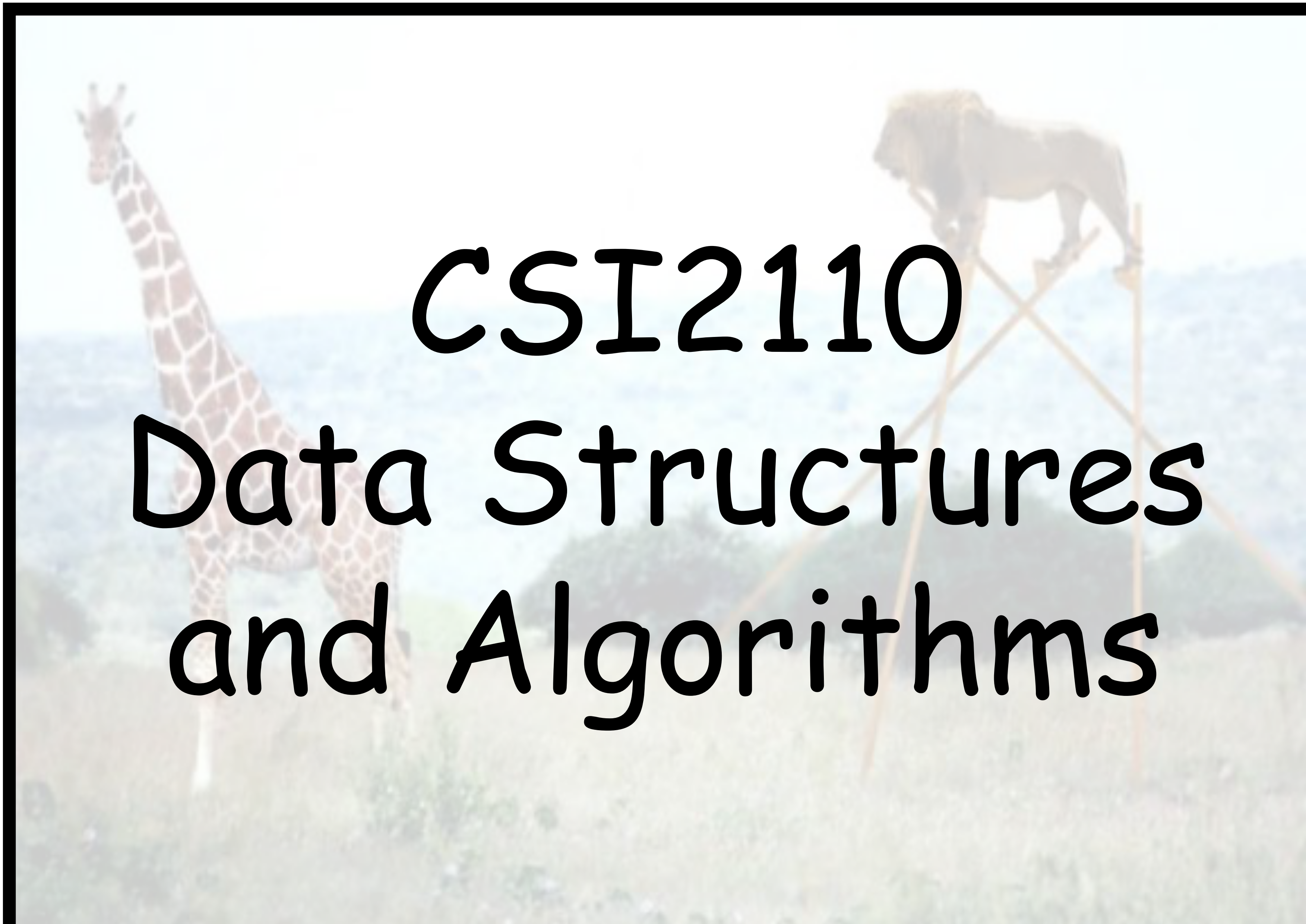
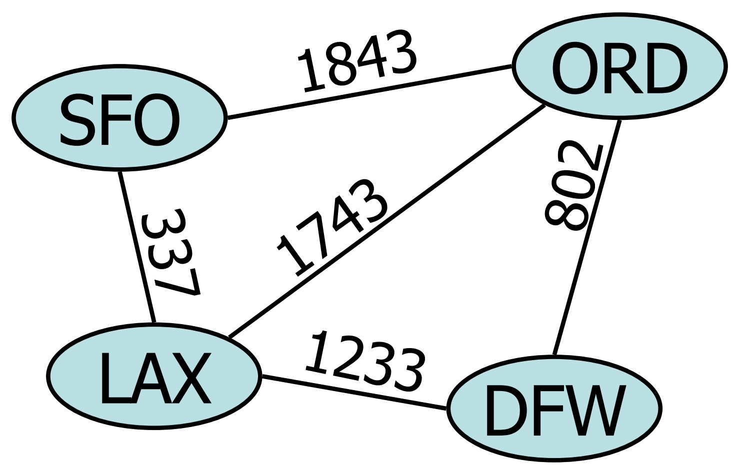


CSI2110
Data Structures and Algorithms

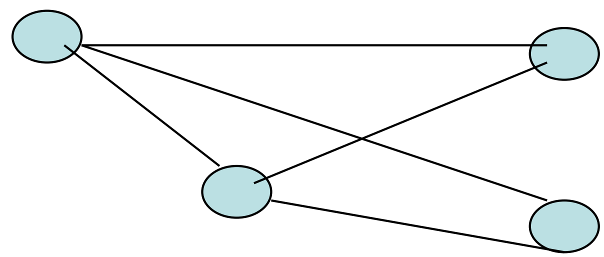
Graphs



Trees

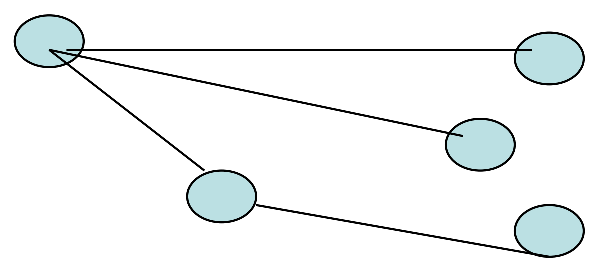
A graph $G = (V,E)$ consists of an set V of VERTICES

and a set E of edges, with $E = \{(u,v): u,v \in V, u \neq v$



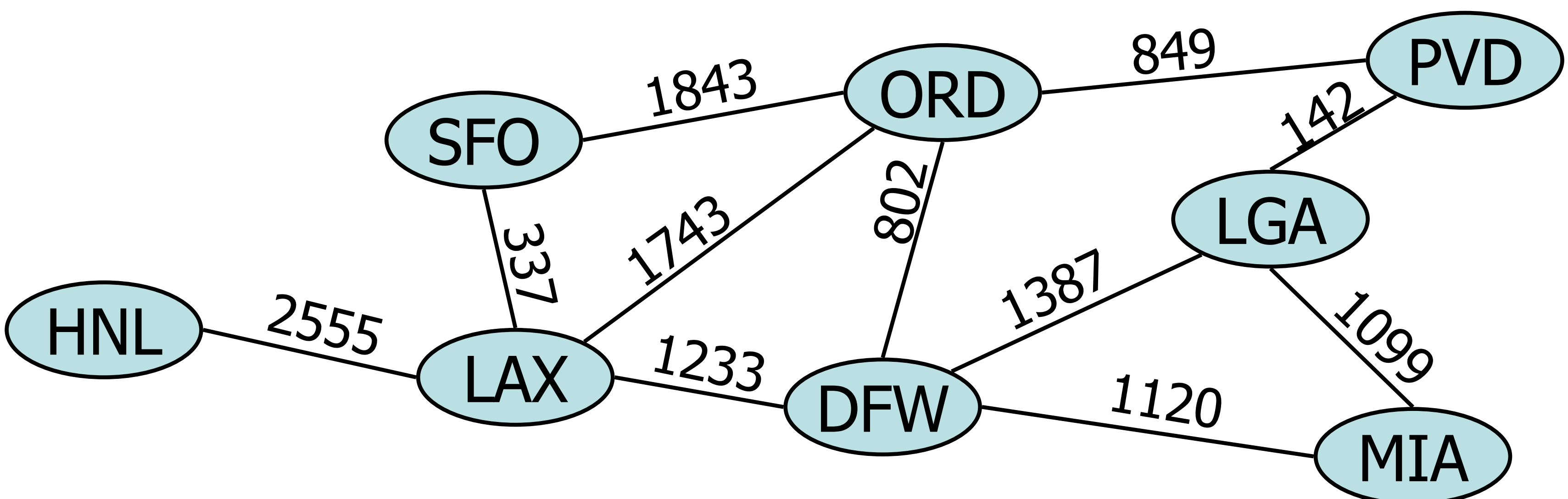
A tree is a connected graph with no cycles.

→ $\exists$ a path between each pair of vertices.



