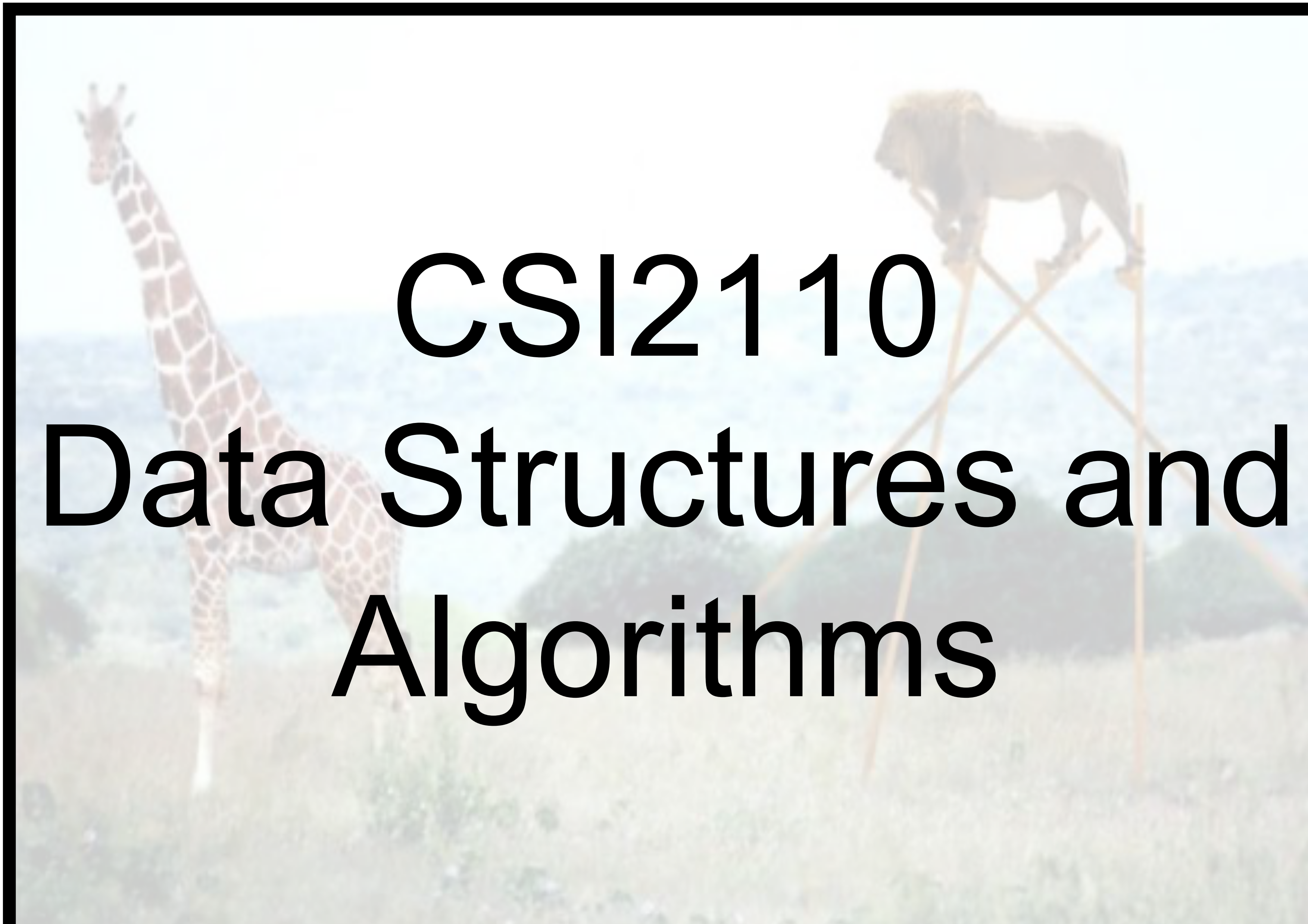
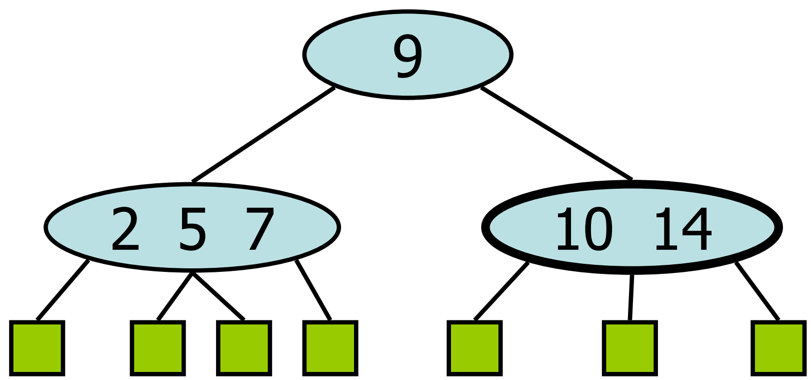


