Underdetermined Low-Complexity Wideband DOA Estimation with Uniform Linear Arrays

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***Abstract*—For a uniform linear array (ULA) with specially designed spacing and system settings, the received signals are first decomposed into different frequency bins via discrete Fourier transform (DFT). By grouping the frequencies of interest into several pairs, a generalized complexity reduction method is proposed to merge the redundant entries in both the auto-correlation matrices at each frequency and the cross-correlation matrices across different frequencies, followed by the group sparsity based low-complexity method to find the directions of arrival (DOAs) of the impinging signals. Simulation results demonstrate that significantly reduced complexity and improved performance can be achieved by the proposed method.**

I. Introduction

By resolving more sources than the number of physical sensors, underdetermined DOA estimation has attracted great attentions in the research community [1]–[4]. Nested arrays [2], co-prime arrays [3], CADiS [5], super nested arrays [6], and thinned co-prime array [7] are representative examples, offering increased degrees of freedom (DOFs) based on the difference co-array concept. By exploiting the high-order statistics, specially designed sparse array structures including multiple nested arrays (ML-NAs) [8], simplified and enhanced ML-NAs (SE-ML-NAs) [9], and other fourth-order cumulant based extensions [10]–[12] have been proposed for underdetermined DOA estimation with improved resolution capacity.

In the narrowband scenario, the co-array MUSIC [2], [13] and the compressive sensing (CS) based method [9], [14] can be employed to exploit the co-array equivalence, and virtual array interpolation [15], [16] can be applied for performance improvement. For underdetermined wideband DOA estimation, group sparsity based methods within the CS framework [17], [18] and focusing based approaches [19], [20] have been proposed, with the Cramér-Rao bound derived in [21].

Among all the aforementioned methods, sparse array configurations are adopted, but the frequency information are not involved for DOFs improvement. In [22], a pair of co-prime frequencies is employed with a single ULA behaving like two equivalent sub-arrays in a co-prime array. This idea is further extended to the wideband case [23], [24] with a higher number of DOFs achieved. However, in that work, the group sparsity concept is utilized twice, generating a block diagonal sensing matrix with a very large dimension, and therefore the extremely high computational complexity becomes an associated problem.

In this paper, after reviewing the formulation for each frequency pair in [23], we first merge the redundant entries in the auto-correlation matrices at each frequency bin for dimension reduction. The combination process of redundant entries in cross-correlation matrices across frequency bins is more complicated due to different selection of the frequency pairs, and a generalized complexity reduction method is then presented, leading to a solution with a lower computational complexity without sacrificing the performance. Then, a group sparsity based low-complexity method is presented, and as will be shown in the simulations, the performance of the proposed method is further improved by optimizing a simplified formulation with a better estimation of the statistics compared with the existing method, and also a shorter computation time is required.

This paper is organized as follows. The system model is presented in Section II. The proposed group sparsity based low-complexity DOA estimation method is proposed in Section III. Simulation results are provided in Section IV, and conclusions are drawn in Section V.

II. System Model Design

For wideband DOA estimation exploiting the co-arrays in the spatio-spectral domain, the correlation property among different frequencies is required, and thus the LFMCW signal [23] and mult-frequency signal [24] are preferred as the transmitted waveform to ensure this property.

In this paper, the LFMCW signal is employed as an example, and the LFMCW signal with a bandwidth B is

$$s(t) = Ae^{j(2\pi f_c t + \pi\alpha\cdot \mathrm{mod}(t,T)^2 + \varphi)} \ , \tag{1}$$

where A is the signal amplitude, f_c is the initial frequency, $\alpha = B/T$ is the chirp rate with T being the modulation period, φ is the initial phase, and $\mathrm{mod}(t,T)$ is the modulo operation.

We simply consider an M-sensor ULA with the inter-element spacing d, and its sensor position set is given as

$$\mathbb{S} = \{md,\ 0 \le m \le M-1, m \in \mathbb{Z}\} \ . \tag{2}$$

The echo signal observed at the m-th sensor is

$$x_m(t) = \sum_{k=1}^{K} \gamma_k(t) \cdot s\left[t - \tau_m(\theta_k)\right] + \bar{n}_m(t) \ , \tag{3}$$

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where $\gamma_k(t)$ is the time-varying reflection coefficient owing to both target motion and radar cross section (RCS) fluctuations. Since the phase delay varies with frequency and the target reflectivity may be different across the signal bandwidth, $\gamma_k(t)$ is in general frequency-dependent. Take the zeroth position as the reference, $\tau_m(\theta_k)$ represents the time delay of the k-th signal impinging from θ_k arriving at the m-th sensor of the array.

