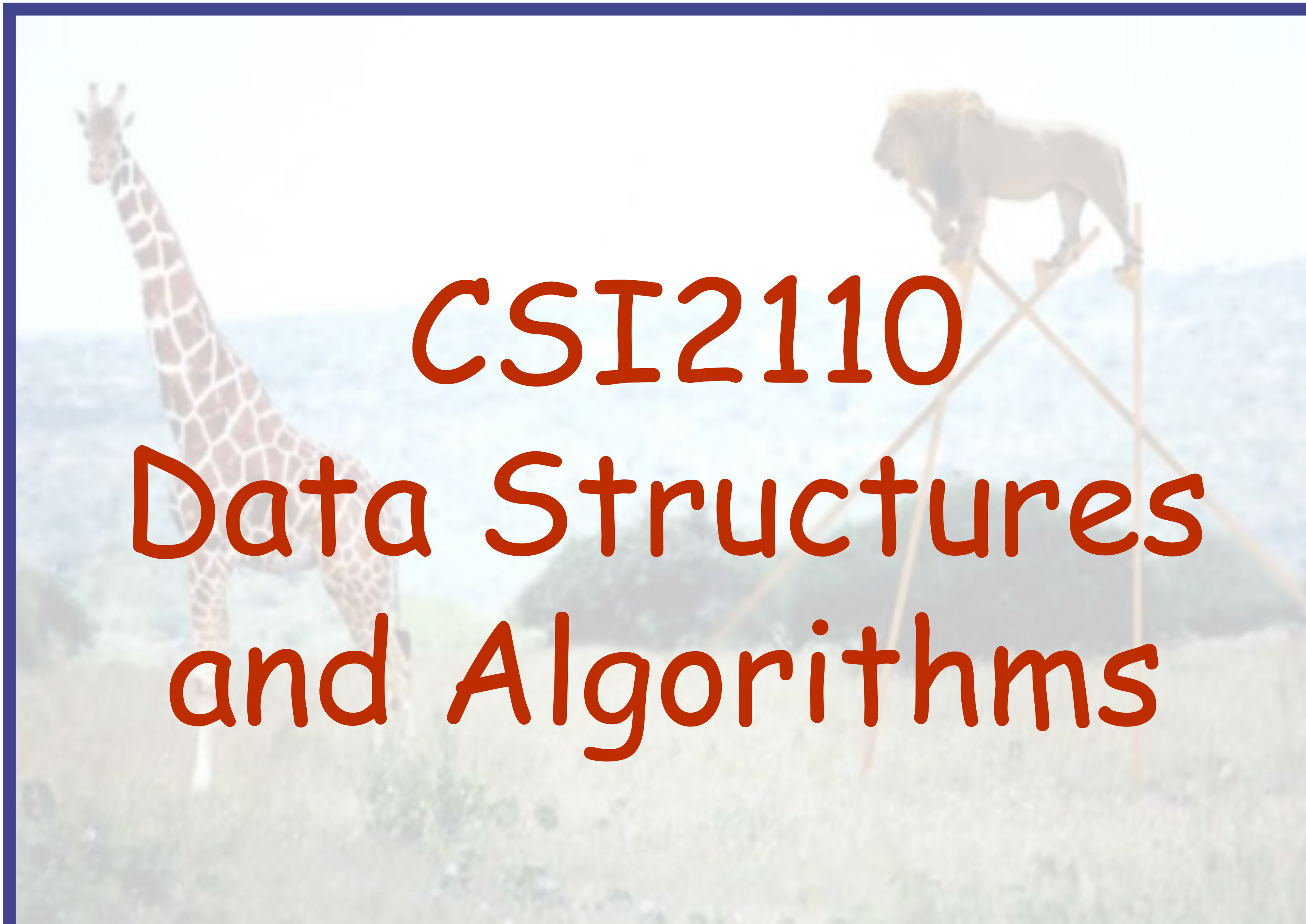
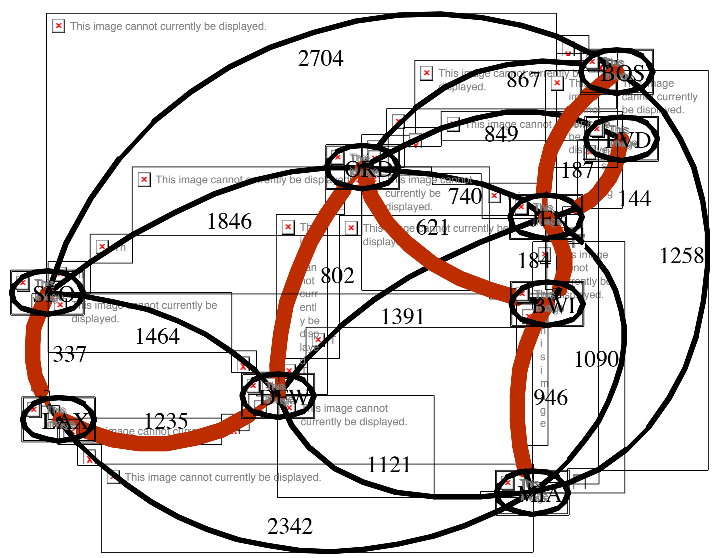


CSI2110 Data Structures and Algorithms

Minimum Spanning Tree

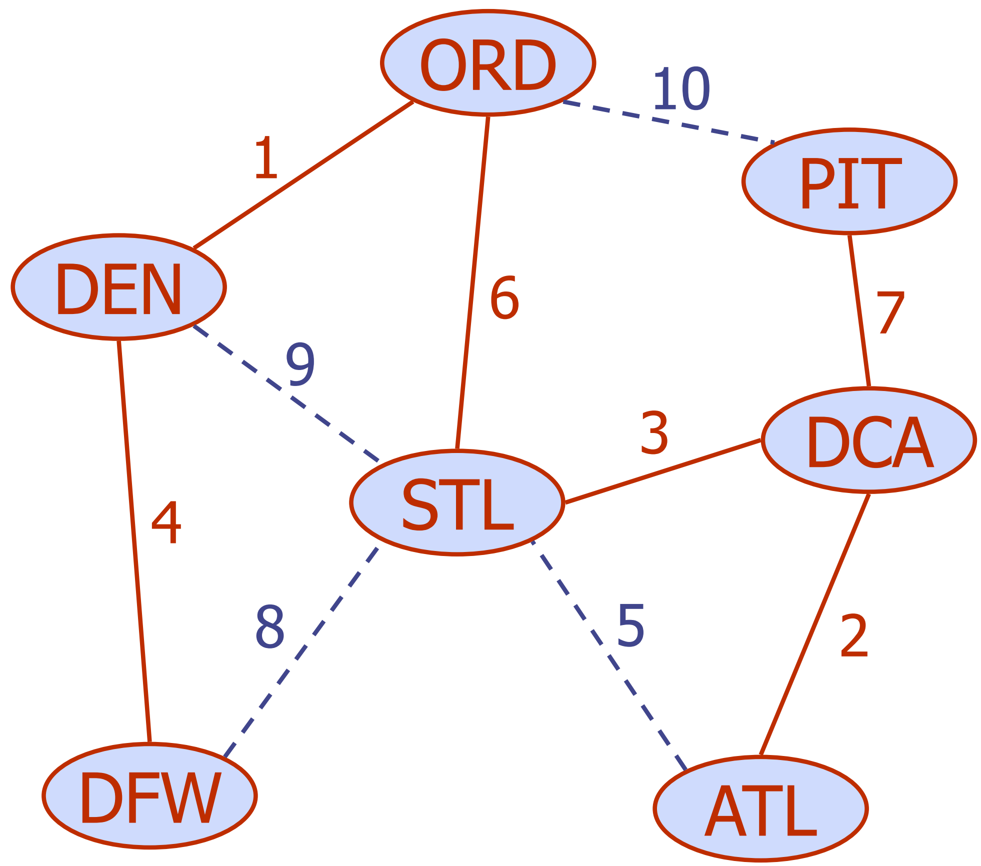


Outline and Reading

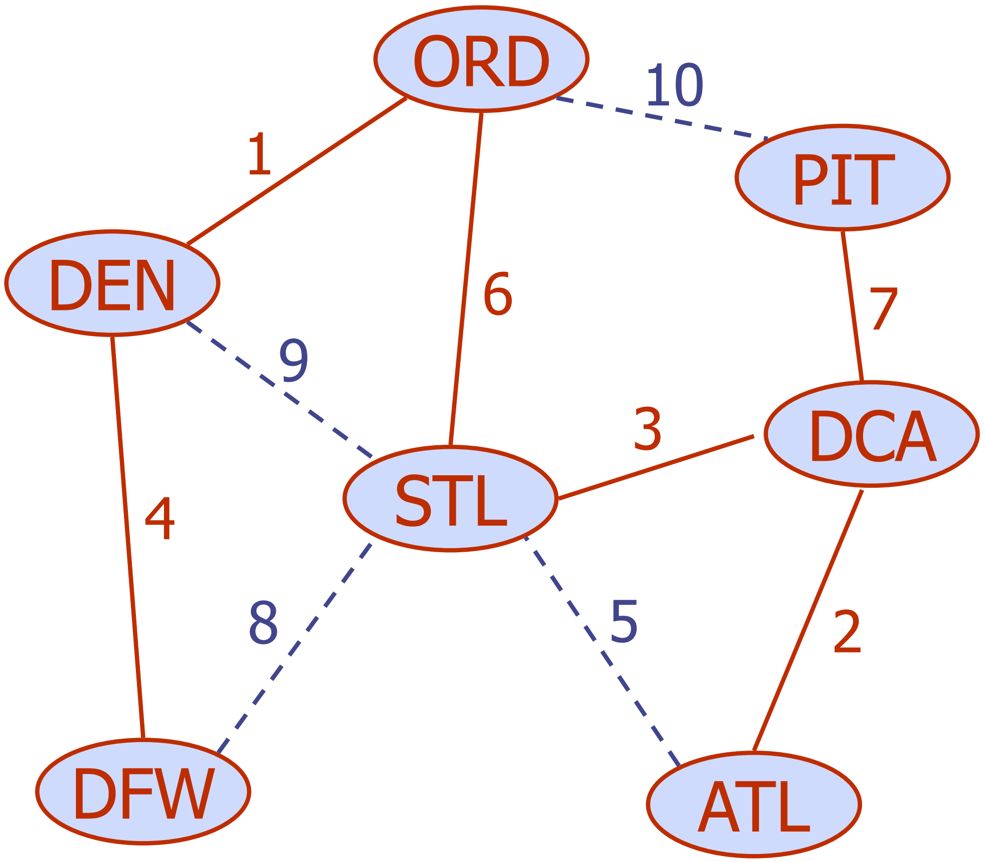
- Minimum Spanning Trees
  - Definitions
  - A crucial fact
- Prim-Jarnik's Algorithm
- Kruskal's Algorithm

Minimum Spanning Tree

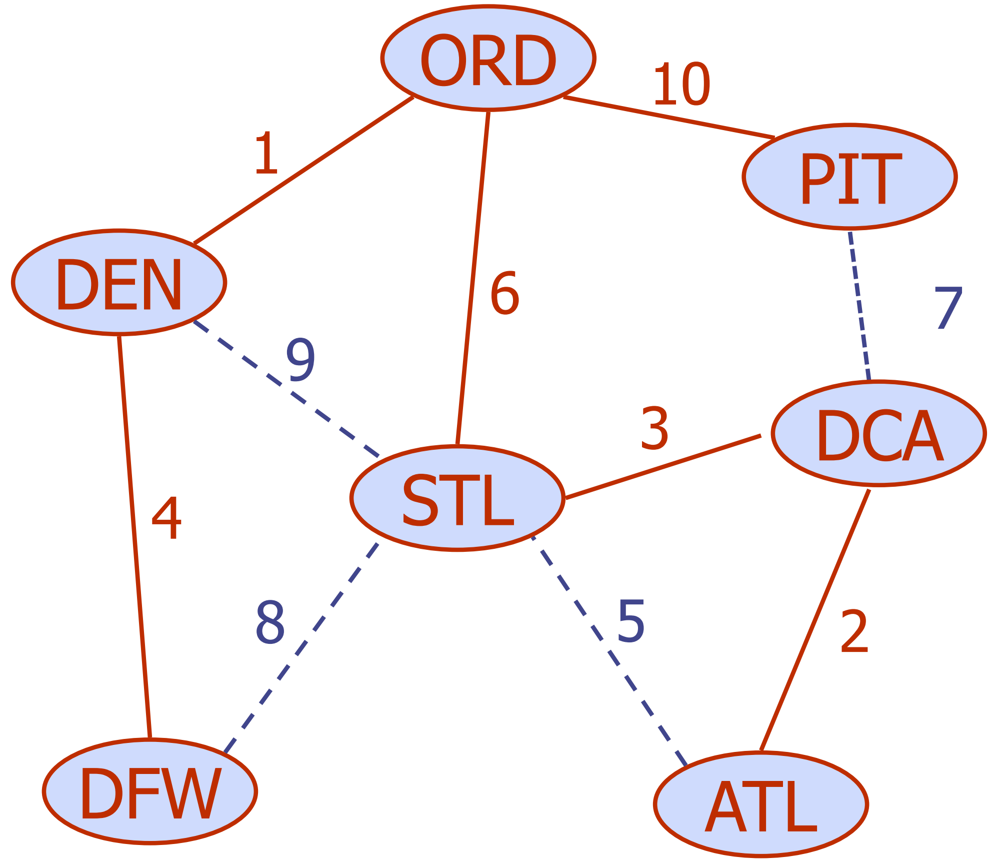
- Spanning subgraph
  - Subgraph of a graph G containing all the vertices of G
- Spanning tree
  - Spanning subgraph that is itself a (free) tree
- Minimum spanning tree (MST)
  - Spanning tree of a weighted graph with minimum total edge weight
- Applications
  - Communications networks
  - Transportation networks



Notice the difference



Minimum Spanning Tree



Shortest Path
Spanning Tree

Cycle Property

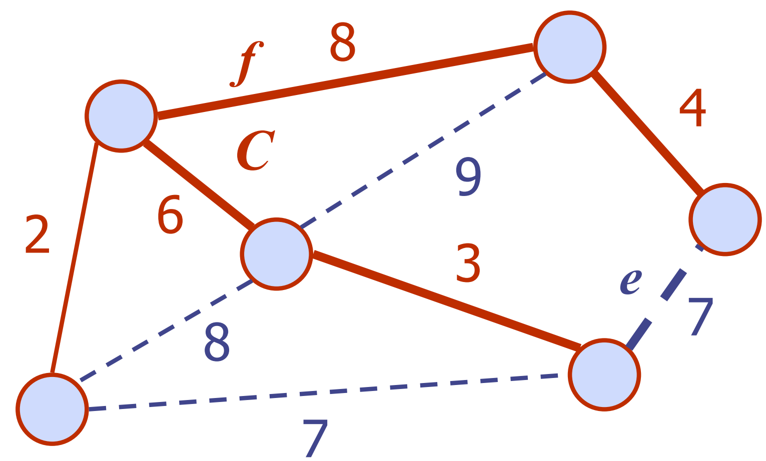
Cycle Property:

- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and let C be the cycle formed by adding e to T
- For every edge f of C, $weight(f) \leq weight(e)$

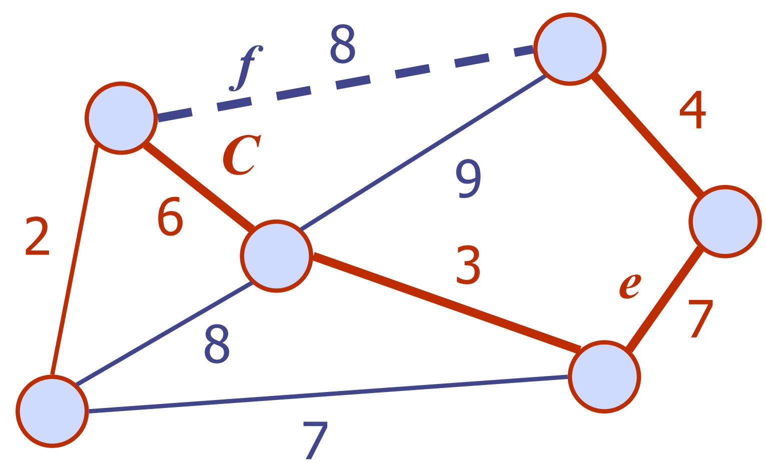
Proof:

By contradiction

- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f



Replacing f with e yields a better spanning tree

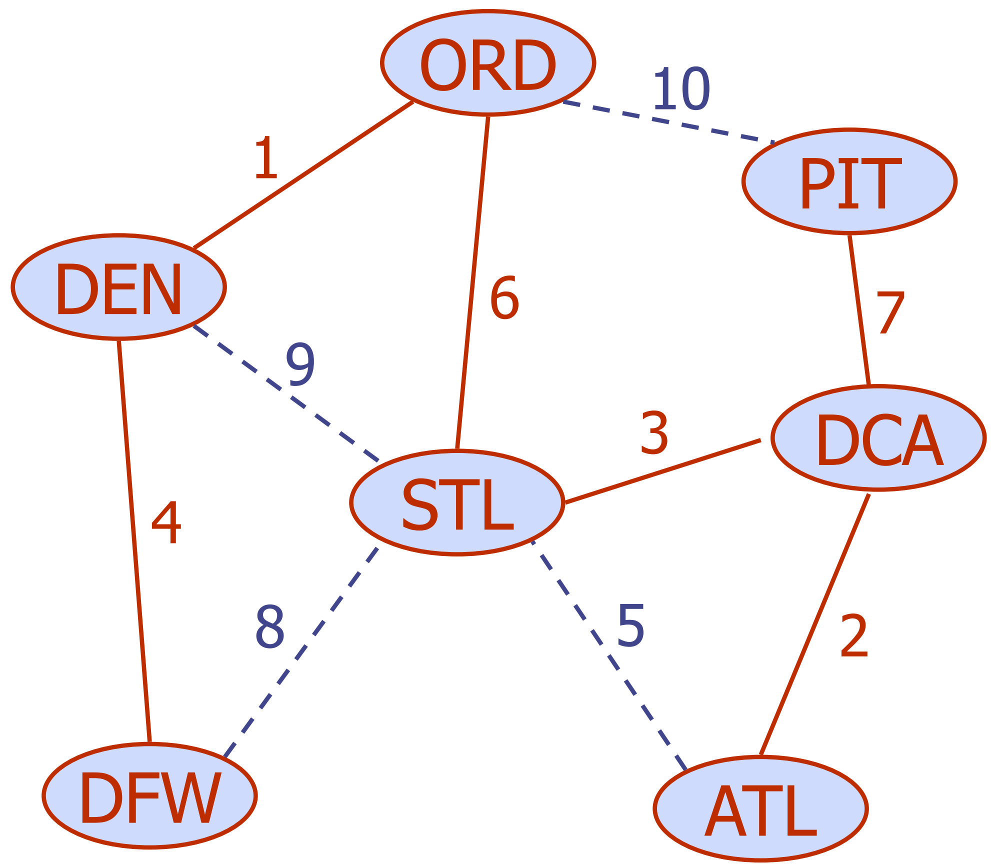


Cycle Property

In other words:

take a MST

in any cycle of the graph the non-spanning tree edge (dotted line) has max weight.



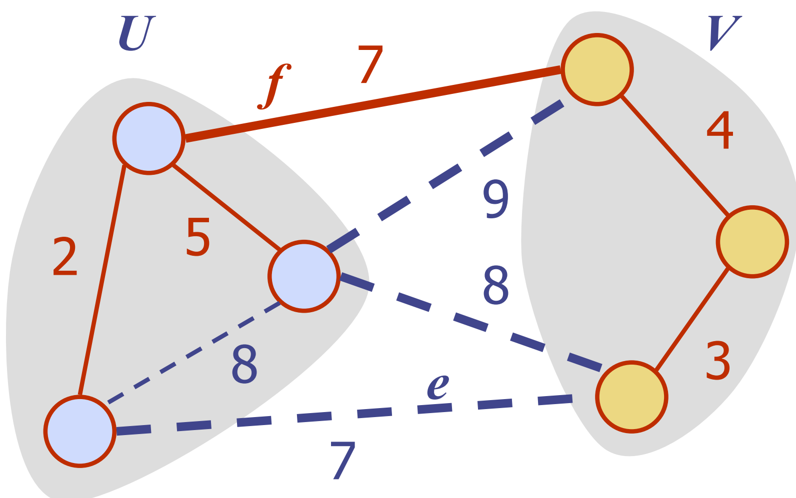
Partition Property

Partition Property:

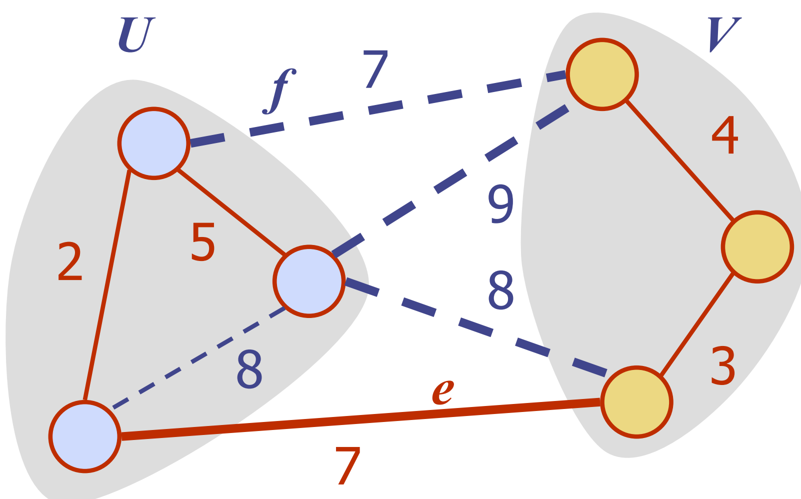
Consider a partition of the vertices of G into subsets U and V. Let e be an edge of minimum weight across the partition. There is a minimum spanning tree of G containing edge e

Proof:

- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property,
$$weight(f) \leq weight(e)$$
- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e



Replacing f with e yields another MST

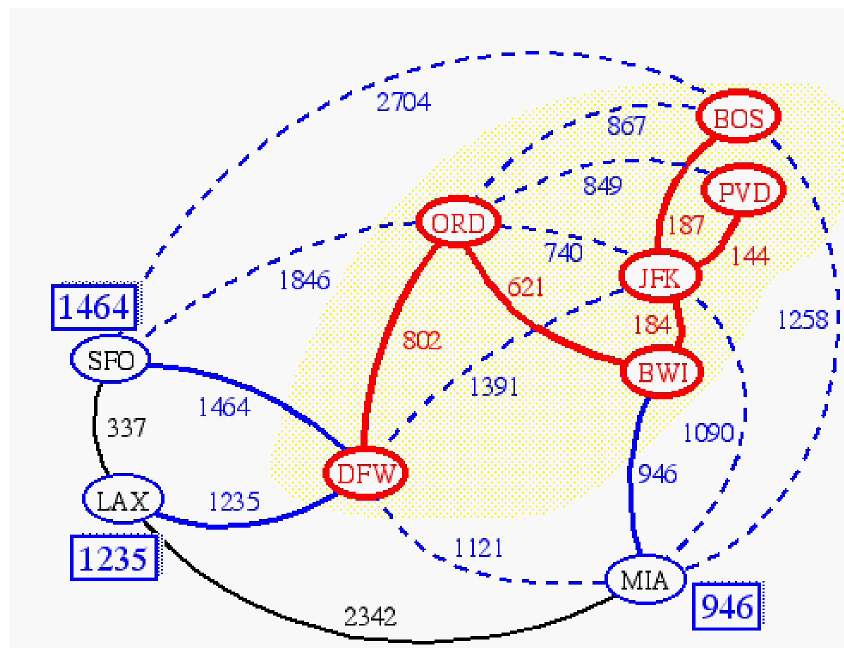


Prim-Jarnik's Algorithm

- Prim-Jarnik's algorithm for computing an MST is similar to Dijkstra's algorithm
- We assume that the graph is connected
- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v a label $d(v)$ representing the smallest weight of an edge connecting v to any vertex in the cloud (as opposed to the total sum of edge weights on a path from the start vertex to u).

Prim-Jarnik's Algorithm

- At each step
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u



- Use a priority queue Q whose keys are D labels, and whose **elements are vertex-edge pairs**.
  - Key: distance
  - Element: vertex
- Any vertex v can be the starting vertex.
- We still initialize all the D[u] values to INFINITE, but we also **initialize E[u] (the edge associated with u) to null.**
- **Return** the minimum-spanning tree T.
- *We can reuse code from Dijkstra's, and we only have to change a few things. Let's look at the pseudocode....*

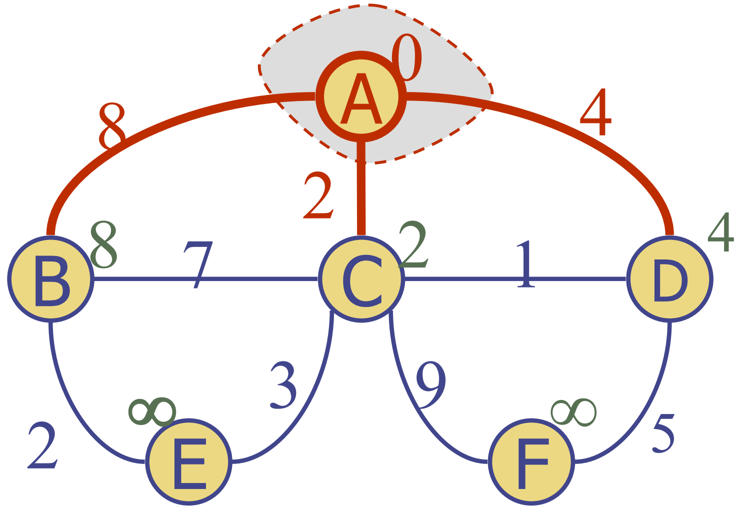
Algorithm PrimJarnik(G):

Input: A weighted graph G.

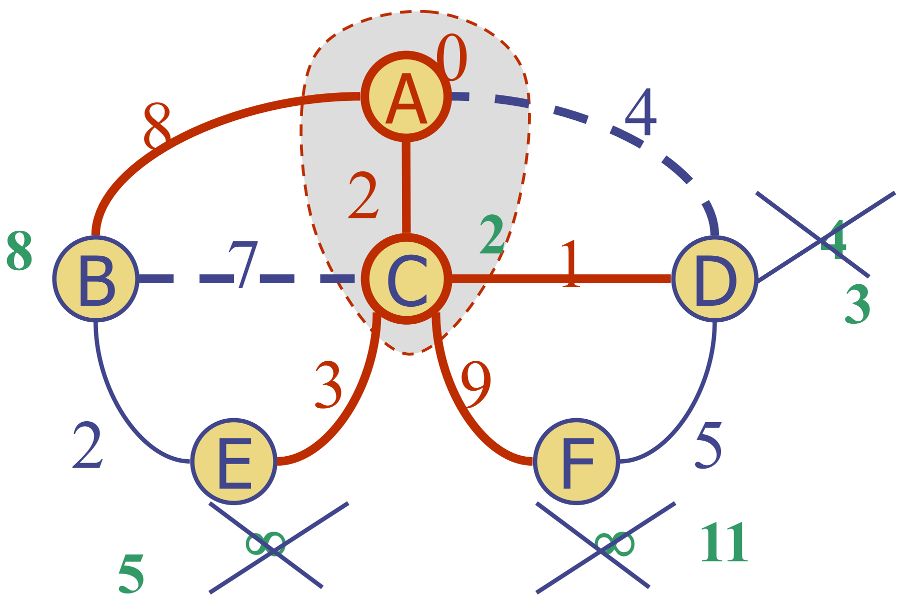
Output: A minimum spanning tree T for G.

```
pick any vertex v of G
{grow the tree starting with vertex v}
 T ← {v}
 D[u] ← 0
 E[u] ← ∅
 for each vertex u ≠ v do
   D[u] ← ∞
 let Q be a priority queue that contains vertices, using the D labels as keys
 while Q ≠ ∅ do  {pull u into the cloud C}
    u ← Q.removeMinElement()
    add vertex u and edge E[u] to T
   for each vertex z adjacent to u do  if z is in Q
    {perform the relaxation operation on edge (u, z) }
          if weight(u, z) < D[z] then
            D[z] ←weight(u, z)
            E[z] ← (u, z) change the key of z in Q to D[z]
 return tree T
```

