# Derivation for the Electric Field Evolution of the Stored Light in $\Lambda$ Atom Memory

Thomas Li

September 2020

## 1 Theory

$\Lambda$ atom scheme:

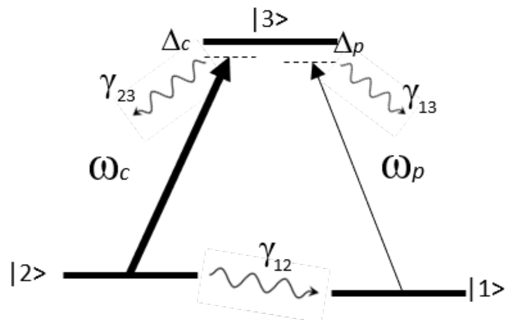


**Figure 1**. $\Lambda$ atom scheme, $\omega_c$: frequency of the pump beam; $\omega_p$: frequency of the probe beam, $\Delta_c$: detuning of the pump frequency (from the $|2\rangle$ to $|3\rangle$ transition frequency), $\Delta_p$: detuning of the probe frequency (from the $|1\rangle$ to $|3\rangle$ transition frequency), $\gamma_{ij}$: decoherence decay term from $|i\rangle$ to $|j\rangle$.

### 1.1 Equation of Motion of the System

State vector of the system:

$$|\psi(t)\rangle = \sum_{i=1}^{3} c_i(t) |i\rangle . \tag{1.1.1}$$

Hamiltonian:

$$\hat{H} = \hat{H}_o + \hat{H}_I. \tag{1.1.2}$$

Bare-atom Hamiltonian has the solution:

$$\hat{H}_o |\psi(t)\rangle = \hbar\omega_{31} |3\rangle + \hbar\omega_{21} |2\rangle ,$$

where $\omega_{31}$ is defined the angular frequency of the $1\longrightarrow 3$ transition and $\omega_{32}$ the $2\longrightarrow 3$ transition; the ground state energy of $|1\rangle$ is offset to zero.

In dipole approximation:

$$\hat{H}_I = -\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{E}}. \tag{1.1.3}$$

Apply matrix notation for n complete and orthonormal states,

$$|1\rangle = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, |n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix},$$

and

$$\hat{\boldsymbol{d}} \equiv \begin{pmatrix} \boldsymbol{d}_{11} & \cdots & \boldsymbol{d}_{1n} \\ \vdots & \ddots & \vdots \\ \boldsymbol{d}_{n1} & \cdots & \boldsymbol{d}_{nn} \end{pmatrix}.$$

The matrix equivalency gives

$$\hat{\boldsymbol{d}} = \boldsymbol{d}_{11} |1\rangle \langle 1| + \boldsymbol{d}_{21} |2\rangle \langle 1| + \cdots + \boldsymbol{d}_{nn} |n\rangle \langle n| .$$

Since $\boldsymbol{d}_{ij}$ denotes the effective dipole moment formed between a relative transitional state and a relative original state, same states which carry no effective dipole distance between each other would bear zero dipole moments, leading to the vanishing of the diagonal entries in the matrix, i.e., $\boldsymbol{d}_{ii} = 0$ for same i's. Hence,

$$\hat{\boldsymbol{d}} = \sum_{j>i} \left( \boldsymbol{d}_{ji} |j\rangle \langle i| + \boldsymbol{d}_{ij} |i\rangle \langle j| \right).$$

Since $\boldsymbol{d}$ is real, $\hat{\boldsymbol{d}}$ is a hermitian operator, and thus $\boldsymbol{d}_{ji} = (\boldsymbol{d}_{ji})^\dagger \equiv \boldsymbol{d}_{ij}^* = \boldsymbol{d}_{ij}$. With the relation $(|j\rangle \langle i|)^\dagger = |i\rangle \langle j|$, the equation is equivalent to:

$$\hat{\boldsymbol{d}} = \sum_{j>i} \left( \boldsymbol{d}_{ji} |j\rangle \langle i| + \text{H.c.} \right).$$

