Problem 1 :

$$\vec{B}(\rho,\phi,z) = F(\rho)\cos(kz-\phi)\,\hat{\rho} + G(\rho)\sin(kz-\phi)\,\hat{\phi} + B_z(\rho,\phi,z)\,\hat{z}$$

On z axis, $$\vec{B}(z) = B_0\cos(kz)\,\hat{x} + B_0\sin(kz)\,\hat{y}$$

$$\nabla\cdot\vec{B} = \frac{dF}{d\rho}\cos(kz-\phi) + \frac{F\cos(kz-\phi)}{\rho} - \frac{G\cos(kz-\phi)}{\rho} + \frac{\partial B_z}{\partial z} = 0$$

$$0 = \nabla\nabla\cdot\vec{B} \Rightarrow \frac{d^2F}{d\rho^2}\cos(kz-\phi) + \frac{dF}{d\rho}\frac{\cos(kz-\phi)}{\rho} - \frac{F}{\rho^2}\cos(kz-\phi) + \frac{G}{\rho^2}\cos(kz-\phi) + \frac{\partial^2 B_z}{\partial\rho\,\partial z} = 0$$

$$\Rightarrow \frac{dF}{d\rho}\sin(kz-\phi) + \frac{F}{\rho}\sin(kz-\phi) - \frac{G}{\rho}\sin(kz-\phi) + \frac{\partial^2 B_z}{\partial\phi\,\partial z} = 0$$

$$\Rightarrow \frac{\partial^2 B_z}{\partial\phi\,\partial z} = -\left(\frac{dF}{d\rho} + \frac{F-G}{\rho}\right)\sin(kz-\phi)$$

$$\Rightarrow \frac{\partial B_z}{\partial z} = \left(\frac{dF}{d\rho} + \frac{F-G}{\rho}\right)\cos(kz-\phi)$$

$$\frac{\partial^2 B}{\partial\rho\,\partial z} = -\left(\frac{d^2F}{d\rho^2} + \frac{1}{\rho}\frac{dF}{d\rho} + \frac{G-F}{\rho^2}\right)\cos(kz-\phi)$$

$$0 = \nabla\times\vec{B} \Rightarrow \frac{1}{\rho}\frac{\partial B_z}{\partial\phi} - Gk\cos(kz-\phi) = 0,$$

$$+Fk\sin(kz-\phi) + \frac{\partial B_z}{\partial\rho} = 0,$$

$$G\sin(kz-\phi) + \rho\frac{dG}{d\rho}\sin(kz-\phi) - F\sin(kz-\phi) = 0$$

$$\frac{\partial B_z}{\partial z} = -\left(\frac{dF}{d\rho} + \frac{F-G}{\rho}\right)\cos(kz-\phi)$$

$$\Rightarrow B_z = -\left(\frac{dF}{d\rho} + \frac{F-G}{\rho}\right)\frac{\sin(kz-\phi)}{k} + C_0 \longrightarrow (1)$$

$$+\frac{1}{\rho}\left(\frac{dF}{d\rho}+\frac{F-G}{\rho}\right)\frac{\cancel{\cos(kz-\phi)}}{k} = Gk\cancel{\cos(kz-\phi)}$$

$$\Rightarrow \quad \frac{dF}{d\rho}+\frac{F-G}{\rho} - k^2\rho G = 0 \longrightarrow ②$$

$$\& \quad -\left(\frac{d^2F}{d\rho^2}+\frac{1}{\rho}\frac{dF}{d\rho}-\frac{1}{\rho}\frac{dG}{d\rho}-\frac{(F-G)}{\rho^2}\right)\frac{\cancel{\sin(kz-\phi)}}{k}$$

$$+ Fk\cancel{\sin(kz-\phi)} = 0$$

$$\Rightarrow \quad \frac{d^2F}{d\rho^2}+\frac{1}{\rho}\frac{dF}{d\rho}-\frac{1}{\rho}\frac{dG}{d\rho}+\frac{G-F}{\rho^2} - Fk^2 = 0 \longrightarrow ③$$

$$G+\rho\frac{dG}{d\rho}-F=0 \Rightarrow \frac{dG}{d\rho}=\frac{F-G}{\rho} \longrightarrow ④$$

④ in ③ $\longrightarrow$

$$\frac{d^2F}{d\rho^2}+\frac{1}{\rho}\frac{dF}{d\rho} \cancel{-\frac{1}{\rho}\frac{(F-G)}{\rho}} + \frac{2(G-F)}{\rho^2} + Fk^2 = 0$$

Using ②, $\cancel{-\frac{(F-G)}{\rho} = \frac{dF}{d\rho}+k^2\rho G}$

$$G\left(k^2\rho+\frac{1}{\rho}\right)=+\frac{dF}{d\rho}+\frac{F}{\rho}$$

$$\Rightarrow \quad \frac{d^2F}{d\rho^2}+\frac{1}{\rho}\frac{dF}{d\rho} - \frac{2}{\rho}\left(\frac{dF}{d\rho}-\frac{F}{\rho}\right)\frac{\cancel{\rho}}{k^2\rho^2-1} - \frac{2F}{\rho^2}+k^2F=0$$

$$\Rightarrow \quad \frac{d^2F}{d\rho^2}+\frac{1}{\rho}\frac{dF}{d\rho}-\frac{2}{\rho(k^2\rho^2-1)}\frac{dF}{d\rho}+\frac{2F(1-k^2\rho^2+1)}{\rho^2(k^2\rho^2-1)}+k^2F=0$$

$$\Rightarrow \quad \frac{d^2F}{d\rho^2}+\frac{1}{\rho}\frac{(k^2\rho^2-3)}{(k^2\rho^2-1)}\frac{dF}{d\rho}+\frac{F(4-2k^2\rho^2+k^4\rho^4-k^2\rho^2)}{\rho^2(k^2\rho^2-1)}=0$$

$$F = G + \rho G'$$

$$F' = 2G' + \rho G''$$

$$\rho F' + F - G - k^2\rho^2 G = 0$$

$$\Rightarrow 2\rho G' + \rho^2 G'' + \cancel{G} + \rho G' - \cancel{G} - k^2\rho^2 G = 0$$

$$\Rightarrow \rho^2 G'' + 3\rho G' - k^2\rho^2 G = 0$$

$$\Rightarrow x^2 G'' + 3xG' - x^2 G = 0$$

$$\cancel{H(x) = \frac{G(x)}{x} \Rightarrow H' = \frac{xG' - G}{x^2}}$$

$$\cancel{H'' = \frac{x^2(G' + xG'' - G') - 2x(xG' - G)}{x^4}}$$

$$H(x) = x\,G(x) \Rightarrow H' = G + xG' \Rightarrow H'' = 2G' + xG''$$

$$\Rightarrow G = \frac{H}{x}\ ; \quad G' = \frac{H' - G}{x}\ , \quad G'' = \frac{H'' - 2G'}{x}$$

then $$x\left(H'' - \frac{2(H' - H/x)}{x}\right) + 3\left(H' - \frac{H}{x}\right) - x^2\frac{H}{x} = 0$$

$$\Rightarrow x^2H'' - 2H'x + 2H + 3H'x - 3H - x^2H = 0$$

$$\Rightarrow x^2H'' + xH' - (1 + x^2)H$$

Sol^n is $$H(x) = c_1 J_1(-ix) + c_2 Y_1(-ix)$$

$$\Rightarrow G(x) = \frac{c_1 J_1(-ix) + c_2 Y_1(-ix)}{x}$$

$$F = G + \rho G'$$