MODEL-FREE BOUNDS FOR MULTI-ASSET OPTIONS USING OPTION-IMPLIED INFORMATION AND THEIR EXACT COMPUTATION
SUPPLEMENTARY MATERIAL

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1. Adjustment of option prices to remove arbitrage opportunities

When preprocessing the real market data in Experiment 4, we adjusted the bid and ask prices slightly to remove arbitrage opportunities. The details of this preprocessing step are explained here.

Let us consider a single asset and European call and put options written on this asset with strike prices $0 < \kappa_1 < \cdots < \kappa_m$. For $j = 1, \ldots, m$, let $\underline{\pi}_j^{\mathrm{call}}$ and $\overline{\pi}_j^{\mathrm{call}}$ denote the bid and ask prices of the call option with strike κ_j, respectively. Similarly, let $\underline{\pi}_j^{\mathrm{put}}$ and $\overline{\pi}_j^{\mathrm{put}}$ denote the bid and ask prices of the put option with strike κ_j, respectively.

To adjust the option prices, we first fix[1] some $\overline{x} > \kappa_m$ and then (approximately) compute a minimizer $\big(\widehat{v}_j^{\mathrm{call}-}, \widehat{v}_j^{\mathrm{call}+}, \widehat{v}_j^{\mathrm{put}-}, \widehat{v}_j^{\mathrm{put}+}\big)_{j=1:m}$ of the following minimization problem:

$$
\begin{aligned}
\min\Big\{ & \textstyle\sum_{j=1}^m \big(v_j^{\mathrm{call}-} + v_j^{\mathrm{call}+} + v_j^{\mathrm{put}-} + v_j^{\mathrm{put}+}\big) : \\
& \exists \mu \in \mathcal{P}([0,\overline{x}]),\ \{0, \kappa_1, \ldots, \kappa_m, \overline{x}\} \subseteq \mathrm{supp}(\mu), \\
& \underline{\pi}_j^{\mathrm{call}} - v_j^{\mathrm{call}-} \leq \textstyle\int_{[0,\overline{x}]} (x-\kappa_j)^+ \,\mu(\mathrm{d}x) \leq \overline{\pi}_j^{\mathrm{call}} + v_j^{\mathrm{call}+}, \\
& \underline{\pi}_j^{\mathrm{put}} - v_j^{\mathrm{put}-} \leq \textstyle\int_{[0,\overline{x}]} (\kappa_j - x)^+ \,\mu(\mathrm{d}x) \leq \overline{\pi}_j^{\mathrm{put}} + v_j^{\mathrm{put}+}, \\
& v_j^{\mathrm{call}-} \geq 0, v_j^{\mathrm{call}+} \geq 0, v_j^{\mathrm{put}-} \geq 0, v_j^{\mathrm{put}+} \geq 0 \ \forall 1 \leq j \leq m \Big\}.
\end{aligned}
\tag{1.1}
$$

Subsequently, we adjust the bid and ask prices of the call option with strike κ_j to $\underline{\pi}_j^{\mathrm{call}} - \widehat{v}_j^{\mathrm{call}-}$ and $\overline{\pi}_j^{\mathrm{call}} + \widehat{v}_j^{\mathrm{call}+}$, respectively, and adjust the bid and ask prices of the put option with strike κ_j to $\underline{\pi}_j^{\mathrm{put}} - \widehat{v}_j^{\mathrm{put}-}$ and $\overline{\pi}_j^{\mathrm{put}} + \widehat{v}_j^{\mathrm{put}+}$, respectively.

The following proposition shows that the above procedure produces option prices that are arbitrage-free.