Employing semi-classical light-matter interaction approach, or, the Rabi Model, $\hat{\boldsymbol{E}} \equiv \boldsymbol{E}$, where the latter is a real function as in the classical case. The real monochromatic plane wave solution to the wave equation of electric field is

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}^{\boldsymbol{0}} \frac{1}{2} \left[ e^{i(\boldsymbol{q}\cdot\boldsymbol{r}(t) - \omega t + \phi)} + e^{-i(\boldsymbol{q}\cdot\boldsymbol{r}(t) - \omega t + \phi)} \right] \\ &= \frac{1}{2} E^0 e^{i(\boldsymbol{q}\cdot\boldsymbol{r}(t) - \omega t + \phi)} \boldsymbol{\epsilon} + c.c.. \end{aligned}$$

In the case of three-level atom in which the 1 to 2 dipole transition is forbidden, i.e., $\boldsymbol{d}_{12} = \boldsymbol{d}_{21} \equiv 0$,

$$\begin{aligned} \hat{\boldsymbol{d}}(t) =& \boldsymbol{d}_{31} |3\rangle \langle 1| + \boldsymbol{d}_{32} |3\rangle \langle 2| + \text{H.c.} \\ =& \sum_{s=1}^{2} \boldsymbol{d}_{3s} |3\rangle \langle s| + H.c.. \end{aligned} \tag{1.1.4}$$

In the case of two monochromatic plane waves, i.e., probe (p, index=1) and pump (c, index=2), with the global phases offset first to zero,

$$\begin{aligned}\hat{\boldsymbol{E}} = \boldsymbol{E_1} + \boldsymbol{E_2} =& \frac{1}{2}E_1^0 e^{i(\boldsymbol{q}_1\cdot\boldsymbol{r}(t)-\omega_1 t)}\boldsymbol{\epsilon}_1 + \frac{1}{2}E_2^0 e^{i(\boldsymbol{q}_2\cdot\boldsymbol{r}(t)-\omega_2 t)}\boldsymbol{\epsilon}_2 + c.c. \\ =& \frac{1}{2}\sum_{s=1}^{2} E_s^0 e^{i(\boldsymbol{q}_s\cdot\boldsymbol{r}(t)-\omega_s t)}\boldsymbol{\epsilon}_s + c.c.. \end{aligned} \tag{1.1.5}$$

To proceed, switch the view into Interaction Picture (denoted with a superscript apostrophe). (2) becomes

$$\begin{aligned}\hat{H}_I' &= e^{i\hat{H}_o/\hbar}\hat{H}_I e^{-i\hat{H}_o/\hbar} \\ &= -e^{i\hat{H}_o/\hbar}\hat{\boldsymbol{d}}\cdot\hat{\boldsymbol{E}}e^{-i\hat{H}_o/\hbar}, \end{aligned} \tag{1.1.6}$$

Combine (4) and (5) into (6). Coupling the transitions with corresponding fields:

$$\begin{aligned}\hat{H}_I' =& -\hbar e^{i\hat{H}_o t/\hbar}\Big[\chi_{31}(t)\left|3\right\rangle\left\langle 1\right| + \chi_{31}^*(t)\left|3\right\rangle\left\langle 1\right| + \chi_{32}(t)\left|3\right\rangle\left\langle 2\right| + \chi_{32}^*(t)\left|3\right\rangle\left\langle 2\right| + \text{H.c.}\Big]e^{-i\hat{H}_o/\hbar} \\ =& -\hbar e^{i\hat{H}_o t/\hbar}\Big[\sum_{s=1}^{2}\left(\chi_{3s}(t)\left|3\right\rangle\left\langle s\right| + \chi_{3s}^*(t)\left|3\right\rangle\left\langle s\right|\right) + \text{H.c.}\Big]e^{-i\hat{H}_o/\hbar}\end{aligned}$$

where the rabi frequency is defined $\chi_{3s} \equiv \frac{\mu_{3s}E_s^0}{2\hbar}$, $\chi_{3s}(t) = \chi_{3s}e^{i(\boldsymbol{q}_s\cdot\boldsymbol{r}(t)-\omega_s t)}$, and $\mu_{3s} = \boldsymbol{d}_{3s}\cdot\boldsymbol{\epsilon}_s$.

