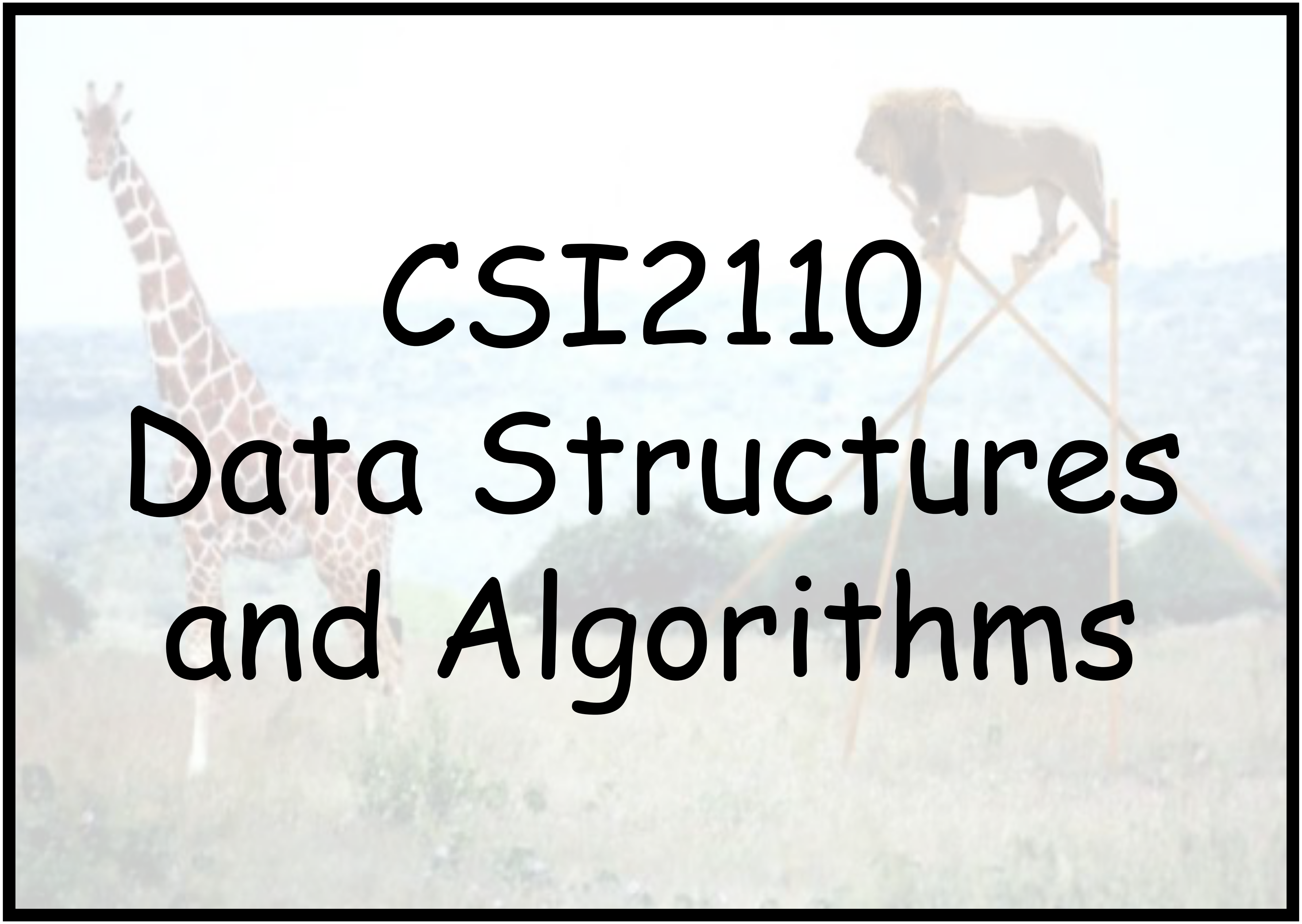
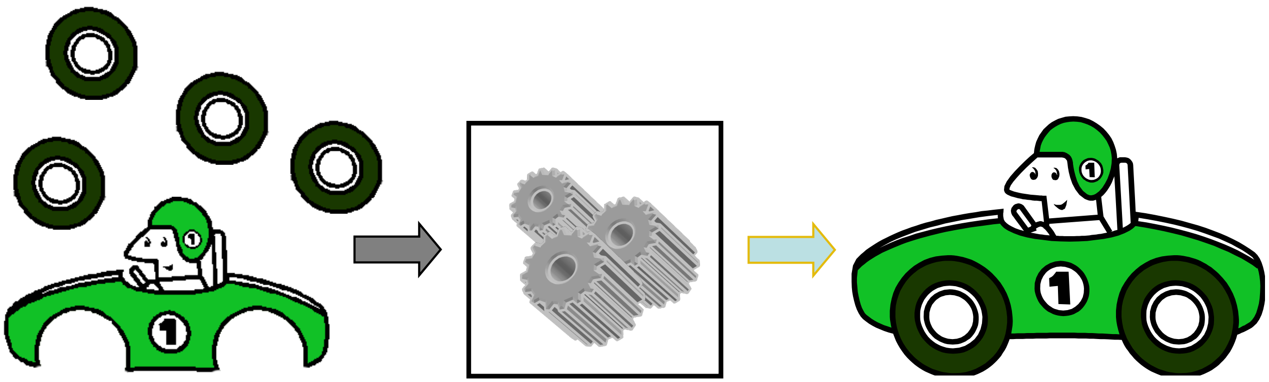

CSI2110
Data Structures
and Algorithms

Algorithms



Input **Algorithm** **Output**

An **algorithm** is a step-by-step procedure for solving a problem in a finite amount of time.

Analyze an algorithm = determine its efficiency

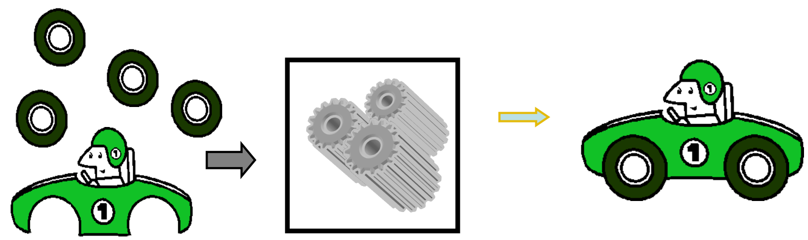
Analyze an algorithm

Analyze an algorithm = determine its efficiency

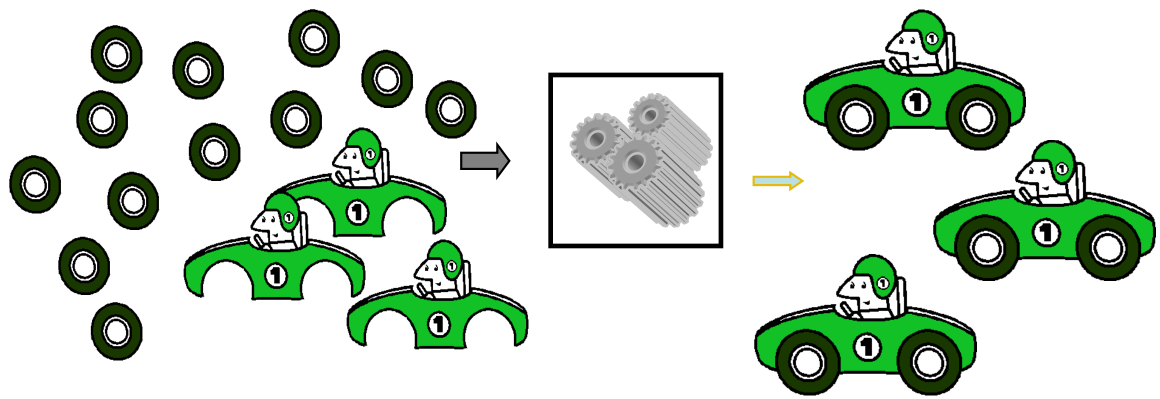
Efficiency ?

- Execution time …
- Memory …
- Quality of the result ….
- Simplicity ….

