The minimal path sum in the 5 by 5 matrix below, by starting in any cell in the left column and finishing in any cell in the right column, and only moving up, down, and right, is indicated in red and bold; the sum is equal to 994.

$$\begin{pmatrix} 131 & 673 & \mathbf{234} & \mathbf{103} & \mathbf{18} \\ \mathbf{201} & \mathbf{96} & \mathbf{342} & 965 & 150 \\ 630 & 803 & 746 & 422 & 111 \\ 537 & 699 & 497 & 121 & 956 \\ 805 & 732 & 524 & 37 & 331 \end{pmatrix}$$

Find the minimal path sum, in matrix.txt (right click and 'Save Link/Target As...'), a 31K text file containing a 80 by 80 matrix, from the left column to the right column.

Problem 83

NOTE: This problem is a significantly more challenging version of Problem 81.

In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by moving left, right, up, and down, is indicated in bold red and is equal to 2297.

$$\begin{pmatrix} \mathbf{131} & 673 & \mathbf{234} & \mathbf{103} & \mathbf{18} \\ \mathbf{201} & \mathbf{96} & \mathbf{342} & 965 & \mathbf{150} \\ 630 & 803 & 746 & \mathbf{422} & \mathbf{111} \\ 537 & 699 & 497 & \mathbf{121} & 956 \\ 805 & 732 & 524 & \mathbf{37} & \mathbf{331} \end{pmatrix}$$

Find the minimal path sum, in matrix.txt (right click and 'Save Link/Target As...'), a 31K text file containing a 80 by 80 matrix, from the top left to the bottom right by moving left, right, up, and down.

Problem 84

In the game, *Monopoly*, the standard board is set up in the following way:

GO	A1	CC1	A2	T1	R1	B1	CH1	B2	B3	JAIL
H2										C1
T2										U1
H1										C2
CH3										C3
R4										R2
G3										D1
CC3										CC2
G2										D2
G1										D3

G2J F3 U2 F2 F1 R3 E3 E2 CH2 E1 FP

A player starts on the GO square and adds the scores on two 6-sided dice to determine the number of squares they advance in a clockwise direction. Without any further rules we would expect to visit each square with equal probability: 2.5%. However, landing on G2J (Go To Jail), CC (community chest), and CH (chance) changes this distribution.

In addition to G2J, and one card from each of CC and CH, that orders the player to go directly to jail, if a player rolls three consecutive doubles, they do not advance the result of their 3rd roll. Instead they proceed directly to jail.

At the beginning of the game, the CC and CH cards are shuffled. When a player lands on CC or CH they take a card from the top of the respective pile and, after following the instructions, it is returned to the bottom of the pile. There are sixteen cards in each pile, but for the purpose of this problem we are only concerned with cards that order a movement; any instruction not concerned with movement will be ignored and the player will remain on the CC/CH square.

- Community Chest (2/16 cards):
  1. Advance to GO
  2. Go to JAIL
- Chance (10/16 cards):
  1. Advance to GO
  2. Go to JAIL
  3. Go to C1
  4. Go to E3
  5. Go to H2
  6. Go to R1
  7. Go to next R (railway company)
  8. Go to next R
  9. Go to next U (utility company)
  10. Go back 3 squares.

The heart of this problem concerns the likelihood of visiting a particular square. That is, the probability of finishing at that square after a roll. For this reason it should be clear that, with the exception of G2J for which the probability of finishing on it is zero, the CH squares will have the lowest probabilities, as 5/8 request a movement to another square, and it is the final square that the player finishes at on each roll that we are interested in. We shall make no distinction between "Just Visiting" and being sent to JAIL, and we shall also ignore the rule about requiring a double to "get out of jail", assuming that they pay to get out on their next turn.

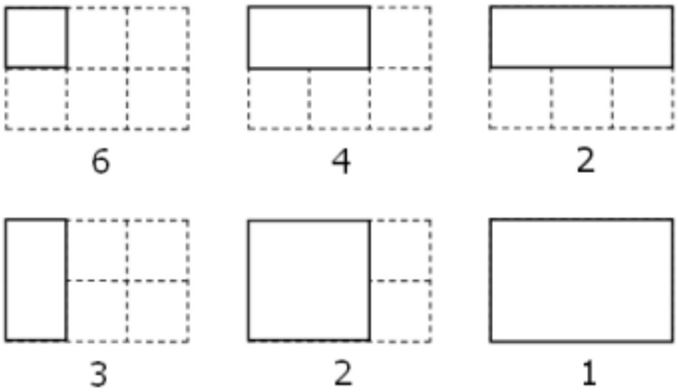
By starting at GO and numbering the squares sequentially from 00 to 39 we can concatenate these two-digit numbers to produce strings that correspond with sets of squares.

Statistically it can be shown that the three most popular squares, in order, are JAIL (6.24%) = Square 10, E3 (3.18%) = Square 24, and GO (3.09%) = Square 00. So these three most popular squares can be listed with the six-digit modal string: 102400.

If, instead of using two 6-sided dice, two 4-sided dice are used, find the six-digit modal string.

Problem 85

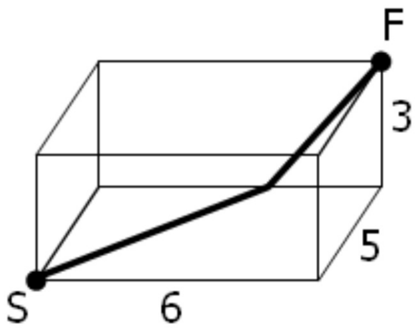
By counting carefully it can be seen that a rectangular grid measuring 3 by 2 contains eighteen rectangles:



Although there exists no rectangular grid that contains exactly two million rectangles, find the area of the grid with the nearest solution.

Problem 86

A spider, S, sits in one corner of a cuboid room, measuring 6 by 5 by 3, and a fly, F, sits in the opposite corner. By travelling on the surfaces of the room the shortest "straight line" distance from S to F is 10 and the path is shown on the diagram.



However, there are up to three "shortest" path candidates for any given cuboid and the shortest route doesn't always have integer length.

By considering all cuboid rooms with integer dimensions, up to a maximum size of M by M by M, there are exactly 2060 cuboids for which the shortest route has integer length when M=100, and this is the least value of M for which the number of solutions first exceeds two thousand; the number of solutions is 1975 when M=99.

Find the least value of M such that the number of solutions first exceeds one million.

Problem 87

The smallest number expressible as the sum of a prime square, prime cube, and prime fourth power is 28. In fact, there are exactly four numbers below fifty that can be expressed in such a way:

$28 = 2^2 + 2^3 + 2^4$
$33 = 3^2 + 2^3 + 2^4$
$49 = 5^2 + 2^3 + 2^4$
$47 = 2^2 + 3^3 + 2^4$

How many numbers below fifty million can be expressed as the sum of a prime square, prime cube, and prime fourth power?

Problem 88

A natural number, N, that can be written as the sum and product of a given set of at least two natural numbers, $\{a_1, a_2, \ldots, a_k\}$ is called a product-sum number: N = $a_1 + a_2 + \ldots + a_k = a_1 \times a_2 \times \ldots \times a_k$.

For example, 6 = 1 + 2 + 3 = 1 × 2 × 3.

For a given set of size, *k*, we shall call the smallest N with this property a minimal product-sum number. The minimal product-sum numbers for sets of size, *k* = 2, 3, 4, 5, and 6 are as follows.

k=2: 4 = 2 × 2 = 2 + 2
k=3: 6 = 1 × 2 × 3 = 1 + 2 + 3
k=4: 8 = 1 × 1 × 2 × 4 = 1 + 1 + 2 + 4
k=5: 8 = 1 × 1 × 2 × 2 × 2 = 1 + 1 + 2 + 2 + 2
k=6: 12 = 1 × 1 × 1 × 1 × 2 × 6 = 1 + 1 + 1 + 1 + 2 + 6

Hence for $2 \le k \le 6$, the sum of all the minimal product-sum numbers is 4+6+8+12 = 30; note that 8 is only counted once in the sum.

In fact, as the complete set of minimal product-sum numbers for $2 \le k \le 12$ is {4, 6, 8, 12, 15, 16}, the sum is 61.

What is the sum of all the minimal product-sum numbers for $2 \le k \le 12000$?

Problem 89

The rules for writing Roman numerals allow for many ways of writing each number (see About Roman Numerals...). However, there is always a "best" way of writing a particular number.

For example, the following represent all of the legitimate ways of writing the number sixteen:

```
IIIIIIIIIIIIIIII
VIIIIIIIIII
VVIIIIII
XIIIIII
VVVI
XVI
```

The last example being considered the most efficient, as it uses the least number of numerals.

The 11K text file, roman.txt (right click and 'Save Link/Target As...'), contains one thousand numbers written in valid, but not necessarily minimal, Roman numerals; that is, they are arranged in descending units and obey the subtractive pair rule (see About Roman Numerals... for the definitive rules for this problem).

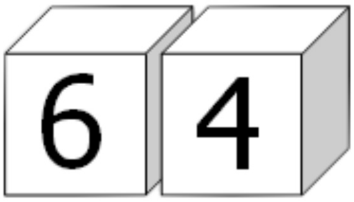
Find the number of characters saved by writing each of these in their minimal form.

Note: You can assume that all the Roman numerals in the file contain no more than four consecutive identical units.

Problem 90

Each of the six faces on a cube has a different digit (0 to 9) written on it; the same is done to a second cube. By placing the two cubes side-by-side in different positions we can form a variety of 2-digit numbers.

For example, the square number 64 could be formed:



In fact, by carefully choosing the digits on both cubes it is possible to display all of the square numbers below one-hundred: 01, 04, 09, 16, 25, 36, 49, 64, and 81.

For example, one way this can be achieved is by placing {0, 5, 6, 7, 8, 9} on one cube and {1, 2, 3, 4, 8, 9} on the other cube.

However, for this problem we shall allow the 6 or 9 to be turned upside-down so that an arrangement like {0, 5, 6, 7, 8, 9} and {1, 2, 3, 4, 6, 7} allows for all nine square numbers to be displayed; otherwise it would be impossible to obtain 09.

In determining a distinct arrangement we are interested in the digits on each cube, not the order.

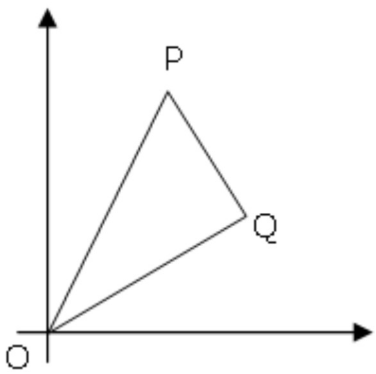
> {1, 2, 3, 4, 5, 6} is equivalent to {3, 6, 4, 1, 2, 5}
> {1, 2, 3, 4, 5, 6} is distinct from {1, 2, 3, 4, 5, 9}

But because we are allowing 6 and 9 to be reversed, the two distinct sets in the last example both represent the extended set {1, 2, 3, 4, 5, 6, 9} for the purpose of forming 2-digit numbers.

How many distinct arrangements of the two cubes allow for all of the square numbers to be displayed?

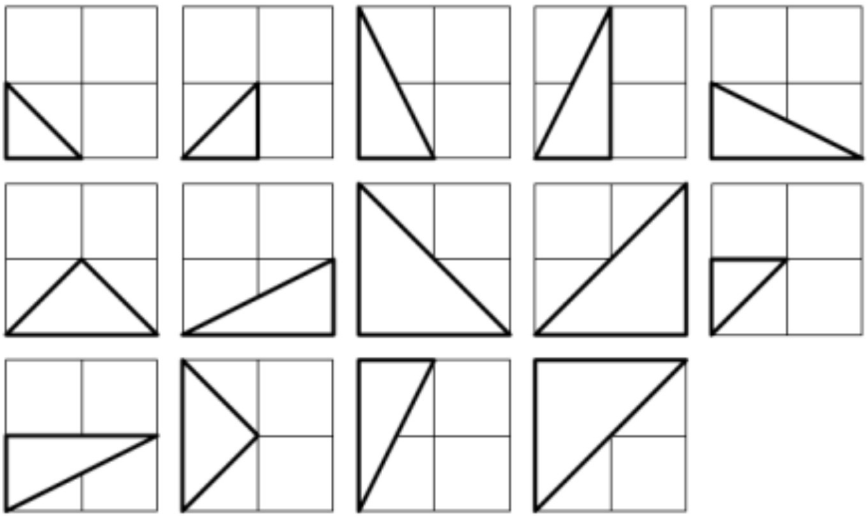
Problem 91

The points P (x_1, y_1) and Q (x_2, y_2) are plotted at integer co-ordinates and are joined to the origin, O(0,0), to form ΔOPQ.



