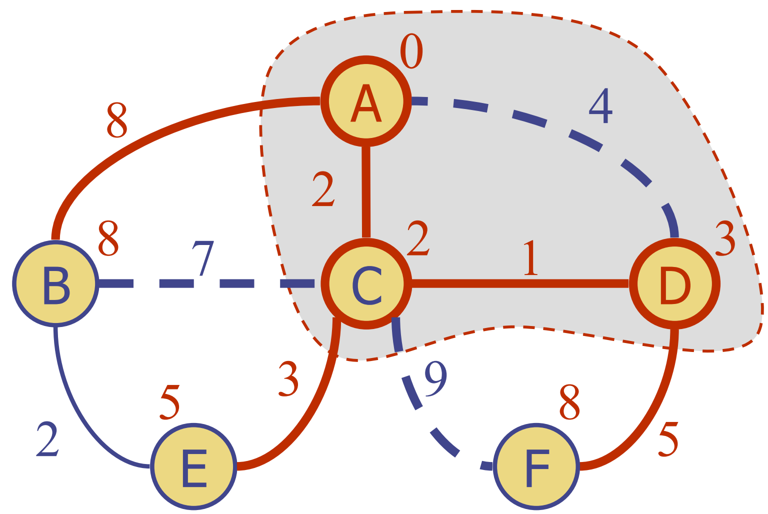
Shortest Path

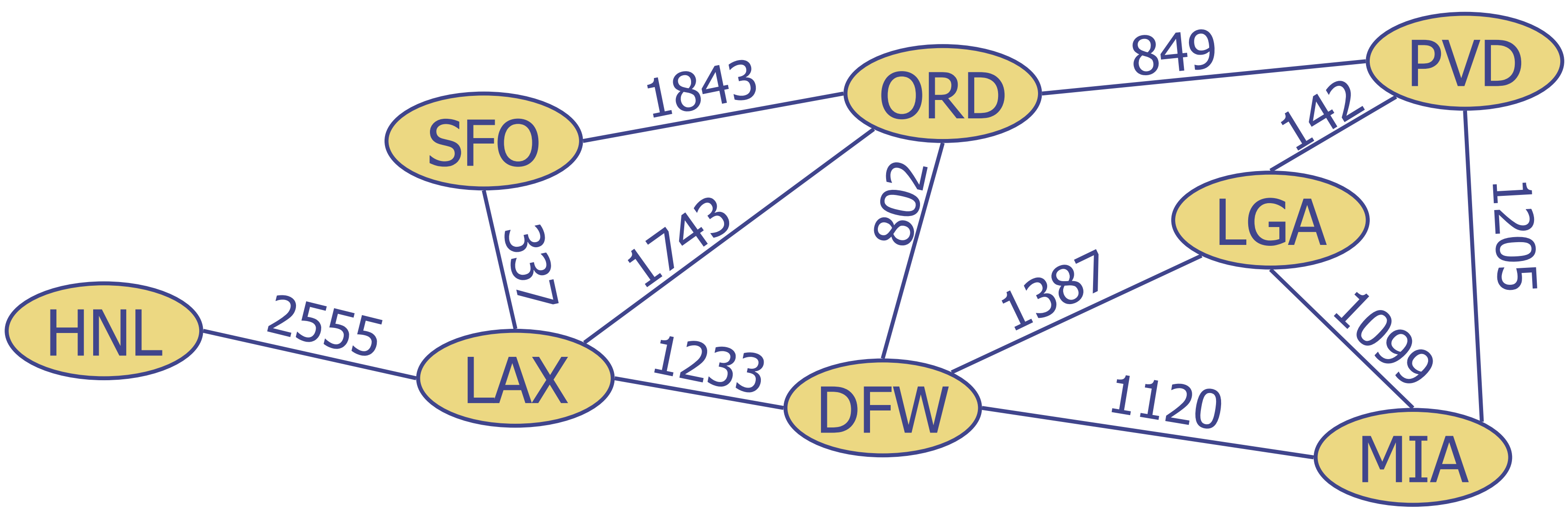


Outline and Reading

- Shortest path
  - Weighted graph
  - Shortest path problem
  - Shortest path properties
- Dijkstra's algorithm
  - Algorithm
  - Edge relaxation
  - Example
  - Analysis

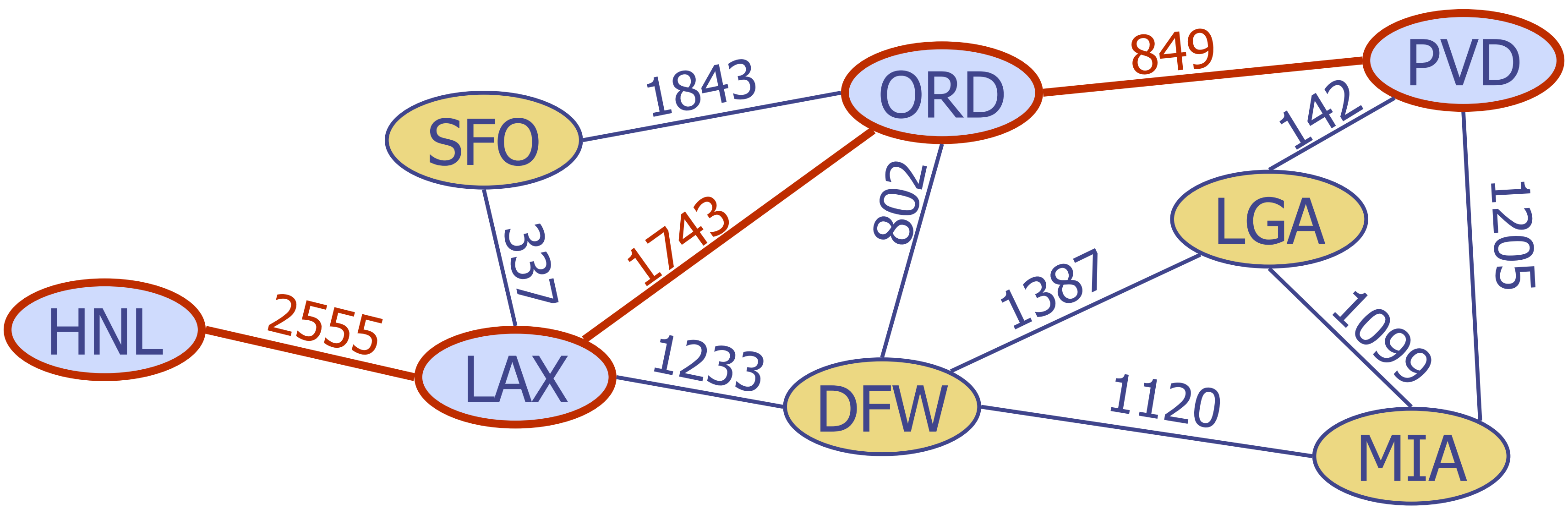
Weighted Graphs

- In a weighted graph, each edge has an associated numerical value, called the weight of the edge
- Edge weights may represent, distances, costs, etc.
- Example:
  - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports



Shortest Path Problem

- Given a weighted graph and two vertices u and v, we want to find a path of minimum total weight between u and v
- Applications
  - Flight reservations
  - Driving directions
  - Internet packet routing
- Example:
  - Shortest path between Providence and Honolulu



Shortest Path Properties

Property 1:

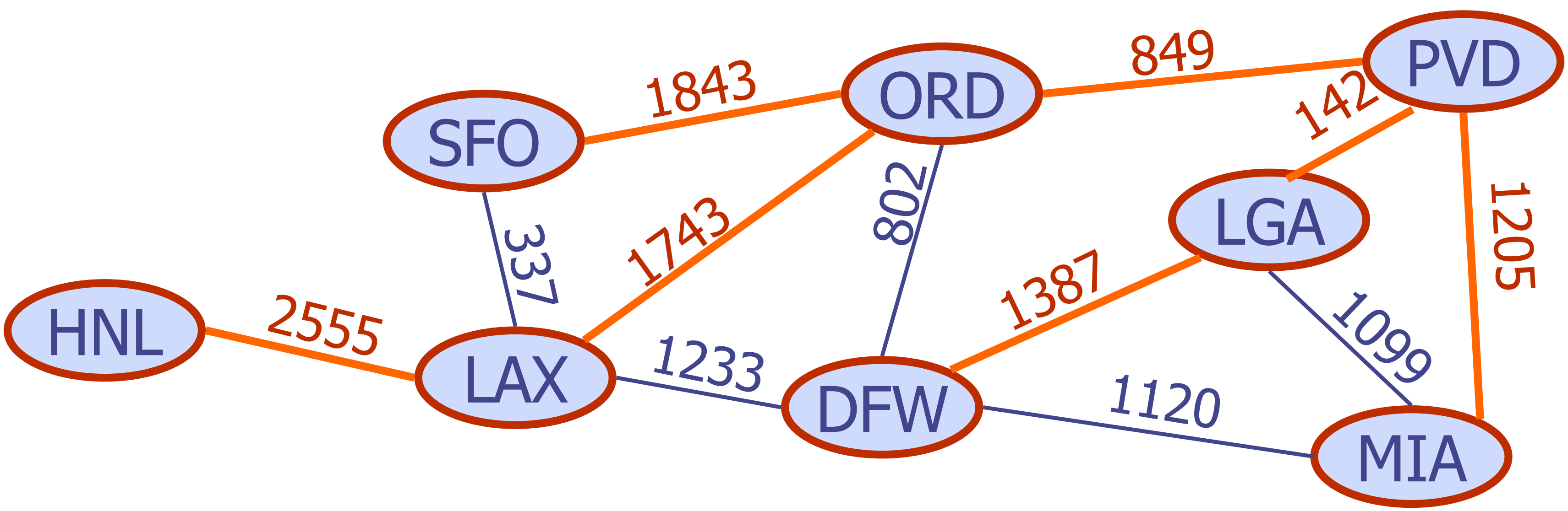
A subpath of a shortest path is itself a shortest path

Property 2:

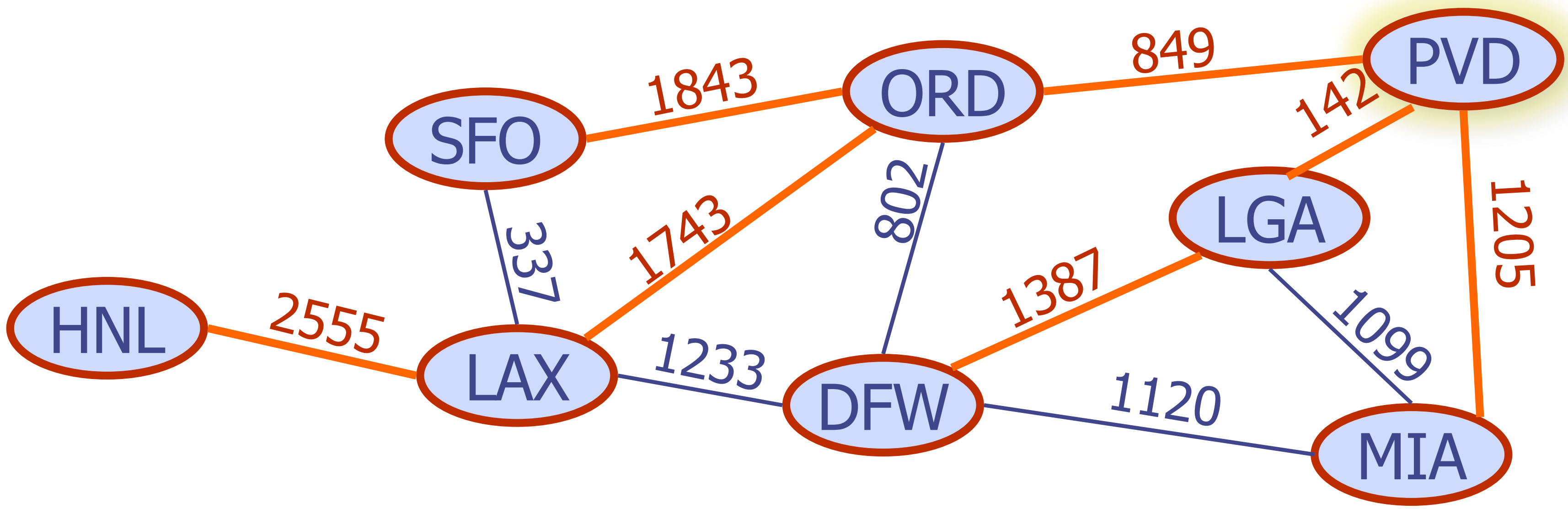
There is a tree of shortest paths from a start vertex to all the other vertices

Example:

Tree of shortest paths from Providence



There is a tree of shortest paths from a start vertex to all the other vertices



Dijkstra's Algorithm

- The distance of a vertex v from a vertex s is the length of a shortest path between s and v
- Dijkstra's algorithm computes the distances of all the vertices from a given start vertex s
- Assumptions:
  - the graph is connected
  - the edge weights are nonnegative

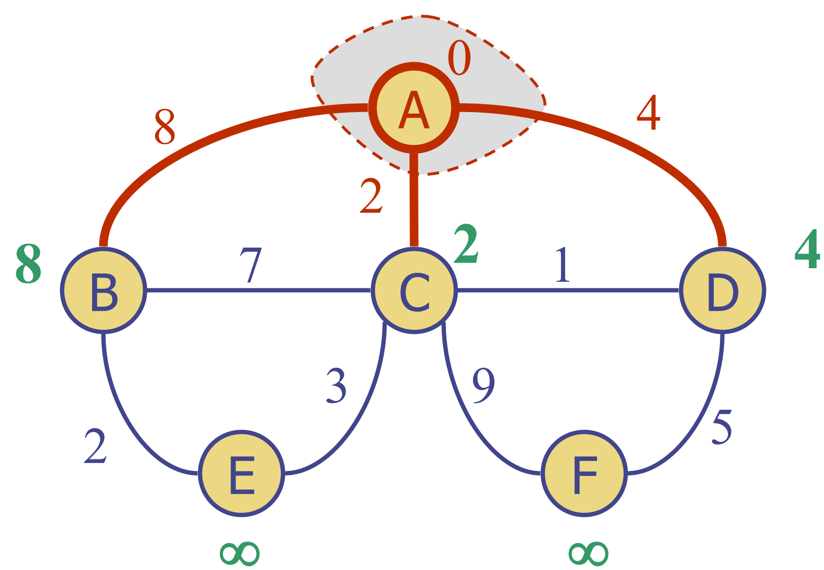
Note: the graph may be directed or undirected.

- We grow a "cloud" of vertices, beginning with s and eventually covering all the vertices

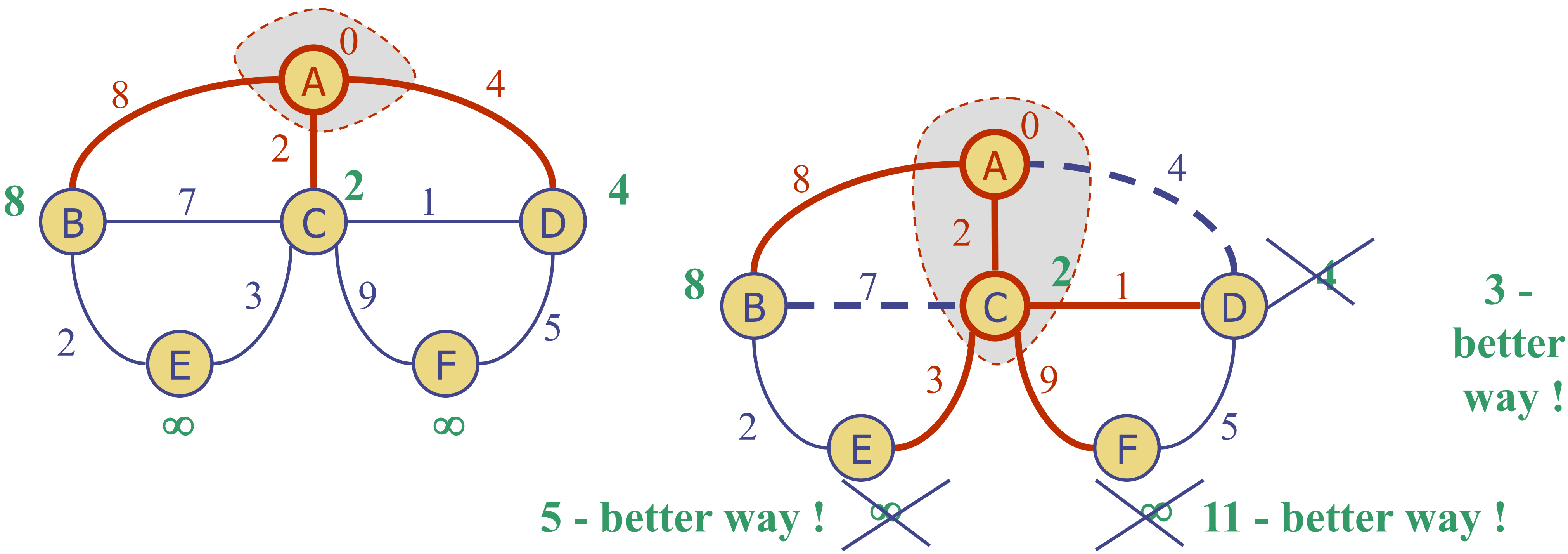
- At each vertex v we store

  d(v) = best distance of v from s **in the subgraph** consisting of the cloud and its adjacent vertices