Graphs

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



Edge Types

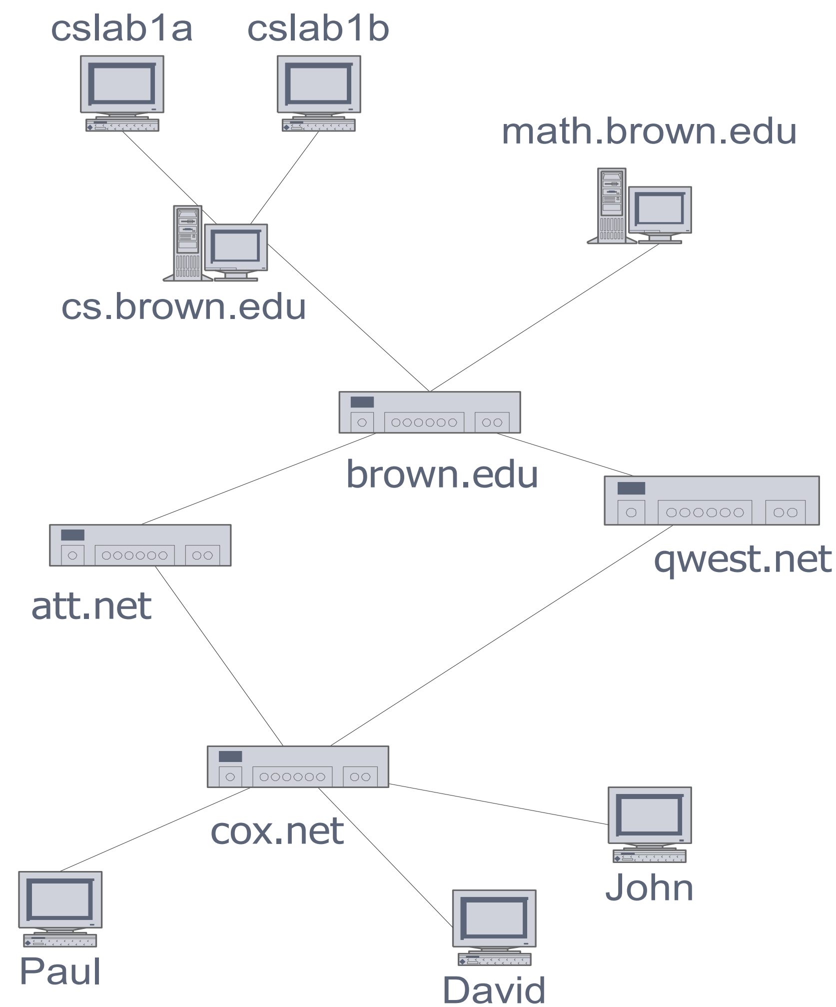
- Directed edge
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight

- Undirected edge
  - unordered pair of vertices (u,v)
  - e.g., a road

- Directed graph
  - all the edges are directed
  - e.g., flight network
- Undirected graph
  - all the edges are undirected
  - e.g., road map

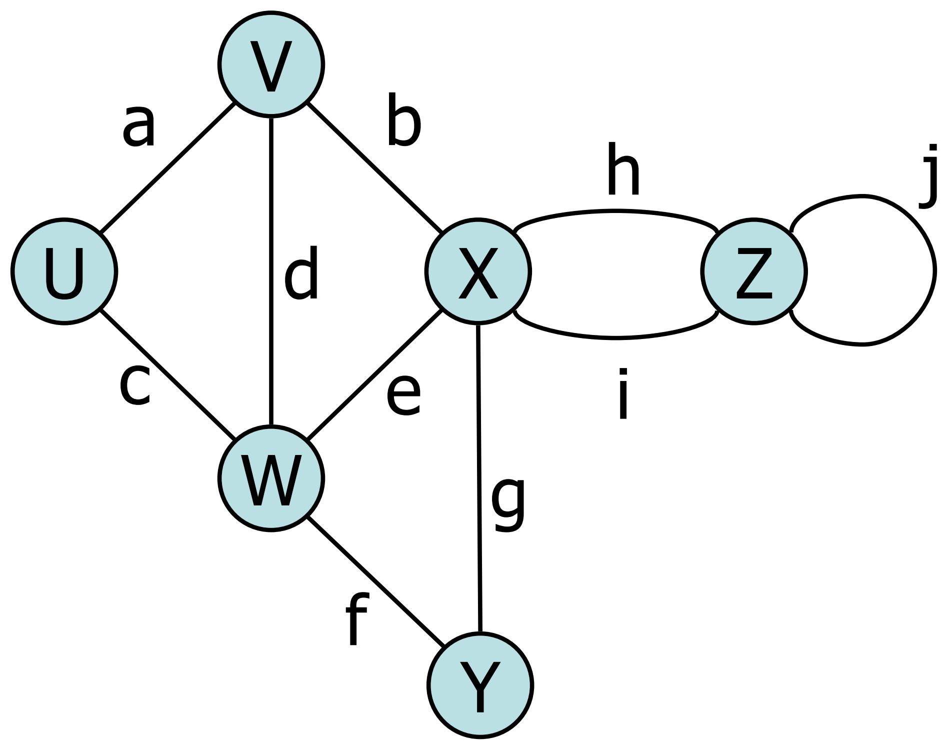
Applications

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



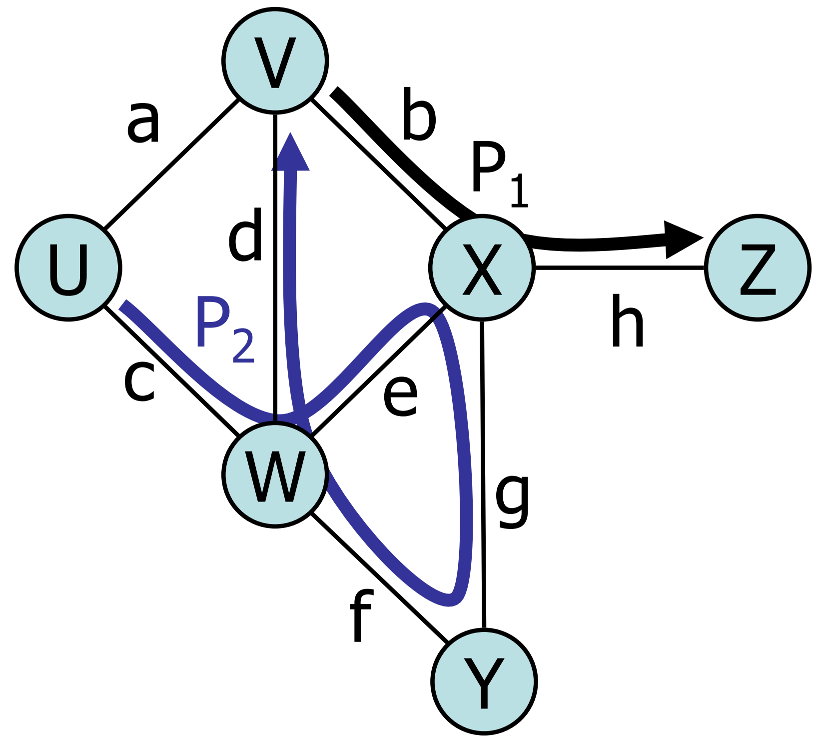
Terminology

- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- Degree of a vertex
  - X has degree 5
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop



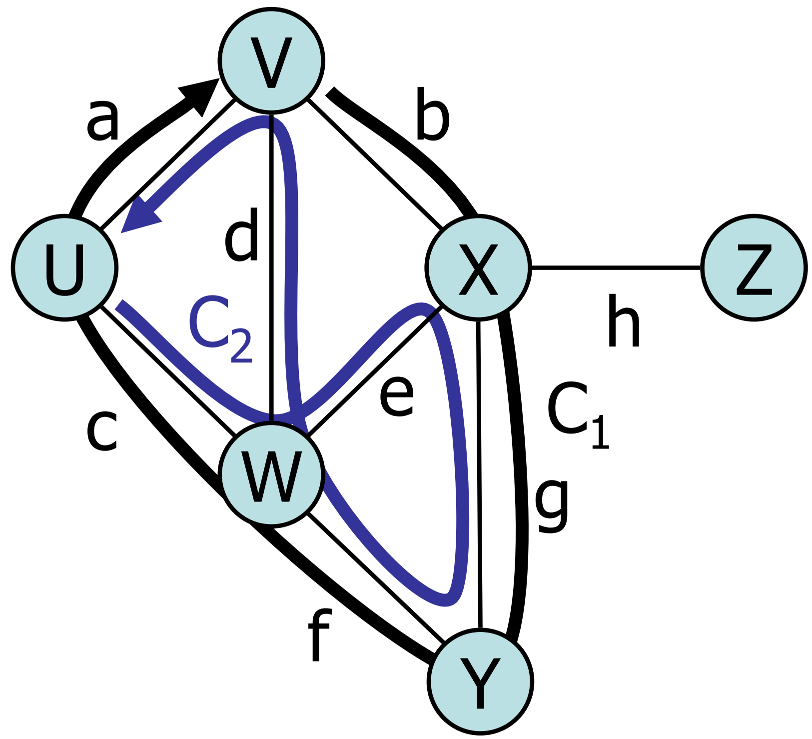
Terminology (cont.)

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



Terminology (cont.)

- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,↵) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,↵) is a cycle that is not simple



Properties

Property 1

$\Sigma_v \deg(\boldsymbol{v}) = 2\boldsymbol{m}$

Proof: each endpoint is counted twice

Property 2

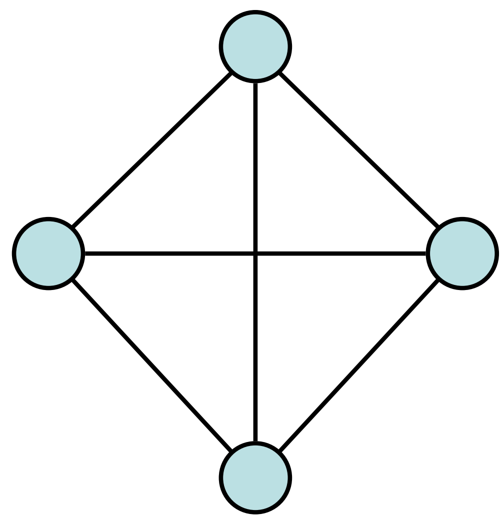
In an undirected graph with no self-loops and no multiple edges

$\boldsymbol{m} \leq \boldsymbol{n}\,(\boldsymbol{n} - 1)/2$

Proof: each vertex has degree at most $(\boldsymbol{n} - 1)$

Notation

$\boldsymbol{n}$	number of vertices
$\boldsymbol{m}$	number of edges
$\deg(\boldsymbol{v})$	degree of vertex $\boldsymbol{v}$