There are exactly fourteen triangles containing a right angle that can be formed when each co-ordinate lies between 0 and 2 inclusive; that is,

$0 \le x_1, y_1, x_2, y_2 \le 2$.



Given that $0 \le x_1, y_1, x_2, y_2 \le 50$, how many right triangles can be formed?

Problem 92

A number chain is created by continuously adding the square of the digits in a number to form a new number until it has been seen before.

For example,

44 → 32 → 13 → 10 → **1** → **1**
85 → **89** → 145 → 42 → 20 → 4 → 16 → 37 → 58 → **89**

Therefore any chain that arrives at 1 or 89 will become stuck in an endless loop. What is most amazing is that EVERY starting number will eventually arrive at 1 or 89.

How many starting numbers below ten million will arrive at 89?

Problem 93

By using each of the digits from the set, {1, 2, 3, 4}, exactly once, and making use of the four arithmetic operations (+, −, *, /) and brackets/parentheses, it is possible to form different positive integer targets.

For example,

```
8 = (4 * (1 + 3)) / 2
14 = 4 * (3 + 1 / 2)
19 = 4 * (2 + 3) − 1
36 = 3 * 4 * (2 + 1)
```

Note that concatenations of the digits, like 12 + 34, are not allowed.

Using the set, {1, 2, 3, 4}, it is possible to obtain thirty-one different target numbers of which 36 is the maximum, and each of the numbers 1 to 28 can be obtained before encountering the first non-expressible number.

Find the set of four distinct digits, $a < b < c < d$, for which the longest set of consecutive positive integers, 1 to n, can be obtained, giving your answer as a string: *abcd*.

Problem 94

It is easily proved that no equilateral triangle exists with integral length sides and integral area. However, the *almost equilateral triangle* 5-5-6 has an area of 12 square units.

We shall define an *almost equilateral triangle* to be a triangle for which two sides are equal and the third differs by no more than one unit.

Find the sum of the perimeters of all *almost equilateral triangles* with integral side lengths and area and whose perimeters do not exceed one billion (1,000,000,000).

Problem 95

The proper divisors of a number are all the divisors excluding the number itself. For example, the proper divisors of 28 are 1, 2, 4, 7, and 14. As the sum of these divisors is equal to 28, we call it a perfect number.

Interestingly the sum of the proper divisors of 220 is 284 and the sum of the proper divisors of 284 is 220, forming a chain of two numbers. For this reason, 220 and 284 are called an amicable pair.

Perhaps less well known are longer chains. For example, starting with 12496, we form a chain of five numbers:

$$12496 \to 14288 \to 15472 \to 14536 \to 14264 \ (\to 12496 \to \ldots)$$

Since this chain returns to its starting point, it is called an amicable chain.

Find the smallest member of the longest amicable chain with no element exceeding one million.

Problem 96

Su Doku (Japanese meaning *number place*) is the name given to a popular puzzle concept. Its origin is unclear, but credit must be attributed to Leonhard Euler who invented a similar, and much more difficult, puzzle idea called Latin Squares. The objective of Su Doku puzzles, however, is to replace the blanks (or zeros) in a 9 by 9 grid in such that each row, column, and 3 by 3 box contains each of the digits 1 to 9. Below is an example of a typical starting puzzle grid and its solution grid.

0 0 3	0 2 0	6 0 0
9 0 0	3 0 5	0 0 1
0 0 1	8 0 6	4 0 0
0 0 8	1 0 2	9 0 0
7 0 0	0 0 0	0 0 8
0 0 6	7 0 8	2 0 0

4 8 3	9 2 1	6 5 7
9 6 7	3 4 5	8 2 1
2 5 1	8 7 6	4 9 3
5 4 8	1 3 2	9 7 6
7 2 9	5 6 4	1 3 8
1 3 6	7 9 8	2 4 5

0	0	2	6	0	9	5	0	0
8	0	0	2	0	3	0	0	9
0	0	5	0	1	0	3	0	0

3	7	2	6	8	9	5	1	4
8	1	4	2	5	3	7	6	9
6	9	5	4	1	7	3	8	2

A well constructed Su Doku puzzle has a unique solution and can be solved by logic, although it may be necessary to employ "guess and test" methods in order to eliminate options (there is much contested opinion over this). The complexity of the search determines the difficulty of the puzzle; the example above is considered *easy* because it can be solved by straight forward direct deduction.

The 6K text file, sudoku.txt (right click and 'Save Link/Target As...'), contains fifty different Su Doku puzzles ranging in difficulty, but all with unique solutions (the first puzzle in the file is the example above).

By solving all fifty puzzles find the sum of the 3-digit numbers found in the top left corner of each solution grid; for example, 483 is the 3-digit number found in the top left corner of the solution grid above.

Problem 97

The first known prime found to exceed one million digits was discovered in 1999, and is a Mersenne prime of the form $2^{6972593}-1$; it contains exactly 2,098,960 digits. Subsequently other Mersenne primes, of the form 2^p-1, have been found which contain more digits.

However, in 2004 there was found a massive non-Mersenne prime which contains 2,357,207 digits: $28433\times2^{7830457}+1$.

Find the last ten digits of this prime number.

Problem 98

By replacing each of the letters in the word CARE with 1, 2, 9, and 6 respectively, we form a square number: $1296 = 36^2$. What is remarkable is that, by using the same digital substitutions, the anagram, RACE, also forms a square number: $9216 = 96^2$. We shall call CARE (and RACE) a square anagram word pair and specify further that leading zeroes are not permitted, neither may a different letter have the same digital value as another letter.

Using words.txt (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, find all the square anagram word pairs (a palindromic word is NOT considered to be an anagram of itself).

What is the largest square number formed by any member of such a pair?

NOTE: All anagrams formed must be contained in the given text file.

Problem 99

Comparing two numbers written in index form like 2^{11} and 3^7 is not difficult, as any calculator would confirm that $2^{11} = 2048 < 3^7 = 2187$.

However, confirming that $632382^{518061} > 519432^{525806}$ would be much more difficult, as both numbers contain over three million digits.

Using base_exp.txt (right click and 'Save Link/Target As...'), a 22K text file containing one thousand lines with a base/exponent pair on each line, determine which line number has the greatest numerical value.

NOTE: The first two lines in the file represent the numbers in the example given above.

Problem 100

If a box contains twenty-one coloured discs, composed of fifteen blue discs and six red discs, and two discs were taken at random, it can be seen that the probability of taking two blue discs, P(BB) = (15/21) $\times$(14/20) = 1/2.

The next such arrangement, for which there is exactly 50% chance of taking two blue discs at random, is a box containing eighty-five blue discs and thirty-five red discs.

By finding the first arrangement to contain over 10^{12} = 1,000,000,000,000 discs in total, determine the number of blue discs that the box would contain.

Problem 101

If we are presented with the first k terms of a sequence it is impossible to say with certainty the value of the next term, as there are infinitely many polynomial functions that can model the sequence.

As an example, let us consider the sequence of cube numbers. This is defined by the generating function,
$u_n = n^3$: 1, 8, 27, 64, 125, 216, ...

Suppose we were only given the first two terms of this sequence. Working on the principle that "simple is best" we should assume a linear relationship and predict the next term to be 15 (common difference 7). Even if we were presented with the first three terms, by the same principle of simplicity, a quadratic relationship should be assumed.

We shall define OP(k, n) to be the n^{th} term of the optimum polynomial generating function for the first k terms of a sequence. It should be clear that OP(k, n) will accurately generate the terms of the sequence for $n \le k$, and potentially the *first incorrect term* (FIT) will be OP(k, k+1); in which case we shall call it a *bad OP* (BOP).

As a basis, if we were only given the first term of sequence, it would be most sensible to assume constancy; that is, for $n \ge 2$, OP(1, n) = u_1.

Hence we obtain the following OPs for the cubic sequence:

OP(1, n) = 1	1, **1**, 1, 1, ...
OP(2, n) = 7n−6	1, 8, **15**, ...

$OP(3, n) = 6n^2-11n+6$ 1, 8, 27, **58**, ...

$OP(4, n) = n^3$ 1, 8, 27, 64, 125, ...

Clearly no BOPs exist for $k \geq 4$.

By considering the sum of FITs generated by the BOPs (indicated in **red** above), we obtain 1 + 15 + 58 = 74.

Consider the following tenth degree polynomial generating function:

$$u_n = 1 - n + n^2 - n^3 + n^4 - n^5 + n^6 - n^7 + n^8 - n^9 + n^{10}$$

Find the sum of FITs for the BOPs.

Problem 102

Three distinct points are plotted at random on a Cartesian plane, for which $-1000 \leq x, y \leq 1000$, such that a triangle is formed.

Consider the following two triangles:

A(-340,495), B(-153,-910), C(835,-947)

X(-175,41), Y(-421,-714), Z(574,-645)

It can be verified that triangle ABC contains the origin, whereas triangle XYZ does not.

Using triangles.txt (right click and 'Save Link/Target As...'), a 27K text file containing the co-ordinates of one thousand "random" triangles, find the number of triangles for which the interior contains the origin.

NOTE: The first two examples in the file represent the triangles in the example given above.

Problem 103

Let S(A) represent the sum of elements in set A of size *n*. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. S(B) ≠ S(C); that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then S(B) > S(C).

If S(A) is minimised for a given *n*, we shall call it an optimum special sum set. The first five optimum special sum sets are given below.

n = 1: {1}
n = 2: {1, 2}
n = 3: {2, 3, 4}
n = 4: {3, 5, 6, 7}
n = 5: {6, 9, 11, 12, 13}

It *seems* that for a given optimum set, A = $\{a_1, a_2, \ldots, a_n\}$, the next optimum set is of the form B = $\{b, a_1+b, a_2+b, \ldots, a_n+b\}$, where *b* is the "middle" element on the previous row.

By applying this "rule" we would expect the optimum set for n = 6 to be A = {11, 17, 20, 22, 23, 24}, with S(A) = 117. However, this is not the optimum set, as we have merely applied an algorithm to provide a near optimum set. The optimum set for n = 6 is A = {11, 18, 19, 20, 22, 25}, with S(A) = 115 and corresponding set string: 111819202225.

Given that A is an optimum special sum set for n = 7, find its set string.

NOTE: This problem is related to problems 105 and 106.

Problem 104

The Fibonacci sequence is defined by the recurrence relation:

$F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1$ and $F_2 = 1$.

It turns out that F_{541}, which contains 113 digits, is the first Fibonacci number for which the last nine digits are 1-9 pandigital (contain all the digits 1 to 9, but not necessarily in order). And F_{2749}, which contains 575 digits, is the first Fibonacci number for which the first nine digits are 1-9 pandigital.

Given that F_k is the first Fibonacci number for which the first nine digits AND the last nine digits are 1-9 pandigital, find k.

Problem 105

Let S(A) represent the sum of elements in set A of size n. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. S(B) ≠ S(C); that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then S(B) > S(C).

For example, {81, 88, 75, 42, 87, 84, 86, 65} is not a special sum set because 65 + 87 + 88 = 75 + 81 + 84, whereas {157, 150, 164, 119, 79, 159, 161, 139, 158} satisfies both rules for all possible subset pair combinations and S(A) = 1286.

Using sets.txt (right click and "Save Link/Target As..."), a 4K text file with one-hundred sets containing seven to twelve elements (the two examples given above are the first two sets in the file), identify all the special sum sets, A_1, A_2, ..., A_k, and find the value of $S(A_1) + S(A_2) + ... + S(A_k)$.

NOTE: This problem is related to problems 103 and 106.

Problem 106

Let S(A) represent the sum of elements in set A of size n. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. S(B) ≠ S(C); that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then S(B) > S(C).

For this problem we shall assume that a given set contains n strictly increasing elements and it already satisfies the second rule.

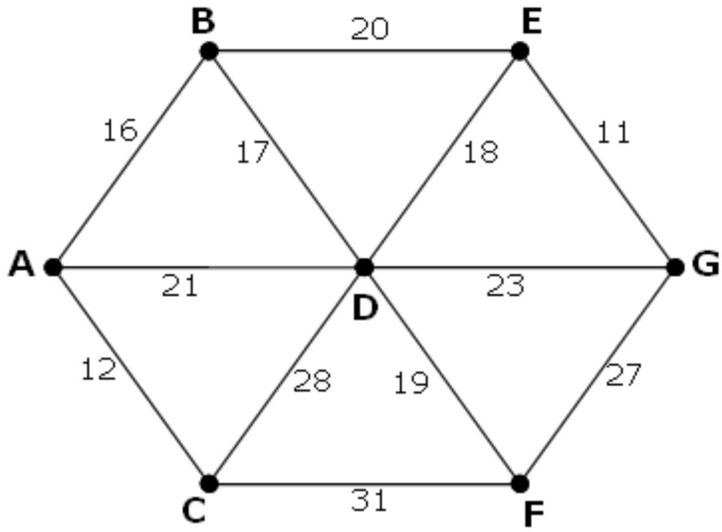
Surprisingly, out of the 25 possible subset pairs that can be obtained from a set for which n = 4, only 1 of these pairs need to be tested for equality (first rule). Similarly, when n = 7, only 70 out of the 966 subset pairs need to be tested.