Example



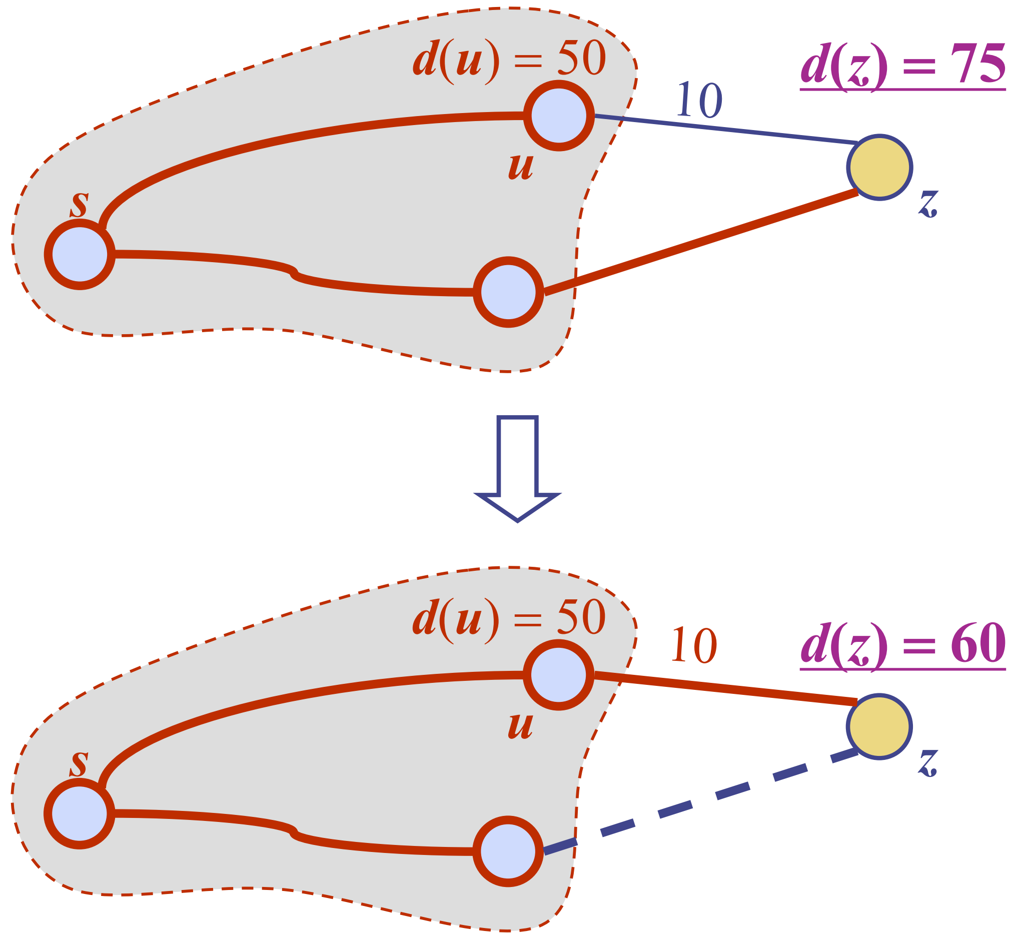
- At each step
  - We add to the cloud the vertex ***u*** outside the cloud with the smallest distance label
  - We **update** the labels of the vertices adjacent to ***u***



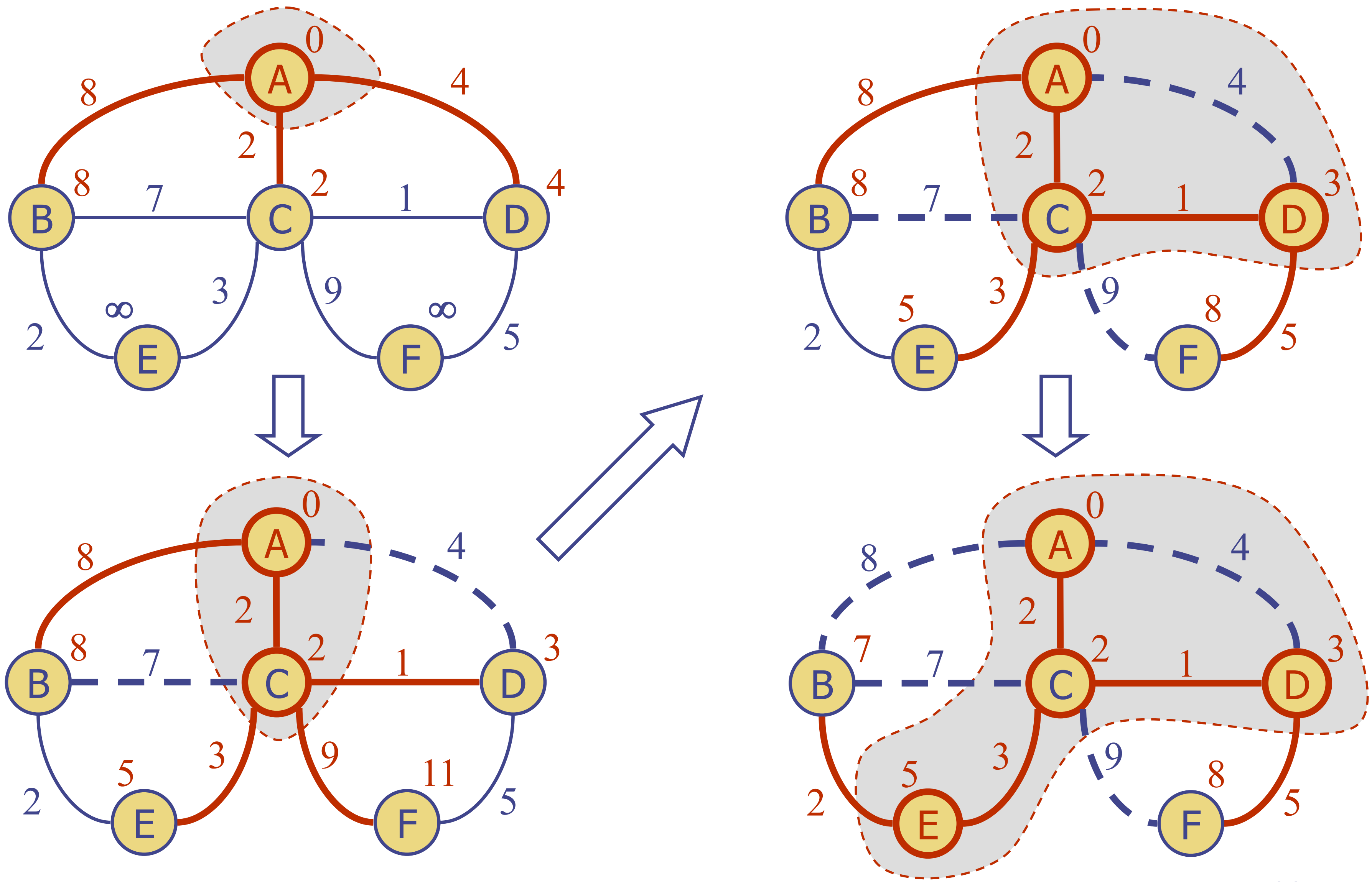
Update = Edge Relaxation

- Consider an edge $e = (u,z)$ such that
  - u is the vertex most recently added to the cloud
  - z is not in the cloud
- The relaxation of edge e updates distance $d(z)$ as follows