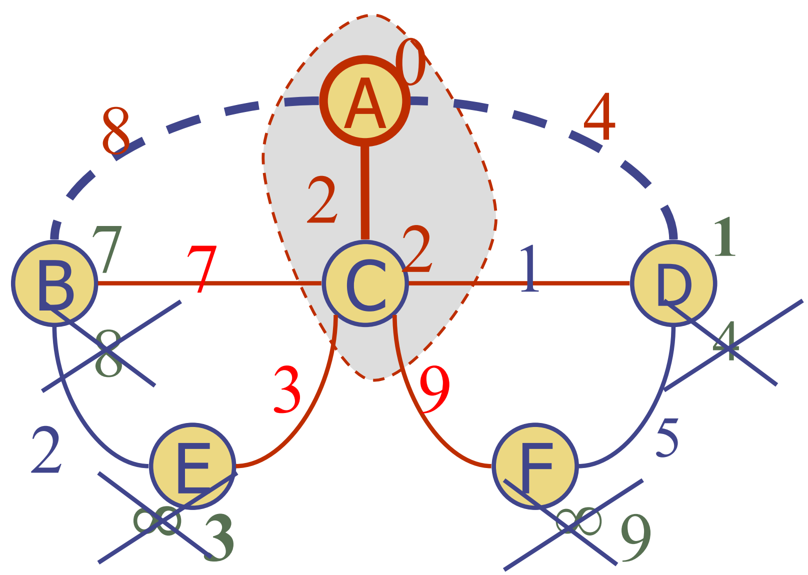
Notice
the difference

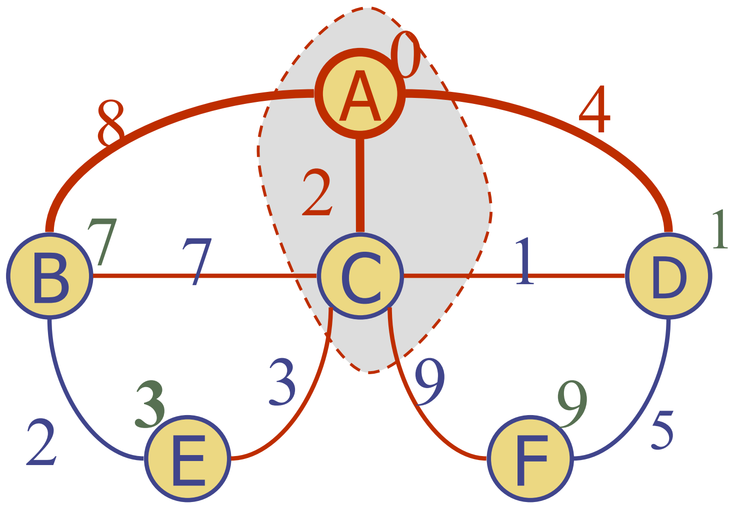


Djkstra
(Shortest Path Spanning Tree)



PRIM
(Minimum Spanning Tree)

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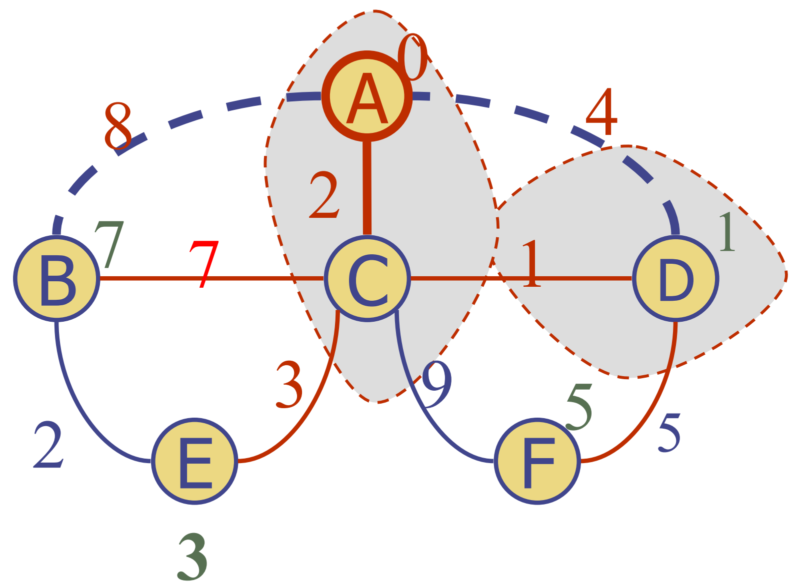
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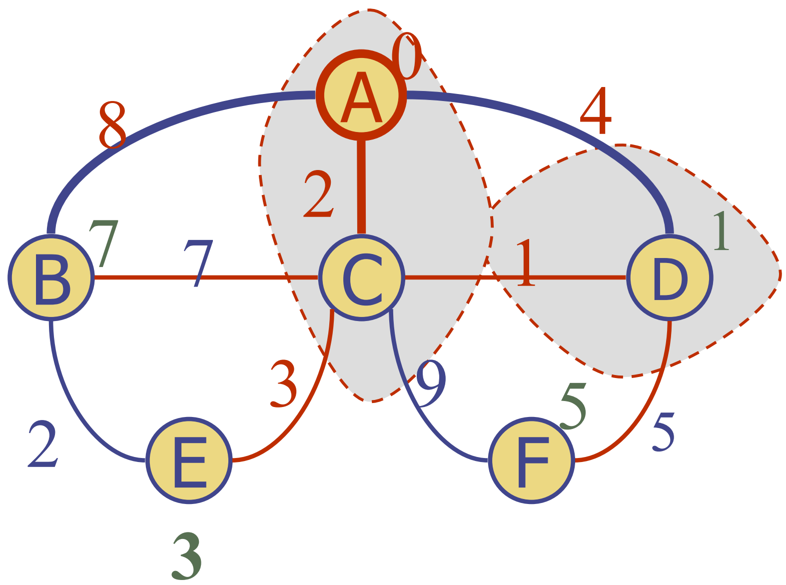
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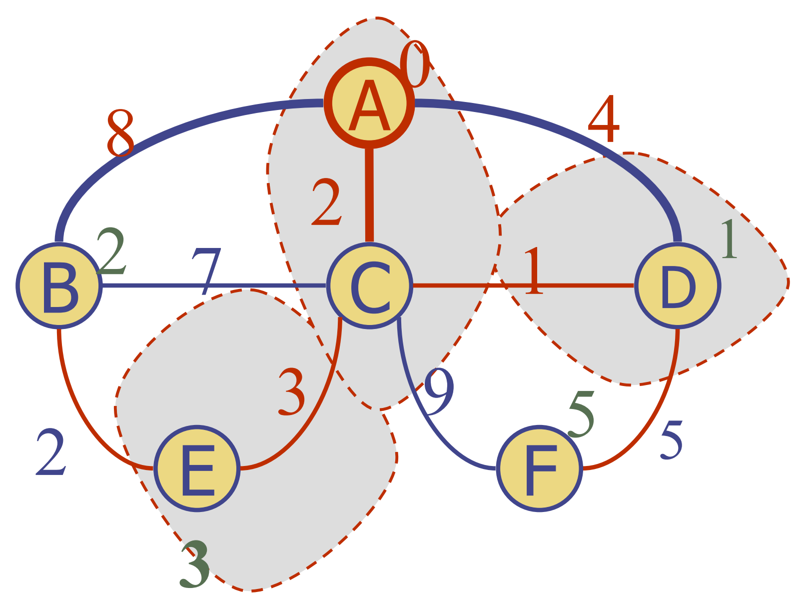
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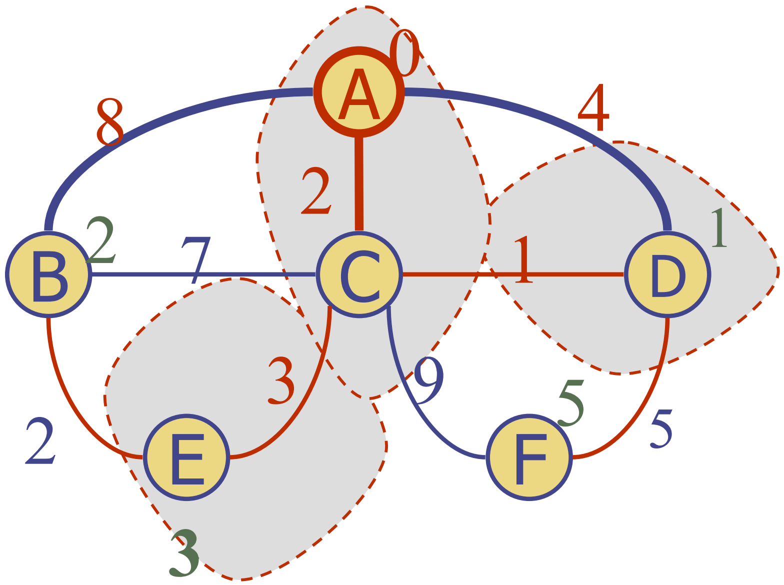
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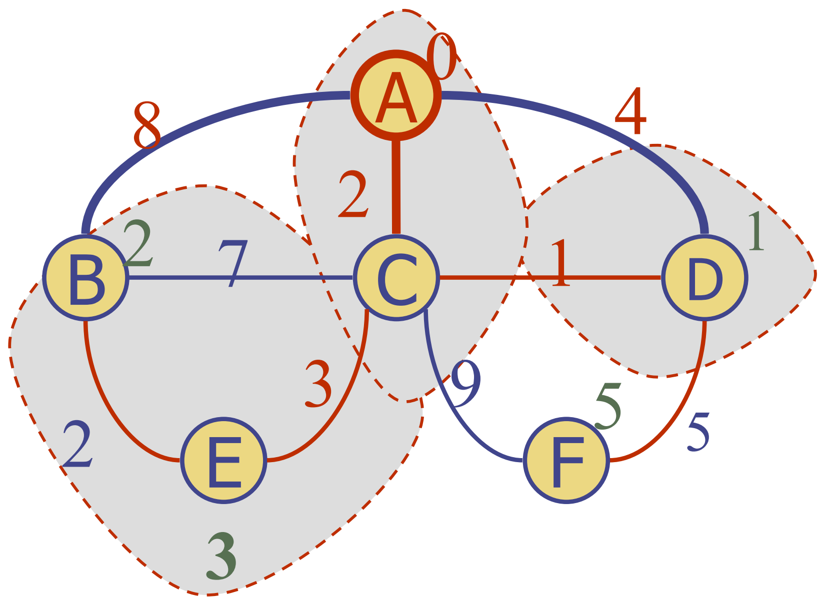
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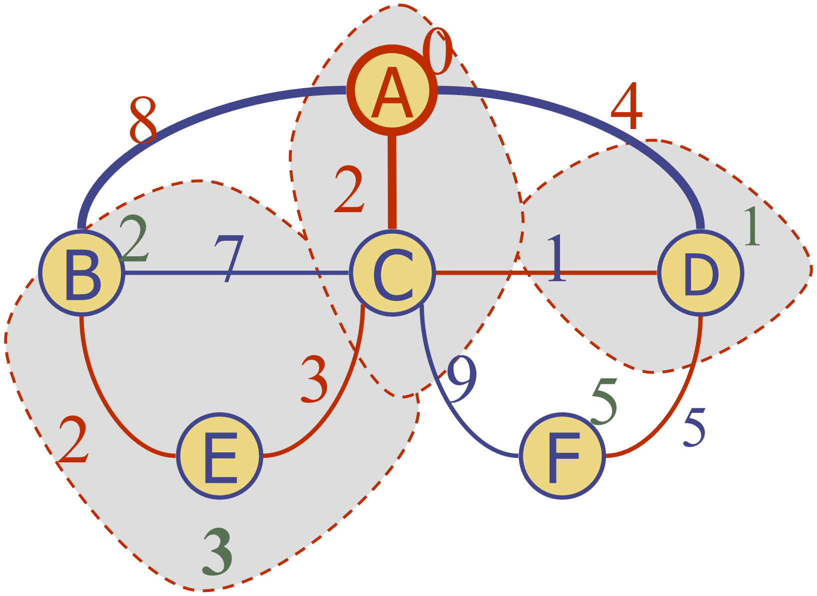
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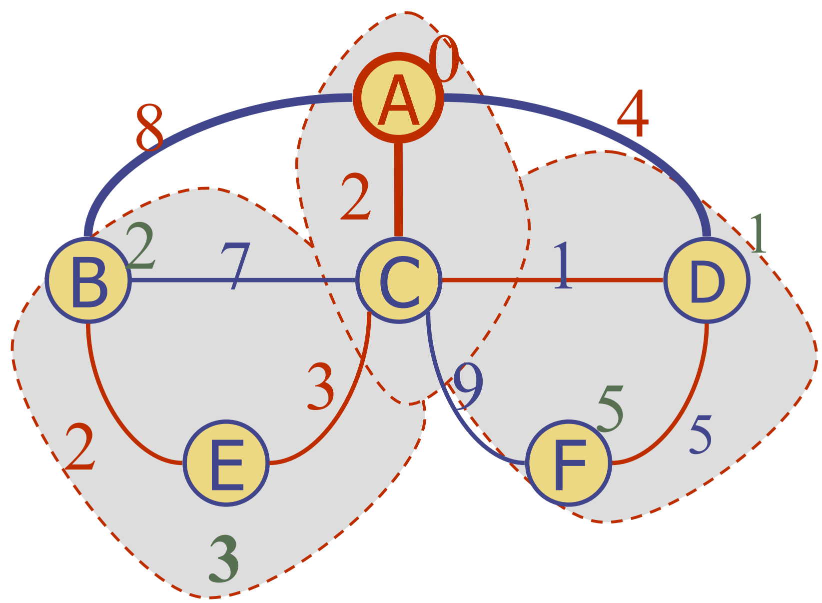
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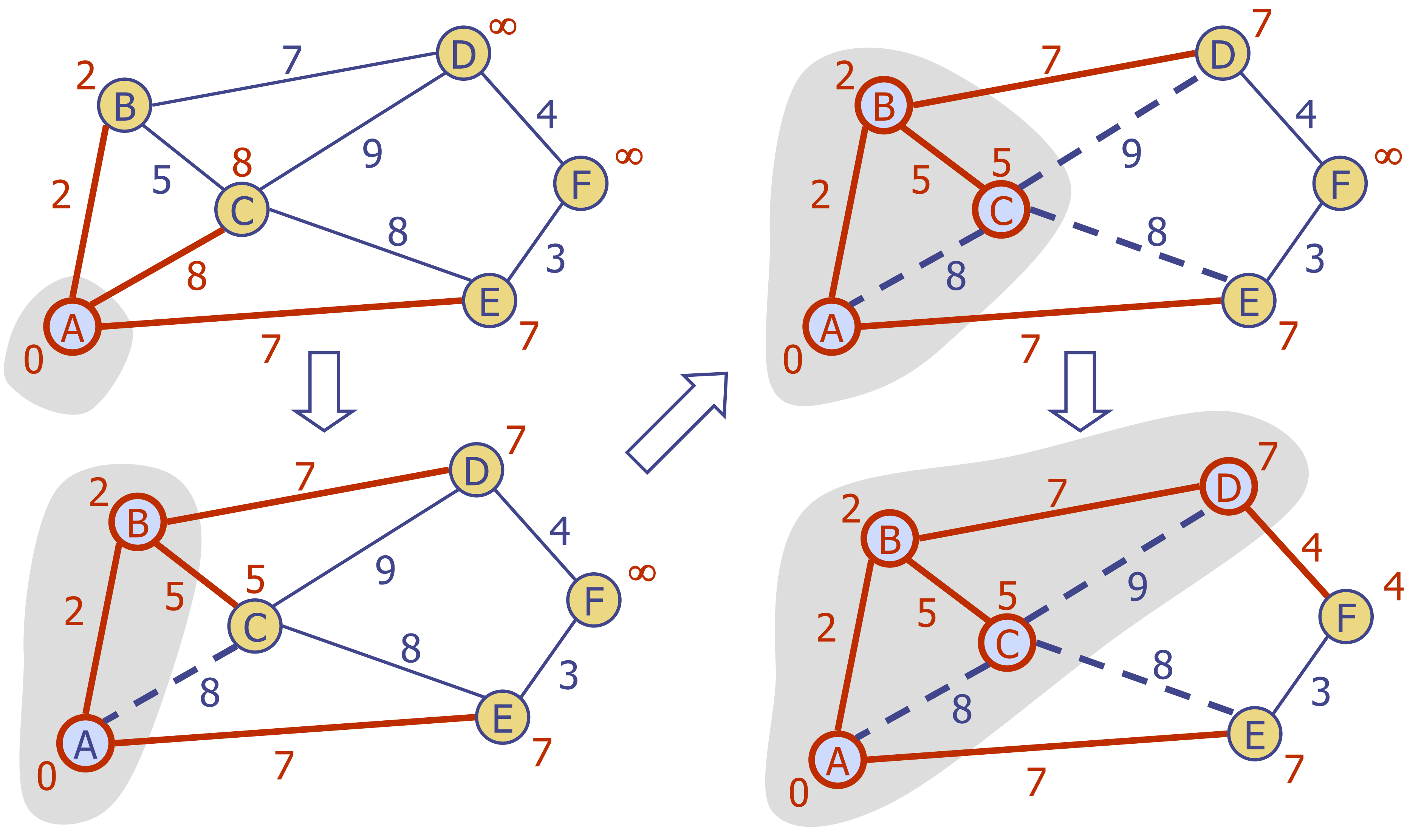
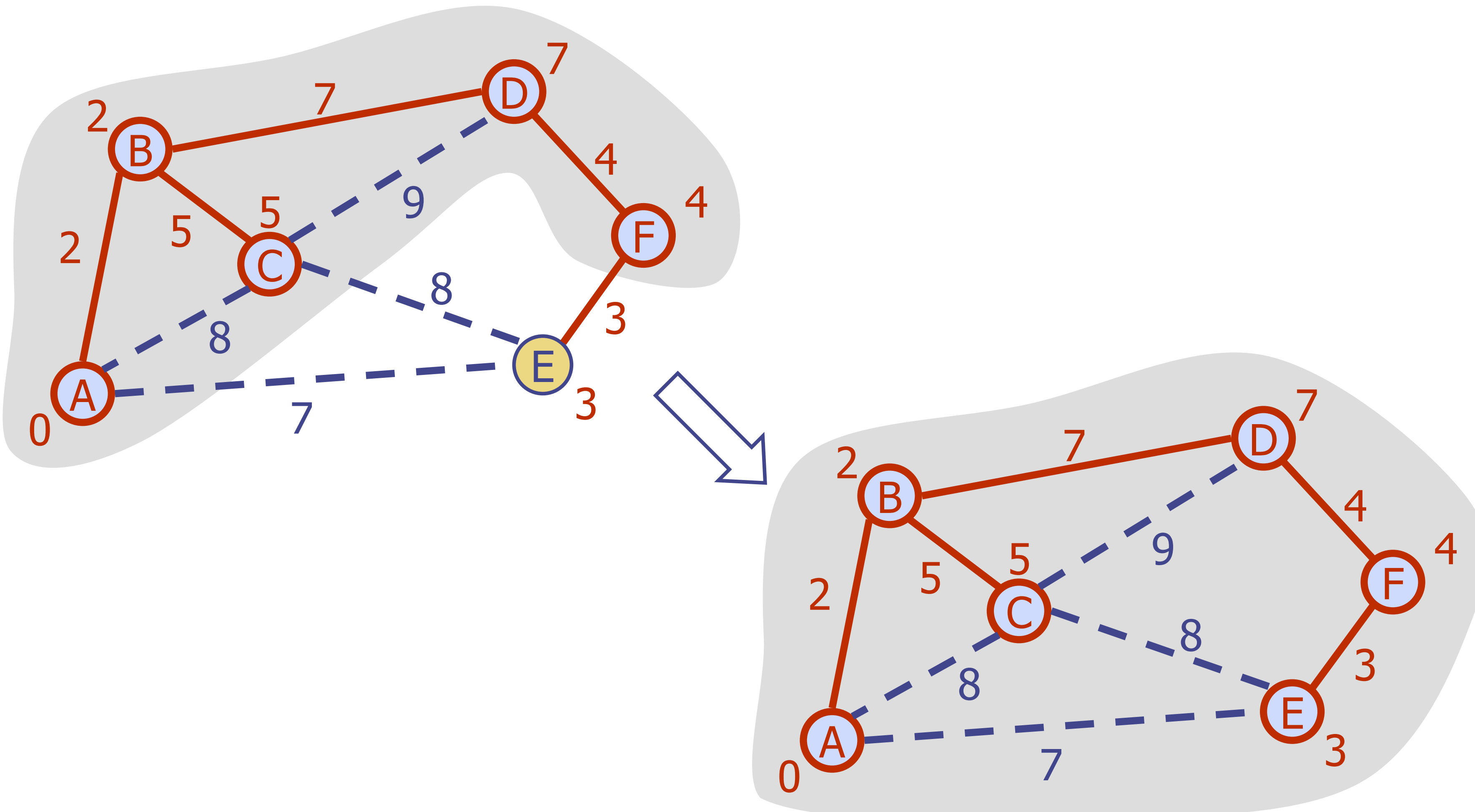
Example