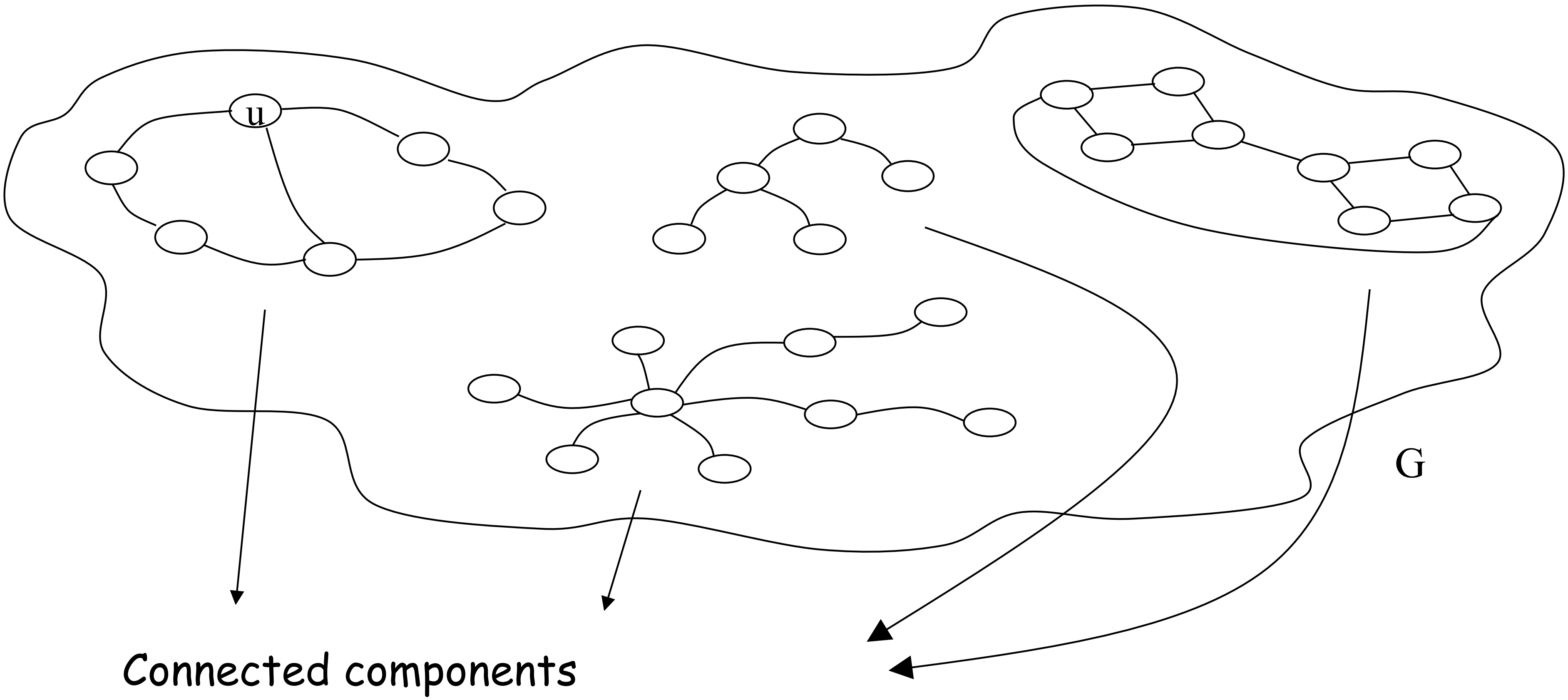


Example

- $\boldsymbol{n} = 4$
- $\boldsymbol{m} = 6$
- $\deg(\boldsymbol{v}) = 3$

Connected Graphs

A (non-directed) graph is connected if there exists a path $\forall$ u, v $\in$ V.



Main Methods of the Graph ADT

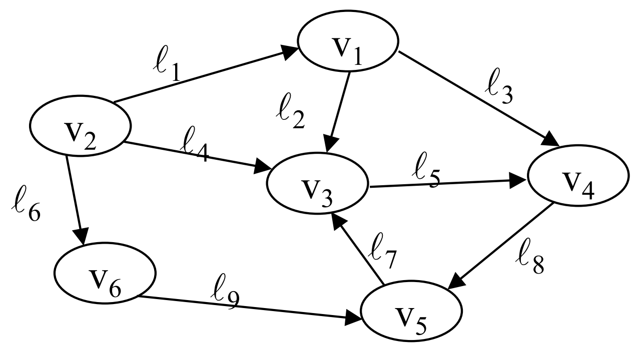
- Vertices and edges
  - are positions
  - store elements
- Accessor methods
  - aVertex()
  - incidentEdges(v)
  - endVertices(e)
  - isDirected(e)
  - origin(e)
  - destination(e)
  - opposite(v, e)
  - areAdjacent(v, w)
- Update methods
  - insertVertex(o)
  - insertEdge(v, w, o)
  - insertDirectedEdge(v, w, o)
  - removeVertex(v)
  - removeEdge(e)
- Generic methods
  - numVertices()
  - numEdges()
  - vertices()
  - edges()

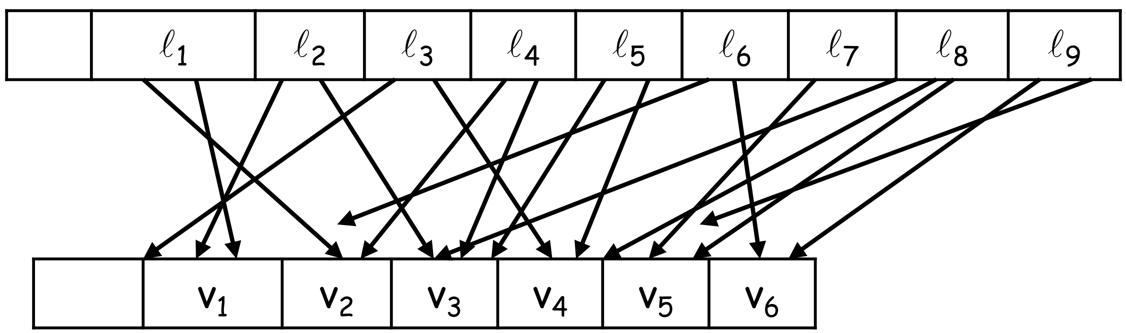
There could be other methods .....

Representations

Edge List
Adjacency List
Adjacency Matrix
Incidence Matrix

Edge List Structure (example)

