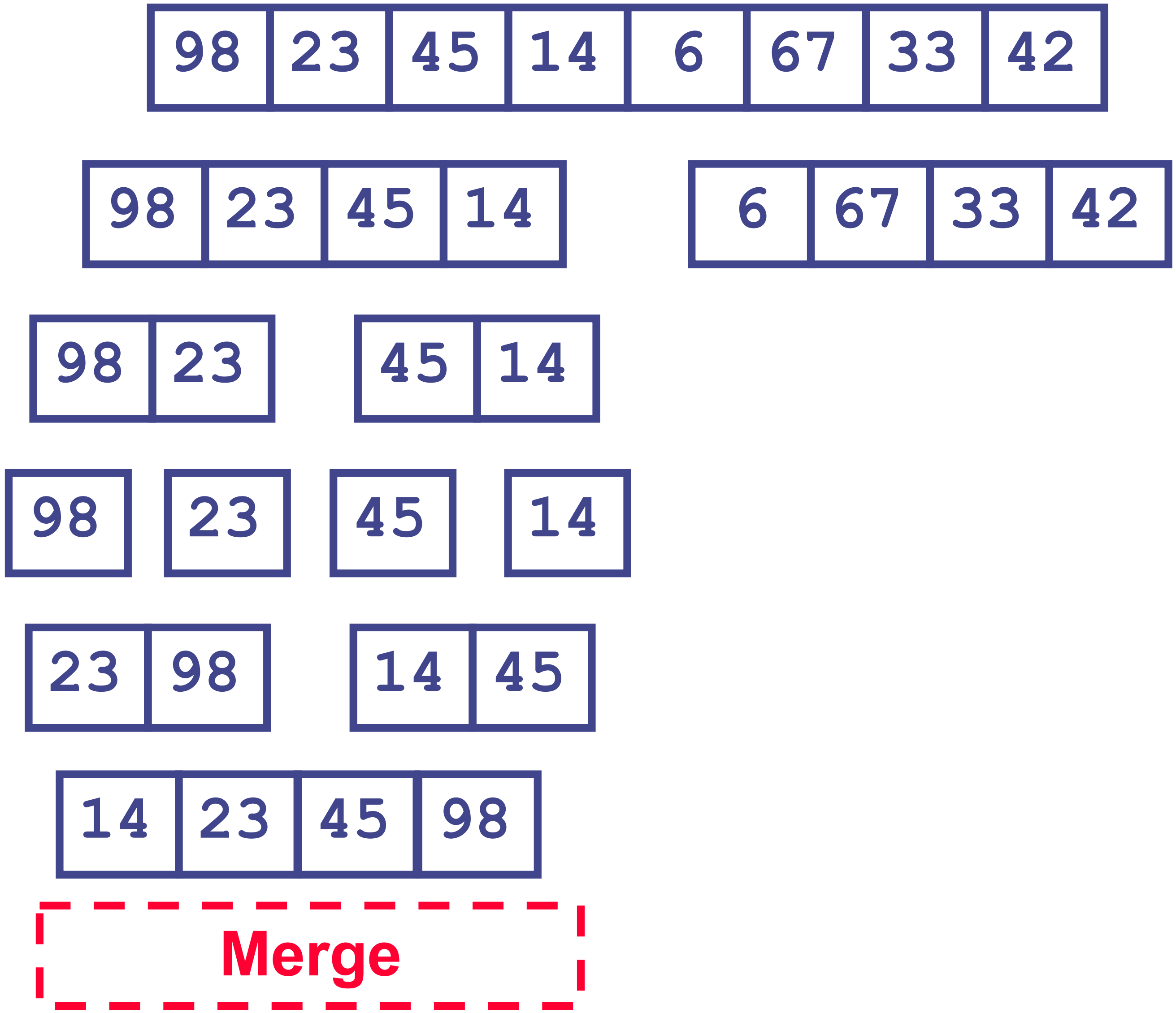
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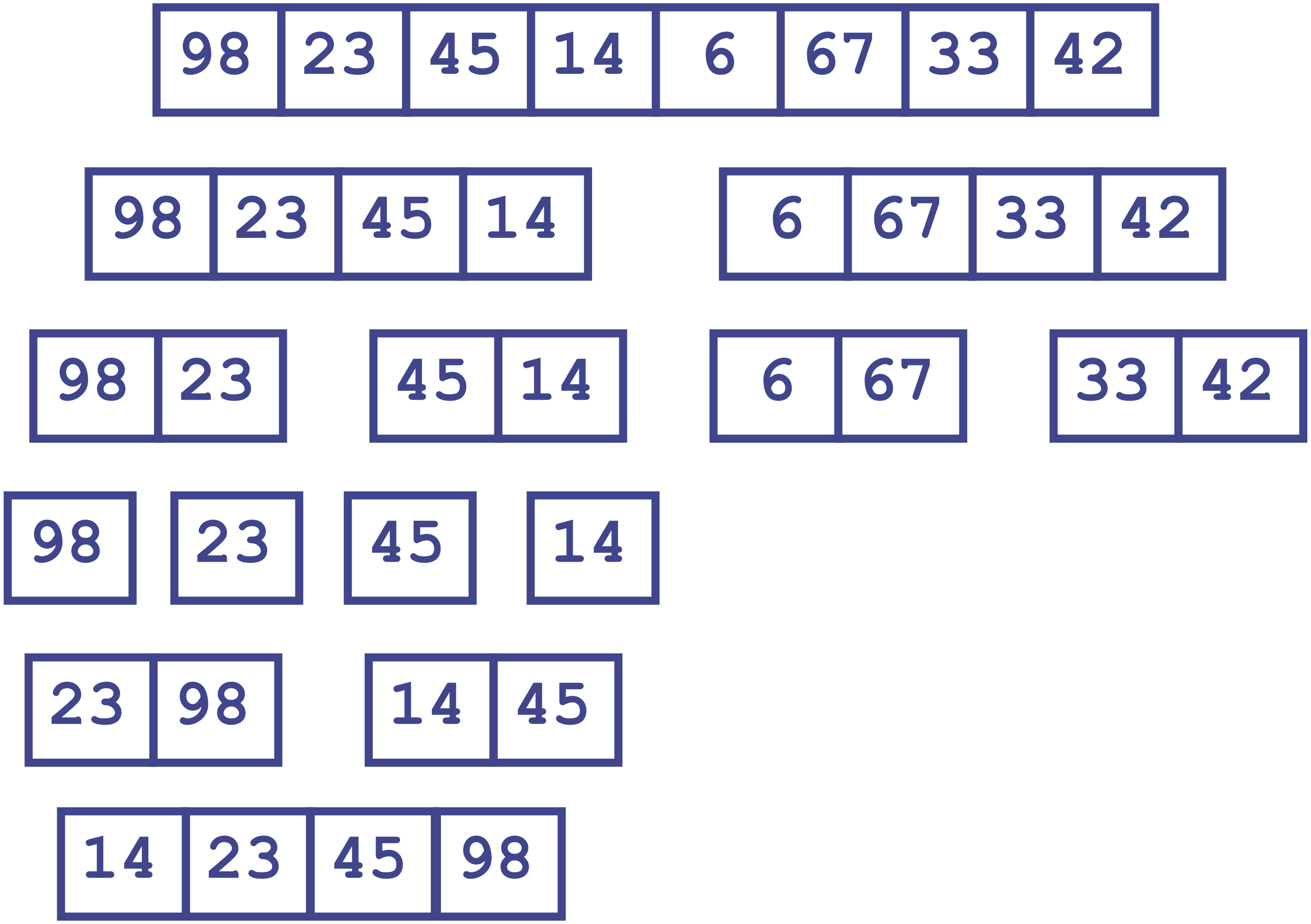
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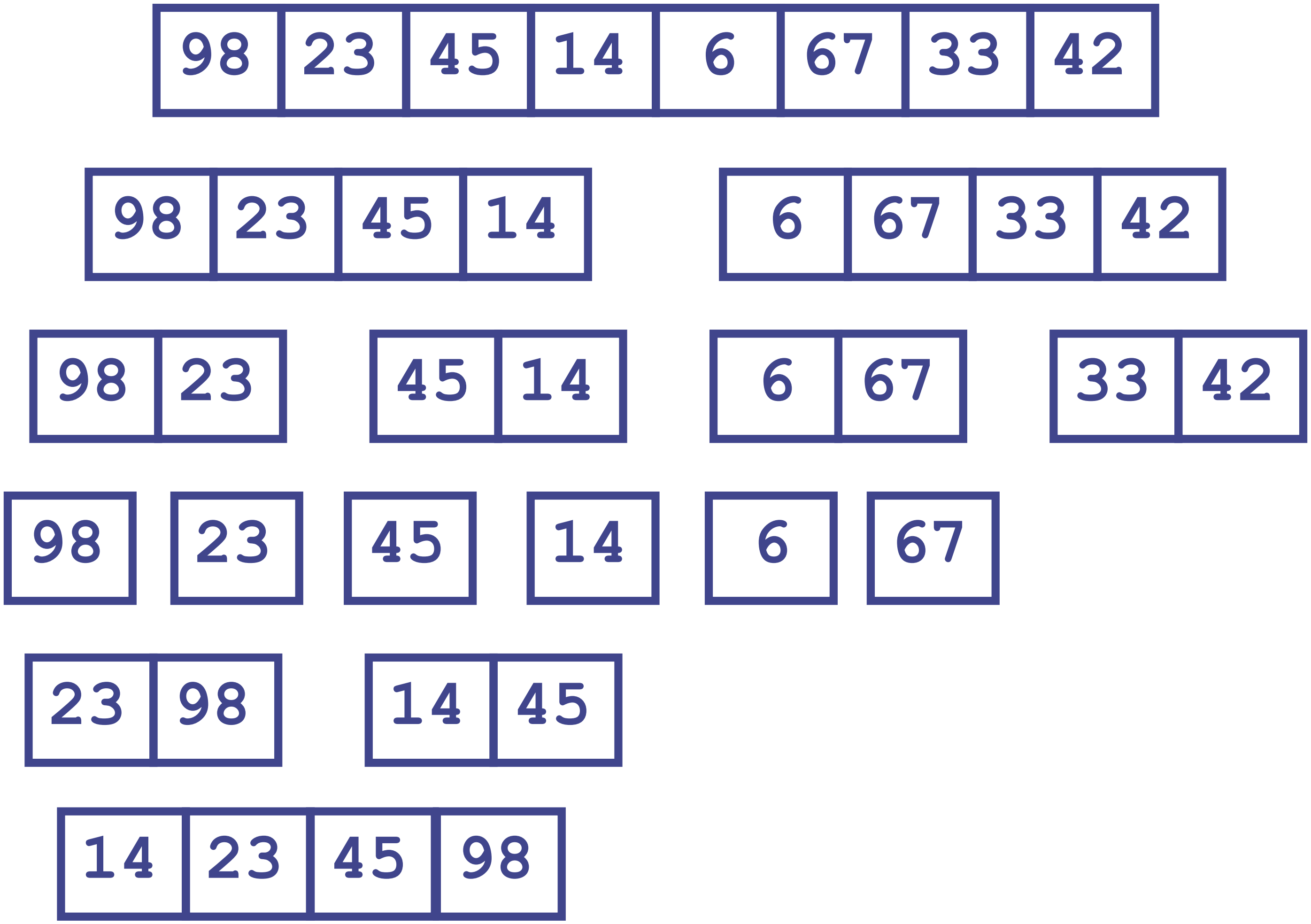
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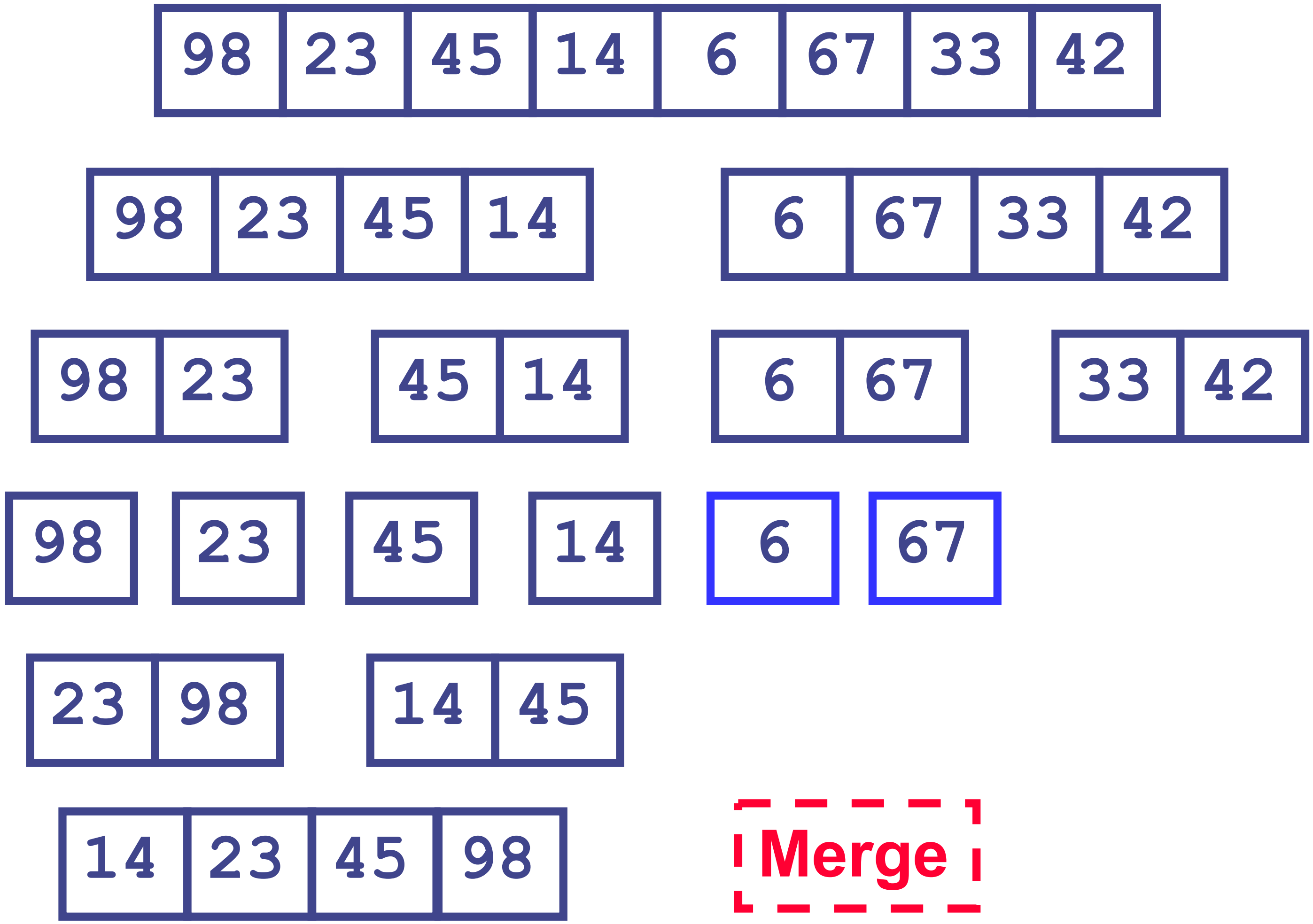
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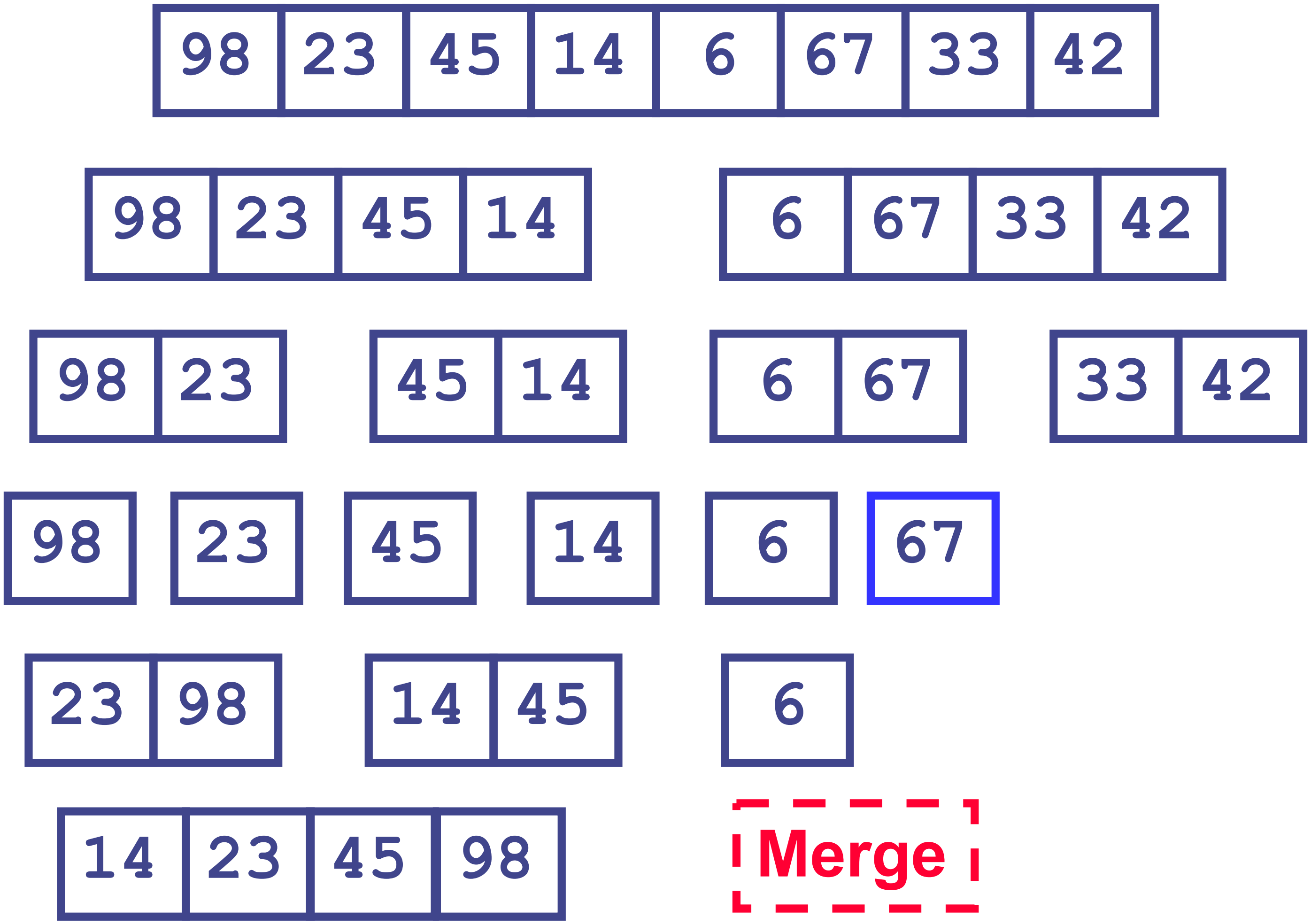
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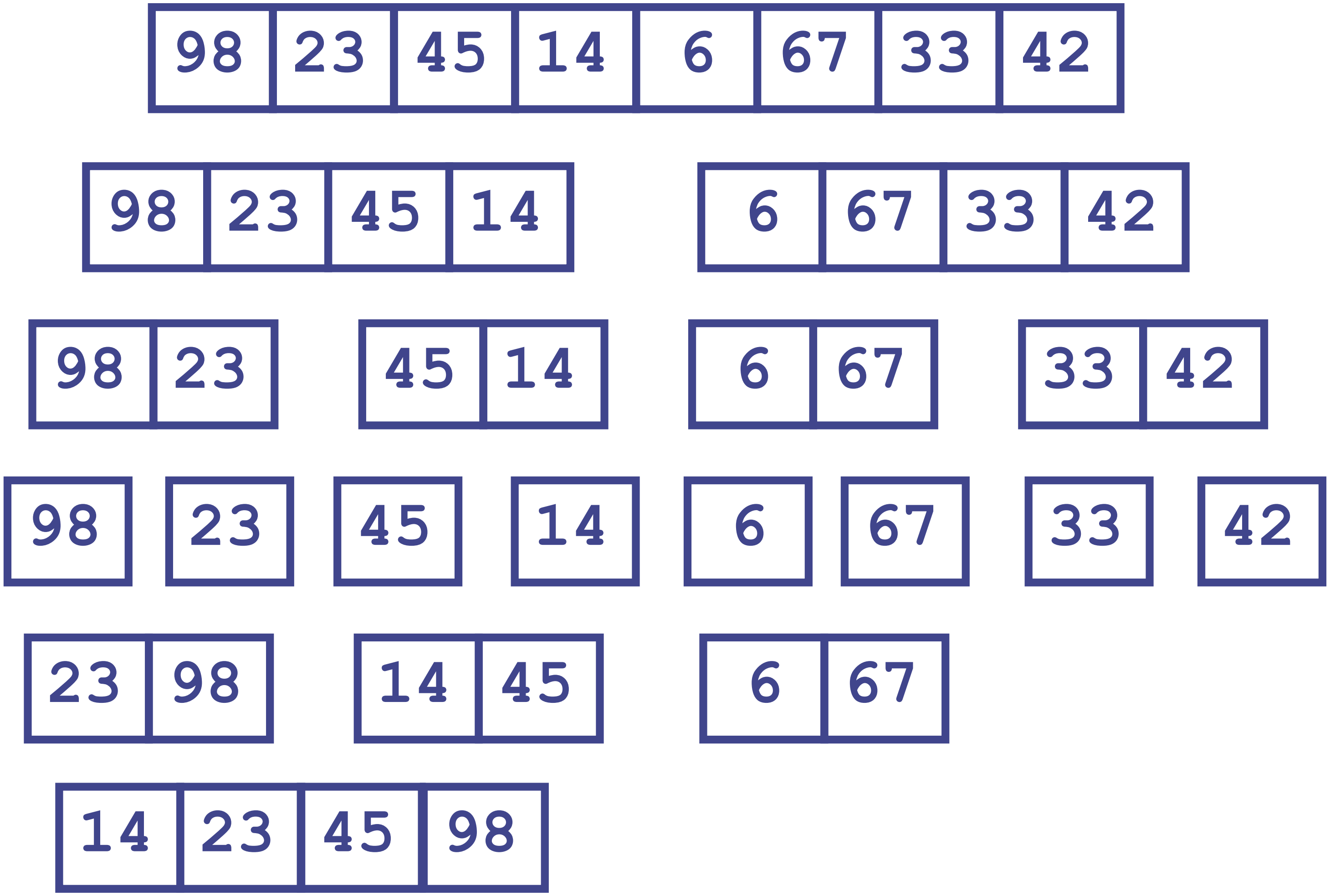
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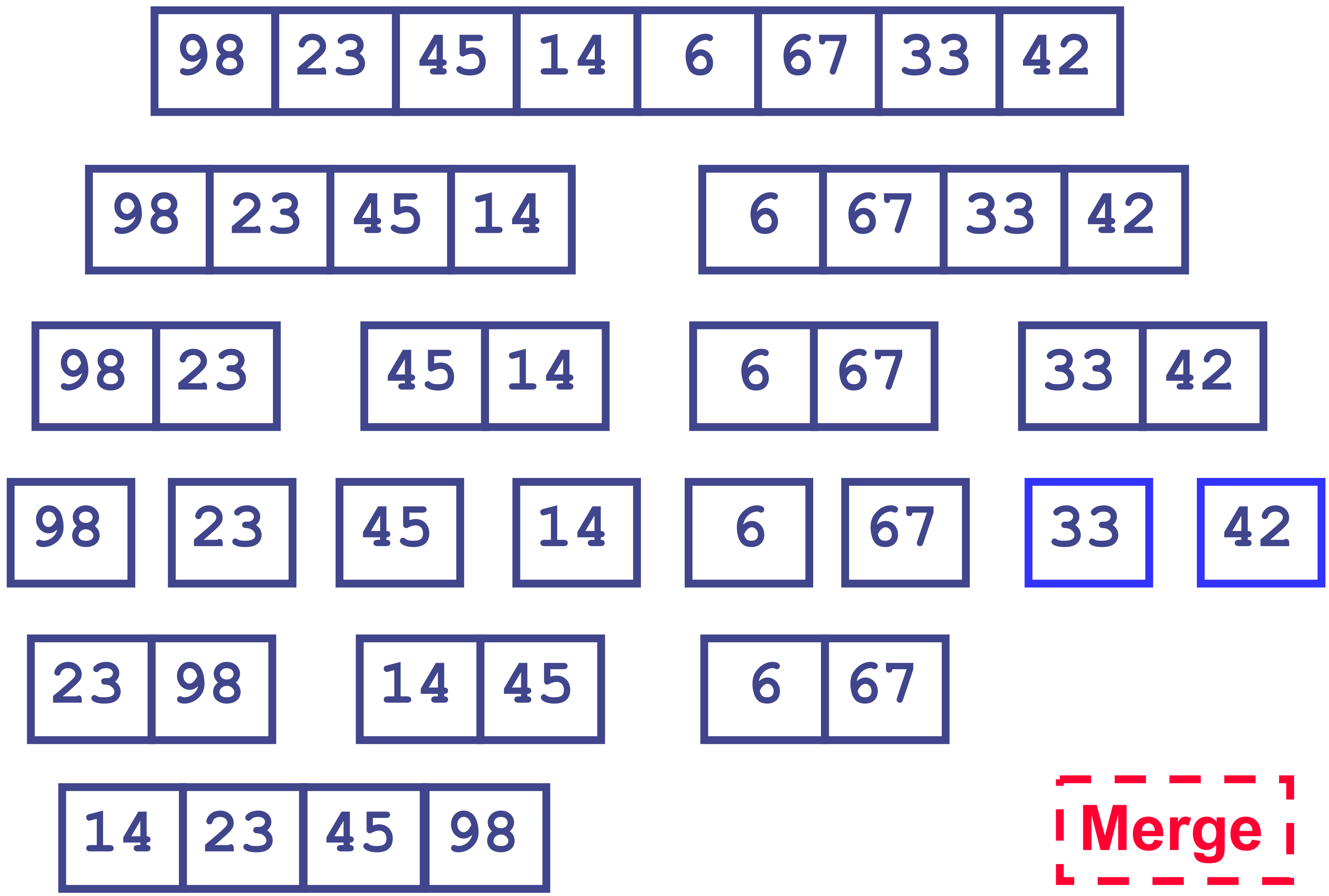