To proceed, use Rotating Wave Approximation (RWA) and drop the terms with the fast-oscillation terms, i.e., those with $i(\omega_{3s}+\omega_s)t$ exponents, for $s = \{1, 2\}$. The equation reduces to

$$\hat{H}_I' = -\hbar\Big[\sum_{s=1}^{2} e^{i\omega_{3s}t}\left(\chi_{3s}(t)\left|3\right\rangle\left\langle s\right| + \text{H.c.}\right)\Big]e^{-i\hat{H}_o/\hbar}.$$

The Schrodinger equation in interaction picture writes:

$$\begin{aligned} i\hbar\frac{\partial\left|\psi(t)'\right\rangle}{\partial t} =& \hat{H}_I'\left|\psi(t)'\right\rangle \\ i\hbar\frac{\partial}{\partial t}\left[e^{i\hat{H}_o t/\hbar}\left|\psi(t)\right\rangle\right] =& -\hbar\Big[\sum_{s=1}^{2} e^{i\omega_{3s}t}\left(\chi_{3s}(t)\left|3\right\rangle\left\langle s\right| + \text{H.c.}\right)\Big]e^{-i\hat{H}_o/\hbar}e^{i\hat{H}_o/\hbar}\left|\psi(t)\right\rangle \\ \frac{\partial}{\partial t}\left[e^{i\hat{H}_o t/\hbar}\left|\psi(t)\right\rangle\right] =& i\Big[\sum_{s=1}^{2} e^{i\omega_{3s}t}\left(\chi_{3s}(t)\left|3\right\rangle\left\langle s\right| + \text{H.c.}\right)\Big]\left|\psi(t)\right\rangle. \end{aligned}$$

The equation leads to a set of preliminary solutions:

$$\begin{aligned}\dot{c}_1 =& i\chi_{31}^*(t)c_3 \\ \dot{c}_2 =& -i\omega_{21}c_2 + i\chi_{32}^*(t)c_3 \\ \dot{c}_3 =& -i\omega_{31}c_3 + i\chi_{31}(t)c_1 + i\chi_{32}(t)c_2.\end{aligned}$$

Next, view the system in the basis of density matrix:

$$\begin{aligned} \hat{\rho} &\equiv |\psi\rangle\langle\psi| \\ \implies \rho_{ij} =& c_i c_j^* \\ \implies \dot{\rho}_{ij} =& c_i \dot{c}_j^* + \dot{c}_i c_j^*. \end{aligned}$$

Hence, $\rho_{ij} = \rho_{ji}^*$, and $\dot{\rho}_{ij} = \dot{\rho}_{ji}^*$. This leads to:

$$\begin{aligned} \dot{\rho}_{21}^0 =& -i\chi_{31}(t)\rho_{23} + i\chi_{32}^*(t)\rho_{31} - i\omega_{21}\rho_{21} \\ \dot{\rho}_{31}^0 =& -i\chi_{31}(t)(\rho_{33} - \rho_{11}) + i\chi_{32}(t)\rho_{21} - i\omega_{31}\rho_{31}, \end{aligned} \tag{1.1.7}$$

where the superscript of 0 denotes the quality of being unperturbed by the environmental factors.