For n = 12, how many of the 261625 subset pairs that can be obtained need to be tested for equality?

NOTE: This problem is related to problems 103 and 105.

Problem 107

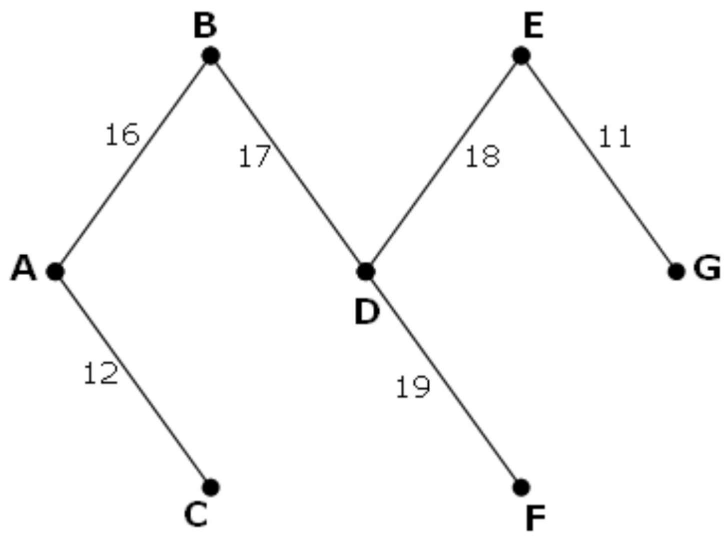
The following undirected network consists of seven vertices and twelve edges with a total weight of 243.



The same network can be represented by the matrix below.

	A	B	C	D	E	F	G
A	-	16	12	21	-	-	-
B	16	-	-	17	20	-	-
C	12	-	-	28	-	31	-
D	21	17	28	-	18	19	23
E	-	20	-	18	-	-	11
F	-	-	31	19	-	-	27
G	-	-	-	23	11	27	-

However, it is possible to optimise the network by removing some edges and still ensure that all points on the network remain connected. The network which achieves the maximum saving is shown below. It has a weight of 93, representing a saving of 243 − 93 = 150 from the original network.



Using network.txt (right click and 'Save Link/Target As...'), a 6K text file containing a network with forty vertices, and given in matrix form, find the maximum saving which can be achieved by removing redundant edges whilst ensuring that the network remains connected.

Problem 108

In the following equation x, y, and n are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

For $n = 4$ there are exactly three distinct solutions:

$$\frac{1}{5} + \frac{1}{20} = \frac{1}{4}$$
$$\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$
$$\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

What is the least value of n for which the number of distinct solutions exceeds one-thousand?

NOTE: This problem is an easier version of problem 110; it is strongly advised that you solve this one first.

Problem 109

In the game of darts a player throws three darts at a target board which is split into twenty equal sized sections numbered one to twenty.

The score of a dart is determined by the number of the region that the dart lands in. A dart landing outside the red/green outer ring scores zero. The black and cream regions inside this ring represent single scores. However, the red/green outer ring and middle ring score double and treble scores respectively.

At the centre of the board are two concentric circles called the bull region, or bulls-eye. The outer bull is worth 25 points and the inner bull is a double, worth 50 points.

There are many variations of rules but in the most popular game the players will begin with a score 301 or 501 and the first player to reduce their running total to zero is a winner. However, it is normal to play a "doubles out" system, which means that the player must land a double (including the double bulls-eye at the centre of the board) on their final dart to win; any other dart that would reduce their running total to one or lower means the score for that set of three darts is "bust".

When a player is able to finish on their current score it is called a "checkout" and the highest checkout is 170: T20 T20 D25 (two treble 20s and double bull).

There are exactly eleven distinct ways to checkout on a score of 6:

D3		
D1	D2	
S2	D2	
D2	D1	
S4	D1	
S1	S1	D2
S1	T1	D1
S1	S3	D1
D1	D1	D1
D1	S2	D1
S2	S2	D1

Note that D1 D2 is considered **different** to D2 D1 as they finish on different doubles. However, the combination S1 T1 D1 is considered the **same** as T1 S1 D1.

In addition we shall not include misses in considering combinations; for example, D3 is the **same** as 0 D3 and 0 0 D3.

Incredibly there are 42336 distinct ways of checking out in total.

How many distinct ways can a player checkout with a score less than 100?

Problem 110

In the following equation x, y, and n are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

It can be verified that when n = 1260 there are 113 distinct solutions and this is the least value of n for which the total number of distinct solutions exceeds one hundred.

What is the least value of n for which the number of distinct solutions exceeds four million?

NOTE: This problem is a much more difficult version of problem 108 and as it is well beyond the limitations of a brute force approach it requires a clever implementation.

Problem 111

Considering 4-digit primes containing repeated digits it is clear that they cannot all be the same: 1111 is divisible by 11, 2222 is divisible by 22, and so on. But there are nine 4-digit primes containing three ones:

1117, 1151, 1171, 1181, 1511, 1811, 2111, 4111, 8111

We shall say that M(*n*, *d*) represents the maximum number of repeated digits for an *n*-digit prime where *d* is the repeated digit, N(*n*, *d*) represents the number of such primes, and S(*n*, *d*) represents the sum of these primes.

So M(4, 1) = 3 is the maximum number of repeated digits for a 4-digit prime where one is the repeated digit, there are N(4, 1) = 9 such primes, and the sum of these primes is S(4, 1) = 22275. It turns out that for *d* = 0, it is only possible to have M(4, 0) = 2 repeated digits, but there are N(4, 0) = 13 such cases.

In the same way we obtain the following results for 4-digit primes.

Digit, *d*	M(4, *d*)	N(4, *d*)	S(4, *d*)
0	2	13	67061
1	3	9	22275
2	3	1	2221
3	3	12	46214

4	3	2	8888
5	3	1	5557
6	3	1	6661
7	3	9	57863
8	3	1	8887
9	3	7	48073

For d = 0 to 9, the sum of all S(4, d) is 273700.

Find the sum of all S(10, d).

Problem 112

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

Clearly there cannot be any bouncy numbers below one-hundred, but just over half of the numbers below one-thousand (525) are bouncy. In fact, the least number for which the proportion of bouncy numbers first reaches 50% is 538.

Surprisingly, bouncy numbers become more and more common and by the time we reach 21780 the proportion of bouncy numbers is equal to 90%.

Find the least number for which the proportion of bouncy numbers is exactly 99%.

Problem 113

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

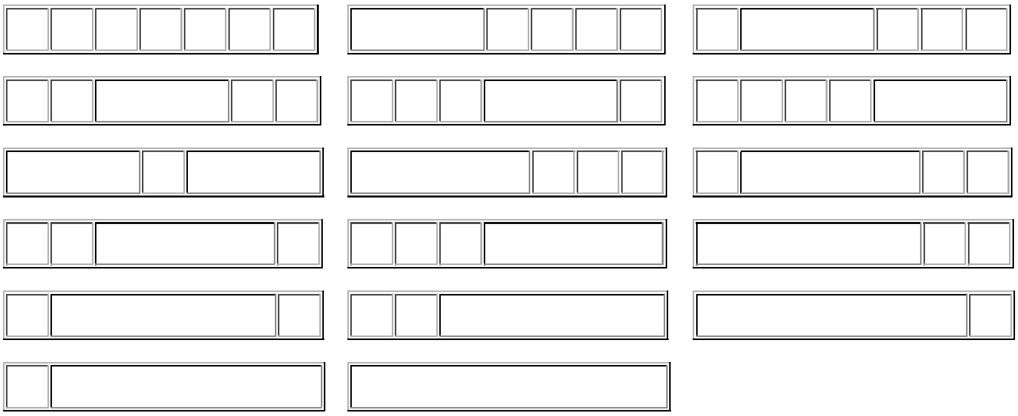
We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

As n increases, the proportion of bouncy numbers below n increases such that there are only 12951 numbers below one-million that are not bouncy and only 277032 non-bouncy numbers below 10^{10}.

How many numbers below a googol (10^{100}) are not bouncy?

Problem 114

A row measuring seven units in length has red blocks with a minimum length of three units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square. There are exactly seventeen ways of doing this.



How many ways can a row measuring fifty units in length be filled?

NOTE: Although the example above does not lend itself to the possibility, in general it is permitted to mix block sizes. For example, on a row measuring eight units in length you could use red (3), black (1), and red (4).

Problem 115

NOTE: This is a more difficult version of problem 114.

A row measuring n units in length has red blocks with a minimum length of m units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square.

Let the fill-count function, F(m, n), represent the number of ways that a row can be filled.

For example, F(3, 29) = 673135 and F(3, 30) = 1089155.

That is, for m = 3, it can be seen that n = 30 is the smallest value for which the fill-count function first exceeds one million.

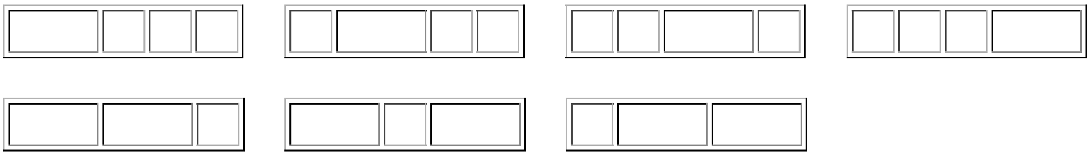
In the same way, for m = 10, it can be verified that F(10, 56) = 880711 and F(10, 57) = 1148904, so n = 57 is the least value for which the fill-count function first exceeds one million.

For m = 50, find the least value of n for which the fill-count function first exceeds one million.

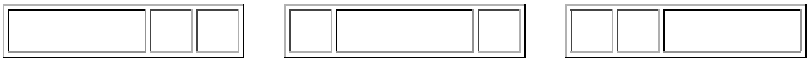
Problem 116

A row of five black square tiles is to have a number of its tiles replaced with coloured oblong tiles chosen from red (length two), green (length three), or blue (length four).

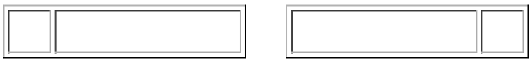
If red tiles are chosen there are exactly seven ways this can be done.



If green tiles are chosen there are three ways.



And if blue tiles are chosen there are two ways.



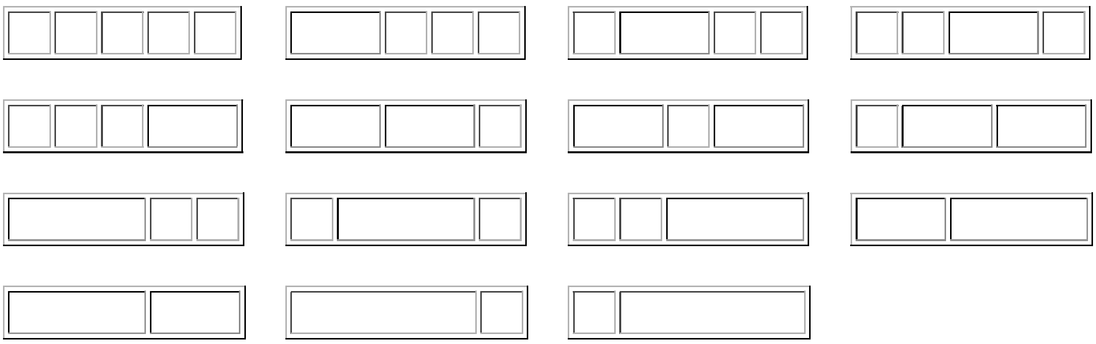
Assuming that colours cannot be mixed there are 7 + 3 + 2 = 12 ways of replacing the black tiles in a row measuring five units in length.

How many different ways can the black tiles in a row measuring fifty units in length be replaced if colours cannot be mixed and at least one coloured tile must be used?

NOTE: This is related to problem 117.

Problem 117

Using a combination of black square tiles and oblong tiles chosen from: red tiles measuring two units, green tiles measuring three units, and blue tiles measuring four units, it is possible to tile a row measuring five units in length in exactly fifteen different ways.



How many ways can a row measuring fifty units in length be tiled?