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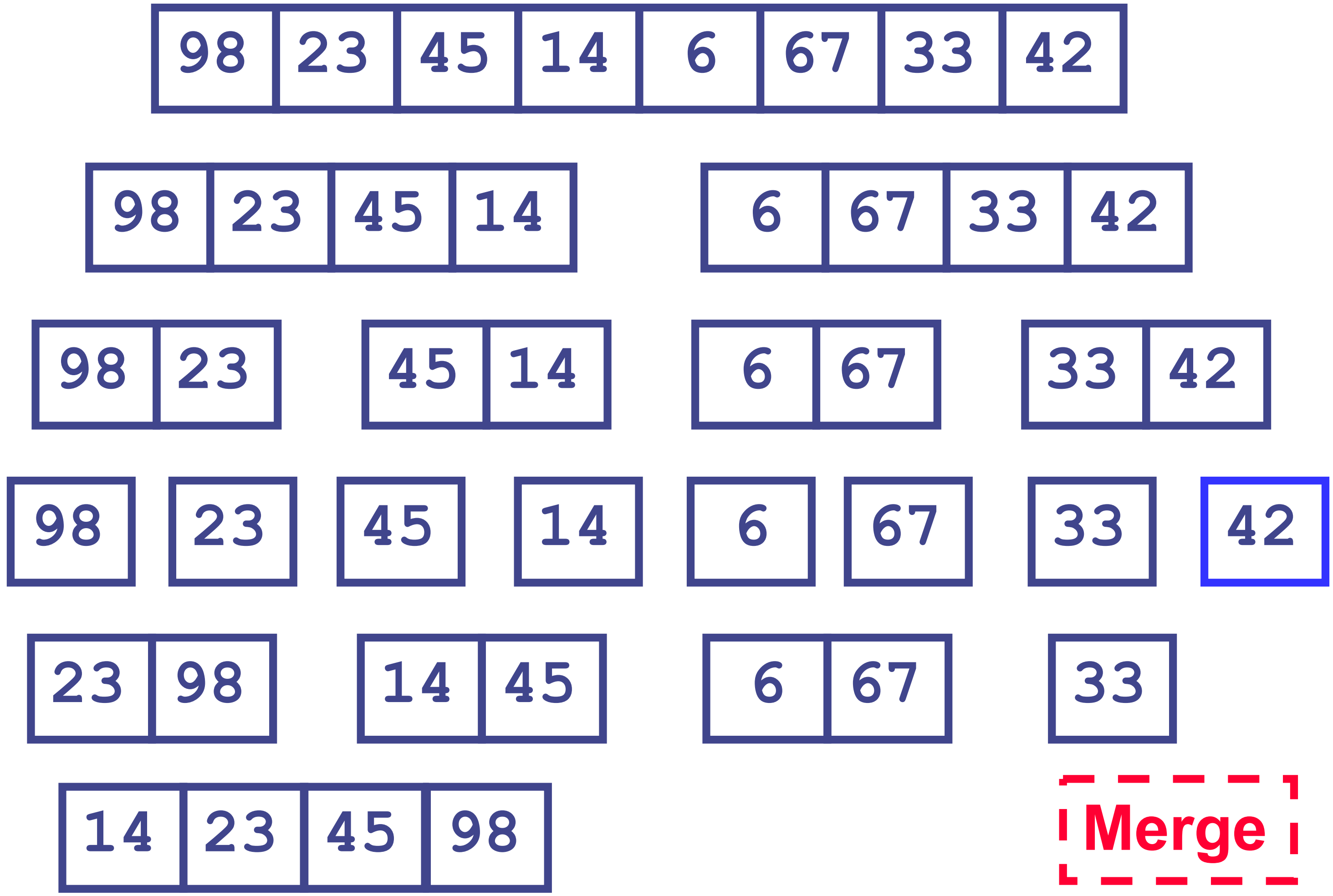
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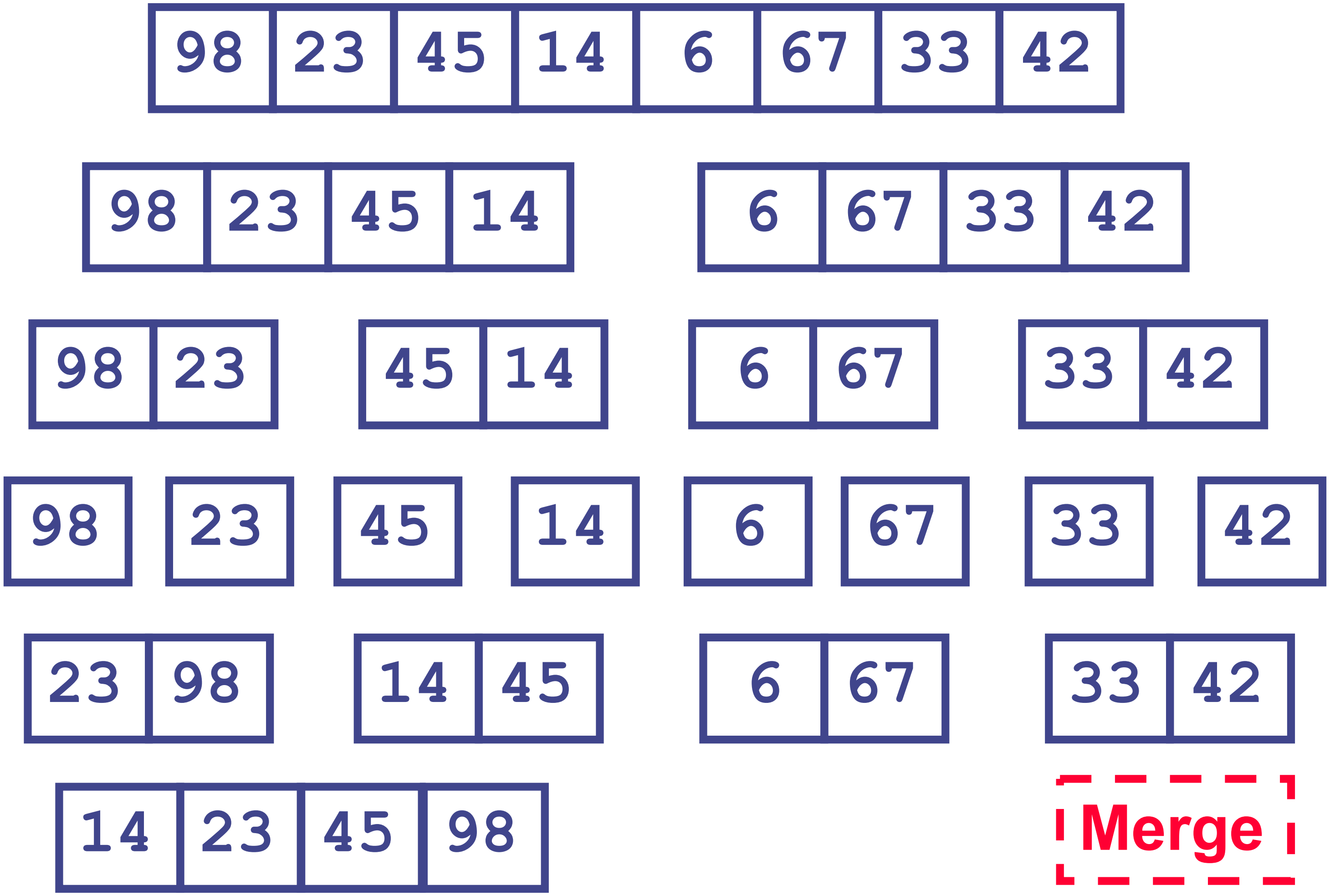
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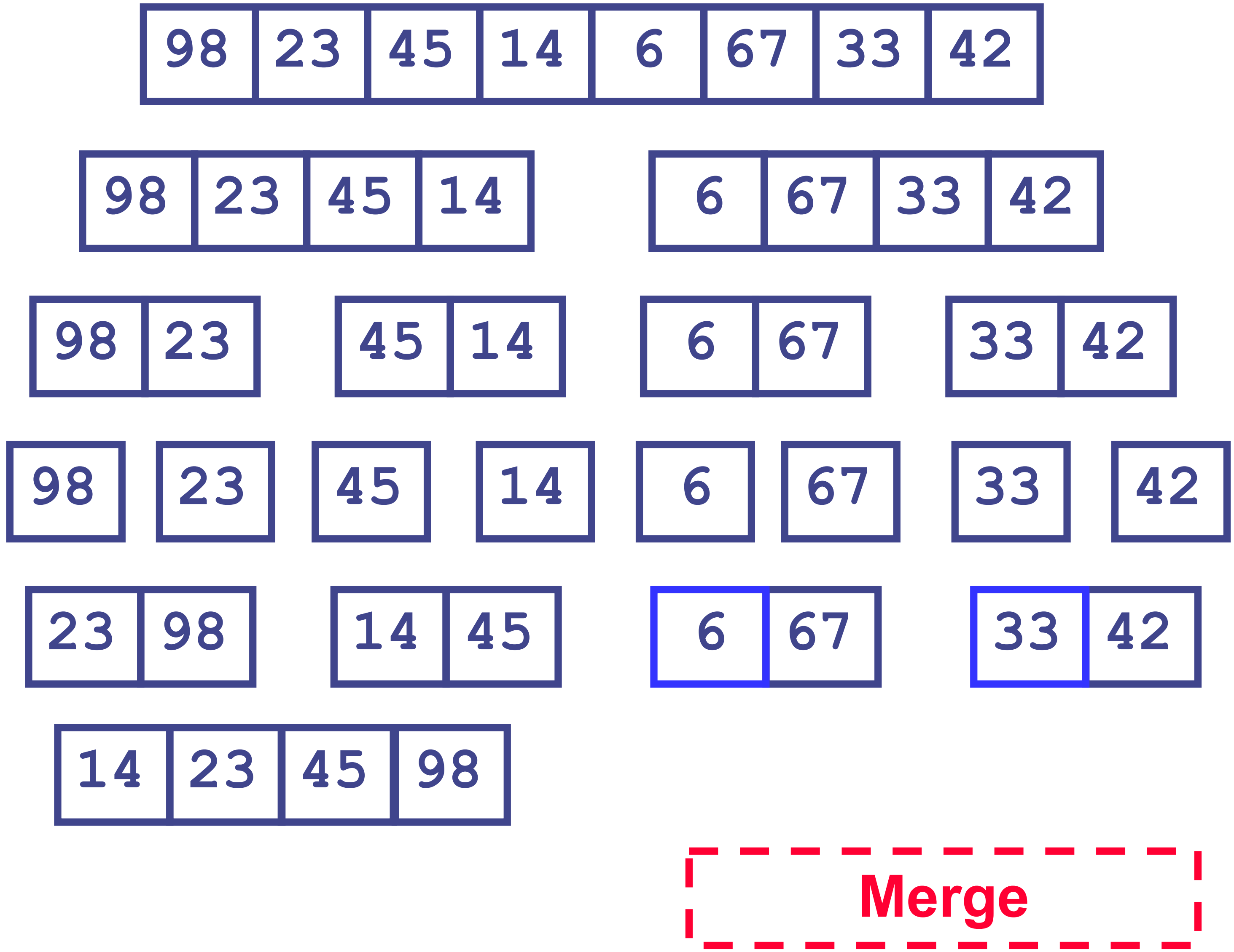
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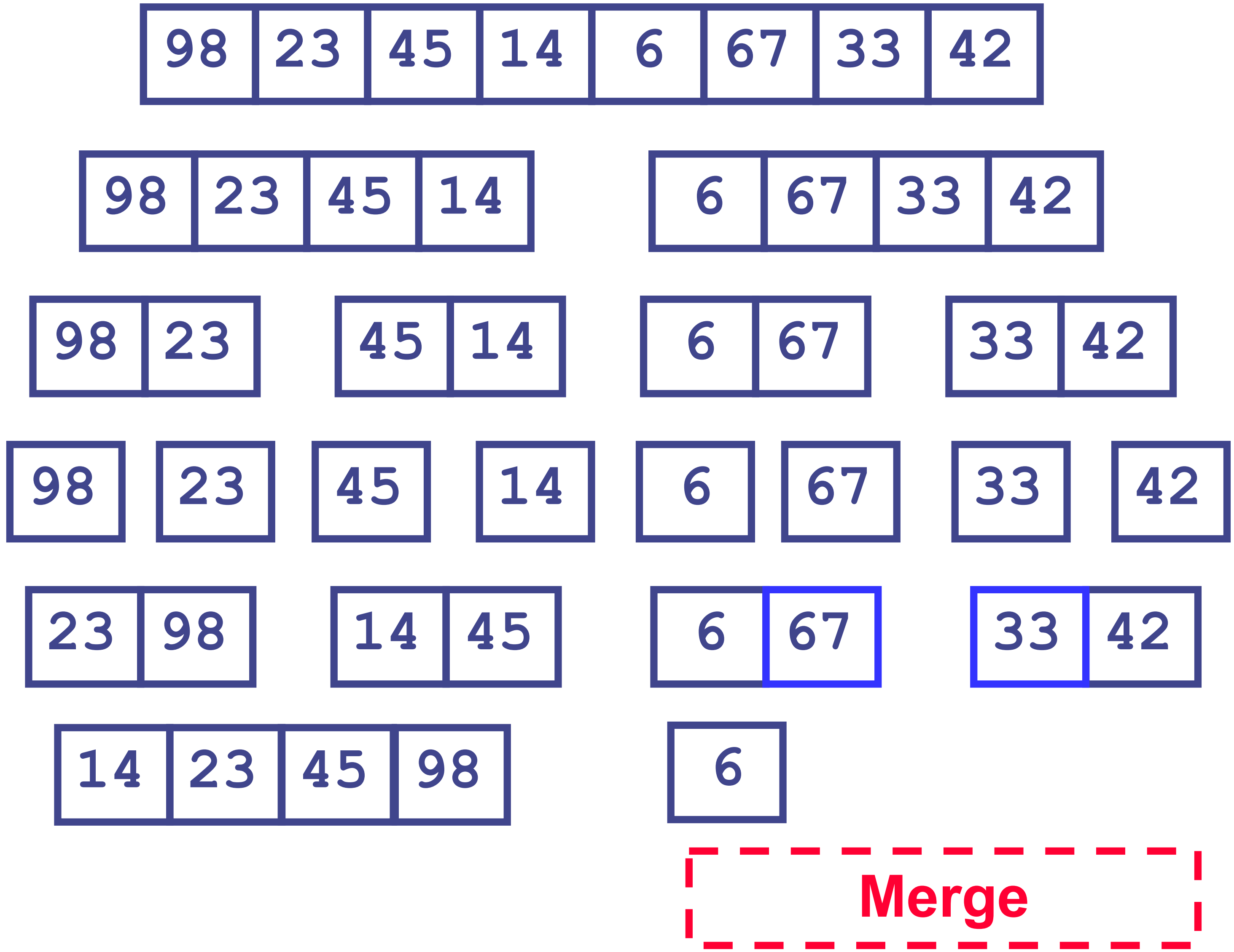
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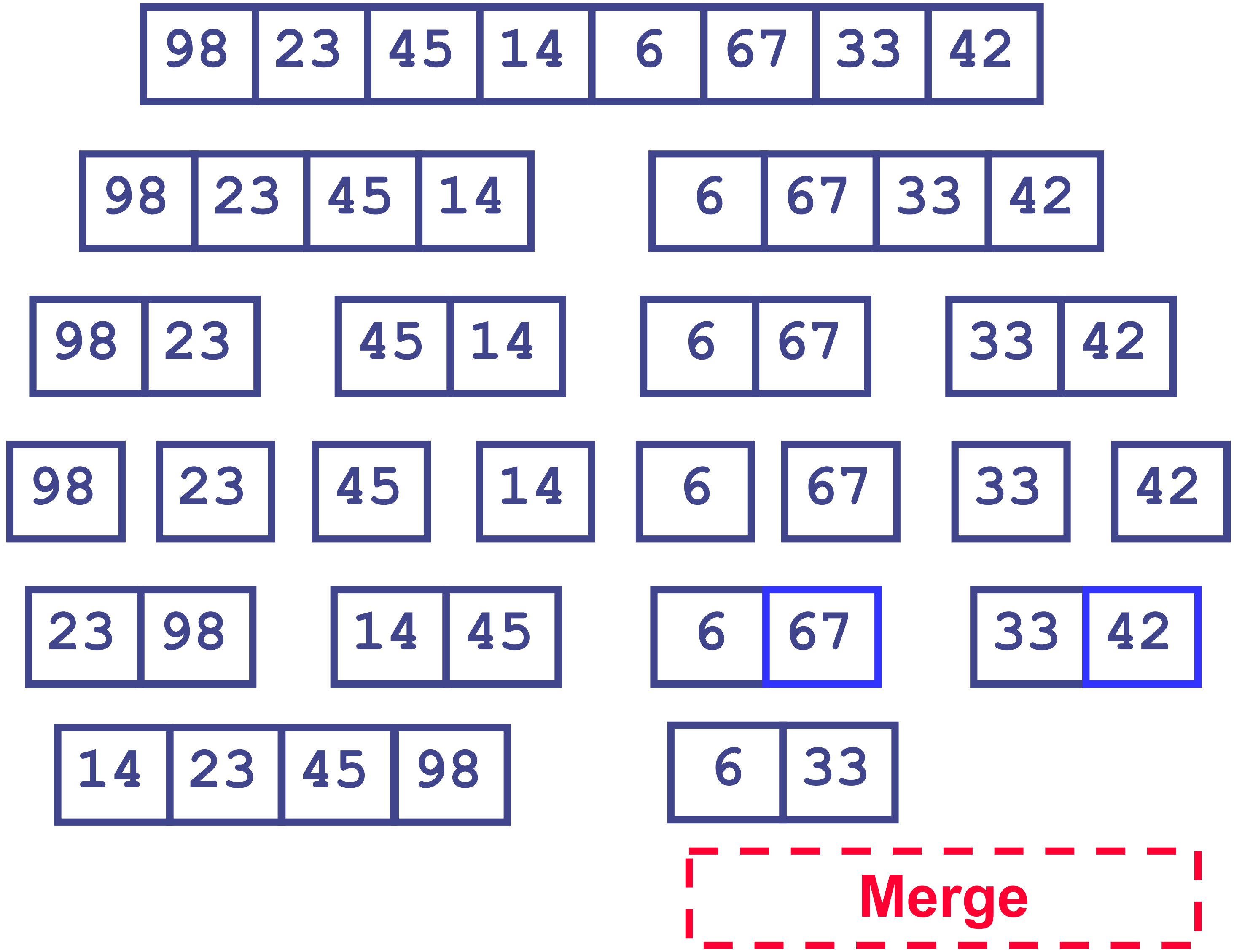
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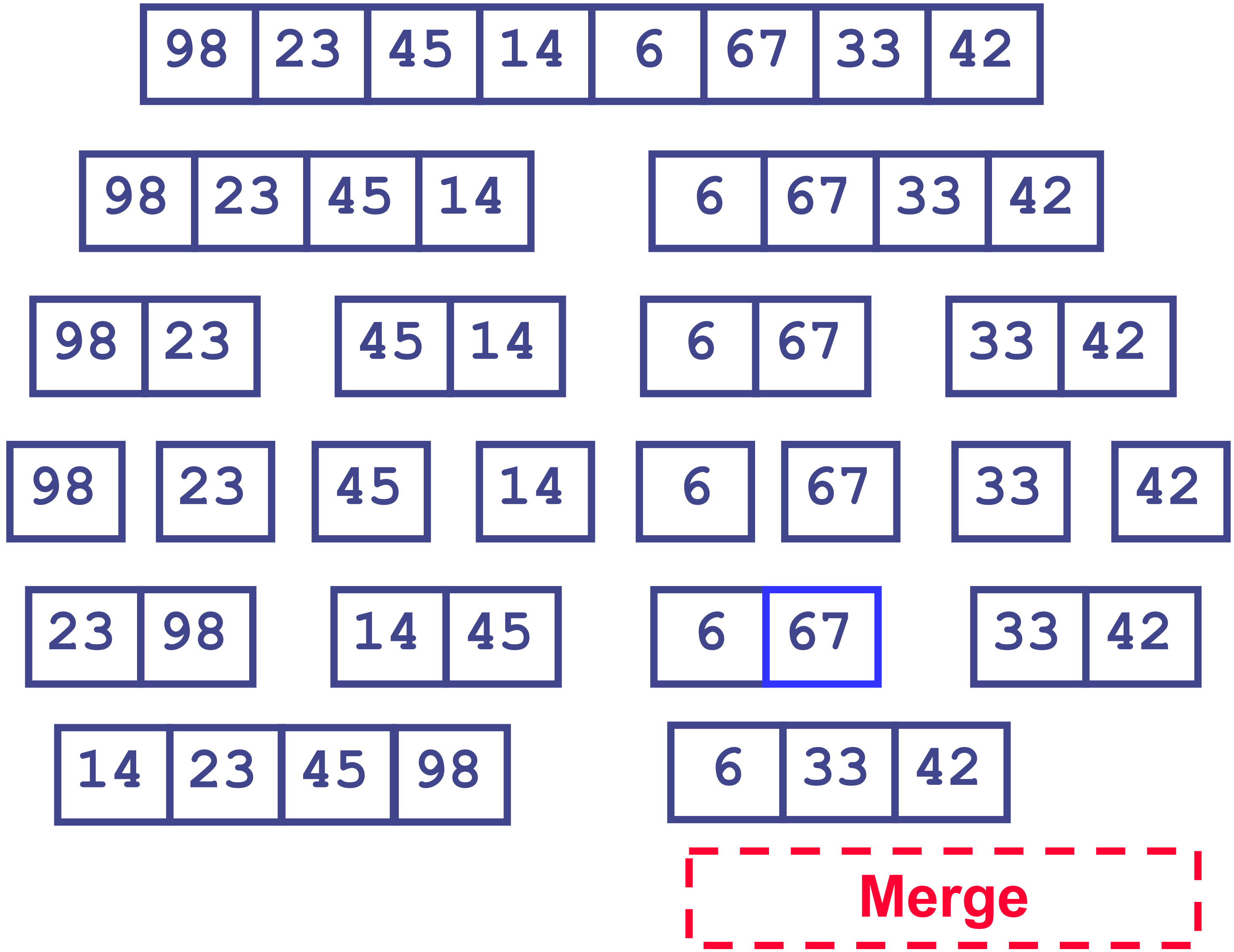