Taking decoherence mechanisms into consideration, express the density elements with a time-dependent exponential decay and it would take the form

$$\rho_{ij} = \rho'_{ij} e^{-\gamma_1 t} e^{-\gamma_2 t} ... e^{-\gamma_n t} = \rho'_{ij} e^{-(\gamma_1 + \gamma_2 + ... + \gamma_n)t} = \rho'_{ij} e^{-\gamma_{ij} t},$$

for some decay mechanisms associated with decay constant $\gamma_1$, $\gamma_2$, ..., $\gamma_n$ combined to the composite decay rate $\gamma_{ij}$. Note that the solutions in (7) have no decay temporal contribution, or, equivalently, the $e^{-\gamma_{ij} t}$ was taken constant (say, $\rho_{ij}^0 = \rho'_{ij} e^{-\gamma_{ij} t_o}$). Hence, the modified EOM for density matrix takes

$$\begin{aligned} \dot{\rho}_{ij} =& \frac{\partial \rho'_{ij} e^{-\gamma_{ij} t_o}}{\partial t} + \rho'_{ij} \frac{\partial e^{-\gamma_{ij} t}}{\partial t} \\ =& \dot{\rho}_{ij}^0 - \gamma_{ij} \rho_{ij}. \end{aligned}$$

This gives the coherence terms:

$$\dot{\rho}_{21} = -i\chi_{31}(t)\rho_{23} + i\chi_{32}^*(t)\rho_{31} - i(\omega_{21} - i\gamma_{21})\rho_{21}, \tag{1.1.8}$$

and all the other coherence terms vanish when the probe field is being stored along with the shutdown of the control field.

## 1.2 $\rho_{21}$ with Arbitrary Input Field Profile and Atmoic Movement Dependence

The input light field is where the information is kept. When the fields for the $\Lambda$ atom meets the transition frequency, and with ideally a much stronger control field, the ground-level coherence of the atomic ensemble can be quickly driven to a coherent state as the input field has. A quasiparticle called a dark-state polariton is formed during this process. Storage of light is essentially linearly mapping the input light (and its states) to the the atomic ground level coherences through the dark-state polariton stage. And this map is [2]:

$$\hat{\boldsymbol{E}}(\boldsymbol{r}) = \frac{g\rho_{21}(\boldsymbol{r})}{\chi_{31}}. \tag{1.2.1}$$

This essentially tells that to find the light profile stored in the atoms in this fashion is to find the time variation of the $\rho_{21}$.

If taken as the ground level coherence dependence on time, then Eq. (1.1.8) does not bear enough spatial information but only depicts an ensemble average. To predict the evolution of the profile of the electric field stored in atoms whose motion follows Maxwell-Boltzmann statistics, an addition of a diffusion term (in the form of diffusion equation) to Eq. (1.1.8) suffices [3]. Now the time variation with spatial dependence of $\rho_{21}$ writes

$$\frac{\partial}{\partial t}\rho_{21}(\boldsymbol{r},t) = D\nabla^2\rho_{21}(\boldsymbol{r},t) - (\Gamma_{21} + i\omega_{21})\rho_{21}(\boldsymbol{r},t). \tag{1.2.2}$$

## 1.3 Solution to stored $\rho_{21}(\boldsymbol{r})$

Corollary: let $\hat{Q}$ be a non-temporal operator, $\widetilde{F}$ a function of $(\boldsymbol{r},t)$, and

$$\frac{\partial}{\partial t}\widetilde{F}(\boldsymbol{r},t) = \hat{Q}\widetilde{F}(\boldsymbol{r},t),$$

then $F(\boldsymbol{r},t) = \widetilde{F}(\boldsymbol{r},t)e^{At}$ for some constant A is a solution to

$$\frac{\partial}{\partial t}F(\boldsymbol{r},t) = \hat{Q}F(\boldsymbol{r},t) + AF(\boldsymbol{r},t).$$

*Proof*:

$$\begin{aligned}\frac{\partial}{\partial t}F = \frac{\partial \widetilde{F}e^{At}}{\partial t} &= e^{At}\frac{\partial}{\partial t}\widetilde{F} + Ae^{At}\widetilde{F}\\ &= e^{At}\hat{Q}\widetilde{F} + AF\\ &= \hat{Q}F + AF.\end{aligned}$$