NOTE: This is related to problem 116.

Problem 118

Using all of the digits 1 through 9 and concatenating them freely to form decimal integers, different sets can be formed. Interestingly with the set {2,5,47,89,631}, all of the elements belonging to it are prime.

How many distinct sets containing each of the digits one through nine exactly once contain only prime elements?

Problem 119

The number 512 is interesting because it is equal to the sum of its digits raised to some power: 5 + 1 + 2 = 8, and 8^3 = 512. Another example of a number with this property is 614656 = 28^4.

We shall define a_n to be the nth term of this sequence and insist that a number must contain at least two digits to have a sum.

You are given that a_2 = 512 and a_{10} = 614656.

Find a_{30}.

Problem 120

Let r be the remainder when $(a-1)^n + (a+1)^n$ is divided by a^2.

For example, if a = 7 and n = 3, then r = 42: $6^3 + 8^3 = 728 \equiv 42 \mod 49$. And as n varies, so too will r, but for a = 7 it turns out that r_{max} = 42.

For $3 \leq a \leq 1000$, find $\sum r_{max}$.

Problem 121

A bag contains one red disc and one blue disc. In a game of chance a player takes a disc at random and its colour is noted. After each turn the disc is returned to the bag, an extra red disc is added, and another disc is taken at random.

The player pays £1 to play and wins if they have taken more blue discs than red discs at the end of the game.

If the game is played for four turns, the probability of a player winning is exactly 11/120, and so the maximum prize fund the banker should allocate for winning in this game would be £10 before they would expect to incur a loss. Note that any payout will be a whole number of pounds and also includes the original £1 paid to play the game, so in the example given the player actually wins £9.

Find the maximum prize fund that should be allocated to a single game in which fifteen turns are played.

Problem 122

The most naive way of computing n^{15} requires fourteen multiplications:

$$n \times n \times \ldots \times n = n^{15}$$

But using a "binary" method you can compute it in six multiplications:

$$n \times n = n^2$$
$$n^2 \times n^2 = n^4$$
$$n^4 \times n^4 = n^8$$
$$n^8 \times n^4 = n^{12}$$
$$n^{12} \times n^2 = n^{14}$$
$$n^{14} \times n = n^{15}$$

However it is yet possible to compute it in only five multiplications:

$$n \times n = n^2$$
$$n^2 \times n = n^3$$
$$n^3 \times n^3 = n^6$$
$$n^6 \times n^6 = n^{12}$$
$$n^{12} \times n^3 = n^{15}$$

We shall define m(k) to be the minimum number of multiplications to compute n^k; for example m(15) = 5.

For $1 \le k \le 200$, find $\sum$ m(k).

Problem 123

Let p_n be the nth prime: 2, 3, 5, 7, 11, ..., and let r be the remainder when $(p_n-1)^n + (p_n+1)^n$ is divided by p_n^2.

For example, when $n = 3$, $p_3 = 5$, and $4^3 + 6^3 = 280 \equiv 5 \bmod 25$.

The least value of n for which the remainder first exceeds 10^9 is 7037.

Find the least value of n for which the remainder first exceeds 10^{10}.

Problem 124

The radical of n, rad(n), is the product of distinct prime factors of n. For example, $504 = 2^3 \times 3^2 \times 7$, so rad(504) = $2 \times 3 \times 7 = 42$.

If we calculate rad(n) for $1 \le n \le 10$, then sort them on rad(n), and sorting on n if the radical values are equal, we get:

Unsorted			Sorted		
n	**rad(*n*)**		***n***	**rad(*n*)**	**k**
1	1		1	1	1

2	2	2	2	2
3	3	4	2	3
4	2	8	2	4
5	5	3	3	5
6	6	9	3	6
7	7	5	5	7
8	2	6	6	8
9	3	7	7	9
10	10	10	10	10

Let E(k) be the kth element in the sorted n column; for example, E(4) = 8 and E(6) = 9.

If rad(n) is sorted for $1 \leq n \leq 100000$, find E(10000).

Problem 125

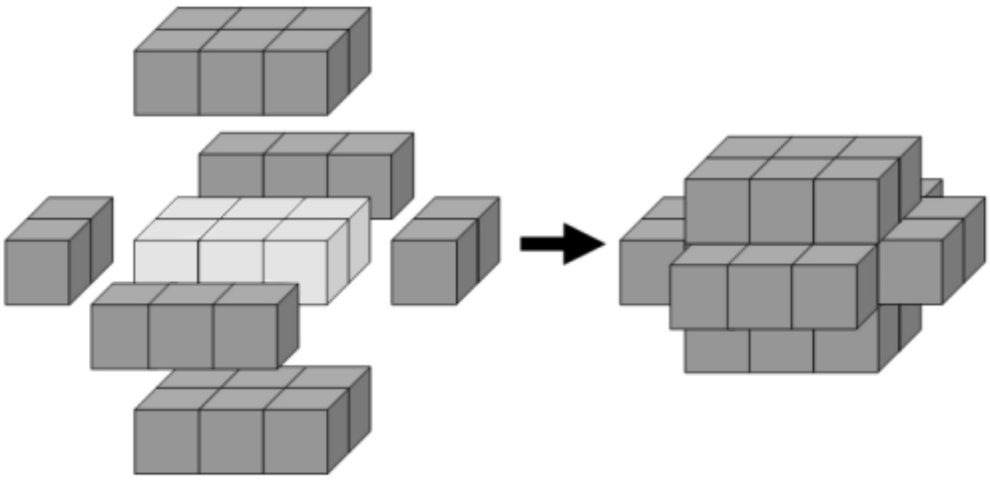
The palindromic number 595 is interesting because it can be written as the sum of consecutive squares: $6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2$.

There are exactly eleven palindromes below one-thousand that can be written as consecutive square sums, and the sum of these palindromes is 4164. Note that $1 = 0^2 + 1^2$ has not been included as this problem is concerned with the squares of positive integers.

Find the sum of all the numbers less than 10^8 that are both palindromic and can be written as the sum of consecutive squares.

Problem 126

The minimum number of cubes to cover every visible face on a cuboid measuring 3 x 2 x 1 is twenty-two.



If we then add a second layer to this solid it would require forty-six cubes to cover every visible face, the third layer would require seventy-eight cubes, and the fourth layer would require one-hundred and eighteen cubes to cover every visible face.

However, the first layer on a cuboid measuring 5 x 1 x 1 also requires twenty-two cubes; similarly the first layer on cuboids measuring 5 x 3 x 1, 7 x 2 x 1, and 11 x 1 x 1 all contain forty-six cubes.

We shall define C(n) to represent the number of cuboids that contain n cubes in one of its layers. So C(22) = 2, C(46) = 4, C(78) = 5, and C(118) = 8.

It turns out that 154 is the least value of n for which C(n) = 10.

Find the least value of n for which C(n) = 1000.

Problem 127

The radical of n, rad(n), is the product of distinct prime factors of n. For example, $504 = 2^3 \times 3^2 \times 7$, so rad(504) = $2 \times 3 \times 7 = 42$.

We shall define the triplet of positive integers (a, b, c) to be an abc-hit if:

1. GCD(a, b) = GCD(a, c) = GCD(b, c) = 1
2. $a < b$
3. $a + b = c$
4. rad(abc) $< c$

For example, (5, 27, 32) is an abc-hit, because:

1. GCD(5, 27) = GCD(5, 32) = GCD(27, 32) = 1
2. $5 < 27$
3. $5 + 27 = 32$
4. rad(4320) = 30 $<$ 32

It turns out that abc-hits are quite rare and there are only thirty-one abc-hits for $c < 1000$, with $\sum c$ = 12523.

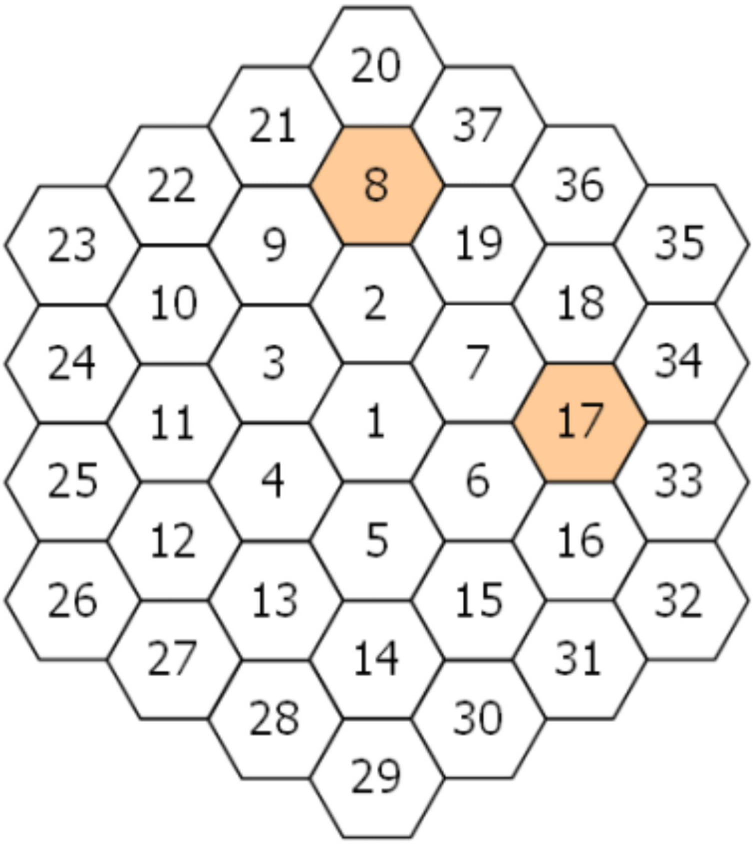
Find $\sum c$ for $c < 120000$.

Note: This problem has been changed recently, please check that you are using the right parameters.

Problem 128

A hexagonal tile with number 1 is surrounded by a ring of six hexagonal tiles, starting at "12 o'clock" and numbering the tiles 2 to 7 in an anti-clockwise direction.

New rings are added in the same fashion, with the next rings being numbered 8 to 19, 20 to 37, 38 to 61, and so on. The diagram below shows the first three rings.



By finding the difference between tile *n* and each its six neighbours we shall define PD(*n*) to be the number of those differences which are prime.

For example, working clockwise around tile 8 the differences are 12, 29, 11, 6, 1, and 13. So PD(8) = 3.

In the same way, the differences around tile 17 are 1, 17, 16, 1, 11, and 10, hence PD(17) = 2.

It can be shown that the maximum value of PD(*n*) is 3.

If all of the tiles for which PD(*n*) = 3 are listed in ascending order to form a sequence, the 10th tile would be 271.

Find the 2000th tile in this sequence.

Problem 129

A number consisting entirely of ones is called a repunit. We shall define R(*k*) to be a repunit of length *k*; for example, R(6) = 111111.

Given that *n* is a positive integer and GCD(*n*, 10) = 1, it can be shown that there always exists a value, *k*, for which R(*k*) is divisible by *n*, and let A(*n*) be the least such value of *k*; for example, A(7) = 6 and A(41) = 5.

The least value of *n* for which A(*n*) first exceeds ten is 17.

Find the least value of *n* for which A(*n*) first exceeds one-million.

Problem 130

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111.

Given that n is a positive integer and GCD(n, 10) = 1, it can be shown that there always exists a value, k, for which R(k) is divisible by n, and let A(n) be the least such value of k; for example, A(7) = 6 and A(41) = 5.

You are given that for all primes, $p > 5$, that $p - 1$ is divisible by A(p). For example, when p = 41, A(41) = 5, and 40 is divisible by 5.

However, there are rare composite values for which this is also true; the first five examples being 91, 259, 451, 481, and 703.

Find the sum of the first twenty-five composite values of n for which
GCD(n, 10) = 1 and $n - 1$ is divisible by A(n).

Problem 131

There are some prime values, p, for which there exists a positive integer, n, such that the expression $n^3 + n^2p$ is a perfect cube.

For example, when p = 19, $8^3 + 8^2 \times 19 = 12^3$.

What is perhaps most surprising is that for each prime with this property the value of n is unique, and there are only four such primes below one-hundred.

How many primes below one million have this remarkable property?

Problem 132

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k.

For example, R(10) = 1111111111 = 11×41×271×9091, and the sum of these prime factors is 9414.

Find the sum of the first forty prime factors of R(10^9).

Problem 133

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111.

Let us consider repunits of the form R(10^n).

Although R(10), R(100), or R(1000) are not divisible by 17, R(10000) is divisible by 17. Yet there is no value of n for which R(10^n) will divide by 19. In fact, it is remarkable that 11, 17, 41, and 73 are the only four primes below one-hundred that can be a factor of R(10^n).

