A Bonus-Malus Framework for Cyber Risk Insurance and Optimal Cybersecurity Provisioning — Notes about the truncated g-and-h distribution

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1 Truncated g-and-h Distribution

The g-and-h distribution is a four-parameter family of distributions, given by the following definition:

$$\begin{aligned} &\widetilde{X} \text{ follows a g-and-h}(\alpha, \varsigma, g, h) \text{ distribution, if} \\ &\widetilde{X} = \alpha + \varsigma Y_{g,h}(Z), \\ &\text{where } Z \sim \text{Normal}(0,1), \\ &Y_{g,h}(z) := \begin{cases} \frac{\exp(gz)-1}{g} \exp\left(\frac{hz^2}{2}\right) & \text{if } g \neq 0, \\ z \exp\left(\frac{hz^2}{2}\right) & \text{if } g = 0, \end{cases} \end{aligned} \tag{1}$$

where $\alpha \in \mathbb{R}$ is the location parameter, $\varsigma > 0$ is the scale parameter, $g \in \mathbb{R}$ is the skewness parameter, and $h \geq 0$ is the kurtosis parameter. By (1), the distribution function of $\widetilde{X}$ is given by

$$F_{\widetilde{X}}(x) := \mathbb{P}[\widetilde{X} \leq x] = \Phi\left(Y_{g,h}^{-1}\left(\tfrac{x-\alpha}{\varsigma}\right)\right), \tag{2}$$

where $Y_{g,h}^{-1}$ denotes the inverse function of $Y_{g,h}$, and Φ denotes the distribution function of the standard normal distribution. Even though $Y_{g,h}^{-1}$ cannot be expressed analytically, it can be efficiently evaluated using a standard root-finding procedure such as the bisection method and the Newton's method. Therefore, we treat $Y_{g,h}^{-1}$ as a tractable function. The g-and-h distribution has the property that the m-th moment of $\widetilde{X}$ exists when $h < \frac{1}{m}$. From now on, we assume that $g > 0$ and $0 \leq h < 1$.

Let us now introduce the truncated g-and-h distribution.

Definition 1.1 (Truncated g-and-h distribution)**.** *For $\alpha \in \mathbb{R}, \varsigma > 0, g > 0, h \in [0,1)$, the random variable X has truncated g-and-h distribution with parameters α, ς, g, h, denoted by $X \sim$ Tr-g-and-h$(\alpha, \varsigma, g, h)$, if*

X has distribution function

$$F_X(x) := \mathbb{P}[X \le x] = \mathbb{P}[\widetilde{X} \le x | \widetilde{X} > 0], \tag{3}$$

where $\widetilde{X} \sim$ g-and-h$(\alpha, \varsigma, g, h)$.

The next lemma shows some useful properties of the truncated g-and-h distribution.

Lemma 1.2. *Suppose that $X \sim$ Tr-g-and-h$(\alpha, \varsigma, g, h)$ for $\alpha \in \mathbb{R}$, $\varsigma > 0$, $g > 0$, $h \in [0, 1)$. Then, the following statements hold.*

(i) The distribution function of X is given by

$$F_X(x) = \begin{cases} \frac{F_{\widetilde{X}}(x) - F_{\widetilde{X}}(0)}{1 - F_{\widetilde{X}}(0)} & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases} \tag{4}$$

where $F_{\widetilde{X}}$ is defined in (1).

(ii) Suppose that $U \sim$ Uniform$[0, 1]$, and let

$$X_U := \alpha + \varsigma Y_{g,h}\Big(\Phi^{-1}\Big(U + (1 - U)F_{\widetilde{X}}(0)\Big)\Big), \tag{5}$$

then $X_U \sim$ Tr-g-and-h$(\alpha, \varsigma, g, h)$.

(iii) For any $\gamma \ge 0$, the expectation $\mathbb{E}\big[(X - \gamma)^+\big]$ is given by:

$$\begin{aligned} \mathbb{E}\big[(X - \gamma)^+\big] = \frac{\varsigma}{(1 - F_{\widetilde{X}}(0))g\sqrt{1 - h}} \Bigg[\exp\left(\frac{g^2}{2(1 - h)}\right) \Phi\left(\left(\frac{g}{1 - h} - Y_{g,h}^{-1}\left(\tfrac{\gamma - \alpha}{\varsigma}\right)\right)\sqrt{1 - h}\right) \\ - \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{\gamma - \alpha}{\varsigma}\right)\sqrt{1 - h}\right)\Bigg] + \frac{(\alpha - \gamma)(1 - F_{\widetilde{X}}(\gamma))}{1 - F_{\widetilde{X}}(0)}. \end{aligned} \tag{6}$$