Apply the corollary to Eq.(1.3.1) and we have

$$\rho_{21}(\boldsymbol{r},t) = \widetilde{\rho}_{21}(\boldsymbol{r},t)e^{-(\Gamma_{21}+i\omega_{21})t}, \tag{1.3.2a}$$

$$\frac{\partial}{\partial t}\widetilde{\rho}_{21}(\boldsymbol{r},t) = D\nabla^2\widetilde{\rho}_{21}(\boldsymbol{r},t). \tag{1.3.2b}$$

Now we seek solutions to Eq.(1.3.2). Set x, y, z as the three spatial degrees of freedoms and apply separation of variables,

$$\widetilde{\rho}_{21}(\boldsymbol{r},t) = \widetilde{\rho}_{21}(x,y,z,t) = X(x,t)Y(y,t)Z(z,t). \tag{1.3.3}$$

Then

$$\widetilde{\rho}_{21}(\boldsymbol{r},0) = X(x,0)Y(y,0)Z(z,0).$$

Substitute Eq.(1.3.3) back to Eq.(1.3.2) and have

$$\frac{\partial X(x,t)}{\partial t}YZ + \frac{\partial Y(y,t)}{\partial t}XZ + \frac{\partial Z(z,t)}{\partial t}XY = DYZ\frac{\partial^2 X(x,t)}{\partial x^2} + DXZ\frac{\partial^2 Y(x,t)}{\partial y^2} + DXY\frac{\partial^2 Z(z,t)}{\partial z^2}.$$

Divide both sides by $XYZ$ and get

$$\frac{1}{X}\frac{\partial X(x,t)}{\partial t} + \frac{1}{Y}\frac{\partial Y(y,t)}{\partial t} + \frac{1}{Z}\frac{\partial Z(z,t)}{\partial t} = D\frac{1}{X}\frac{\partial^2 X(x,t)}{\partial x^2} + D\frac{1}{Y}\frac{\partial^2 Y(y,t)}{\partial y^2} + D\frac{1}{Z}\frac{\partial^2 Z(z,t)}{\partial z^2},$$

which gives

$$\frac{\partial X(x,t)}{\partial t} = D\frac{\partial^2 X(x,t)}{\partial x^2}, \tag{1.3.4a}$$

$$\frac{\partial Y(y,t)}{\partial t} = D\frac{\partial^2 Y(y,t)}{\partial y^2}, \tag{1.3.4b}$$

$$\frac{\partial Z(z,t)}{\partial t} = D\frac{\partial^2 Z(z,t)}{\partial z^2}. \tag{1.3.4c}$$

We first seek the solution to $X(x,t)$. Let $X(x,0) = f(x)$. To solve the Eq.(1.3.4a), take Fourier transform of $X(x,t)$ and have

$$\begin{aligned}
\widehat{X}(k,t) &\equiv \int_{-\infty}^{\infty} X(x,t)e^{-2\pi ixk}dx, \\
X(x,t) &= \int_{-\infty}^{\infty} \widehat{X}(k,t)e^{2\pi ixk}dk.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial X(x,t)}{\partial t} &= \int_{-\infty}^{\infty} \frac{\partial \widehat{X}(x,t)}{\partial t}e^{2\pi ixk}dk, \\
\frac{\partial^2 X(x,t)}{\partial x^2} &= -4\pi^2k^2 \int_{-\infty}^{\infty} \widehat{X}(k,t)e^{2\pi ixk}dk.
\end{aligned}$$

Plug back in Eq.(1.3.4a) and we have

$$\frac{\partial \widehat{X}(k,t)}{\partial t} = -4\pi^2Dk^2\widehat{X}(k,t).$$

Now we are returned with a first-order linear homogeneous differential equation, which has the solution

$$\begin{aligned}
\widehat{X}(k,t) &= \widehat{X}(k,0)e^{-4\pi^2Dk^2t} \\
&= \int_{-\infty}^{\infty} X(x,0)e^{-2\pi ixk}dxe^{-4\pi^2Dk^2t} \\
&= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixk}dxe^{-4\pi^2Dk^2t} \\
&= \widehat{f}(k)e^{-4\pi^2Dk^2t}. 
\end{aligned} \tag{1.3.5}$$