Example (contd.)



Prim-Jarnik… Why It Works

- This is an application of the Cycle Property!
- Let the minimum edge at some iteration be (u,v). If there is an MST not containing (u,v), then (u,v) completes a cycle. Since (u,v) was considered before some other edge connecting v to the cluster, it must have weight equal to or lower than that other edge. A new MST can be formed by swapping.

Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$
- The running time is $O(m \log n)$ since the graph is connected

Dijkstra vs. Prim-Jarnik

```
Algorithm DijkstraShortestPaths(G, s)
    Q ← new heap-based priority queue

    for all v ∈ G.vertices()
        if v = s
            setDistance(v, 0)
        else
            setDistance(v, ∞)
        setParent(v, ∅)
        l ← Q.insert(getDistance(v), v)
        setLocator(v,l)
    while ¬Q.isEmpty()
        u ← Q.removeMin()
        for all e ∈ G.incidentEdges(u)
            z ← G.opposite(u,e)
            r ← getDistance(u) + weight(e)
            if r < getDistance(z)
                setDistance(z,r)
                setParent(z,e)
                Q.replaceKey(getLocator(z),r)
```

```
Algorithm PrimJarnikMST(G)
    Q ← new heap-based priority queue
    s ← a vertex of G
    for all v ∈ G.vertices()
        if v = s
            setDistance(v, 0)
        else
            setDistance(v, ∞)
        setParent(v, ∅)
        l ← Q.insert(getDistance(v), v)
        setLocator(v,l)
    while ¬Q.isEmpty()
        u ← Q.removeMin()
        for all e ∈ G.incidentEdges(u)
            z ← G.opposite(u,e)
            r ← weight(e)
            if r < getDistance(z)
                setDistance(z,r)
                setParent(z,e)
                Q.replaceKey(getLocator(z),r)
```

Kruskal's Algorithm

Kruskal's Algorithm

- Each vertex is initially stored as its own cluster.
- At each iteration, the minimum weight edge is added to the spanning tree if it joins 2 distinct clusters.
- The algorithm ends when all the vertices are in the same cluster.

Kruskal's Algorithm… Why It Works

- This is an application of the Partition Property!
- If the minimum edge at some iteration is (u,v), then if we consider a partition of G with u in one cluster and v in the other, then the partition property says that there must be an MST containing (u,v).

Kruskal's Algorithm

- A priority queue stores the edges outside the cloud
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - We are left with one cloud that encompasses the MST
  - A tree *T* which is our MST

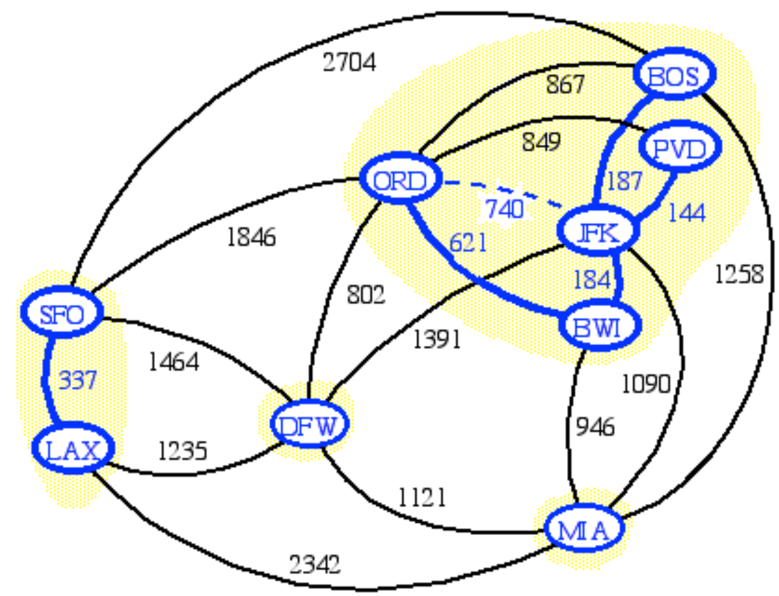
Algorithm ***KruskalMST(G)***
    for each vertex *V* in ***G*** **do**
        define a ***Cloud(v)*** of $\leftarrow$ {*v*}
    let ***Q*** be a priority queue.
    Insert all edges into ***Q*** using their weights as the key
    T $\leftarrow \varnothing$
    while ***T*** has fewer than ***n***-1 edges **do**
    edge ***e*** = ***Q.removeMin()***
        Let ***u***, ***v*** be the endpoints of *e*
        if ***Cloud(v)*** $\neq$ ***Cloud(u)*** **then**
            Add edge ***e*** to ***T***
            Merge ***Cloud(v)*** and ***Cloud(u)***
    return ***T***

Data Structure for Kruskal Algortihm

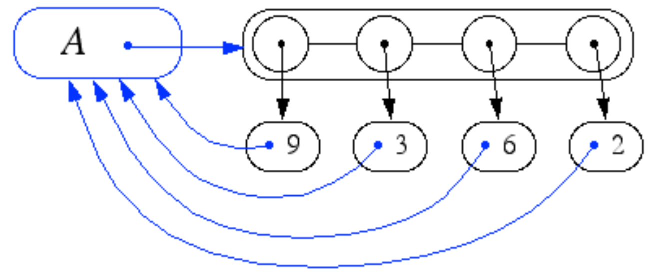
- The algorithm maintains a forest of trees
- An edge is accepted it if connects distinct trees
- We need a data structure that maintains a **partition**, i.e., a collection of disjoint sets, with the operations:

  -**find**(u): return the set storing u

  -**union**(u,v): replace the sets storing u and v with their union



Representation of a Partition



- Each set is stored in a sequence
- Each element has a reference back to the set
  - operation find(u) takes O(1) time, and returns the set of which u is a member.
  - in operation union(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation union(u,v) is $\min(n_u,n_v)$, where n_u and n_v are the sizes of the sets storing u and v
- Whenever an element is processed, it goes into a set of size at least double, hence each element is processed at most log n times

Partition-Based Implementation

- A oretion-based version of Kruskal's Algorithm performs cloud merges as unions and tests as finds.

Algorithm **Kruskal(*G*)**:

Input: A weighted graph ***G***.

Output: a set of edges T of a MST for ***G***, if ***G*** is connected;

a set of edges T of a MS forest ***T*** for ***G***, if ***G*** is not connected;

Let ***P*** be a partition of the vertices of ***G***, where each vertex forms a separate set.

Let ***Q*** be a priority queue storing the edges of ***G***, organized by their weights

Let ***T*** be an initially-empty set of edges

while (***Q*** is not empty) and (T.size() < n-1) **do**

(***u***,***v***) ← ***Q***.removeMinElement()

if ***P***.find(***u***) != ***P***.find(***v***) **then**

Add (***u***,***v***) to ***T***

P.union(***u***,***v***)

return ***T***

Running time:

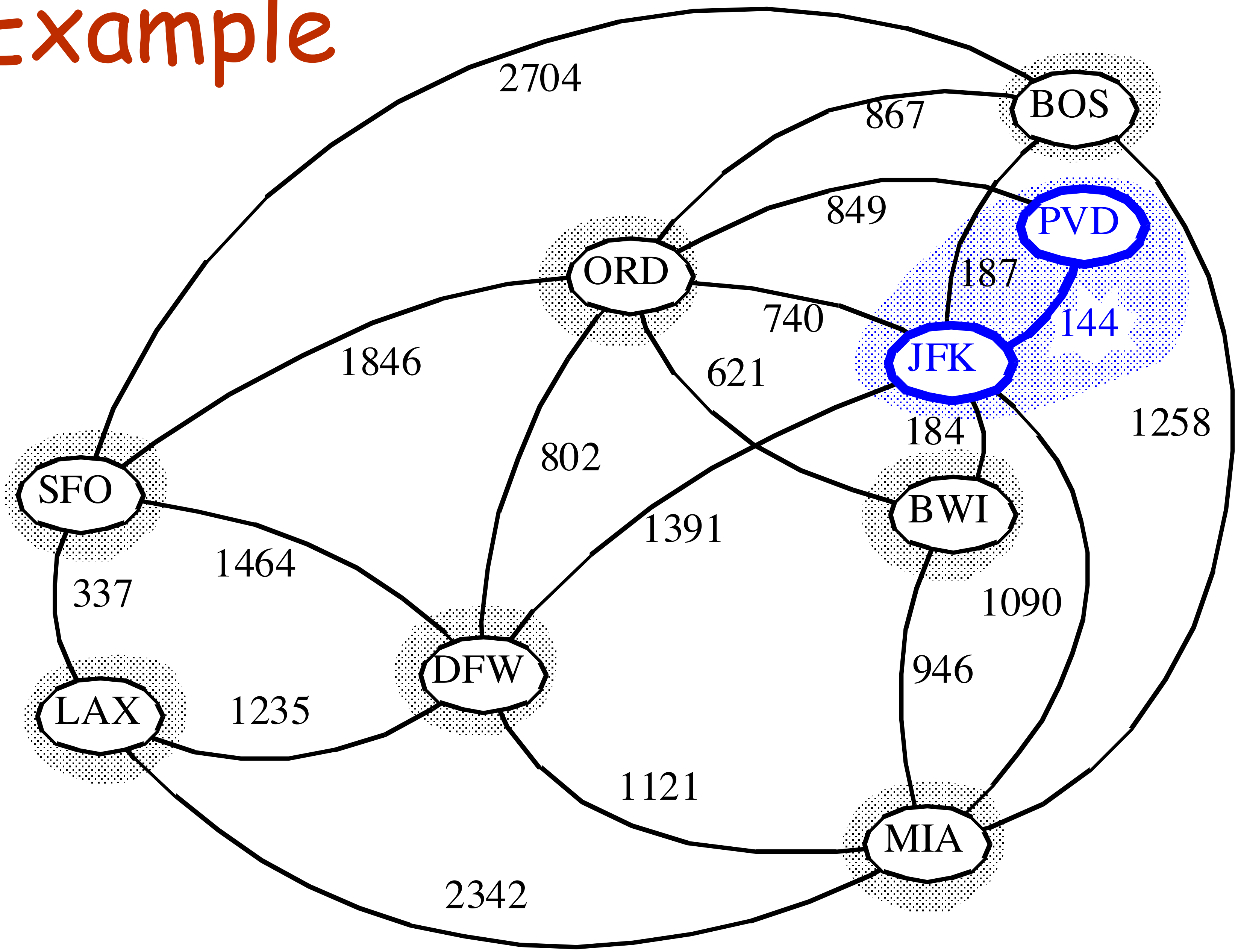
O((n+m)log n)

More about Kruskal's running time

- Building a priority queue Q with m elements takes O(m log m) or O(m) using buildheap.
- Operation ***Q***.removeMinElement() takes time O(log m) and is repeated up to m times for a total of O(m log m)
- Operation ***P***.find(x) takes O(1) and is repeated at most 2m times.
- Operation ***P***.union(***u***,***v***) takes time proportional to the smallest of the two parts and is repeated at most n-1 times. As discussed before each of the n elements gets moved to another list at most log n times over all unions. So the total cost for all operations ***P***.union(***u***,***v***) is O(n log n).