Each received signal is divided into non-overlapping groups with length L. Then, the output array signal model at the l-th frequency bin and the p-th DFT group after an L-point DFT can be expressed as

$$\mathbf{X}[l,p] = \mathbf{A}(l,\boldsymbol{\theta})\mathbf{S}[l,p] + \overline{\mathbf{N}}[l,p] \ , \tag{4}$$

where $\mathbf{S}[l,p] = \left[S_1[l,p], \ldots, S_K[l,p]\right]^T$ with $S_k[l,p]$ being the DFT of the received echo signals $\gamma_k[i]s[i]$. The steering vector in the steering matrix $\mathbf{A}(l,\boldsymbol{\theta})$ is

$$\mathbf{a}(l,\theta_k) = \left[1, e^{-j\frac{2\pi d}{\lambda_l}\sin(\theta_k)}, \ldots, e^{-j\frac{2\pi(M-1)d}{\lambda_l}\sin(\theta_k)}\right]^T .$$

For a certain array structure with a given inter-element spacing d, we design the frequency interval f_Δ to satisfy

$$f_\Delta = \frac{f_s}{L} = \frac{c}{2d} \ , \tag{5}$$

Accordingly, $\mathbf{a}(l,\theta_k)$ can be updated to

$$\mathbf{a}(l,\theta_k) = \left[1, e^{-j\pi l\sin(\theta_k)}, \ldots, e^{-j\pi(M-1)l\sin(\theta_k)}\right]^T . \tag{6}$$

III. Underdetermined Low-Complexity DOA Estimation Based on Difference Co-Arrays

A. Difference co-array generation in the spatio-spectral domain

Assume that the bandwidth of the echo signals covers Q frequency bins in the DFT domain with indexes $l_q \in \Phi_l$, $0 \le q \le Q-1$. These frequency bins of interest are divided into N pairs, with l_{n_1} and l_{n_2} being the frequency components of the n-th pair ($l_{n_1} \in \Phi_l$, $l_{n_2} \in \Phi_l$, and $l_{n_1} \neq l_{n_2}$).

Denote $\alpha, \beta \in \{l_{n_1}, l_{n_2}\}$, and we use α and β for convenience of formulations at a later stage. The correlation matrices (including both auto-correlation matrices and cross-correlation matrices) of the two frequencies can be obtained by

$$\begin{aligned}&\mathbf{R_x}[\alpha,\beta] = \mathrm{E}\left\{\mathbf{X}[\alpha,p]\cdot\mathbf{X}^H[\beta,p]\right\}\\ =&\sum_{k=1}^{K}\sigma_k^2[\alpha,\beta]\mathbf{a}(\alpha,\theta_k)\mathbf{a}^H(\beta,\theta_k) + \sigma_{\bar{n}}^2[\alpha,\beta]\mathbf{I}_M\cdot\delta(\alpha-\beta) \ ,\end{aligned}$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix. For $\alpha = \beta$, the parameters $\sigma_k^2[\alpha,\alpha]$, representing both $\sigma_k^2[l_{n_1}, l_{n_1}]$ and $\sigma_k^2[l_{n_2}, l_{n_2}]$, denotes the power of the k-th signal at the α-th frequency bin, whereas $\sigma_{\bar{n}}^2[\alpha,\alpha]$ is the corresponding noise power. For $\alpha \neq \beta$, $\sigma_k^2[\alpha,\beta]$ is not zero due to correlation among different frequencies offered by the LFMCW waveforms, and $\sigma_{\bar{n}}^2[\alpha,\beta] = 0$. $\delta(\alpha-\beta)$ is the Kronecker delta function.

It is worth nothing that both $\sigma_k^2[\alpha,\alpha]$ and $\sigma_{\bar{n}}^2[\alpha,\alpha]$ are real and positive, while $\sigma_k^2[l_{n_1}, l_{n_2}]$ and $\sigma_k^2[l_{n_2}, l_{n_1}]$ are in general complex values owing to the phase shift between different frequency bins caused by the LFMCW echo signal and the reflection coefficient.