Proposition 1.1. *Let $m \in \mathbb{N}$ and $0 < \kappa_1 < \cdots < \kappa_m < \overline{x}$. For $j = 1, \ldots, m$, let $\underline{\pi}_j^{\mathrm{call}} < \overline{\pi}_j^{\mathrm{call}}$, $\underline{\pi}_j^{\mathrm{put}} < \overline{\pi}_j^{\mathrm{put}}$ be fixed, and define*

$$
\begin{aligned}
\pi\big(y_1^{\mathrm{call}}, \ldots, y_m^{\mathrm{call}}, y_1^{\mathrm{put}}, \ldots, y_m^{\mathrm{put}}\big) := \sum_{j=1}^m & \big(y_j^{\mathrm{call}} \vee 0\big)\overline{\pi}_j^{\mathrm{call}} - \big(-y_j^{\mathrm{call}} \vee 0\big)\underline{\pi}_j^{\mathrm{call}} \\
& + \big(y_j^{\mathrm{put}} \vee 0\big)\overline{\pi}_j^{\mathrm{put}} - \big(-y_j^{\mathrm{put}} \vee 0\big)\underline{\pi}_j^{\mathrm{put}}
\end{aligned}
$$

for $\big(y_j^{\mathrm{call}} \in \mathbb{R}\big)_{j=1:m}$, $\big(y_j^{\mathrm{put}} \in \mathbb{R}\big)_{j=1:m}$. Suppose there exists $\mu \in \mathcal{P}([0,\overline{x}])$ with $\{0, \kappa_1, \ldots, \kappa_m, \overline{x}\} \subset \mathrm{supp}(\mu)$, such that for $j = 1, \ldots, m$,

$$
\begin{aligned}
\underline{\pi}_j^{\mathrm{call}} &\leq \int_{[0,\overline{x}]} (x-\kappa_j)^+ \,\mu(\mathrm{d}x) \leq \overline{\pi}_j^{\mathrm{call}}, \\
\underline{\pi}_j^{\mathrm{put}} &\leq \int_{[0,\overline{x}]} (\kappa_j - x)^+ \,\mu(\mathrm{d}x) \leq \overline{\pi}_j^{\mathrm{put}}.
\end{aligned}
\tag{1.2}
$$

[1] In the MATLAB implementation we take $\overline{x} := 2\kappa_m$.

Then, for any $y_1^{\text{call}}, \ldots, y_m^{\text{call}}, y_1^{\text{put}}, \ldots, y_m^{\text{put}}$ such that

$$\left[\sum_{j=1}^{m} y_j^{\text{call}}(x-\kappa_j)^+ + y_j^{\text{put}}(\kappa_j - x)^+\right] - \pi\big(y_1^{\text{call}}, \ldots, y_m^{\text{call}}, y_1^{\text{put}}, \ldots, y_m^{\text{put}}\big) \geq 0 \quad \forall x \in \mathbb{R}_+, \tag{1.3}$$

it holds that

$$\left[\sum_{j=1}^{m} y_j^{\text{call}}(x-\kappa_j)^+ + y_j^{\text{put}}(\kappa_j - x)^+\right] - \pi\big(y_1^{\text{call}}, \ldots, y_m^{\text{call}}, y_1^{\text{put}}, \ldots, y_m^{\text{put}}\big) = 0 \quad \forall x \in \mathbb{R}_+. \tag{1.4}$$

Proof of Proposition 1.1. Suppose, for the sake of contradiction, that there exist $y_1^{\text{call}}, \ldots, y_m^{\text{call}}, y_1^{\text{put}}, \ldots, y_m^{\text{put}}$ such that (1.3) holds but (1.4) does not hold. Let $h : \mathbb{R}_+ \to \mathbb{R}$ denote the left-hand side of the inequality in (1.3) as a function of x, *i.e.*

$$h(x) := \left[\sum_{j=1}^{m} y_j^{\text{call}}(x-\kappa_j)^+ + y_j^{\text{put}}(\kappa_j - x)^+\right] - \pi\big(y_1^{\text{call}}, \ldots, y_m^{\text{call}}, y_1^{\text{put}}, \ldots, y_m^{\text{put}}\big) \quad \forall x \in \mathbb{R}_+.$$