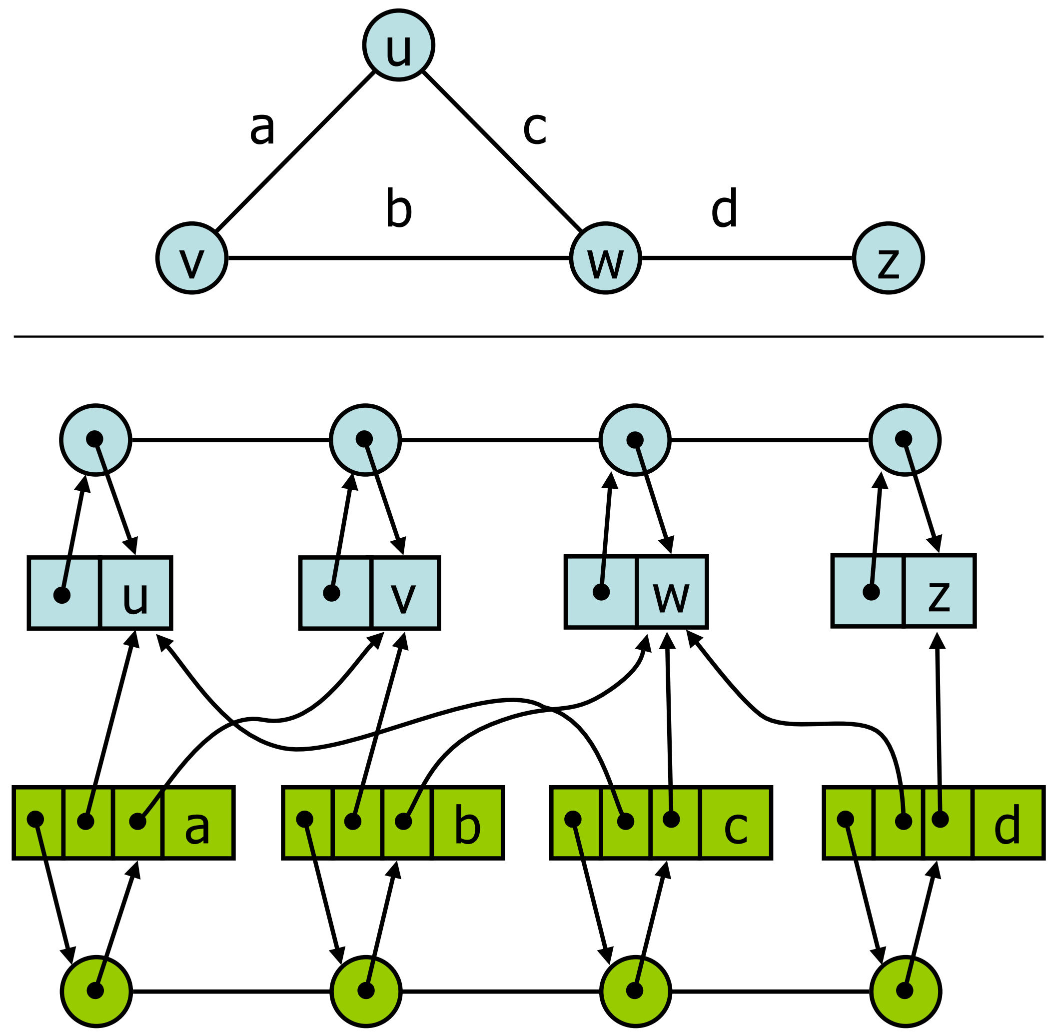




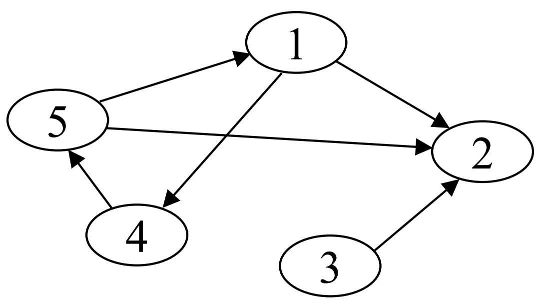
Space: $\boxed{n + m}$

Edge List Structure

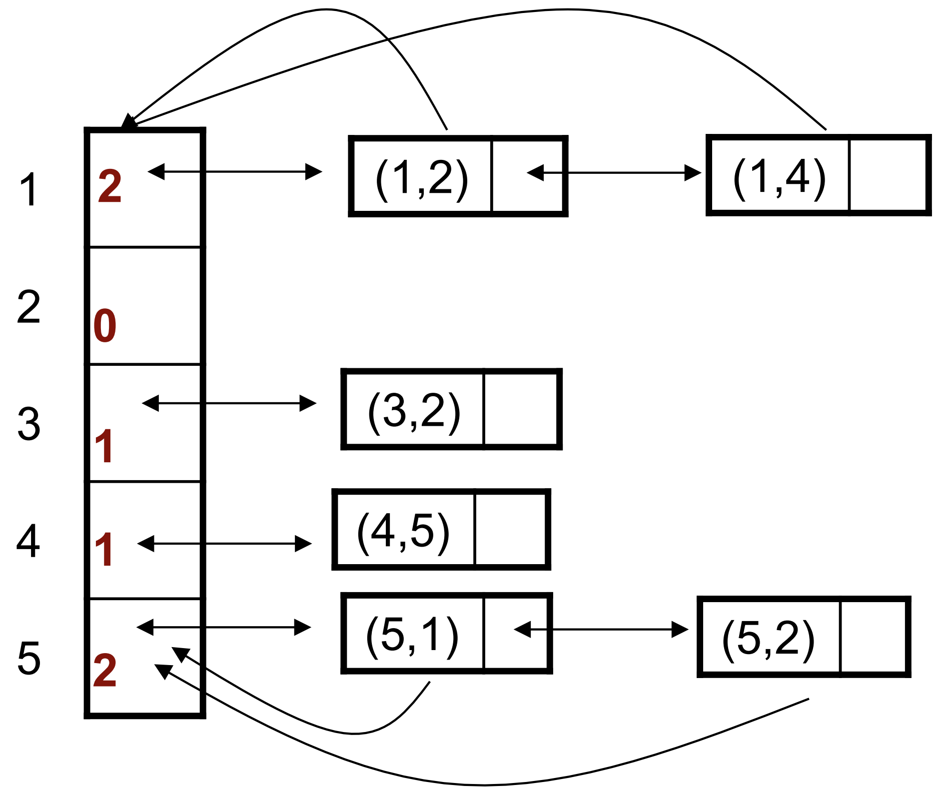
- Vertex object
  - element
  - reference to position in vertex sequence
- Edge object
  - element
  - origin vertex object
  - destination vertex object
  - reference to position in edge sequence
- Vertex sequence
  - sequence of vertex objects
- Edge sequence
  - sequence of edge objects



Adjacency List (example)

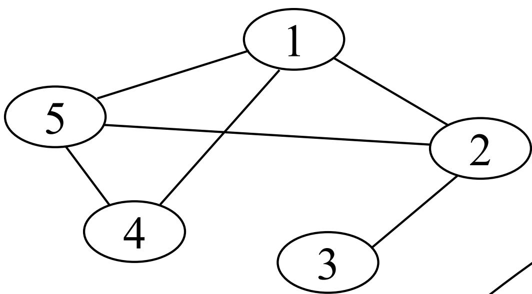


Often, the node out-degree is kept at the node.



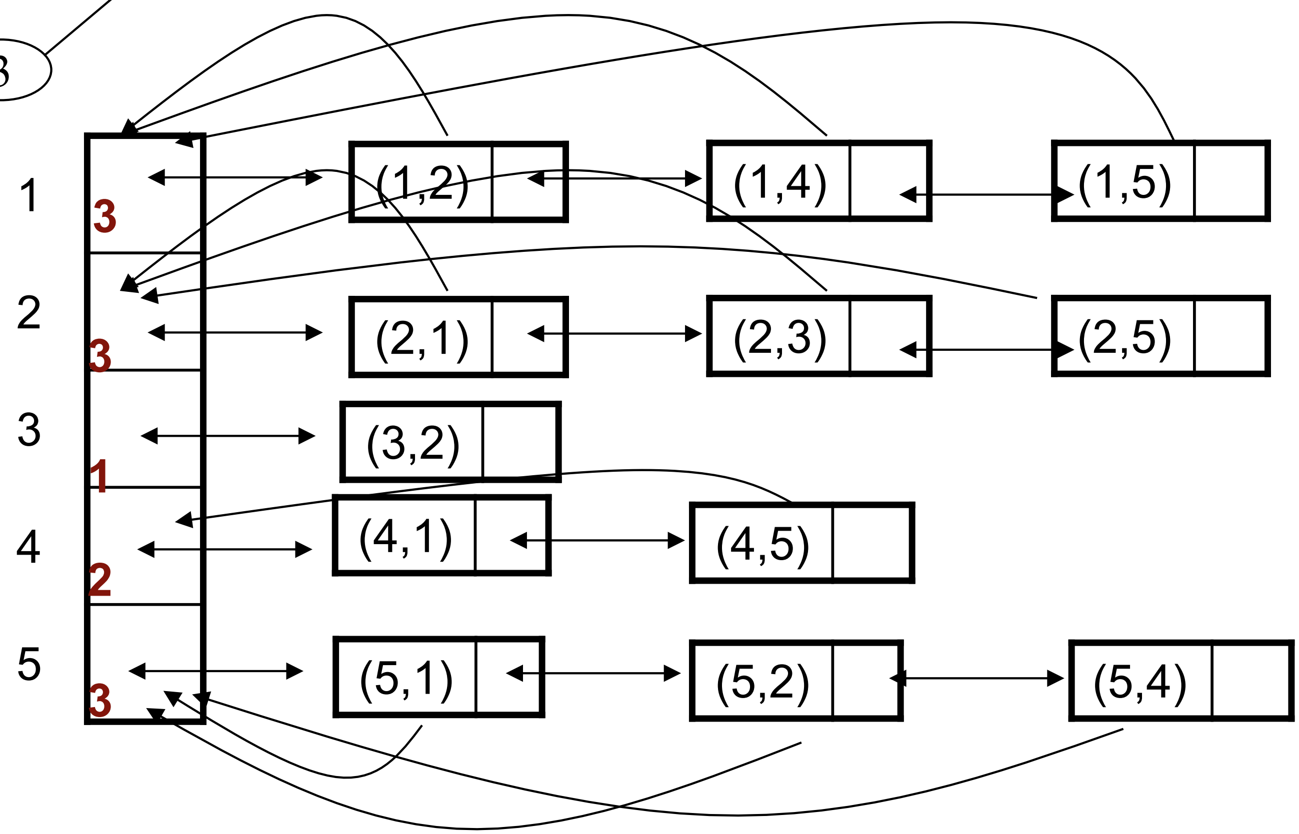
Edges augmented by link to node

Adjacency List
(another example -with undirected edges)

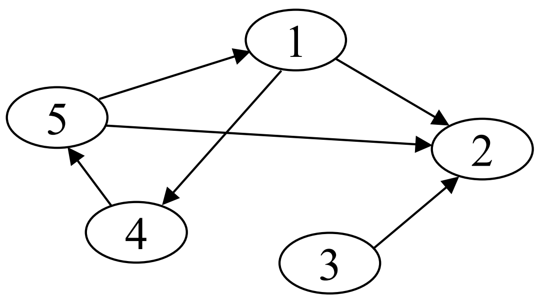


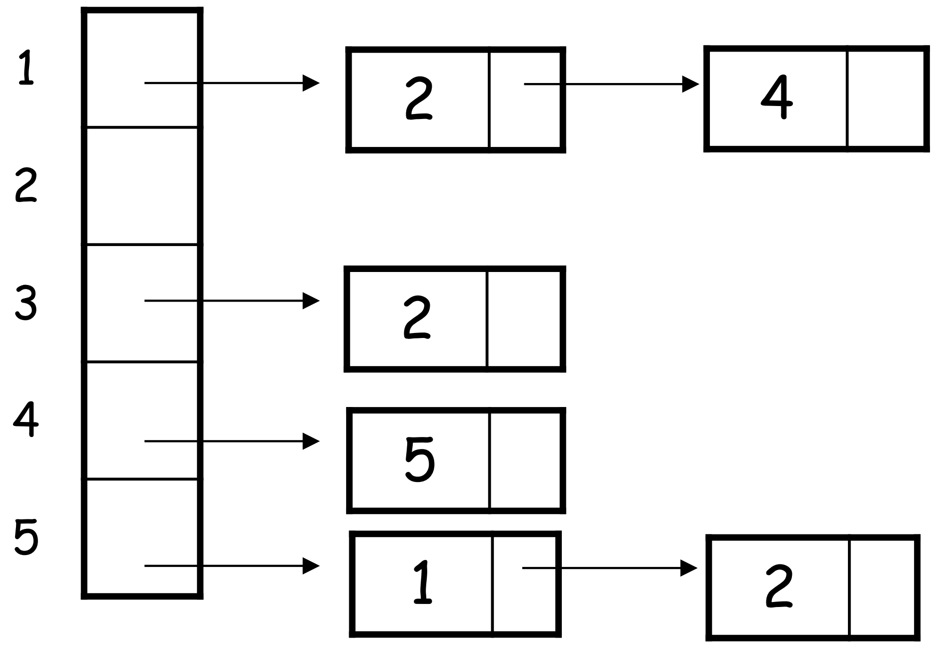
Edges augmented by link to node

degree is kept at the node



Adjacency List (example)

