# Fibonacci numbers and matrices

R C Johnson[1]

bob.johnson@dur.ac.uk

www.dur.ac.uk/bob.johnson/fibonacci/

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[1]*Maths Dept, Durham University, Durham City, DH1 3LE, UK*

# Contents

# Chapter 1

# Introduction

The Fibonacci numbers [1, *A000045*] $f_k = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ for $k = 0, 1, 2, \ldots$ obey by definition the recurrence rule

$$f_{k+1} = f_k + f_{k-1}. \tag{1.1}$$

However this is not their only obvious pattern. For example, in the list above we notice that the square of each $f_k$ differs by 1 from the product of its neighbours — *eg.* $8^2$ and $5 \times 13$. Indeed, induction as in Fig. 1.1 quickly establishes the '*Cassini formula*' [2] for $k \geq 1$:

$$f_{k+1}f_{k-1} - f_k^2 = (-1)^k. \tag{1.2}$$

It turns out that this result is only one of a great profusion of Fibonacci



Figure 1.1: the induction step, using Fibonacci recurrence.

properties [3, 4, 5] most of which seem less immediate to discover. But in fact many emerge elegantly and easily from a matrix formulation, as follows.

Start by writing eq. (1.1) as

$$\begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} = Q \begin{pmatrix} f_k \\ f_{k-1} \end{pmatrix} \qquad \text{where} \qquad Q \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

So with $f_k = 0, 1, 1$ for $k = 0, 1, 2$ we have

$$\begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} = Q^{k-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} f_k \\ f_{k-1} \end{pmatrix} = Q^{k-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and these together give [6]

$$\begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} = Q^k \qquad \text{for} \qquad k = 1, 2, 3, \ldots \qquad (1.3)$$

Eq. (1.3) embodies the recurrence rule eq. (1.1) with the initial values $(0, 1)$ while making explicit the 2-dimensional linear context.

Then for example remembering that $\det(Q^k) = (\det Q)^k$, the observation that $\det Q = -1$ at once gives Cassini eq. (1.2) for $k = 1, 2, 3, \ldots$

$$\det(Q^k) = (-1)^k = \begin{vmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{vmatrix} = f_{k+1}f_{k-1} - f_k^2. \qquad (1.4)$$

It turns out that similar standard matrix properties lead to corresponding Fibonacci results. Also, generalisations become natural.

Chap. 2 is about Fibonacci numbers and Chap. 3 deals with Lucas and related numbers. Chap. 4 extends to tribonacci and higher recurrences, where a $3 \times 3$ or larger matrix replaces $Q$.

Chap. 5 covers some aspects of Fibonacci, Lucas, *etc* modulo $m$.

Appendix A summarises results from the matrix formulation of a general 2-term recurrence, and Appendix B illustrates how some additional Fibonacci and Lucas formulas emerge.

# Chapter 2

# Fibonacci

## 2.1 Basic results

To extend the Fibonacci sequence to all $k = 0, \pm 1, \pm 2, \ldots$ use

$$Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{and} \qquad Q^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with the normal index rules. Then $\det Q = -1$ plus eq. (1.3) give

$$Q^{-k} = (Q^k)^{-1} \qquad \text{as} \qquad \begin{pmatrix} f_{-k+1} & f_{-k} \\ f_{-k} & f_{-k-1} \end{pmatrix} = \frac{1}{(-1)^k} \begin{pmatrix} f_{k-1} & -f_k \\ -f_k & f_{k+1} \end{pmatrix}$$

and so

$$f_{-k} = (-1)^{k+1} f_k. \tag{2.1}$$

**Ex:** check that therefore Cassini eq. (1.2) holds for negative $k$ too.

Two basic addition formulas follow easily.

First, the 21 matrix entry[1] of $\quad Q^{k+l} = Q^k Q^l \quad$ gives for all $k$, $l$

$$f_{k+l} = f_k f_{l+1} + f_{k-1} f_l, \tag{2.2}$$

where eq. (1.1) is the case $l = 1$, since $f_2 = f_1 = 1$.

Then with help from eq. (1.1) the 21 entry of $\quad Q^{j+k+l} = Q^j Q^k Q^l \quad$ gives[2]

$$f_{j+k+l} = f_{j+1} f_{k+1} f_{l+1} + f_j f_k f_l - f_{j-1} f_{k-1} f_{l-1} \qquad \text{for all } j, k, l. \tag{2.3}$$

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[1]Other entries give the same information with indices re-labelled.

[2]Compare this direct construction with the lengthy inductive manouevres in ref. [7].

## 2.2 Bilinear index-reduction formula

If $a + b = c + d$ then $\quad Q^a Q^b = Q^c Q^d \quad$ where the 22 matrix elements read

$$f_a f_b + f_{a-1} f_{b-1} = f_c f_d + f_{c-1} f_{d-1}.$$

This re-arranges to

$$f_a f_b - f_c f_d = (-1)(f_{a-1} f_{b-1} - f_{c-1} f_{d-1}),$$

when iteration gives[3]

$$f_a f_b - f_c f_d = (-1)^r (f_{a-r} f_{b-r} - f_{c-r} f_{d-r}), \qquad (2.4)$$

which holds for $a, b, c, d, r = 0, \pm 1, \pm 2, \ldots$, given $a + b = c + d$.

This '*index-reduction formula*' (IRF) [9] at once provides a framework for standard bilinear Fibonacci identities. Each is a particular case of either this formula or its generalisation eq. (3.23) below, given $(f_0, f_1) = (0, 1)$ plus in some cases the $k \rightarrow -k$ symmetry eq. (2.1).

For instance Cassini eq. (1.2) follows from choosing

$$a = k + 1, \; b = k - 1, \; c = d = r = k, \qquad \text{given} \qquad f_1 = f_{-1} = 1, \; f_0 = 0.$$

Likewise the addition result of eq. (2.2) is just the case where

$$(a, \, b, \, c, \, d, \, r) \rightarrow (k, \, -l - 1, \, k - 1, \, -l, \, -l),$$

and even the fundamental recurrence itself eq. (1.1) can be recovered with eg.

$$(a, \, b, \, c, \, d, \, r) \rightarrow (k + 1, \, 1, \, k, \, 2, \, 2).$$

Appendix B includes a few more examples of derived identities.

**Ex:** put $\quad (a, \, b, \, c, \, d, \, r) = (n + k, \, n + k, \, 2n, \, 2k, \, 2k) \quad$ to get a difference-of-squares formula [10]; replace $(c, \, d, \, r)$ with $(2n + 1, \, 2k - 1, \, 2k - 1)$ to get his similar sum-of-squares result. Establish the rest of the formulas in ref. [10] from eq. (2.4) or eq. (3.10) below.

For others see ref. [4, Chap.3 and pps. *176–84*] as well as ref. [5, pps. *87–93*].

**Ex:** if $a + b + c = d + e + f$ then what follows from $Q^a Q^b Q^c = Q^d Q^e Q^f$?

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[3]A much longer inductive proof of essentially this result is in ref. [8].

## 2.3   Eigenvalues and eigenvectors

Symmetric matrix $Q$ has real eigenvalues $\lambda_\pm$ where

$$\det Q = \lambda_+ \lambda_- = -1 \qquad \text{and} \qquad \operatorname{tr} Q = \lambda_+ + \lambda_- = 1. \tag{2.5}$$

So its characteristic equation   $\lambda^2 - \lambda \operatorname{tr} Q + \det Q = 0$   is

$$\lambda^2 = \lambda + 1 \tag{2.6}$$

with roots   $\lambda_\pm = (1 \pm \sqrt{5})/2 \approx 1.618,\ -0.618$.

The orthogonal eigenvectors $(y,\ x)^T$ of $Q$ have $y = \lambda_\pm x$ and the rotation that diagonalises $Q$:

$$Q = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \tag{2.7}$$

has   $s = \{\lambda_+(\lambda_+ - \lambda_-)\}^{-1/2}$   and   $c = \{\lambda_-(\lambda_- - \lambda_+)\}^{-1/2} = \lambda_+ s$.

Rotation in the $(x,\ y)$-plane through angle[4] $\gamma = \arctan(1/\lambda_+) \approx 31.72°$ brings the eigen-line $y = \lambda_+ x$ to the $y$-axis, and $y = \lambda_- x$ to the $x$-axis.

From eq. (2.7) to the $k$-th power and eq. (1.3), read off

$$f_k = \frac{\lambda_+^k - \lambda_-^k}{\lambda_+ - \lambda_-} \tag{2.8}$$

which is the '*Binet formula*' [12] since $\lambda_+ - \lambda_- = \sqrt{5}$.

Eq. (2.8) provides one prescription [13] for $f_k \to f(k)$ with non-integer $k$, as well as the estimate   $f_k \approx (\lambda_+)^k / \sqrt{5}$   for large positive $k$.

The power method [11] gives the leading eigenvalue of a matrix from the limit of the ratio of corresponding elements of successive powers. Applied to $Q$ via eq. (1.3), we have

$$\lambda_+ = (f_{k+1}/f_k)_{k \to \infty}, \tag{2.9}$$

consistent with Binet eq. (2.8).

In addition, however, from Cassini eq. (1.2) we have for $k \geq 2$:

$$f_{k+1}/f_k - f_k/f_{k-1} = (-1)^k / f_k f_{k-1}, \tag{2.10}$$

so that with the limit in eq. (2.9) an infinite sum over $k$ telescopes to give[5]

$$\lambda_+ = 1 + \sum_2^\infty \frac{(-1)^k}{f_k f_{k-1}}. \tag{2.11}$$

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[4] Note $\tan 2\gamma = 2$, and $4\gamma$ is the external angle opposite the '4' side of a 3–4–5 triangle.

[5] An alternative right-hand side is $1 + \sum_1^\infty (f_{2k+1} f_{2k-1})^{-1}$, using eq. (2.4) with $a = k+2$, $b = k - 2$, $c = d = r = k$ plus $f_3 = 2$ and $\lambda_+^2 = \lambda_+ + 1$.

This is formula 102 of ref. [4, p. *103*]. Truncation error for such an alternating sum is bounded by the magnitude of the first omitted term, and we note that *eg.* $(f_{10}f_9)^{-1} \approx 0.0005$.

To see the similar approach to the limit in eq. (2.9), write the characteristic equation eq. (2.6) as $\quad \lambda = 1 + 1/\lambda \quad$ when $\quad \lambda > 0 \quad$ gives successively

$$\lambda > 1, \quad \lambda < 2, \quad \lambda > \tfrac{3}{2}, \quad \lambda < \tfrac{5}{3}, \quad \lambda > \tfrac{8}{5}, \quad \ldots.$$

A simple induction then leads to:

$$\frac{f_{k+1}}{f_k} > \lambda_+ > \frac{f_k}{f_{k-1}} \qquad \text{for} \qquad k = 2, 4, 6, \ldots, \qquad (2.12)$$

where from eq. (2.10), upper and lower bounds differ by $(f_k f_{k-1})^{-1}$.

Evidently there is a rapidly-narrowing corridor in the positive quadrant of the $(x, y)$-plane of integer-coordinate points $(x, y) = (f_k, f_{k+1})$ lying alternately on opposite sides of the eigen-line $y = \lambda_+ x$ of $Q$.

**Ex:** draw the corresponding diagram.

The sequence of rational estimates $\lambda_+ \approx f_{k+1}/f_k$ is directly seen to be optimal: that is[6], if $m$ and $n$ are positive integers with $m \leq f_k$ and $n \leq f_{k+1}$ then the ratio $n/m$ is closest to $\lambda_+$ when $m = f_k$ and $n = f_{k+1}$.

## 2.4 Golden ratio

The leading eigenvalue $\lambda_+$ of $Q$ has wider significance, as follows.

Ancient philosophers asserted [14, 15] that the most pleasing division of a line is such that the ratio of its length $a + b$ to its larger part $a$ is equal to the ratio of the larger to the smaller part. That is, $\quad (a + b)/a = a/b$.