Proof of Lemma 1.2. Statement (i) follows by checking the following:

$$F_X(x) = \mathbb{P}[\widetilde{X} \le x | \widetilde{X} > 0] = \frac{\mathbb{P}[0 < \widetilde{X} \le x]}{\mathbb{P}[\widetilde{X} > 0]} = \begin{cases} \frac{F_{\widetilde{X}}(x) - F_{\widetilde{X}}(0)}{1 - F_{\widetilde{X}}(0)} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Statement (ii) can be verified directly by checking that $\mathbb{P}[X_U \le x] = F_X(x)$ for all $x \in \mathbb{R}$.

Finally, statement (iii) can be derived from (4) as follows:

$$
\begin{aligned}
&\mathbb{E}\big[(X-\gamma)^+\big]\\
&= \int_\gamma^\infty (x-\gamma)\, F_X(\mathrm{d}x)\\
&= \frac{1}{1-F_{\widetilde{X}}(0)} \left[\int_\gamma^\infty x\, F_{\widetilde{X}}(\mathrm{d}x) - \gamma(1-F_{\widetilde{X}}(\gamma))\right]\\
&= \frac{\varsigma}{1-F_{\widetilde{X}}(0)} \int_{Y_{g,h}^{-1}(\frac{\gamma-\alpha}{\varsigma})}^\infty Y_{g,h}(z)\, \Phi(\mathrm{d}z) + \frac{(\alpha-\gamma)(1-F_{\widetilde{X}}(\gamma))}{1-F_{\widetilde{X}}(0)}\\
&= \frac{\varsigma}{1-F_{\widetilde{X}}(0)} \frac{1}{g} \int_{Y_{g,h}^{-1}(\frac{\gamma-\alpha}{\varsigma})}^\infty (\exp(gz)-1) \exp\left(\frac{hz^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z + \frac{(\alpha-\gamma)(1-F_{\widetilde{X}}(\gamma))}{1-F_{\widetilde{X}}(0)}\\
&= \frac{\varsigma}{1-F_{\widetilde{X}}(0)} \frac{1}{g\sqrt{2\pi}} \left[\int_{Y_{g,h}^{-1}(\frac{\gamma-\alpha}{\varsigma})}^\infty \exp\left(-\frac{(1-h)z^2}{2} + gz\right) - \exp\left(-\frac{(1-h)z^2}{2}\right) \mathrm{d}z\right] + \frac{(\alpha-\gamma)(1-F_{\widetilde{X}}(\gamma))}{1-F_{\widetilde{X}}(0)}\\
&= \frac{\varsigma}{1-F_{\widetilde{X}}(0)} \frac{1}{g\sqrt{2\pi}} \left[\int_{Y_{g,h}^{-1}(\frac{\gamma-\alpha}{\varsigma})}^\infty \exp\left(\frac{g^2}{2(1-h)}\right) \exp\left(-\frac{(1-h)}{2}\left(z-\frac{g}{1-h}\right)^2\right) \mathrm{d}z\right.\\
&\qquad\qquad\qquad\left. - \int_{Y_{g,h}^{-1}(\frac{\gamma-\alpha}{\varsigma})}^\infty \exp\left(-\frac{(1-h)z^2}{2}\right) \mathrm{d}z\right] + \frac{(\alpha-\gamma)(1-F_{\widetilde{X}}(\gamma))}{1-F_{\widetilde{X}}(0)}\\
&= \frac{\varsigma}{(1-F_{\widetilde{X}}(0))g\sqrt{1-h}} \left[\exp\left(\frac{g^2}{2(1-h)}\right) \Phi\left(\left(\frac{g}{1-h} - Y_{g,h}^{-1}\left(\tfrac{\gamma-\alpha}{\varsigma}\right)\right)\sqrt{1-h}\right)\right.\\
&\qquad\qquad\qquad\left. - \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{\gamma-\alpha}{\varsigma}\right)\sqrt{1-h}\right)\right] + \frac{(\alpha-\gamma)(1-F_{\widetilde{X}}(\gamma))}{1-F_{\widetilde{X}}(0)},
\end{aligned}
$$

where the last equality is obtained by noticing that both integrals are Gaussian integrals after a change of variable. The proof is now complete. □

The next lemma presents the variance of the truncated g-and-h distribution when $0 \le h < \frac{1}{2}$.