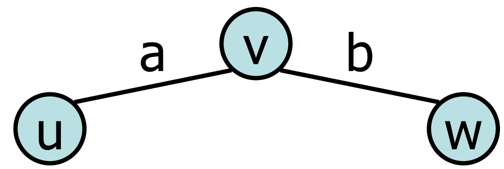


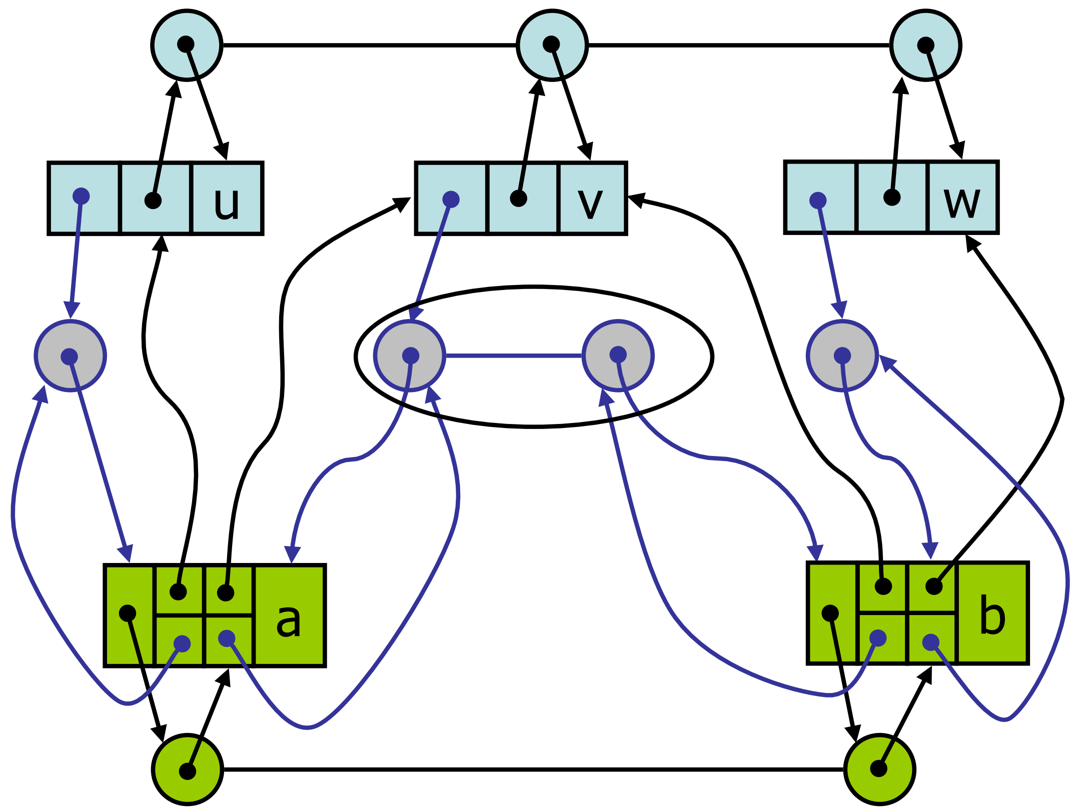


or

1	2	2,4
2	0	0
3	1	2
4	1	5
5	2	1,2

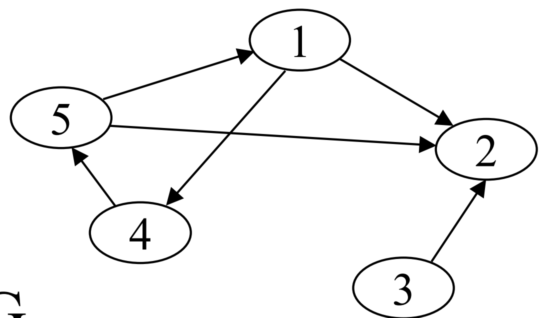
Adjacency List Structure





- Edge list structure
- Incidence sequence for each vertex
  - sequence of references to edge objects of incident edges
- Augmented edge objects
  - references to associated positions in incidence sequences of end vertices

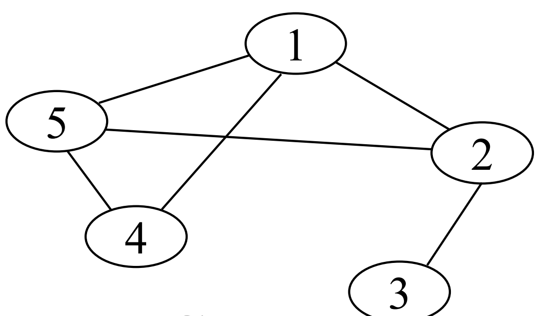
Adjacency Matrix (example)



G

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 1 \\
5 & 1 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

If G is non-directed



G

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 1 & 0 & 1 \\
3 & 0 & 1 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 1 \\
5 & 1 & 1 & 0 & 1 & 0 \\
\hline
\end{array}
$$

symmetric matrix

Adjacency Matrix (observation)

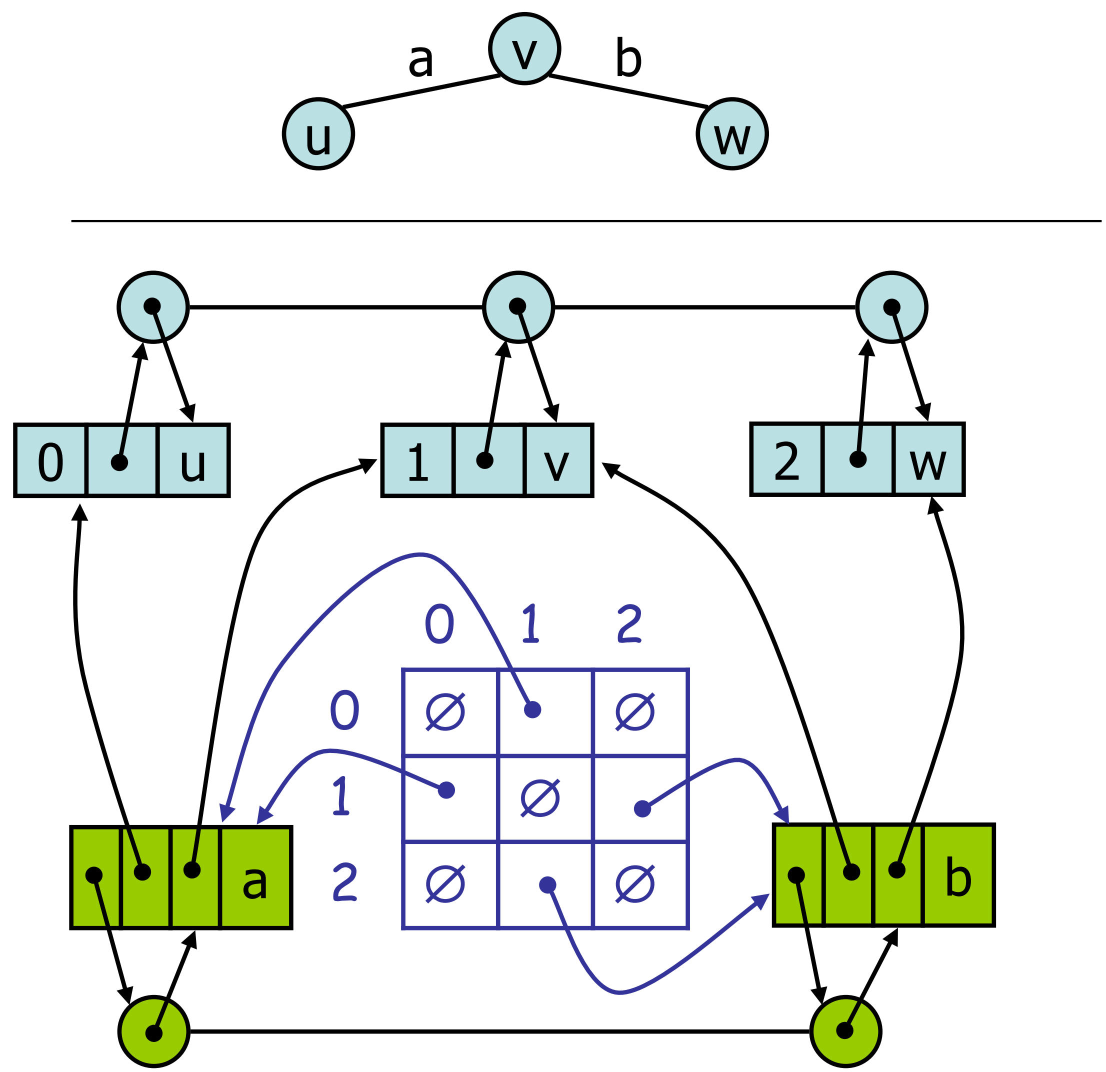
Space: $\boxed{n \times n}$

Lots of waste space if the matrix is SPARSE …

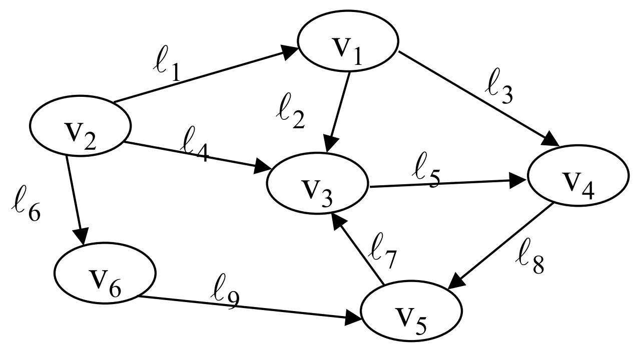
$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Adjacency Matrix Structure

- Edge list structure
- Augmented vertex objects
  - Integer key (index) associated with vertex
- 2D-array adjacency array
  - Reference to edge object for adjacent vertices
  - Null for non nonadjacent vertices



Incidence Matrix (1)

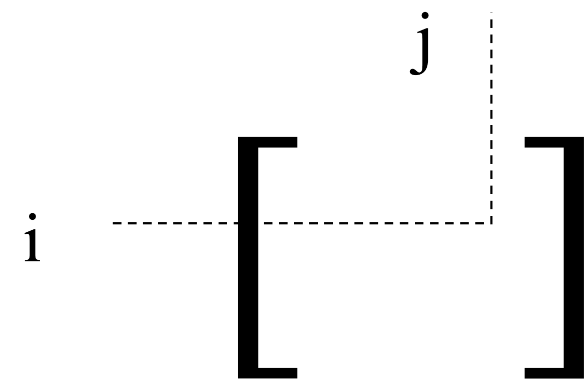


	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9
v_1	-1	1	1	0	0	0	0	0	0
v_2	1	0	0	1	0	1	0	0	0
v_3	0	-1	0	-1	1	0	-1	0	0
v_4	0	0	-1	0	-1	0	0	1	0
v_5	0	0	0	0	0	0	1	-1	-1
v_6	0	0	0	0	0	-1	0	0	1

Space: $\boxed{n \times m}$

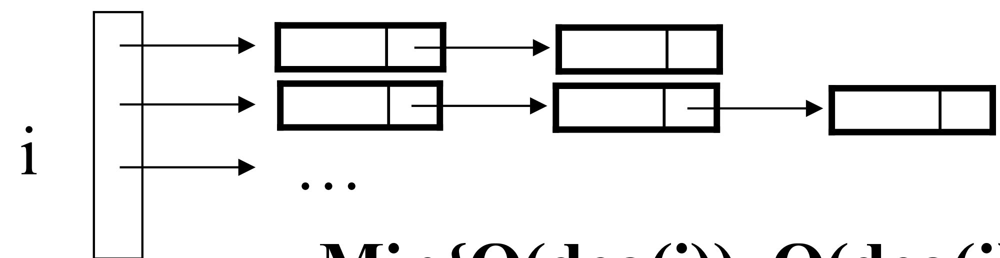
Is (v_i, v_j) an edge?