This aesthetic whimsy was called '*division in divine proportion*' or in '*mean and extreme ratio*'. A more recent term is '*golden section*', and the number $\tau = a/b > 1$ is called the '*golden ratio*'.

From its definition, the golden ratio obeys $\quad \tau^2 = \tau + 1$, which coincides with the characteristic equation of $Q$, eq. (2.6). So we conclude that $\tau \equiv \lambda_+$.

Then Binet eq. (2.8) relates Fibonacci numbers to the golden ratio:

$$f_k = (\tau^k - (-\tau)^{-k})/\sqrt{5},$$

while the large-$k$ limit in eq. (2.9), the sum rule in eq. (2.11) and the bounds in eq. (2.12) are inverse relations.

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[6] It's manifest for $k = 1, 2, 3$ — then by contradiction using Cassini eq. (1.2).

## 2.5   Generating functions

The matrix generating function is just the expansion

$$(I - \xi Q)^{-1} = \sum_k \xi^k Q^k, \qquad \text{where} \qquad \det(I - \xi Q) = 1 - \xi - \xi^2 \quad (2.13)$$

(*cf.* the characteristic equation eq. (2.6)). The 21 and 12 entries give at once the Fibonacci generating function

$$f_0 + \xi f_1 + \xi^2 f_2 + \cdots = \frac{\xi}{1 - \xi - \xi^2} \qquad \text{for} \qquad |\xi| < 1/\lambda_+ \approx 0.618. \quad (2.14)$$

Equating coefficients of $\xi^k$ gives for $k \geq 0$ an explicit solution of eq. (1.1):

$$f_{k+1} = \sum_{r=0}^{[k/2]} \binom{k-r}{r}, \tag{2.15}$$

which sums along a shallow diagonal of Pascal's triangle.

From $(I - \xi Q^k)^{-1}$ likewise we have

$$f_k + \xi f_{2k} + \xi^2 f_{3k} + \cdots = \frac{f_k}{1 - \xi \ell_k + \xi^2 (-1)^k}, \tag{2.16}$$

using Cassini eq. (1.2) and where

$$\ell_k \stackrel{\text{def}}{=} \mathrm{tr}(Q^k) = f_{k+1} + f_{k-1}. \tag{2.17}$$

These turn out to be Lucas numbers — Sec. 3 below.

The series of eq. (2.16) converges for $|\xi| < \lambda_+^{-k}$. For instance, with $k = 2$:

$$f_2 + \xi f_4 + \xi^2 f_6 + \cdots = \frac{1}{1 - 3\xi + \xi^2} \qquad \text{for} \qquad |\xi| < 1/\lambda_+^2 \approx 0.382.$$

Equating coefficients of $\xi^n$ in eq. (2.16) gives the explicit factorisation

$$f_{nk} = f_k \sum_{r=0}^{[(n-1)/2]} \binom{n-r-1}{r} (\ell_k)^{n-2r-1} (-1)^{(k+1)r}. \tag{2.18}$$

## 2.6   The GCD theorem

The Binet formula eq. (2.8) shows that if $a|b$ then $f_a|f_b$, while eq. (2.18) gives an expression for the quotient. So if $m$ and $n$ have a common factor $k$, then $f_m$ and $f_n$ have the common factor $f_k$.

In fact we have that
$$(f_m, \, f_n) = f_{(m, \, n)} \tag{2.19}$$
where the parentheses denote greatest common divisor.

To prove this, suppose $m > n \geq 1$. Then $\quad m = qn + r \quad$ where $q \geq 1$ and $n > r \geq 0$. Now the key to the familiar Euclid algorithm [16] is the



Figure 2.1: the essence of Euclid's algorithm.

observation that
$$(m, \, n) = (n, \, r) \tag{2.20}$$
as in Fig. 2.1 — and the proof of eq. (2.19) relies on the parallel result:
$$(f_m, \, f_n) = (f_n, \, f_r) \, . \tag{2.21}$$
This follows because the addition formula eq. (2.2) gives
$$(f_m, \, f_n) = (f_{qn+1} f_r + f_{qn} f_{r-1}, \, f_n)$$
and then besides $\quad f_n | f_{qn} \quad$ we have also $\quad (f_n, \, f_{qn+1}) = 1$ — which comes at once from the basic recurrence eq. (1.1) for otherwise (absurdly) all Fibonacci numbers have a common factor[7].

Then as eq. (2.20) gives the descent to $(m, \, n)$, eq. (2.21) leads to $f_{(m, \, n)}$.

**Ex:** what about Lucas numbers? (See Sec. 3.)

## 2.7 A surprising sum

From the standard matrix identity[8] $\quad \exp(\operatorname{tr} A) = \det(\exp A) \quad$ with
$$\sum_{r=1}^{\infty} \frac{1}{r} A^r = -\ln(I - A) \qquad \text{if} \qquad ||A|| < 1$$

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[7]Indeed any three successive Fibonacci numbers are co-prime. What about Lucas, *etc*?
[8]Recall tr and det in terms of sum and product of eigenvalues.

we have

$$\exp\left(\sum_1^\infty \frac{1}{r} \operatorname{tr}(A^r)\right) = \frac{1}{\det(I-A)}. \tag{2.22}$$

Then putting $A = \xi Q^k$ for $k \neq 0$, and with the definition of $\ell_k$ eq. (2.17) plus the generating function eq. (2.16), we find

$$\exp\left(\sum_1^\infty \frac{\xi^r}{r} \ell_{rk}\right) = \frac{1}{1 - \xi\ell_k + \xi^2(-1)^k} = \frac{1}{f_k}(f_k + \xi f_{2k} + \xi^2 f_{3k} + \cdots)$$

for $|\xi| < (\lambda_+)^{-k}$. Notice that as $f_k \mid f_{rk}$, the coefficients on the right-hand side are integers!

This surprising formula comes naturally in this matrix context. And it generalises the $k = 1$ case plucked from the air in *eg.* ref. [17, Chap. 8, Sec. 11].

## 2.8 Finite sums

The identity

$$I + A + A^2 + \cdots + A^n = (I - A^{n+1})(I - A)^{-1}$$

with $A = \xi Q^k$ gives

$$f_k + \xi f_{2k} + \cdots + \xi^{n-1} f_{nk} = \frac{f_k - \xi^n f_{(n+1)k} + \xi^{n+1}(-1)^k f_{nk}}{1 - \xi\ell_k + \xi^2(-1)^k}, \tag{2.23}$$

where the IRF eq. (2.4) is used to simplify the coefficient of $\xi^{n+1}$.

**Ex:** check that $n \to \infty$ gives eq. (2.16) if $|\xi| < \lambda_+^{-k-1}$.

## 2.9 Binomial sums

Cayley-Hamilton implies that matrix powers are not independent — *ie,* because $Q^2 = Q + I$ we must have $Q^k = a_k Q + b_k I$ for some $(a_k, b_k)$.

Indeed, replacing the 11 element of eq. (1.3) with the recurrence eq. (1.1) gives by inspection

$$Q^k = f_k Q + f_{k-1} I \tag{2.24}$$

for all $k = 0, \pm 1, \pm 2, \ldots$. Therefore

$$Q^{kn} = (f_k Q + f_{k-1} I)^n = \sum_{r=0}^n \binom{n}{r} (f_k)^r (f_{k-1})^{n-r} Q^r, \tag{2.25}$$

with 21 element

$$f_{kn} = \sum_{r=0}^{n} \binom{n}{r} (f_k)^r (f_{k-1})^{n-r} f_r. \tag{2.26}$$

The cases $k = 2$ and $k = 3$ respectively of course give the standard results [17, Chap. 8, Sec. 6]

$$f_{2n} = \sum_{1}^{n} \binom{n}{r} f_r \qquad \text{and} \qquad f_{3n} = \sum_{1}^{n} \binom{n}{r} 2^r f_r,$$

remembering that $f_0 = 0$.

**Ex:** write out the cases $k = 4$ and $k = 5$.

Eq. (2.24) with the $k \to -k$ symmetry eq. (2.1) is

$$Q^{-k} = (-1)^{k+1} (f_k Q - f_{k+1} I),$$

and this leads likewise to

$$Q^{-kn} = (-1)^{nk} \sum_{r=0}^{n} \binom{n}{r} (f_k)^r (f_{k+1})^{n-r} (-1)^r Q^r.$$

So (using eq. (2.1) again) we have in addition

$$f_{kn} = \sum_{r=0}^{n} \binom{n}{r} (f_k)^r (f_{k+1})^{n-r} (-1)^{r+1} f_r. \tag{2.27}$$

With $k = 2$ for instance this gives the superficially strange result:

$$f_{2n} = -2^n \sum_{1}^{n} \binom{n}{r} (-\tfrac{1}{2})^r f_r.$$

**Ex:** write out the cases $k = 3$ and $k = 4$.

Contrast these simple and general matrix constructions with the contrivance of just the $k = 3$ case of eq. (2.26) in ref. [17, Chap. 8, Sec. 6].

- The sums in eq. (2.26) and eq. (2.27) effectively start at $r = 1$ because $f_0 = 0$, and then the factor $f_k$ of $f_{kn}$ is explicit once more.

- A small generalisation — multiply eq. (2.25) by $Q^p$ to give in eq. (2.26)

$$(f_{kn}, \ f_r) \to (f_{kn+p}, \ f_{r+p}).$$

With $Q^{-p}$ for eq. (2.27): $(f_{kn}, (-1)^{r+1} f_r) \to (f_{kn+p}, (-1)^{r+p+1} f_{r-p})$. The sums in Sec. 2.5 and Sec. 2.8 generalise likewise [5, pps. 85–7].

# Chapter 3

# Lucas *etc*

By definition, the numbers $\ell_k = \text{tr}(Q^k)$ of eq. (2.17) are related to Fibonacci numbers by

$$\ell_k = f_{k+1} + f_{k-1}. \tag{3.1}$$

They are '*Lucas numbers*' [1, *A000032*] $\ell_k = 2, 1, 3, 4, 7, 11, 18, 29\ldots$.

Evidently Lucas numbers too obey $\quad \ell_{k+1} = \ell_k + \ell_{k-1} \quad$ and, with the Fibonacci $k \to -k$ symmetry eq. (2.1), they are defined for $k < 0$ by

$$\ell_{-k} = (-1)^k \ell_k. \tag{3.2}$$

Since $\ell_0 = 2$ and $\ell_1 = 1$, the matrix relation corresponding to eq. (1.3) — and likewise valid for all integers $k$ — is

$$\begin{pmatrix} \ell_{k+1} & \ell_k \\ \ell_k & \ell_{k-1} \end{pmatrix} = Q^k \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}. \tag{3.3}$$

Then for example multiplying matrices as for eq. (2.2):

$$\ell_{k+l} = f_k \ell_{l+1} + f_{k-1} \ell_l \qquad \text{for all } k \text{ and } l, \tag{3.4}$$

which for $k = 2$ recovers the basic recurrence, and for $l = 0$ gives eq. (3.1).

Inversion of eq. (3.3):

$$Q^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} \ell_{k+1} & \ell_k \\ \ell_k & \ell_{k-1} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \tag{3.5}$$

gives the inverse of eq. (3.1) as

$$5f_k = \ell_{k+1} + \ell_{k-1}. \tag{3.6}$$

## 3.1  Index-reduction formulas

An IRF parallel to eq. (2.4) is

$$f_a\ell_b - f_c\ell_d = (-1)^r(f_{a-r}\ell_{b-r} - f_{c-r}\ell_{d-r}) \tag{3.7}$$

for $a + b = c + d$. This comes either from $Q^aQ^b = Q^cQ^d$ applied to $(1,\, 2)^T$ or from eq. (2.4) itself by addition using the definition eq. (3.1).

**Ex:** choose values of $(a,\, b,\, c,\, d,\, r)$ to get    $f_{n+1}\ell_{n-1} - f_n\ell_n = (-1)^{n+1}$

— and what if you interchange $f$ and $\ell$?