Since $\mathbf{R_x}[l_{n_1}, l_{n_2}] = \mathbf{R_x}^H[l_{n_2}, l_{n_1}]$, we only use the former in the estimation process for complexity reduction.

Vectorizing $\mathbf{R_x}[\alpha,\beta]$ yields a virtual array model

$$\begin{aligned}\mathbf{z}[\alpha,\beta] &= \mathrm{vec}\left\{\mathbf{R_x}[\alpha,\beta]\right\}\\ &= \widetilde{\mathbf{A}}[\alpha,\beta]\widetilde{\mathbf{s}}[\alpha,\beta] + \sigma_{\bar{n}}^2[\alpha,\beta]\widetilde{\mathbf{I}}_{M^2}\cdot\delta(\alpha-\beta) \ ,\end{aligned} \tag{7}$$

with the equivalent steering matrix

$$\widetilde{\mathbf{A}}[\alpha,\beta] = \left[\widetilde{\mathbf{a}}(\alpha,\beta,\theta_1), \ldots, \widetilde{\mathbf{a}}(\alpha,\beta,\theta_K)\right] \ , \tag{8}$$

where $\widetilde{\mathbf{a}}(\alpha,\beta,\theta_k) = \mathbf{a}^*(\beta,\theta_k) \otimes \mathbf{a}(\alpha,\theta_k)$ with $\otimes$ as the Kronecker product, and $\widetilde{\mathbf{s}}[\alpha,\beta] = \left[\sigma_1^2[\alpha,\beta], \ldots, \sigma_K^2[\alpha,\beta]\right]^T$. $\widetilde{\mathbf{I}}_{M^2} = \mathrm{vec}\{\mathbf{I}_M\}$ is an $M^2 \times 1$ column vector.

For $\alpha = \beta$, (7) characterize a virtual array model corresponding to the self-difference co-array lags

$$\{(\alpha m_1 - \alpha m_2), 0 \le m_1, m_2 \le M-1\} \ , \tag{9}$$

while for $\alpha \neq \beta$, the cross-difference co-array lags in the spatio-spectral domain is generated as

$$\{\pm(\alpha m_1 - \beta m_2), 0 \le m_1, m_2 \le M-1\} \ . \tag{10}$$

It is noted that by exploiting the co-arrays in (9) and (10) for DOA estimation, more sources than the number of sensors can be resolved due to a larger number of virtual sensors compared with the physical sensor number M. Specifically, l_{n_1} and l_{n_2} can be chosen to be co-prime, and then the decomposed signals at two frequency bins are equivalent to the received signals of two uniform linear sub-arrays in a co-prime array with $2M - 1 - \mathrm{floor}\{\frac{M-1}{\max(l_{n_1}, l_{n_2})}\}$ sensors [22], where $\mathrm{floor}\{\cdot\}$ returns the largest integer not exceeding the argument and $\max\{\cdot\}$ returns the maximum value of the input vector.

B. Complexity reduction

$\widetilde{\mathbf{s}}[l_{n_1}, l_{n_1}]$, $\widetilde{\mathbf{s}}[l_{n_2}, l_{n_2}]$, and $\widetilde{\mathbf{s}}[l_{n_1}, l_{n_2}]$ are in general different and therefore the three virtual array models characterized by (7) are unique and should be treated separately. Furthermore, $\widetilde{\mathbf{s}}[l_{n_1}, l_{n_2}]$ is generally complex, indicating that the entries related to the opposite co-array lags in the spatio-spectral domain are also unique. As a result, the redundancies combination method in [17] can not be adopted.

Note that the auto-correlation matrix $\mathbf{R_x}[\alpha,\alpha]$ is both Hermitian and Toeplitz. Since $\mathbf{R_x}[\alpha,\alpha] = \mathbf{R_x}^H[\alpha,\alpha]$, the complex conjugate part can be removed for complexity reduction. Further complexity reduction can be achieved by averaging the redundant entries with the same lag along the diagonal direction according to the Toeplitz property to form a more accurate estimation of the statistical expectation.

After removing the redundant lags in $\mathbf{R_x}[\alpha,\alpha]$, we obtain a column vector $\mathbf{z_c}[\alpha,\alpha]$ with its merged m-th entry given by

$$z_c^m[\alpha,\alpha] = \sum\nolimits_{\hat{m}=m}^{M-1} R_x^{\hat{m},\hat{m}-m}[\alpha,\alpha] \ , \tag{11}$$

where $m = 0, 1, \ldots, M-1$, and the superscripts of matrices denotes the corresponding row and column indexes.