Adjacency Matrix:



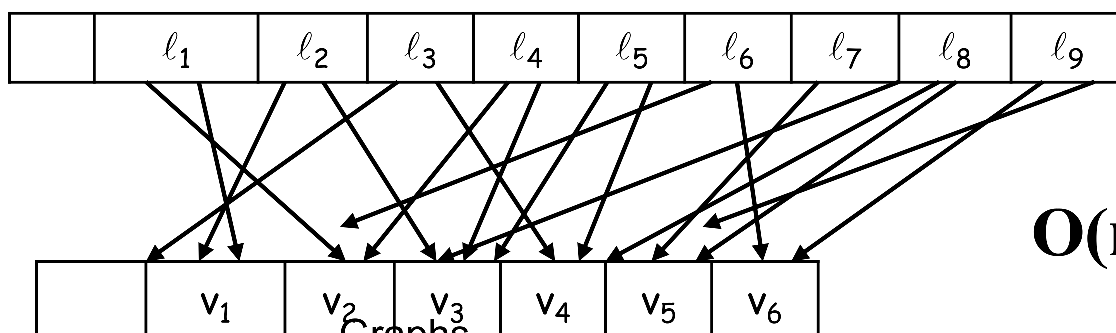
O(1)

Adjacency List:



Min{O(deg(i)), O(deg(j))}

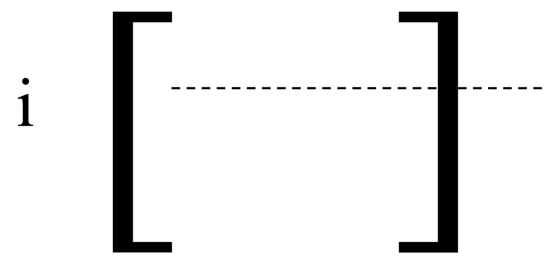
Edge List:



O(m)

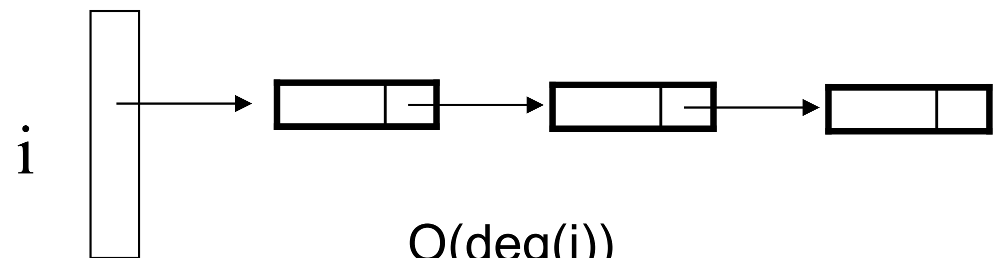
Which nodes are adjacent to v_i?

Adjacency Matrix:



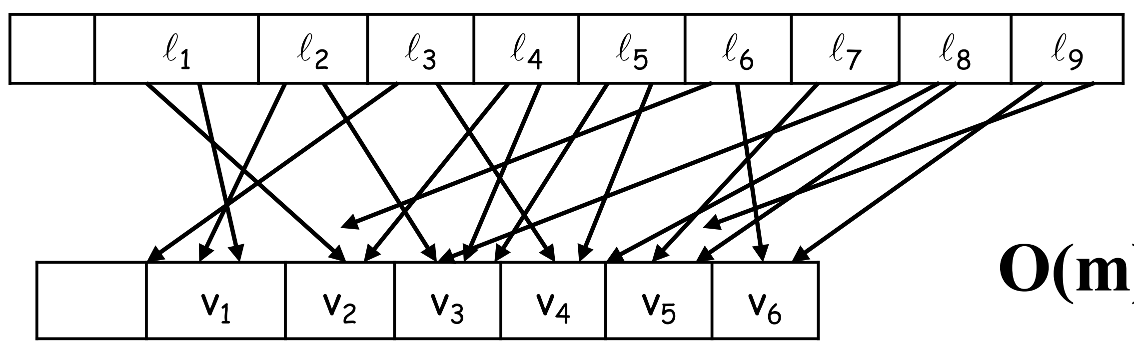
O(n)

Adjacency List:



O(deg(i))

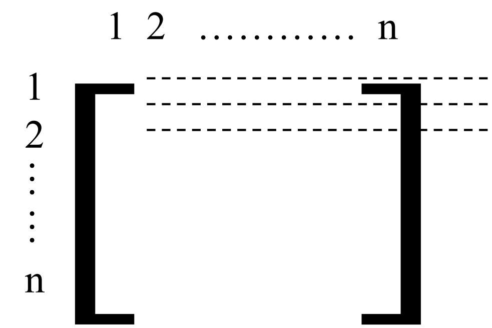
Edge List:



O(m)

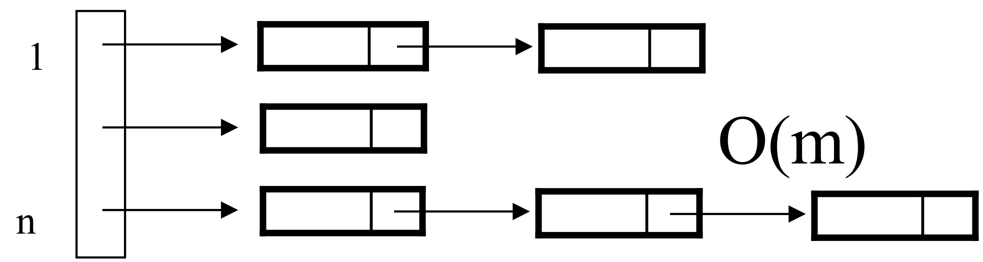
Mark all Edges

Adjacency Matrix:



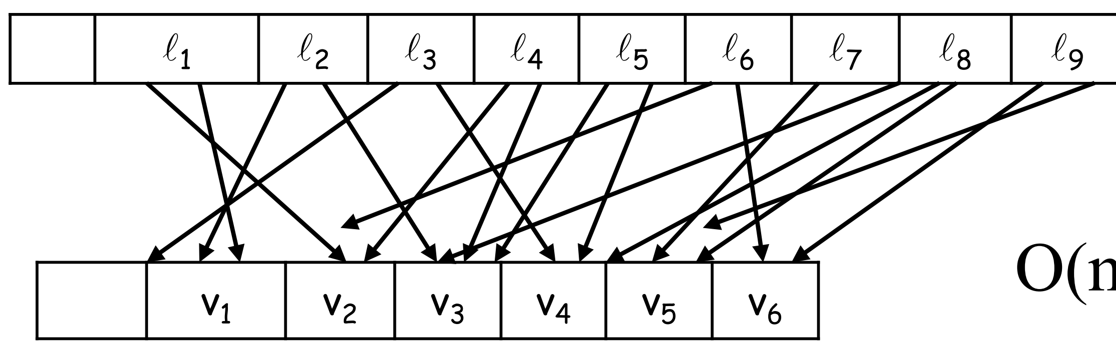
$O(n^2)$

Adjacency List:



$O(m)$

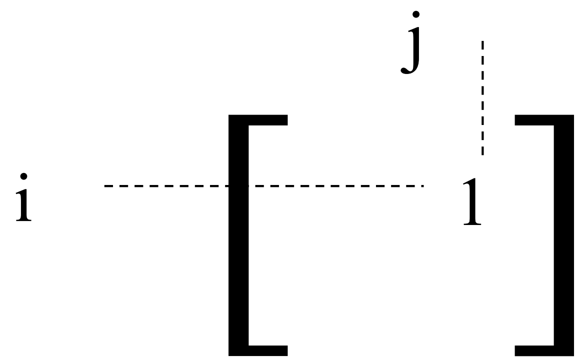
Edge List



$O(m)$

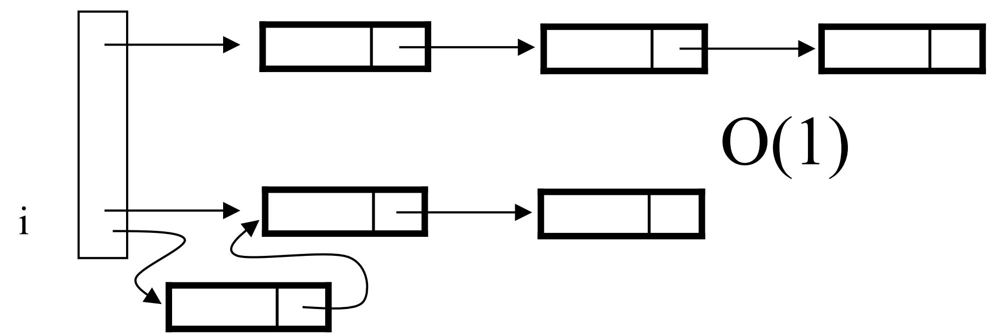
Add an Edge (v_i, v_j)

Adjacency Matrix:



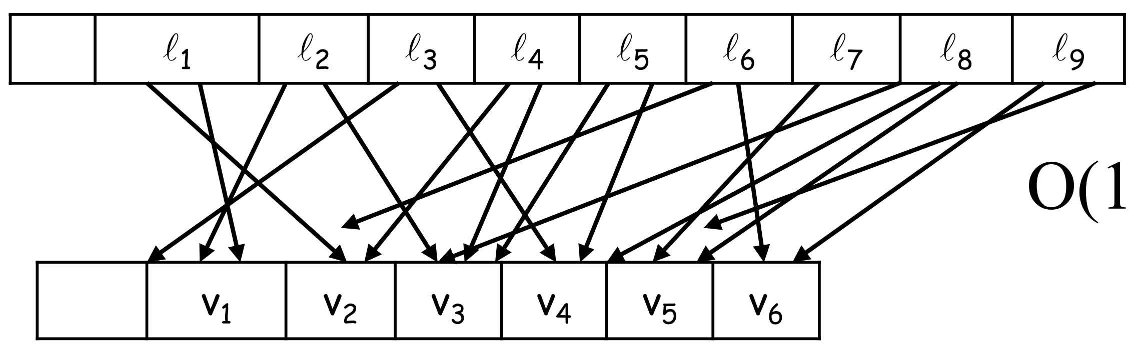
O(1)

Adjacency List :



O(1)