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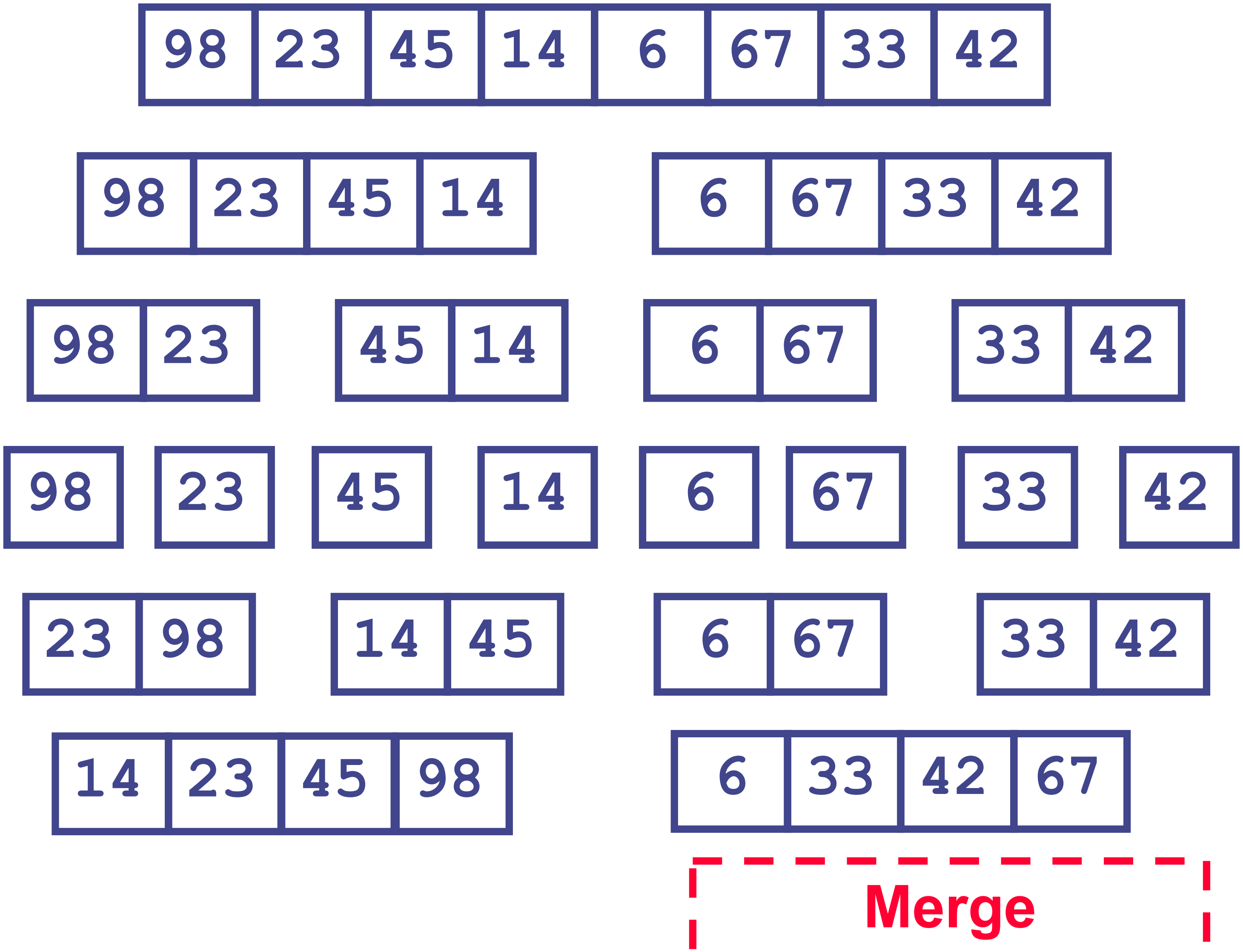
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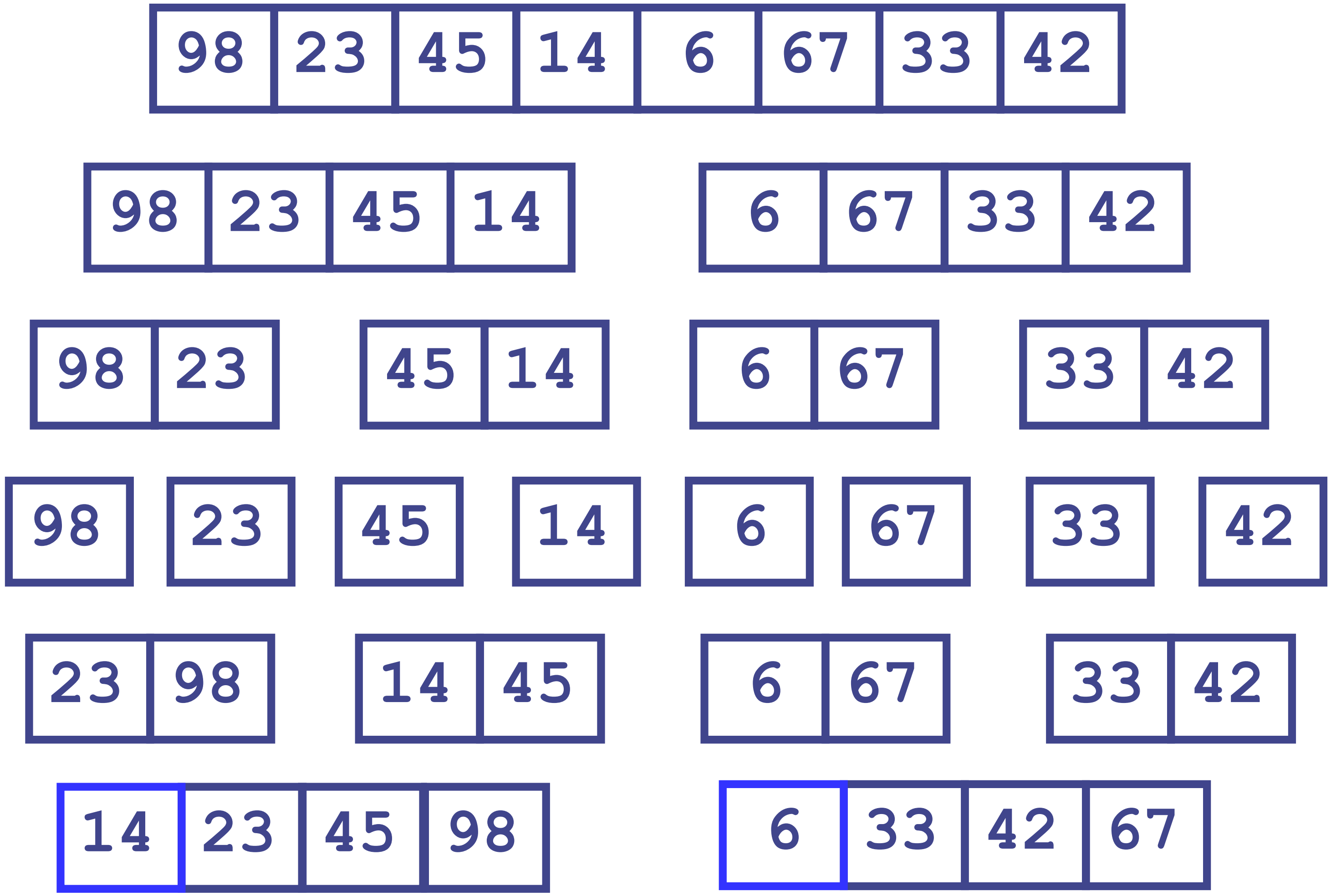
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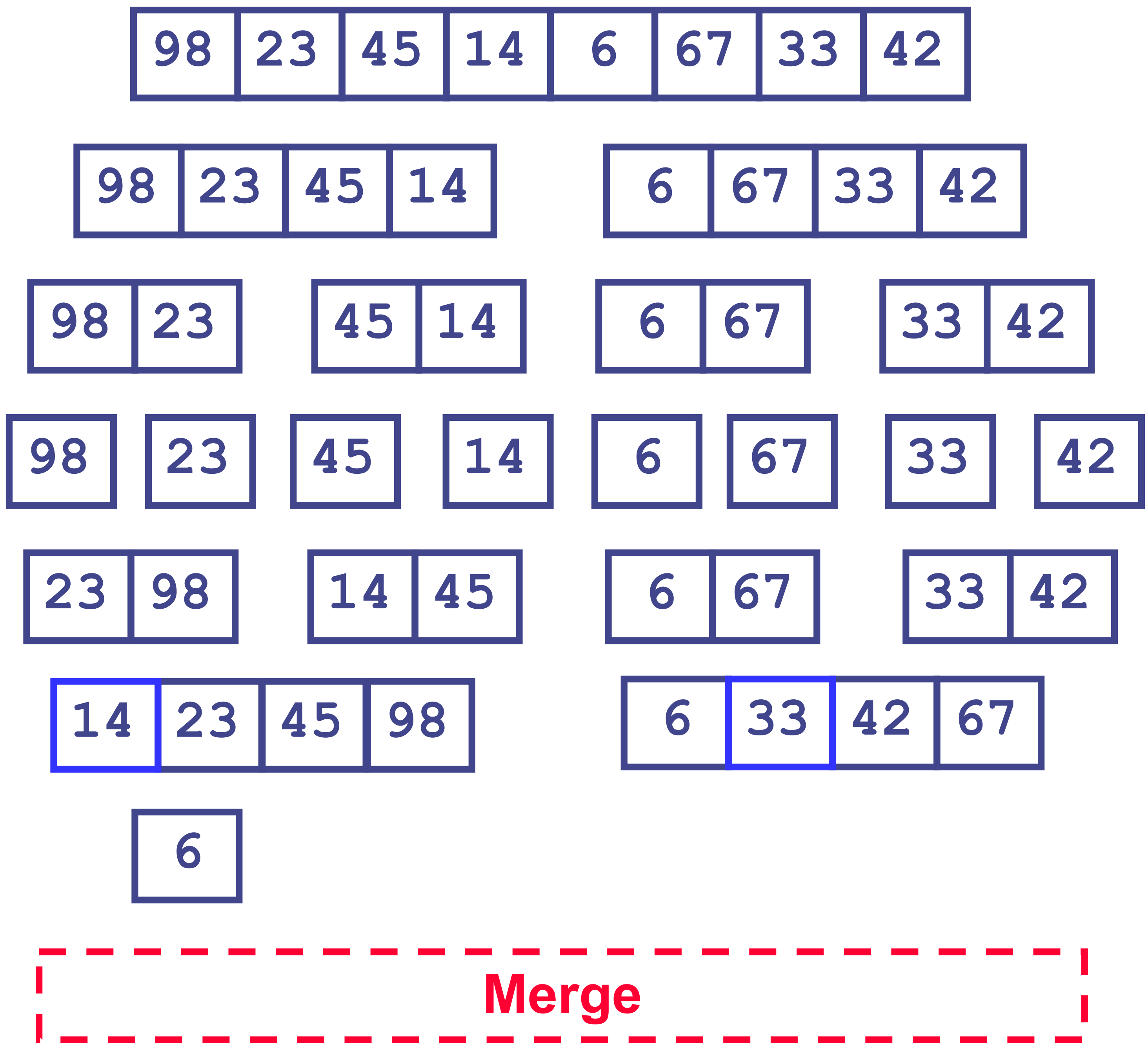
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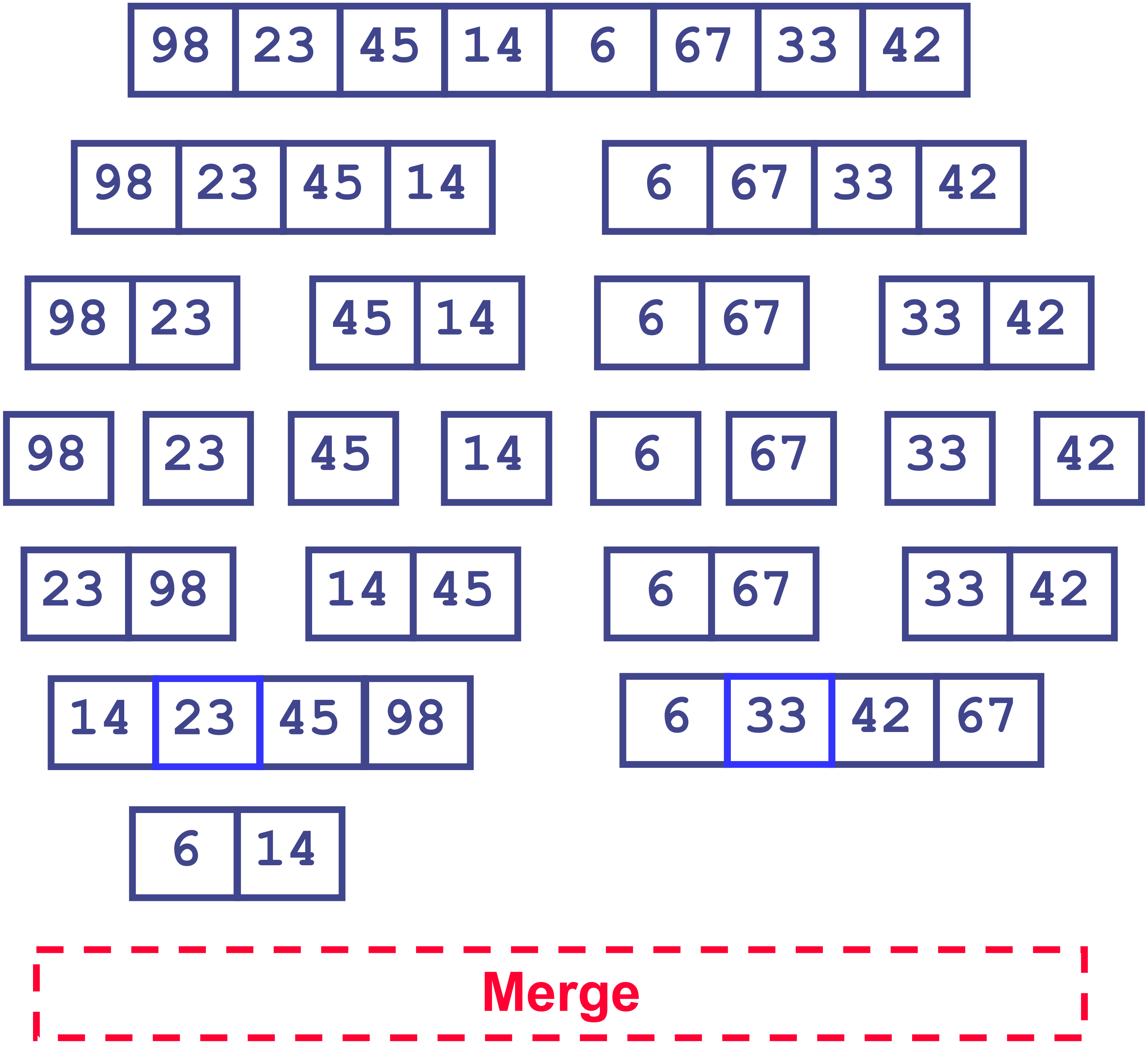
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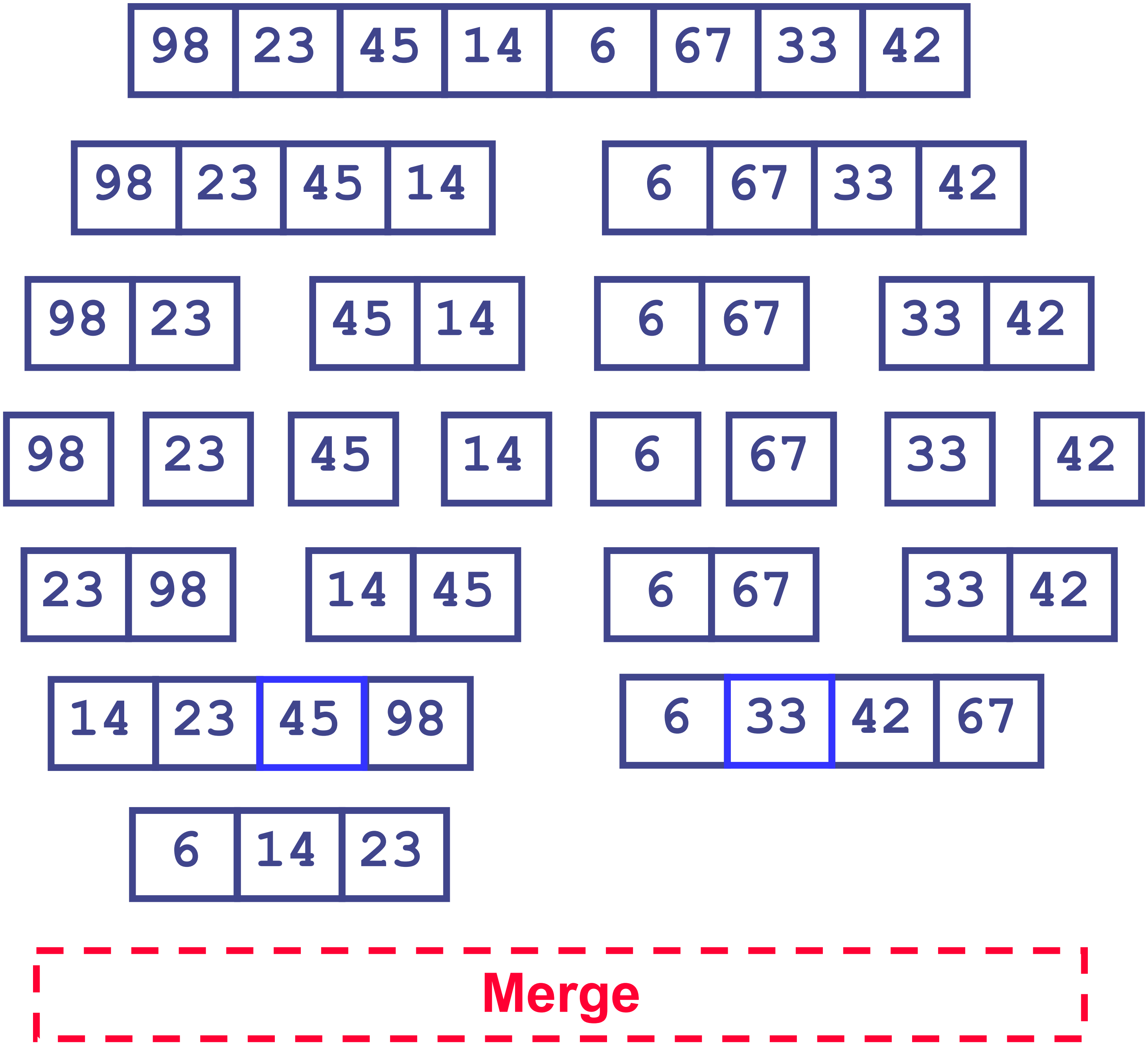
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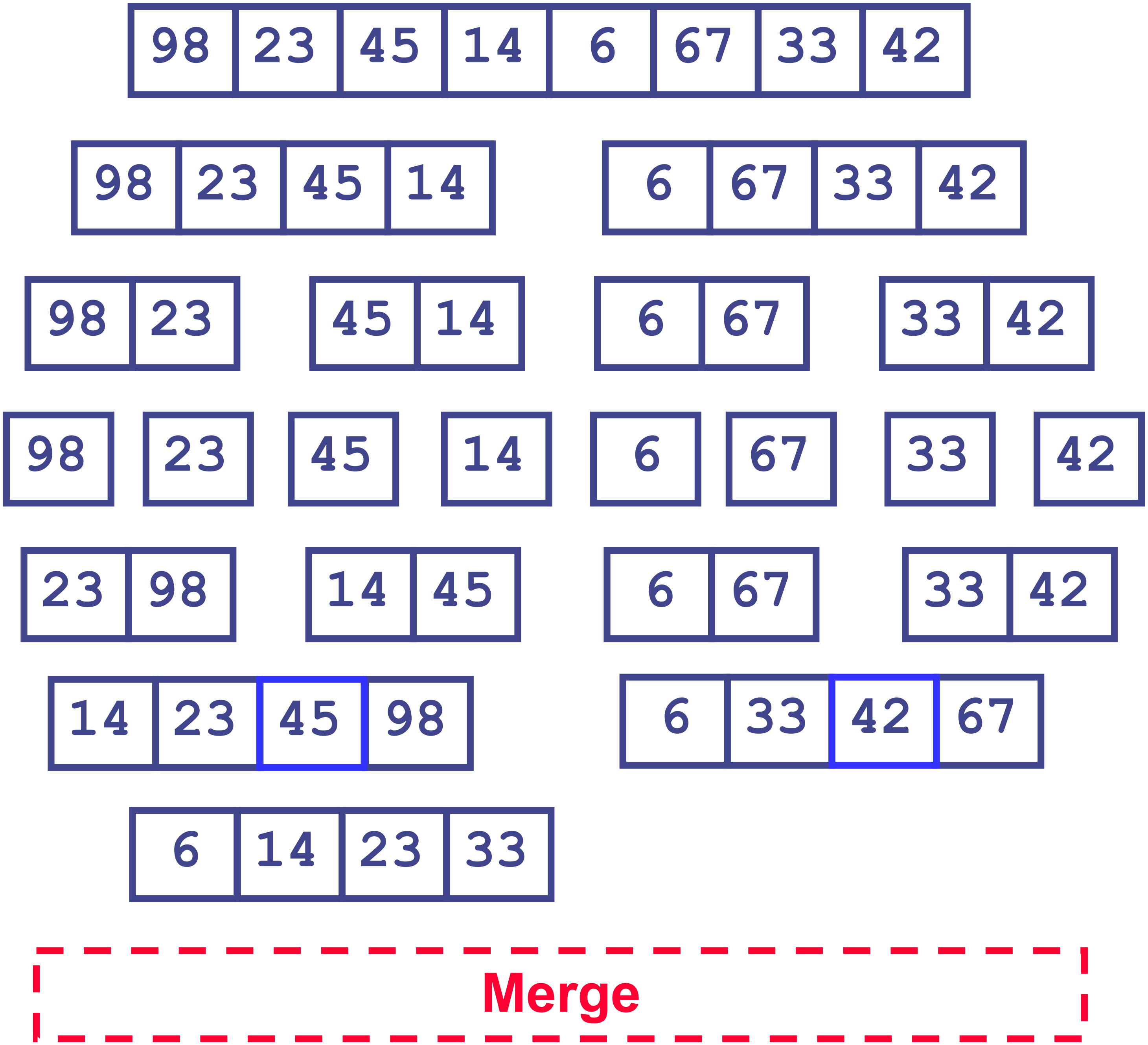