The m_1-th row and m_2-th column of the cross-correlation matrix $R_x^{m_1,m_2}[l_{n_1}, l_{n_2}]$ is given by

$$R_x^{m_1,m_2}[l_{n_1}, l_{n_2}] = \sum_{k=1}^{K} \sigma_k^2[l_{n_1}, l_{n_2}] e^{-j\pi(l_{n_1}m_1 - l_{n_2}m_2)\sin\theta_k} \ .$$

For the indexes (m_1, m_2) and $(\hat{m}_1, \hat{m}_2)$, $R_x^{m_1,m_2}[l_{n_1}, l_{n_2}]$ and $R_x^{\hat{m}_1,\hat{m}_2}[l_{n_1}, l_{n_2}]$ are equal if and only if $l_{n_1}m_1 - l_{n_2}m_2 = l_{n_1}\hat{m}_1 - l_{n_2}\hat{m}_2$, which can be modified into

$$l_{n_1}(m_1 - \hat{m}_1) = l_{n_2}(m_2 - \hat{m}_2) \ , \tag{12}$$

where $0 \leq m_1, m_2, \hat{m}_1, \hat{m}_2 \leq M-1$.

Assume that η_n is the greatest common divisor between l_{n_1} and l_{n_2}, and then (12) can be updated to

$$\frac{l_{n_1}}{\eta_n}(m_1 - \hat{m}_1) = \frac{l_{n_2}}{\eta_n}(m_2 - \hat{m}_2) \ , \tag{13}$$

where $\frac{l_{n_1}}{\eta_n}$ and $\frac{l_{n_2}}{\eta_n}$ are co-prime.

The necessary and sufficient condition of (13) is

$$m_1 - \hat{m}_1 = \bar{k}\frac{l_{n_2}}{\eta_n} \bigcap m_2 - \hat{m}_2 = \bar{k}\frac{l_{n_1}}{\eta_n} \ , \ \bar{k} \in \mathbb{Z} \ . \tag{14}$$

Then, we have

$$\begin{aligned} R_x^{\hat{m}_1,\hat{m}_2}[l_{n_1}, l_{n_2}] &= R_x^{m_1,m_2}[l_{n_1}, l_{n_2}] \\ &= R_x^{\bar{k}\frac{l_{n_2}}{\eta_n}+\hat{m}_1, \bar{k}\frac{l_{n_1}}{\eta_n}+\hat{m}_2}[l_{n_1}, l_{n_2}] \ . \end{aligned} \tag{15}$$

Thus, we can obtain a new smoothed cross-correlation matrix $\mathbf{R_c}[l_{n_1}, l_{n_2}]$ by combining the equal entries together, with its m_1-th row and m_2-th column given by

$$R_c^{m_1,m_2}[l_{n_1}, l_{n_2}] = \frac{\sum_{\bar{k}=\hat{k}_{\min}}^{\hat{k}_{\max}} R_x^{\bar{k}\frac{l_{n_2}}{\eta_n}+m_1, \bar{k}\frac{l_{n_1}}{\eta_n}+m_2}[l_{n_1}, l_{n_2}]}{\hat{k}_{\text{len}}}, \tag{16}$$

with

$$\begin{aligned} &\hat{k}_{\text{len}} = \hat{k}_{\max} - \hat{k}_{\min} + 1 \ , \\ &\hat{k}_{\min} = \text{ceil}\left\{\max\left\{\frac{-m_1\eta_n}{l_{n_2}}, \frac{-m_2\eta_n}{l_{n_1}}\right\}\right\} \ , \\ &\hat{k}_{\max} = \text{floor}\left\{\min\left\{\frac{\eta_n(M-1-m_1)}{l_{n_2}}, \frac{\eta_n(M-1-m_2)}{l_{n_1}}\right\}\right\} \ , \end{aligned} \tag{17}$$

where $\text{ceil}\{\cdot\}$ returns the smallest integer exceeding the argument, while $\min\{\cdot\}$ returns the minimum value.