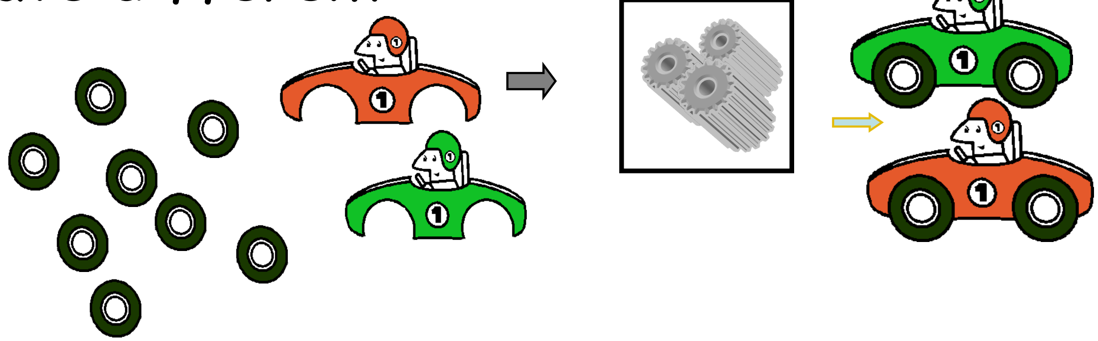
Running Time



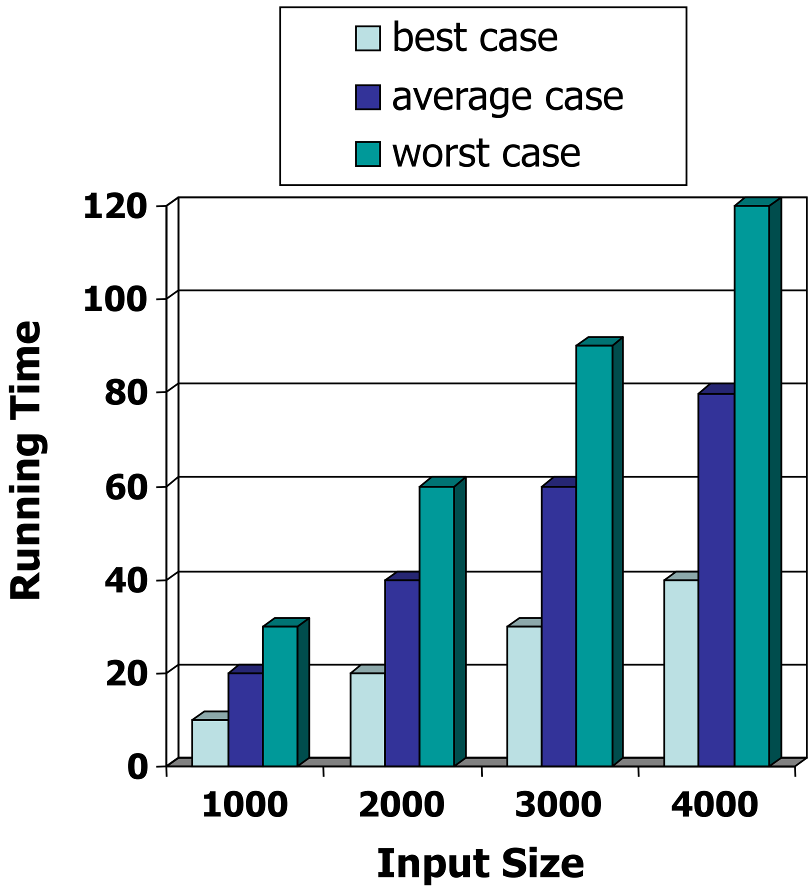
The running time depends on the input size



It also depends on the input data: Different inputs can have different running times

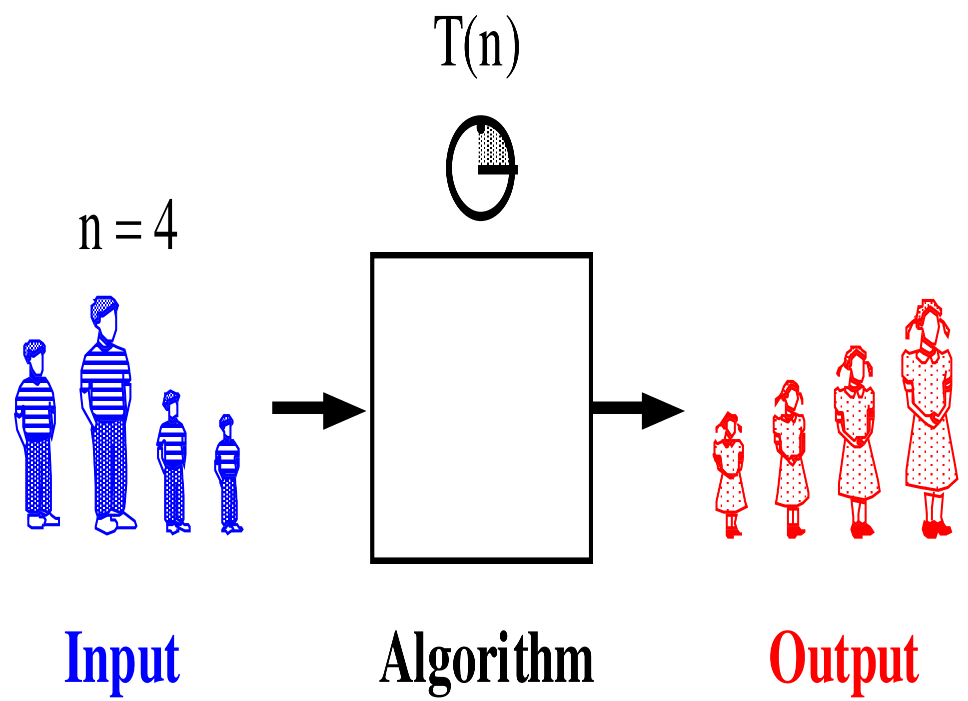


Running time



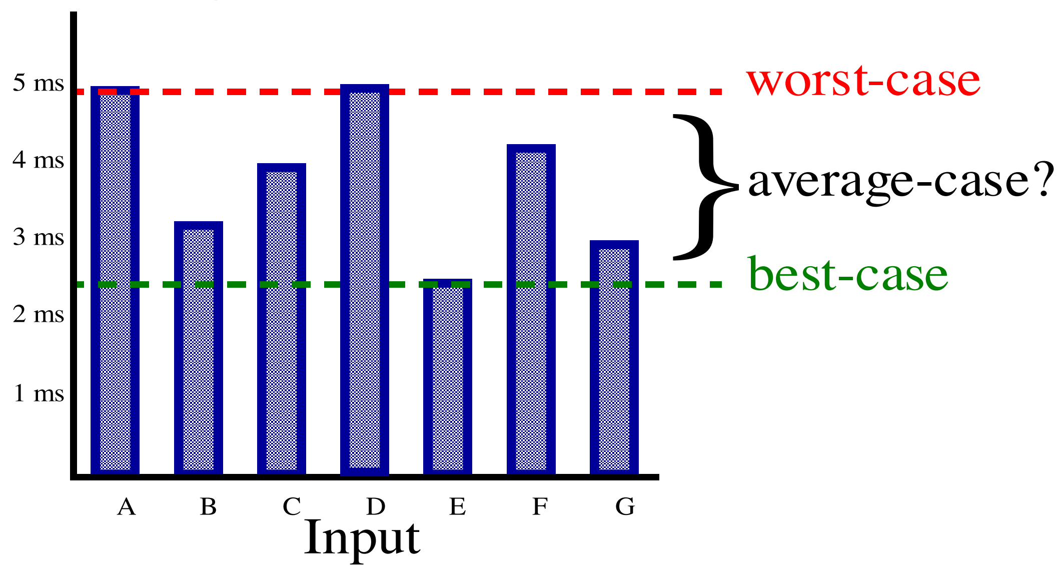
Analysis of Algorithms

- Running Time
- Upper Bounds
- Lower Bounds
- Examples
- Mathematical facts

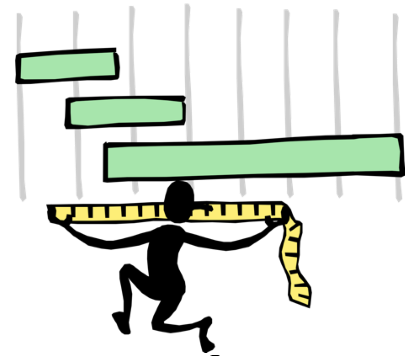


Average Case vs. Worst Case Running Time of an algorithm

- Finding the average case can be very difficult
- Knowing the worst-case time complexity can be important. We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games and robotics