Lemma 1.3. *Suppose that $X \sim$ Tr-g-and-h$(\alpha, \varsigma, g, h)$ for $\alpha \in \mathbb{R}$, $\varsigma > 0$, $g > 0$, $h \in \left[0, \frac{1}{2}\right)$. Then, the variance of X is given by*

$$
\begin{aligned}
\mathrm{Var}[X] = &\frac{\varsigma^2}{(1-F_{\widetilde{X}}(0))g^2\sqrt{1-2h}} \left[\exp\left(\frac{2g^2}{(1-2h)}\right) \Phi\left(\left(\tfrac{2g}{1-2h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\right)\sqrt{1-2h}\right)\right.\\
&\left. - 2\exp\left(\frac{g^2}{2(1-2h)}\right) \Phi\left(\left(\tfrac{g}{1-2h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\right)\sqrt{1-2h}\right) + \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\sqrt{1-2h}\right)\right]\\
&- \frac{\varsigma^2}{(1-F_{\widetilde{X}}(0))^2 g^2(1-h)} \left[\exp\left(\frac{g^2}{2(1-h)}\right) \Phi\left(\left(\tfrac{g}{1-h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\right)\sqrt{1-h}\right)\right.\\
&\qquad\qquad\left. - \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\sqrt{1-h}\right)\right]^2.
\end{aligned}
$$

Proof of Lemma 1.3. It follows from (4) that

$$
\begin{aligned}
\mathbb{E}[X^2] &= \int_0^\infty x^2\, F_X(\mathrm{d}x) \\
&= \frac{1}{1-F_{\widetilde{X}}(0)} \int_0^\infty x^2\, F_{\widetilde{X}}(\mathrm{d}x) \\
&= \frac{1}{1-F_{\widetilde{X}}(0)} \int_{Y_{g,h}^{-1}(\frac{-\alpha}{\varsigma})}^\infty \big(\alpha + \varsigma Y_{g,h}(z)\big)^2\, \Phi(\mathrm{d}z) \\
&= \alpha^2 + \frac{2\alpha\varsigma}{1-F_{\widetilde{X}}(0)} \int_{Y_{g,h}^{-1}(\frac{-\alpha}{\varsigma})}^\infty Y_{g,h}(z)\, \Phi(\mathrm{d}z) + \frac{\varsigma^2}{1-F_{\widetilde{X}}(0)} \int_{Y_{g,h}^{-1}(\frac{-\alpha}{\varsigma})}^\infty \big(Y_{g,h}(z)\big)^2\, \Phi(\mathrm{d}z).
\end{aligned}
\tag{7}
$$

We have by the same derivation as the proof of Lemma 1.2(iii) that

$$
\begin{aligned}
&\frac{2\alpha\varsigma}{1-F_{\widetilde{X}}(0)} \int_{Y_{g,h}^{-1}(\frac{-\alpha}{\varsigma})}^\infty Y_{g,h}(z)\, \Phi(\mathrm{d}z) \\
&= \frac{2\alpha\varsigma}{(1-F_{\widetilde{X}}(0))g\sqrt{1-h}} \left[\exp\left(\frac{g^2}{2(1-h)}\right) \Phi\left(\left(\frac{g}{1-h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\right)\sqrt{1-h}\right) \right. \\
&\qquad \left. - \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\sqrt{1-h}\right)\right].
\end{aligned}
\tag{8}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{\varsigma^2}{1-F_{\widetilde{X}}(0)} \int_{Y_{g,h}^{-1}(\frac{-\alpha}{\varsigma})}^\infty \big(Y_{g,h}(z)\big)^2\, \Phi(\mathrm{d}z) \\
&= \frac{\varsigma^2}{(1-F_{\widetilde{X}}(0))g^2\sqrt{2\pi}} \int_{Y_{g,h}^{-1}(\frac{-\alpha}{\varsigma})}^\infty \big(\exp(2gz) - 2\exp(gz) + 1\big) \exp(hz^2) \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z.
\end{aligned}
\tag{9}
$$