Observe that h is a continuous function which is piece-wise affine on intervals $[0,\kappa_1]$, $[\kappa_1,\kappa_2]$, ..., $[\kappa_{m-1},\kappa_m]$, and $[\kappa_m,\infty)$. By our assumption, h is non-negative on $\mathbb{R}_+$ but is not everywhere zero. Hence, there exists $\widehat{x} \in \{0,\kappa_1,\ldots,\kappa_m,\overline{x}\}$ such that $h(\widehat{x}) = \alpha > 0$. This implies by the continuity of h that there exists an open interval $E \subset \mathbb{R}$ with $\widehat{x} \in E$, such that $h(x) > \frac{\alpha}{2}$ for all $x \in E \cap \mathbb{R}_+$. Moreover, by the assumption that $\{0,\kappa_1,\ldots,\kappa_m,\overline{x}\} \subset \operatorname{supp}(\mu)$, we have $\widehat{x} \in \operatorname{supp}(\mu)$ and thus $\mu\big(E \cap [0,\overline{x}]\big) > 0$. Hence,

$$\int_{[0,\overline{x}]} h \,\mathrm{d}\mu \geq \int_{E\cap[0,\overline{x}]} h \,\mathrm{d}\mu \geq \frac{\alpha}{2}\mu\big(E \cap [0,\overline{x}]\big) > 0. \tag{1.5}$$

On the other hand, by (1.2), it holds for $j = 1,\ldots,m$ that

$$y_j^{\text{call}} \int_{[0,\overline{x}]} (x-\kappa_j)^+ \,\mu(\mathrm{d}x) \leq \big(y_j^{\text{call}} \vee 0\big)\overline{\pi}_j^{\text{call}} - \big(-y_j^{\text{call}} \vee 0\big)\underline{\pi}_j^{\text{call}}, \tag{1.6}$$

$$y_j^{\text{put}} \int_{[0,\overline{x}]} (\kappa_j - x)^+ \,\mu(\mathrm{d}x) \leq \big(y_j^{\text{put}} \vee 0\big)\overline{\pi}_j^{\text{put}} - \big(-y_j^{\text{put}} \vee 0\big)\underline{\pi}_j^{\text{put}}. \tag{1.7}$$

Adding up both sides of (1.6) and (1.7) and then adding over $j = 1,\ldots,m$, we get $\int_{[0,\overline{x}]} h \,\mathrm{d}\mu \leq 0$, which contradicts (1.5). The proof is now complete. □

In order to numerically solve the minimization problem (1.1), we need the following proposition.

Proposition 1.2. *Let $m \in \mathbb{N}$ and $0 < \kappa_1 < \cdots < \kappa_m < \overline{x}$. For $j = 1,\ldots,J$ where $J \in \mathbb{N}$, let $g_j : [0,\overline{x}] \to \mathbb{R}$ be a continuous function which is piece-wise affine on the intervals $[0,\kappa_1]$, $[\kappa_1,\kappa_2]$, ..., $[\kappa_{m-1},\kappa_m]$, $[\kappa_m,\overline{x}]$. For $j = 1,\ldots,J$, let $\pi_j \in \mathbb{R}$ be fixed. Suppose there exists $\mu \in \mathcal{P}([0,\overline{x}])$ with $\{0,\kappa_1,\ldots,\kappa_m,\overline{x}\} \subset \operatorname{supp}(\mu)$, such that $\int_{[0,\overline{x}]} g_j \,\mathrm{d}\mu = \pi_j$ for $j = 1,\ldots,J$. Then, there exists a probability measure $\widehat{\mu} \in \mathcal{P}([0,\overline{x}])$ with $\operatorname{supp}(\widehat{\mu}) = \{0,\kappa_1,\ldots,\kappa_m,\overline{x}\}$ such that $\int_{[0,\overline{x}]} g_j \,\mathrm{d}\widehat{\mu} = \pi_j$ for $j = 1,\ldots,J$.*

Proof of Proposition 1.2. Let $\kappa_0 := 0$ and $\kappa_{m+1} := \overline{x}$. Let $\widetilde{g}_0, \dots, \widetilde{g}_{m+1} : [0, \overline{x}] \to \mathbb{R}$ be defined as follows:

$$\begin{aligned}\widetilde{g}_0(x) :=& \frac{(\kappa_1 - x)^+}{\kappa_1},\\ \widetilde{g}_j(x) :=& \frac{(x-\kappa_{j-1})^+}{\kappa_j - \kappa_{j-1}} \wedge \frac{(\kappa_{j+1} - x)^+}{\kappa_{j+1} - \kappa_j} \quad \text{for } j = 1, \dots, m,\\ \widetilde{g}_{m+1}(x) :=& \frac{(x-\kappa_m)^+}{\overline{x} - \kappa_m}.\end{aligned}$$