Measuring the Running Time

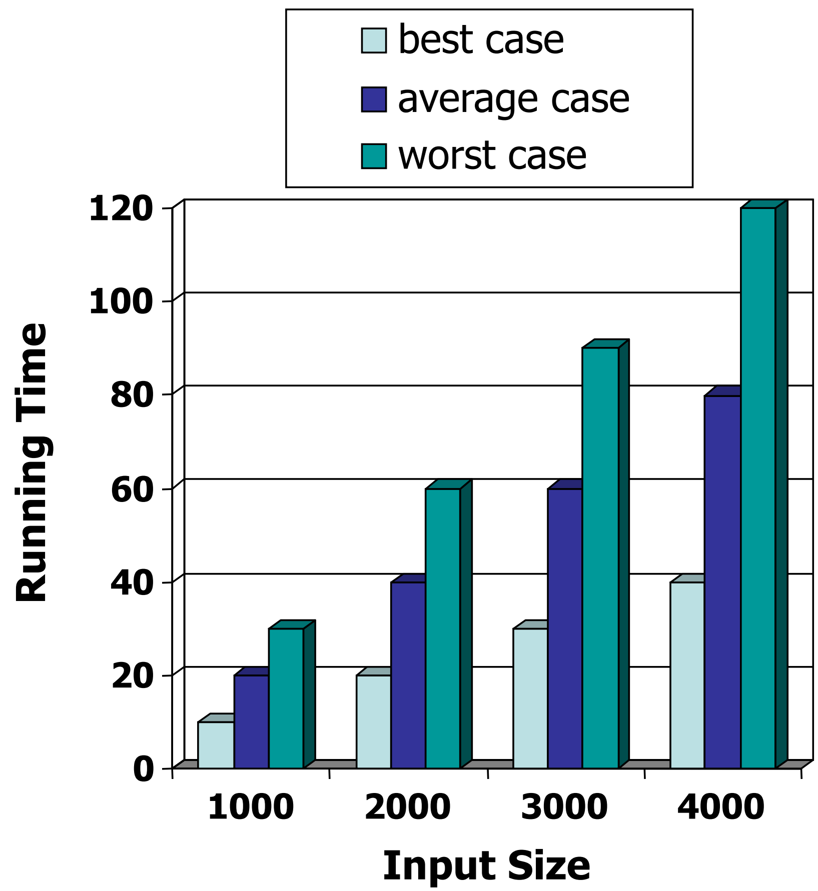


- How should we measure the running time of an algorithm?

- Approach 1: Experimental Study
  - Write a program which implement the algorithm
  - Run the program with all possible input sets of data of various size and contents
  - Use a method (system.currentTimeMillis()) to measure exact running time

Measuring the Running Time

- Approach 1: Experimental Study
- The measurement of the experiment can be like this…



Beyond Experimental Studies

- Experimental studies have several limitations:
  - need to implement
  - limited set of inputs
  - hardware and software environments.

Theoretical Analysis

- We need a general methodology which:
  - is independent of implementation.
    - Uses a high-level description of the algorithm
  - takes into account all possible inputs.
    - Characterizes running time as a function of the input size.
  - is independent of the hardware and software environment.

Analysis of Algorithms

- **Primitive Operations:** Low-level computations independent from the programming language can be identified in pseudocode.

- Examples:
  - calling a method and returning from a method
  - arithmetic operations (e.g. addition)
  - comparing two numbers, etc.

- By inspecting the pseudo-code, we can count the number of primitive operations executed by an algorithm.

Pseudo-code

- Mixture of natural language and high-level programming concept
  - to tell the general idea behind the data structure or algorithm implementation
- The pseudo-code is an description of algorithm which is
  - more structured than ordinary prose, but
  - less definite than programming languages

Pseudo-code

- Expressions: use standard mathematical symbols to describe Boolean and numerical expressions
  - use ← for allocations ("=" en Java)
  - use = for equal relation ("==" en Java)
- Method declaration
  - **Algorithm** nom(param1, param2)

Pseudo-code

- Programming element:
  - decision: if.. Then.. [else..]
  - Loop while: while…do
  - Loop repeat: repeat … until..
  - Loop for: for.. Do
  - Vector index: A[i]
- methods:
  - call: object method (args)
  - return: return value

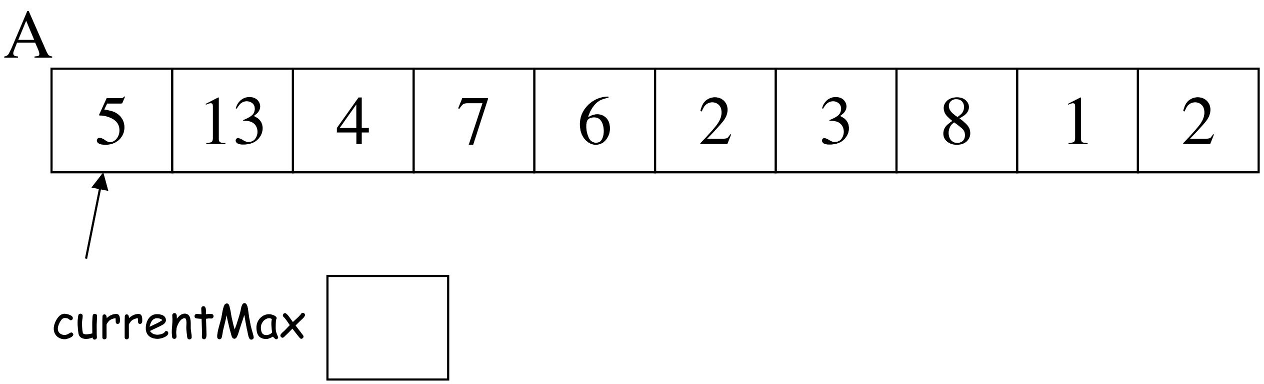
Example:

Find the maximum element of a vector (array)

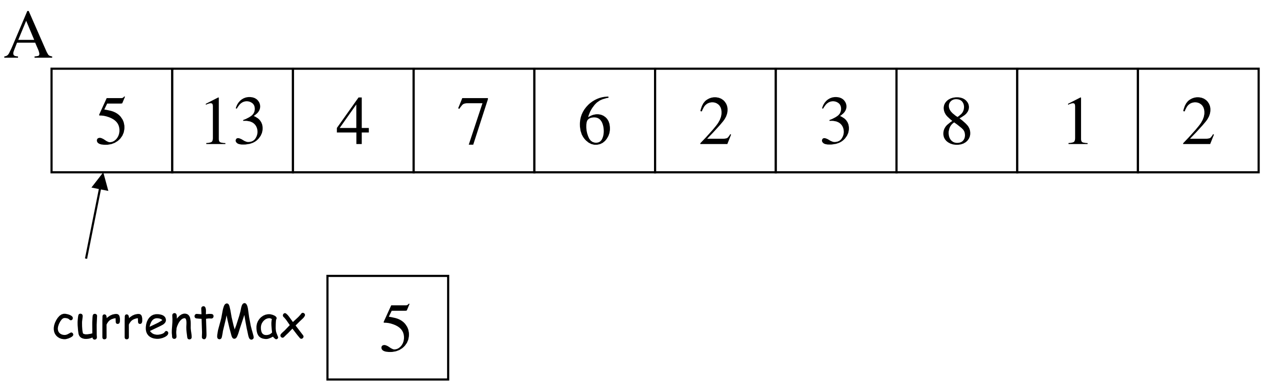
Algorithm arrayMax(*A*, n):