CSI2110 Data Structures and Algorithms

- http://cs.armstrong.edu/liang/animation/web/24Tree.html

(2,4) Trees

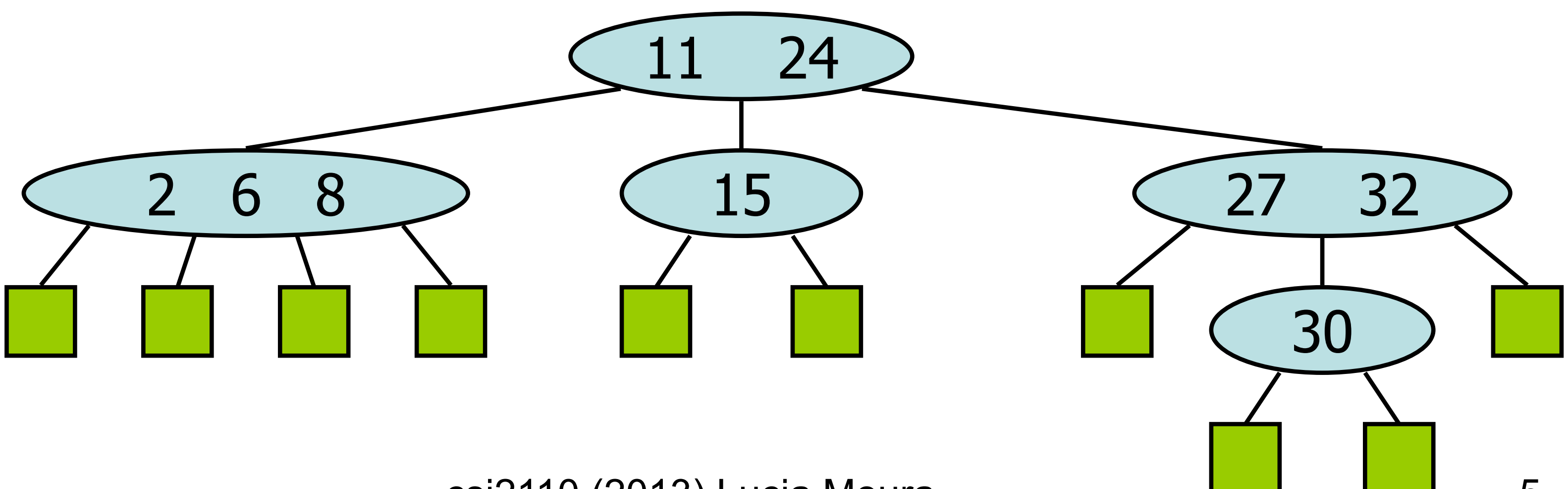


Outline and Reading

- Multi-way search tree
  - Definition
  - Search
- (2,4) tree
  - Definition
  - Search
  - Insertion
  - Deletion

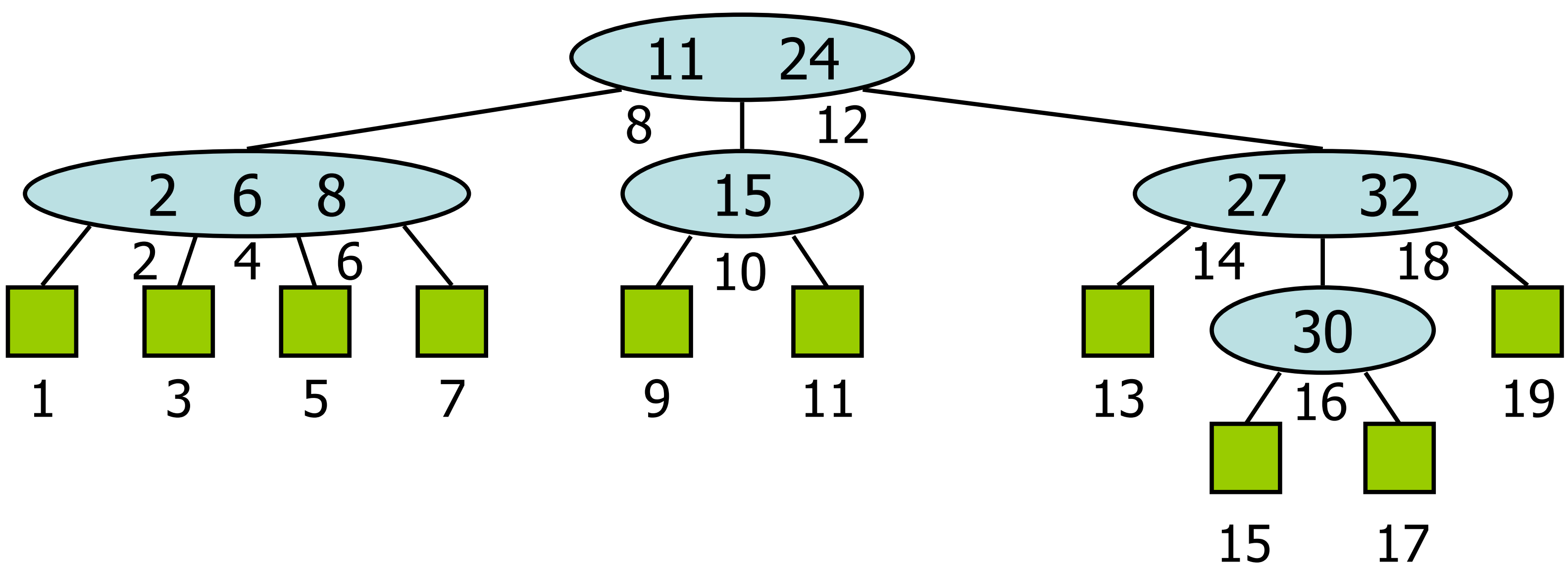
Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores ***d*-1** key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\ v_2\ \ldots\ v_d$ storing keys $k_1\ k_2\ \ldots\ k_{d-1}$
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i ($i = 2, \ldots, d - 1$)
    - keys in the subtree of v_d are greater than k_{d-1}
  - The leaves store no items and serve as placeholders