Define two sets of indexes (m_1, m_2), $m_1, m_2 \in \mathbb{Z}$, given by

$$\begin{aligned} &\Phi_M = \{(m_1, m_2), 0 \leq m_1, m_2 < M\} \ , \\ &\widetilde{\Phi}_n = \left\{(m_1, m_2), \frac{l_{n_2}}{\eta_n} \leq m_1 < M \bigcap \frac{l_{n_1}}{\eta_n} \leq m_2 < M\right\} \ . \end{aligned}$$

Note that $\widetilde{\Phi}_n = \emptyset$ is an empty set when $\frac{l_{n_2}}{\eta_n} \geq M \bigcup \frac{l_{n_1}}{\eta_n} \geq M$. Then, the set $\Phi_n = \Phi_M - \widetilde{\Phi}_n$ represents the unique entries without redundancy in the smoothed cross-correlation matrix $\mathbf{R_c}[l_{n_1}, l_{n_2}]$. For the n-th frequency pair, the number of unique entries without redundancy in $\mathbf{R_c}[l_{n_1}, l_{n_2}]$ is

$$M_n = \begin{cases} \frac{M(l_{n_1}+l_{n_2})}{\eta_n} - \frac{l_{n_1}l_{n_2}}{\eta_n^2}, & \widetilde{\Phi}_n \neq \emptyset, \\ M^2, & \widetilde{\Phi}_n = \emptyset. \end{cases} \tag{18}$$

C. Group sparsity based low-complexity DOA estimation

According to (11), the auto-correlation matrix $\mathbf{R_x}[\alpha, \alpha]$ is simplified into a vector, rewritten as

$$\mathbf{z_c}[\alpha, \alpha] = \mathbf{A_c}[\alpha, \alpha]\widetilde{\mathbf{s}}[\alpha, \alpha] + \sigma_{\bar{n}}^2[\alpha, \alpha]\mathbf{v}_M, \tag{19}$$

where $\mathbf{A_c}[\alpha, \alpha] = [\mathbf{a}(\alpha, \theta_1), \ldots, \mathbf{a}(\alpha, \theta_K)]$, and $\mathbf{v}_M$ has a size of $M \times 1$, being all zeroes except for a 1 at the zeroth entry.

By vectorizing $\mathbf{R_c}[l_{n_1}, l_{n_2}]$, we obtain

$$\begin{aligned} \mathbf{z_c}[l_{n_1}, l_{n_2}] &= \text{vec}\{\mathbf{R_c}[l_{n_1}, l_{n_2}]\} \\ &= \widetilde{\mathbf{A}}[l_{n_1}, l_{n_2}]\widetilde{\mathbf{s}}[l_{n_1}, l_{n_2}] \ . \end{aligned} \tag{20}$$

where $\widetilde{\mathbf{A}}[l_{n_1}, l_{n_2}]$ is given in (8).

For the n-th frequency pair, with the same search grid of K_g potential incident angles $\theta_{g,0}, \cdots, \theta_{g,K_g-1}$, we construct

$$\widetilde{\mathbf{A}}_{\mathbf{cg}}[\alpha, \alpha] = \left[\mathbf{a}(\alpha, \theta_{g,0}), \ldots, \mathbf{a}(\alpha, \theta_{g,K_g-1})\right] \ . \tag{21}$$

We use $z_{c,\bar{m}}[l_{n_1}, l_{n_2}]$, $0 \leq \bar{m} \leq M^2 - 1$, to denote the $\bar{m}$-th entry in the column vector $\mathbf{z_c}[l_{n_1}, l_{n_2}]$, and row vectors $\widetilde{\mathbf{a}}_{\mathbf{r},\bar{m}}[l_{n_1}, l_{n_2}]$ and $\widetilde{\mathbf{a}}_{\mathbf{g},\bar{m}}[l_{n_1}, l_{n_2}]$ are the $\bar{m}$-th row of the matrices $\widetilde{\mathbf{A}}[l_{n_1}, l_{n_2}]$ and $\widetilde{\mathbf{A}}_{\mathbf{g}}[l_{n_1}, l_{n_2}] = \left[\widetilde{\mathbf{a}}(l_{n_1}, l_{n_2}, \theta_{g,0}), \ldots, \widetilde{\mathbf{a}}(l_{n_1}, l_{n_2}, \theta_{g,K_g-1})\right]$, respectively.