$$d(z) \leftarrow \min(d(z), d(u) + weight(e))$$

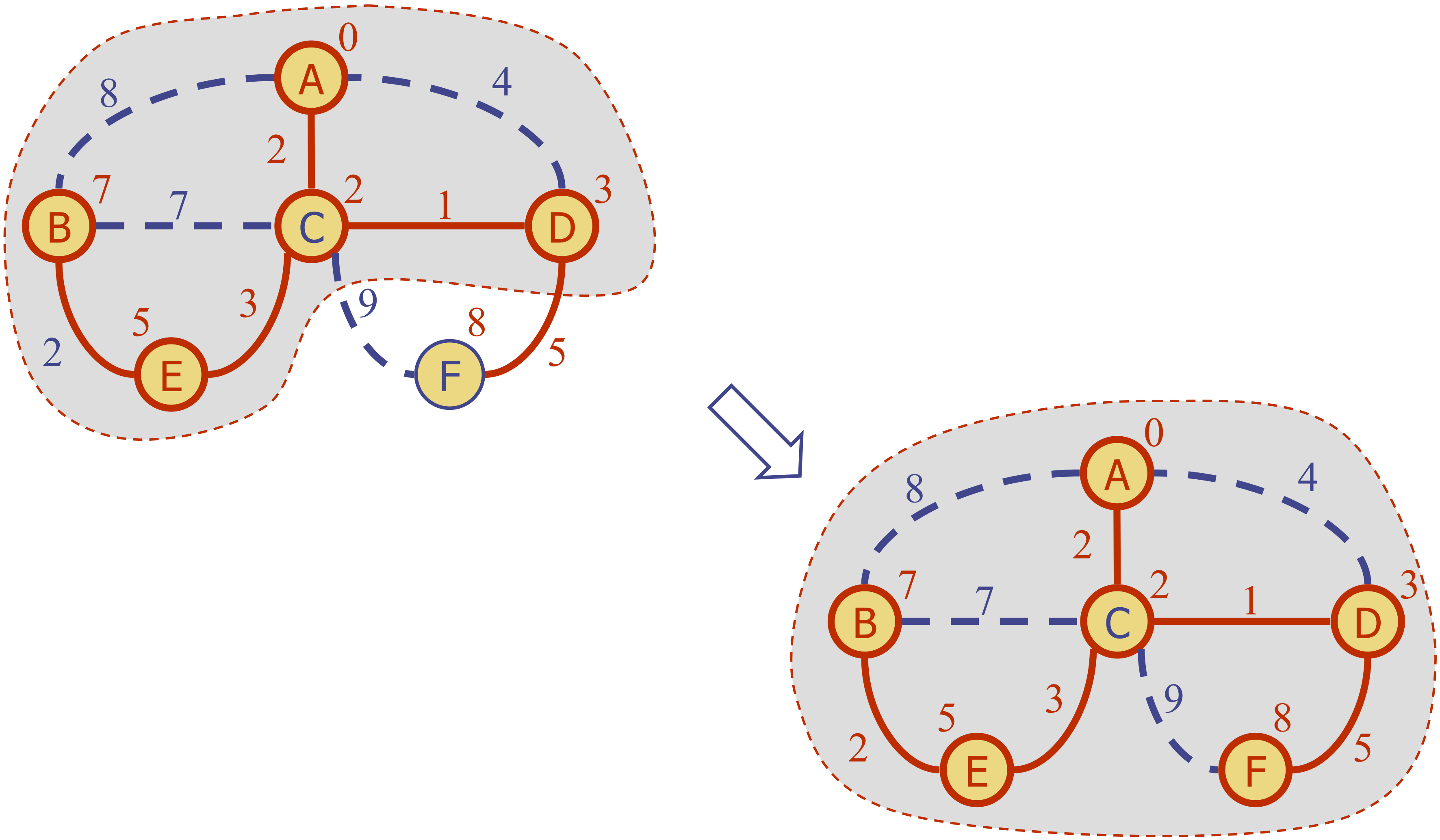


Example



0
A
8
4
2
8
7
2
1
4
B
C
D
∞
3
9
∞
2
E
F
5
0
A
8
4
2
8
7
2
1
3
B
C
D
5
3
9
8
2
E
F
5
0
A
8
4
2
8
7
2
1
3
B
C
D
5
3
9
11
2
E
F
5
0
A
8
4
2
7
7
2
1
3
B
C
D
5
3
9
8
2
E
F
5

Example (cont)



Dijkstra's Algorithm

we use a priority queue Q to store
the vertices not in the cloud,
where D[v] is the key of a vertex v in Q

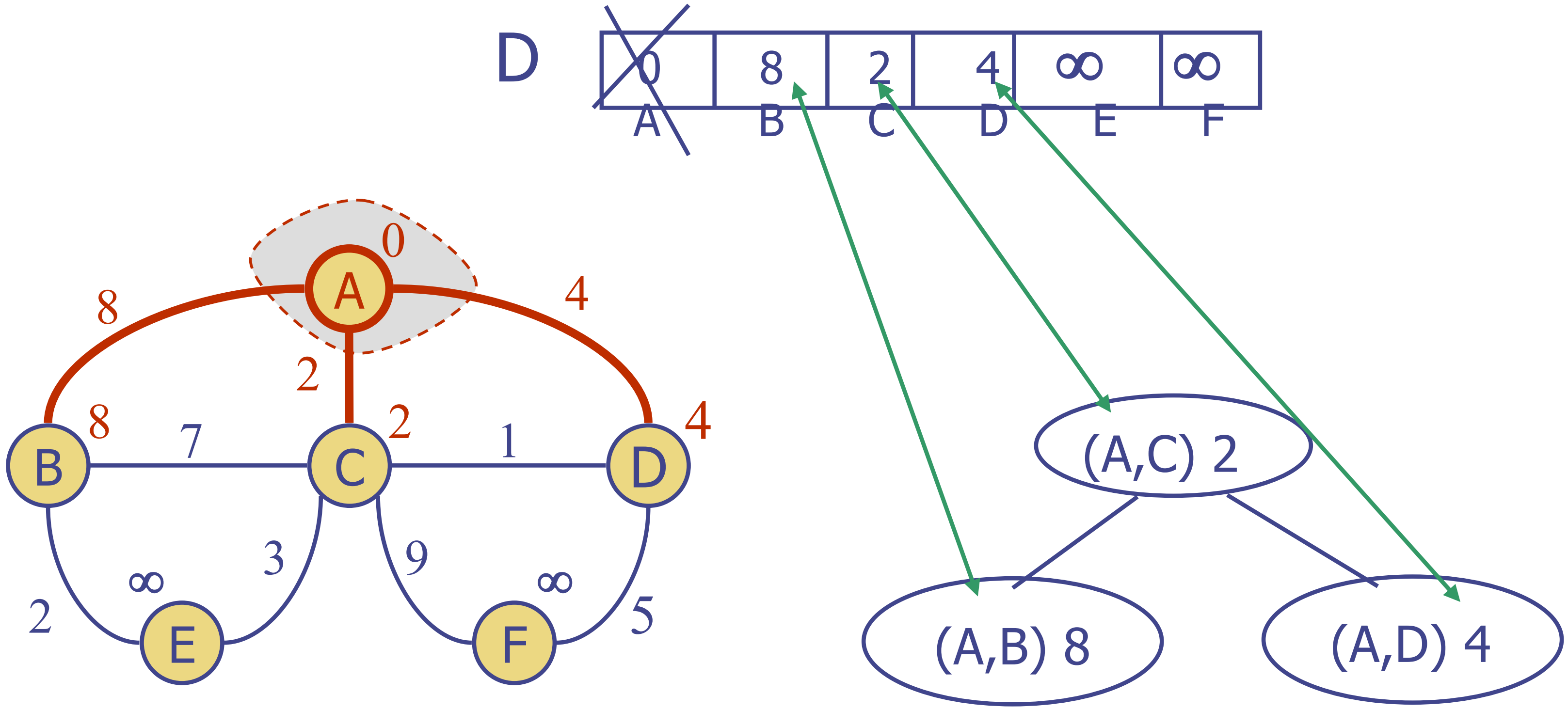
Algorithm ShortestPath(G, v):

Input: A weighted graph G and a distinguished vertex v of G.