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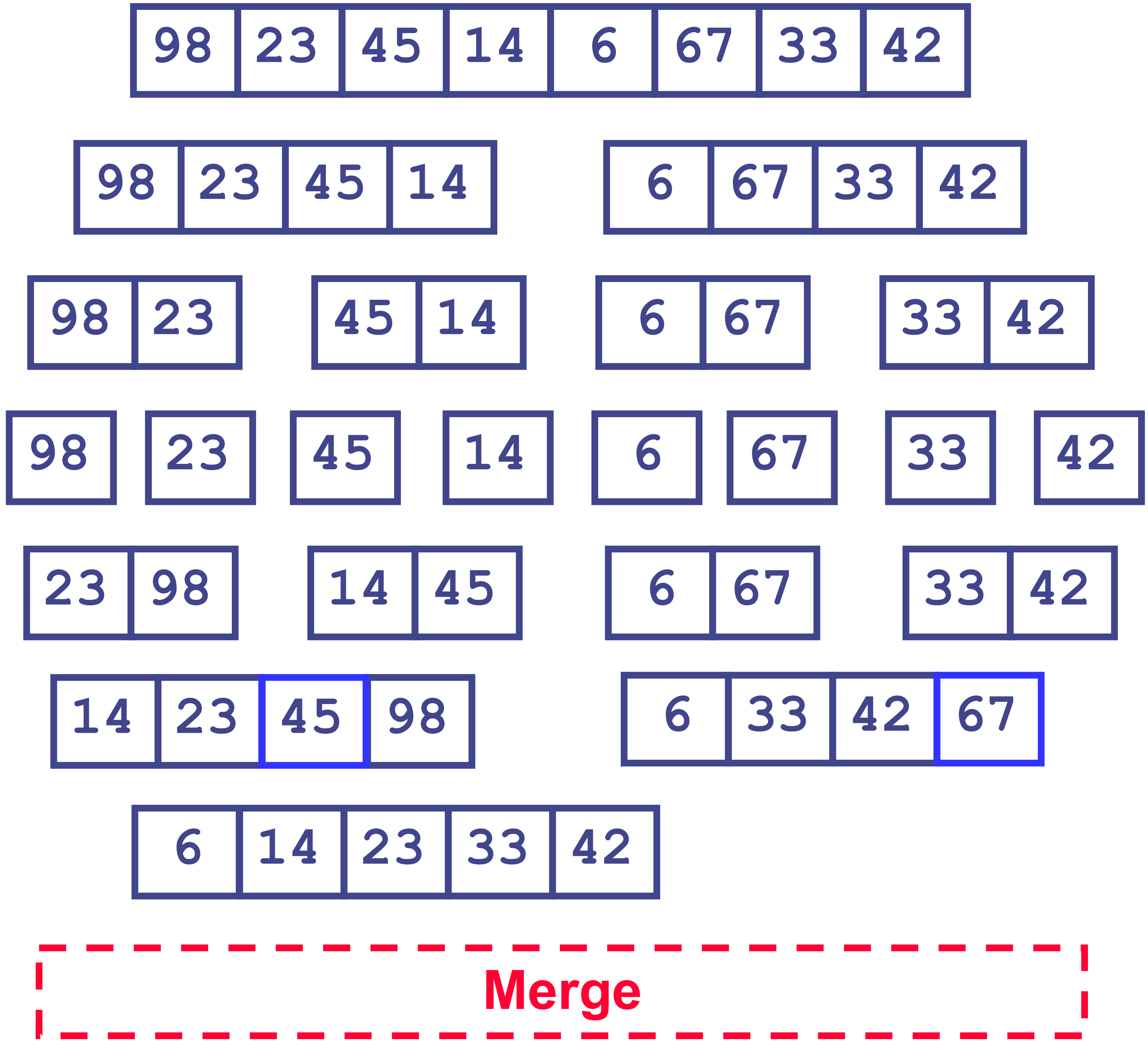
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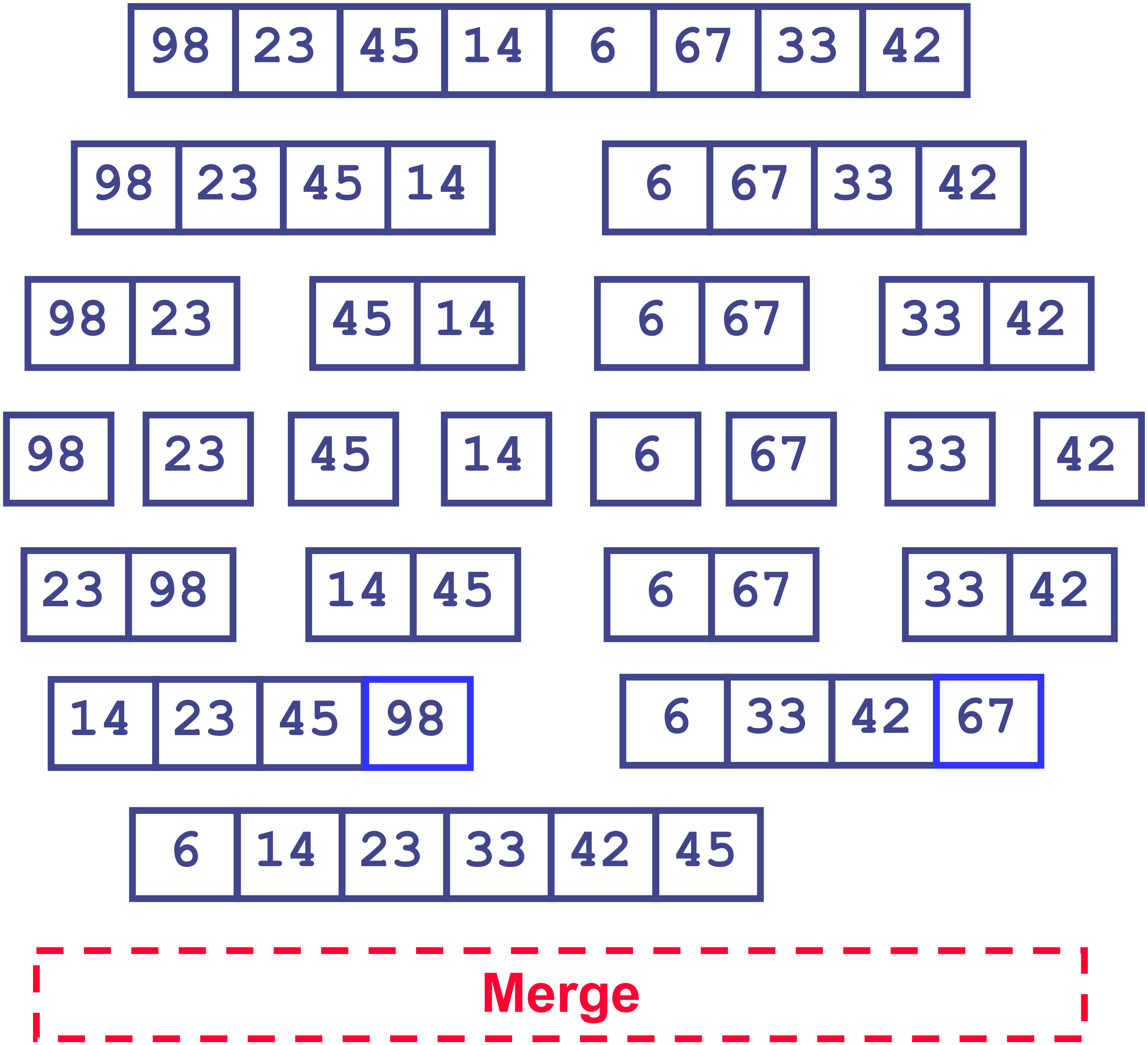
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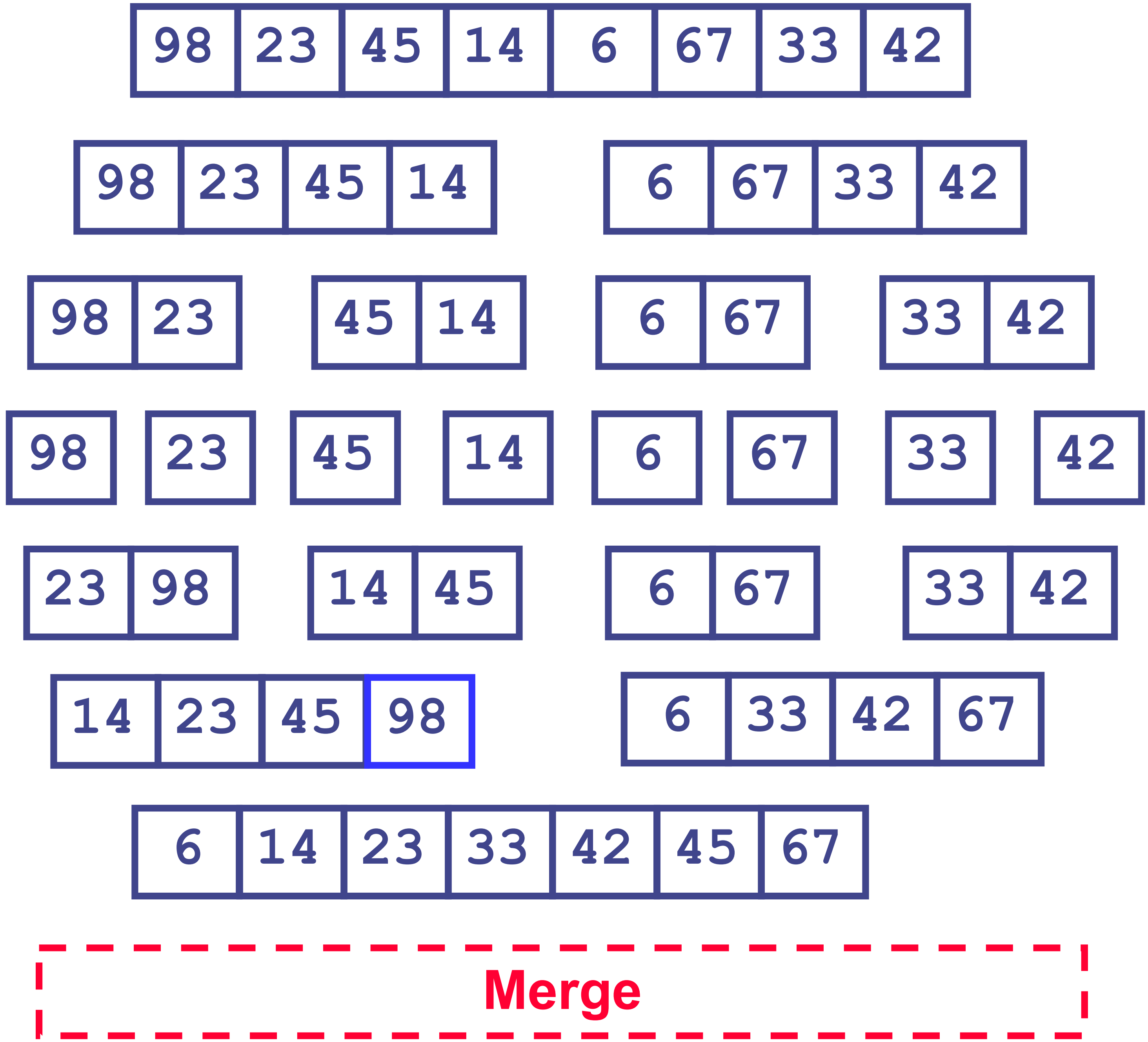
Merge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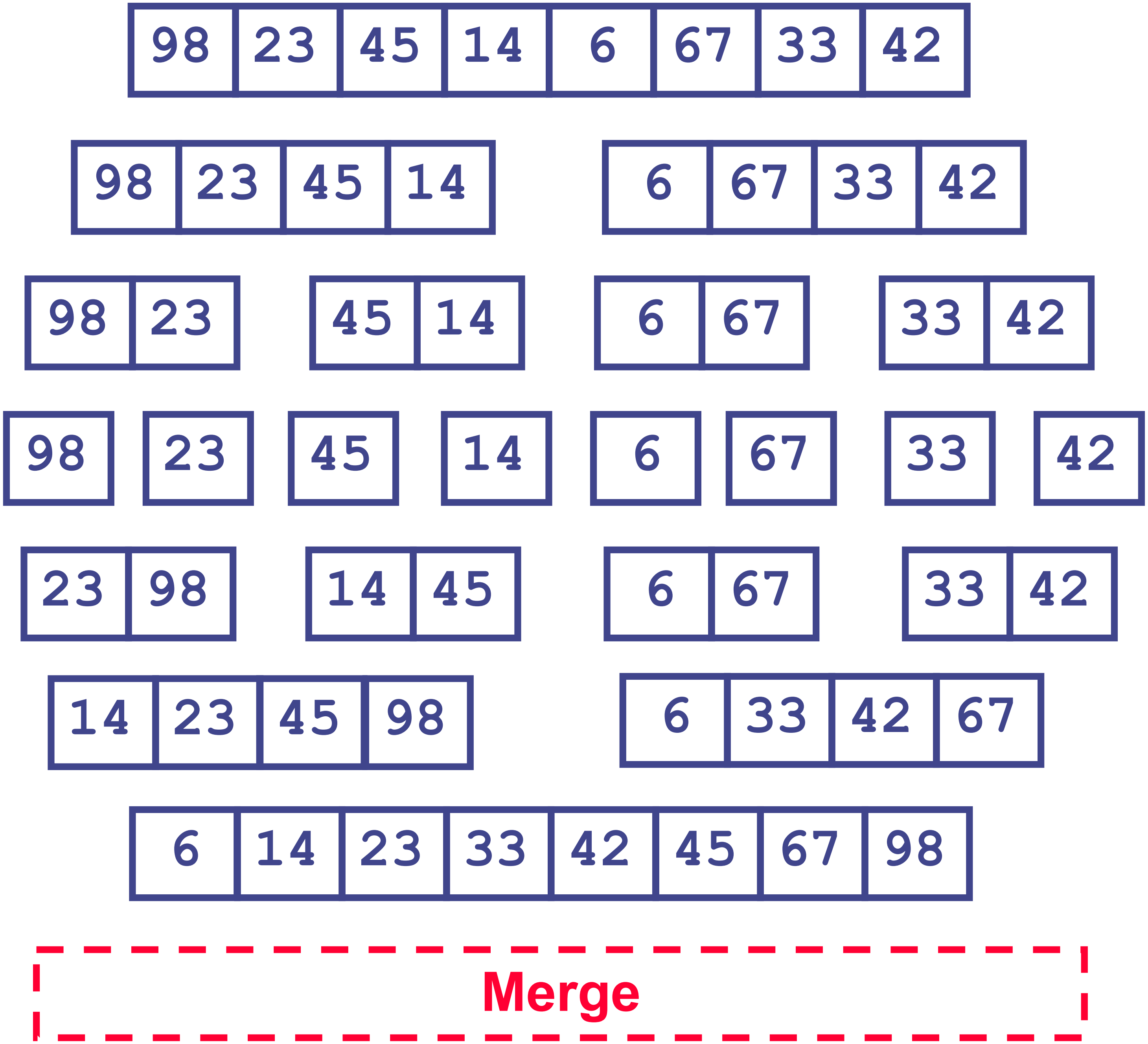
Merge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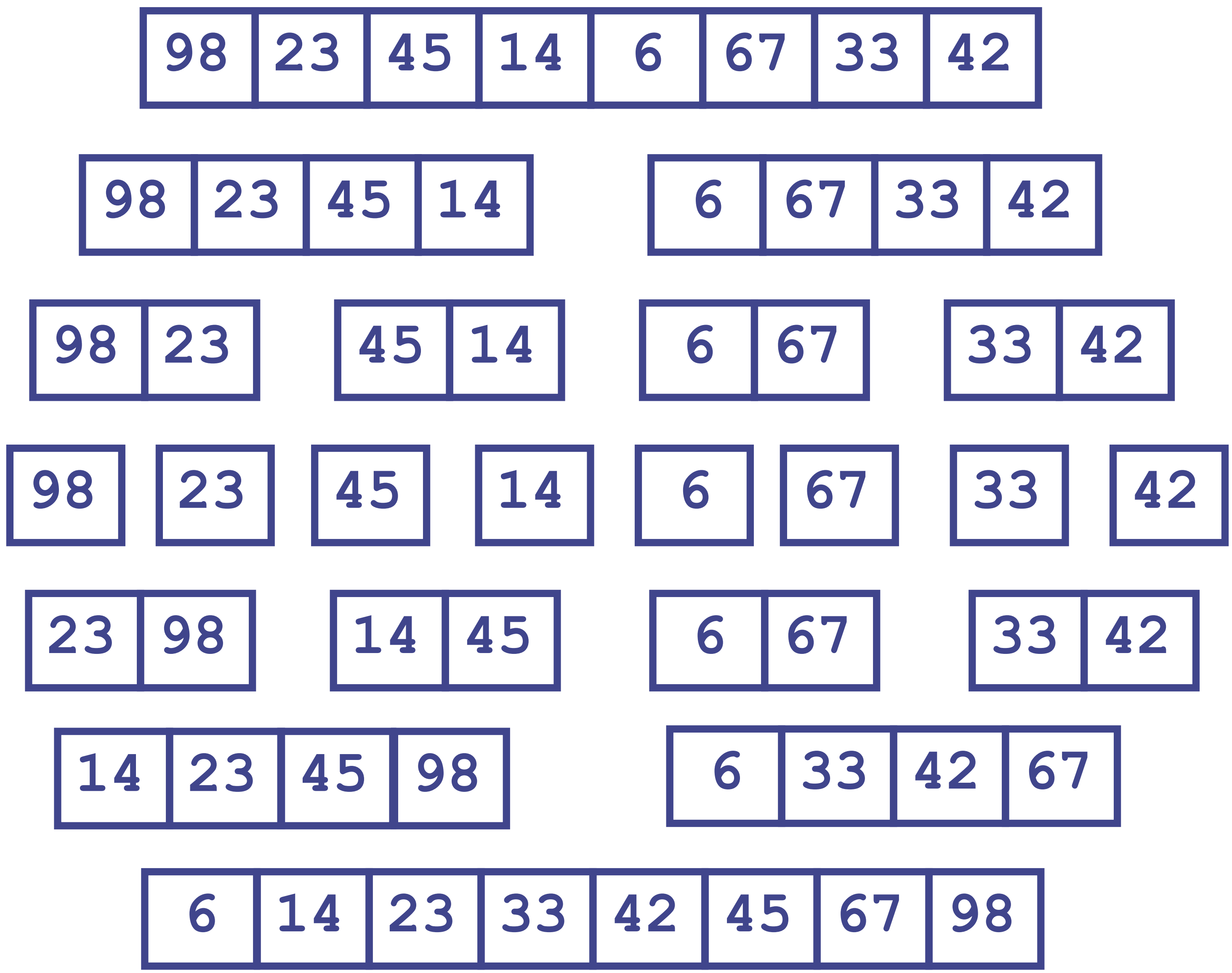
Merge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33 42
14 23 45 98
6 33 42 67
6 14 23 33 42 45 67 98