Likewise there is an IRF with $f \to \ell$ by adding again. In fact these are just $f\ell$ and $\ell\ell$ versions of the general IRF eq. (3.23) below.

**Ex:** from both the $\ell\ell$ IRF and from eq. (3.3) prove that

$$\ell_{k+1}\ell_{k-1} - \ell_k^2 = 5(-1)^{k+1} \tag{3.8}$$

— the Lucas version of Cassini.

For a different bilinear IRF, observe that the matrix of eq. (3.3)

$$X \overset{\text{def}}{=} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

obeys:

$$X^{-1} = \tfrac{1}{5}X \qquad \text{and} \qquad QX = XQ. \tag{3.9}$$

Then if $a + b = c + d$, we have not only $Q^aQ^b = Q^cQ^d$ but also

$$5Q^aQ^b = Q^cXQ^dX,$$

which gives the mixed result [10, formula D] for $r = 0, \pm1, \pm2, \ldots$

$$5f_af_b - \ell_c\ell_d = (-1)^r(5f_{a-r}f_{b-r} - \ell_{c-r}\ell_{d-r}). \tag{3.10}$$

**Ex:** put    $(a,\, b,\, c,\, d,\, r) = (n,\, n,\, n,\, n,\, n)$   to get    $\ell_n^2 - 5f_n^2 = 4(-1)^n$.

This identity is central to analysis of a variant Pell equation [18] as in ref. [17, Chap. 8, Sec. 11]. See also Sec. A.2.1 in Appendix A.

## 3.2 Sums

Operating on $(1, 2)^T$ with $(I - \xi Q)^{-1}$ gives the Lucas generating function

$$\ell_0 + \xi \ell_1 + \xi^2 \ell_2 + \cdots = \frac{2 - \xi}{1 - \xi - \xi^2}, \tag{3.11}$$

while the analogue of eq. (2.23) reads

$$\ell_0 + \xi \ell_k + \cdots + \xi^n \ell_{nk} = \frac{2 - \xi \ell_k - \xi^{n+1} \ell_{(n+1)k} + \xi^{n+2}(-1)^k \ell_{nk}}{1 - \xi \ell_k + \xi^2(-1)^k}, \tag{3.12}$$

where $n \to \infty$ for $|\xi| < \lambda_+^{-k}$ leaves numerator $2 - \xi \ell_k$ (compare eq. (2.16)).

The Fibonacci binomial sums of Sec. 2.9 have straightforward Lucas analogues — *eg.* take the tr of eq. (2.25) and use eq. (2.17) to get a version of eq. (2.26) with $(f_{kn}, f_r) \to (\ell_{kn}, \ell_r)$. Likewise for eq. (2.27), with also $(-1)^{r+1} \to (-1)^r$.

## 3.3 Lucas and Binet

Read off from the trace definition of $\ell_k$ eq. (2.17) the Binet formula for Lucas numbers:

$$\ell_k = \lambda_+^k + \lambda_-^k. \tag{3.13}$$

Then as for Fibonacci eq. (2.9) we have

$$\lambda_+ = (\ell_{k+1}/\ell_k)_{k \to \infty}. \tag{3.14}$$

Such a limit clearly depends only on the $Q$-matrix — *ie,* the recurrence formula — and not on the values of the first two sequence-members.

**Ex:** Divide Lucas-Cassini eq. (3.8) by $\ell_k \ell_{k-1}$, sum over $k$ and use the limit in eq. (3.14) to derive another sum rule for the golden ratio:

$$\tau = \tfrac{1}{2} + 5 \sum_1^\infty \frac{(-1)^{k+1}}{\ell_k \ell_{k-1}}.$$

Comparing Lucas-Binet eq. (3.13) with Binet for Fibonacci numbers eq. (2.8) we have at once

$$\ell_k = f_{2k}/f_k. \tag{3.15}$$

This generalises —

$$\begin{aligned}
\frac{f_{nk}}{f_k} &= \frac{\lambda_+^{nk} - \lambda_-^{nk}}{\lambda_+^k - \lambda_-^k} \\
&= \lambda_+^{(n-1)k} + \lambda_+^{(n-2)k}\lambda_-^k + \lambda_+^{(n-3)k}\lambda_-^{2k} + \cdots \\
&\qquad \cdots + \lambda_+^{2k}\lambda_-^{(n-3)k} + \lambda_+^k\lambda_-^{(n-2)k} + \lambda_-^{(n-1)k} \\
&= \ell_{(n-1)k} + (-1)^k \ell_{(n-3)k} + \cdots,
\end{aligned}$$

including only Lucas numbers of non-negative index in the last line.

The connection to the factorisation result eq. (2.18) coming from the Fibonacci generating function in Sec. 2.5 is via $\quad (\ell_k)^p = (\lambda_+^k + \lambda_-^k)^p$.

## 3.4   General initial values

Consider [19] $\quad g_{k+1} = g_k + g_{k-1} \quad$ for general $(g_0, g_1)$.

Then

$$\begin{pmatrix} g_{k+1} & g_k \\ g_k & g_{k-1} \end{pmatrix} = Q^k \begin{pmatrix} g_1 & g_0 \\ g_0 & g_1 - g_0 \end{pmatrix}, \tag{3.16}$$

which generalises Cassini as

$$g_{k+1}g_{k-1} - g_k^2 = (-1)^k(g_1^2 - g_0 g_1 - g_0^2). \tag{3.17}$$

**Ex:** From eq. (3.17) prove that generally

$$\tau = \frac{g_1}{g_0} + (g_1^2 - g_0 g_1 - g_0^2) \sum_1^\infty \frac{(-1)^k}{g_k g_{k-1}}.$$

Deal separately with any case of a zero denominator.

The addition formulas eq. (2.2) and eq. (3.4) generalise to

$$g_{k+l} = f_k g_{l+1} + f_{k-1} g_l, \tag{3.18}$$

while eq. (2.3) becomes

$$g_{j+k+l} = f_{j+1}f_{k+1}g_{l+1} + f_j f_k g_l - f_{j-1}f_{k-1}g_{l-1}.$$

Operating on $(g_1, g_0)^T$ with $(I - \xi Q)^{-1}$ gives the generating function

$$g_0 + \xi g_1 + \xi^2 g_2 + \cdots = \frac{g_0 + (g_1 - g_0)\xi}{1 - \xi - \xi^2}. \tag{3.19}$$

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Comparing with the Fibonacci generating function eq. (2.14):

$$g_k = g_0 f_{k+1} + (g_1 - g_0) f_k = g_0 f_{k-1} + g_1 f_k, \qquad (3.20)$$

which is the 21 entry of the matrix expression eq. (3.16) and a special case of the addition formula eq. (3.18).

From this, with eq. (2.8) and eq. (3.13), comes the Binet generalisation

$$g_k = \left( g_0 (\lambda_+^{k-1} - \lambda_-^{k-1}) + g_1 (\lambda_+^k - \lambda_-^k) \right) / (\lambda_+ - \lambda_-),$$

which of course leads to $\lambda_+ = (g_{k+1}/g_k)_{k\to\infty}$.

Operating with powers of $Q$ on $(g_1, g_0)^T$ easily gives generalisations of *eg.* the finite sum eq. (2.23) and the binomial sum eq. (2.26), but eq. (2.27) needs also symmetry under $k \to -k$.

Changing $k$ to $-k$ in the matrix expression eq. (3.16) and proceeding as in Sec. 2.1 gives

$$g_{-k} = (-1)^k (g_0 f_{k+1} - g_1 f_k), \qquad (3.21)$$

consistent with eq. (3.20) plus the Fibonacci symmetry eq. (2.1). Lucas has $(g_0, g_1) = (2, 1)$, recovering its symmetry eq. (3.2).

Fibonacci and Lucas are the two choices of $(g_0, g_1)$ (up to a factor) that lead to simple $k \to -k$ symmetry as in eq. (2.1) and eq. (3.2). This singles out the Fibonacci-Lucas basis:

$$g_k = (g_1 - \tfrac{1}{2} g_0) f_k + \tfrac{1}{2} g_0 \ell_k. \qquad (3.22)$$

In this basis the generalisation of the IRFs eq. (2.4) and eq. (3.7) follows by addition

$$g_a h_b - g_c h_d = (-1)^r (g_{a-r} h_{b-r} - g_{c-r} h_{d-r}) \qquad (3.23)$$

where $a, b, c, d, r = 0, \pm 1, \pm 2, \dots$ and $a + b = c + d$. This holds for sequences $\{g_k\}$ and $\{h_k\}$ that are different or the same, but obey the Fibonacci recurrence rule embodied in the matrix expression eq. (3.16).

Eq. (3.23) is the case $a_1 = a_2 = 1$ of eq. (A.10) in Sec. A.2 below, where it is established *ab initio*.

Compare the simple derivations of eq. (3.23) and its generalisation eq. (A.10) with the painful manipulations in ref. [4, pps. *27–8*] to obtain just the special case listed as formula 18 [4, p. *177*].

## 3.5 Other matrices

Besides $Q$, are there any other $2 \times 2$ matrices (say $K$) whose powers $K^n$ have entries involving Fibonacci numbers?

If there are, then (Binet) they must have eigenvalues $\lambda_\pm = (1 \pm \sqrt{5})/2$ and so the characteristic equation $\quad \lambda^2 = \lambda + 1$. Equivalently,

$$\operatorname{tr} K = \lambda_+ + \lambda_- = 1 \qquad \text{and} \qquad \det K = \lambda_+ \lambda_- = -1.$$

Therefore if

$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $a + d = 1$ and $ad - bc = -1$ ... and hence

$$K = \begin{pmatrix} a & b \\ (1 + a - a^2)/b & 1 - a \end{pmatrix}$$

in terms of two suitable numbers $a$, $b$.

To calculate $K^n$, use Cayley-Hamilton. That is, matrix $K$ obeys its characteristic equation $\quad K^2 = K + I \quad$ where $I$ is the $2 \times 2$ unit.

So by induction (exercise! — and compare eq. (2.24))

$$K^n = f_n K + f_{n-1} I$$

... leading to

$$K^n = \begin{pmatrix} a f_n + f_{n-1} & b f_n \\ (1 + a - a^2) f_n / b & (1 - a) f_n + f_{n-1} \end{pmatrix}. \tag{3.24}$$

The matrix $Q$ has $a = b = 1$.

Choosing $(a, b) = \left(\frac{1}{2}, \frac{5}{2}\right)$ gives the case mentioned by Demirtürk [20] —

$$\begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^n = \begin{pmatrix} \frac{1}{2}\ell_n & \frac{5}{2}f_n \\ \frac{1}{2}f_n & \frac{1}{2}\ell_n \end{pmatrix}, \tag{3.25}$$

with Lucas numbers $\ell_n \equiv f_{n+1} + f_{n-1} = f_n + 2 f_{n-1}$.

Other examples include $eg$.

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}^n = \begin{pmatrix} \frac{1}{2}f_{n-3} & \frac{1}{2}f_n \\ \frac{1}{2}f_n & \frac{1}{2}f_{n+3} \end{pmatrix} \tag{3.26}$$

and

$$\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}^n = \begin{pmatrix} f_{n+2} & f_n \\ -f_n & -f_{n-2} \end{pmatrix} \tag{3.27}$$

and so on.

Taking determinants in eq. (3.24) gives Cassini eq. (1.2) for all $(a, b)$. Then doing likewise in eqs. (3.25)–(3.27) gives variants. Note that eq. (3.25) leads to the important identity in the example immediately following eq. (3.10).