input: A vector *A* containing *n* entries
output: the maxim element of *A*

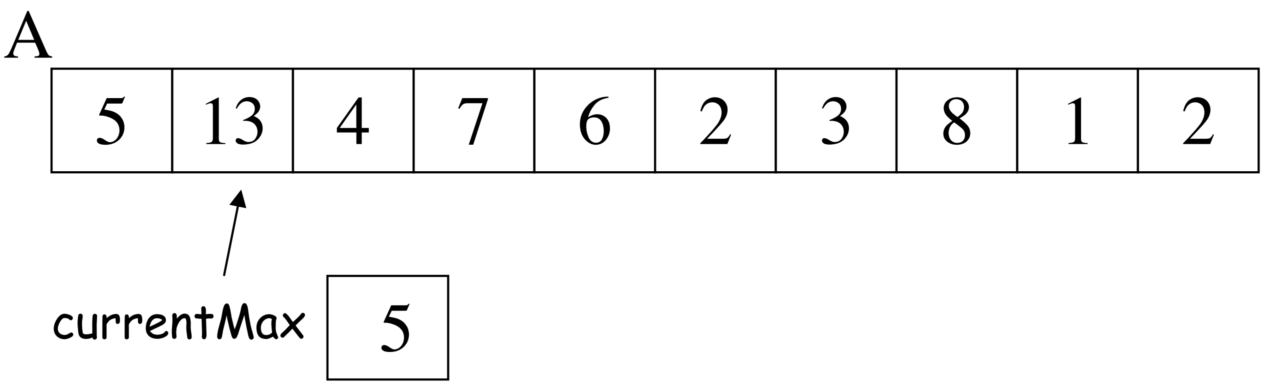
currentMax $\leftarrow$ *A*[0]
for *i* $\leftarrow$ 1 **to** *n -1* **do**
if *currentMax* < *A[i]* **then**
currentMax $\leftarrow$ *A[i]*
return *currentMax*




```
currentMax ← A[0]
for i ← 1 to n -1 do
if  currentMax < A[i] then
            currentMax ← A[i]


return currentMax
```

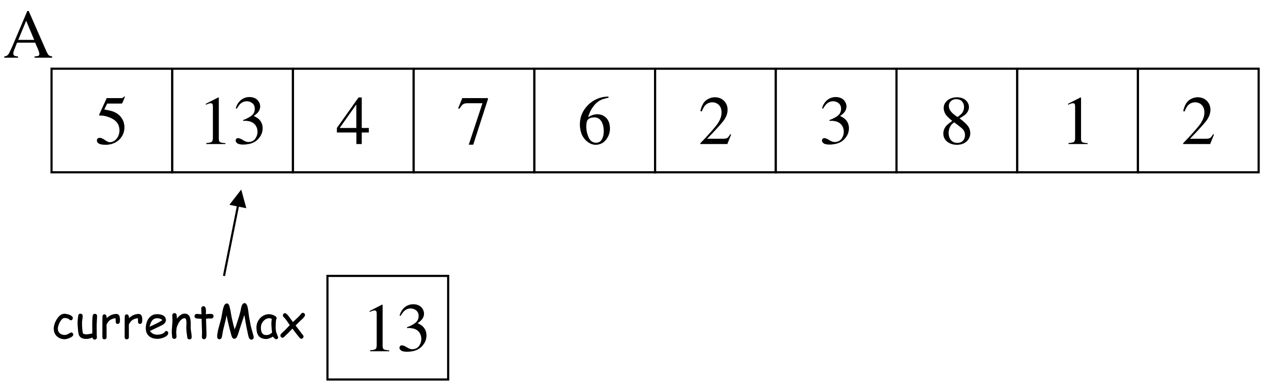



```
currentMax ← A[0]
for i ← 1 to n -1 do
if  currentMax < A[i] then
          currentMax ← A[i]


return currentMax
```




```
currentMax ← A[0]
for i ← 1 to n -1 do
if  currentMax < A[i] then
            currentMax ← A[i]


return currentMax
```

What are the primitive operations to count



Comparisons
Assignments to currentMax

```
currentMax ← A[0]
for i ← 1 to n -1 do
if  currentMax < A[i] then
            currentMax ← A[i]

return currentMax
```

```
currentMax ← A[0]                      - - - - - - - - - - - - - - -> 1 assignment
for i ← 1 to n -1 do
if currentMax < A[i] then                        n-1 comparisons
        currentMax ← A[i]               - - ->   n-1  assignments (worst case)

return currentMax
```

5	7	8	10	11	12	14	16	17	20

In the best case ?

15	1	12	3	9	7	6	4	2	11

```
currentMax ← A[0]                    ----------------→ 1 assign
for i ← 1 to n -1 do
if currentMax < A[i] then
          currentMax ← A[i]          ---→ n-1 comparaisons
                                          0  assign
return currentMax
```

Summarizing:

Worst Case:
n-1 comparisons
n assignments

Best Case:
n-1 comparisons
1 assignment

Another Example

Looking for the rank of an element in *A*
(size of *A* is size*A*)

$i \leftarrow 0$ ----------------→ 1 assignment

while $(A[i] \neq \text{element})$

$\quad i \leftarrow i+1$ ----------------→ size*A* checks & assignment
(if we are not lucky)

return i

Worst Case

Upper Bound

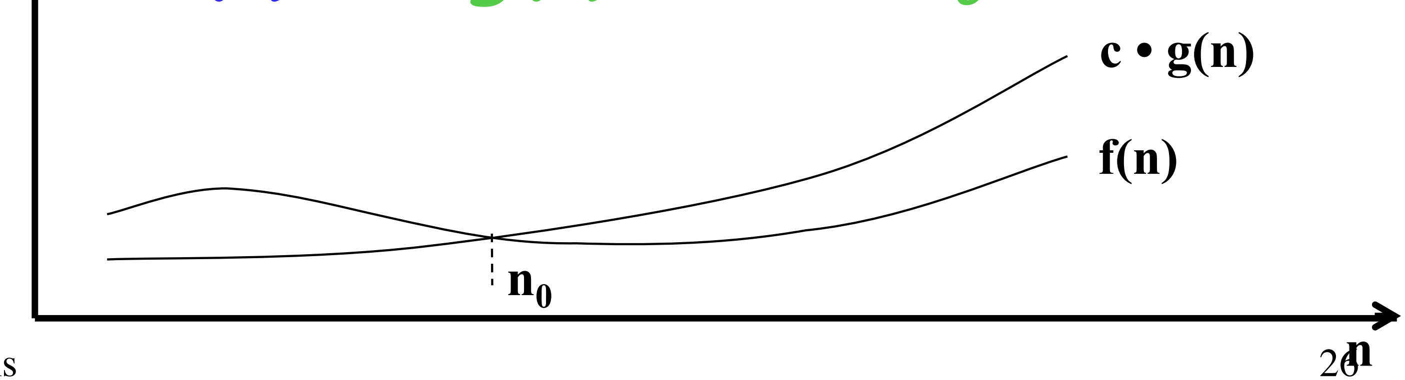
The "Big-Oh" Notation:

- given functions $f(n)$ and $g(n)$, we say that

$$f(n) \text{ is } O(g(n))$$

if and only if there are positive constants c and n_0 such that

$$f(n) \leq c\, g(n) \text{ for } n \geq n_0$$



An Example

prove that $f(n) \leq c\ g(n)$ for some $n \geq n_0$

$$f(n) = 60n^2 + 5n + 1$$

$$g(n) = n^2$$

prove that $f(n) \leq c\ n^2$

$$\leq$$

$$\underbrace{60n^2 + 5n^2 + n^2}$$

for $n \geq 1$

$$= 66n^2$$

$\Rightarrow$ $c = 66 \quad n_0 = 1$

$$f(n) \leq c\ n^2 \quad \forall\ n \geq n_0$$

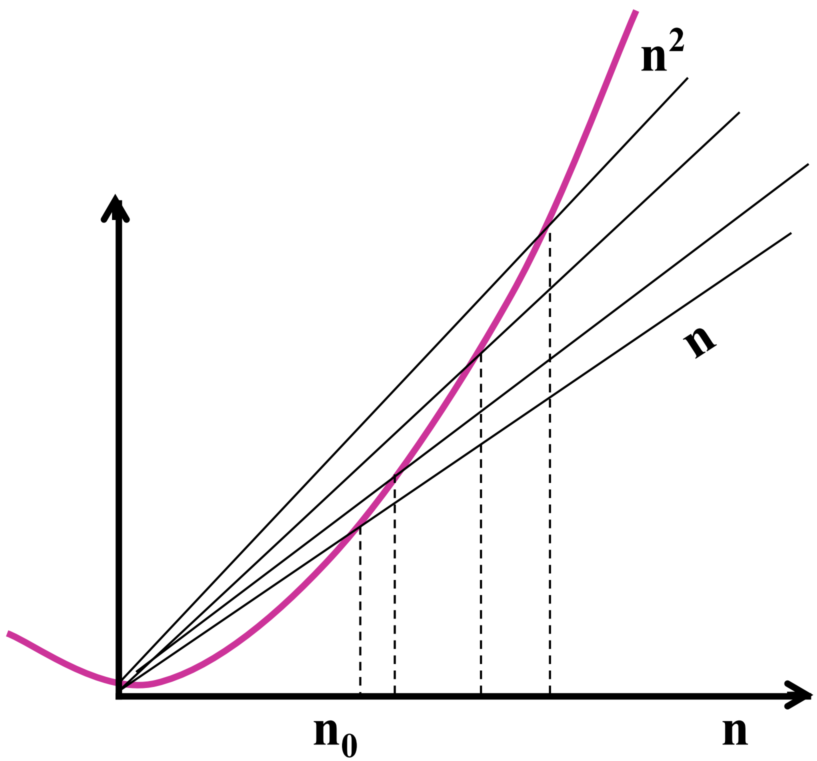
$\Rightarrow$ $f(n) = O(n^2)$

On the other hand…

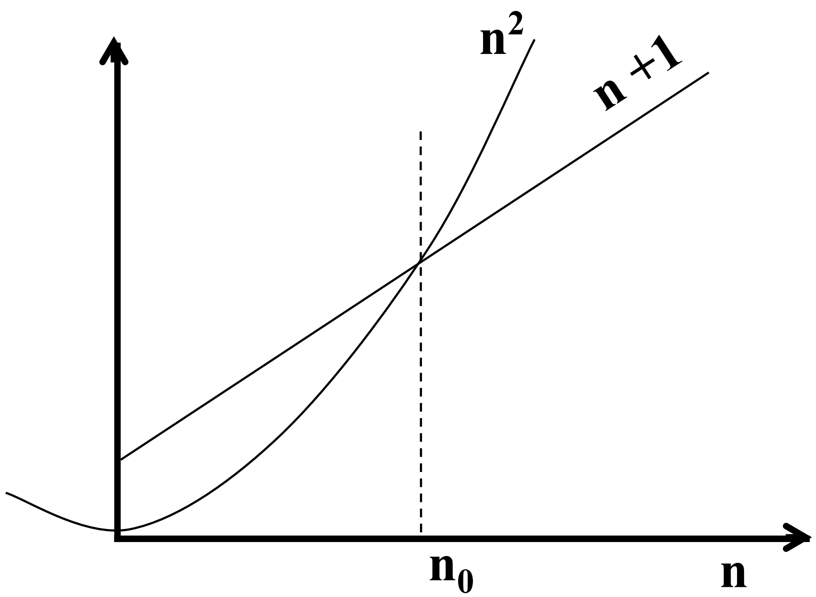
n^2 is not $O(n)$ because there is no c and n_0 such that:

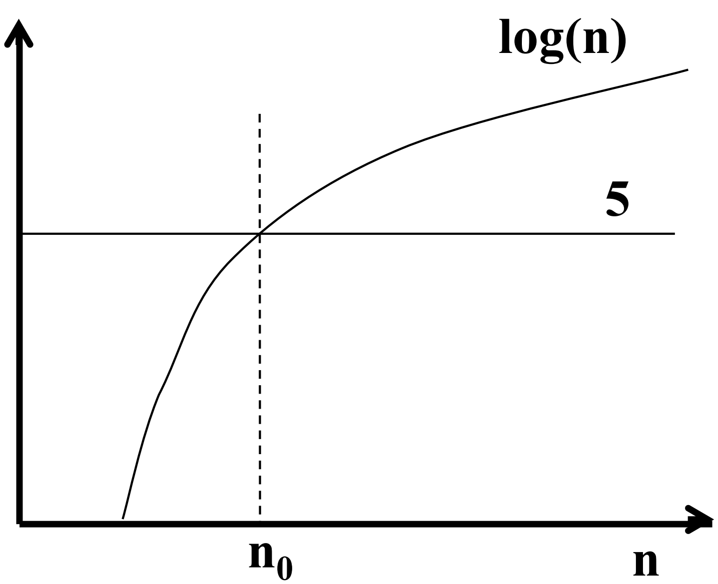
$$n^2 \leq cn \text{ for } n \geq n_0$$

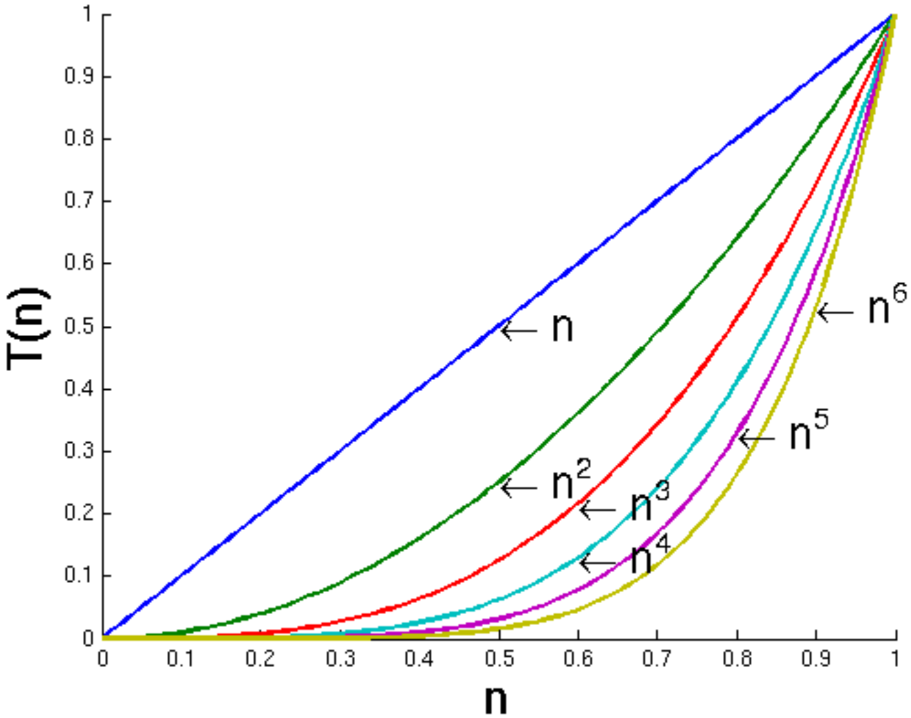
(no matter how large a c is chosen there is an n big enough that $n^2 > c\,n$) .