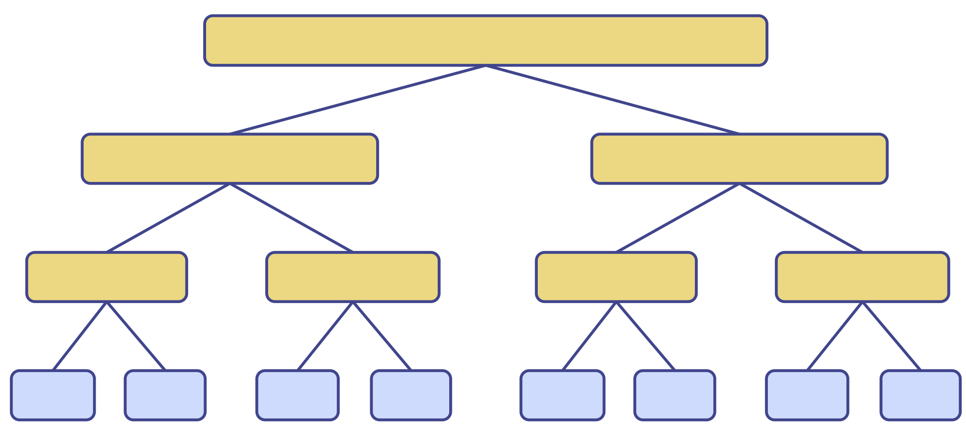
98	23	45	14	6	67	33	42

6	14	23	33	42	45	67	98

Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide in half the sequence,
- The overall amount or work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