# Chapter 4

# Tribonacci *etc*

Here $\quad t_k = 0, 0, 1, 1, 2, 4, 7, 13, 24, \ldots \quad$ obey [1, A000073]

$$t_{k+1} = t_k + t_{k-1} + t_{k-2} \qquad \text{where} \qquad t_0 = t_1 = 0, \ t_2 = 1.$$

Introducing

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad M^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \qquad (4.1)$$

gives

$$M^k = \begin{pmatrix} t_{k+2} & t_{k+1} + t_k & t_{k+1} \\ t_{k+1} & t_k + t_{k-1} & t_k \\ t_k & t_{k-1} + t_{k-2} & t_{k-1} \end{pmatrix} \qquad \text{for} \qquad k = 0, \pm 1, \pm 2, \ldots. \quad (4.2)$$

Using $M^{-k} = (M^k)^{-1}$ as for $Q$ above:

$$t_{-k} = \begin{vmatrix} t_{k+1} & t_{k+2} \\ t_k & t_{k+1} \end{vmatrix} \qquad (4.3)$$

— rather less simple than for Fibonacci eq. (2.1).

From $\det M = 1$ read off

$$\begin{vmatrix} t_k & t_{k+1} & t_{k+2} \\ t_{k-1} & t_k & t_{k+1} \\ t_{k-2} & t_{k-1} & t_k \end{vmatrix} = 1, \qquad (4.4)$$

corresponding to Cassini eq. (1.2). Like eq. (2.2), from $M^{k+l} = M^k M^l$ there is the addition rule:

$$t_{k+l} = t_{k+1} t_{l+1} + t_k t_l + t_{k-1} t_l + t_k t_{l-1},$$

and there are further multilinear relations as for Fibonacci.

**Ex:** if $a + b = c + d$ then what follows from $M^a M^b = M^c M^d$?

The matrix generating function $(I - \xi M)^{-1}$ gives

$$t_0 + \xi t_1 + \xi^2 t_2 + \cdots = \frac{\xi^2}{1 - \xi - \xi^2 - \xi^3}.$$

Equating coefficients of powers of $\xi$ gives explicitly for $k \geq 0$

$$t_{k+2} = \sum_{p=0}^{[k/2]} \sum_{q=0}^{q_{max}} \binom{k-p-q}{p} \binom{p}{q}$$

where $q_{max} = \min(p, k - 2p)$. This is the analogue of the Fibonacci formula eq. (2.15) but has no straightforward interpretation in terms of Pascal's triangle.

## 4.1   Trucas

Introduce $u_k \overset{\text{def}}{=} \text{tr}(M^k)$ with $M$ as in eq. (4.1), and so from eq. (4.2)

$$u_k = t_{k+1} + 2t_k + 3t_{k-1}. \tag{4.5}$$

From the trace definition:

$$u_k = \lambda_1^k + \lambda_2^k + \lambda_3^k, \tag{4.6}$$

where $\lambda_{1,2,3}$ are the eigenvalues of $M$.

Writing the characteristic equation $\lambda^3 = \lambda^2 + \lambda + 1$ as

$$\lambda^3 = (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\lambda + \lambda_1\lambda_2\lambda_3$$

read off from eq. (4.6) with $k = 0, 1, 2$ that $u_0 = 3$, $u_1 = 1$ and

$$u_2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 - 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) = 1 - 2(-1) = 3.$$

So the Trucas numbers are [1, A001644]

$$\ldots 23, 3, -15, 11, -1, -5, 5, -1, -1, \mathbf{3}, \mathbf{1}, \mathbf{3}, 7, 11, 21, 39, 71, 131, 241, \ldots$$

where for $k = 0, \pm 1, \pm 2, \ldots$

$$\begin{pmatrix} u_{k+2} & u_{k+1} + u_k & u_{k+1} \\ u_{k+1} & u_k + u_{k-1} & u_k \\ u_k & u_{k-1} + u_{k-2} & u_{k-1} \end{pmatrix} = M^k \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{pmatrix}.$$

Taking determinants, the equivalent of 'Cassini' eq. (4.4) is

$$\begin{vmatrix} u_k & u_{k+1} & u_{k+2} \\ u_{k-1} & u_k & u_{k+1} \\ u_{k-2} & u_{k-1} & u_k \end{vmatrix} = 44.$$

The inverse is

$$22M^k = \begin{pmatrix} u_{k+2} & u_{k+1} + u_k & u_{k+1} \\ u_{k+1} & u_k + u_{k-1} & u_k \\ u_k & u_{k-1} + u_{k-2} & u_{k-1} \end{pmatrix} \begin{pmatrix} 2 & 1 & 5 \\ 5 & -3 & 4 \\ -4 & 9 & 1 \end{pmatrix},$$

whose 31 entry gives an inverse of eq. (4.5):

$$t_k = \tfrac{1}{22}(2u_k + u_{k-1} + 5u_{k-2}). \tag{4.7}$$

An alternative with successive $u$'s is $t_k = \tfrac{1}{22}(5u_{k+1} - 3u_k - 4u_{k-1})$ but eq. (4.7) involves only positive coefficients.

   With this equation and its inverse eq. (4.5), plus the result for $t_{-k}$ eq. (4.3), a complicated expression for $u_{-k}$ in terms of $u_k$ *etc* is possible. But the simplicity of Fibonacci's eq. (2.1) and Lucas' eq. (3.2) — and their 2-dimensional generalisations eq. (A.3) and eq. (A.18) — is gone in higher dimensions. Any naturalness of Trucas is eq. (4.6), from which

$$u_{-k} = (\lambda_1\lambda_2)^k + (\lambda_2\lambda_3)^k + (\lambda_3\lambda_1)^k. \tag{4.8}$$

The generating function

$$u_0 + \xi u_1 + \xi^2 u_2 + \cdots = \frac{3 - 2\xi - \xi^2}{1 - \xi - \xi^2 - \xi^3}$$

is the 31 element of $(I - \xi M)^{-1}$ multiplied into $(3, 1, 3)^T$.

## 4.2   General $N$-term recurrence

Consider $\{g_k\}$ from

$$g_0 = g_1 = \cdots = g_{N-2} = 0 \qquad \text{and} \qquad g_{N-1} = 1 \tag{4.9}$$

via the constant-coefficient recurrence relation

$$g_k = a_1 g_{k-1} + a_2 g_{k-2} + \cdots + a_N g_{k-N} \qquad \text{with} \qquad a_N \neq 0. \tag{4.10}$$

Define $g_k$ for $k = 0, \pm 1, \pm 2, \ldots$ at the foot of the first column of an $n$-square matrix:

$$g_k \stackrel{\text{def}}{=} (M^k)_{N1} \qquad \text{where} \qquad M = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \qquad (4.11)$$

Here $\det M = (-1)^{N+1} a_N$, $M^0 = I$, and

$$M^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1/a_N & -a_1/a_N & \cdots & \cdots & -a_{N-1}/a_N \end{pmatrix}.$$

Then for $k = 0, \pm 1, \pm 2, \ldots$ the whole first column of $M^k$ is

$$(g_{k+N-1}, \ldots, g_{k+1}, g_k)^T,$$

and following columns are linear combinations with successive $a_j$ coefficients of respectively $N - 1$, $N - 2$, $\ldots$, 2, 1 successive elements of the first.
   For $eg.$ $N = 3$:

$$M^k = \begin{pmatrix} g_{k+2} & a_2 g_{k+1} + a_3 g_k & a_3 g_{k+1} \\ g_{k+1} & a_2 g_k + a_3 g_{k-1} & a_3 g_k \\ g_k & a_2 g_{k-1} + a_3 g_{k-2} & a_3 g_{k-1} \end{pmatrix}. \qquad (4.12)$$

Families of multilinear relations (including extension to $k < 0$) are read off from matrix-multiplication and determinant identities. For $eg.$ $N = 3$, from $\det(M^k) = a_3^k$:

$$\begin{vmatrix} g_k & g_{k+1} & g_{k+2} \\ g_{k-1} & g_k & g_{k+1} \\ g_{k-2} & g_{k-1} & g_k \end{vmatrix} = a_3^{k-2}.$$

Now generally

$$M^k = a_1 M^{k-1} + a_2 M^{k-2} + \cdots + a_N M^{k-N}$$

corresponding to characteristic equation

$$\lambda^N = a_1 \lambda^{N-1} + a_2 \lambda^{N-2} \cdots + a_{N-1} \lambda + a_N. \qquad (4.13)$$

The matrix generating function $(I - \xi M)^{-1}$ gives

$$g_0 + \xi g_1 + \xi^2 g_2 + \cdots = \frac{\xi^{N-1}}{1 - a_1 \xi - a_2 \xi^2 - \cdots - a_N \xi^N}.$$

### 4.2.1 General initial values

For any values of $h_0$ to $h_{N-1}$, define $h_k$ for $k = 0, \pm 1, \pm 2, \ldots$ as the N1 entry of the matrix product $M^k (h_{N-1}, \ldots, h_0)^T$ where $M$ is as in eq. (4.11).

For *eg.* $N = 3$:

$$h_k = h_2 g_k + h_1(a_2 g_{k-1} + a_3 g_{k-2}) + h_0 a_3 g_{k-1}.$$

Follow Sec. 3.4 to get for any $N$:

$$h_0 + \xi h_1 + \xi^2 h_2 + \cdots = \frac{h_0 + \sum_1^{N-1} d_r \xi^r}{1 - a_1 \xi - a_2 \xi^2 - \cdots - a_N \xi^N}$$

where

$$d_r = h_r - \sum_0^{r-1} h_i a_{r-i}.$$

For *eg.* $N = 4$ the generating function has numerator

$$h_0 + (h_1 - h_0 a_1)\xi + (h_2 - h_1 a_1 - h_0 a_2)\xi^2 + (h_3 - h_2 a_1 - h_1 a_2 - h_0 a_3)\xi^3.$$

### 4.2.2 Lucas-aid

Let $\quad \gamma_k \overset{\text{def}}{=} \text{tr}(M^k) \quad$ with $M$ as in eq. (4.10). Then

$$\gamma_k = \sum_{i=1}^{N} \lambda_i^k \tag{4.14}$$

where $\{\lambda_i\}$ are the eigenvalues of $M$.

As for Trucas in Sec. 4.1, read off from eq. (4.14) and eq. (4.13)

$$\gamma_0 = N, \quad \gamma_1 = a_1, \quad \gamma_2 = a_1^2 + 2a_2, \quad \ldots$$

and so on up to $\gamma_{N-1}$. Then the results of Sec. 4.2.1 apply, including a formula for $\gamma_{-k}$ in terms of $\lambda$s similar to eq. (4.8) but with $\det M = (-1)^{N+1} a_N$.

### 4.2.3 Complete set

If $N > 2$ then there are $N - 2$ other independent sequences. The $N$th elements of each of the $N - 2$ central columns of $M^k$ will do.

For *eg.* $N = 3$, from eq. (4.12), we have $h_k = a_2 g_{k-1} + a_3 g_{k-2}$. Then for instance to go with Tribonacci and Trucas there is $v_k = t_{k+1} - t_k$, namely [1, A001590]

$$\ldots -4, 5, -2, -1, , 2, -1, \mathbf{0}, \mathbf{1}, \mathbf{0}, 1, 2, 3, 6, 11, 20, \ldots.$$

## 4.3　Multinacci

Includes Fibo- ($N = 2$), Tribo- ($N = 3$), Tetra- ($N = 4$) [21] naccis [22] where the $N$-step recurrence eq. (4.10) with coefficients $a_1 = a_2 = \cdots = a_N = 1$ applies to simple initial values eq. (4.9).

Here

$$\lambda = \left( \frac{g_{k+1}}{g_k} \right)_{k \to \infty}$$

obeys

$$\lambda = 2 - \frac{1}{\lambda^N} \qquad \text{where} \qquad \lambda > 1$$

(multiply the characteristic equation by $\lambda - 1$). Since $|\det M| = 1$, matrix $M$ has at least one other eigenvalue $\lambda$ with $|\lambda| < 1$.

In fact the graphs of the smooth functions $\lambda$ and $2 - 1/\lambda^N$ show that for $N = 1, 2, \ldots$ there is exactly one real eigenvalue $\lambda \in (1, 2)$ plus, for even $N$ only, exactly one real eigenvalue $\lambda \in (-1, 0)$.