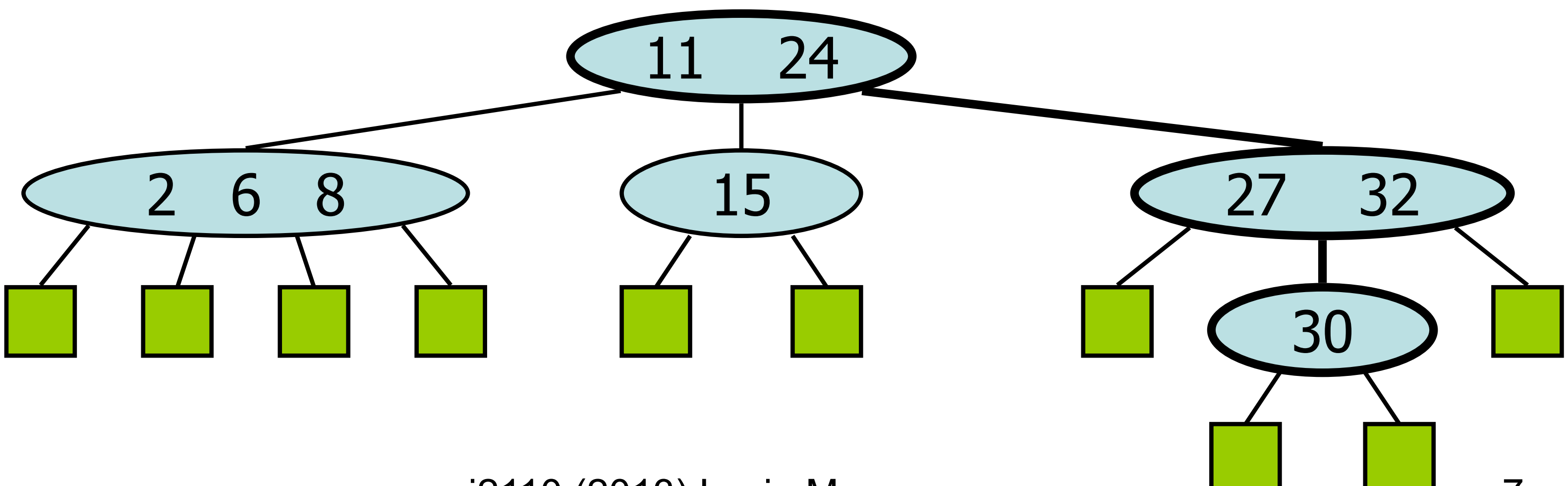
Multi-Way Inorder Traversal

- We can extend the notion of inorder traversal from binary trees to multi-way search trees
- Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- An inorder traversal of a multi-way search tree visits the keys in increasing order