Denote $\bar{m}_{m_0} \in \phi_n$, $0 \leq m_0 \leq M_n - 1$ as the row indexes corresponding to the unique co-array lags without redundancy, where $\phi_n = \{m_1 + m_2 M, (m_1, m_2) \in \Phi_n\}$ with M_n elements. By keeping all the row indexes $\bar{m}_{m_0}$ associated with unique entries, the following matrices can be generated

$$\begin{aligned} \bar{\mathbf{z}}_{\mathbf{c}}[l_{n_1}, l_{n_2}] &= \left[z_{c,\bar{m}_0}[l_{n_1}, l_{n_2}], \ldots, z_{c,\bar{m}_{M_n-1}}[l_{n_1}, l_{n_2}]\right]^T, \\ \widetilde{\mathbf{A}}_{\mathbf{c}}[l_{n_1}, l_{n_2}] &= \left[\widetilde{\mathbf{a}}_{\mathbf{r},\bar{m}_0}^T[l_{n_1}, l_{n_2}], \ldots, \widetilde{\mathbf{a}}_{\mathbf{r},\bar{m}_{M_n-1}}^T[l_{n_1}, l_{n_2}]\right]^T, \\ \widetilde{\mathbf{A}}_{\mathbf{cg}}[l_{n_1}, l_{n_2}] &= \left[\widetilde{\mathbf{a}}_{\mathbf{g},\bar{m}_0}^T[l_{n_1}, l_{n_2}], \ldots, \widetilde{\mathbf{a}}_{\mathbf{g},\bar{m}_{M_n-1}}^T[l_{n_1}, l_{n_2}]\right]^T. \end{aligned}$$

Therefore, the model in (20) can be simplified into

$$\bar{\mathbf{z}}_{\mathbf{c}}[l_{n_1}, l_{n_2}] = \widetilde{\mathbf{A}}_{\mathbf{c}}[l_{n_1}, l_{n_2}]\widetilde{\mathbf{s}}[l_{n_1}, l_{n_2}] \ . \tag{22}$$

After constructing a block diagonal matrix $\widetilde{\mathbf{A}}_{\mathbf{cg}}[n] = \text{blkdiag}\left\{\widetilde{\mathbf{A}}_{\mathbf{cg}}[l_{n_1}, l_{n_1}], \widetilde{\mathbf{A}}_{\mathbf{cg}}[l_{n_2}, l_{n_2}], \widetilde{\mathbf{A}}_{\mathbf{cg}}[l_{n_1}, l_{n_2}]\right\}$, and a $K_g \times 3$ matrix $\widetilde{\mathbf{S}}_{\mathbf{g}}[n]$ with each column vector representing the potential signals over the predefined search grid, we obtain the low-complexity virtual array model under the CS framework, given by

$$\mathbf{z_c}[n] = \widetilde{\mathbf{A}}_{\mathbf{cg}}[n]\widetilde{\mathbf{s}}_{\mathbf{g}}[n] + \mathbf{V}\mathbf{w}[n] = \widetilde{\mathbf{A}}_{\mathbf{cg}}^{\circ}[n]\widetilde{\mathbf{s}}_{\mathbf{g}}^{\circ}[n] \ , \tag{23}$$

where $\mathbf{z_c}[n] = \left[\mathbf{z}_{\mathbf{c}}^T[l_{n_1}, l_{n_1}], \mathbf{z}_{\mathbf{c}}^T[l_{n_2}, l_{n_2}], \bar{\mathbf{z}}_{\mathbf{c}}^T[l_{n_1}, l_{n_2}]\right]^T$. The matrix $\mathbf{V} = [\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{v}}_2]^T$ has a size of $(2M + M_n) \times 2$ with $\widetilde{\mathbf{v}}_1 = \left[\mathbf{v}_M^T, \mathbf{0}_M^T, \mathbf{0}_{M_n}^T\right]^T$ and $\widetilde{\mathbf{v}}_2 = \left[\mathbf{0}_M^T, \mathbf{v}_M^T, \mathbf{0}_{M_n}^T\right]$, where $\mathbf{0}_M^T$ denotes an $M \times 1$ column vector consisting of all zeros. $\widetilde{\mathbf{A}}_{\mathbf{cg}}^{\circ}[n] = \left[\widetilde{\mathbf{A}}_{\mathbf{cg}}[n], \mathbf{V}\right]$, $\widetilde{\mathbf{s}}_{\mathbf{g}}[n] = \text{vec}\left\{\widetilde{\mathbf{S}}_{\mathbf{g}}[n]\right\}$, $\mathbf{w}[n] = \left[\sigma_{\bar{n}}^2[l_{n_1}, l_{n_1}], \sigma_{\bar{n}}^2[l_{n_2}, l_{n_2}]\right]^T$, and $\widetilde{\mathbf{s}}_{\mathbf{g}}^{\circ}[n] = \left[\widetilde{\mathbf{s}}_{\mathbf{g}}^T[n], \mathbf{w}^T[n]\right]^T$.