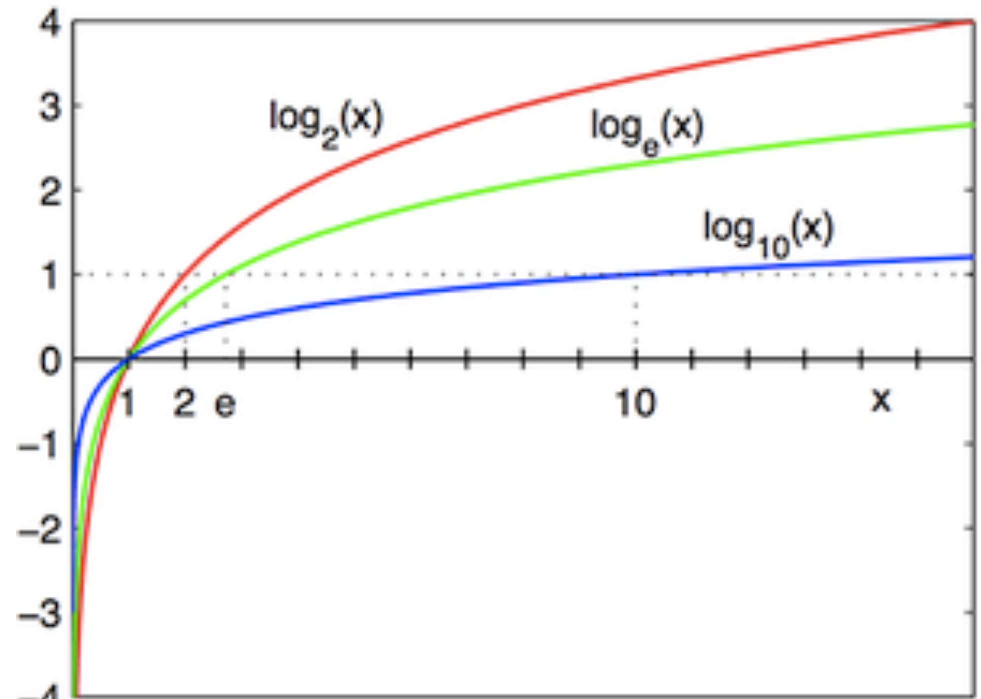
$O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n)$ … *remember !!*