So for even $N$ there are $N - 2$ complex eigenvalues and for odd $N$ there are $N - 1$ (when there is at least a pair inside the unit circle). Direct from the characteristic equation, there is no eigenvalue $\lambda = \pm 1$ for any $N > 1$.

In fact Rouché's Theorem shows [23] that all complex eigenvalues are inside the unit circle, and numerical experiments show them approaching it as $N$ increases.

An eigenvector belonging to any eigenvalue $\lambda$ of $M$ is

$$(\lambda^{N-1}, \lambda^{N-2}, \ldots, \lambda, 1)^T.$$

This is significant in the context of the next section.

## 4.4　Chaotic maps

The Arnol'd cat map on $T^2$

$$x \to x' = Kx \quad (\text{mod } 1) \tag{4.15}$$

involves [24, pps. *92–8*]

$$K = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2.$$

Generalise to $N$ dimensions with $N \times N$ matrix

$$K \stackrel{\text{def}}{=} M^N \qquad \text{where} \qquad M = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

as for multinacci, Sec. 4.3. This map is volume- and orientation-preserving since

$$\det K = (\det M)^N = ((-1)^{N+1})^N = +1,$$

and on $T^N$ is an example of an Anosov hyperbolic automorphism [25, pps. *190–201*].

Examples — $N = 3$, 4:

$$K = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad K = \begin{pmatrix} 8 & 7 & 6 & 4 \\ 4 & 4 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Each $K$ has at least one irrational eigenvalue $\mu > 1$ (where $\mu = \lambda^N$ with $\lambda$ as in Sec. 4.3) plus at least one with $|\mu| < 1$, from $\det K = 1$ (and there is none with $\mu = \pm 1$). So every fixed point in phase space $T^N$ is unstable.

There are unstable periodic points of all periods. Every rational point is periodic and a $p$-periodic point has rational coordinates with denominator $|\det(K^p - I)|$.

So the matrices $K$ provide a family of linear congruential random-number generators, via eq. (4.15) in $T^N$.

# Chapter 5

# Fibonacci modulo $m$

Simulations of the Arnol'd cat map (Sec. 4.4) necessarily discretise phase space $[0, 1) \times [0, 1)$ (mod 1) and replace it with an $m$-square toroidal lattice. Then the linear map as in Sec. 4.4 with matrix $K = Q^2$ leads to consideration of Fibonacci numbers modulo $m$.

Lattice-points $(x, y) = (f_r, f_{r+1})$ (mod $m$) are on the trajectory generated by

$$\begin{pmatrix} y \\ x \end{pmatrix} \to Q \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{(mod } m) \tag{5.1}$$

and based at $(x, y) = (0, 1)$. As in eq. (1.3) the coordinates are columns of

$$Q^r \quad \text{(mod } m) \qquad \text{for} \qquad r \geq 1.$$

Because $Q$ is invertible and there are only $m^2 - 1$ distinct available lattice points (excluding the fixed point $(0, 0)$) the trajectory closes to a cycle after $r = \pi(m) \leq m^2 - 1$ steps. In other words, every integer $m > 1$ is a factor of some $f_k$ with $k < m^2 - 1$.

Thus for each $m \geq 2$ there is a minimum positive integer $\pi(m)$ such that

$$Q^{\pi(m)} \equiv I \quad \text{(mod } m). \tag{5.2}$$

Taking determinants: $(-1)^{\pi(m)} \equiv 1$ and so $\pi(m)$ is even for $m \geq 3$. Note that $-1 \equiv 1$ (mod 2), and indeed $\pi(2) = 3$.

The periods $\pi(m)$ comprise the Pisano sequence [1, A001175] which starts

$$(m, \pi(m)) = (2, 3), (3, 8), (4, 6), (5, 20), (6, 24), (7, 16), (8, 12), \ldots.$$

See ref. [26] for more. It has no overall regularity, but Fig. 5.1 shows obvious linear structure. Prominent lines include $\pi \sim 3m, \sim 2m, \sim m$ and it is known that $\pi(m)/m \leq 6$ with equality iff $m = 2 \times 5^n$ for $n = 1, 2, 3, \ldots$. Values $\pi(250) = 1500$ and $\pi(1250) = 7500$ stand out in Fig. 5.1.
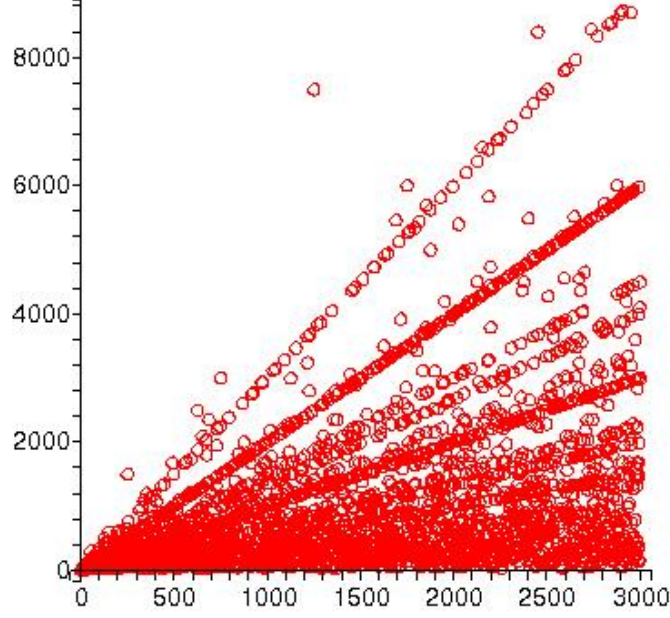
Figure 5.1: Pisano period $\pi(m)$ versus $m$ for $2 \leq m \leq 3000$.

Also we have (*eg.* ref. [27]) $\pi(m) = 3m/2$ for $m = 2^k$, $\pi(m) = 8m/3$ for $m = 3^k$ and $\pi(m) = 4m$ for $m = 5^k$, *etc*, but such cases contribute little to Fig. 5.1.

Since $Q^{\pi(m)} = I + mA$ where $A$ is some $2 \times 2$ symmetric matrix, if $n \mid m$ then $Q^{\pi(m)} \equiv I \pmod{n}$ too. Thus $n \mid m$ implies $\pi(n) \mid \pi(m)$. However there is no implication that $\pi(m)/\pi(n) = m/n$.

## 5.1 Other orbits

Pairs $(x, y) = (g_r, g_{r+1}) \pmod{m}$ are likewise lattice-points on the trajectory based at $(g_0, g_1)$, and they appear as columns of

$$Q^r G \pmod{m} \qquad \text{for} \qquad r \geq 1. \tag{5.3}$$

Here

$$G \overset{\text{def}}{=} \begin{pmatrix} g_1 & g_0 \\ g_0 & g_1 - g_0 \end{pmatrix}$$

as in eq. (3.16).

Clearly from eq. (5.2) any such trajectory repeats after $r = \pi(m)$. But its cycle may have a shorter minimum period $\pi_{jk}(m) < \pi(m)$ depending on $(g_0, g_1) = (j, k)$ when

$$Q^{\pi_{jk}(m)}G \equiv G \pmod{m}. \tag{5.4}$$

That is, the cycle through $(j, k)$ comprises $\pi_{jk}$ lattice points of which $(j, k)$ is a representative. With this notation the Pisano period itself is $\pi_{01}(m)$.

For consistency with eq. (5.2), $\pi_{jk}(m) \mid \pi_{01}(m)$. And for given $(j, k)$, $n \mid m$ implies $\pi_{jk}(n) \mid \pi_{jk}(m)$ by the same logic as for $\pi_{01}$.

Of course this periodicity is just Poincaré recurrence in the context of the discrete cat map [28, p. 98]. Such shorter cycles in a pixel-array lead to *eg.* the ghostly multiple images in diagram 48 of ref. [28, Fig. 6.107].

Because $Q$ is invertible, each lattice-point belongs to one and only one orbit and so there is the sum rule

$$\sum_{jk} \pi_{jk}(m) = m^2 \tag{5.5}$$

where the sum is over all distinct cycles including the fixed point $(0, 0)$ of period 1.

From eq. (5.5) a linear rise of the Pisano period $\pi(m)$ implies a linear increase in the number of distinct cycles $N(m)$.

Indeed the bound $\pi_{01}(m)/m \leq 6$, plus $\pi_{jk}(m) \leq \pi_{01}$, implies via eq. (5.5) that $N(m) \geq \frac{1}{6}m$. However this can be improved by counting zeroes, as follows.

## 5.2 Zeroes

The Fibonacci trajectory $(x, y) = (f_r, f_{r+1}) \pmod{m}$ closes on $(0, 1)$ after $\pi(m)$ terms but in general its orbit includes $(0, a)$ after $\alpha(m) \leq \pi(m)$.

Let $r = \alpha(m)$ be the smallest $r > 0$ such that [1, A001177]

$$f_r \equiv 0 \pmod{m}.$$

Then with $f_{\alpha(m)\pm 1} \equiv a \pmod{m}$ we have

$$Q^{\alpha(m)} \equiv aI \pmod{m}. \tag{5.6}$$

Taking determinants

$$(-1)^{\alpha(m)} \equiv a^2 \pmod{m} \tag{5.7}$$

and so either $a^2 \equiv 1$ or $a^2 \equiv m - 1 \pmod{m}$.

Taking powers:

$$Q^{2\alpha(m)} \equiv bI, \quad Q^{3\alpha(m)} \equiv cI, \quad Q^{4\alpha(m)} \equiv dI, \quad \ldots \quad (\text{mod } m)$$

so that

$$a^2 \equiv b, \quad a^3 \equiv c, \quad a^4 \equiv d, \quad \ldots, \quad (\text{mod } m). \tag{5.8}$$

From eq. (5.2) some multiple of $\alpha$ equals $\pi(m)$ and the corresponding coefficient in the set $a, b, c, d, \ldots$ is 1; then the sequence repeats.

Given eq. (5.7) and eq. (5.8) the possibilities are

- $a = b = \cdots = 1$, when $\pi(m) = \alpha(m)$ and zero-number $z(m) = 1$;

- $a \neq 1$ and $a^2 \equiv 1$, then $b = 1$ and so $\pi(m) = 2\alpha(m)$ and $z(m) = 2$;

- $a \neq 1$ and $a^2 \equiv m - 1$, then $b = m - 1$, $d \equiv 1$ and so $\pi(m) = 4\alpha(m)$ with $z(m) = 4$. Also $c = m - a$.

In the second two cases $4 \mid \pi(m)$ with an odd quotient if $z = 4$.

No other case arises — the Fibonacci cycle has $z(m) = 1, 2$ or 4 only.

**Example** $m = f_K$ $(K > 2)$:   here $\alpha = K$, $a = f_{K-1}$ and

$$a^2 \equiv (-1)^K \quad (\text{mod } f_K).$$

Therefore   $\pi(f_K) = 2K$ and $z(f_K) = 2$ if $K$ is even

$$\pi(f_K) = 4K \text{ and } z(f_K) = 4 \text{ if } K \text{ is odd.}$$

For odd $K$:   $b = f_K - 1, \quad c = f_{K-2}$.

Here $\pi(m)$ rises very slowly (logarithmically) with $m$.

A distinct trajectory based at $(g_0, g_1) = (j, k) \neq (0, 1)$ may or may not have a zero. For instance the Lucas sequence modulo 5 has none:   $z_{21}(5) = 0$. That is, no Lucas number is divisible by 5.

Consider however an orbit, distinct from the Fibonacci cycle, with at least one zero. Such orbits exist, eg. $m = 7$ with $(g_0, g_1) = (2, 4)$.

Then choose $(g_0, g_1) = (0, g)$, that is, $G = gI$ with $g \neq 1$.

If there is another zero then for some $\beta(m) > 0$

$$gQ^{\beta(m)} \equiv aI \quad (\text{mod } m) \tag{5.9}$$

when taking determinants gives

$$g^2(-1)^{\beta(m)} \equiv a^2 \quad (\text{mod } m). \tag{5.10}$$

Here there are 2 solutions with $g = a$, $\pi = \beta$ and $z = 1$ — namely

- $\beta = $ even;

- $\beta = 3$, $g = m/2$ with $m = $ even.

