Homework 2.

1. "The shortest distance between two points is a taxi"

$$\begin{array}{ccccccccc} 3 & 8 & 8 & 7 & 3 & 6 & 2 & 1 & 4 \end{array}$$

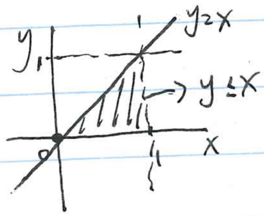
So Y take value 1, 2, 3, 4, 6, 7, 8

$$f(Y) = \begin{cases} \frac{1}{9} & Y=1 \\ \frac{1}{9} & Y=2 \\ \frac{2}{9} & Y=3 \end{cases} \quad \begin{array}{ll} \frac{1}{9} & Y=4 \\ \frac{1}{9} & Y=6 \\ \frac{1}{9} & Y=7 \end{array} \quad \begin{array}{ll} \frac{2}{9} & Y=8 \end{array}$$

So we have

$$E(Y) = \frac{1}{9}\times 1 + \frac{1}{9}\times 2 + \frac{2}{9}\times 3 + \frac{1}{9}\times 4 + \frac{1}{9}\times 6 + \frac{1}{9}\times 7 + \frac{2}{9}\times 8$$
$$= \frac{42}{9}$$
$$= \frac{14}{3}$$

2.



$0 \le x \le 1$
$0 \le y \le x$

$$E(XY) = \int_0^1 \int_0^x f(x,y)\cdot xy \, dy\, dx$$
$$= \int_0^1 \int_0^x 12y^2 \cdot xy \, dy\, dx$$
$$= \int_0^1 \int_0^x 12xy^3 \, dy\, dx$$
$$= \int_0^1 3x^5 \, dx$$
$$= 3\cdot\frac{1}{6}\cdot x^6 \Big|_0^1$$
$$= \frac{1}{2}$$

3. $X_1, X_2, X_3 \sim \text{Uniform}[0,1]$

$$E(X_1) = E(X_2) = E(X_3) = \frac{0+1}{2} = \frac{1}{2}$$
$$E(X_1^2) = E(X_2^2) = E(X_3^2) = \int_0^1 \frac{1}{b-a} x^2 \, dx = \frac{1}{3}$$
$$E(X_1X_2) = E(X_1X_3) = E(X_2X_3) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$$

$$E[(X_1 - 2X_2 + X_3)^2] = E[X_1^2 - 2X_1(2X_2 - X_3) + (2X_2 - X_3)^2]$$
$$= E[X_1^2 - 4X_1X_2 + 2X_1X_3 + 4X_2^2 - 4X_2X_3 + X_3^2]$$
$$= E(X_1^2) - 4E(X_1X_2) + 2E(X_1X_3) + 4E(X_2^2) - 4(X_2X_3) + E(X_3^2)$$
$$= \frac{1}{3} - 1 + \frac{1}{2} + \frac{4}{3} - 1 + \frac{1}{3}$$
$$= \frac{1}{2}$$

4.

$$E(Y) = \int_0^\infty e^{\frac{3}{4}x} \cdot f(x)\, dx = \int_0^\infty e^{\frac{3}{4}x} \cdot e^{-x}\, dx = \int_0^\infty e^{-\frac{1}{4}x}\, dx$$
$$= -4e^{-\frac{1}{4}x}\Big|_0^\infty$$
$$= -4 \times 0 - (-4) \times 1$$
$$= 4$$

5.

$X = 1, \dots, 6 \qquad P(X_1) = P(X_2) = \cdots = P(X_6) = \frac{1}{6}$

$$E(X) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 1 = \frac{7}{3}$$
$$E(X^2) = 1\times\frac{1}{6} + 4\times\frac{1}{6} + 9\times\frac{1}{6} + 16\times\frac{1}{6} + 25\times\frac{1}{6} + 36\times\frac{1}{6} = \frac{91}{6}$$
$$E(Y) = E[2X^2 + 1] = 2E(X^2) + 1 = \frac{91}{3} + 1 = \frac{94}{3}$$

6.

$$E(Y^2) = E[(2X+1)^2] = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1$$
$$E(X) = \int_0^1 x f(x)\, dx = \int_0^1 x(2-2x)\, dx = \frac{1}{3}$$
$$E(X^2) = \int_0^1 x^2 f(x)\, dx = \int_0^1 x^2(2-2x)\, dx = \frac{1}{6}$$
$$\therefore E(Y^2) = 4\times\frac{1}{6} + 4\times\frac{1}{3} + 1 = 3$$

7.

$$E[(aX+b)^n] = E\left[\sum_{k=0}^{n} \binom{n}{k} (aX)^{n-k} (b)^k\right] = E\left[\sum_{i=0}^{n} \binom{n}{i} a^{n-i} X^{n-i} b^i\right]$$
$$= \sum_{i=0}^{n} \binom{n}{i} E(a^{n-i} X^{n-i} b^i)$$
$$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$$

8. $E(X) = np \quad E(Y) = n(1-p) = n - np$

$E(X-Y) = E(X) - E(Y) = np - (n - np) = 2np - n$

$n = 20 \quad p = 0.05$

$E(X-Y) = 2 \times 20 \times 0.05 - 20 = -18$

For a sample size of 20, defective rate 0.05, we expect that the defective parts is 18 units less than the good parts.