Moreover, it holds for all $b \in \mathbb{R}$ that

$$
\begin{aligned}
&\int_{Y_{g,h}^{-1}(\frac{-\alpha}{\varsigma})}^\infty \exp(bz)\exp(hz^2)\exp\left(-\frac{z^2}{2}\right) \mathrm{d}z \\
&= \int_{Y_{g,h}^{-1}(\frac{-\alpha}{\varsigma})}^\infty \exp\left(-\frac{(1-2h)z^2}{2} + bz\right) \mathrm{d}z \\
&= \frac{\sqrt{2\pi}}{\sqrt{1-2h}} \exp\left(\frac{b^2}{2(1-2h)}\right) \Phi\left(\left(\frac{b}{1-2h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\right)\sqrt{1-2h}\right).
\end{aligned}
\tag{10}
$$

Letting $b = 2g$, $b = g$, and $b = 0$ in (10) and substituting the results into (9) yields

$$
\begin{aligned}
&\frac{\varsigma^2}{1-F_{\widetilde{X}}(0)} \int_{Y_{g,h}^{-1}(\frac{-\alpha}{\varsigma})}^\infty \big(Y_{g,h}(z)\big)^2\, \Phi(\mathrm{d}z) \\
&= \frac{\varsigma^2}{(1-F_{\widetilde{X}}(0))g^2\sqrt{1-2h}} \left[\exp\left(\frac{2g^2}{(1-2h)}\right) \Phi\left(\left(\frac{2g}{1-2h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\right)\sqrt{1-2h}\right) \right. \\
&\quad \left. - 2\exp\left(\frac{g^2}{2(1-2h)}\right) \Phi\left(\left(\frac{g}{1-2h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\right)\sqrt{1-2h}\right) + \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right)\sqrt{1-2h}\right)\right].
\end{aligned}
\tag{11}
$$

Next, combining (7), (8), and (11) yields

$$\begin{aligned}\mathbb{E}[X^2] &= \alpha^2 + \frac{1}{1-F_{\widetilde{X}}(0)} \\ &\quad \times \left[\frac{2\alpha\varsigma}{g\sqrt{1-h}} \left(\exp\left(\frac{g^2}{2(1-h)}\right) \Phi\left(\left(\tfrac{g}{1-h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \right) \sqrt{1-h} \right) - \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \sqrt{1-h} \right) \right) \right. \\ &\qquad + \frac{\varsigma^2}{g^2\sqrt{1-2h}} \left(\exp\left(\frac{2g^2}{(1-2h)}\right) \Phi\left(\left(\tfrac{2g}{1-2h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \right) \sqrt{1-2h} \right) \right. \\ &\qquad \left. \left. - 2\exp\left(\frac{g^2}{2(1-2h)}\right) \Phi\left(\left(\tfrac{g}{1-2h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \right) \sqrt{1-2h} \right) + \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \sqrt{1-2h} \right) \right) \right].\end{aligned} \tag{12}$$

Finally, we have by Lemma 1.2(iii) that

$$\begin{aligned}\mathbb{E}[X] = \alpha + \frac{\varsigma}{(1-F_{\widetilde{X}}(0))g\sqrt{1-h}} &\left[\exp\left(\frac{g^2}{2(1-h)}\right) \Phi\left(\left(\tfrac{g}{1-h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \right) \sqrt{1-h} \right) \right. \\ &\left. - \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \sqrt{1-h} \right) \right].\end{aligned} \tag{13}$$

Combining (12) and (13), we obtain

$$\begin{aligned}\mathrm{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \frac{\varsigma^2}{(1-F_{\widetilde{X}}(0))g^2\sqrt{1-2h}} \left[\exp\left(\frac{2g^2}{(1-2h)}\right) \Phi\left(\left(\tfrac{2g}{1-2h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \right) \sqrt{1-2h} \right) \right. \\ &\quad \left. - 2\exp\left(\frac{g^2}{2(1-2h)}\right) \Phi\left(\left(\tfrac{g}{1-2h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \right) \sqrt{1-2h} \right) + \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \sqrt{1-2h} \right) \right] \\ &\quad - \frac{\varsigma^2}{(1-F_{\widetilde{X}}(0))^2 g^2 (1-h)} \left[\exp\left(\frac{g^2}{2(1-h)}\right) \Phi\left(\left(\tfrac{g}{1-h} - Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \right) \sqrt{1-h} \right) \right. \\ &\qquad \left. - \Phi\left(-Y_{g,h}^{-1}\left(\tfrac{-\alpha}{\varsigma}\right) \sqrt{1-h} \right) \right]^2.\end{aligned}$$

The proof is now complete. $\square$