One can check that for $i = 0, \dots, m+1$ and $j = 0, \dots, m+1$, it holds that

$$\widetilde{g}_i(\kappa_j) = \begin{cases} 1 & \text{if } i = j,\\ 0 & \text{if } i \neq j.\end{cases} \tag{1.8}$$

Consequently, for each $j = 1, \dots, J$, it holds that

$$g_j(x) = \sum_{i=0}^{m+1} g_j(\kappa_i)\widetilde{g}_i(x) \quad \forall x \in [0, \overline{x}]. \tag{1.9}$$

Now, let us fix some $\mu \in \mathcal{P}([0, \overline{x}])$ with $\{0, \kappa_1, \dots, \kappa_m, \overline{x}\} \subset \operatorname{supp}(\mu)$ and $\int_{[0,\overline{x}]} g_j \,\mathrm{d}\mu = \pi_j$ for $j = 1, \dots, J$. For $i = 0, \dots, m+1$, let $\widetilde{\pi}_i := \int_{[0,\overline{x}]} \widetilde{g}_i \,\mathrm{d}\mu$. It follows from the assumption $\{0, \kappa_1, \dots, \kappa_m, \overline{x}\} \subset \operatorname{supp}(\mu)$ that $\widetilde{\pi}_i > 0$ for $i = 0, \dots, m+1$. Moreover, since $\sum_{i=0}^{m+1} \widetilde{g}_i(x) = 1$ for all $x \in [0, \overline{x}]$, it holds that $\sum_{i=0}^{m+1} \widetilde{\pi}_i = 1$. Subsequently, let us define $\widehat{\pi} \in \mathcal{P}([0, \overline{x}])$ as follows:

$$\widehat{\pi}(\mathrm{d}x) := \sum_{i=0}^{m+1} \widetilde{\pi}_i \delta_{\kappa_i}(\mathrm{d}x), \tag{1.10}$$

where $\delta_{\kappa_i}(\mathrm{d}x)$ denotes the Dirac measure at κ_i. Thus, $\operatorname{supp}(\widehat{\mu}) = \{0, \kappa_1, \dots, \kappa_m, \overline{x}\}$. Finally, combining (1.9), (1.10), (1.8), and the assumption on μ, we have for $j = 1, \dots, J$ that

$$\begin{aligned}\int_{[0,\overline{x}]} g_j \,\mathrm{d}\widehat{\mu} =& \sum_{i=0}^{m+1} g_j(\kappa_i) \int_{[0,\overline{x}]} \widetilde{g}_i \,\mathrm{d}\widehat{\mu}\\ =& \sum_{i=0}^{m+1} g_j(\kappa_i) \left(\sum_{l=0}^{m+1} \widetilde{g}_i(\kappa_l)\widetilde{\pi}_l \right)\\ =& \sum_{i=0}^{m+1} g_j(\kappa_i)\widetilde{\pi}_i\\ =& \sum_{i=0}^{m+1} g_j(\kappa_i) \int_{[0,\overline{x}]} \widetilde{g}_i \,\mathrm{d}\mu\\ =& \int_{[0,\overline{x}]} \textstyle\sum_{i=0}^{m+1} g_j(\kappa_i)\widetilde{g}_i \,\mathrm{d}\mu\\ =& \int_{[0,\overline{x}]} g_j \,\mathrm{d}\mu\\ =& \pi_j.\end{aligned}$$

The proof is now complete. $\square$

Notice that the payoff functions of the call and put options are all continuous and piece-wise affine on the intervals $[0, \kappa_1]$, $[\kappa_1, \kappa_2]$, …, $[\kappa_{m-1}, \kappa_m]$, $[\kappa_m, \overline{x}]$. Therefore, Proposition 1.2 implies that for fixed $\big(v_j^{\text{call}-} \geq 0, v_j^{\text{call}+} \geq 0, v_j^{\text{put}-} \geq 0, v_j^{\text{put}+} \geq 0\big)_{j=1:m}$, there exists $\mu \in \mathcal{P}([0, \overline{x}])$ with