Find the sum of all the primes below one-hundred thousand that will never be a factor of $R(10^n)$.

Problem 134

Consider the consecutive primes $p_1 = 19$ and $p_2 = 23$. It can be verified that 1219 is the smallest number such that the last digits are formed by p_1 whilst also being divisible by p_2.

In fact, with the exception of $p_1 = 3$ and $p_2 = 5$, for every pair of consecutive primes, $p_2 > p_1$, there exist values of n for which the last digits are formed by p_1 and n is divisible by p_2. Let S be the smallest of these values of n.

Find $\sum S$ for every pair of consecutive primes with $5 \le p_1 \le 1000000$.

Problem 135

Given the positive integers, x, y, and z, are consecutive terms of an arithmetic progression, the least value of the positive integer, n, for which the equation, $x^2 - y^2 - z^2 = n$, has exactly two solutions is $n = 27$:

$$34^2 - 27^2 - 20^2 = 12^2 - 9^2 - 6^2 = 27$$

It turns out that $n = 1155$ is the least value which has exactly ten solutions.

How many values of n less than one million have exactly ten distinct solutions?

Problem 136

The positive integers, x, y, and z, are consecutive terms of an arithmetic progression. Given that n is a positive integer, the equation, $x^2 - y^2 - z^2 = n$, has exactly one solution when $n = 20$:

$$13^2 - 10^2 - 7^2 = 20$$

In fact there are twenty-five values of n below one hundred for which the equation has a unique solution.

How many values of n less than fifty million have exactly one solution?

Problem 137

Consider the infinite polynomial series $A_F(x) = xF_1 + x^2F_2 + x^3F_3 + \ldots$, where F_k is the kth term in the Fibonacci sequence: 1, 1, 2, 3, 5, 8, ... ; that is, $F_k = F_{k-1} + F_{k-2}$, $F_1 = 1$ and $F_2 = 1$.

For this problem we shall be interested in values of x for which $A_F(x)$ is a positive integer.

Surprisingly $A_F(1/2) = (1/2).1 + (1/2)^2.1 + (1/2)^3.2 + (1/2)^4.3 + (1/2)^5.5 + ...$
$= 1/2 + 1/4 + 2/8 + 3/16 + 5/32 + ...$
$= 2$

The corresponding values of x for the first five natural numbers are shown below.

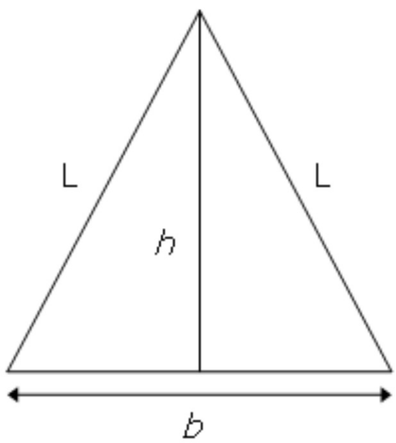
x	$A_F(x)$
$\sqrt{2}-1$	1
1/2	2
$(\sqrt{13}-2)/3$	3
$(\sqrt{89}-5)/8$	4
$(\sqrt{34}-3)/5$	5

We shall call $A_F(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 10th golden nugget is 74049690.

Find the 15th golden nugget.

Problem 138

Consider the isosceles triangle with base length, b = 16, and legs, L = 17.



By using the Pythagorean theorem it can be seen that the height of the triangle, $h = \sqrt{(17^2 - 8^2)} = 15$, which is one less than the base length.

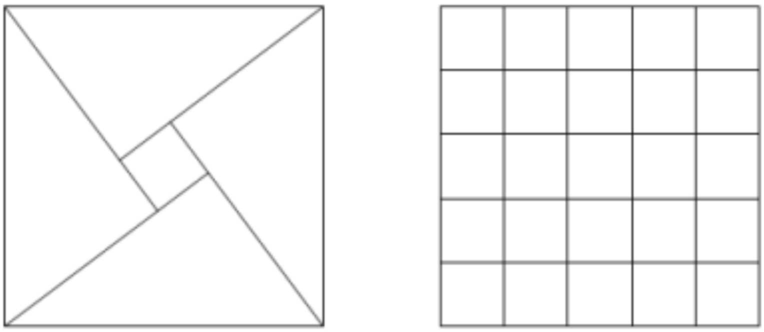
With b = 272 and L = 305, we get h = 273, which is one more than the base length, and this is the second smallest isosceles triangle with the property that $h = b \pm 1$.

Find $\sum$ L for the twelve smallest isosceles triangles for which $h = b \pm 1$ and b, L are positive integers.

Problem 139

Let (a, b, c) represent the three sides of a right angle triangle with integral length sides. It is possible to place four such triangles together to form a square with length c.

For example, (3, 4, 5) triangles can be placed together to form a 5 by 5 square with a 1 by 1 hole in the middle and it can be seen that the 5 by 5 square can be tiled with twenty-five 1 by 1 squares.



However, if (5, 12, 13) triangles were used then the hole would measure 7 by 7 and these could not be used to tile the 13 by 13 square.

Given that the perimeter of the right triangle is less than one-hundred million, how many Pythagorean triangles would allow such a tiling to take place?

Problem 140

Consider the infinite polynomial series $A_G(x) = xG_1 + x^2G_2 + x^3G_3 + \ldots$, where G_k is the kth term of the second order recurrence relation $G_k = G_{k-1} + G_{k-2}$, $G_1 = 1$ and $G_2 = 4$; that is, 1, 4, 5, 9, 14, 23, ...

For this problem we shall be concerned with values of x for which $A_G(x)$ is a positive integer.

The corresponding values of x for the first five natural numbers are shown below.

x	$A_G(x)$
$(\sqrt{5}-1)/4$	1
2/5	2
$(\sqrt{22}-2)/6$	3
$(\sqrt{137}-5)/14$	4
1/2	5

We shall call $A_G(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 20th golden nugget is 211345365.

Find the sum of the first thirty golden nuggets.

Problem 141

A positive integer, n, is divided by d and the quotient and remainder are q and r respectively. In addition d, q, and r are consecutive positive integer terms in a geometric sequence, but not necessarily in that order.

For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that 4, 6, 9 are consecutive terms in a geometric sequence (common ratio 3/2).

We will call such numbers, *n*, progressive.

Some progressive numbers, such as 9 and 10404 = 102^2, happen to also be perfect squares.
The sum of all progressive perfect squares below one hundred thousand is 124657.

Find the sum of all progressive perfect squares below one trillion (10^{12}).

Problem 142

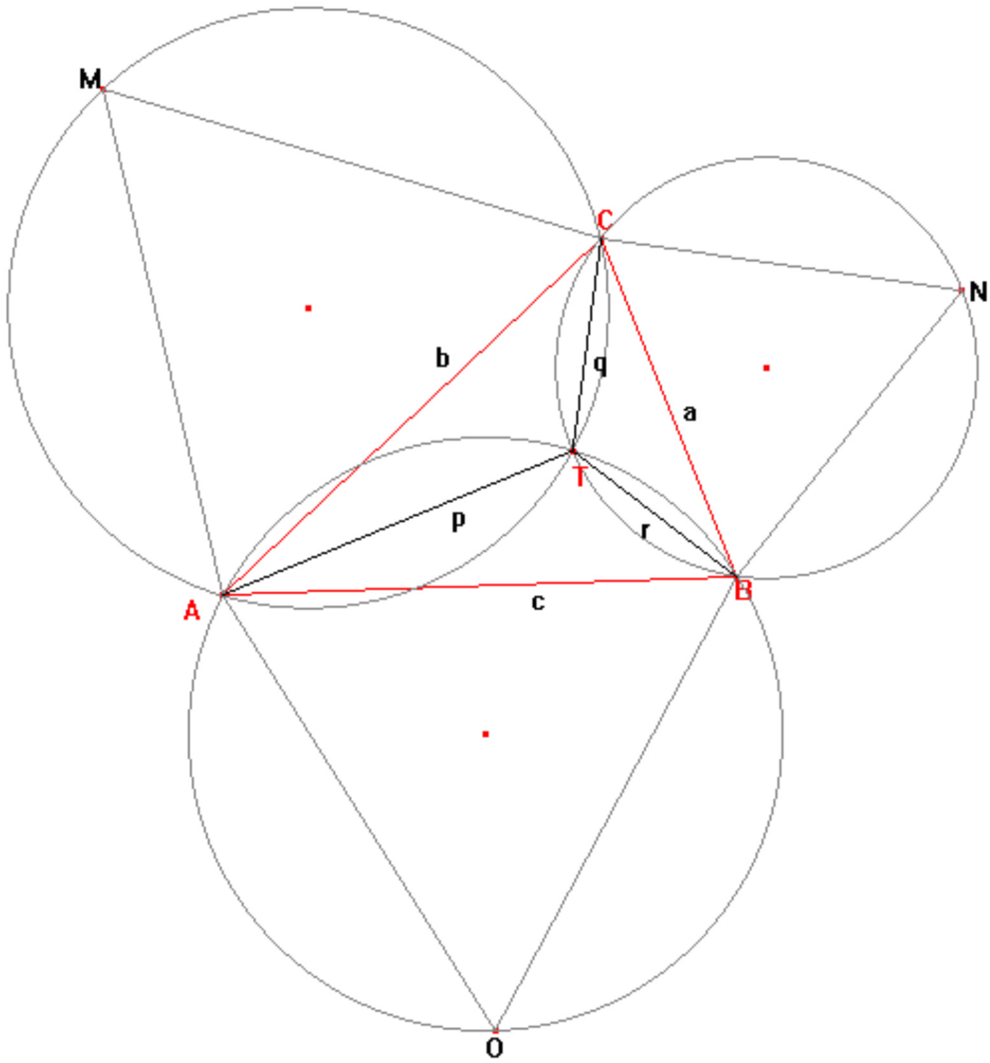
Find the smallest x + y + z with integers x > y > z > 0 such that x + y, x − y, x + z, x − z, y + z, y − z are all perfect squares.

Problem 143

Let ABC be a triangle with all interior angles being less than 120 degrees. Let X be any point inside the triangle and let XA = p, XB = q, and XC = r.

Fermat challenged Torricelli to find the position of X such that p + q + r was minimised.

Torricelli was able to prove that if equilateral triangles AOB, BNC and AMC are constructed on each side of triangle ABC, the circumscribed circles of AOB, BNC, and AMC will intersect at a single point, T, inside the triangle. Moreover he proved that T, called the Torricelli/Fermat point, minimises p + q + r. Even more remarkable, it can be shown that when the sum is minimised, AN = BM = CO = p + q + r and that AN, BM and CO also intersect at T.



If the sum is minimised and a, b, c, p, q and r are all positive integers we shall call triangle ABC a Torricelli triangle. For example, a = 399, b = 455, c = 511 is an example of a Torricelli triangle, with p + q + r = 784.

Find the sum of all distinct values of p + q + r ≤ 120000 for Torricelli triangles.

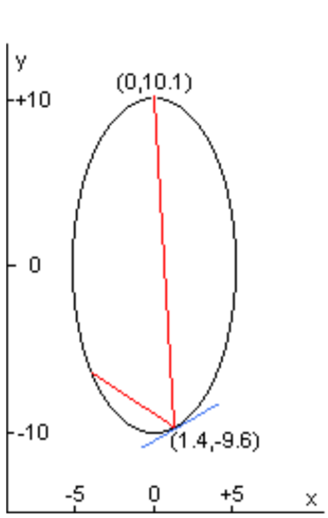
Note: This problem has been changed recently, please check that you are using the right parameters.

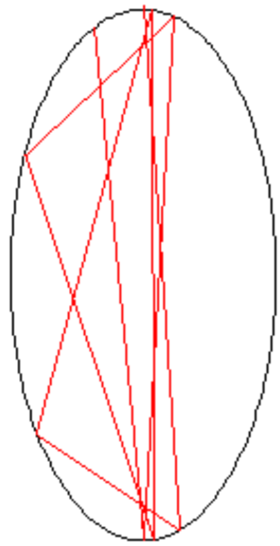
Problem 144

In laser physics, a "white cell" is a mirror system that acts as a delay line for the laser beam. The beam enters the cell, bounces around on the mirrors, and eventually works its way back out.

The specific white cell we will be considering is an ellipse with the equation $4x^2 + y^2 = 100$

The section corresponding to $-0.01 \leq x \leq +0.01$ at the top is missing, allowing the light to enter and exit through the hole.





The light beam in this problem starts at the point (0.0,10.1) just outside the white cell, and the beam first impacts the mirror at (1.4,-9.6).