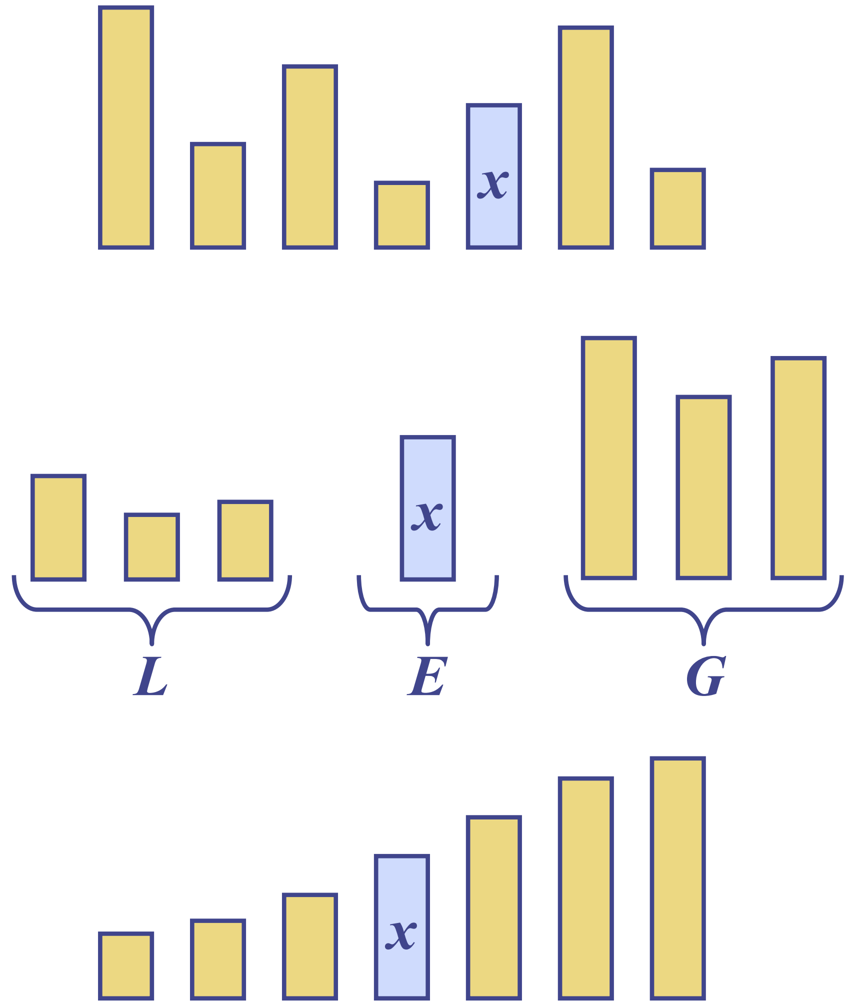
depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…



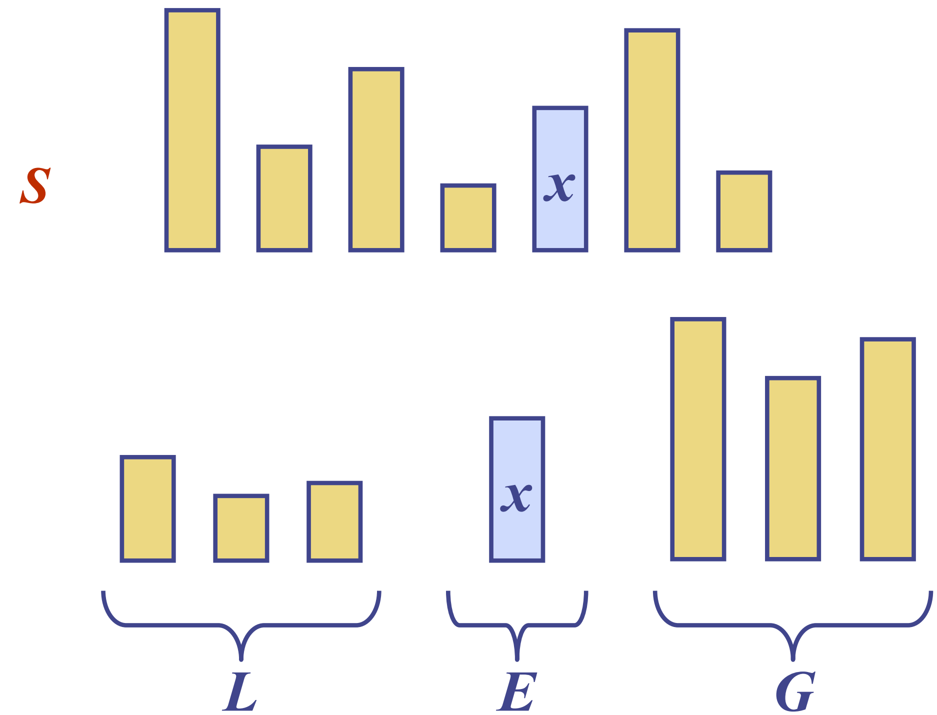
Quick-Sort

Quick-Sort

- Quick-sort is a sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick an element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



Not In-Place



QuickSort(S)

$i \leftarrow$ PIVOT
$x \leftarrow S.elemAtRank(i)$
$(L,E,G) \leftarrow Partition(S,x)$
QuickSort(L)
QuickSort(G)
combine L,E,G

In this example the PIVOT is chosen randomly, but we could decide always to choose the first element of the array, or the last.

Partition

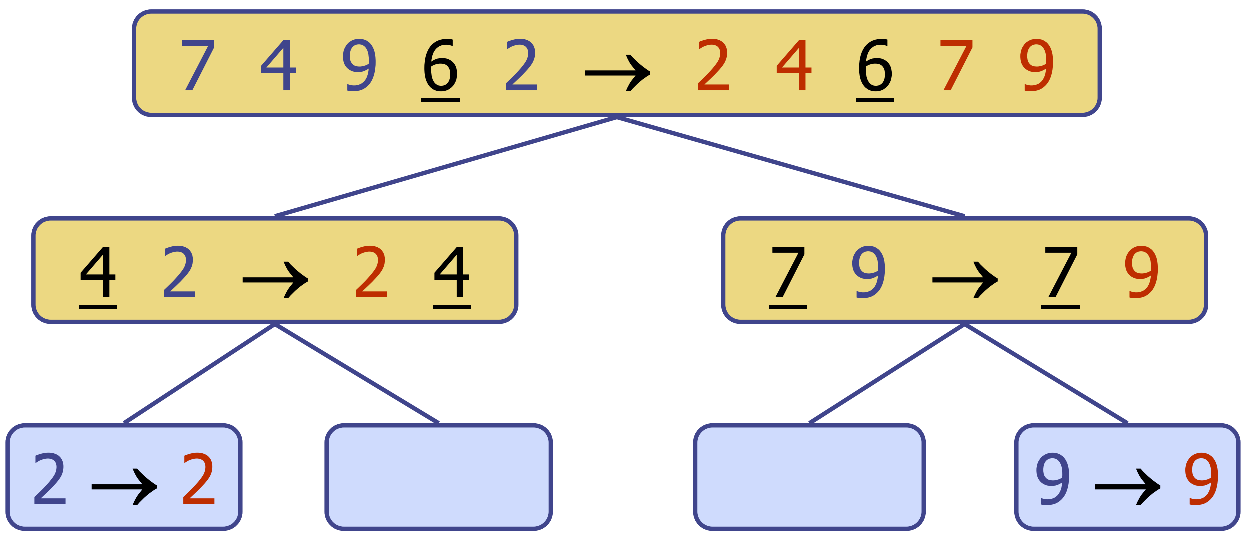
Not in-place

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

Algorithm *partition*(S, p)
  Input sequence S, position p of pivot
  Output subsequences L, E, G of the elements of S less than, equal to, or greater than the pivot, resp.
  $L, E, G \leftarrow$ empty sequences
  $x \leftarrow S.remove(p)$
  while $!S.isEmpty()$
    $y \leftarrow S.remove(S.first())$
    if $y < x$
      $L.insertLast(y)$
    else if $y = x$
      $E.insertLast(y)$
    else $\{ y > x \}$
      $G.insertLast(y)$
  return L, E, G

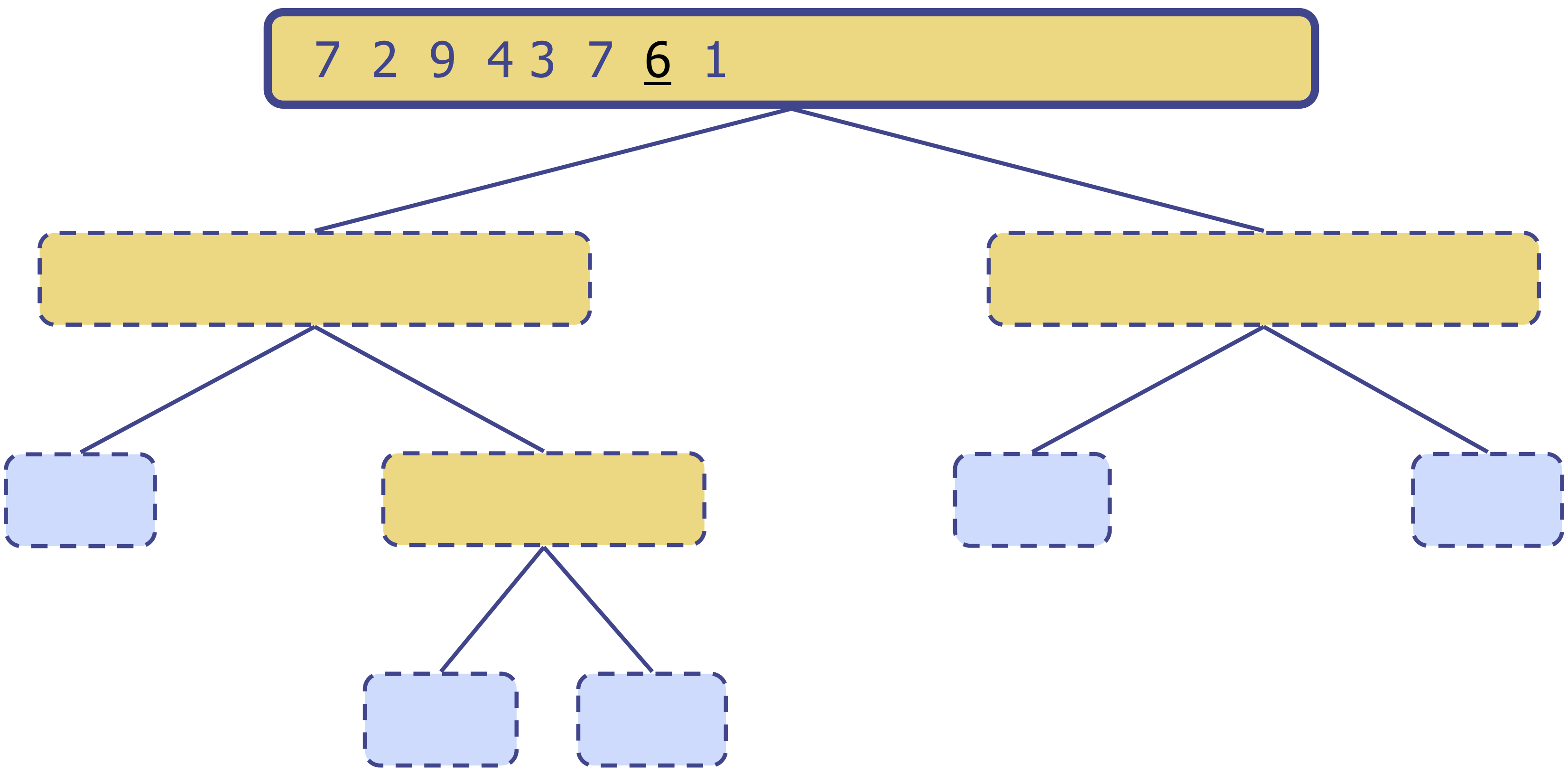
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



