MLE & Sufficiency

1. (a) total = 43 marked = 5

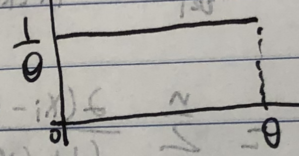
MLE $\hat{p} = \frac{5}{43}$

(b) total = 58 marked = 3

MLE $\hat{p} = \frac{3}{58}$

2. $f(x_i) = \frac{1}{\theta}$

$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} = \frac{1}{\theta^n}$



$\ell(\theta) = -n \log \theta$

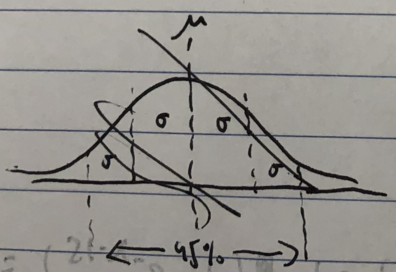
$\frac{\partial \ell(\theta)}{\partial \theta} = -\frac{n}{\theta}$

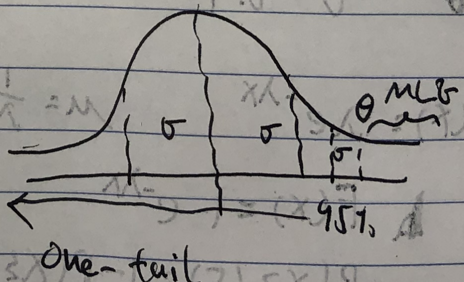
$L(\theta)$ is a decreasing function and it is maximized at $\theta = X_n$

$\therefore \hat{\theta} = X_n$

The minimal value of θ is the maximal of $X_1, \ldots, X_n$ since $X_1, \ldots, X_n \le \theta$

3.





$P(X < \theta) = 0.95 \Rightarrow P(X < \theta) = P\left(\frac{X-\mu}{\sigma} < \frac{\theta - \mu}{\sigma}\right)$

we know the 0.95 quantile of standard normal is 1.645

therefore, $\frac{\theta - \mu}{\sigma} = 1.645$ $\theta = \mu + 1.645\sigma$

4. $P(X>2) = P\left(\frac{X-\mu}{\sigma} > \frac{2-\mu}{\sigma}\right) = 1 - \phi\left(\frac{2-\mu}{\sigma}\right)$

5. $n = 20$ $\quad f(x) = \frac{1}{\pi(1+(x-\theta)^2)}$

$$L(\theta) = \prod_i^n f(x_i) = \frac{1}{\pi^n (1-(x_i-\theta)^2)^n} = \frac{1}{\pi^n} \cdot \frac{1}{(1-(x_i-\theta)^2)^n}$$

$$l(\theta) = -n\log\pi - \sum_{i=1}^n \log(1+(x_i-\theta)^2)$$

$$\frac{\partial l(\theta)}{\partial\theta} = \sum_{i=1}^n \frac{2(x_i-\theta)}{1+(x_i-\theta)^2}$$

$$\frac{\partial^2 l(\theta)}{\partial\theta^2} = \sum_{i=1}^n \frac{2(x_i-\theta)^2-2}{(1+(x_i-\theta)^2)^2}$$

To obtain MLE of θ, solve $g(\theta) = \frac{\partial l}{\partial\theta} = 0$, $g'(\theta) = \frac{\partial^2 l}{\partial\theta^2}$

Using newton-Raphson

$$\hat\theta^{t+1} = \hat\theta^t - \frac{g(\hat\theta^t)}{g'(\hat\theta^t)} ~~= \hat\theta^t - \frac{\partial^2 l(\hat\theta^t)}{\partial\hat\theta^{t^2}}~~$$