Each time the laser beam hits the surface of the ellipse, it follows the usual law of reflection "angle of incidence equals angle of reflection." That is, both the incident and reflected beams make the same angle with the normal line at the point of incidence.

In the figure on the left, the red line shows the first two points of contact between the laser beam and the wall of the white cell; the blue line shows the line tangent to the ellipse at the point of incidence of the first bounce.

The slope m of the tangent line at any point (x,y) of the given ellipse is: $m = -4x/y$

The normal line is perpendicular to this tangent line at the point of incidence.

The animation on the right shows the first 10 reflections of the beam.

How many times does the beam hit the internal surface of the white cell before exiting?

Problem 145

Some positive integers n have the property that the sum [n + reverse(n)] consists entirely of odd (decimal) digits. For instance, 36 + 63 = 99 and 409 + 904 = 1313. We will call such numbers *reversible*; so 36, 63, 409, and 904 are reversible. Leading zeroes are not allowed in either n or reverse(n).

There are 120 reversible numbers below one-thousand.

How many reversible numbers are there below one-billion (10^9)?

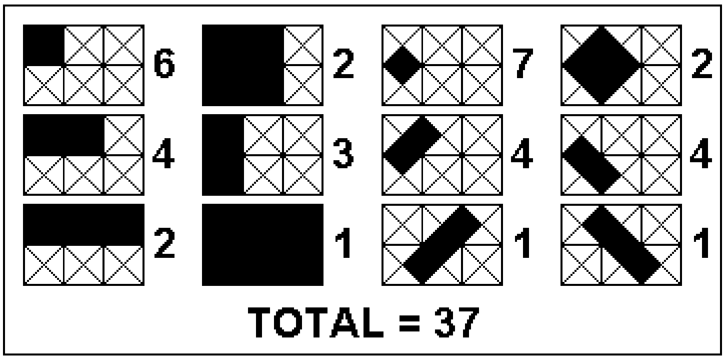
Problem 146

The smallest positive integer *n* for which the numbers n^2+1, n^2+3, n^2+7, n^2+9, n^2+13, and n^2+27 are consecutive primes is 10. The sum of all such integers *n* below one-million is 1242490.

What is the sum of all such integers *n* below 150 million?

Problem 147

In a 3x2 cross-hatched grid, a total of 37 different rectangles could be situated within that grid as indicated in the sketch.



There are 5 grids smaller than 3x2, vertical and horizontal dimensions being important, i.e. 1x1, 2x1, 3x1, 1x2 and 2x2. If each of them is cross-hatched, the following number of different rectangles could be situated within those smaller grids:

1x1: 1
2x1: 4
3x1: 8
1x2: 4
2x2: 18

Adding those to the 37 of the 3x2 grid, a total of 72 different rectangles could be situated within 3x2 and smaller grids.

How many different rectangles could be situated within 47x43 and smaller grids?

Problem 148

We can easily verify that none of the entries in the first seven rows of Pascal's triangle are divisible by 7:

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5   10  10  5   1
1   6   15  20  15  6   1
```

However, if we check the first one hundred rows, we will find that only 2361 of the 5050 entries are *not* divisible by 7.

Find the number of entries which are *not* divisible by 7 in the first one billion (10^9) rows of Pascal's triangle.

Problem 149

Looking at the table below, it is easy to verify that the maximum possible sum of adjacent numbers in any direction (horizontal, vertical, diagonal or anti-diagonal) is 16 (= 8 + 7 + 1).

−2	5	3	2
9	−6	5	1
3	2	7	3
−1	8	−4	8

Now, let us repeat the search, but on a much larger scale:

First, generate four million pseudo-random numbers using a specific form of what is known as a "Lagged Fibonacci Generator":

For $1 \le k \le 55$, $s_k = [100003 - 200003k + 300007k^3]$ (modulo 1000000) − 500000.
For $56 \le k \le 4000000$, $s_k = [s_{k-24} + s_{k-55} + 1000000]$ (modulo 1000000) − 500000.

Thus, $s_{10} = -393027$ and $s_{100} = 86613$.

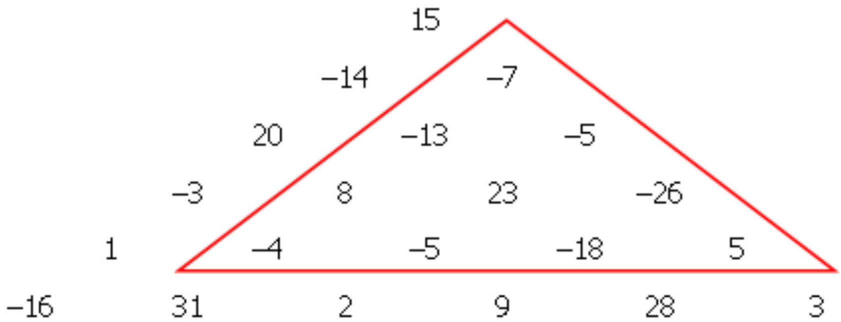
The terms of s are then arranged in a 2000×2000 table, using the first 2000 numbers to fill the first row (sequentially), the next 2000 numbers to fill the second row, and so on.

Finally, find the greatest sum of (any number of) adjacent entries in any direction (horizontal, vertical, diagonal or anti-diagonal).

Problem 150

In a triangular array of positive and negative integers, we wish to find a sub-triangle such that the sum of the numbers it contains is the smallest possible.

In the example below, it can be easily verified that the marked triangle satisfies this condition having a sum of −42.



We wish to make such a triangular array with one thousand rows, so we generate 500500 pseudo-random numbers s_k in the range $\pm 2^{19}$, using a type of random number generator (known as a Linear Congruential Generator) as follows:

$t := 0$
for k = 1 up to k = 500500:
$t := (615949*t + 797807) \text{ modulo } 2^{20}$
$s_k := t - 2^{19}$

Thus: $s_1 = 273519$, $s_2 = -153582$, $s_3 = 450905$ etc

Our triangular array is then formed using the pseudo-random numbers thus:

$$\begin{array}{c} s_1 \\ s_2 \;\; s_3 \\ s_4 \;\; s_5 \;\; s_6 \\ s_7 \;\; s_8 \;\; s_9 \;\; s_{10} \\ \cdots \end{array}$$

Sub-triangles can start at any element of the array and extend down as far as we like (taking-in the two elements directly below it from the next row, the three elements directly below from the row after that, and so on).
The "sum of a sub-triangle" is defined as the sum of all the elements it contains.
Find the smallest possible sub-triangle sum.

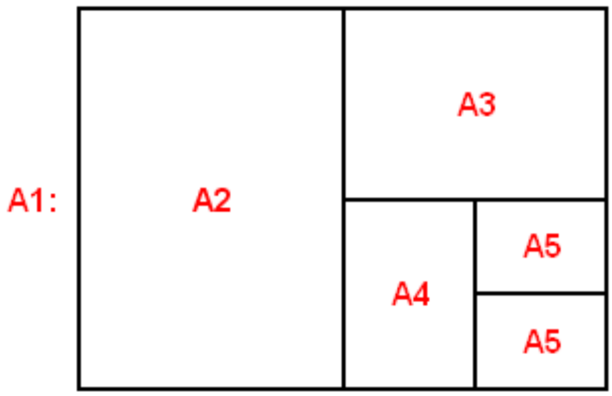
Problem 151

A printing shop runs 16 batches (jobs) every week and each batch requires a sheet of special colour-proofing paper of size A5.

Every Monday morning, the foreman opens a new envelope, containing a large sheet of the special paper with size A1.

He proceeds to cut it in half, thus getting two sheets of size A2. Then he cuts one of them in half to get two sheets of size A3 and so on until he obtains the A5-size sheet needed for the first batch of the week.

All the unused sheets are placed back in the envelope.



At the beginning of each subsequent batch, he takes from the envelope one sheet of paper at random. If it is of size A5, he uses it. If it is larger, he repeats the 'cut-in-half' procedure until he has what he needs and any remaining sheets are always placed back in the envelope.

Excluding the first and last batch of the week, find the expected number of times (during each week) that the foreman finds a single sheet of paper in the envelope.

Give your answer rounded to six decimal places using the format x.xxxxxx .

Problem 152

There are several ways to write the number 1/2 as a sum of inverse squares using *distinct* integers.

For instance, the numbers {2,3,4,5,7,12,15,20,28,35} can be used:

$$\frac{1}{2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2} + \frac{1}{28^2} + \frac{1}{35^2}$$

In fact, only using integers between 2 and 45 inclusive, there are exactly three ways to do it, the remaining two being: {2,3,4,6,7,9,10,20,28,35,36,45} and {2,3,4,6,7,9,12,15,28,30,35,36,45}.

How many ways are there to write the number 1/2 as a sum of inverse squares using distinct integers between 2 and 80 inclusive?

Problem 153

As we all know the equation x^2=-1 has no solutions for real x.
If we however introduce the imaginary number i this equation has two solutions: $x=i$ and $x=-i$.
If we go a step further the equation $(x\text{-}3)^2$=-4 has two complex solutions: x=3+2i and x=3-2i.
x=3+2i and x=3-2i are called each others' complex conjugate.
Numbers of the form $a+bi$ are called complex numbers.
In general $a+bi$ and $a–bi$ are each other's complex conjugate.

A Gaussian Integer is a complex number $a+bi$ such that both a and b are integers.
The regular integers are also Gaussian integers (with b=0).
To distinguish them from Gaussian integers with $b \neq 0$ we call such integers "rational integers."
A Gaussian integer is called a divisor of a rational integer n if the result is also a Gaussian integer.
If for example we divide 5 by 1+2i we can simplify $\frac{5}{1+2i}$ in the following manner:
Multiply numerator and denominator by the complex conjugate of 1+2i: 1–2i.
The result is $\frac{5}{1+2i} = \frac{5}{1+2i}\frac{1-2i}{1-2i} = \frac{5(1-2i)}{1-(2i)^2} = \frac{5(1-2i)}{1-(-4)} = \frac{5(1-2i)}{5} = 1-2i$.
So 1+2i is a divisor of 5.
Note that 1+i is not a divisor of 5 because $\frac{5}{1+i} = \frac{5}{2} - \frac{5}{2}i$.
Note also that if the Gaussian Integer ($a+bi$) is a divisor of a rational integer n, then its complex conjugate ($a–bi$) is also a divisor of n.

In fact, 5 has six divisors such that the real part is positive: {1, 1 + 2i, 1 – 2i, 2 + i, 2 – i, 5}.
The following is a table of all of the divisors for the first five positive rational integers:

n	Gaussian integer divisors with positive real part	Sum s(n) of these divisors
1	1	1
2	1, 1+i, 1-i, 2	5
3	1, 3	4
4	1, 1+i, 1-i, 2, 2+2i, 2-2i,4	13
5	1, 1+2i, 1-2i, 2+i, 2-i, 5	12

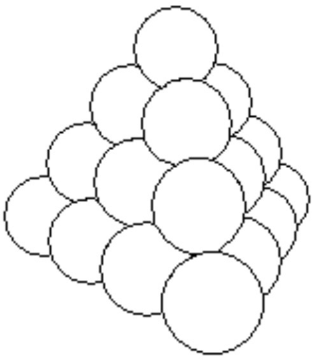
For divisors with positive real parts, then, we have: $\sum_{n=1}^{5} s(n) = 35$.

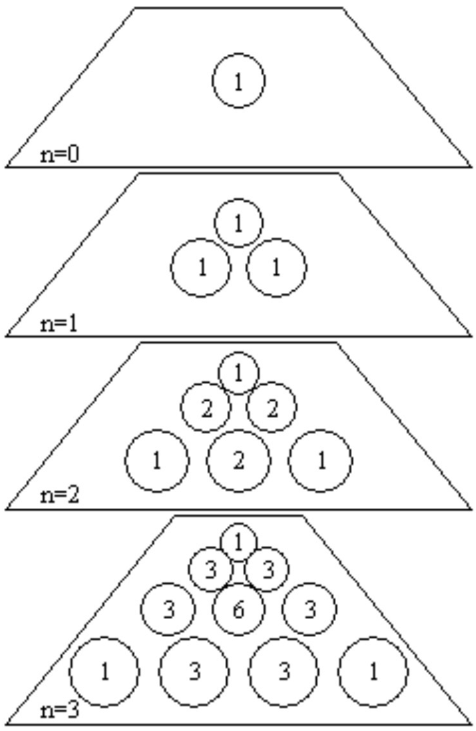
For $1 \leq n \leq 10^5$, $\sum$ s(n)=17924657155.

What is $\sum$ s(n) for $1 \leq n \leq 10^8$?

Problem 154

A triangular pyramid is constructed using spherical balls so that each ball rests on exactly three balls of the next lower level.





Then, we calculate the number of paths leading from the apex to each position:

A path starts at the apex and progresses downwards to any of the three spheres directly below the current position.

Consequently, the number of paths to reach a certain position is the sum of the numbers immediately above it (depending on the position, there are up to three numbers above it).

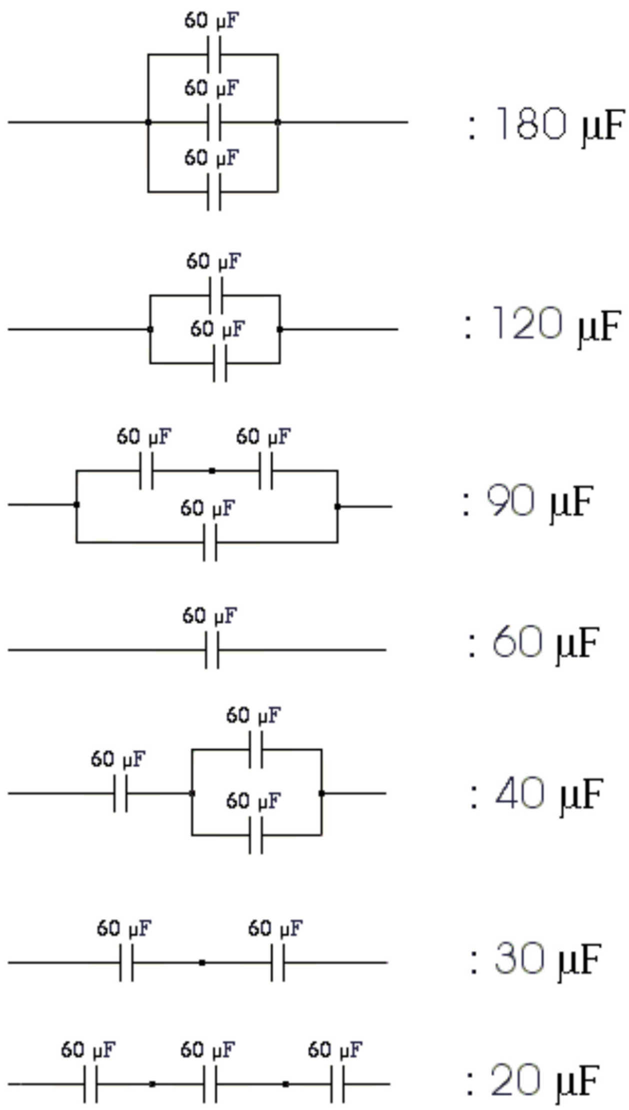
The result is *Pascal's pyramid* and the numbers at each level n are the coefficients of the trinomial expansion $(x + y + z)^n$.

How many coefficients in the expansion of $(x + y + z)^{200000}$ are multiples of 10^{12}?

Problem 155

An electric circuit uses exclusively identical capacitors of the same value C.
The capacitors can be connected in series or in parallel to form sub-units, which can then be connected in series or in parallel with other capacitors or other sub-units to form larger sub-units, and so on up to a final circuit.

Using this simple procedure and up to n identical capacitors, we can make circuits having a range of different total capacitances. For example, using up to n=3 capacitors of 60 μF each, we can obtain the following 7 distinct total capacitance values:



If we denote by $D(n)$ the number of distinct total capacitance values we can obtain when using up to n equal-valued capacitors and the simple procedure described above, we have: $D(1)$=1, $D(2)$=3, $D(3)$=7 …

Find $D(18)$.

Reminder : When connecting capacitors C_1, C_2 etc in parallel, the total capacitance is $C_T = C_1 + C_2 +\ldots$, whereas when connecting them in series, the overall capacitance is given by: $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots$

Problem 156

Starting from zero the natural numbers are written down in base 10 like this:
0 1 2 3 4 5 6 7 8 9 10 11 12....

Consider the digit d=1. After we write down each number n, we will update the number of ones that have occurred and call this number $f(n,1)$. The first values for $f(n,1)$, then, are as follows:

n	$f(n,1)$
0	0
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	2
11	4
12	5

Note that $f(n,1)$ never equals 3.
So the first two solutions of the equation $f(n,1)=n$ are n=0 and n=1. The next solution is n=199981.

In the same manner the function $f(n,d)$ gives the total number of digits d that have been written down after the number n has been written.
In fact, for every digit $d \neq 0$, 0 is the first solution of the equation $f(n,d)=n$.

Let $s(d)$ be the sum of all the solutions for which $f(n,d)=n$.
You are given that $s(1)$=22786974071.

Find $\sum s(d)$ for $1 \leq d \leq 9$.

Note: if, for some n, $f(n,d)=n$ for more than one value of d this value of n is counted again for every value of d for which $f(n,d)=n$.

Problem 157

Consider the diophantine equation $^1/_a+^1/_b= ^p/_{10^n}$ with a, b, p, n positive integers and $a \leq b$.

For n=1 this equation has 20 solutions that are listed below:

$^1/_1+^1/_1=^{20}/_{10}$	$^1/_1+^1/_2=^{15}/_{10}$	$^1/_1+^1/_5=^{12}/_{10}$	$^1/_1+^1/_{10}=^{11}/_{10}$	$^1/_2+^1/_2=^{10}/_{10}$
$^1/_2+^1/_5=^7/_{10}$	$^1/_2+^1/_{10}=^6/_{10}$	$^1/_3+^1/_6=^5/_{10}$	$^1/_3+^1/_{15}=^4/_{10}$	$^1/_4+^1/_4=^5/_{10}$
$^1/_4+^1/_{20}=^3/_{10}$	$^1/_5+^1/_5=^4/_{10}$	$^1/_5+^1/_{10}=^3/_{10}$	$^1/_6+^1/_{30}=^2/_{10}$	$^1/_{10}+^1/_{10}=^2/_{10}$
$^1/_{11}+^1/_{110}=^1/_{10}$	$^1/_{12}+^1/_{60}=^1/_{10}$	$^1/_{14}+^1/_{35}=^1/_{10}$	$^1/_{15}+^1/_{30}=^1/_{10}$	$^1/_{20}+^1/_{20}=^1/_{10}$

How many solutions has this equation for $1 \leq n \leq 9$?

Problem 158

Taking three different letters from the 26 letters of the alphabet, character strings of length three can be formed.
Examples are 'abc', 'hat' and 'zyx'.
When we study these three examples we see that for 'abc' two characters come lexicographically after its neighbour to the left.
For 'hat' there is exactly one character that comes lexicographically after its neighbour to the left. For 'zyx' there are zero characters that come lexicographically after its neighbour to the left.
In all there are 10400 strings of length 3 for which exactly one character comes lexicographically after its neighbour to the left.

We now consider strings of $n \leq 26$ different characters from the alphabet.
For every n, p(n) is the number of strings of length n for which exactly one character comes lexicographically after its neighbour to the left.

What is the maximum value of p(n)?

Problem 159

A composite number can be factored many different ways. For instance, not including multiplication by one, 24 can be factored in 7 distinct ways:

24 = 2x2x2x3
24 = 2x3x4
24 = 2x2x6
24 = 4x6
24 = 3x8
24 = 2x12
24 = 24

Recall that the digital root of a number, in base 10, is found by adding together the digits of that number, and repeating that process until a number is arrived at that is less than 10. Thus the digital root of 467 is 8.

We shall call a Digital Root Sum (DRS) the sum of the digital roots of the individual factors of our number.
The chart below demonstrates all of the DRS values for 24.

Factorisation	Digital Root Sum
2x2x2x3	9
2x3x4	9
2x2x6	10
4x6	10
3x8	11
2x12	5
24	6

The maximum Digital Root Sum of 24 is 11.
The function mdrs(n) gives the maximum Digital Root Sum of n. So mdrs(24)=11.
Find $\sum$mdrs(n) for 1 $<$ n $<$ 1,000,000.

Problem 160

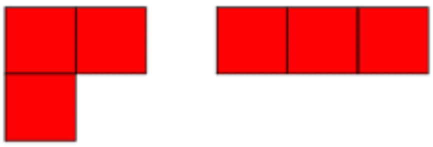
For any N, let f(N) be the last five digits before the trailing zeroes in N!.
For example,

9! = 362880 so f(9)=36288
10! = 3628800 so f(10)=36288
20! = 2432902008176640000 so f(20)=17664

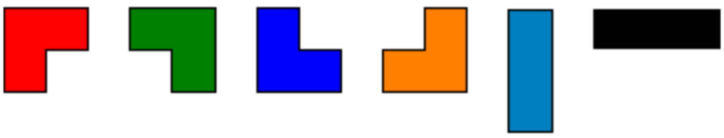
Find f(1,000,000,000,000)

Problem 161

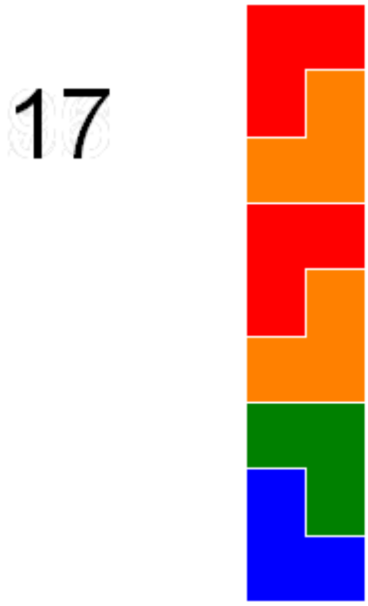
A triomino is a shape consisting of three squares joined via the edges. There are two basic forms:



If all possible orientations are taken into account there are six:



Any n by m grid for which nxm is divisible by 3 can be tiled with triominoes.
If we consider tilings that can be obtained by reflection or rotation from another tiling as different there are 41 ways a 2 by 9 grid can be tiled with triominoes:



In how many ways can a 9 by 12 grid be tiled in this way by triominoes?

Problem 162

In the hexadecimal number system numbers are represented using 16 different digits:

0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

The hexadecimal number AF when written in the decimal number system equals 10x16+15=175.

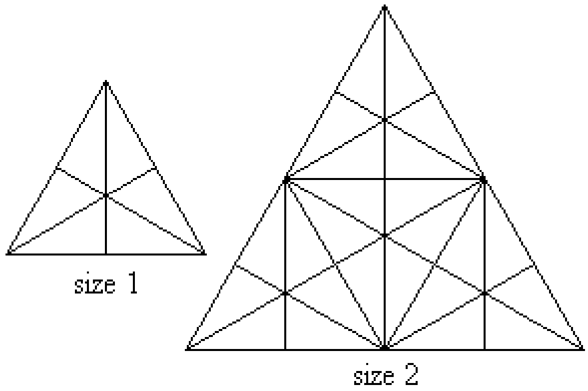
In the 3-digit hexadecimal numbers 10A, 1A0, A10, and A01 the digits 0,1 and A are all present.
Like numbers written in base ten we write hexadecimal numbers without leading zeroes.

How many hexadecimal numbers containing at most sixteen hexadecimal digits exist with all of the digits 0,1, and A present at least once?
Give your answer as a hexadecimal number.

(A,B,C,D,E and F in upper case, without any leading or trailing code that marks the number as hexadecimal and without leading zeroes , e.g. 1A3F and not: 1a3f and not 0x1a3f and not $1A3F and not #1A3F and not 0000001A3F)

Problem 163

Consider an equilateral triangle in which straight lines are drawn from each vertex to the middle of the opposite side, such as in the *size 1* triangle in the sketch below.



Sixteen triangles of either different shape or size or orientation or location can now be observed in that triangle. Using *size 1* triangles as building blocks, larger triangles can be formed, such as the *size 2* triangle in the above sketch. One-hundred and four triangles of either different shape or size or orientation or location can now be observed in that *size 2* triangle.

It can be observed that the *size 2* triangle contains 4 *size 1* triangle building blocks. A *size 3* triangle would contain 9 *size 1* triangle building blocks and a *size n* triangle would thus contain n^2 *size 1* triangle building blocks.

If we denote T(n) as the number of triangles present in a triangle of *size* n, then

T(1) = 16
T(2) = 104

Find T(36).

Problem 164

How many 20 digit numbers n (without any leading zero) exist such that no three consecutive digits of n have a sum greater than 9?

Problem 165

A segment is uniquely defined by its two endpoints.
By considering two line segments in plane geometry there are three possibilities:
the segments have zero points, one point, or infinitely many points in common.

Moreover when two segments have exactly one point in common it might be the case that that common point is an endpoint of either one of the segments or of both. If a common point of two segments is not an endpoint of either of the segments it is an interior point of both segments.
We will call a common point T of two segments L_1 and L_2 a true intersection point of L_1 and L_2 if T is the only common point of L_1 and L_2 and T is an interior point of both segments.

Consider the three segments L_1, L_2, and L_3:

L_1: (27, 44) to (12, 32)
L_2: (46, 53) to (17, 62)
L_3: (46, 70) to (22, 40)

It can be verified that line segments L_2 and L_3 have a true intersection point. We note that as the one of the end points of L_3: (22,40) lies on L_1 this is not considered to be a true point of intersection. L_1 and L_2 have no common point. So among the three line segments, we find one true intersection point.

Now let us do the same for 5000 line segments. To this end, we generate 20000 numbers using the so-called "Blum Blum Shub" pseudo-random number generator.

$s_0 = 290797$

$s_{n+1} = s_n \times s_n$ (modulo 50515093)

$t_n = s_n$ (modulo 500)

To create each line segment, we use four consecutive numbers t_n. That is, the first line segment is given by:

(t_1, t_2) to (t_3, t_4)

The first four numbers computed according to the above generator should be: 27, 144, 12 and 232. The first segment would thus be (27,144) to (12,232).

How many distinct true intersection points are found among the 5000 line segments?

Problem 166

A 4x4 grid is filled with digits d, $0 \leq d \leq 9$.

It can be seen that in the grid