Let $\widehat{g}(k,t) \equiv e^{-4\pi^2Dk^2t}$, then

$$\begin{aligned}
g(x,t) &= \int_{-\infty}^{\infty} e^{-4\pi^2Dk^2t}e^{2\pi ixk}dk \\
&= \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}.
\end{aligned}$$

From Eq. 1.3.5 we have $\widehat{X}(k,t) = \widehat{f}(k)\widehat{g}(k,t)$, hence

$$\begin{aligned}
X(x,t) &= \int_{-\infty}^{\infty} \widehat{f}(k)\widehat{g}(k,t)e^{2\pi ixk}dk \\
&= \int_{-\infty}^{\infty} \widehat{g}(k,t)\left[\int_{-\infty}^{\infty} \widehat{f}(x')e^{-2\pi ikx'}dx'\right]e^{2\pi ixk}dk \\
&= \int_{-\infty}^{\infty} f(x')\left[\int_{-\infty}^{\infty} \widehat{g}(k,t)e^{2\pi ik(x-x')}dk\right]dx' \\
&= \int_{-\infty}^{\infty} f(x')g(x-x')dx' \\
&= \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} f(x')e^{-\frac{(x-x')^2}{4Dt}}dx',
\end{aligned}$$

which is the solution to $X(x,t)$. Since thermal motion of atoms bears no axial preference, the solutions for $Y(y,t)$ and $Z(z,t)$ should take the same form. Eq. (1.3.3) now writes

$$\widetilde{\rho}_{21}(\boldsymbol{r},t) = (4\pi Dt)^{-\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')f(y')f(z')e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4Dt}}dx'dy'dz'.$$

Since $f(\boldsymbol{r}) = \widetilde{\rho}_{21}(\boldsymbol{r},0) = X(x,0)Y(y,0)Z(z,0) = f(x')f(y')f(z')$, we have

$$\widetilde{\rho}_{21}(\boldsymbol{r},t) = (4\pi Dt)^{-\frac{3}{2}}\prod_{i=1}^{3}\int_i \kappa_i' d\xi_i' f(\boldsymbol{r'})e^{-\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{4Dt}}, \tag{1.3.6}$$

where $\kappa_i$'s and $d\xi_i's$ for $i = \{1,2,3\}$ are the position characteristic functions and orthogonal infinitesimal displacements, respectively, of a spatial coordinate system.[1] Incorporate the equation into Eqn.(1.3.2b) and have

$$\rho_{21}(\boldsymbol{r},t) = e^{-(\Gamma_{21}+i\omega_{21})t}(4\pi Dt)^{-\frac{3}{2}}\prod_{i=1}^{3}\int_i \kappa_i' d\xi_i' f(\boldsymbol{r'})e^{-\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{4Dt}}. \tag{1.3.7}$$

This is the final solution to the ground level coherence of atoms to which the electric field of light $f(\boldsymbol{r})$ along with its states has been mapped after storage time $t$. The retrieval of the light stored in here is also a linear map following Eq.(1.2.1).

# 2 References

[1] David J. Griffiths, Introduction to Electrodynamics, fourth Edition, pp.576
[2] Fleischhauer, M., amp; Lukin, M. D. (2000). Dark-State Polaritons in Electromagnetically Induced Transparency. Physical Review Letters, 84(22), 5094-5097.
[3] Firstenberg, O., Shuker, M., Pugatch, R., Fredkin, D. R., Davidson, N., amp; Ron, A. (2008). Theory of thermal motion in electromagnetically induced transparency: Effects of diffusion, Doppler broadening, and Dicke and Ramsey narrowing. Physical Review A, 77(4).