For DOA estimation across the frequency range of interest with N pairs, a $K_g \times 3N$ matrix $\mathbf{R_g}$, and a $(3K_g + 2)N \times 1$ column vector $\mathbf{r}_{\mathbf{g}}^{\circ}$, are constructed as

$$\begin{aligned} \mathbf{R_g} &= \left[\widetilde{\mathbf{S}}_{\mathbf{g}}[0], \widetilde{\mathbf{S}}_{\mathbf{g}}[1], \ldots, \widetilde{\mathbf{S}}_{\mathbf{g}}[N-1]\right] \ , \\ \mathbf{r}_{\mathbf{g}}^{\circ} &= \left[\widetilde{\mathbf{s}}_{\mathbf{g}}^{\circ T}[0], \widetilde{\mathbf{s}}_{\mathbf{g}}^{\circ T}[1], \ldots, \widetilde{\mathbf{s}}_{\mathbf{g}}^{\circ T}[N-1]\right]^T \ . \end{aligned} \tag{24}$$

TABLE I
NUMBER OF ENTRIES IN VECTORS/MATRICES

Vector / Matrix	Theoretical results		Example	
	Existing method (Single Pair)	LC method (Single Pair)	Existing method (Single Pair)	LC method (Single Pair)
$\widetilde{\mathbf{s}}^{\circ}_{\mathbf{g}}[n]$	$3K_g+2$	$3K_g+2$	5405	5405
$\mathbf{z_c}[n]$	$3M^2$	$2M+M_n$	147	61
$\widetilde{\mathbf{A}}^{\circ}_{\mathbf{cg}}[n]$	$3M^2(3K_g+2)$	$(2M+M_n)(3K_g+2)$	794535	329705
	Existing method (Wideband)	LC method (Wideband)	Existing method (Wideband)	LC method (Wideband)
$\mathbf{r}^{\circ}_{\mathbf{g}}$	$(3K_g+2)N$	$(3K_g+2)N$	27025	27025
$\mathbf{z_{cg}}$	$3M^2\cdot N$	$2MN+\sum_{n=1}^{N} M_n$	735	259
$\widetilde{\mathbf{B}}^{\circ}_{\mathbf{cg}}$	$3(3K_g+2)(MN)^2$	$(2MN+\sum_{n=1}^{N} M_n)(3K_g+2)N$	19863375	6999475
Computation Time			116.5474 s	55.2154 s

Denote $\hat{\mathbf{r}}_{\mathbf{g}} = \left[\left\|\mathbf{r}_{\mathbf{g},0}\right\|_2, \left\|\mathbf{r}_{\mathbf{g},2}\right\|_2, \ldots, \left\|\mathbf{r}_{\mathbf{g},K_g-1}\right\|_2\right]^T$ with the row vector $\mathbf{r}_{\mathbf{g},k_g}$, $0 \le k_g \le K_g-1$ representing the k_g-th row of the matrix $\mathbf{R_g}$. Finally, the group sparsity based low-complexity wideband DOA estimation method for multiple frequency pairs can be expressed as

$$\min_{\mathbf{r}^{\circ}_{\mathbf{g}}} \quad \|\hat{\mathbf{r}}_{\mathbf{g}}\|_1, \quad \text{subject to} \quad \left\|\mathbf{z_{cg}} - \widetilde{\mathbf{B}}^{\circ}_{\mathbf{cg}}\mathbf{r}^{\circ}_{\mathbf{g}}\right\|_2 \le \varepsilon \ , \tag{25}$$