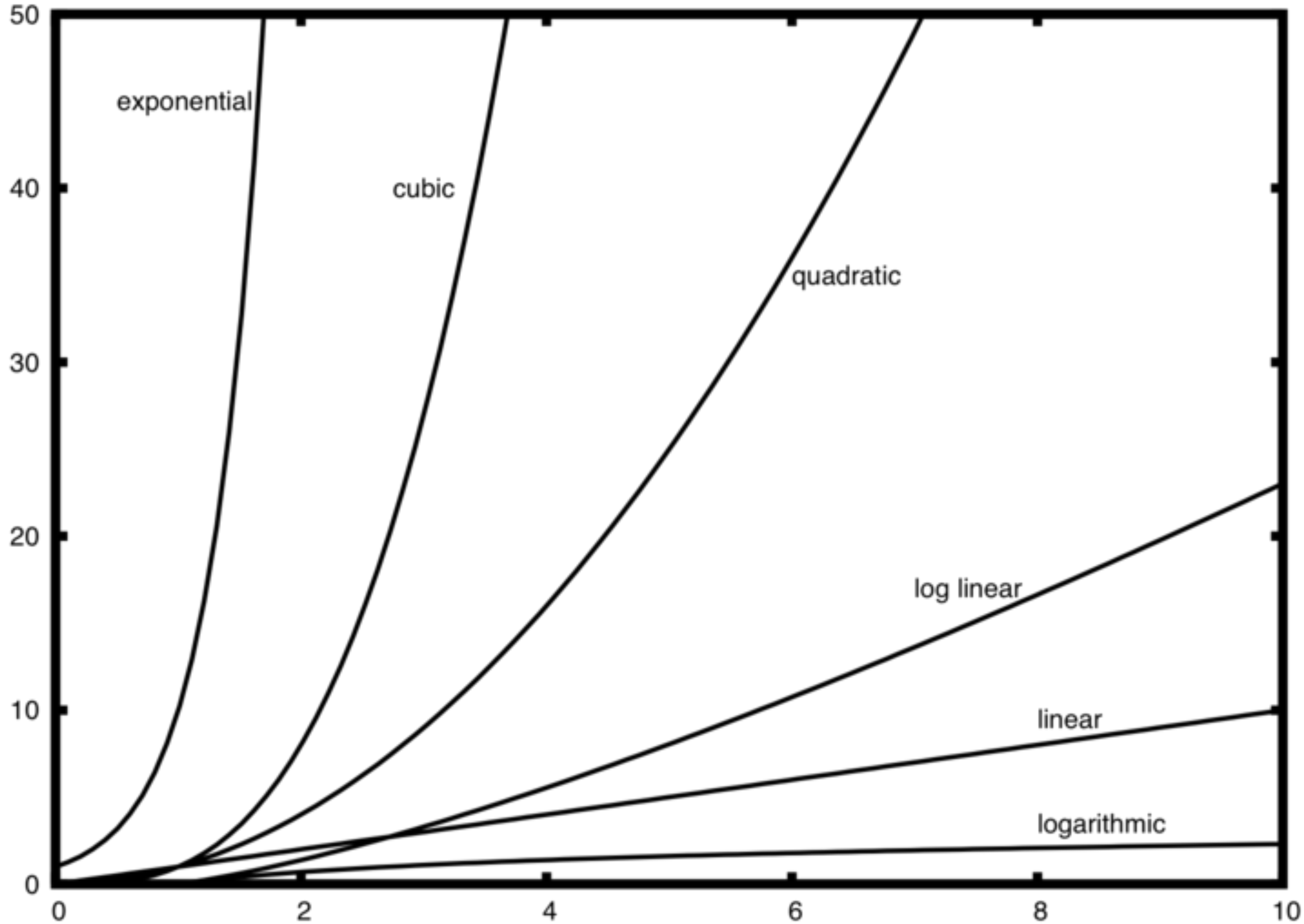




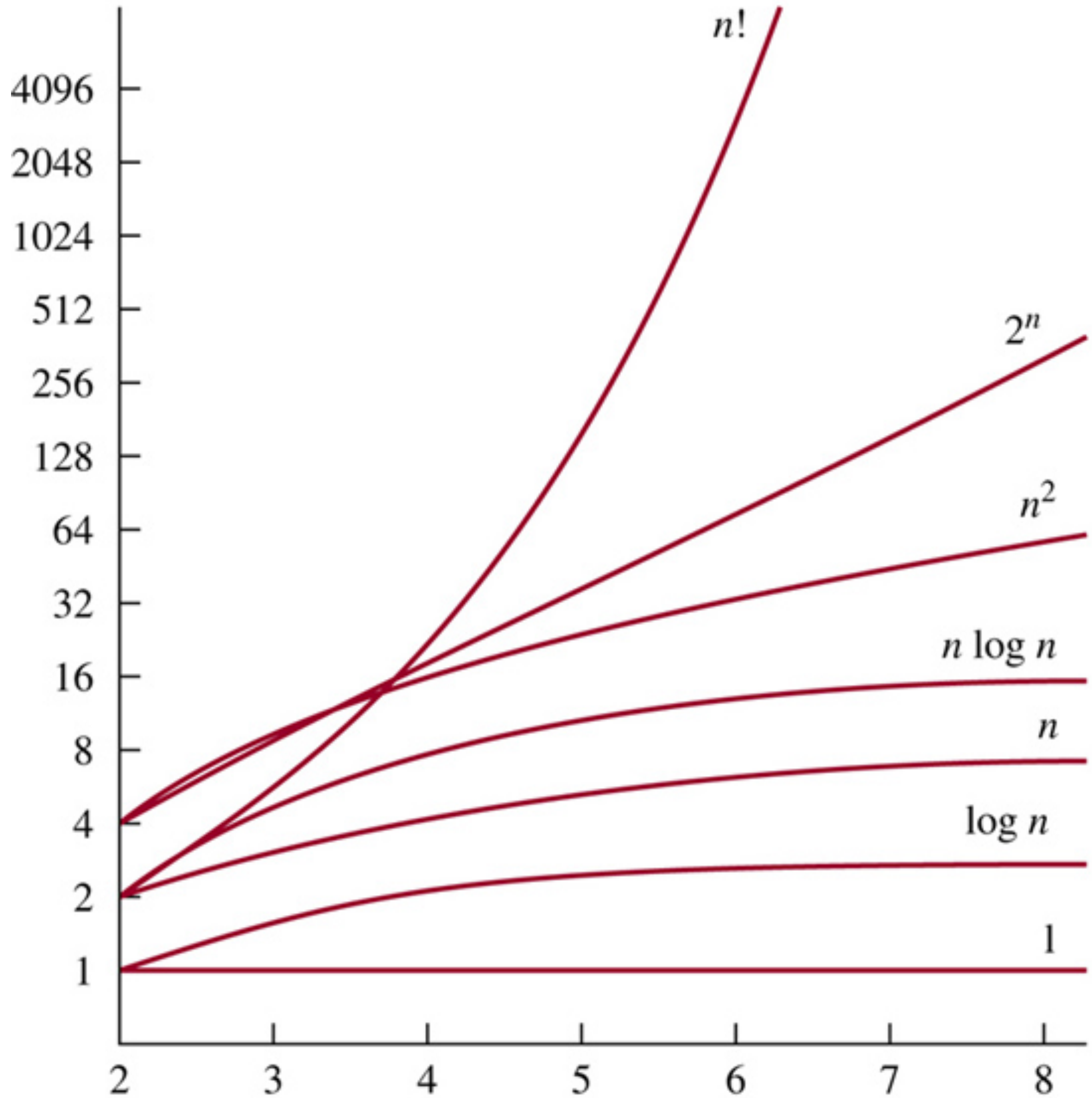
T(n)
n
← n
← n²
← n³
← n⁴
← n⁵
← n⁶



log₂(x)
logₑ(x)
log₁₀(x)
x



50
40
30
20
10
0
0
2
4
6
8
10
exponential
cubic
quadratic
log linear
linear
logarithmic



4096
2048
1024
512
256
128
64
32
16
8
4
2
1
2
3
4
5
6
7
8
$n!$
2^n
n^2
$n \log n$
n
$\log n$
1

$O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n)$ … *remember !!*

n =	2	16	256	1024
log log n	0	2	3	3.32
log n	1	4	8	10
n	2	16	256	1024
n log n	2	64	448	10 200
n^2	4	256	65 500	$1.05 * 10^6$
n^3	8	4 100	16 800 800	$1.07 * 10^9$
2^n	4	35 500	$11.7 * 10^6$	$1.80 * 10^{308}$

Asymptotic Notation (cont.)

- Note: Even though it is correct to say
- "$7n - 3$ is $O(n^3)$",
- a better statement is
- "$7n - 3$ is $O(n)$", that is,
- one should make the approximation as tight as possible

Theorem:

If $g(n)$ is $O(f(n))$, then for any constant $c > 0$

$g(n)$ is also $O(c\, f(n))$

Theorem:

$O(f(n) + g(n)) = O(\max(f(n), g(n)))$

Ex 1:

$$2n^3 + 3n^2 = O(\max(2n^3, 3n^2))$$

$$= O(2n^3) = O(n^3)$$

Ex 2:

$$n^2 + 3 \log n - 7 = O(\max(n^2, 3 \log n - 7))$$

$$= O(n^2)$$

Simple Big Oh Rule:

Drop lower order terms and constant factors

$7n-3$ is $O(n)$

$8n^2\log n + 5n^2 + n$ is $O(n^2\log n)$

$12n^3 + 5000n^2 + 2n^4$ is $O(n^4)$

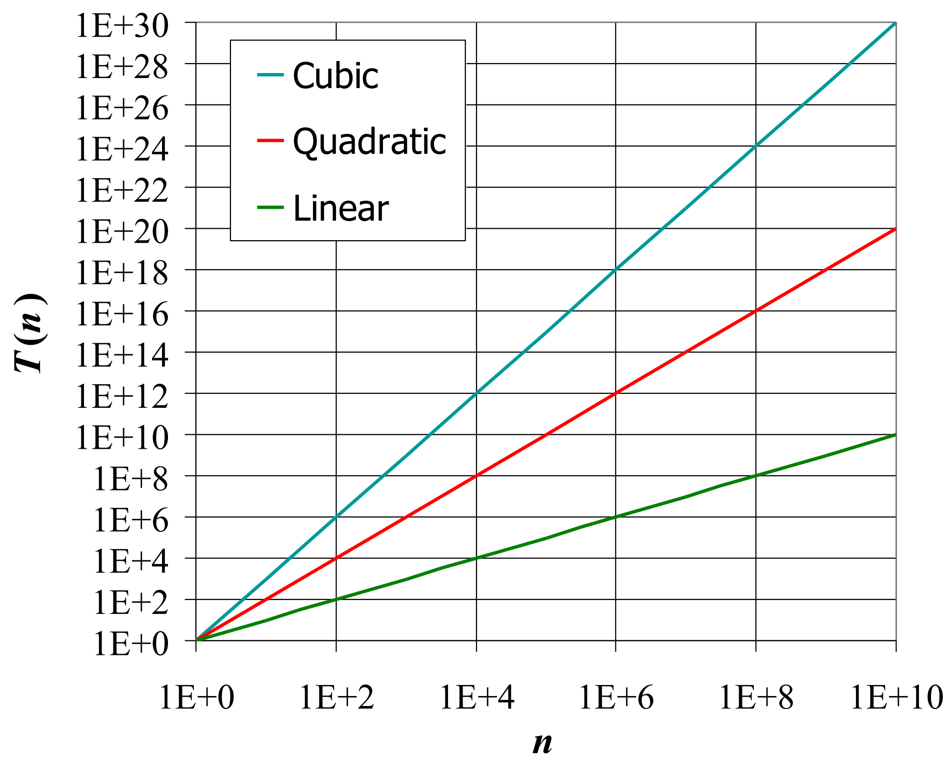
Other Big Oh Rules:

- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Notation (terminology)

- Special classes of algorithms:

constant:	$O(1)$
logarithmic:	$O(\log n)$
linear:	$O(n)$
quadratic:	$O(n^2)$
cubic:	$O(n^3)$
polynomial:	$O(n^k)$, $k \geq 1$
exponential:	$O(a^n)$, $n > 1$



Asymptotic Analysis and execution time

- Use the Big-O notation
  - to indicate the number of primitive operations executed according to the entry size
- For example, we say that algorithm arrayMax has an execution time O(n)
- While comparing the asymptotic execution times
  - O(log n) is better than O(n)
  - O(n) is better than $O(n^2)$
  - **$\log n \ll n^{-2} \ll n \ll n \log n \ll n^2 \ll n^3 \ll 2^n$**

Asymptotic Analysis and execution time

- Use the Big-O notation
  - to indicate the number of primitive operations executed according to the entry size
- For example, we say that algorithm arrayMax has an execution time $O(n)$
- While comparing the asymptotic execution times
  - $O(\log n)$ is better than $O(n)$
  - $O(n)$ is better than $O(n^2)$
  - $\mathbf{\log n \ll n \ll n \log n \ll n^2 \ll n^3 \ll 2^n}$

Example of Asymptotic Analysis

An algorithm for computing prefix averages

The i-th prefix average of an array X is average of the first $(i + 1)$ elements of X