The second case includes $m = 2$, $g = 1$ as well as $m = 4k$, $g^2 \equiv 0 \pmod{m}$.

Otherwise if $z > 1$

$$gQ^{2\beta(m)} \equiv bI, \quad gQ^{3\beta(m)} \equiv cI, \quad gQ^{4\beta(m)} \equiv dI, \quad \ldots \pmod{m}$$

when, multiplying by $g$, $g^2$ and $g^3$ respectively, and using eq. (5.9),

$$a^2 \equiv gb, \quad a^3 \equiv g^2c, \quad a^4 \equiv g^3d, \quad \ldots, \pmod{m}. \tag{5.11}$$

Again from eq. (5.4), some multiple of $\beta$ equals $\pi_{0g}(m)$ when the corresponding coefficient $a, b, c, d, \ldots$ is $g$.

From eq. (5.10) and eq. (5.11) together, the possibilities are

- $\beta = $ even, $a^2 \equiv g^2$, $b = g$ and so $z = 2$, $\pi = 2\beta$,

- $\beta = $ odd, $a^2 \equiv -g^2$, $b = m - g$, $c = m - a$ and so $z = 4$, $\pi = 4\beta$.

Here $g = 1$ recovers the Fibonacci case.

In summary, non-Fibonacci cycles may have either 0, 1, 2 or 4 zeroes.

On the $m$-square lattice there are $m$ points $(0, g)$, including $g = 0$ and $g = 1$. Each lies on exactly one orbit. Therefore if the cycle from $(j, k)$ has $z_{jk}(m)$ zeroes there is the sum rule

$$\sum_{jk} z_{jk}(m) = m \tag{5.12}$$

where the sum is over $N(m)$ distinct cycles as in eq. (5.5).

Since no cycle has more than 4 zeroes, $N(m) \geq \frac{1}{4}m$. This is the improvement advertised above. Or, given eq. (5.5), we have independently that $\pi(m)$ increases no more than linearly with $m$.

## 5.3   Aside — divisibility of sums

The sum of any 10 consecutive Fibonacci numbers is divisible by 11. This is because $\pi(11) = 10$.

It is one of an infinite family of such results that follow from eq. (B.1) taken mod $m$, whose right-hand side is then zero if $q = p - 1 + \pi_{g_0 g_1}(m)$.

So for instance the sum of any 28 consecutive Lucas numbers is divisible by 13 because $\pi_{21}(13) = 28$, *etc.*

**Ex:** write down a factor of the sum of any 24 consecutive Lucas numbers.

| $m$ | $j, k$ | $\pi$ | $C$ | $w$ | $z$ |
|---|---|---|---|---|---|
| 2 | (0, 1) | 3 | 1 | 1 | 1 |
| 3 | (0, 1) | 8 | 1 | 3 | 2 |
| 4 | (0, 1) | 6 | 1 | 2 | 1 |
|   | (0, 2) | 3 | 0 | 1 | 1 |
|   | (0, 3) | 6 | 1 | 3 | 1 |
| 5 | (0, 1) | 20 | 1 | 8 | 4 |
|   | (2, 1) | 4 | 0 | 2 | 0 |
| 6 | (0, 1) | 24 | 1 | 11 | 2 |
|   | (0, 2) | 8 | 2 | 3 | 2 |
|   | (0, 3) | 3 | 3 | 1 | 1 |
| 7 | (0, 1) | 16 | 1 | 7 | 2 |
|   | (0, 2) | 16 | 3 | 7 | 2 |
|   | (0, 3) | 16 | 2 | 7 | 2 |
| 8 | (0, 1) | 12 | 1 | 4 | 2 |
|   | (0, 2) | 6 | 4 | 2 | 1 |
|   | (0, 3) | 12 | 1 | 6 | 2 |
|   | (0, 4) | 3 | 0 | 1 | 1 |
|   | (0, 6) | 6 | 4 | 3 | 1 |
|   | (2, 1) | 12 | 3 | 6 | 0 |
|   | (3, 1) | 12 | 3 | 6 | 0 |
| 9 | (0, 1) | 24 | 1 | 11 | 2 |
|   | (0, 2) | 24 | 4 | 11 | 2 |
|   | (0, 3) | 8 | 0 | 3 | 2 |
|   | (0, 4) | 24 | 2 | 11 | 2 |
| 10 | (0, 1) | 60 | 1 | 28 | 4 |
|   | (0, 2) | 20 | 4 | 8 | 4 |
|   | (0, 5) | 3 | 5 | 1 | 1 |
|   | (2, 1) | 12 | 5 | 6 | 0 |
|   | (4, 2) | 4 | 0 | 2 | 0 |

| $m$ | $j, k$ | $\pi$ | $C$ | $w$ | $z$ |
|---|---|---|---|---|---|
| 11 | (0, 1) | 10 | 1 | 3 | 1 |
|   | (0, 2) | 10 | 4 | 4 | 1 |
|   | (0, 3) | 10 | 2 | 4 | 1 |
|   | (0, 4) | 10 | 5 | 5 | 1 |
|   | (0, 5) | 10 | 3 | 5 | 1 |
|   | (0, 6) | 10 | 3 | 4 | 1 |
|   | (0, 7) | 10 | 5 | 4 | 1 |
|   | (0, 8) | 10 | 2 | 5 | 1 |
|   | (0, 9) | 10 | 4 | 5 | 1 |
|   | (0, 10) | 10 | 1 | 6 | 1 |
|   | (2, 5) | 10 | 0 | 5 | 0 |
|   | (2, 8) | 5 | 0 | 3 | 0 |
|   | (3, 1) | 5 | 0 | 2 | 0 |
| 12 | (0, 1) | 24 | 1 | 9 | 2 |
|   | (0, 2) | 24 | 4 | 11 | 2 |
|   | (0, 3) | 6 | 3 | 2 | 1 |
|   | (0, 4) | 8 | 4 | 3 | 2 |
|   | (0, 6) | 3 | 0 | 1 | 1 |
|   | (0, 7) | 24 | 1 | 13 | 2 |
|   | (0, 9) | 6 | 3 | 3 | 1 |
|   | (2, 1) | 24 | 5 | 12 | 0 |
|   | (4, 1) | 24 | 5 | 12 | 0 |
| 13 | (0, 1) | 28 | 1 | 12 | 4 |
|   | (0, 2) | 28 | 4 | 12 | 4 |
|   | (0, 4) | 28 | 3 | 12 | 4 |
|   | (2, 1) | 28 | 5 | 14 | 0 |
|   | (2, 5) | 28 | 2 | 14 | 0 |
|   | (2, 8) | 28 | 6 | 14 | 0 |

Table 5.1: all distinct proper orbits for $m = 2$ to 13. Each is identified by one point $(j, k)$ followed by period $\pi$, Cassini number $C$, winding number $w$ and number of zeroes $z$. Note that eq. (5.5) and eq. (5.12) are obeyed.

## 5.4   Other invariants

The sequence of numbers of cycles [1, A015134] begins

$$
\begin{aligned}
(m,\,N(m)) \;=\; & (2,\,2)\,,(3,\,2)\,,(4,\,4)\,,(5,\,3)\,,(6,\,4)\,,(7,\,4)\,,(8,\,8)\,,(9,\,5)\,, \\
& (10,\,6)\,,(11,\,14)\,,(12,\,10)\,,(13,\,7)\,,(14,\,8)\,,(15,\,12)\,,\dots
\end{aligned}
$$

and, like the sequence of periods $\pi_{01}(m)$, it is irregular.

The examples in Table 5.1 include other invariants which help to distinguish cycles — *ie,* Cassini number and winding number.

### 5.4.1   Cassini number

Referring to eq. (5.3) let

$$
\det G \equiv \Delta \pmod{m} \qquad \text{when} \qquad 0 \le \Delta \le m-1.
$$

Since $\det G = g_1^2 - g_0 g_1 - g_0^2$ appears in the generalised Cassini formula, eq. (3.17), define the '*Cassini number*' of an orbit from eq. (5.3) as

$$
C \overset{\text{def}}{=} \min(\Delta, m - \Delta) \tag{5.13}
$$

when   $0 \le C \le m/2$   if $m$ is even and   $m/2 \to (m-1)/2$   if $m$ is odd.

Since $\det Q = -1$ each cycle comprises points $(x,\,y)$ where

$$
y^2 - xy - x^2 \equiv \pm C \pmod{m}
$$

with sign alternating. This is the equivalent of a first integral for a continuous-time dynamical system. The Fibonacci cycle has $C = 1$ for all $m$.

Values of $C$ partition phase space to some extent — see Table 5.1. But counting by $C$ is not clear-cut because the range of about $\frac{1}{2}m$ $C$-values is not always realised, and two distinct cycles may have the same $C$-value. There seems to be nothing special about $C = 0$.

### 5.4.2   Fibonacci tiles

Fig. 5.2 shows two '*Fibonacci tiles*' [29], where each cycle of $Q$ is a distinct colour. Identifying opposite edges to make a torus, these patterns are the square pixel-arrays invariant under the $Q$-map and the discrete cat-map.
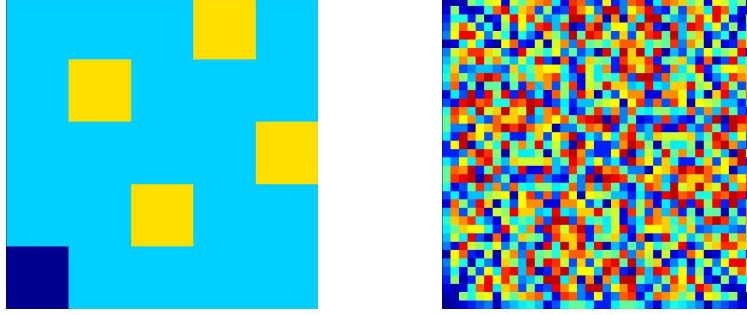
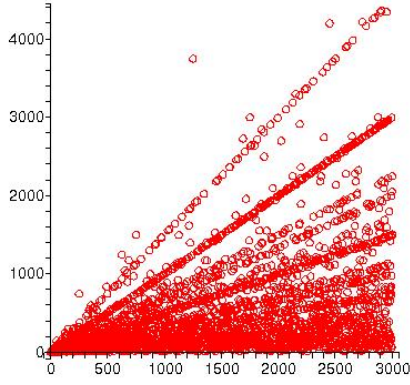Figure 5.2: orbits distinguished by colour, $m = 5$ ($N = 3$), $m = 37$ ($N = 19$).



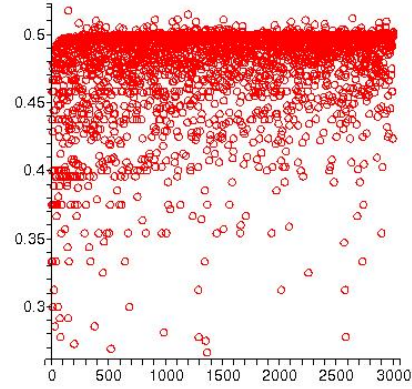Figure 5.3: winding number $w_{01}$ versus $m$ for $2 \leq m \leq 3000$.



Figure 5.4: ratio $w_{01}/\pi_{01}$ versus $m$ for $2 \leq m \leq 3000$.

### 5.4.3 Winding number

The sequence of lattice points generated by eq. (5.1) closes on $(0, 1)$ after a non-zero number $w(m)$ of modulo-$m$ operations — *ie*, the cycle winds round the toroidal lattice $w(m)$ times with $1 \leq w(m) < \pi(m)$.