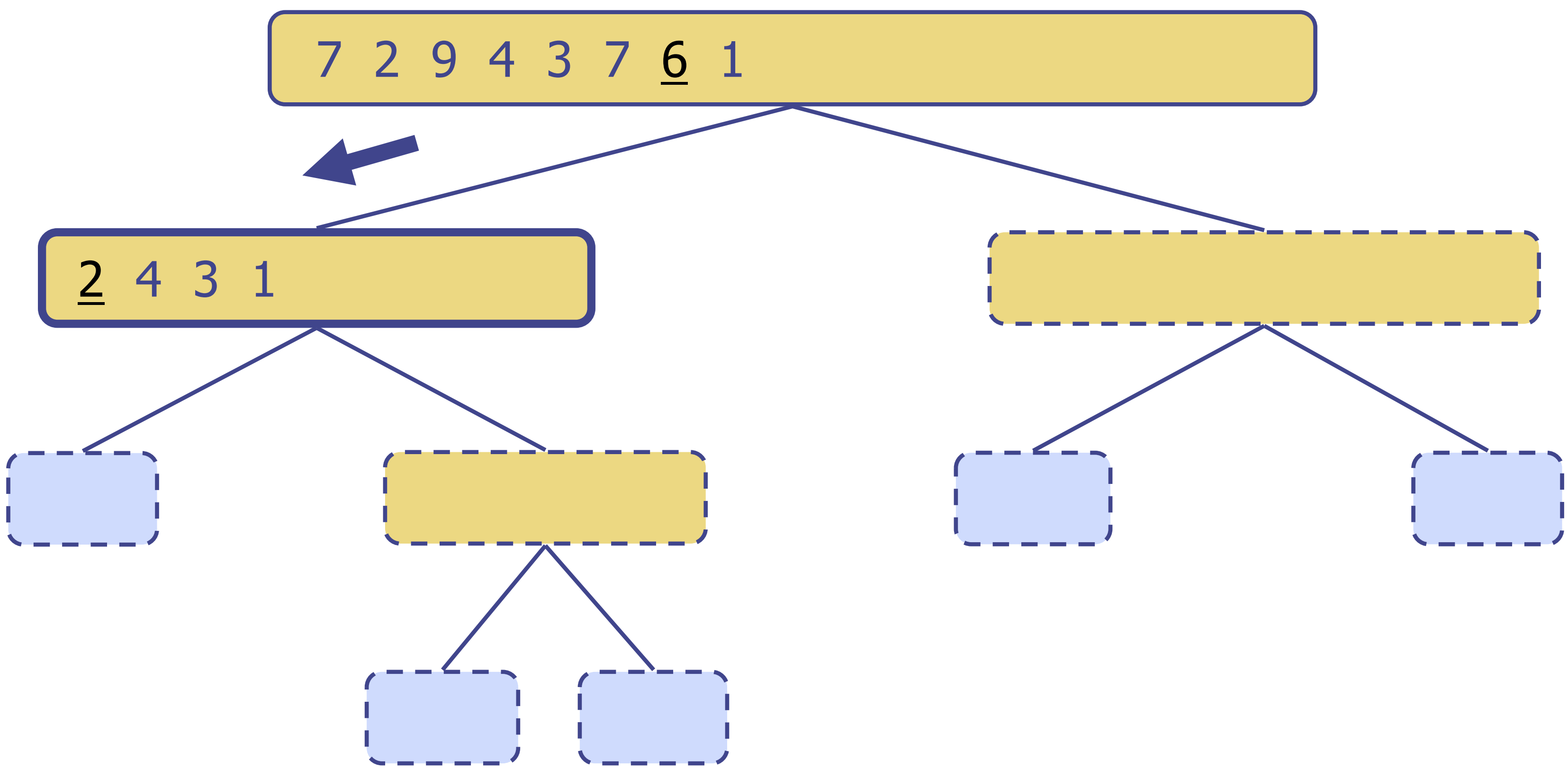
Execution Example

- Pivot selection



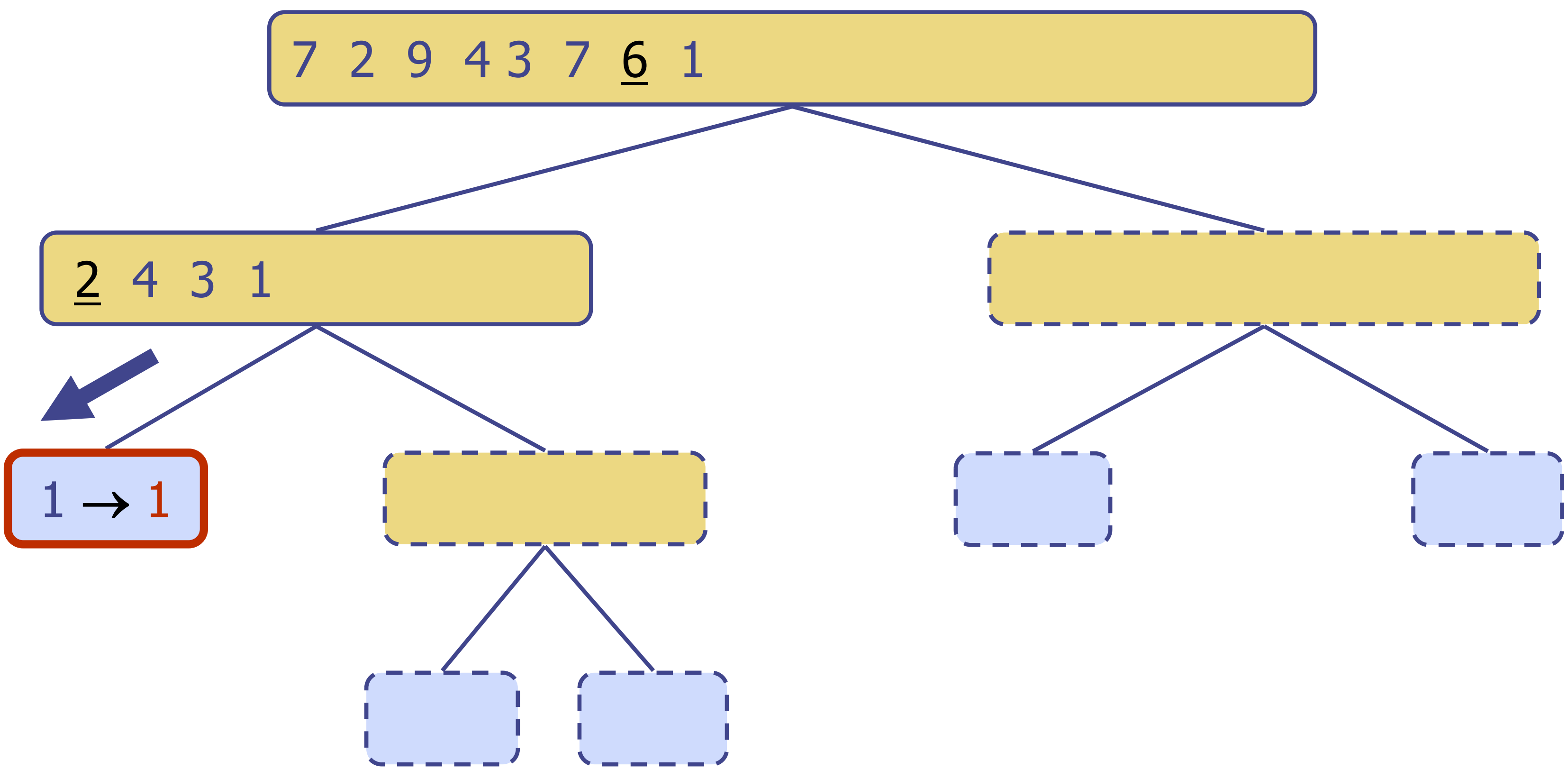
Execution Example (cont.)

- Partition, recursive call, pivot selection



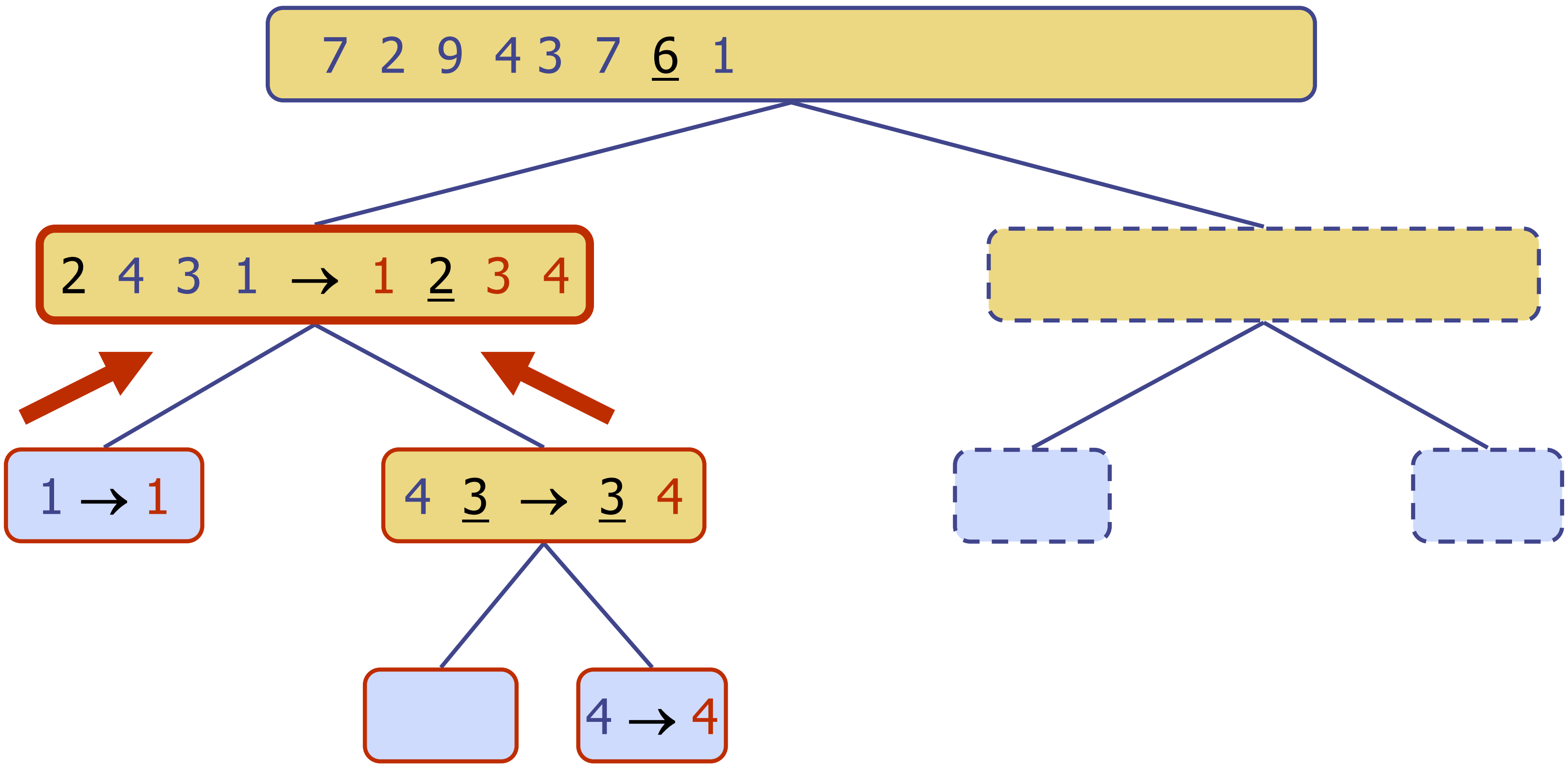
Execution Example (cont.)

- Partition, recursive call, base case



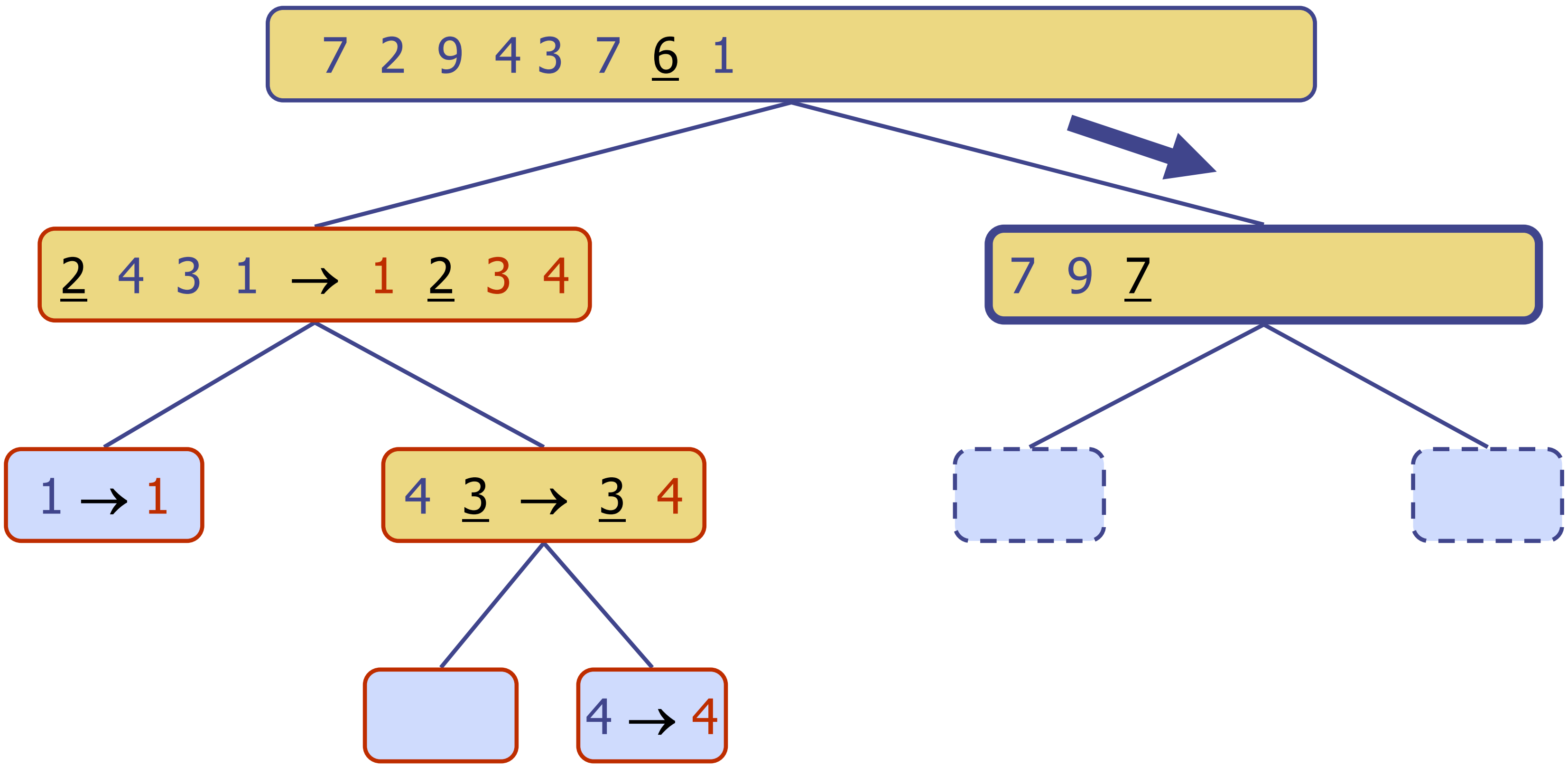
Execution Example (cont.)

- Recursive call, …, base case, join



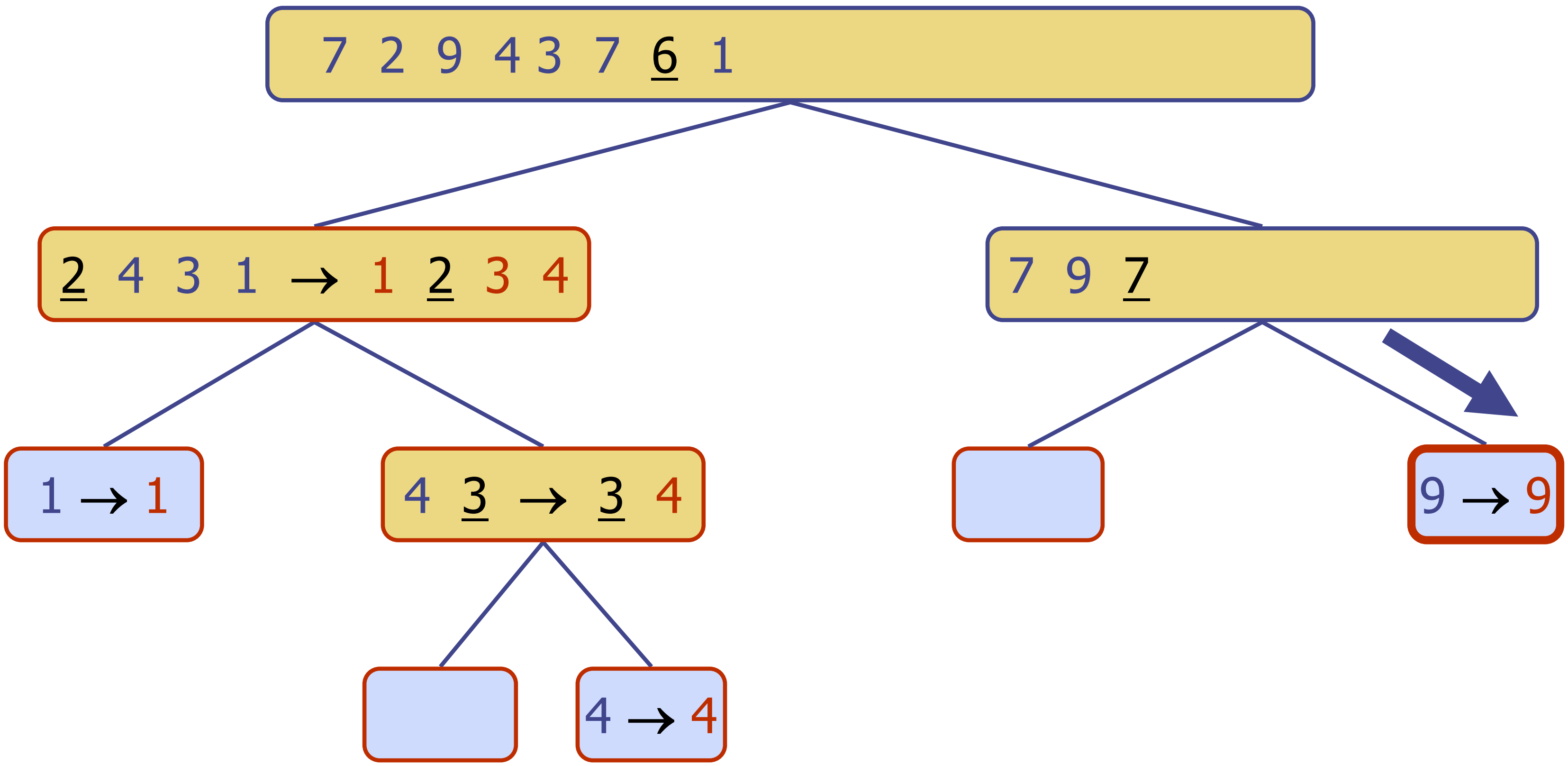
Execution Example (cont.)

