Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2020.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

Proof. • Assume there is not any prime satisfying the inequation, which means every integer a_i in set $A = \{n < p < n!\}$ is not a prime.

Then we select $a_1 = n! - 1$, which is in A. That's to say, a_1 is not a prime.

Therefore, there must exsit a prime q ($q < a_1$) satisfy: $a_1 \bmod q = 0$. Because q is a prime, it should be out of the set A. Considering $q < a_1 = n! - 1 < n!$, we can get: $q \leq n$.

Since $q \leq n$, $n! = 1 \times 2 \times ... \times q \times ... \times n$. So: $n! \bmod q = 0$. Therefore, $(n! - 1) \bmod q \neq 0$.

So, a_1 is not a prime, which contradicts the assumption that every a_i in set A is a prime. So the statement has been proved.

□

2. Use the minimal counterexample principle to prove that for any integer $n > 17$, there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$.

Proof. If the statement is not true for every $n > 17$,then there are values of n which make the statement false, and there must be a smallest such value, say $n = k$.

Since when $n = 18 = 1 \times 4 + 2 \times 7$, $i_{18} = 1$, $j_{18} = 2$, we have $k \geq 19$, and $(k - 1) \geq 18$.

Since k is the smalleset value among the false-values, $k - 1$ should make the correspond statement right. Thus, $k - 1 = i_{k-1} \times 4 + j_{k-1} \times 7$.

(1) If $j_{k-1} \geq 1$,Then, we can get :

$$k = (i_{k-1} + 2) \times 4 + (j_{k-1} - 1) \times 7$$

Then, we can easily get:$i_k = i_{k-1} + 2 \geq 0$ and $j_k = j_{k-1} - 1 \geq 0$, so $n = k$ makes the statement true.

(2) If $j_{k-1} < 1$,which means $j_{k-1} = 0$. Then, we can let

$$k = (j_{k-1} - 5) \times 4 + (j_{k-1} + 3) \times 7$$

Considering that $(k - 1) \geq 18$ and $(k - 1) \bmod 4 = 0$, we can get: $i_{k-1} \geq 5$, because $5 \times 4 = 20 > 18$ while $4 \times 4 = 16 < 18$. Then, let:$i_k = i_{k-1} - 5 \geq 0$ and $j_k = j_{k-1} + 3 \geq 0$, so $n = k$ makes the statement true.

To sum up, $n = k$ can always make the statement true. We have derived a contradiction, which allows us to conclude that our original assumption is false.

□

3. Let $P = \{p_1, p_2, \cdots\}$ the set of all primes. Suppose that $\{p_i\}$ is monotonically increasing, i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\cdots$. Please prove: $p_n < 2^{2^n}$. (Hint: $p_i \nmid (1 + \prod_{j=1}^{n} p_j), i = 1, 2, \cdots, n.$)

Proof. Let $P(n)$ mean: $p_n < 2^{2^n}$.

(1)**Basis step.** Obviously, $P(1)$ is true.

(2)**Induction hypothesis.** For $k \geq 1$ and $1 \leq n \leq k$, $P(n)$ is true.

(3)**Proof of induction step.** Then prove $P(k+1)$

We know that if a interger is not divisiable by any smaller prime, then it must be a prime. Consider that:

$$p_i \nmid (1 + \prod_{j=1}^{k} p_j), i = 1, 2, \cdots, k$$

Let $a = (1 + \prod_{j=1}^{k} p_j)$, and a must be a prime.

That's to say a is a prime which is bigger than p_k, and p_{k+1} is the smallest prime of the ones which are bigger than p_k, then we can get: $p_{k+1} \leq a$.

Then,

$$(1 + \prod_{j=1}^{k} p_j) = 1 + 2^{2^1} \times 2^{2^2} \times \cdots \times 2^{2^k} < 2^{2^1} \times 2^{2^1} \times 2^{2^2} \times \cdots \times 2^{2^k} = 2^{2^1+2^1+2^2+\cdots+2^k} = 2^{2^{k+1}}$$

So $p_{k+1} \leq a < 2^{2^{k+1}}$. Thus, $P(k+1)$ is true and the statement has been proved.

□

4. Prove that a plane divided by n lines can be colored with only 2 colors, and the adjacent regions have different colors.

Proof. Let $P(n)$ mean: the statement is true when there are n lines.

(1)**Basis step.** Obviously, $P(1)$ is true.

(2)**Induction hypothesis.** For $n = k$, $P(k)$ is true.

(3)**Proof of induction step.** Then prove when $n = k + 1$, $P(k+1)$ is true

As we assume, the plane has been divided by k lines, and has been colored successfully. Then we add anothor line and name it λ, now we call the part at the right of λ "Rpart", and the other one is "Lpart".

Then, we invert the color of every cell in Rpart as the fig1 (the yellow line is λ). As you can see, the new graph is still colored successfully.

Now, we will prove its rationality. Because we only propose the Rpart, so every adjacent cells in Lpart stay the same and have different colors. Also, all of the cells in Rpart change their color, so every adjacent cells in Rpart still have different colors too.

Then, about the ones next to the λ.They used to be unbroken, but are separated by the λ. Since we change the colors of the ones in Rpart, these changed cells have different colors with the left-adjacent ones.

Now, every adjacent cells have different colors and $P(k+1)$ is true.

□

Remark: You need to include your .pdf and .tex files in your uploaded .rar or .zip file.