In summary, all above considered we get O(n log n + m log m); but since m=O(n^2), O(log m) is O(2 log n)=O(log n). So the running time can be written as O((m+n) log n).

Example



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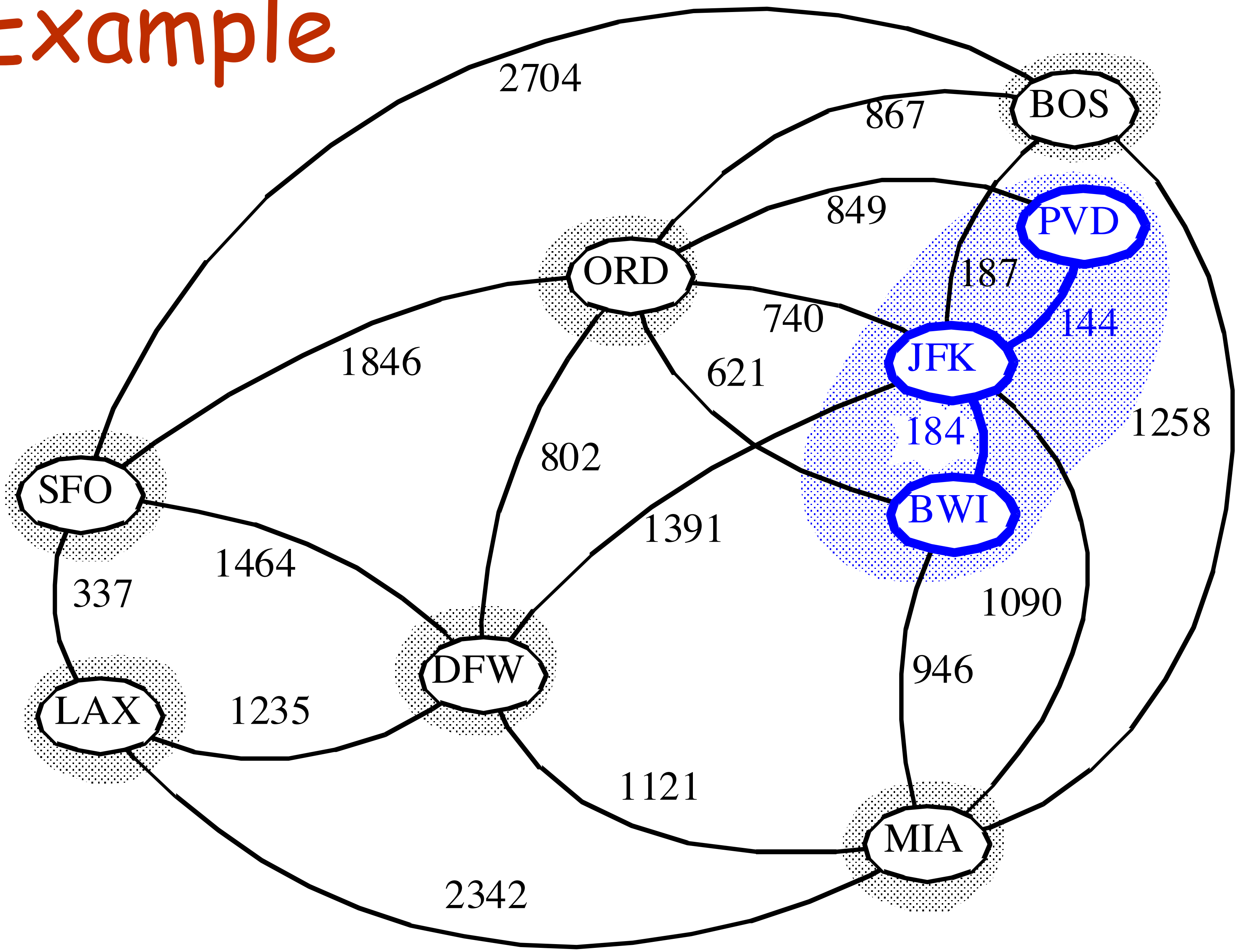
Example



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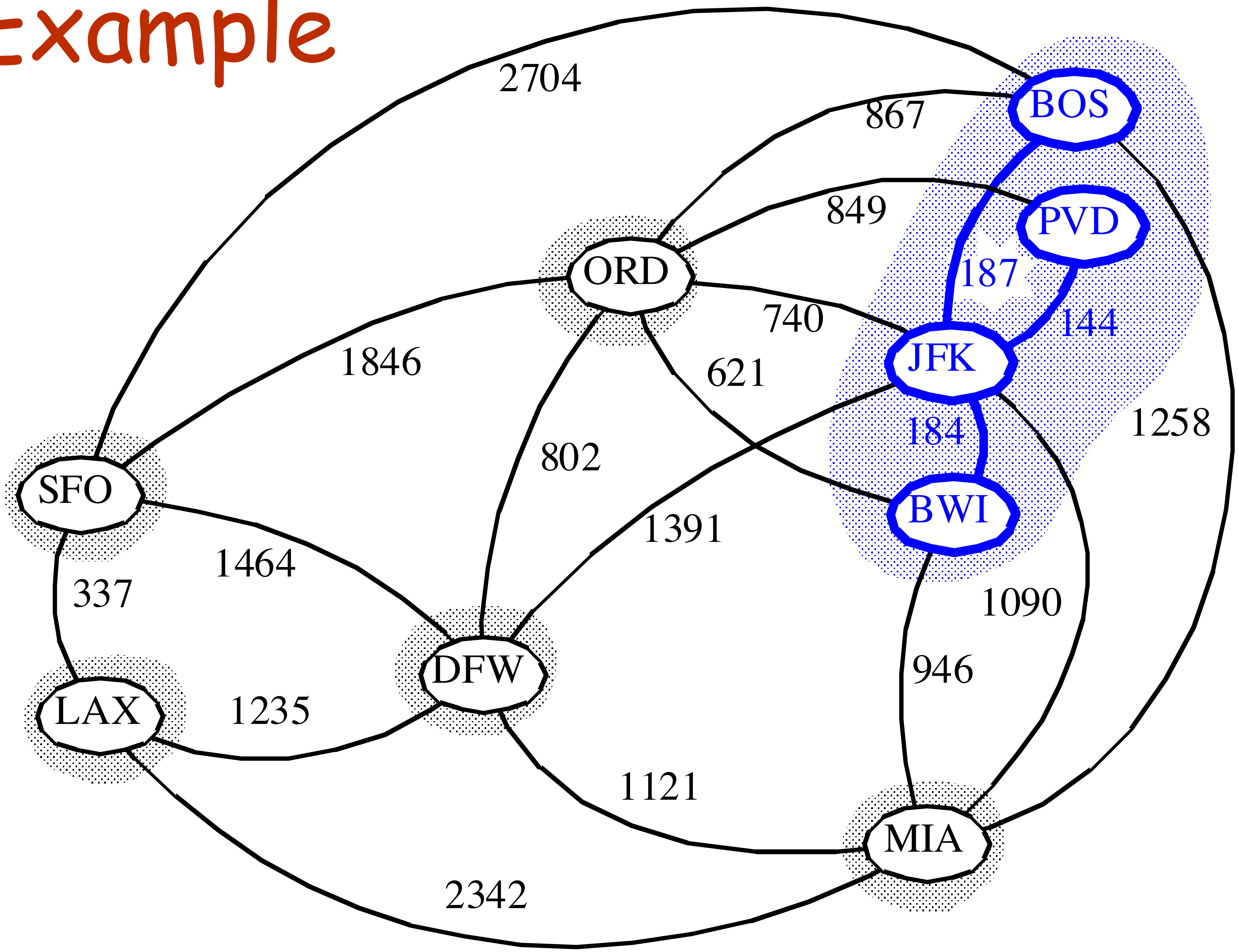

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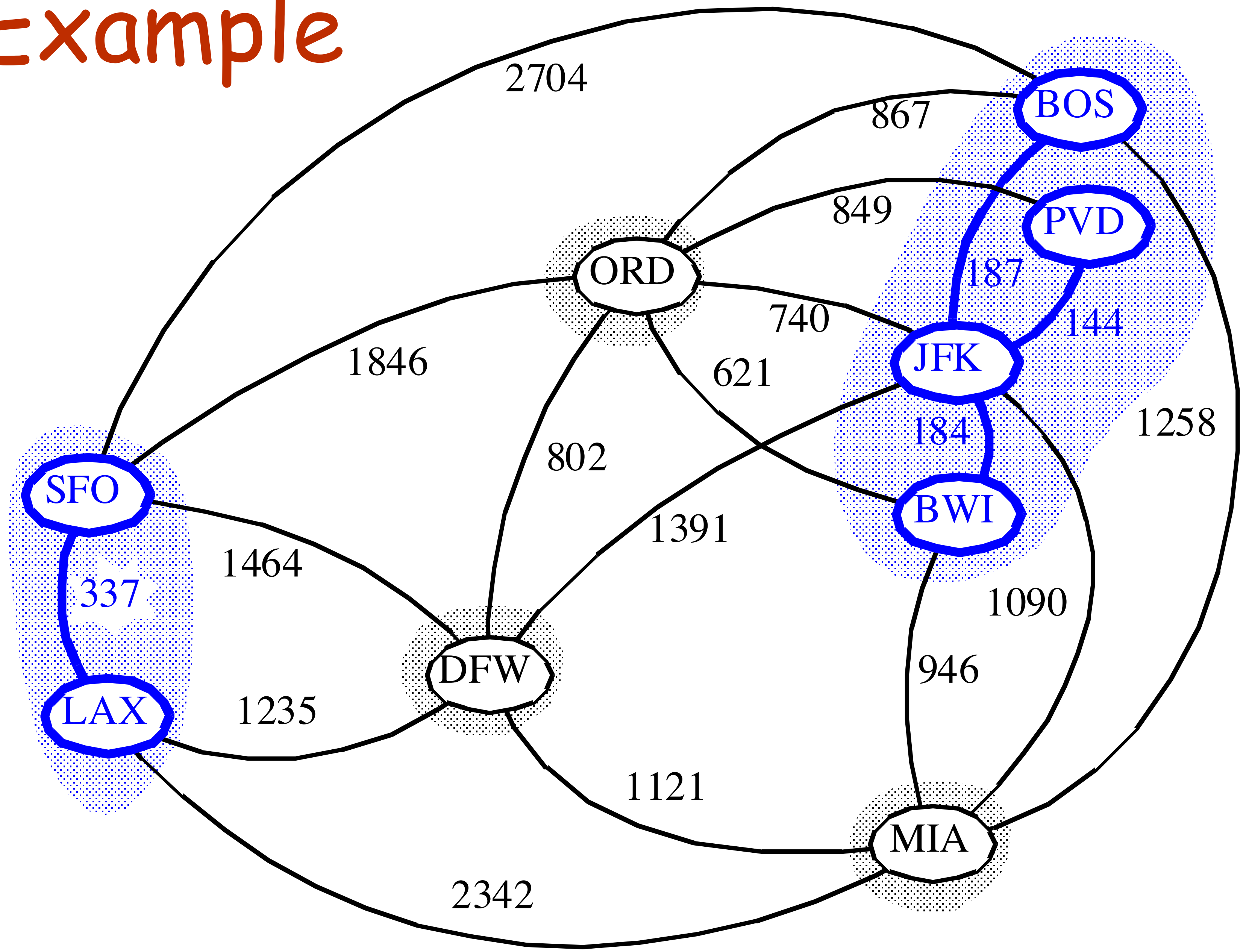

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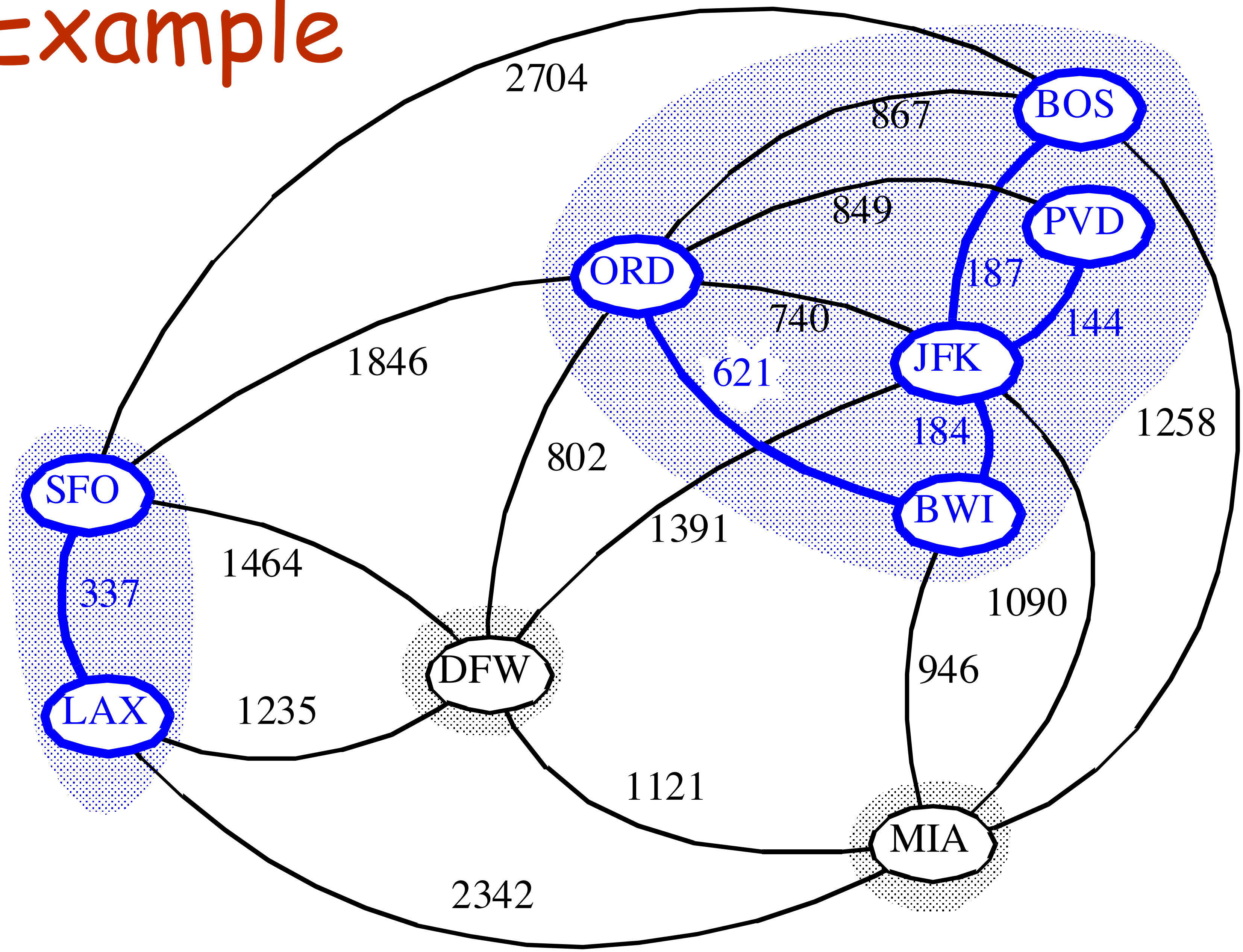