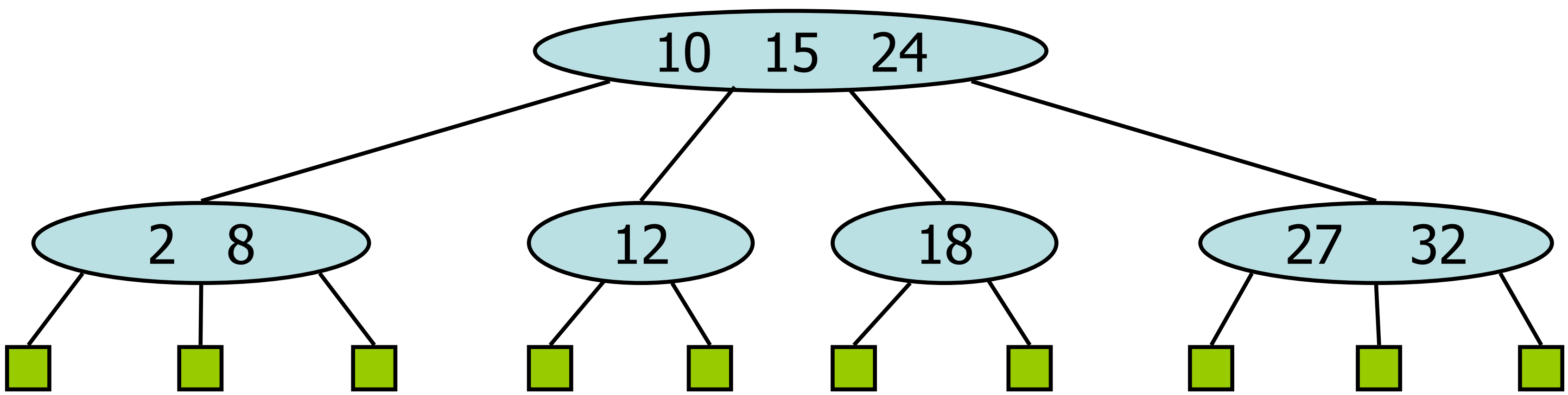
Multi-Way Searching

- Similar to search in a binary search tree
- At each internal node with children $v_1\ v_2\ \ldots\ v_d$ and keys $k_1\ k_2\ \ldots\ k_{d-1}$
  - $k = k_i$ ($i = 1, \ldots, d - 1$): the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ ($i = 2, \ldots, d - 1$): we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully
- Example: search for 30



(2,4) Tree

- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node

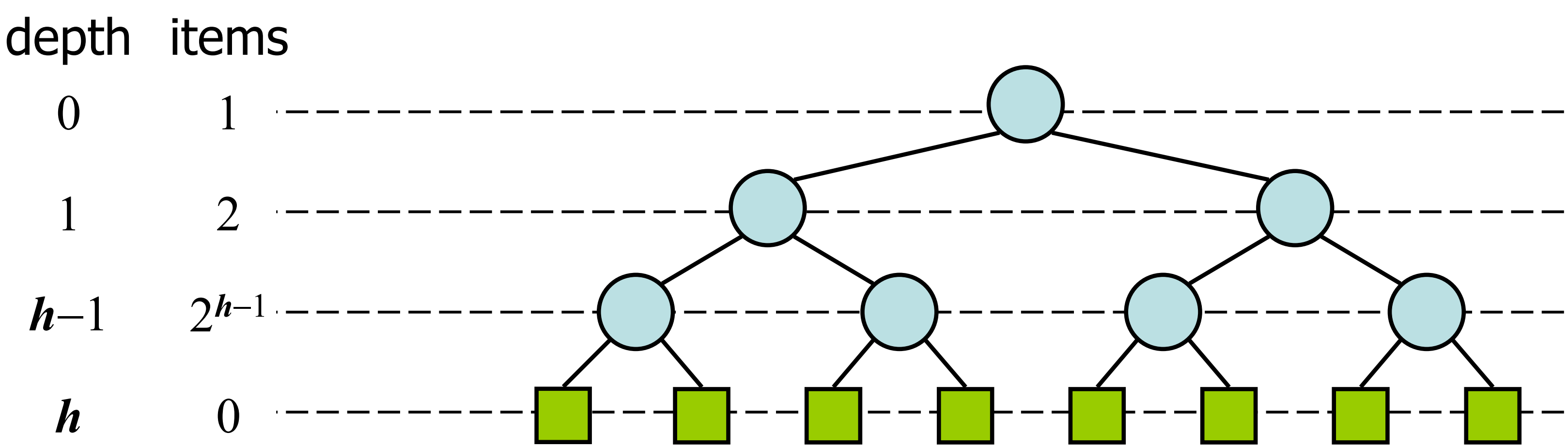


Height of a (2,4) Tree

- Theorem: A (2,4) tree storing n items has height $O(\log n)$

  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \ldots, h - 1$ and no items at depth h, we have

    $$n \geq 1 + 2 + 4 + \ldots + 2^{h-1} = 2^h - 1$$
  - Thus, $h \leq \log(n + 1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time



Height of a (2,4) Tree

Min # of items n:

When all internal nodes have 1 key and 2 children

$n = 2^{h+1}-1 \qquad h = O(\log n)$

"perfect" binary tree

Max # of items n:

When all internal nodes have 3 keys and 4 children

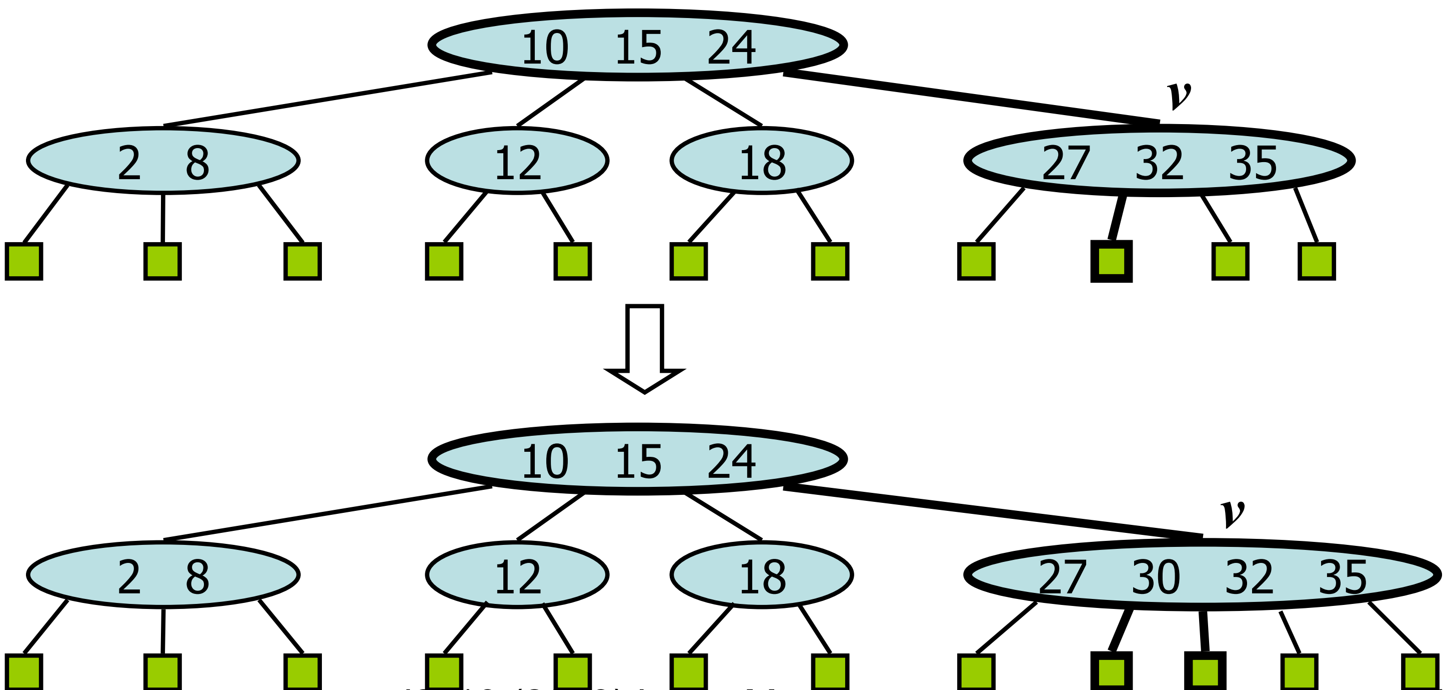
$$n = \sum_{i=0}^{h} 4^i = (4^{h+1}-1)$$

$n = 4^{h+1}-1 \quad h = O(\log_4 n)$

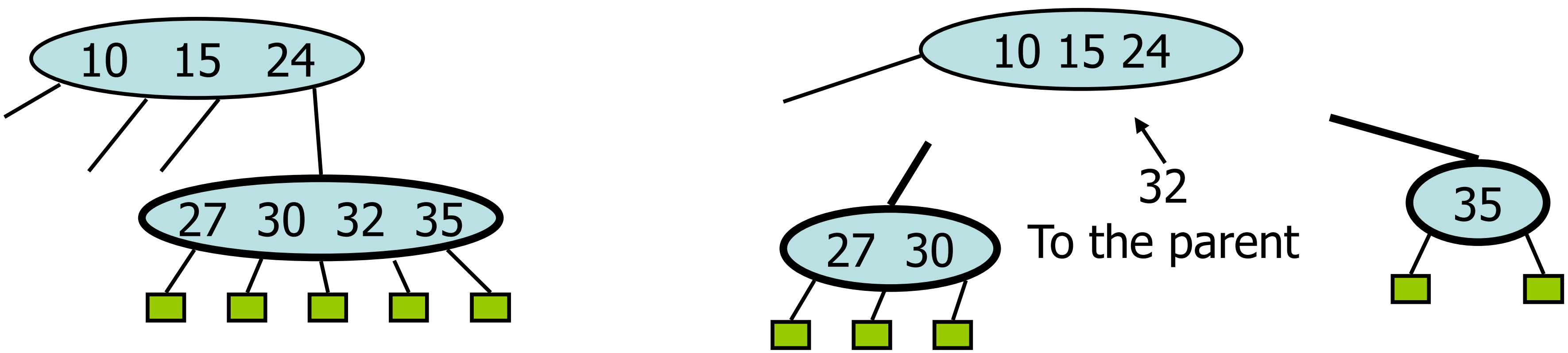
→ Search $O(\log n)$

Insertion

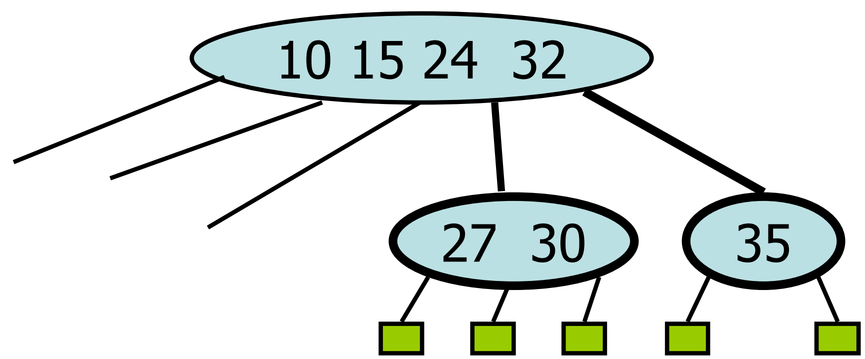
- We insert a new item (***k***, ***o***) at the parent ***v*** of the leaf reached by searching for ***k***
  - We preserve the depth property but
  - We may cause an overflow (i.e., node ***v*** may become a 5-node)
- Example: inserting key 30 causes an overflow