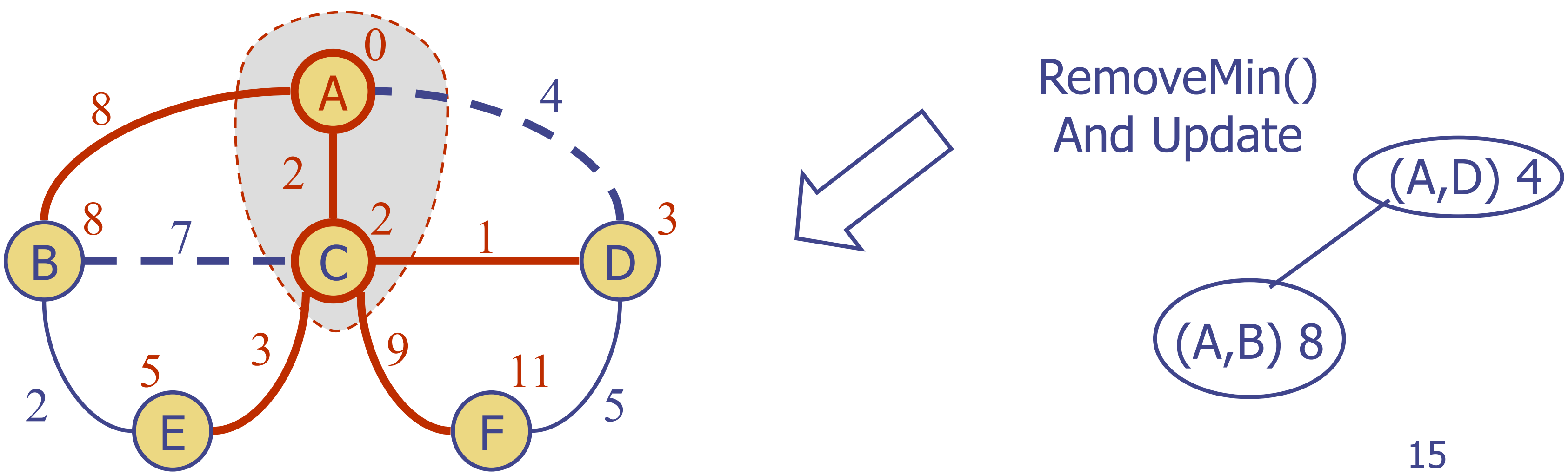
Output: A label D[u], for each vertex that u of G, such that D[u] is the length of a shortest path from v to u in G.

```
initialize D[v] ← 0 and D[u] ← ∞ for each
        vertex v ≠ u
let Q be a priority queue that contains all of the
        vertices of G using the D labels as keys.
while Q ≠ ∅ do {pull u into the cloud C}
        u ← Q.removeMinElement()
        for each vertex z adjacent to (out of) u such that z is in Q do
                {perform the relaxation operation on edge (u, z) }
                if D[u] + w((u, z)) < D[z] then
                        D[z] ←D[u] + w((u, z))
                        change the key value of z in Q to D[z]
return the label D[u] of each vertex u.
```

Same Example – Using heap



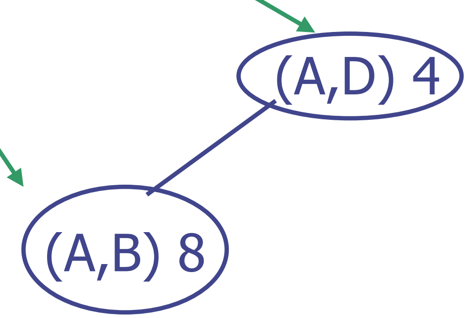
In the book: location-aware priority queue

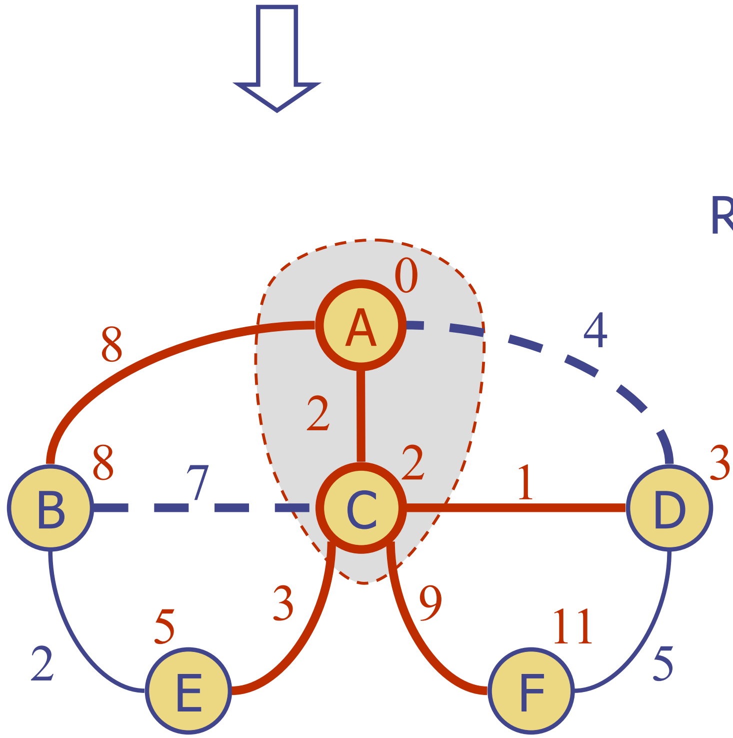


Update cont.

	A	B	C	D	E	F
D	~~0~~	8	~~2~~	4	∞	∞

RemoveMin()





Relaxation:

Update:

Neighbors of node C

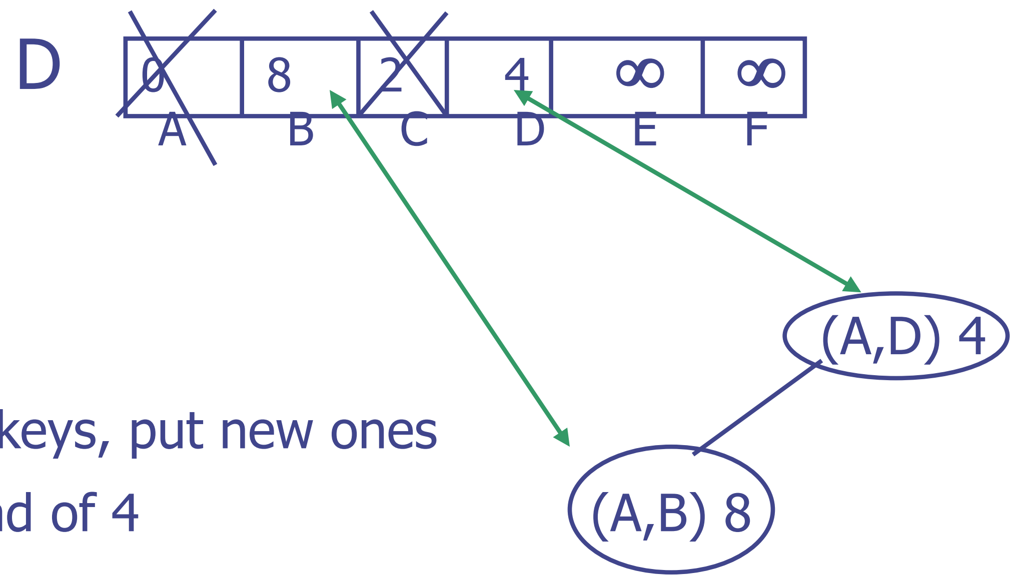
(C,D) 3 YES (3 < 4)

(C,E) 5 YES (5 < ∞)

(C,F) 11 YES (11 < ∞)

(C,B) 9 NO (9>8)

Update cont.