$A[i] = X[0] + X[1] + \ldots + X[i]$

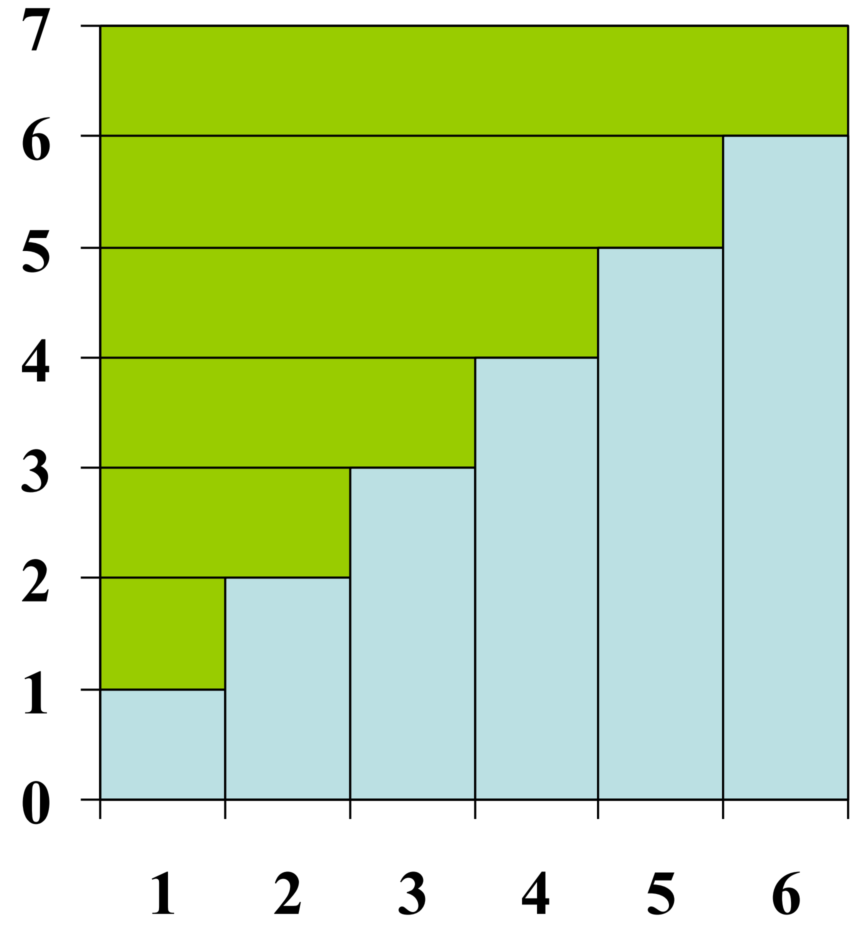
Example of Asymptotic Analysis

Algorithm ***prefixAverages1***(X, n)

Input array X of n integers
Output array A of prefix averages of X #operations

$A \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n - 1$ **do**	
$\quad s \leftarrow X[0]$	n
$\quad$ **for** $j \leftarrow 1$ **to** i **do**	
$\qquad s \leftarrow s + X[j]$	$1 + 2 + \ldots + (n - 1)$
$\quad A[i] \leftarrow s / (i + 1)$	n
return A	1

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time



Another Example

- A better algorithm for computing prefix averages:

Algorithm prefixAverages2(X):

Input: An *n*-element array X of numbers.

Output: An *n* -element array A of numbers such that A[*i*] is the average of elements X[0], ... , X[*i*].

Let X be an array of ***n*** numbers.	# operations
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** ***n*** **do**	n
$s \leftarrow s + X[i]$	n
$A[i] \leftarrow s/(i+1)$	n
return array A	1

O(n) time

Lower Bound

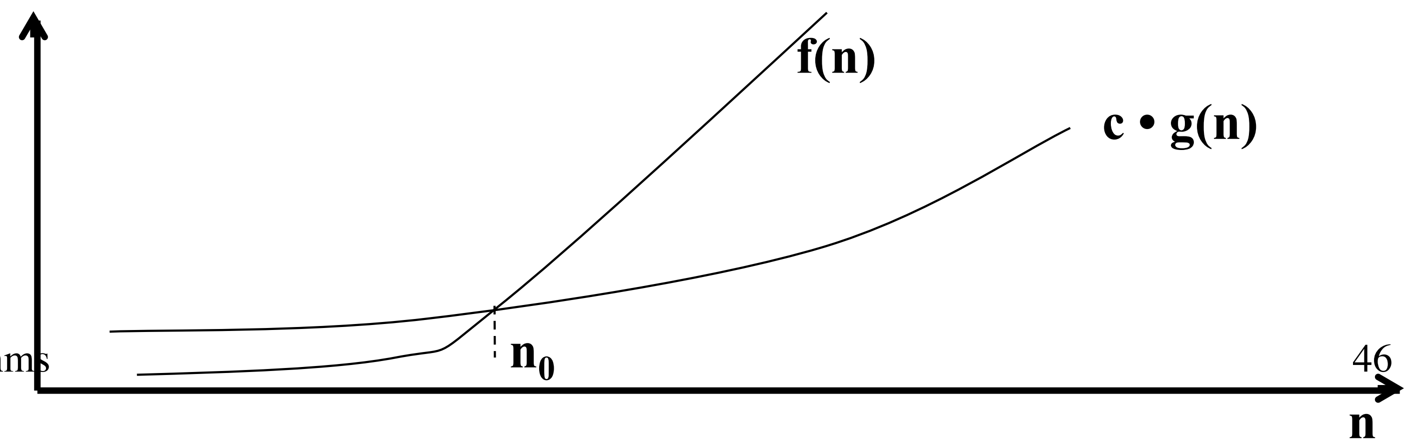
… is big omega …

$f(n)$ is $\Omega(g(n))$

if there exist $c > 0$ and $n_0 > 0$ such that

$f(n) \geq c \cdot g(n)$ for all $n \geq n_0$

(thus, $f(n)$ is $\Omega(g(n))$ iff $g(n)$ is $O(f(n))$)



Tight Bound

… is big theta …

g(n) is Θ(f(n))

<===>

if g(n) $\in$ O(f(n))

AND

f(n) $\in$ O(g(n))

is an element of (set membership)

Mathematical notation instead of "is"

An Example

We have seen that

$f(n) = 60n^2 + 5n + 1$ is $O(n^2)$

but $60n^2 + 5n + 1 \geq 60n^2$ for $n \geq 1$

So: with $c = 60$ and $n_0 = 1$

$f(n) \geq c \bullet n^2$ for all $n \geq 1$ $\Rightarrow$ $f(n)$ is $\Omega(n^2)$

Therefore:

$$\boxed{\begin{array}{c} f(n) \text{ is } O(n^2) \\ \text{AND} \\ f(n) \text{ is } \Omega(n^2) \end{array}}$$

$\downarrow$

$f(n)$ is $\Theta(n^2)$

Intuition for Asymptotic Notation

Big-Oh

- f(n) is $O(g(n))$ if f(n) is asymptotically **less than or equal** to $g(n)$

big-Omega

- f(n) is $\Omega(g(n))$ if f(n) is asymptotically **greater than or equal** to $g(n)$

big-Theta

- f(n) is $\Theta(g(n))$ if f(n) is asymptotically **equal** to $g(n)$

Math You Need to Review

Logarithms and Exponents

properties of logarithms:

$\log_b(xy) = \log_b x + \log_b y$

$\log_b (x/y) = \log_b x - \log_b y$

$\log_b x^a = a\log_b x$

$\log_b a = \log_x a / \log_x b$

properties of exponentials:

$a^{(b+c)} = a^b a^c$

$a^{bc} = (a^b)^c$

$a^b / a^c = a^{(b-c)}$

$b = a^{\log_a b}$

$b^c = a^{c*\log_a b}$

More Math to Review

- Floor: $\lfloor x \rfloor$ = the largest integer $\leq x$ $\lfloor 2.3 \rfloor = 2$
- Ceiling: $\lceil x \rceil$ = the smallest integer $\geq x$ $\lceil 2.3 \rceil = 3$
- Summations:
  - General definition:

$$\sum_{i=s}^{t} f(i) = f(s) + f(s+1) + f(s+2) + .. + f(t)$$

  - where *f* is a function, *s* is the starting index, and *t* is the ending index

More Math to Review

- Arithmetic Progression : $f(i) = i\,a$

$$S = \sum_{i=0}^{n} id = 0 + d + 2d + \ldots + nd$$

$$= nd+(n-1)d+(n-2)d + \ldots + 0$$

$$2S = nd + nd + nd + \ldots + nd$$

$$= (n+1)\,nd$$

$$S = d/2\; n(n+1)$$

$$\text{for } d=1,\; S = 1/2\; n(n+1)$$

More Math to Review

- Geometric Sum : $f(i) = a^i$
- The geometric progressions have an exponential growth

$$S = \sum_{i=0}^{n} r^i = 1 + r + r^2 + \ldots + r^n$$

$$rS = r + r^2 + \ldots + r^n + r^{n+1}$$

$$rS - S = (r-1)S = r^{n+1} - 1$$

$$S = (r^{n+1}-1)/(r-1)$$

$$\text{If } r=2,\ S = (2^{n+1}-1)$$

"Dear Andy: How have you been?
Your mother and I are fine. We miss you.
Please sign off your computer and come
downstairs for something to eat. Love, Dad."