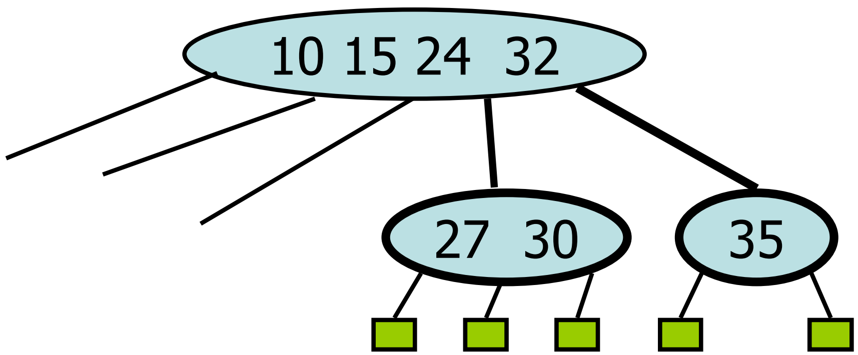
Overflow and Split

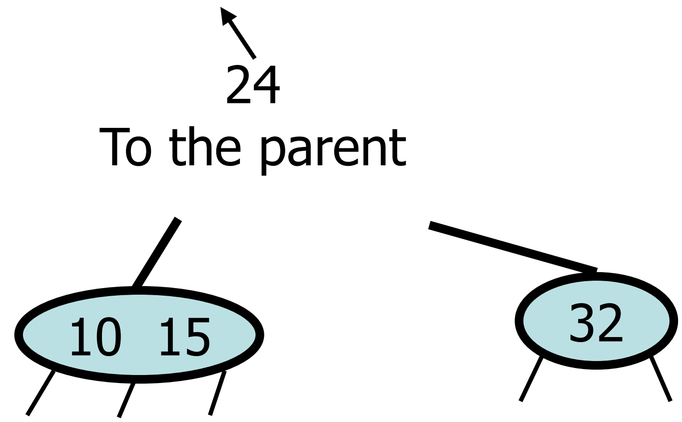


Overflow again!