Update means: remove old keys, put new ones

(C,D) **3** Instead of 4

(C,E) 5 Instead of ∞

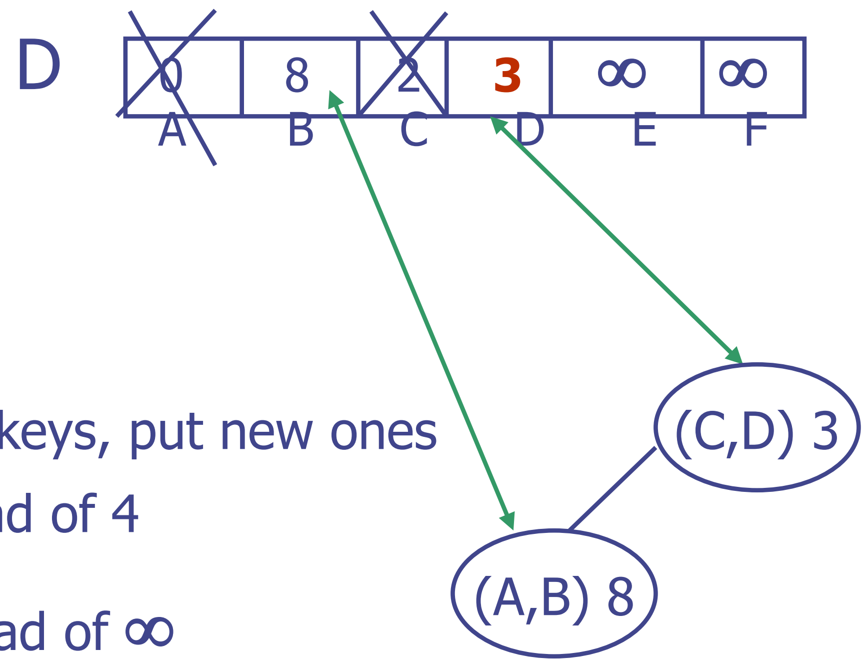
(C,F) 11 Instead of ∞

Replace (A,D) 4 with (C,D) 3

Insert (C,E) 5

Insert (C,F) 11

Update cont.



Update means: remove old keys, put new ones

(C,D) **3** Instead of 4

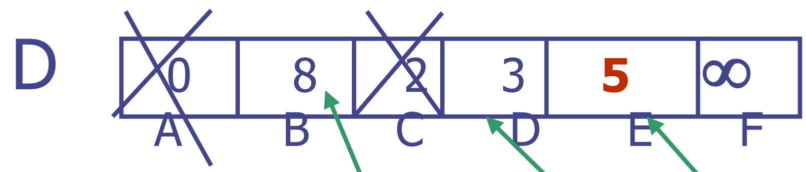
(C,E) 5 Instead of ∞

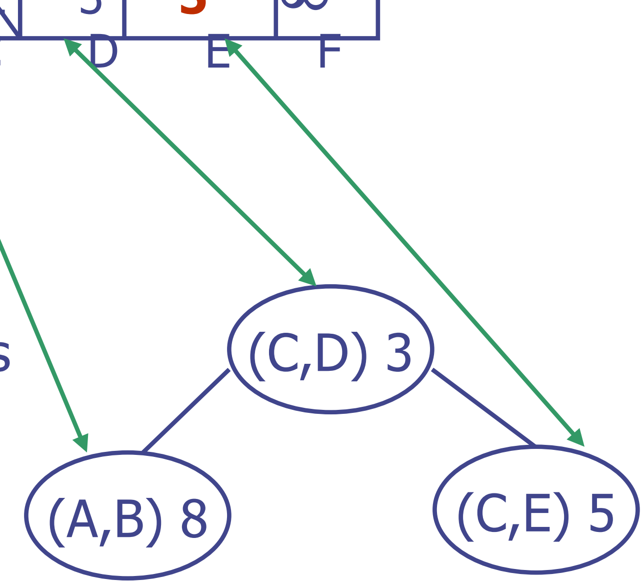
(C,F) 11 Instead of ∞

Replace (A,D) 4 with (C,D) 3
Insert (C,E) 5
Insert (C,F) 11

When replacing you might need to rearrange the heap (not in this example).

Update cont.





Update means: remove old keys, put new ones

(C,D) **3** Instead of 4

(C,E) 5 Instead of ∞

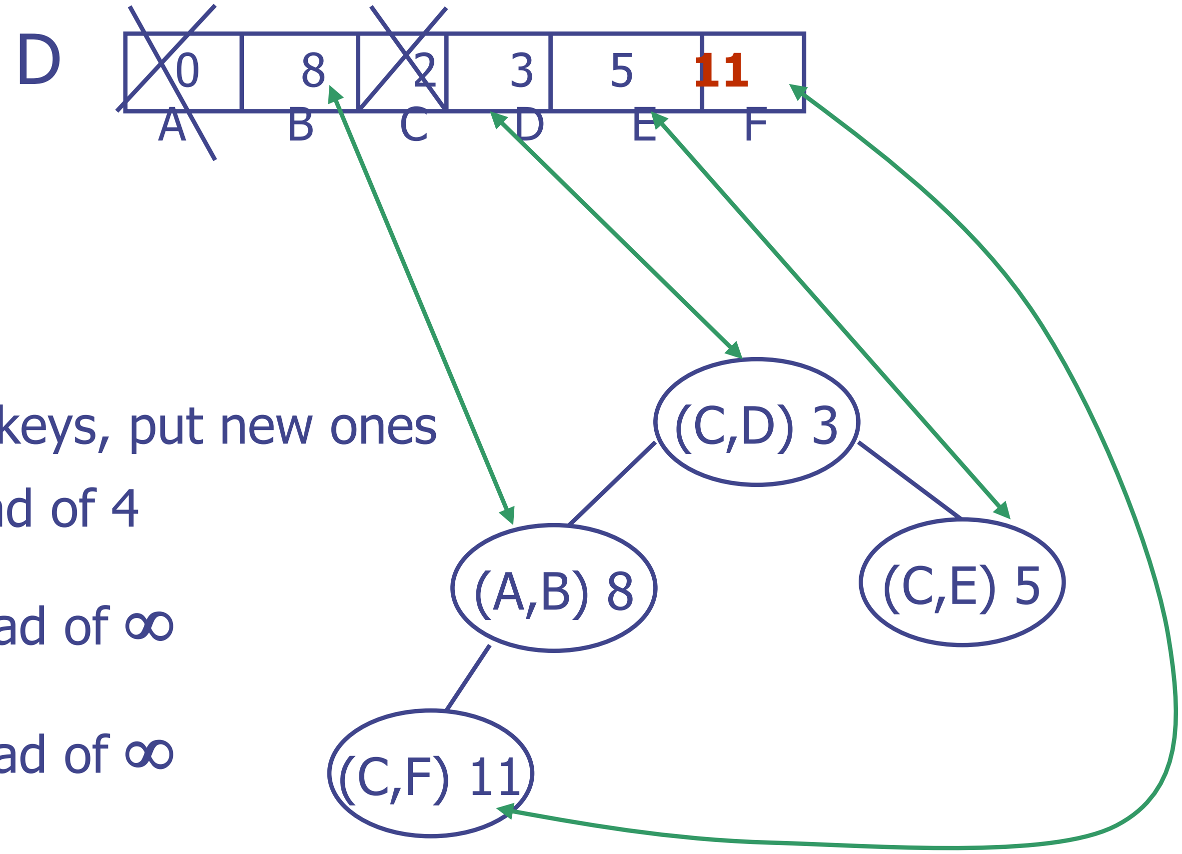
(C,F) 11 Instead of ∞

Replace (A,D) 4 with (C,D) 3

Insert (C,E) 5

Insert (C,F) 11

Update cont.



