

MLE & Sufficiency

1. (a) total = 43 marked = 5

$$\text{MLE } \hat{P} = \frac{5}{43}$$

(b) total = 58 marked = 3

$$\text{MLE } \hat{P} = \frac{3}{58}$$

$$((\theta - \hat{\theta}) + 1) \frac{1}{N} - \pi_{\text{pc}} / N = (\theta) \downarrow$$

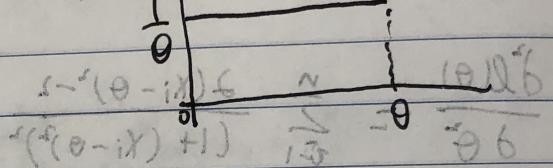
$$f(x_i) = \frac{1}{\theta}$$

$$L(\theta) = \prod_{i=1}^n \frac{1}{\theta} = \frac{1}{\theta^n}$$

$$l(\theta) = -n \log \theta$$

$$\frac{\partial l(\theta)}{\partial \theta} = -\frac{n}{\theta}$$

$$\frac{(\theta - \hat{\theta})}{(\theta - \hat{\theta}) + 1} \cdot \frac{N}{N} = \frac{(\theta) \downarrow}{\theta \downarrow}$$



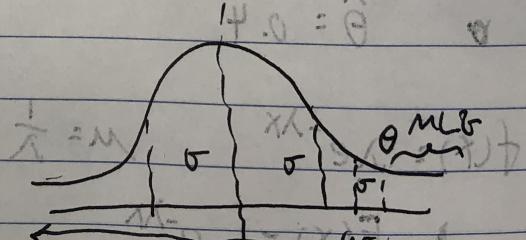
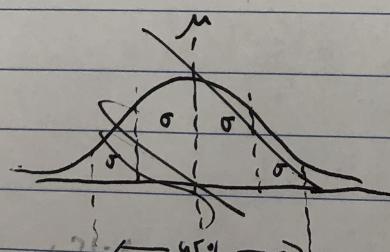
$\frac{\partial l(\theta)}{\partial \theta} = -\frac{n}{\theta}$ L(θ) is a decreasing function and it is maximized

at $\theta = x_n$

$$\therefore \hat{\theta} = x_n$$

The minimal value of θ is the maximal of x_1, \dots, x_n
since $x_1, \dots, x_n \leq \theta$

3.



$$\theta = (\bar{x} - 1.96\sigma) \rightarrow (\bar{x} - 1.96\sigma)$$

$$P(X < \theta) = 0.95 \Rightarrow P(X < \theta) = P\left(\frac{X - \mu}{\sigma} < \frac{\theta - \mu}{\sigma}\right)$$

we know the 0.95 quantile of standard normal is 1.645

therefore,

$$\frac{\theta - \mu}{\sigma} = 1.645 \quad \theta = \mu + 1.645\sigma$$

$$P(X > 2) = P\left(\frac{X-\mu}{\sigma} > \frac{2-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{2-\mu}{\sigma}\right)$$

4. $n = 20$ $f(x) = \frac{1}{\pi(1+(x-\theta)^2)}$

$$L(\theta) = \prod_{i=1}^n f(x_i) = \frac{1}{\pi^n (1+(x_i-\theta)^2)^n} = \frac{1}{\pi^n} \cdot \frac{1}{(1+(x_i-\theta)^2)^n}$$

$$\ell(\theta) = -n \log \pi - \sum_{i=1}^n \log(1+(x_i-\theta)^2)$$

$$\frac{\partial \ell(\theta)}{\partial \theta} = \sum_{i=1}^n \frac{2(x_i-\theta)}{1+(x_i-\theta)^2}$$

$$\frac{\partial^2 \ell(\theta)}{\partial \theta^2} = \sum_{i=1}^n \frac{2(1-(x_i-\theta)^2)}{(1+(x_i-\theta)^2)^2}$$

$$\frac{1}{\theta} = (\hat{\theta})$$

$$\frac{1}{\theta} = \frac{1}{\hat{\theta}} = (\theta)$$

To obtain MLE of θ , solve $g(\theta) = \frac{d\ell}{d\theta} = 0$, $g'(\theta) = \frac{\partial^2 \ell}{\partial \theta^2}$