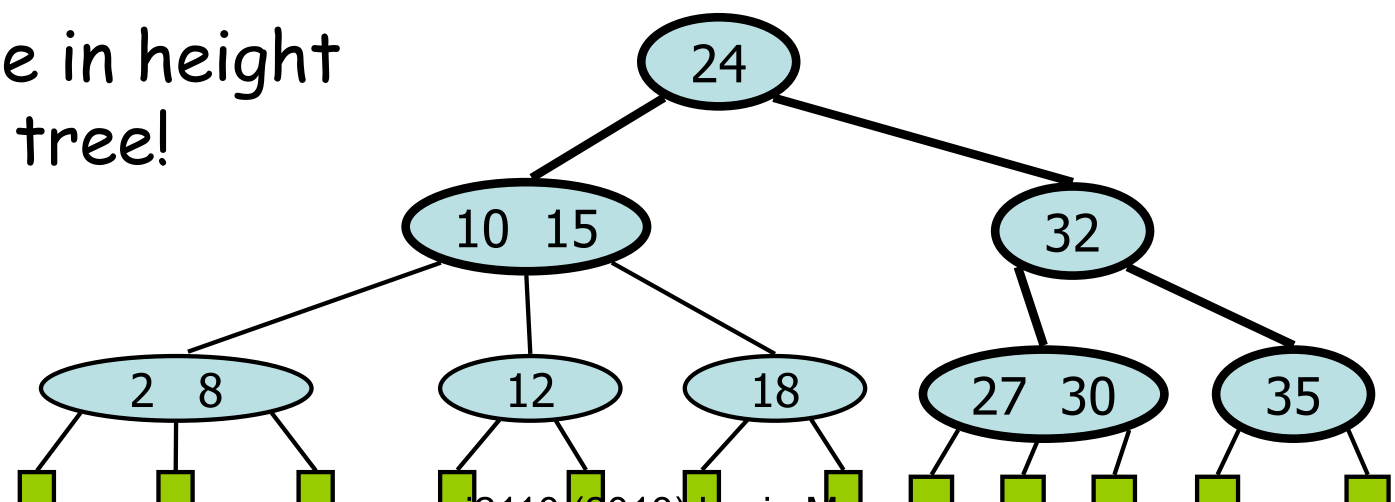


Overflow and Split Again



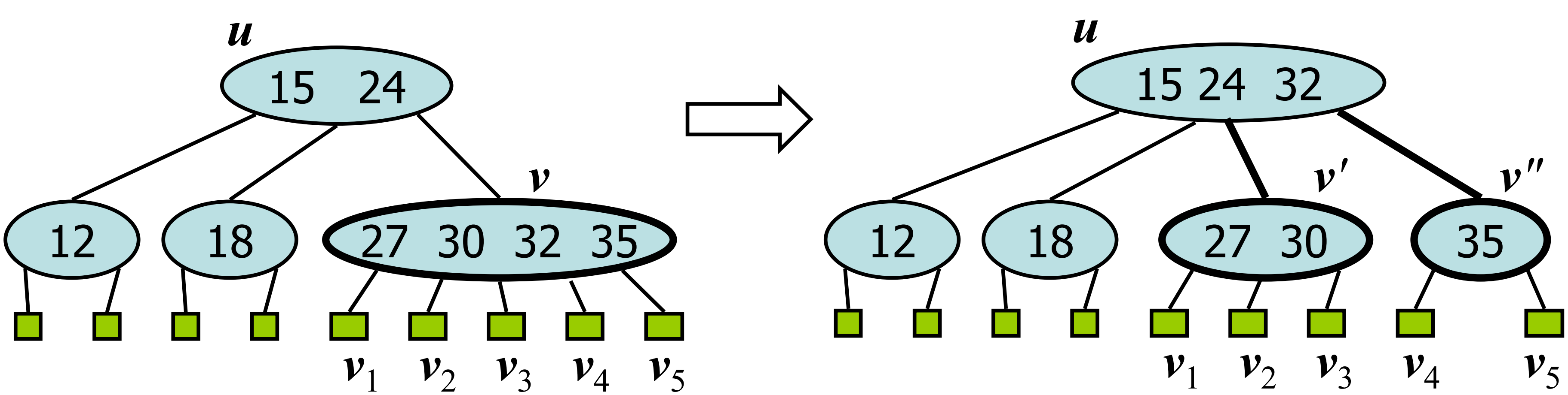


Increase in height of tree!



Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced by nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent of node u



Analysis of Insertion

Algorithm ***insertItem***(***k***, ***o***)

1. We search for key ***k*** to locate the insertion node ***v***

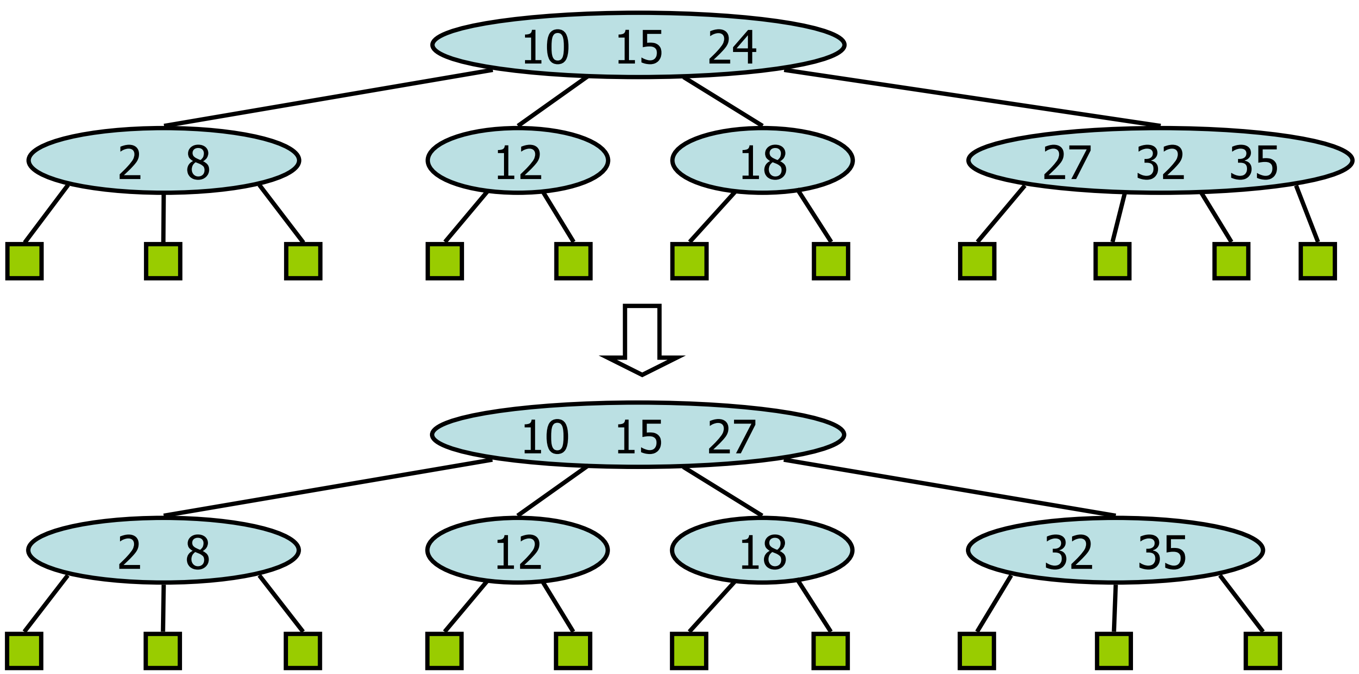
2. We add the new item (***k***, ***o***) at node ***v***

3. **while** ***overflow***(***v***)
   if ***isRoot***(***v***)
   create a new empty root above ***v***
   v ← ***split***(***v***)

- Let ***T*** be a (2,4) tree with ***n*** items
  - Tree ***T*** has ***O***(log ***n***) height
  - Step 1 takes ***O***(log ***n***) time because we visit ***O***(log ***n***) nodes
  - Step 2 takes ***O***(1) time
  - Step 3 takes ***O***(log ***n***) time because each split takes ***O***(1) time and we perform ***O***(log ***n***) splits
- Thus, an insertion in a (2,4) tree takes ***O***(log ***n***) time