Implementing functions in R to calculate,

$\hat\theta = 0.4$

6. $f(x) = \lambda e^{-\lambda x}$ $\quad \mu = \frac{1}{\lambda}$

$F(x) = 1 - e^{-\lambda x}$

$P(X>15) = 1 - P(X \le 15) = 1 - (1 - e^{-\lambda\cdot 15}) = e^{-15\lambda}$

$$L(\lambda) = \prod_i^{20} \lambda e^{-\lambda x} \cdot e^{-15\lambda} = \lambda^{20} e^{-120\lambda} \cdot e^{-15\lambda} = \lambda^{20} e^{-135\lambda}$$

$$l(\lambda) = 20\log\lambda + (-135)\lambda = -20\log\mu - \frac{135}{\mu}$$

$$\frac{\partial l(\lambda)}{\partial\lambda} = -\frac{20}{\mu} + 135 = 0 \qquad \hat\mu = 6.75$$

7. $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$L(\lambda) = \frac{\lambda^{\Sigma x_i} e^{-n\lambda}}{\prod_i^n x_i!}$

$\ell(\lambda) = \Sigma x_i \log \lambda - n\lambda - \log \sum_i^n x_i!$

$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{\Sigma x_i}{\lambda} - n = 0 \quad \hat{\lambda} = \frac{\Sigma x_i}{n}$

$\hat{sd} = \sqrt{\hat{\lambda}} = \sqrt{\frac{\Sigma x_i}{n}}$

8. $f(x) = \lambda e^{-\lambda x}$ $\cancel{\mu = \frac{1}{\lambda}}$ Median $= \frac{\ln 2}{\lambda}$

$L(\lambda, x) = \lambda^n e^{-\lambda \Sigma x_i}$

$\ell(\lambda, x) = n \log \lambda - \lambda \Sigma x_i$

$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \Sigma x_i = 0$

$\hat{\lambda} = \frac{n}{\Sigma x_i}$ $\quad \beta = \sqrt{\frac{1}{\lambda}} \Rightarrow \lambda = \sqrt{\frac{1}{\beta}}$

$\cancel{\mu = \frac{1}{\lambda} = \frac{\Sigma x_i}{n}}$

$\cancel{\hat{\text{Median}} = \frac{\Sigma x_i}{n} \cdot \ln 2}$

$\cancel{\hat{\text{Median}} = \sqrt{\beta} \cdot \ln 2}$

$\hat{\text{Median}} = \frac{\Sigma x_i}{n} \ln 2$

Sufficiency

1. $T = \sum_i^n X_i \qquad f(x) = p^x (1-p)^{1-x}$

$L(p) = p^{\sum x_i} (1-p)^{n - \sum x_i}$

$= p^T (1-p)^{n-T}$

2. $T = \sum_i^n X_i \qquad f(x) = (1-p)^{x-1} p$

$L(p) = (1-p)^{\sum x_i - n} p^n = (1-p)^{T-n} p^n$

3. $T = \sum_i^n X_i \qquad f(k) = \binom{k+r-1}{k} (1-p)^r p^k$

$L(r, p; k) = \prod \binom{k+r-1}{k} (1-p)^{\sum r_i} p^{\sum k_i}$

$= \cancel{\prod (}$

$= \prod \binom{k+r-1}{k} (1-p)^{\sum r_i} p^T$

4. $T = \sum_i^n X_i \qquad f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$L(\alpha, \beta; x) = \frac{\beta^{\alpha n}}{[\Gamma(\alpha)]^n} \prod_i^n x_i^{\alpha-1} e^{-\beta \sum x_i}$

$= \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \right)^n \prod_i^n x_i^{\alpha-1} e^{-\beta T}$

$\prod_i^n x_i^{\alpha-1} = \exp\{(\alpha-1) \sum_i^n \ln x_i\}$

$T = \sum_i^n \ln X_i \qquad \exp\{T\} = \prod_i^n X_i$

$f(x) = \prod_i^n x_i^{\alpha-1} \qquad g(\alpha, \beta) = \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \right)^n \qquad h(\alpha, \beta) T(x) = -\beta \sum X_i$

5. $T = \prod_i^n X_i \qquad f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta \Sigma x_i}$

$$L(\alpha, \beta; x) = \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n \prod_i^n x_i^{\alpha-1} e^{-\beta \Sigma x_i}$$

$$= \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n T^{\alpha-1} e^{-\beta \Sigma x_i}$$

$$U(x) = e^{-\beta \Sigma x_i} \qquad V(T(x), \alpha) = \prod_i^n X_i^{\alpha-1}$$