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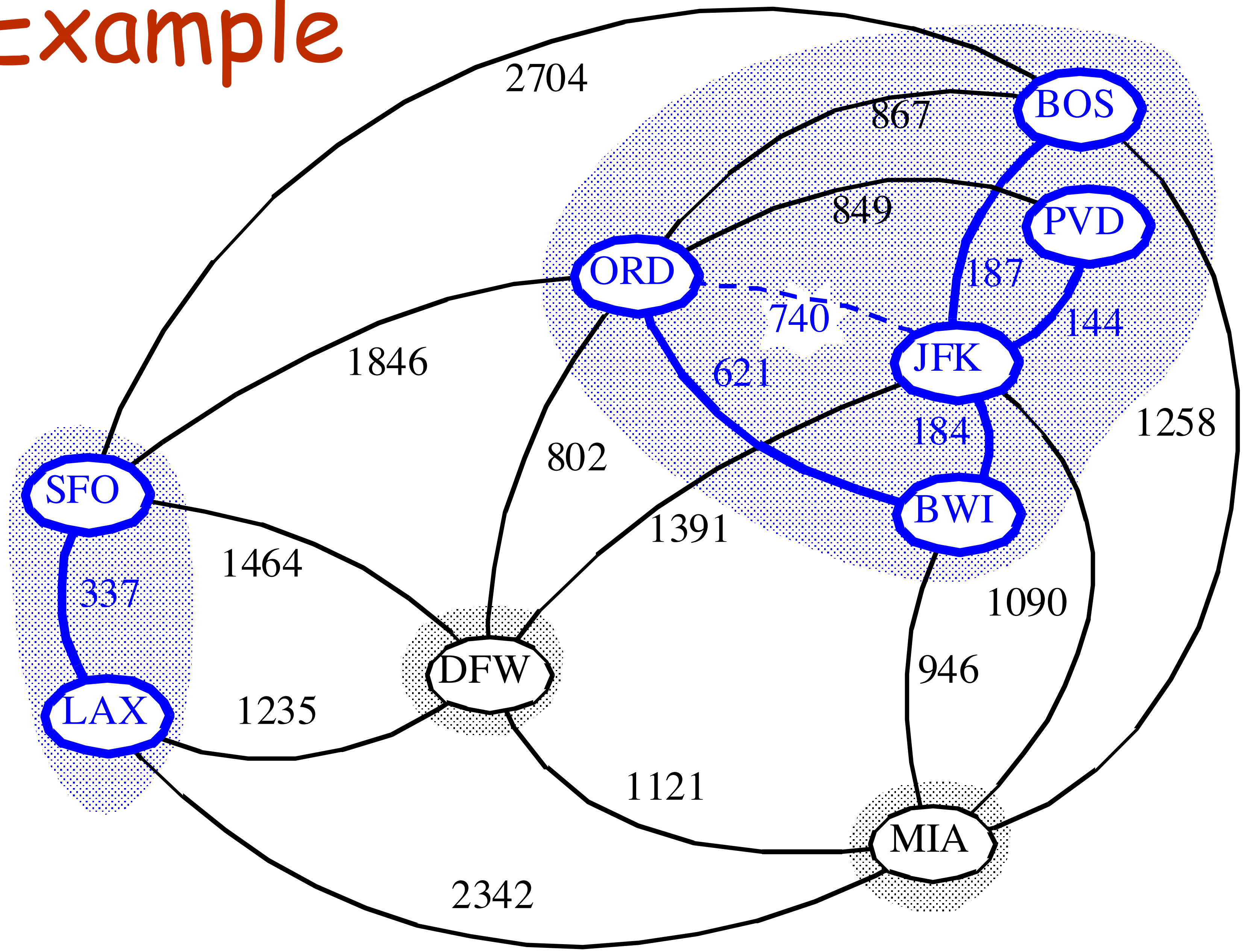

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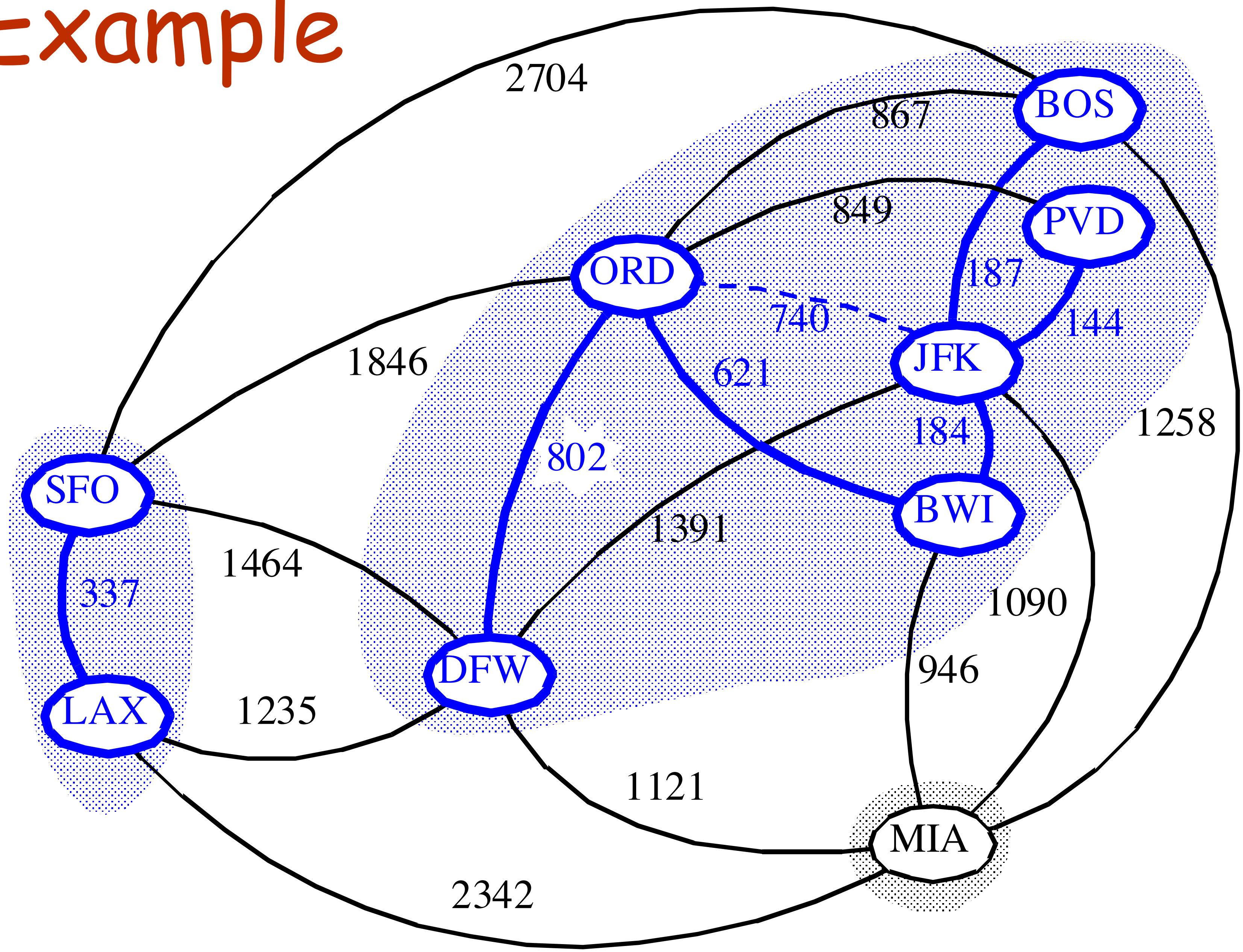

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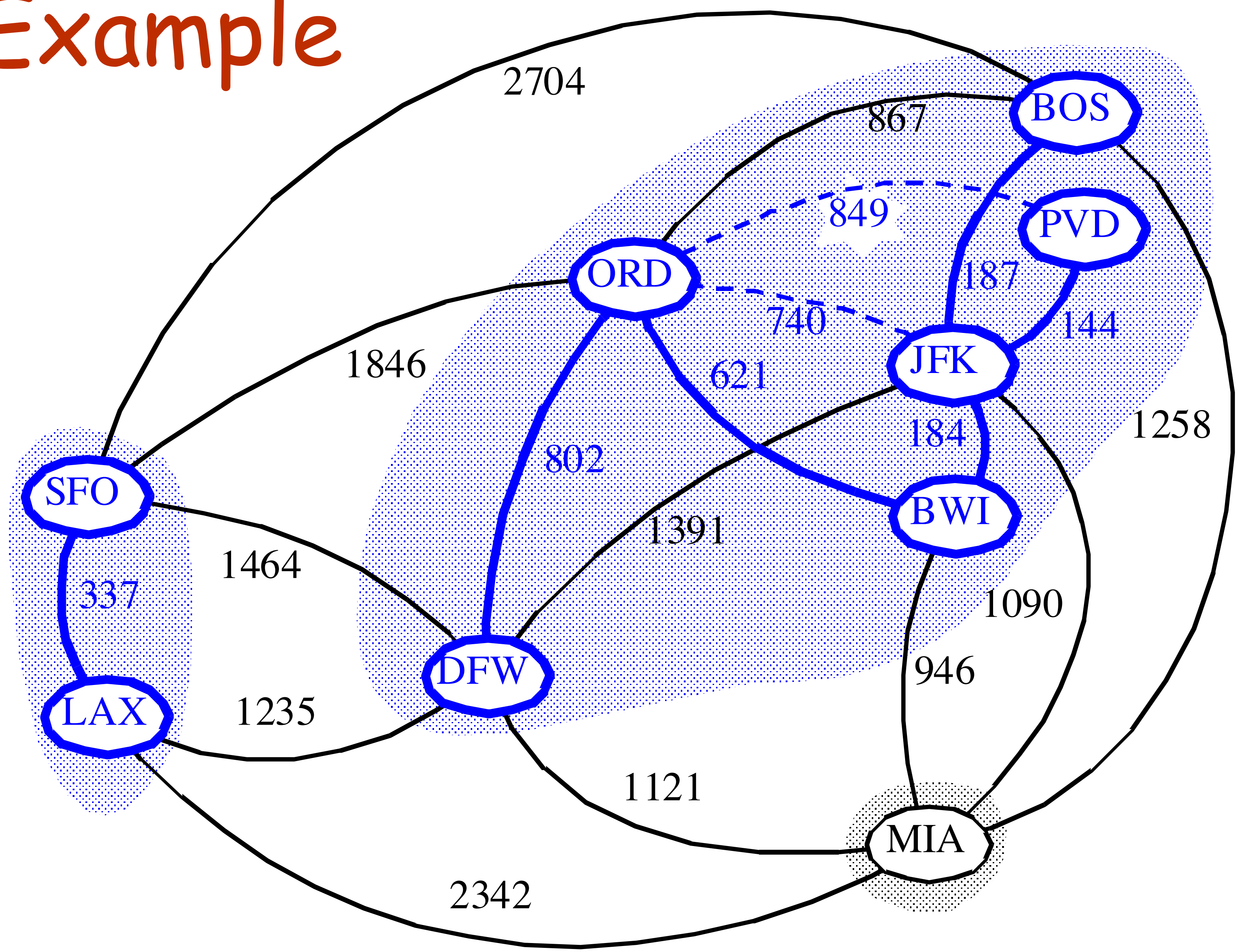
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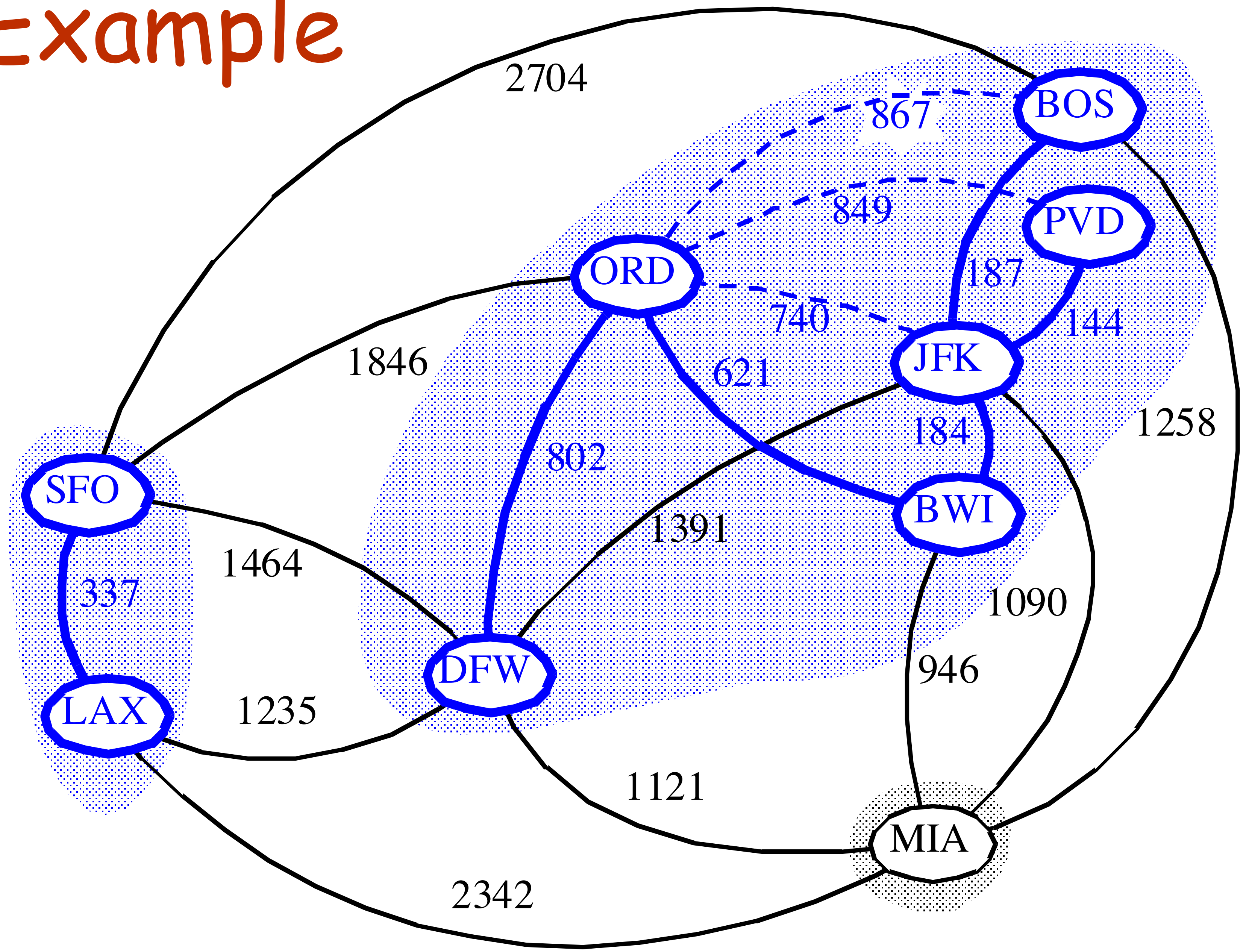