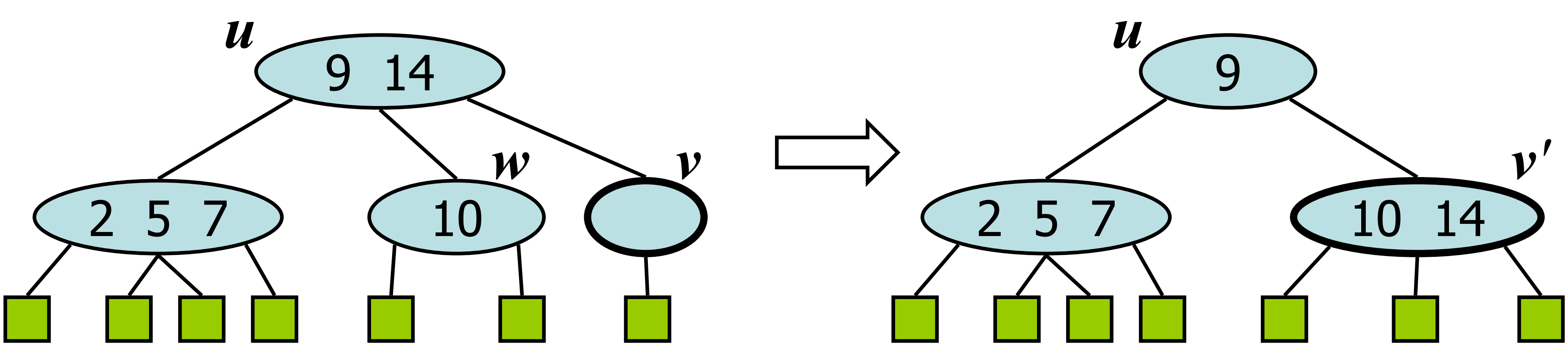
Deletion

- We reduce deletion of an item to the case where the item is at the node with leaf children
- Otherwise, we replace the item with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter item
- Example: to delete key 24, we replace it with 27 (inorder successor)



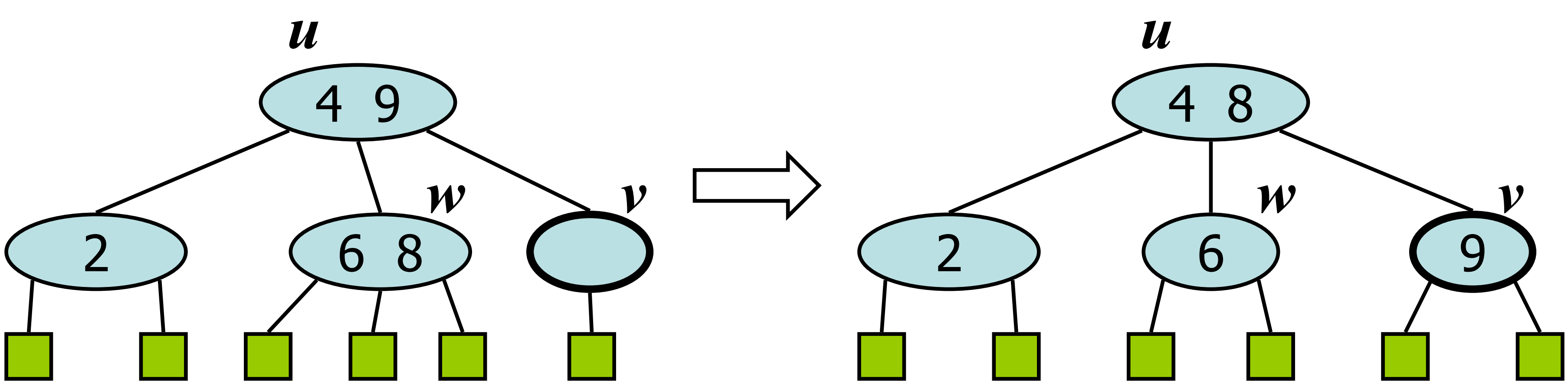
Underflow and Fusion

- Deleting an item from a node ***v*** may cause an underflow, where node ***v*** becomes a 1-node with one child and no keys
- To handle an underflow at node ***v*** with parent ***u***, we consider two cases
- Case 1: the adjacent siblings of ***v*** are 2-nodes
  - Fusion operation: we merge ***v*** with an adjacent sibling ***w*** and move an item from ***u*** to the merged node ***v'***
  - After a fusion, the underflow may propagate to the parent ***u***



Underflow and Transfer

- To handle an underflow at node ***v*** with parent ***u***, we consider two cases
- Case 2: an adjacent sibling ***w*** of ***v*** is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of ***w*** to ***v***
    2. we move an item from ***u*** to ***v***
    3. we move an item from ***w*** to ***u***
  - After a transfer, no underflow occurs



Analysis of Deletion

- Let *T* be a (2,4) tree with *n* items
  - Tree *T* has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the item
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time