- Recursive call, pivot selection



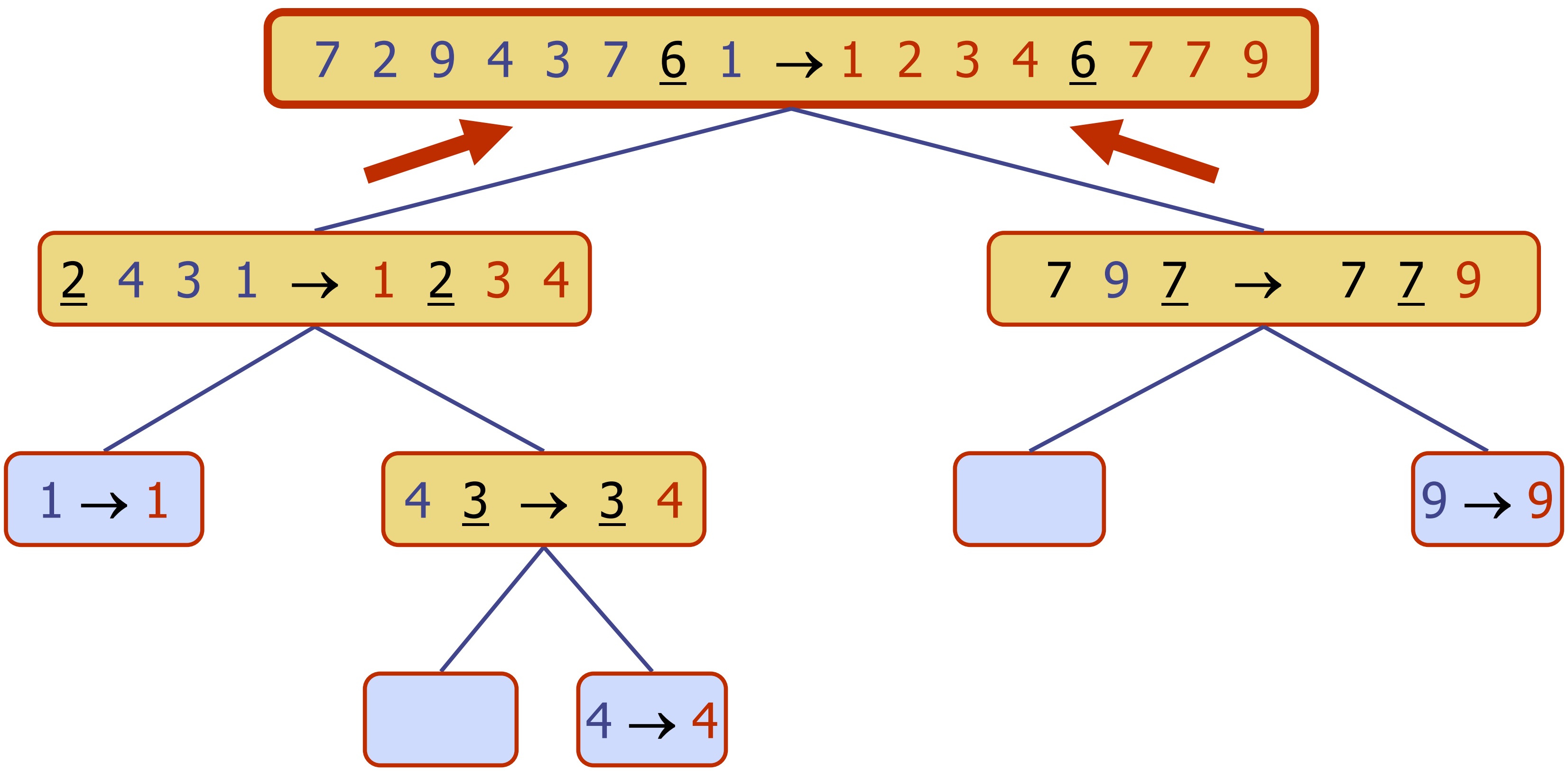
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join



In-Place Quick-Sort

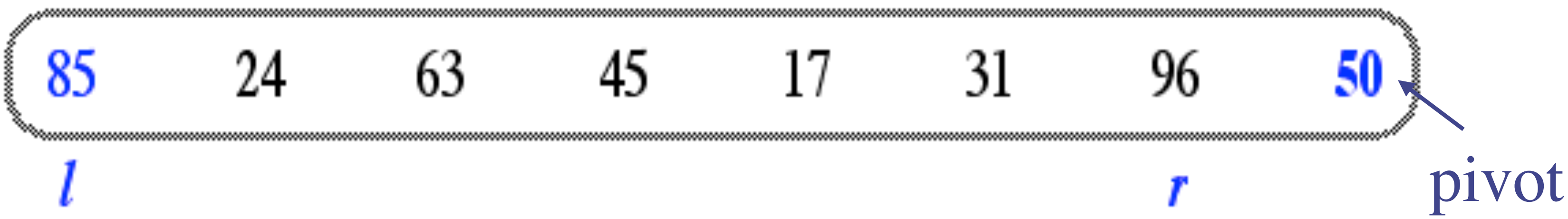
In the partition step, we use replace operations to rearrange the elements of the input sequence such that

- the elements less than the pivot have rank less than h
- the elements equal to the pivot have rank between h and k
- the elements greater than the pivot have rank greater than k

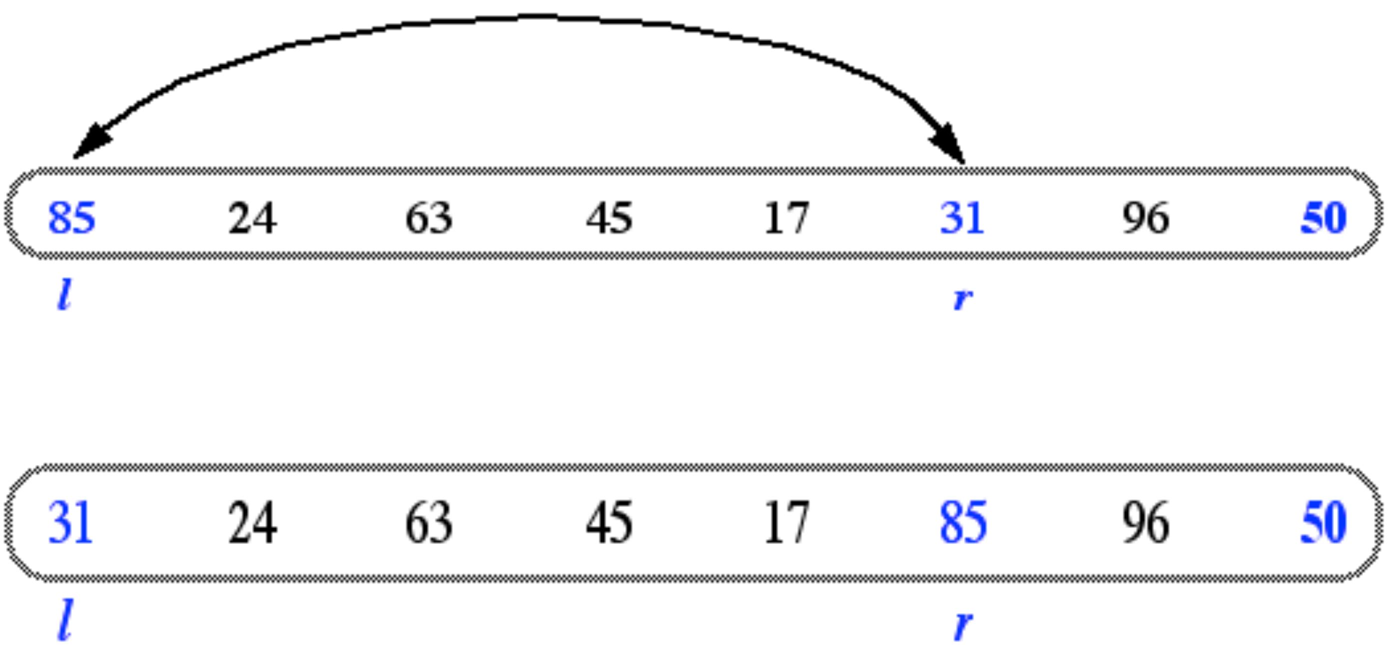
- The recursive calls consider
  - elements with rank less than h
  - elements with rank greater than k

In-Place Quick-Sort

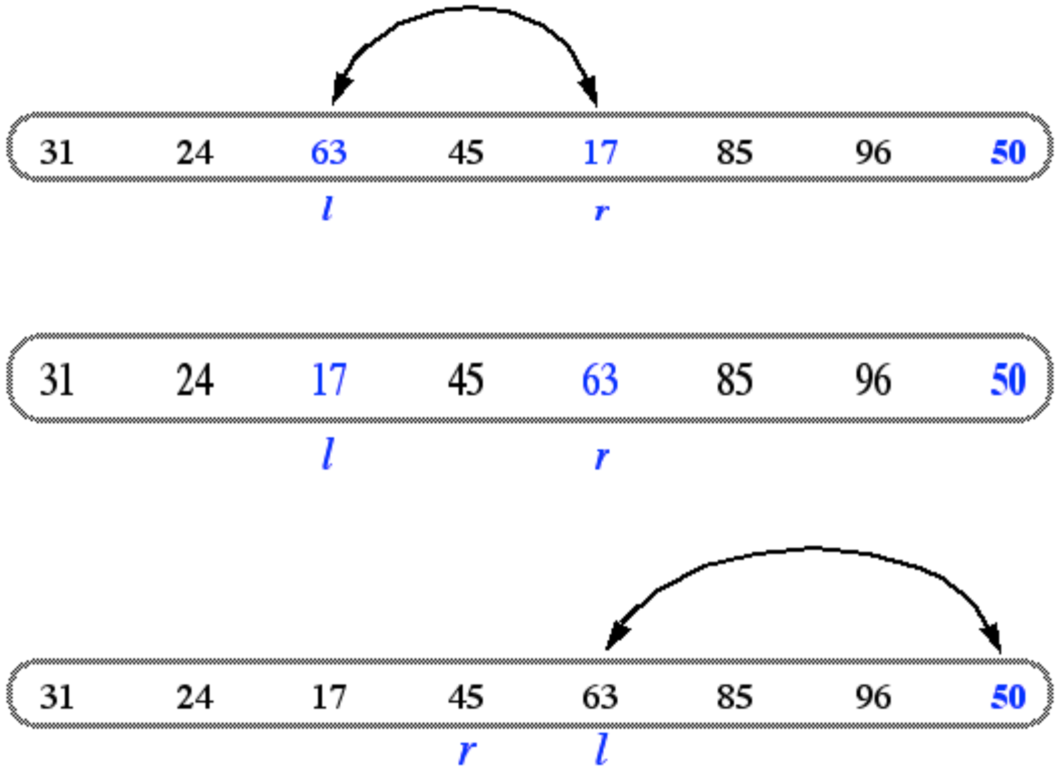
Divide step: *l* scans the sequence from the left, and *r* from the right.



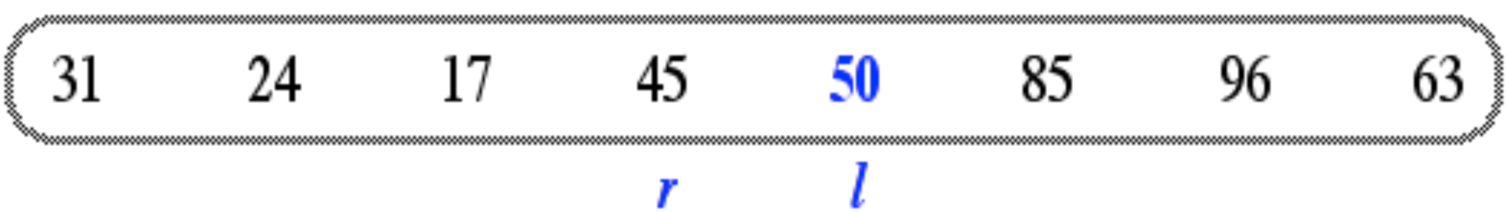
A swap is performed when l is at an element larger than the pivot and r is at one smaller than the pivot.



In Place Quick Sort (contd.)



A final swap with the pivot completes the divide step



In-Place Quick-Sort

Algorithm *inPlaceQuickSort*(S, l, r)
    Input sequence S, ranks l and r
    Output sequence S with the elements of rank between l and r rearranged in increasing order
    if $l \geq r$
        return
    $i \leftarrow$ a random integer between l and r
    $(h, k) \leftarrow$ *inPlacePartition*(i, l, r)
    inPlaceQuickSort($S, l, h - 1$)
    inPlaceQuickSort($S, k + 1, r$)

In Place Partition

- Repeat until l and r cross:
  - l traverse the array from left to right until it finds and element ≥ pivot
  - r traverse the array from right to left until it finds an element < pivot
  - Swap elements at indices l and r

Algorithm *inPlacePartition*(p,s,e)

Input: position ***p*** of the pivot; s and e are the sequence limits

Output: l and r such that:

r-1=index of the last element smaller than the pivot

l+1=index of the first element larger than the pivot

$l \leftarrow s$, $r \leftarrow e-1$

swap $S[p]$ **with** $S[e]$, $p \leftarrow e$

while $l \leq r$

    while $S[l] < S[p]$ **and** $r \geq l$

        $l \leftarrow l+1$

    while $S[r] \geq S[p]$ **and** $r \geq l$

        $r \leftarrow r-1$

    if $r > l$ **swap** $S[r]$ **with** $S[l]$

swap $S[l]$ **with** $S[p]$

return $r+1, l$

In Place Quick-sort

```
/** Sort the subarray S[a..b] inclusive. */
private static <K> void quickSortInPlace(K[ ] S, Comparator<K> comp,
                                                                int a, int b) {
  if (a >= b) return;            // subarray is trivially sorted
  int left = a;
  int right = b-1;
  K pivot = S[b];
  K temp;                        // temp object used for swapping
  while (left <= right) {
    // scan until reaching value equal or larger than pivot (or right marker)
    while (left <= right && comp.compare(S[left], pivot) < 0) left++;
    // scan until reaching value equal or smaller than pivot (or left marker)
    while (left <= right && comp.compare(S[right], pivot) > 0) right--;
    if (left <= right) {         // indices did not strictly cross
      // so swap values and shrink range
      temp = S[left]; S[left] = S[right]; S[right] = temp;
      left++; right--;
    }
  }
  // put pivot into its final place (currently marked by left index)
  temp = S[left]; S[left] = S[b]; S[b] = temp;
  // make recursive calls
  quickSortInPlace(S, comp, a, left - 1);
  quickSortInPlace(S, comp, left + 1, b);
}
```

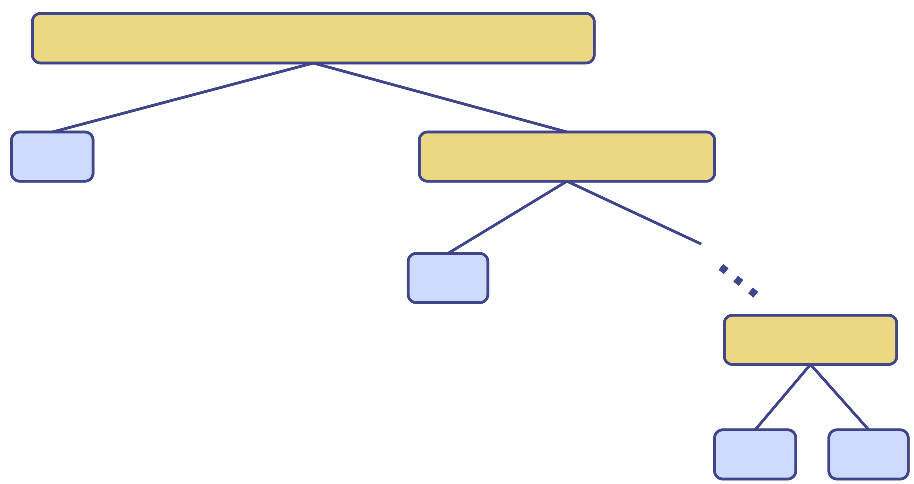
Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to the sum

$$n + (n - 1) + \ldots + 2 + 1$$

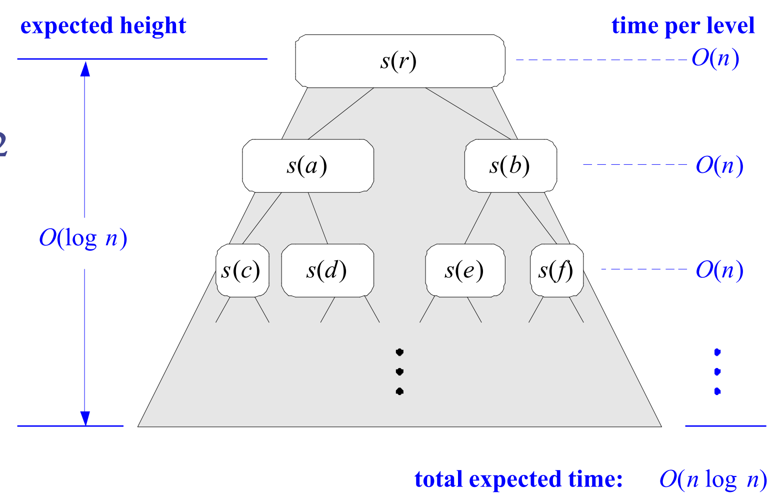
- Thus, the worst-case running time of quick-sort is $O(n^2)$

depth	time
0	n
1	$n - 1$
…	…
$n - 1$	1



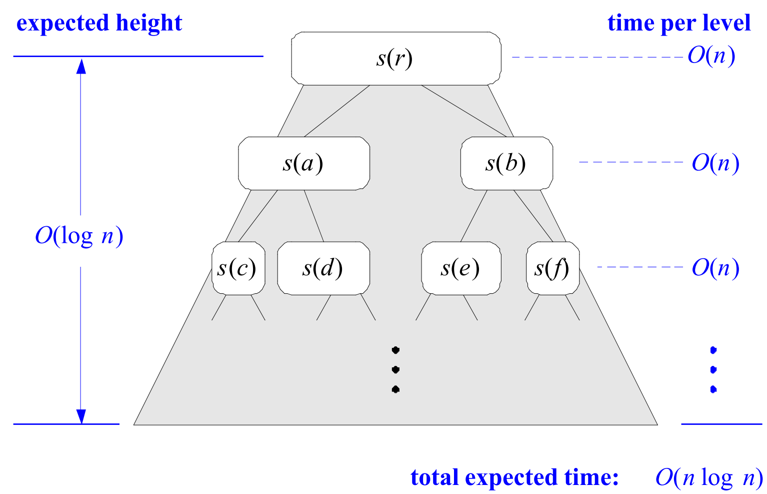
Expected Running Time

- Consider a recursive call of quick-sort on a sequence of size s
  - Good call: the sizes of L and G are each less than $3s/4$
  - Bad call: one of L and G has size greater than $3s/4$
- A call is good with probability ½
  (for an element, the expected number of calls until a good call is 2)
- Hence, for a node of depth i, we expect that
  - $i/2$ ancestor nodes are associated with good calls
  - the expected size of the input sequence for the current call is at most $(3/4)^{i/2}n$



Expected Running Time

- Thus, we have
  - For a node of depth $2\log_{4/3}\boldsymbol{n}$, the expected size of the input sequence is one $((3/4)^{\wedge}((2\log_{4/3}n)/2)\ n = 1)$
  - The expected height of the quick-sort tree is $\boldsymbol{O}(\log \boldsymbol{n})$
- The overall amount or work done at the nodes of the same depth of the quick-sort tree is $\boldsymbol{O}(\boldsymbol{n})$
- Thus, the expected running time of quick-sort is $\boldsymbol{O}(\boldsymbol{n} \log \boldsymbol{n})$



Algorithm	Time	Notes
selection-sort	$O(n^2)$ w.c. and av.	◆ in-place ◆ slow (good for small inputs)
insertion-sort	$O(n^2)$ w.c. and av.	◆ in-place ◆ slow (good for small inputs)
quick-sort	$O(n^2)$ *w.c.* $O(n \log n)$ average	◆ in-place, randomized ◆ fastest (good for large inputs)
heap-sort	$O(n \log n)$ w.c. and av.	◆ in-place ◆ fast (good for large inputs)
merge-sort	$O(n \log n)$ w.c. and av.	◆ sequential data access ◆ fast (good for huge inputs)