Table 5.1 gives the winding number for each proper orbit included and the first few Fibonacci winding numbers $w_{01}(m)$ are [1, A088551]

$$
\begin{aligned}
(m,\, w(m)) \;=\; & (2,\, 1)\,, (3,\, 3)\,, (4,\, 2)\,, (5,\, 8)\,, (6,\, 11)\,, (7,\, 7)\,, (8,\, 4)\,, \\
& (9,\, 11)\,, (10,\, 28)\,, (11,\, 3)\,, (12,\, 9)\,, (13,\, 12)\,, (14,\, 23) \\
& (15,\, 19)\,, (16,\, 9)\,, (17,\, 16)\,, (18,\, 11)\,, (19,\, 7)\,, \ldots
\end{aligned}
$$

Again there is no regularity — but Fig. 5.3 is strikingly similar to Fig. 5.1. The prominent straight lines have half the slope, however. Indeed each step in eq. (5.1) moves round the lattice $\frac{1}{2}m(m + 1)$ points out of the $m^2 - 1$

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available, so that $w(m) \sim \pi(m)/2$ is expected.

Fig. 5.4 shows that there is considerable variation, consistent with the bound

$$w(m) > \tfrac{1}{4}\pi(m) \tag{5.14}$$

— *ie*, that the Fibonacci cycle has on average fewer than 4 points per transit round the lattice.

# Chapter 6

# Conclusion

To explore systematically the properties of Fibonacci numbers *etc*, recognise the 2-dimensional (or $N$-dimensional) linear context, formulate definitions in the corresponding natural language of matrices, and look at the implications of each interesting matrix result.

For instance the straightforward multiplication of matrices leads to index-reduction formulas that unify the host of bilinear Fibonacci-Lucas identities.

However these introductory notes do not deal with every possibility — there's plenty left as exercises for the student.

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# Appendix A

# General two-term recurrence

## A.1  Simple initial values

Consider
$$\varphi_{k+1} = a_1\varphi_k + a_2\varphi_{k-1} \tag{A.1}$$
with non-zero constants $(a_1,\ a_2)$ and values
$$\varphi_0 = 0, \quad \varphi_1 = 1.$$
Then the matrix expression corresponding to the Fibonacci case eq. (1.3) is

$$\begin{pmatrix} \varphi_{k+1} & a_2\varphi_k \\ \varphi_k & a_2\varphi_{k-1} \end{pmatrix} = P^k \qquad \text{where} \qquad P \overset{\text{def}}{=} \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix}. \tag{A.2}$$

Following the same manipulations as for Fibonacci numbers in Sec. 2.1 we have:
$$\varphi_{-k} = -\varphi_k/(-a_2)^k \tag{A.3}$$
and along with the Cassini-equivalent

$$\varphi_{k+1}\varphi_{k-1} - \varphi_k^2 = -(-a_2)^{k-1}. \tag{A.4}$$

Also
$$\varphi_{k+l} = \varphi_k\varphi_{l+1} + a_2\varphi_{k-1}\varphi_l \tag{A.5}$$
and
$$a_1\varphi_{j+k+l} = \varphi_{j+1}\varphi_{k+1}\varphi_{l+1} + a_1a_2\varphi_j\varphi_k\varphi_l - a_2^3\varphi_{j-1}\varphi_{k-1}\varphi_{l-1}.$$

Equating coefficients of $\xi^k$ in the generating-function expansion

$$\varphi_0 + \xi\varphi_1 + \xi^2\varphi_2 + \cdots = \frac{\xi}{1 - a_1\xi - a_2\xi^2}$$

gives for $k \geq 0$ an explicit solution of eq. (A.1) in terms of $a_1$ and $a_2$:

$$\varphi_{k+1} = \sum_{r=0}^{[k/2]} \binom{k-r}{r} a_1^{k-2r} a_2^r. \tag{A.6}$$

This generalises the Fibonacci result eq. (2.15), summing along a shallow diagonal of Pascal's triangle with a binomial coefficient from row $n$ multiplied by just those powers of $a_1$ and $a_2$ that accompany it in the expansion of $(a_1 + a_2)^n$.

## A.2    General initial values

If $g_{k+1} = a_1 g_k + a_2 g_{k-1}$ for general $(g_0, g_1)$ then

$$\begin{pmatrix} g_{k+1} & a_2 g_k \\ g_k & a_2 g_{k-1} \end{pmatrix} = P^k \begin{pmatrix} g_1 & a_2 g_0 \\ g_0 & g_1 - a_1 g_0 \end{pmatrix}, \tag{A.7}$$

with $P$ as in eq. (A.2) and so *eg.*

$$g_{k+1} g_{k-1} - g_k^2 = -(-a_2)^{k-1} (g_1^2 - a_1 g_0 g_1 - a_2 g_0^2) \tag{A.8}$$

and

$$g_{k+l} = \varphi_k g_{l+1} + a_2 \varphi_{k-1} g_l.$$

With $l = 0$ relate to $\{\varphi_k\}$:

$$g_k = g_1 \varphi_k + a_2 g_0 \varphi_{k-1}, \tag{A.9}$$

where the inverse follows from inverting eq. (A.7), as for Lucas.

Also:

$$a_1 g_{j+k+l} = \varphi_{j+1} \varphi_{k+1} g_{l+1} + a_1 a_2 \varphi_j \varphi_k g_l - a_2^3 \varphi_{j-1} \varphi_{k-1} g_{l-1}.$$

### A.2.1    Example

Solutions $(x_n, y_n)$ to the Pell equation [18]:

$$x^2 - Dy^2 = 1,$$

where $(x, y, D)$ are positive integers and $D$ is non-square, are given by

$$\begin{pmatrix} x_n & Dy_n \\ y_n & x_n \end{pmatrix} = \begin{pmatrix} \alpha & D\beta \\ \beta & \alpha \end{pmatrix}^n \qquad \text{for} \qquad n = 0, 1, 2, \ldots.$$

Here $(x_1, y_1) = (\alpha, \beta)$ is a minimal nontrivial solution — ie,

$$\alpha^2 - D\beta^2 = 1 \quad \text{and} \quad (\alpha, \beta) \neq (x_0, y_0) = (1, 0).$$

From the characteristic equation $\{x_n\}$ and $\{y_n\}$ obey $\quad g_{n+1} = 2\alpha g_n - g_{n-1}$ with $(g_0, g_1) = (1, \alpha)$ and $(g_0, g_1) = (0, \beta)$ respectively.

So this example has $(a_1, a_2) = (2\alpha, -1)$.

Then alongside $\quad x_n y_{n+1} - x_{n+1} y_n = \beta \quad$ we have eg. from eq. (A.8)

$$x_{n+1} x_{n-1} - x_n^2 = \alpha^2 - 1 \quad \text{and} \quad y_{n+1} y_{n-1} - y_n^2 = -\beta^2.$$

## A.2.2 Bilinear index-reduction formula

Any matrix $R$ that commutes with the matrix $P$ of eq. (A.2) has the form

$$R = \begin{pmatrix} g_1 & a_2 g_0 \\ g_0 & g_1 - a_1 g_0 \end{pmatrix}$$

as in eq. (A.7).

Therefore if $a + b = c + d$ not only does $P^a P^b = P^c P^d$ hold, but also

$$P^a R P^b S = P^c R P^d S \quad \text{where} \quad S \stackrel{\text{def}}{=} \begin{pmatrix} h_1 & a_2 h_0 \\ h_0 & h_1 - a_1 h_0 \end{pmatrix}.$$

Using eq. (A.7) to define both $\{g_k\}$ from $R$ and $\{h_k\}$ from $S$, the 22 entries on each side rearrange to

$$g_a h_b - g_c h_d = (-a_2)(g_{a-1} h_{b-1} - g_{c-1} h_{d-1})$$

when iteration as for the Fibonacci IRF eq. (2.4) gives

$$g_a h_b - g_c h_d = (-a_2)^r (g_{a-r} h_{b-r} - g_{c-r} h_{d-r}) \tag{A.10}$$

where $\quad a, b, c, d, r = 0, \pm 1, \pm 2, \ldots \quad$ and $\quad a + b = c + d.$

## A.2.3 Sums

The $N = 2$ generating function is

$$g_0 + \xi g_1 + \xi^2 g_2 + \cdots = \frac{g_0 + (g_1 - g_0 a_1)\xi}{1 - a_1 \xi - a_2 \xi^2},$$

and the analogue of eq. (2.16) is

$$g_0 + \xi g_k + \xi^2 g_{2k} + \cdots = \frac{g_0 + \xi(g_1 \varphi_k - g_0 \varphi_{k+1})}{1 - \xi(\varphi_{k+1} + a_2 \varphi_{k-1}) + \xi^2(-a_2)^k}.$$

The generalisation of the finite sum in eq. (2.23) is

$$g_0 + \xi g_k + \cdots + \xi^n g_{nk} = \frac{G_n}{1 - \xi(\varphi_{k+1} + a_2\varphi_{k-1}) + \xi^2(-a_2)^k},$$ (A.11)

where the numerator

$$
\begin{aligned}
G_n \equiv\ & g_0 + \xi(g_1\varphi_k - g_0\varphi_{k+1}) \\
& - \xi^{n+1}(g_1\varphi_{(n+1)k} + g_0 a_2\varphi_{(n+1)k-1}) \\
& + \xi^{n+2}(-a_2)^k(g_1\varphi_{nk} + g_0 a_2\varphi_{nk-1})
\end{aligned}
$$

has been simplified with the help of eq. (A.10) for $g = h = \varphi$.

The limit $n \to \infty$ exists for $|\xi| < |\sigma|^{-k}$ where $\sigma$ is the leading eigenvalue of matrix $P$ defined in eq. (A.2). If $\sigma$ is real (ie, if $a_1^2 + 4a_2 > 0$) then

$$\sigma = (\varphi_{k+1}/\varphi_k)_{k\to\infty}$$ (A.12)

and, from eq. (A.4), a generalisation of eq. (2.11) is

$$\sigma = a_1 - \sum_{1}^{\infty} \frac{(-a_2)^k}{\varphi_k\varphi_{k+1}}.$$

The analogue of eq. (2.24) is

$$P^k = \varphi_k P + a_2\varphi_{k-1} I$$

and, acting on $(g_1,\ g_0)^T$ with powers of $P$, eq. (2.26) generalises to

$$g_{kn} = \sum_{r=0}^{n} \binom{n}{r} (\varphi_k)^r (a_2\varphi_{k-1})^{n-r} g_r.$$ (A.13)

With eq. (A.3):

$$P^{-k} = \frac{1}{(-a_2)^k} \{-\varphi_k P + \varphi_{k+1} I\},$$

and so in the case $(g_1,\ g_0) = (1,\ 0)$ there is also:

$$\varphi_{kn} = \sum_{r=0}^{n} \binom{n}{r} (\varphi_k)^r (\varphi_{k+1})^{n-r} (-1)^{r+1} \varphi_r.$$ (A.14)

Evidently $\varphi_k \mid \varphi_{kn}$ since $\varphi_0 = 0$.

## A.3  Glucas

The denominator in eq. (A.11), compared to that in eq. (2.23) *etc*, suggests that

$$\nu_k \stackrel{\text{def}}{=} \operatorname{tr}(P^k) = \varphi_{k+1} + a_2\varphi_{k-1} \tag{A.15}$$

relates to $\varphi_k$ as Lucas to Fibonacci. Following Sec. 4.2.2, such Glucas numbers have $\nu_0 = 2$, $\nu_1 = a_1$ and so

$$\nu_0 + \xi\nu_1 + \xi^2\nu_2 + \cdots = \frac{2 - a_1\xi}{1 - a_1\xi - a_2\xi^2}.$$

Inverting this case of eq. (A.7):

$$\begin{pmatrix} \varphi_{k+1} & a_2\varphi_k \\ \varphi_k & a_2\varphi_{k-1} \end{pmatrix} = \frac{1}{a_1^2 + 4a_2} \begin{pmatrix} \nu_{k+1} & a_2\nu_k \\ \nu_k & a_2\nu_{k-1} \end{pmatrix} \begin{pmatrix} a_1 & 2a_2 \\ 2 & -a_1 \end{pmatrix}$$

leads to

$$(a_1^2 + 4a_2)\varphi_k = \nu_{k+1} + a_2\nu_{k-1} \tag{A.16}$$

to compare with eq. (3.6).