Using Newton-Raphson $g(\hat{\theta}^t)$

$$\text{when } \hat{\theta}^{t+1} = \hat{\theta}^t - \frac{g'(\hat{\theta}^t)}{\frac{\partial^2 \ell(\hat{\theta}^t)}{\partial \theta^2}}$$

Implementing functions in R to calculate,

$$\hat{\theta} = 0.4$$

6. $f(x) = \lambda e^{-\lambda x}$ $\mu = \frac{1}{\lambda}$

$$F(x) = 1 - e^{-\lambda x}$$

$$P(X > 15) = 1 - P(X \leq 15) = 1 - (1 - e^{-\lambda \cdot 15}) = e^{-15\lambda}$$

$$L(\lambda) = \prod_{i=1}^{20} \lambda e^{-\lambda x_i} \cdot e^{-15\lambda} = \lambda^{20} e^{-120\lambda} e^{-15\lambda} = \lambda^{20} e^{-135\lambda}$$

$$\ell(\lambda) = 20 \log \lambda + (-135)\lambda = 20 \log \lambda - \frac{135}{\lambda}$$

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = -\frac{20}{\lambda} + 135 \Rightarrow \hat{\lambda} = 6.75$$

7.

$$f(x) = \frac{x^{\lambda} e^{-\lambda}}{x!} \quad i x \stackrel{n}{\sum} = T$$

$$L(\lambda) = \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod x_i!} \quad i x \stackrel{n}{\sum} (q-1)^q = (x) \uparrow$$

$$\ell(\lambda) = \sum x_i \log \lambda - n\lambda - \tau \log \left(\frac{\lambda}{\prod x_i!} \right) + q =$$

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{\sum x_i}{\lambda} - n(1-x)(q-1) = (\lambda) = \hat{\lambda} = \frac{\sum x_i}{n} \quad i x \stackrel{n}{\sum} = T$$

$$\hat{s}d = \sqrt{\hat{\lambda}^{n-1} \left(\frac{\sum x_i}{n} \right)} = n q^{n-\frac{1}{2}} (q-1) = (q) \downarrow$$

8.

$$f(x) = \lambda e^{-\lambda} x \quad \text{Median} = \frac{\ln 2}{\lambda} \quad i x \stackrel{n}{\sum} = T$$

$$L(\lambda, x) = \lambda^n e^{-\lambda \sum x_i} \pi = (x; \lambda, \beta) \downarrow$$

$$\ell(\lambda, x) = n \log \lambda - \lambda \sum x_i$$

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum x_i =$$

$$\hat{\lambda} = \frac{n}{\sum x_i} = (x) \beta = \sqrt{\frac{1}{\lambda} i x \stackrel{n}{\sum}} \Rightarrow \lambda = \frac{1}{\beta}$$

$$\text{Median} = \frac{\sum x_i}{n} = (x; \lambda, \beta) \downarrow$$

$$\text{Median} = \sqrt{\beta \cdot \ln 2}$$

$$\hat{\text{Median}} = \frac{\sum x_i}{n} \ln 2 \left(\frac{1}{\lambda} \right) =$$

$$(ixn) \frac{n}{\sum} (1-\lambda) \uparrow q \uparrow = 1 - \lambda \frac{n}{\sum} = T$$

$$ix \frac{n}{\sum} = QT \uparrow q \uparrow \quad ix \frac{n}{\sum} = T$$

Sufficiency

1. $T = \sum_i x_i$ $f(x) = p^x (1-p)^{1-x}$ $\frac{\partial \ln f(x)}{\partial T} = (x-1)$