```
6 3 3 0
5 0 4 3
0 7 1 4
1 2 4 5
```

the sum of each row and each column has the value 12. Moreover the sum of each diagonal is also 12.

In how many ways can you fill a 4x4 grid with the digits d, $0 \leq d \leq 9$ so that each row, each column, and both diagonals have the same sum?

Problem 167

For two positive integers a and b, the Ulam sequence U(a,b) is defined by $U(a,b)_1 = a$, $U(a,b)_2 = b$ and for $k > 2$, $U(a,b)_k$ is the smallest integer greater than $U(a,b)_{(k-1)}$ which can be written in exactly one way as the sum of two distinct previous members of U(a,b).

For example, the sequence U(1,2) begins with
1, 2, 3 = 1 + 2, 4 = 1 + 3, 6 = 2 + 4, 8 = 2 + 6, 11 = 3 + 8;
5 does not belong to it because 5 = 1 + 4 = 2 + 3 has two representations as the sum of two previous members, likewise 7 = 1 + 6 = 3 + 4.

Find $\sum U(2,2n+1)_k$ for $2 \le n \le 10$, where $k = 10^{11}$.

Problem 168

Consider the number 142857. We can right-rotate this number by moving the last digit (7) to the front of it, giving us 714285.
It can be verified that 714285=5×142857.
This demonstrates an unusual property of 142857: it is a divisor of its right-rotation.

Find the last 5 digits of the sum of all integers n, $10 < n < 10^{100}$, that have this property.

Problem 169

Define f(0)=1 and f(n) to be the number of different ways n can be expressed as a sum of integer powers of 2 using each power no more than twice.

For example, f(10)=5 since there are five different ways to express 10:

1 + 1 + 8
1 + 1 + 4 + 4
1 + 1 + 2 + 2 + 4
2 + 4 + 4
2 + 8

What is f(10^{25})?

Problem 170

Take the number 6 and multiply it by each of 1273 and 9854:

6 × 1273 = 7638
6 × 9854 = 59124

By concatenating these products we get the 1 to 9 pandigital 763859124. We will call 763859124 the "concatenated product of 6 and (1273,9854)". Notice too, that the concatenation of the input numbers, 612739854, is also 1 to 9 pandigital.

The same can be done for 0 to 9 pandigital numbers.

What is the largest 0 to 9 pandigital 10-digit concatenated product of an integer with two or more other integers, such that the concatenation of the input numbers is also a 0 to 9 pandigital 10-digit number?

Problem 171

For a positive integer n, let f(n) be the sum of the squares of the digits (in base 10) of n, e.g.

f(3) = 3^2 = 9,
f(25) = $2^2 + 5^2$ = 4 + 25 = 29,
f(442) = $4^2 + 4^2 + 2^2$ = 16 + 16 + 4 = 36

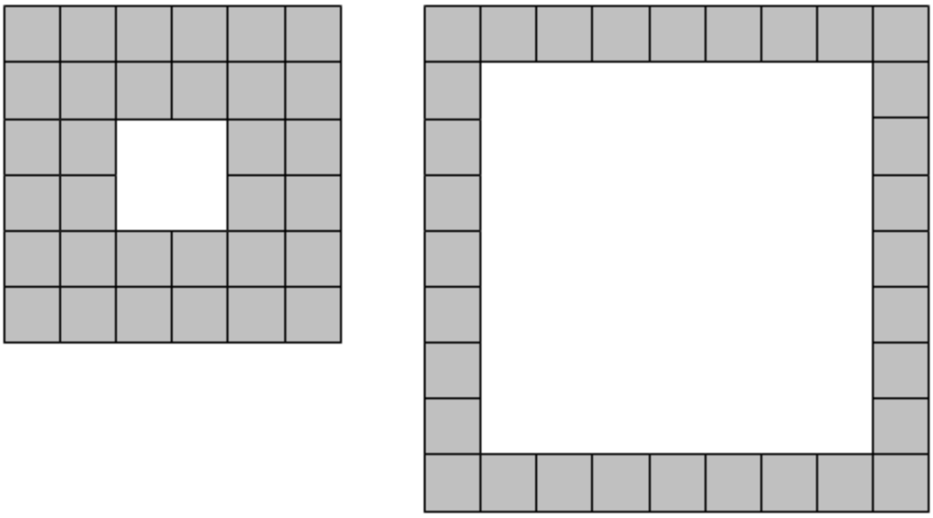
Find the last nine digits of the sum of all n, $0 < n < 10^{20}$, such that f(n) is a perfect square.

Problem 172

How many 18-digit numbers n (without leading zeros) are there such that no digit occurs more than three times in n?

Problem 173

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry. For example, using exactly thirty-two square tiles we can form two different square laminae:



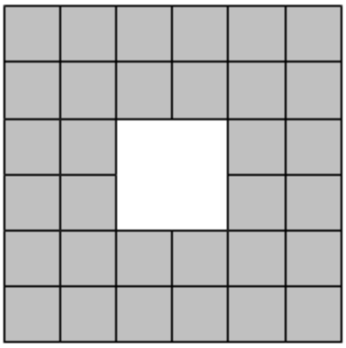
With one-hundred tiles, and not necessarily using all of the tiles at one time, it is possible to form forty-one different square laminae.

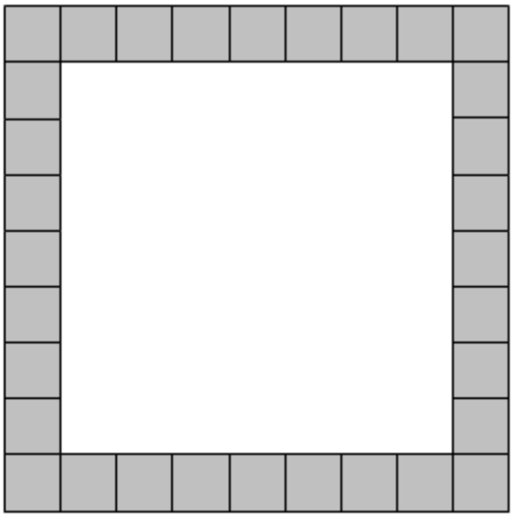
Using up to one million tiles how many different square laminae can be formed?

Problem 174

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry.

Given eight tiles it is possible to form a lamina in only one way: 3x3 square with a 1x1 hole in the middle. However, using thirty-two tiles it is possible to form two distinct laminae.





If t represents the number of tiles used, we shall say that t = 8 is type L(1) and t = 32 is type L(2).

Let N(n) be the number of $t \leq$ 1000000 such that t is type L(n); for example, N(15) = 832.

What is $\sum$ N(n) for 1 $\leq n \leq$ 10?

Problem 175

Define f(0)=1 and f(n) to be the number of ways to write n as a sum of powers of 2 where no power occurs more than twice.

For example, f(10)=5 since there are five different ways to express 10:
10 = 8+2 = 8+1+1 = 4+4+2 = 4+2+2+1+1 = 4+4+1+1

It can be shown that for every fraction p/q (p>0, q>0) there exists at least one integer n such that f(n)/f(n-1)=p/q.

For instance, the smallest n for which f(n)/f(n-1)=13/17 is 241.
The binary expansion of 241 is 11110001.
Reading this binary number from the most significant bit to the least significant bit there are 4 one's, 3 zeroes and 1 one. We shall call the string 4,3,1 the *Shortened Binary Expansion* of 241.

Find the Shortened Binary Expansion of the smallest n for which
f(n)/f(n-1)=123456789/987654321.

Give your answer as comma separated integers, without any whitespaces.

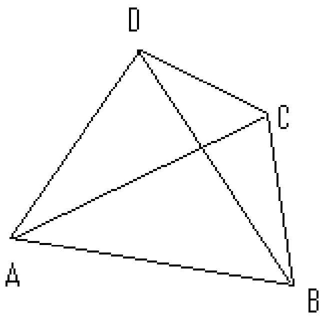
Problem 176

The four rectangular triangles with sides (9,12,15), (12,16,20), (5,12,13) and (12,35,37) all have one of the shorter sides (catheti) equal to 12. It can be shown that no other integer sided rectangular triangle exists with one of the catheti equal to 12.

Find the smallest integer that can be the length of a cathetus of exactly 47547 different integer sided rectangular triangles.

Problem 177

Let ABCD be a convex quadrilateral, with diagonals AC and BD. At each vertex the diagonal makes an angle with each of the two sides, creating eight corner angles.



For example, at vertex A, the two angles are CAD, CAB.

We call such a quadrilateral for which all eight corner angles have integer values when measured in degrees an "integer angled quadrilateral". An example of an integer angled quadrilateral is a square, where all eight corner angles are 45°. Another example is given by DAC = 20°, BAC = 60°, ABD = 50°, CBD = 30°, BCA = 40°, DCA = 30°, CDB = 80°, ADB = 50°.

What is the total number of non-similar integer angled quadrilaterals?

Note: In your calculations you may assume that a calculated angle is integral if it is within a tolerance of 10^{-9} of an integer value.

Problem 178

Consider the number 45656.
It can be seen that each pair of consecutive digits of 45656 has a difference of one.
A number for which every pair of consecutive digits has a difference of one is called a step number.
A pandigital number contains every decimal digit from 0 to 9 at least once.
How many pandigital step numbers less than 10^{40} are there?

Problem 179

Find the number of integers $1 < n < 10^7$, for which n and $n + 1$ have the same number of positive divisors. For example, 14 has the positive divisors 1, 2, 7, 14 while 15 has 1, 3, 5, 15.

Problem 180

For any integer n, consider the three functions

$$f_{1,n}(x,y,z) = x^{n+1} + y^{n+1} - z^{n+1}$$
$$f_{2,n}(x,y,z) = (xy + yz + zx)*(x^{n-1} + y^{n-1} - z^{n-1})$$
$$f_{3,n}(x,y,z) = xyz*(x^{n-2} + y^{n-2} - z^{n-2})$$

and their combination

$$f_n(x,y,z) = f_{1,n}(x,y,z) + f_{2,n}(x,y,z) - f_{3,n}(x,y,z)$$

We call (x,y,z) a golden triple of order k if x, y, and z are all rational numbers of the form a / b with $0 < a < b \le k$ and there is (at least) one integer n, so that $f_n(x,y,z) = 0$.

Let $s(x,y,z) = x + y + z$.
Let $t = u / v$ be the sum of all distinct $s(x,y,z)$ for all golden triples (x,y,z) of order 35.
All the $s(x,y,z)$ and t must be in reduced form.

Find $u + v$.

Problem 181

Having three black objects B and one white object W they can be grouped in 7 ways like this:

(BBBW) (B,BBW) (B,B,BW) (B,B,B,W) (B,BB,W) (BBB,W) (BB,BW)

In how many ways can sixty black objects B and forty white objects W be thus grouped?

Problem 182

The RSA encryption is based on the following procedure:

Generate two distinct primes p and q.
Compute $n=pq$ and $\varphi=(p-1)(q-1)$.
Find an integer e, $1<e<\varphi$, such that $\gcd(e,\varphi)=1$.

A message in this system is a number in the interval $[0,n-1]$.
A text to be encrypted is then somehow converted to messages (numbers in the interval $[0,n-1]$).
To encrypt the text, for each message, m, $c=m^e \bmod n$ is calculated.

To decrypt the text, the following procedure is needed: calculate d such that $ed=1 \bmod \varphi$, then for each encrypted message, c, calculate $m=c^d \bmod n$.