Update means: remove old keys, put new ones

(C,D) **3** Instead of 4

(C,E) 5 Instead of ∞

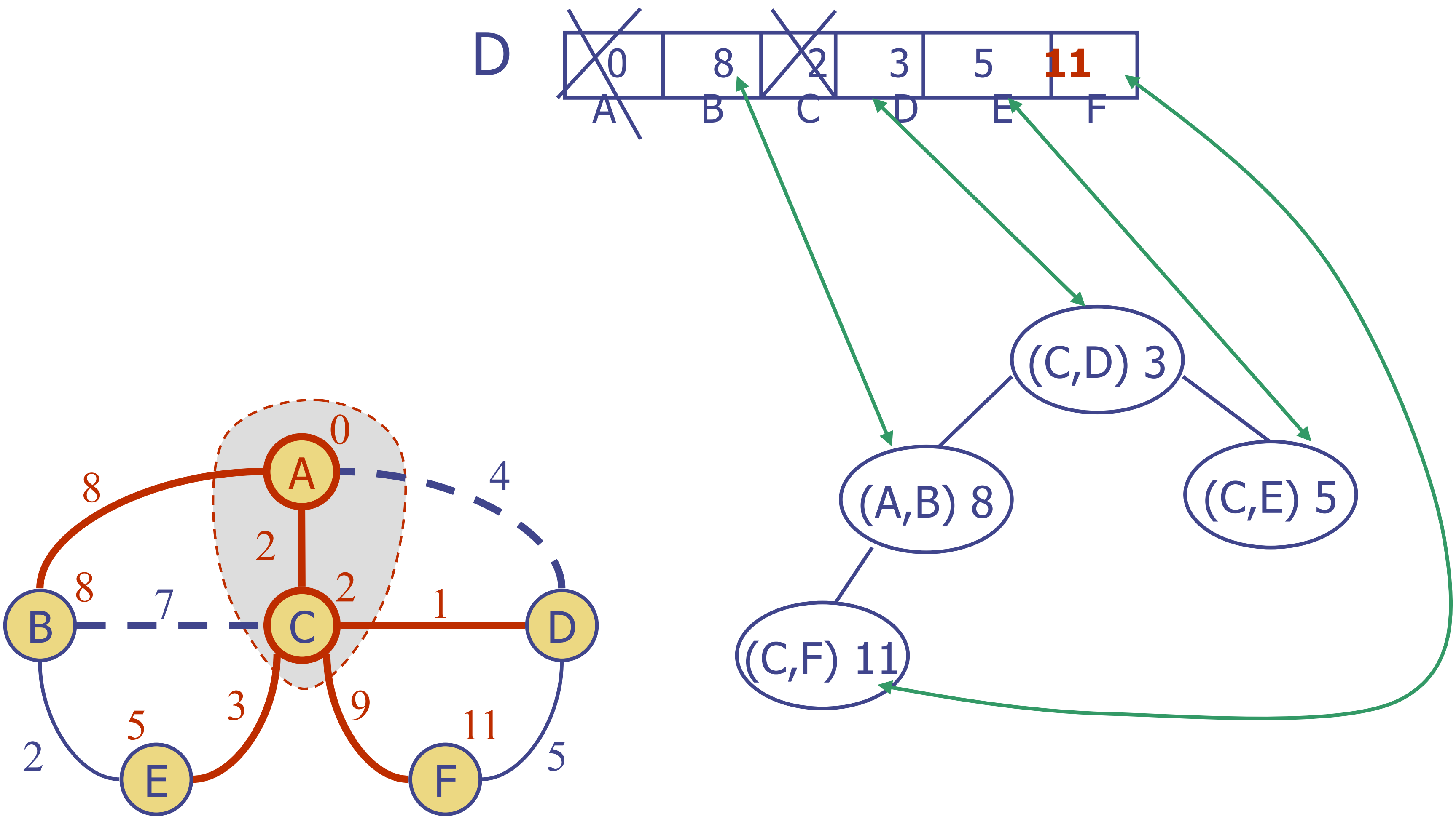
(C,F) 11 Instead of ∞

Replace (A,D) 4 with (C,D) 3

Insert (C,E) 5

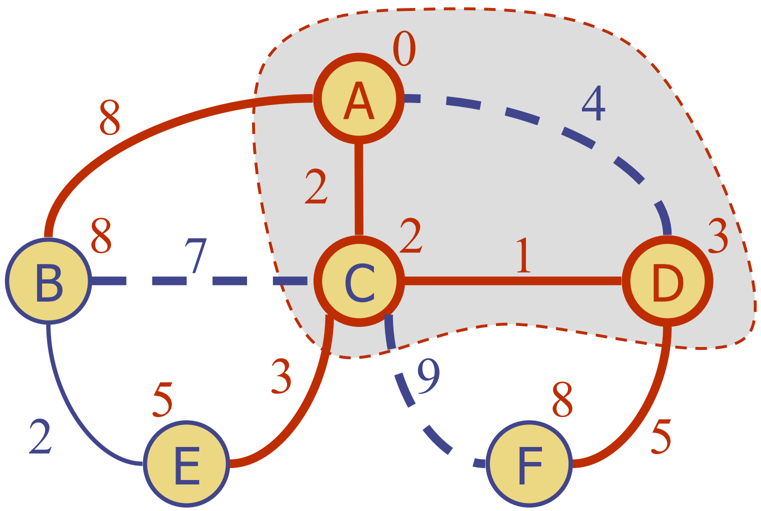
Insert (C,F) 11

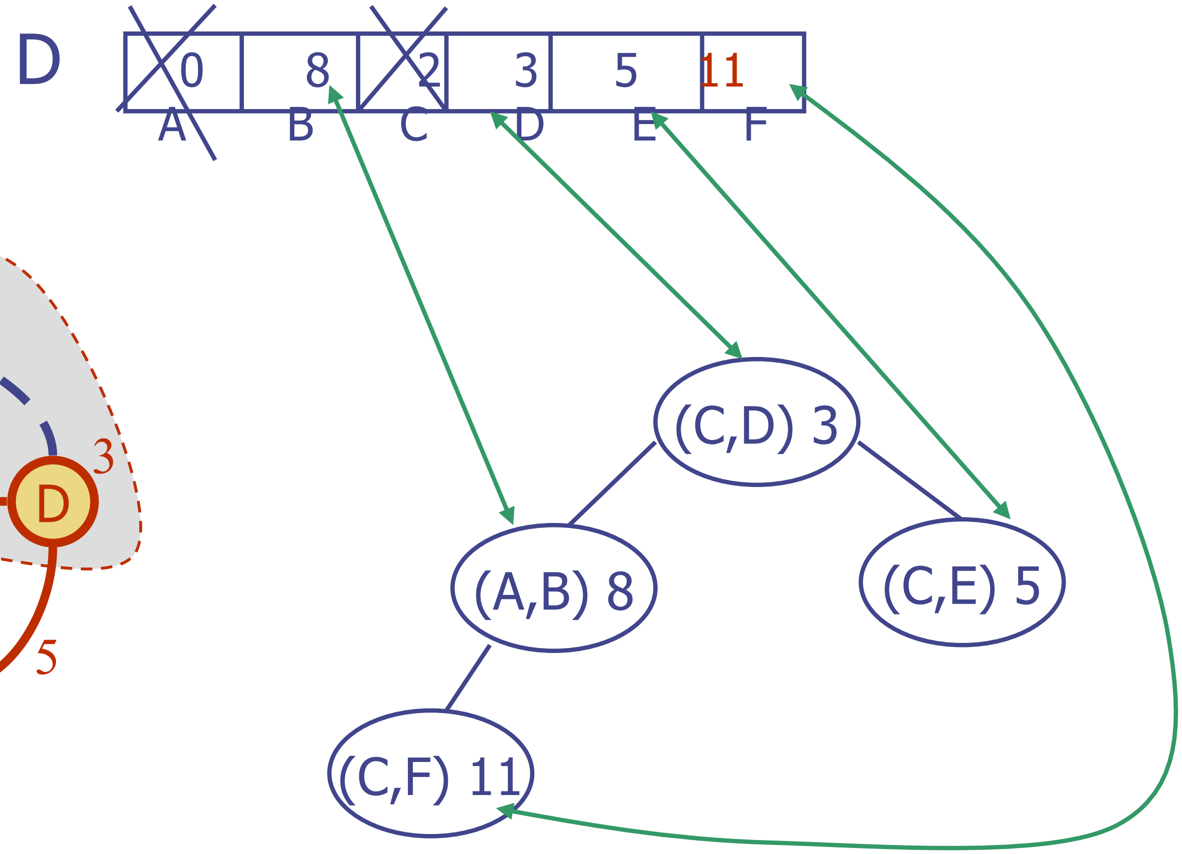
Now node C is added to the Cloud

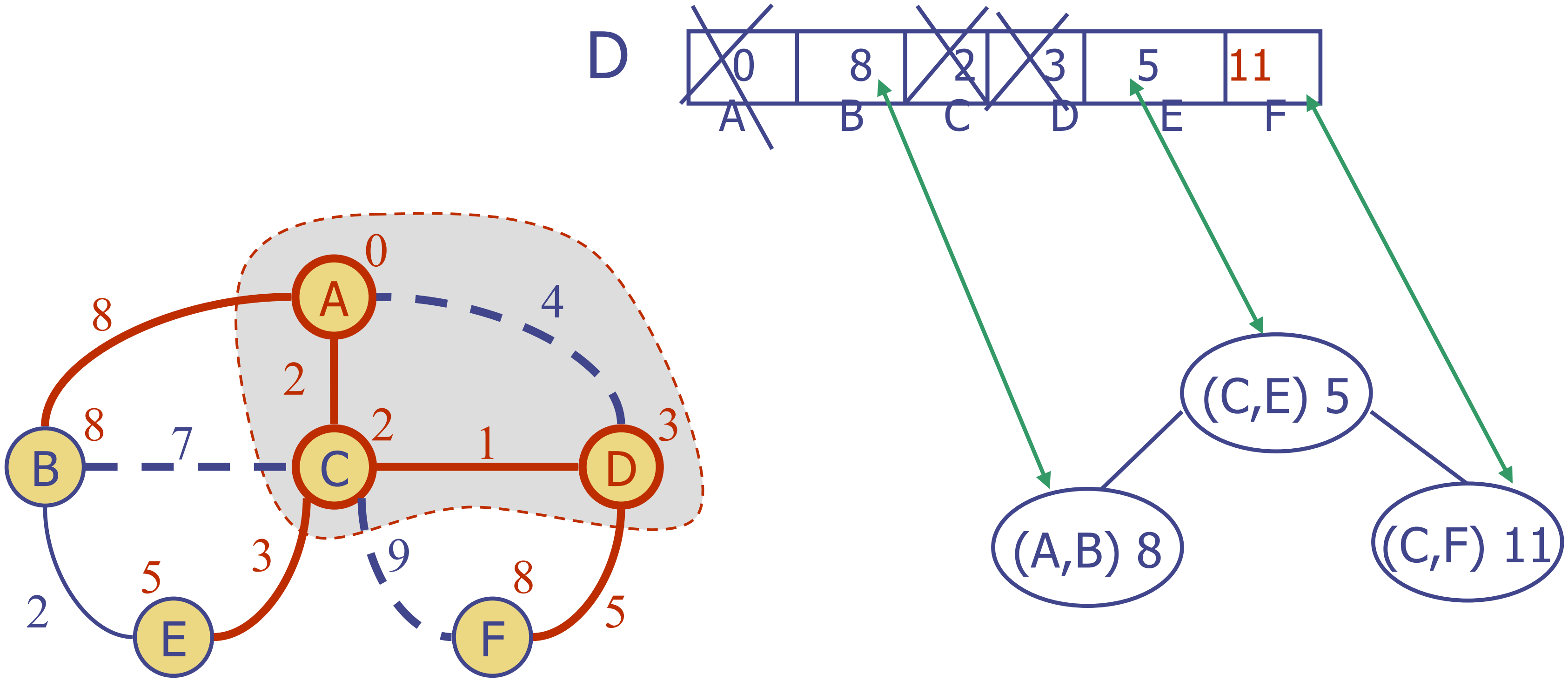


Next step ….

RemoveMin()
Update

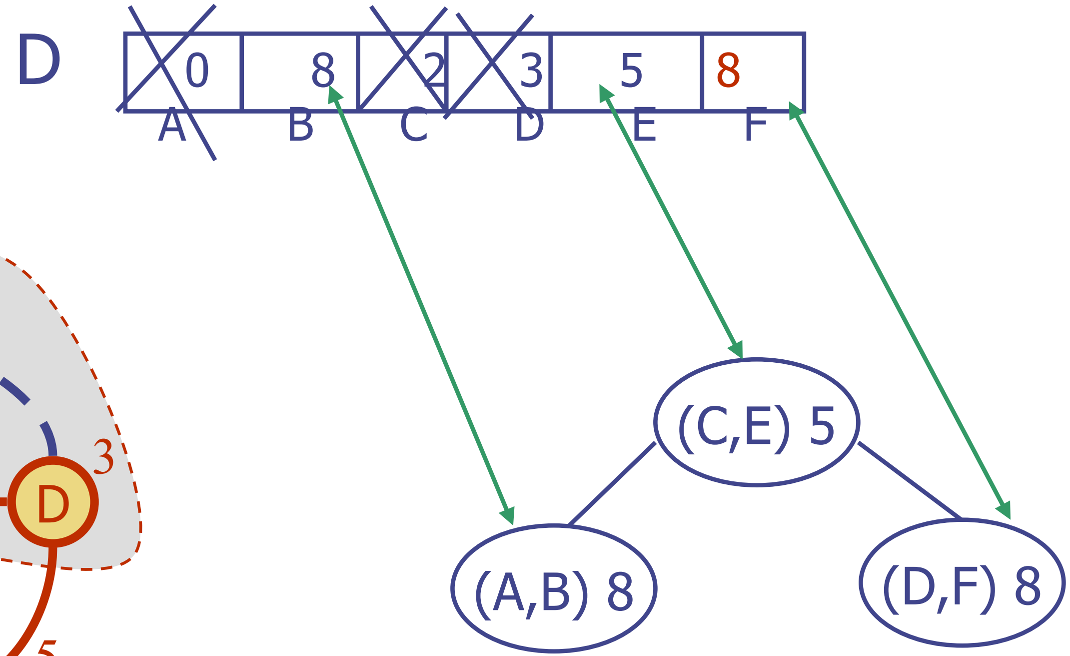


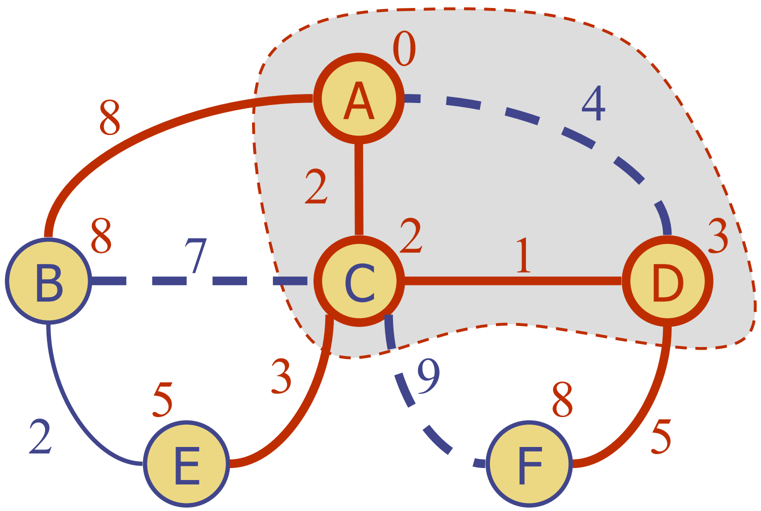




RemoveMin()
Update (D,F) 8 ?   Yes 8 < 11

Replace (C,F) 11
with (D,F) 8





RemoveMin()

Update (D,F) 8 ? Yes 8 < 11

Replace (C,F) 11
with (D,F) 8

while $Q \neq \emptyset$ do {pull u into the cloud C}

u ← Q.removeMinElement() → **O(log n)**

deg(u) of them

for each vertex z adjacent to u such that z is in Q do

{perform the relaxation operation on edge (u, z) }

if D[u] + w((u, z)) < D[z] then

D[z] ←D[u] + w((u, z))

change the key value of z in Q to D[z] → **O(log n)**

O(deg(u) log n)

$$\sum_{u \in G} (1 + deg(u)) \log n = O((n+m) \log n) = O(m \log n)$$

Running Time

If we represent G with an adjacency list. We can then step through all the vertices adjacent to u in time proportional to **deg(u)**

- <u>**The priority queue Q**</u>

  A Heap:

  while $Q \neq \emptyset$ do {pull u into the cloud C}

  at each iteration:

  - extraction of element with the smallest distance label: **O(log n)**.
  - key updates: **O(log n)** for each update (replace and insert keys).

  So, after each extraction: **O(deg(u) log n)**

  in total: $\sum_{u \in G} (1 + deg(u)) \log n = O((n+m) \log n) = O(m \log n)$

  worst case: $O(n^2 \log n)$

An Unsorted Sequence:

$O(n)$ when we extract minimum elements,

but fast key updates ($O(1)$).

There are only n-1 extractions and m updates.

The running time is $O(n^2+m) = \mathbf{O(n^2)}$

In conclusion:

Heap	Sequence
$O(m \log n)$	**$O(n^2)$**