 $L(p) = p^{\sum x_i} (1-p)^{n-\sum x_i}$ $\frac{\partial \ln L(p)}{\partial T} = (x-1)$
 $= p^T (1-p)^{n-T}$ $\ln L(p) = T \ln p + (n-T) \ln(1-p)$

2. $T = \sum_i x_i$ $f(x) = (1-p)^{x-1} p^x$ $\frac{\partial \ln f(x)}{\partial T} = \frac{x}{1-p} = \frac{1}{\lambda}$

 $L(p) = (1-p)^{\sum x_i - n} p^n$ $\frac{\partial \ln L(p)}{\partial T} = \frac{n}{1-p} = \lambda$

3. $T = \sum_i x_i$ $f(x) = \binom{k+r-1}{k} (1-p)^r p^k$ $\frac{\partial \ln f(x)}{\partial T} = (x-1)$

 $L(r, k; x) = \prod_i \binom{k+r-1}{k} (1-p)^{x_i} p^{k_i}$ $\frac{\partial \ln L(r, k; x)}{\partial T} = (x-1)$
 $= \prod_i \binom{k+r-1}{k} (1-p)^{\sum x_i} p^T$ $\frac{\partial \ln L(r, k; x)}{\partial T} = (x-1)$

4. $\frac{\partial}{\partial \beta} \ln f(x) = \sum_i x_i = T$ $f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$L(\alpha, \beta; x) = \frac{\beta^{\alpha n}}{[\Gamma(\alpha)]^n} \prod_i \frac{1}{i} x_i^{\alpha-1} e^{-\beta \sum x_i}$
 $= \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \right)^n \frac{1}{\prod_i i} \frac{x^{\alpha-1}}{n!} e^{-\beta T}$

$\frac{1}{\prod_i i} x_i^{\alpha-1} = \exp \left\{ (\alpha-1) \sum_i (\ln x_i) \right\}$

$T = \sum_i \ln x_i$ $\exp(T) = \prod_i x_i$

$f(x) = \prod_i x_i^{\alpha-1}$ $g(\alpha, \beta) = \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \right)^n h(\alpha, \beta) T(x)$
 $= -\beta \sum x_i$

$$J. \quad T = \prod_i^n X_i \quad f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta \sum x_i}$$

$$L(\alpha, \beta; x) = \left[\frac{\beta^\alpha}{\Gamma(\alpha)} \right]^n \prod_i^n x_i^{\alpha-1} e^{-\beta \sum x_i}$$

$$= \left[\frac{\beta^\alpha}{\Gamma(\alpha)} \right]^n T^{\alpha-1} e^{-\beta \sum x_i}$$

$$v(x) = e^{-\beta \sum x_i} \quad v(T(x), \beta) = \prod_i^n x_i^{\alpha-1}$$

$$(T, \cdot) \text{ ist } \mathbb{R}^n \text{ auf } \mathbb{R}^n \hookrightarrow \mathbb{R}^n \quad \leftarrow \quad M = \mathbb{R}^n \quad (\text{S})$$

$$i\pi \bar{z} / i\bar{z} = i\pi \text{ gleich}$$

$$(T, \cdot) \text{ ist } \mathbb{R}^n \text{ auf } \mathbb{R}^n \hookrightarrow \mathbb{R}^n \quad \leftarrow \quad \mathbb{R}^n = \mathbb{R}^n \quad (\text{S})$$

$$\text{rechts} \rightarrow \text{es negativen ent in fest normen} \text{ kann nicht}$$

$$i\pi p_1 + i\pi p_2 + i\pi p_3 = i\pi p_0 \subseteq i\pi \cdot i\pi n = [i\pi] \mathbb{Z} : \downarrow$$

$$i\pi + i\pi + i\pi = i\pi n = [i\pi] \mathbb{Z} : \downarrow$$

$$i(\pi x) + i\pi + i\pi + i\pi = i\pi(p_1 + p_2 + p_3) = i\pi p_0$$

~~stetig~~

$$0 = g(\pi x) : \mathbb{H} \subset$$