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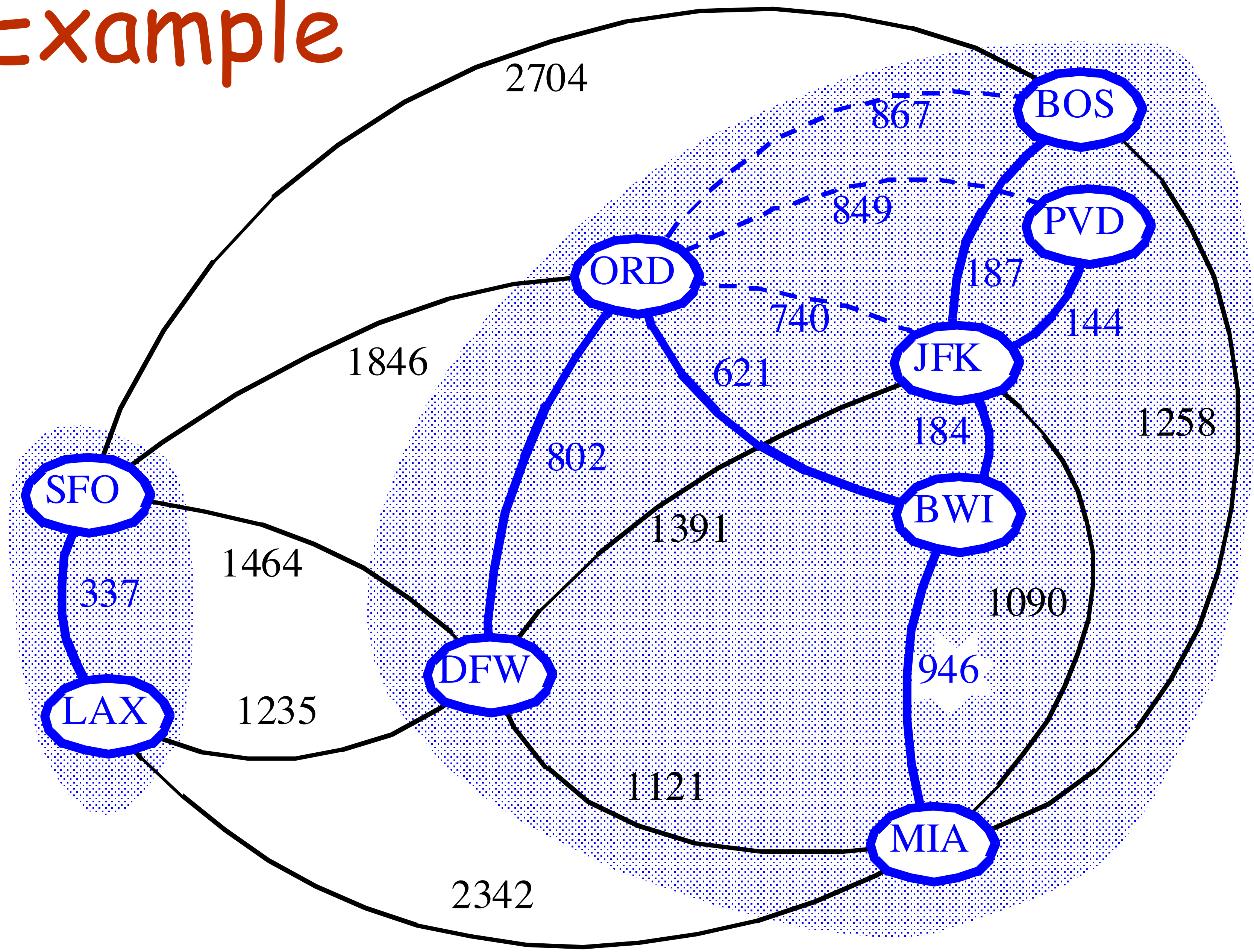
Example



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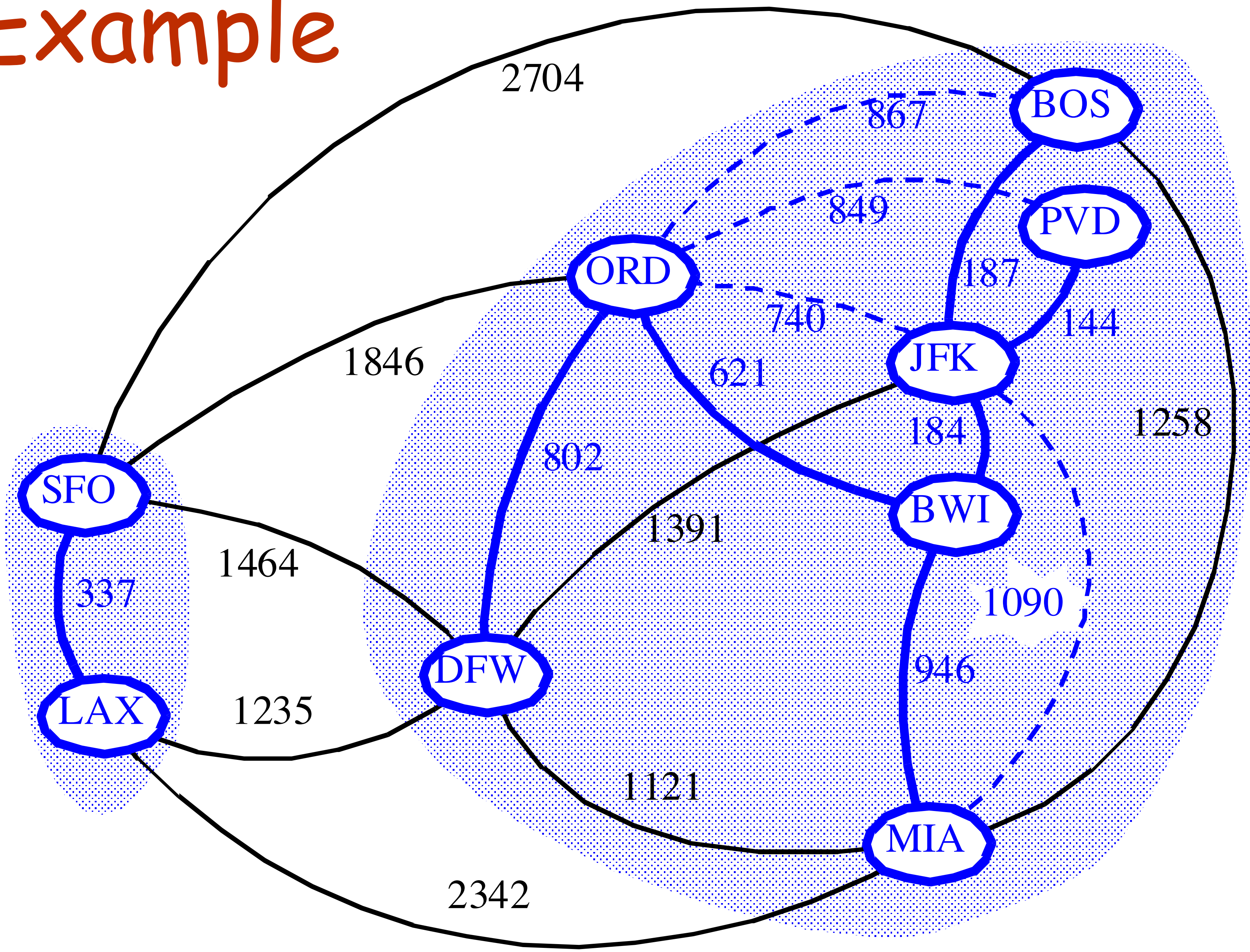
Example



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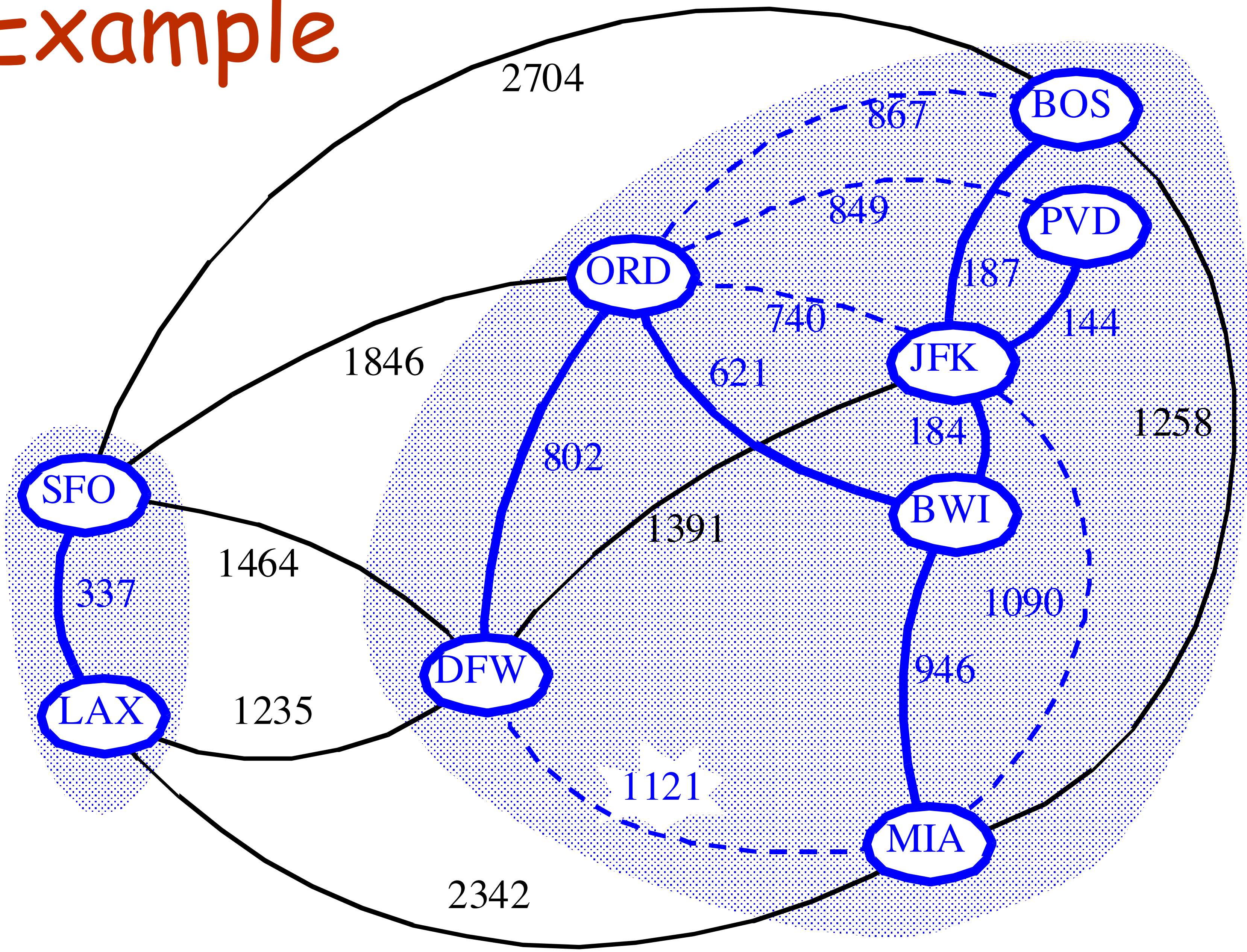
Example