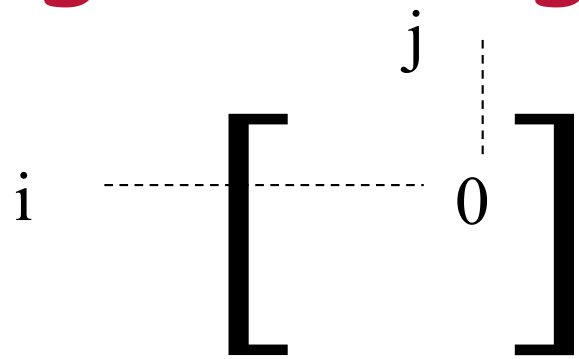
Edge List:



O(1)

Remove a given Edge $e=(v_i, v_j)$

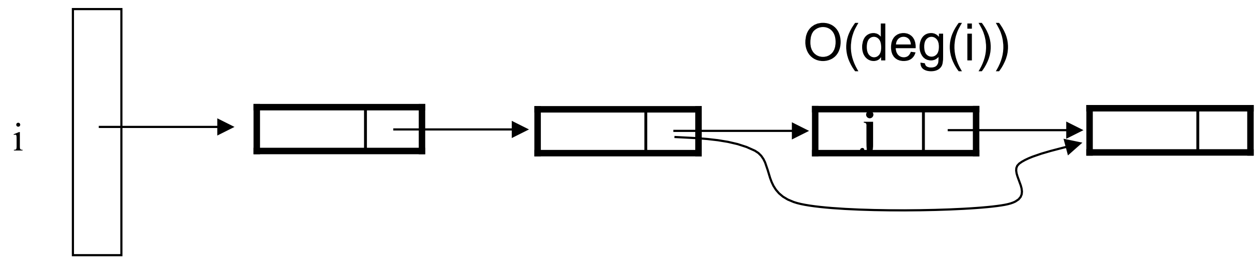
Adjacency Matrix:



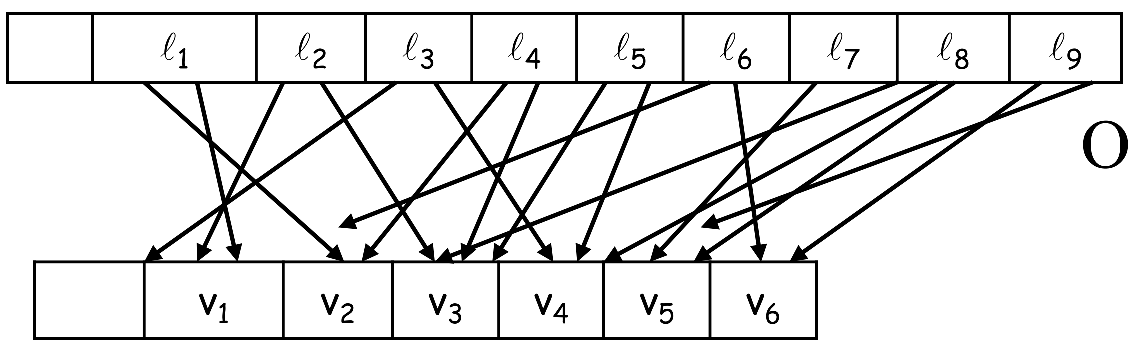
O(1)

O(deg(i))

Adjacency List:



Edge List



O(1)

	Adjacency Matrix	Adjacency List
Is (v_i, v_j) an edge?	O(1)	O(deg(i))
Which nodes are adjacent to v_i?	O(n)	O(deg(i))
Mark all edges	$O(n^2)$	O(m)
Add edge (v_i, v_j)	O(1)	O(1)
Remove edge (v_i, v_j)	O(1)	O(1)

O(deg(i)) = OUT-degree of node vi

G is directed

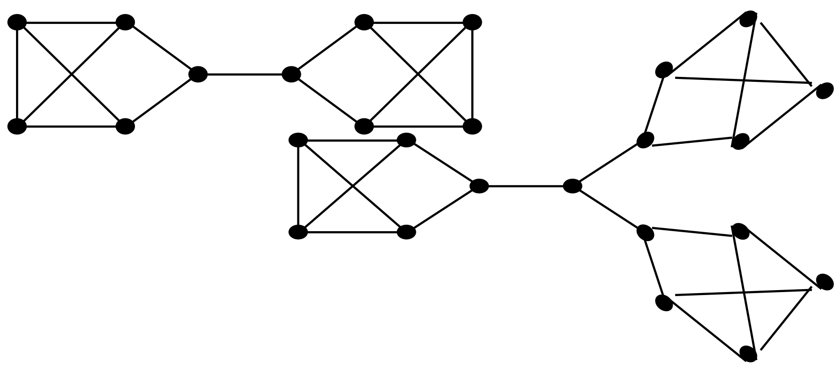
What are the predecessors of v_i?

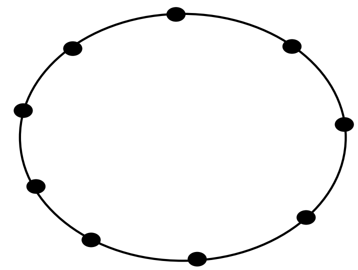
Performance

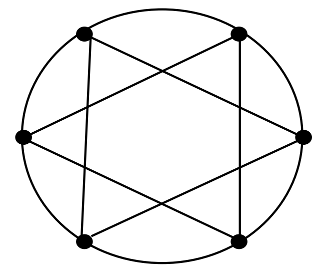
• n vertices • m edges • no parallel edges • no self-loops	Edge List	Adjacency List	Adjacency Matrix
Space	$n + m$	$n + m$	n^2
incidentEdges(v)	m	deg(v)	n
areAdjacent (v, w)	m	min(deg(v), deg(w))	1
insertVertex(o)	1	1	n^2
insertEdge(v, w, o)	1	1	1
removeVertex(v)	m	deg(v)	n^2
removeEdge(e)	1	O(deg(i))	1

Special Graphs

Regular Graphs

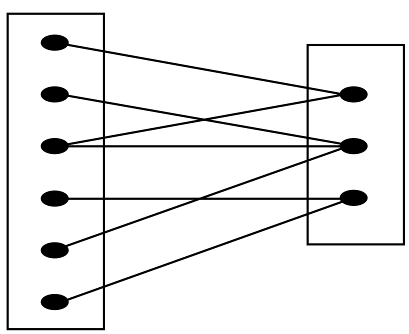




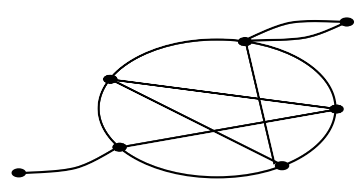


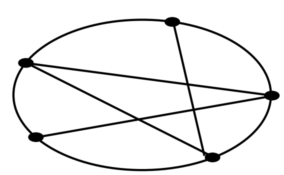
$\forall v_i, v_j \in V \qquad d(v_i) = d(v_j)$

Bipartite Graphs

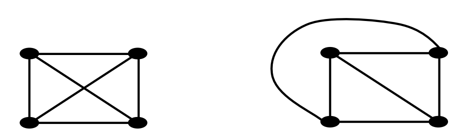


Planar Graphs





Cannot have



$$\boxed{\begin{array}{c} n - 1 \le m \le \dfrac{n(n-1)}{2} \\ 1 \le d_i \le n - 1 \end{array}}$$

connected,
non-directed

degree

$n = |V| \qquad m = |E|$

$$\boxed{\begin{array}{c} n - 1 \le m \le n(n - 1) \\ 1 \le d_i \le n - 1 \end{array}}$$

connected,
directed

OUT-degree

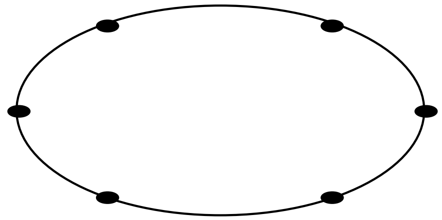
Special Graphs

Biconnected graph: A *connected graph* that is not broken into disconnected pieces by deleting any single *vertex* (and incident *edges*).