where $\mathbf{z_{cg}} = \left[\mathbf{z}_{\mathbf{c}}^T[0], \mathbf{z}_{\mathbf{c}}^T[1], \ldots, \mathbf{z}_{\mathbf{c}}^T[N-1]\right]^T$. $\|\cdot\|_1$ is the ℓ_1 norm, $\|\cdot\|_2$ is the ℓ_2 norm, and the $(2MN + \sum_{n=0}^{N-1} M_n) \times (3K_g+2)N$ block diagonal matrix $\widetilde{\mathbf{B}}^{\circ}_{\mathbf{cg}}$ is generated by

$$\widetilde{\mathbf{B}}^{\circ}_{\mathbf{cg}} = \text{blkdiag}\left[\widetilde{\mathbf{A}}^{\circ}_{\mathbf{cg}}[0], \widetilde{\mathbf{A}}^{\circ}_{\mathbf{cg}}[1], \ldots, \widetilde{\mathbf{A}}^{\circ}_{\mathbf{cg}}[N-1]\right] \ . \tag{26}$$

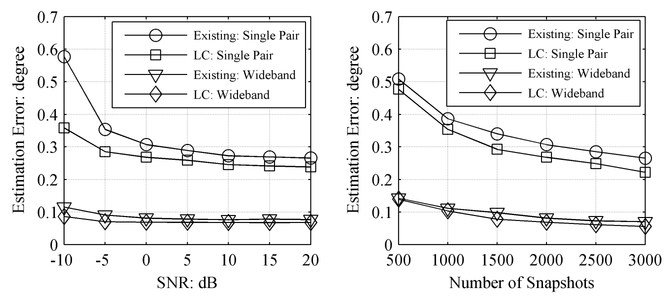
IV. SIMULATION RESULTS

Consider an example of ULA with $M = 7$ sensors, and a DFT of $L = 64$ points is applied with the inter-element spacing and the frequency interval $f_\triangle$ chosen according to (5). The bandwidth of the employed LFMCW signals is $B = 10f_\triangle$, covering $Q = 10$ frequency bins with the index set $\Phi_l = [1, 2, \ldots, 10]$.

The frequency pair consisting of the 5-th and the 6-th frequency bins is utilized as the single pair case for comparison, while for the wideband case, these $Q = 10$ frequency bins are divided into $N = 5$ pairs with 5 and 6, 1 and 10, 2 and 7, 3 and 9, as well as 4 and 8. Under the CS framework, a search grid of $K_g = 1801$ incident angles is formed within the angle range from $-90°$ to $90°$, where the step size is $0.1°$. The allowable error bound ε is chosen to give the best result through trial-and-error in every experiment, and the CVX package [25], [26] is used to solve these optimization problems.

Table I shows the number of entries in the vectors/matrices involved in different DOA estimation methods. Clearly, the number of entries of the proposed low-complexity (LC) solution is less than that of the existing method, leading to reduced complexity due to less multiplicative and additive operations in solving these formulations. The computation time under the environment of Intel CPU I5-3470 and 16GB RAM using the CVX package is also listed in Table I. Obviously, a shorter computation time has been achieved by our group sparsity based LC method for both the single frequency pair case and the wideband case.

Then, we set $K = 16$ targets with incident angles uniformly distributed between $-60°$ and $60°$. The root mean square error



(a) RMSE results vs input SNR. (b) RMSE results vs snapshot number

Fig. 1. RMSE results obtained by different estimation methods.

(RMSE) results based on 500 Monte-Carlo simulation runs with respect to the input SNR are shown in Fig. 1(a), where the number of snapshots in the frequency domain is 2000, and Fig. 1(b) gives the RMSE results versus the number of snapshots with the input SNR fixed at 0 dB. Owing to a more precise covariance matrix estimation and less optimization effort requirements in the LC solution, it is clear that a better performance has been achieved by the proposed low-complexity methods in both the single frequency pair case and the wideband case compared with the existing methods.

V. CONCLUSION

An underdetermined low-complexity DOA estimation method based on the group sparsity concept has been proposed employing a widebnad ULA. The received signals were first decomposed into different frequencies by DFT, and these frequency bins were divided into several pairs for co-array generation in the spatio-spectral domain. A generalized complexity reduction method was then proposed to merge the redundant entries in both the auto-correlation matrices at each frequency and the cross-correlation matrices across frequencies, leading to a series of simplified virtual array models with reduced dimension, followed by the group sparsity based method for wideband DOA estimation with reduced complexity achieved. As shown by simulations, the proposed low-complexity estimation method outperformed the existing method in terms of both estimation accuracy and computation time.

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