$\{0, \kappa_1, \ldots, \kappa_m, \overline{x}\} \subset \operatorname{supp}(\mu)$, such that for $j = 1, \ldots, J$,

$$\underline{\pi}_j^{\text{call}} - v_j^{\text{call}-} \leq \int_{[0,\overline{x}]} (x - \kappa_j)^+ \, \mu(\mathrm{d}x) \leq \overline{\pi}_j^{\text{call}} + v_j^{\text{call}+},$$
$$\underline{\pi}_j^{\text{put}} - v_j^{\text{put}-} \leq \int_{[0,\overline{x}]} (\kappa_j - x)^+ \, \mu(\mathrm{d}x) \leq \overline{\pi}_j^{\text{put}} + v_j^{\text{put}+},$$

if and only if there exists a probability measure $\widehat{\mu} \in \mathcal{P}([0, \overline{x}])$ with $\operatorname{supp}(\widehat{\mu}) = \{0, \kappa_1, \ldots, \kappa_m, \overline{x}\}$ such that for $j = 1, \ldots, J$,

$$\underline{\pi}_j^{\text{call}} - v_j^{\text{call}-} \leq \int_{[0,\overline{x}]} (x - \kappa_j)^+ \, \widehat{\mu}(\mathrm{d}x) \leq \overline{\pi}_j^{\text{call}} + v_j^{\text{call}+},$$
$$\underline{\pi}_j^{\text{put}} - v_j^{\text{put}-} \leq \int_{[0,\overline{x}]} (\kappa_j - x)^+ \, \widehat{\mu}(\mathrm{d}x) \leq \overline{\pi}_j^{\text{put}} + v_j^{\text{put}+}.$$

Thus, the minimization problem (1.1) can be reformulated as follows:

$$\begin{aligned}
\min \Big\{ & \textstyle\sum_{j=1}^m \left(v_j^{\text{call}-} + v_j^{\text{call}+} + v_j^{\text{put}-} + v_j^{\text{put}+} \right) : \\
& \exists \widetilde{\pi}_0 > 0, \widetilde{\pi}_1 > 0, \ldots, \widetilde{\pi}_m > 0, \widetilde{\pi}_{m+1} > 0, \ \textstyle\sum_{i=0}^{m+1} \widetilde{\pi}_i = 1, \\
& \underline{\pi}_j^{\text{call}} - v_j^{\text{call}-} \leq [\textstyle\sum_{i=1}^m \widetilde{\pi}_i (\kappa_i - \kappa_j)^+] + \widetilde{\pi}_{m+1}(\overline{x} - \kappa_j) \leq \overline{\pi}_j^{\text{call}} + v_j^{\text{call}+}, \\
& \underline{\pi}_j^{\text{put}} - v_j^{\text{put}-} \leq \widetilde{\pi}_0 \kappa_j + [\textstyle\sum_{i=1}^m \widetilde{\pi}_i (\kappa_j - \kappa_i)^+] \leq \overline{\pi}_j^{\text{put}} + v_j^{\text{put}+}, \\
& v_j^{\text{call}-} \geq 0, v_j^{\text{call}+} \geq 0, v_j^{\text{put}-} \geq 0, v_j^{\text{put}+} \geq 0 \ \forall 1 \leq j \leq m \Big\}.
\end{aligned} \tag{1.11}$$

Now, problem (1.11) can be formulated into a linear programming problem (the constraints $\widetilde{\pi}_0 > 0$, $\widetilde{\pi}_1 > 0, \ldots, \widetilde{\pi}_m > 0$, $\widetilde{\pi}_{m+1} > 0$ need to be approximated by $\widetilde{\pi}_0 \geq \eta$, $\widetilde{\pi}_1 \geq \eta, \ldots, \widetilde{\pi}_m \geq \eta$, $\widetilde{\pi}_{m+1} \geq \eta$ for $\eta > 0$ very small) which can be solved efficiently.

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