Any graph containing a node of degree 1 cannot be biconnected

Some Regular Graphs

— Ring —



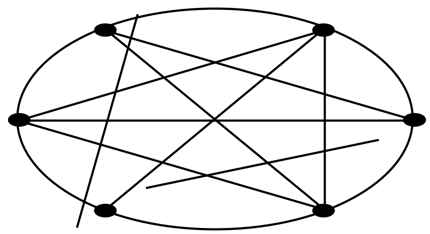
$m = n$

$d_i = 2 \quad \forall i$

Tree

$m = O(n)$

— Complete Graph —



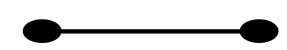
$$m = \frac{n(n-1)}{2}$$

$d_i = n - 1 \quad \forall i$

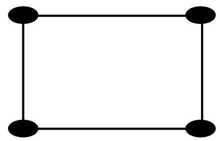
$m = O(n^2)$

— Hypercube —

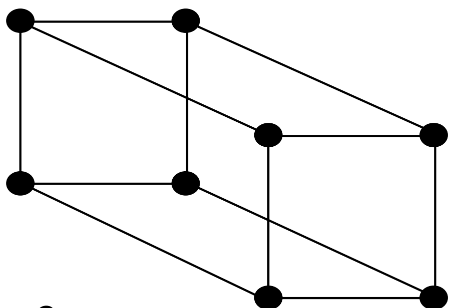
dim 0



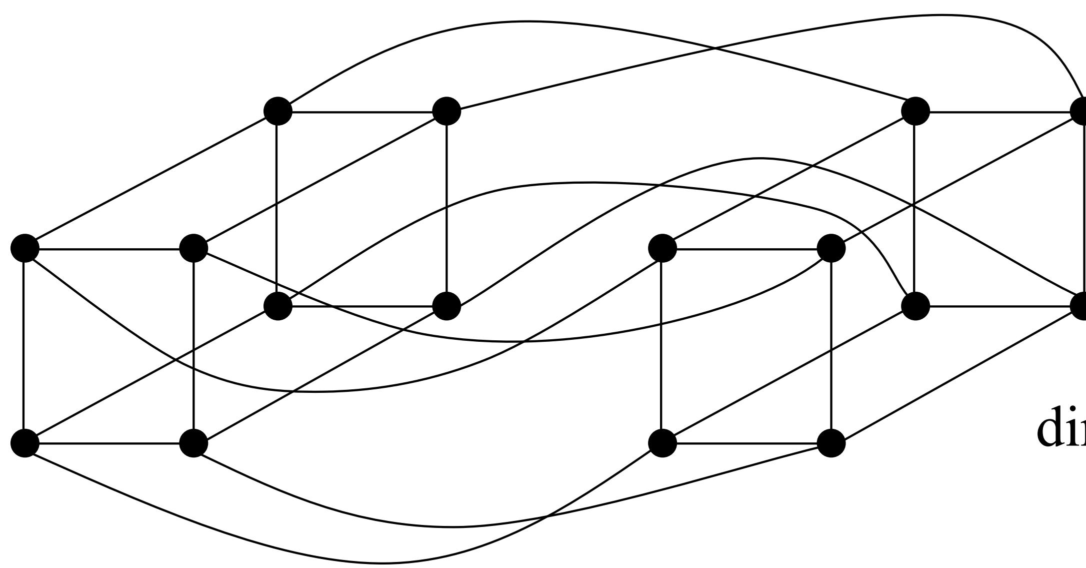
dim 1



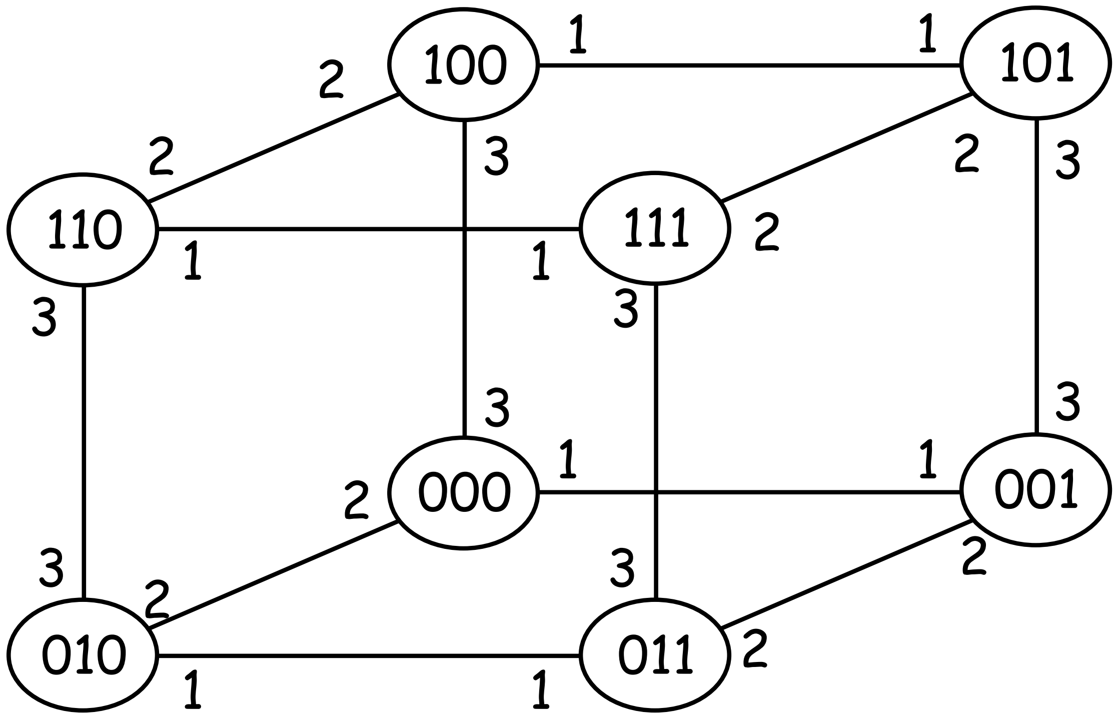
dim 2



dim 3



dim 4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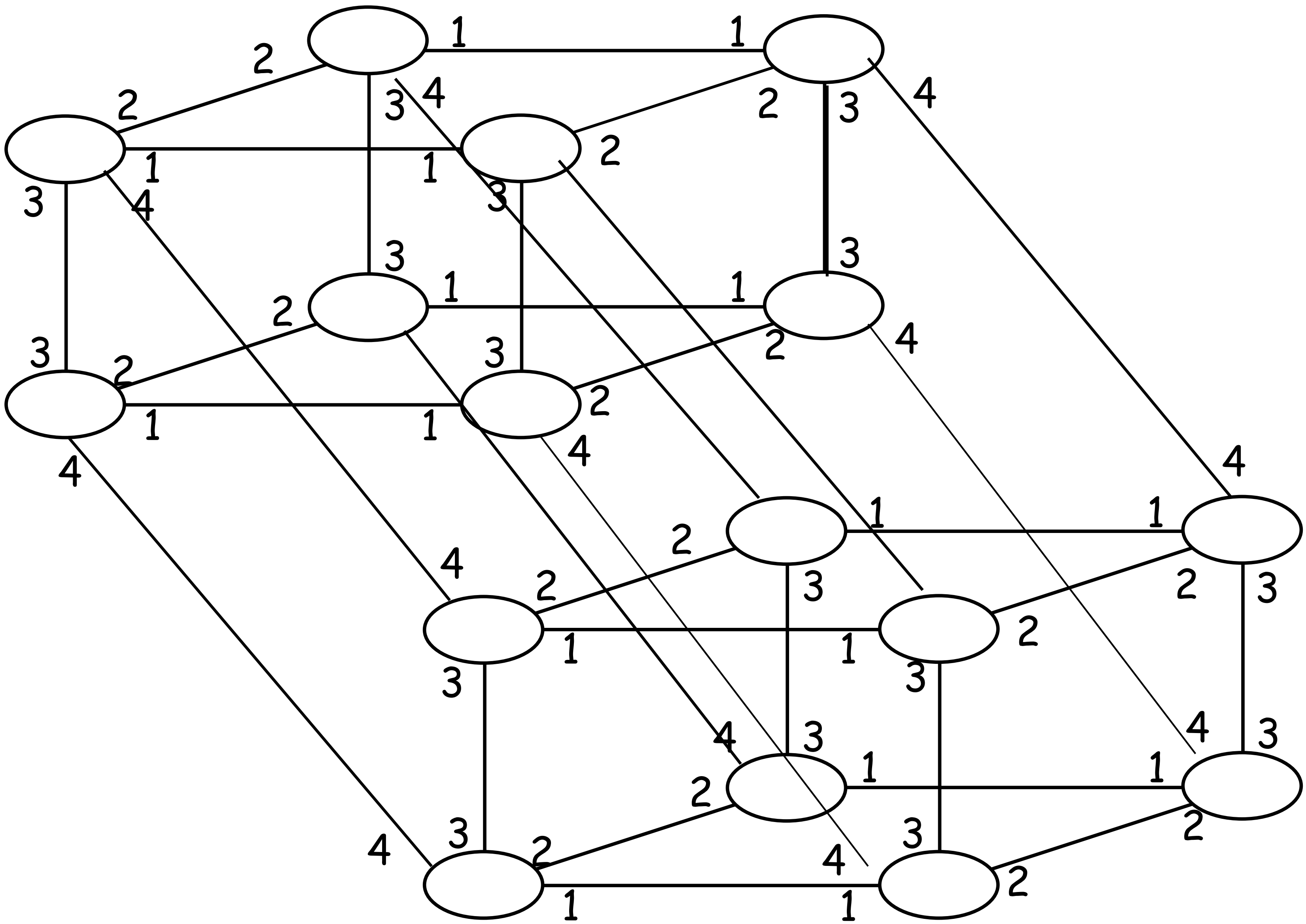


Hypercube

	n	m
h_0	1	0
h_1	2	1
h_2	4	1x2 + 2 = 4
h_3	8	4x2 + 4 = 12
h_4	16	12x2 + 8 = 32

$m_i = i \cdot 2^{i-1}$

h_i:

$n_0 = 1$
$n_i = 2\, n_{i-1} \longrightarrow n_i = 2^i$

$$\left\{\begin{array}{l} n_0 = 1 \\ n_i = 2\, n_{i-1} \end{array}\right. \longrightarrow n_i = 2^i$$

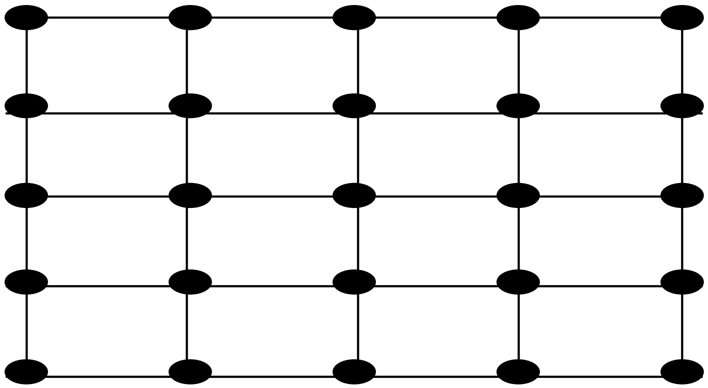
$$m_i = i \bullet 2^{i-1}$$

$$m = \frac{n \log n}{2}$$

degree = log n

$$m = O(n \log n)$$

— Grid —

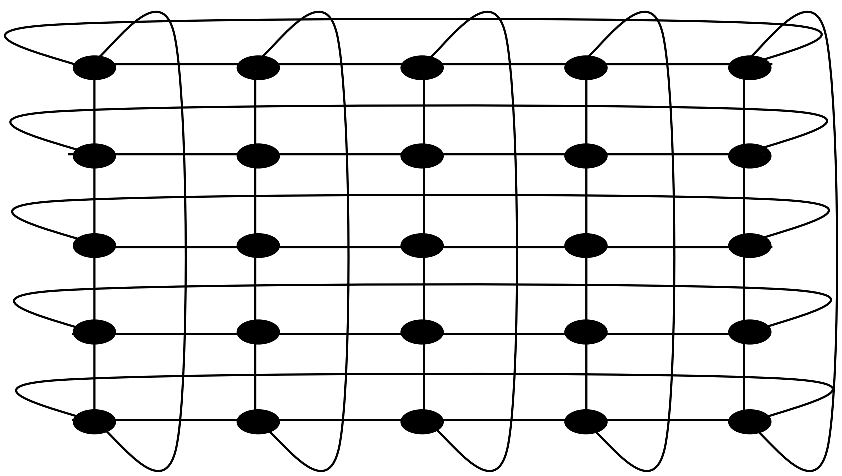


Not regular

m = O(n)

deg(i) = 4 v_i = internal
deg(i) = 3 v_i = border
deg(i) = 2 v_i = corner

— Torus —



m = O(n)

deg(i) = 4 $\forall i$