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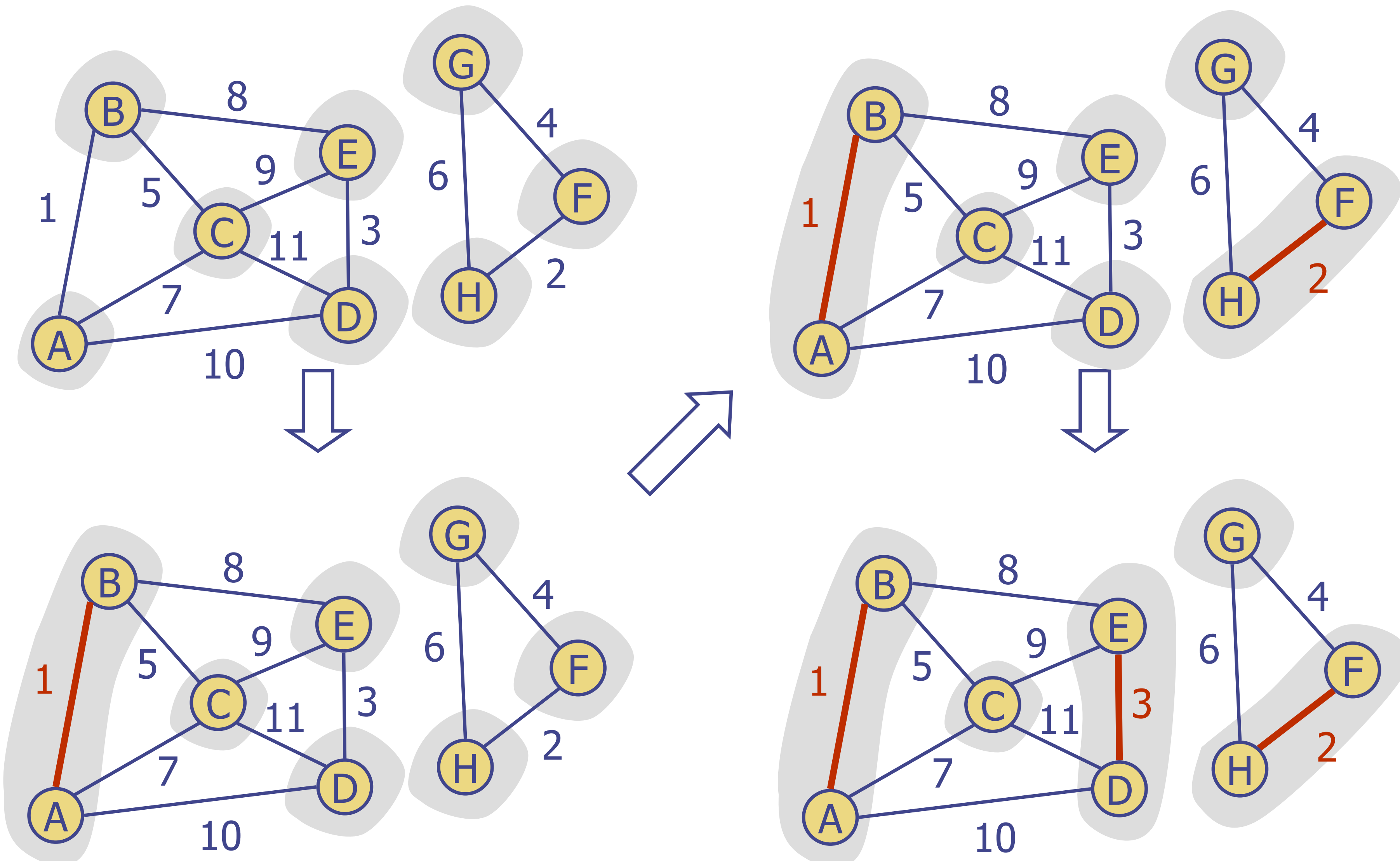
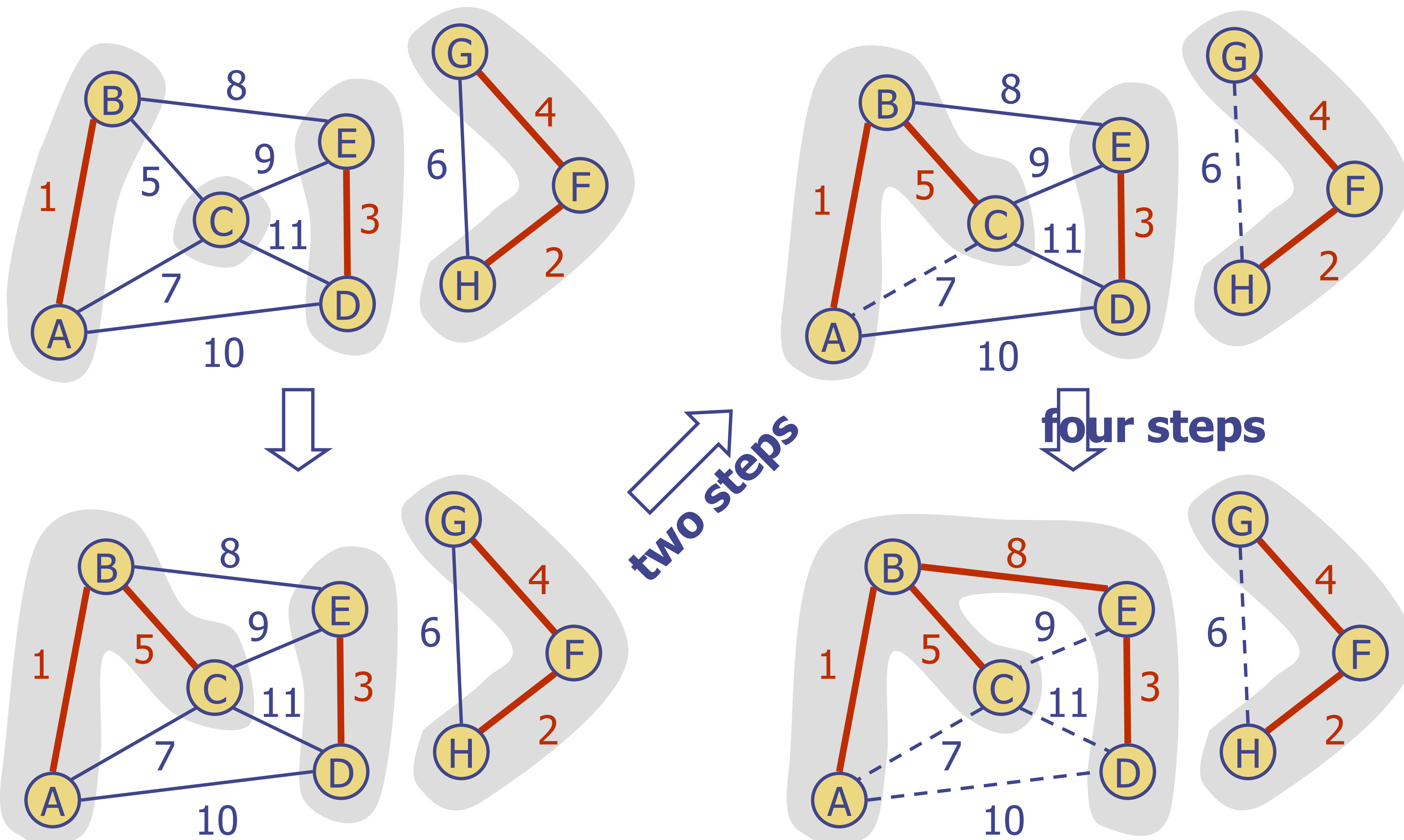
Example



Example (contd.)



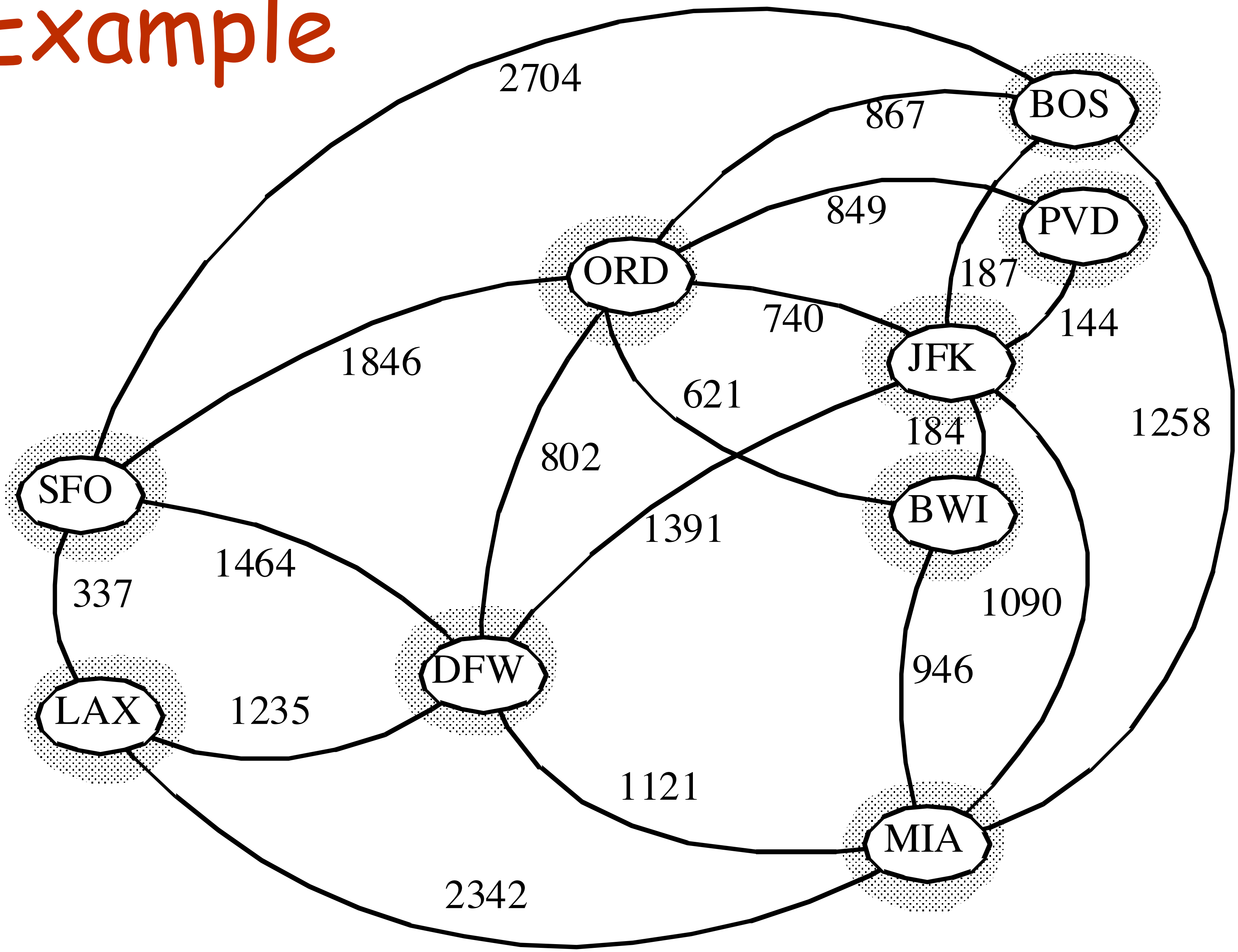
Baruvka's Algorithm

- Like Kruskal's Algorithm, Baruvka's algorithm grows many "clouds" at once.

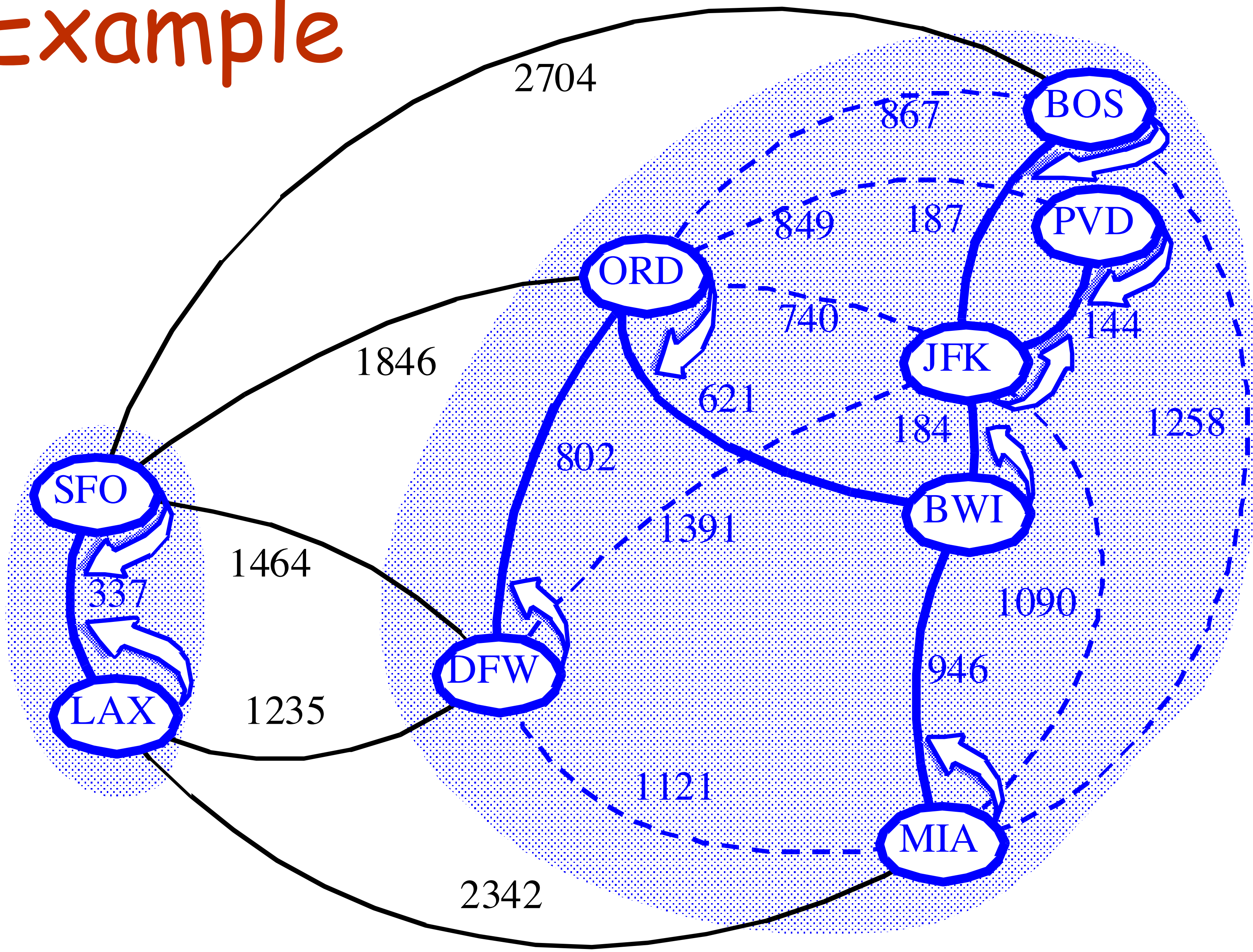
```
Algorithm BaruvkaMST(G)
    T ← V  {just the vertices of G}
  while T has fewer than n-1 edges do
    for each connected component C in T do
      Let edge e be the smallest-weight edge from C to another component in T.
      if e is not already in T then
        Add edge e to T
  return T
```

- Each iteration of the while-loop halves the number of connected compontents in T.
  - The running time is $O(m \log n)$.

Baruvka Example



Example



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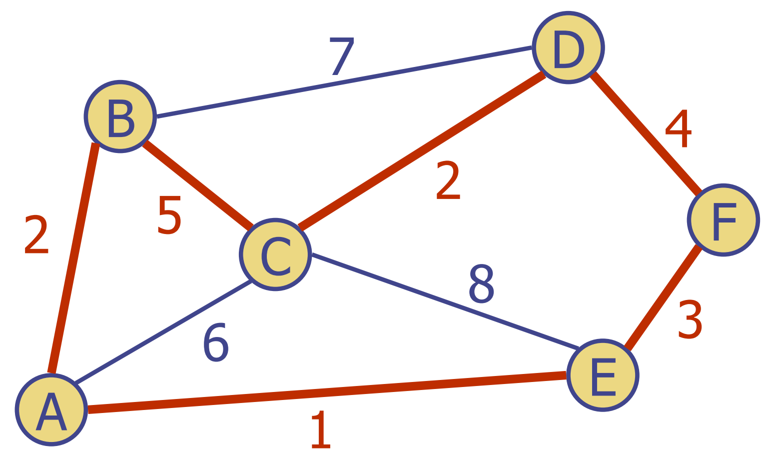
Traveling Salesperson Problem

- A tour of a graph is a spanning cycle (e.g., a cycle that goes through all the vertices)
- A traveling salesperson tour of a weighted graph is a tour that is simple (i.e., no repeated vertices or edges) and has has minimum weight
- No polynomial-time algorithms are known for computing traveling salesperson tours
- The traveling salesperson problem (TSP) is a major open problem in computer science
  - Find a polynomial-time algorithm computing a traveling salesperson tour or prove that none exists



Example of traveling salesperson tour (with weight 17)

TSP Approximation

- We can approximate a TSP tour with a tour of at most twice the weight for the case of Euclidean graphs
  - Vertices are points in the plane
  - Every pair of vertices is connected by an edge
  - The weight of an edge is the length of the segment joining the points
- Approximation algorithm
  - Compute a minimum spanning tree
  - Form an Eulerian circuit around the MST
  - Transform the circuit into a tour