Either by taking determinants or direct from eq. (A.8), we find a Cassini-equivalent (compare eq. (3.8))

$$\nu_{k+1}\nu_{k-1} - \nu_k^2 = (-a_2)^{k-1}(a_1^2 + 4a_2).$$

**Ex:** and so obtain another sum rule for $\sigma$ (eq. (A.12)).

The trace definition of $\nu_k$ gives as analogue of eq. (3.13)

$$\nu_k = \sigma_+^k + \sigma_-^k, \qquad \text{where} \qquad \sigma_+ + \sigma_- = a_1, \quad \sigma_+\sigma_- = -a_2. \tag{A.17}$$

Eigenvalues $\sigma_\pm$ of $P$ are defined with $\pm$ signs in the usual formula; the (real) leading eigenvalue $\sigma$ in eq. (A.12) could be either.

Then eq. (A.16) gives

$$\varphi_k = \frac{\sigma_+^k - \sigma_-^k}{\sigma_+ - \sigma_-}$$

as analogue of eq. (2.8). Note that $\sigma_+ - \sigma_- = \sqrt{a_1^2 + 4a_2}$. The formal limit $\sigma_+ \to \sigma_-$ deals with equality.

Alongside eq. (A.3) we have

$$\nu_{-k} = \nu_k/(-a_2)^k \tag{A.18}$$

so that the simple-symmetry basis is

$$g_k = (g_1 - \tfrac{1}{2}g_0a_1)\varphi_k + \tfrac{1}{2}g_0\nu_k,$$

to compare with eq. (3.22).

Eq. (A.18) also means that besides eq. (A.13) and eq. (A.14) we have

$$\nu_{kn} = \sum_{r=0}^{n} \binom{n}{r} (\varphi_k)^r (\varphi_{k+1})^{n-r} (-1)^r \nu_r.$$

And from eq. (2.22) with $A = \xi P^k$ $(k \neq 0)$

$$\exp\left(\sum_{1}^{\infty} \frac{\xi^r}{r} \nu_{rk}\right) = \frac{1}{1 - \xi \nu_k + \xi^2 (-a_2)^k} = \frac{1}{\varphi_k} (\varphi_k + \xi \varphi_{2k} + \xi^2 \varphi_{3k} + \cdots)$$

for $|\xi| < |\sigma|^{-k}$, and where $\varphi_k \mid \varphi_{rk}$.

The explicit factorisation analogous to eq. (2.18) is

$$\varphi_{rk} = \varphi_k \sum_{p=0}^{[(r-1)/2]} \binom{r-p-1}{p} (\nu_k)^{r-2p-1} (-1)^p (-a_2)^{kp} \qquad \text{(A.19)}$$

where eq. (A.6) is a special case.

## A.4    Mixed bilinear IRF

The generalisation of the Fibonacci-Lucas IRF eq. (3.10) is

$$(a_1^2 + 4a_2)\varphi_a \varphi_b - \nu_c \nu_d = (-a_2)^r [(a_1^2 + 4a_2)\varphi_{a-r}\varphi_{b-r} - \nu_{c-r}\nu_{d-r}] \qquad \text{(A.20)}$$

where $a + b = c + d$ and $a, b, c, d, r = 0, \pm 1, \pm 2, \ldots$.

This relies on

$$Y \stackrel{\text{def}}{=} \begin{pmatrix} a_1 & 2a_2 \\ 2 & -a_1 \end{pmatrix}$$

obeying both $(a_1^2 + 4a_2)Y^{-1} = Y$ and $PY = YP$ — compare eq. (3.9).

## A.5    Divisibility

Consider $\{\varphi_k\}$ with $k \geq 0$, obeying eq. (A.1) with non-zero *integers* $(a_1, a_2)$ and $\varphi_0 = 0$, $\varphi_1 = 1$. We have from eq. (A.14) and eq. (A.19) that $\varphi_k \mid \varphi_{rk}$.

To show the converse, *ie*, that $\varphi_k \mid \varphi_m \Rightarrow k \mid m$, use eq. (A.5) as

$$\varphi_m = \varphi_{m-k}\varphi_{k+1} + a_2 \varphi_{m-k-1}\varphi_k.$$

Then if $\varphi_k$ and $\varphi_{k+1}$ are coprime

$$\varphi_k \mid \varphi_m \quad \Rightarrow \quad \varphi_k \mid \varphi_{m-k} \quad \Rightarrow \quad \varphi_k \mid \varphi_{m-2k} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad \varphi_k \mid \varphi_r,$$

where $m \equiv r \pmod{k}$ — a contradiction unless $r = 0$.

From *eg.* eq. (A.6): $\gcd(\varphi_k, \varphi_{k+1}) = 1 \iff \gcd(a_1, a_2) = 1$. So finally we have that if $a_1$ and $a_2$ are coprime, then $\varphi_k \mid \varphi_m \iff k \mid m$. This includes the Fibonacci case, of course.

## A.6 For pedestrians

A solution $g_k$ of $g_{k+1} = a_1 g_k + a_2 g_{k-1}$ is proportional to $\sigma_+^k + A\sigma_-^k$ with $A = const$ and with distinct[1] $\sigma_+$ and $\sigma_-$ as in eq. (A.17); hence a dreary routine way to verify any candidate identity.

For eq. (A.10) for instance, consider

$$(\sigma_+^a + A\sigma_-^a)(\sigma_+^b + B\sigma_-^b) - (\sigma_+^c + A\sigma_-^c)(\sigma_+^d + B\sigma_-^d)$$

which equals

$$
\begin{aligned}
\sigma_+^{a+b} - \sigma_+^{c+d} \quad &+ \quad AB(\sigma_-^{a+b} - \sigma_-^{c+d}) \\
&+ \quad (\sigma_+\sigma_-)^b (A\sigma_-^{a-b} + B\sigma_+^{a-b}) \\
&- \quad (\sigma_+\sigma_-)^d (A\sigma_-^{c-d} + B\sigma_+^{c-d})
\end{aligned}
$$

The first terms cancel if $a + b = c + d$, and then the substitution

$$(a,\, b,\, c,\, d) \to (a - r,\, b - r,\, c - r,\, d - r)$$

gives an overall factor $(\sigma_+\sigma_-)^{-r}$ to be compensated by $(-a_2)^r$.

**Ex:** verify eq. (A.20) this way.

## A.7 Eigen-line

If the coefficients $(a_1, a_2)$ in eq. (A.1) are positive integers, then $y = \sigma_+ x$ generalises the eigen-line of Sec. 2.3 in that for $x > 0$ it passes through a corridor of integer points $(x, y) = (\varphi_k, \varphi_{k+1})$ on alternate sides.

From eq. (A.4) the corridor narrows according to

$$\frac{a_2^{k-1}}{\varphi_k \varphi_{k-1}} \to \frac{\sigma_-^{k-1}}{\sigma_+^k} \to 0 \qquad \text{as} \qquad k \to \infty.$$

The slowest-narrowing, corresponding to the maximum of $\sigma_-/\sigma_+$ over positive integers $(a_1, a_2)$, is when $a_1 = a_2 = 1$ — *ie*, the Fibonacci case.

---

[1]The case of equality is an easy extension.

## A.8 Example

Illustrate with $(a_1, a_2) = (-1, 2)$ — Fybonacci:

$$\varphi_{k+1} = -\varphi_k + 2\varphi_{k-1}, \quad \varphi_0 = 0, \varphi_1 = 1,$$

giving

$$\tfrac{1}{3} \leftarrow \cdots \tfrac{21}{64}, \tfrac{11}{32}, \tfrac{5}{16}, \tfrac{3}{8}, \tfrac{1}{4}, \tfrac{1}{2}, \mathbf{0}, \mathbf{1}, -1, 3, -5, 11, -21, \cdots,$$

and Glucas:

$$\nu_{k+1} = -\nu_k + 2\nu_{k-1}, \quad \nu_0 = 2, \nu_1 = -1,$$

giving

$$1 \leftarrow \cdots \tfrac{65}{64}, \tfrac{31}{32}, \tfrac{17}{16}, \tfrac{7}{8}, \tfrac{5}{4}, \tfrac{1}{2}, \mathbf{2}, -\mathbf{1}, 5, -7, 17, -31, 65, \cdots.$$

The symmetries of eq. (A.3) and eq. (A.18) are evident.

In closed form

$$\varphi_k = \tfrac{1}{3}[1 - (-2)^k] \qquad \text{and} \qquad \nu_k = 1 + (-2)^k$$

— ie, $\sigma_+ = 1$ and $\sigma_- = \sigma = -2$ .

## A.9 Polynomials

An $N = 2$ example with $a_1 = x$ and $a_2 = 1$.

For Fibonacci polynomials [30], $\varphi_k \to \varphi_k(x)$ obeying

$$\varphi_{k+1}(x) = x\varphi_k(x) + \varphi_{k-1}(x) \qquad \text{with} \qquad \varphi_0(x) = 0, \quad \varphi_1(x) = 1.$$

Then

$$\varphi_2(x) = x, \quad \varphi_3(x) = x^2 + 1, \quad \varphi_4(x) = x^3 + 2x, \quad \ldots,$$

ie, polynomials $\varphi_k(x)$ of degree $k - 1$ with $\varphi_k(-x) = (-1)^{k-1}\varphi_k(x)$.

We have now

$$P(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad [P(x)]^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -x \end{pmatrix}$$

where $\det P(x) = -1$ and from eq. (A.3)

$$\varphi_{-k}(x) = (-1)^{k+1}\varphi_k(x).$$

So more generally $\varphi_k(x)$ is a polynomial of degree $|k| - 1$.

Expansion of $(I - \xi P)^{-1}$ gives the generating function

$$\varphi_0(x) + \xi\varphi_1(x) + \xi^2\varphi_2(x) + \cdots = \frac{\xi}{1 - x\xi - \xi^2}$$

— leading for $k \geq 0$ to

$$\varphi_{k+1}(x) = \sum_{r=0}^{[k/2]} \binom{k-r}{r} x^{k-2r}, \qquad (A.21)$$

which is a special case of eq. (A.6).

From Sec. A, the corresponding Lucas polynomials [31] are $\nu_k \to \nu_k(x)$ with $\nu_0(x) = 2$ plus $\nu_1(x) = x$. Then

$$\nu_2(x) = x^2 + 2, \quad \nu_3(x) = x^3 + 3x, \quad \dots,$$

and $\nu_{-k}(x) = \nu_k(-x) = (-1)^k \nu_k(x)$. Evidently $\nu_k(x)$ has degree $|k|$.

The generating function is

$$\nu_0(x) + \xi\nu_1(x) + \xi^2\nu_2(x) + \cdots = \frac{2 - x\xi}{1 - x\xi - \xi^2},$$

from which the analog of eq. (A.21) can be written down.

In fact, from eq. (A.15) and eq. (A.16):

$$\nu_k(x) = \varphi_{k+1}(x) + \varphi_{k-1}(x) \qquad \text{and} \qquad (x^2 + 4)\varphi_k(x) = \nu_{k+1}(x) + \nu_{k-1}(x).$$

At each fixed $x$, the Fibonacci and Lucas polynomials obey the same multilinear relations and reduction and summation formulas as the numbers $\varphi_k$ and $\nu_k$ in Sec. A (simplified by $a_2 = 1$) and have closed forms in terms of

$$\sigma_\pm = \tfrac{1}{2}(x \pm \sqrt{x^2 + 4}).$$

A wider '$w$-polynomial' context [32] has $a_1 = p(x)$ and $a_2 = q(x)$.

Obvious generalisation: to 'tribonacci polynomials' with

$$\begin{pmatrix} x^2 & x & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \frac{\xi^2}{1 - x^2\xi - x\xi^2 - \xi^3},$$

... and so on.

# Appendix B

# Formulas

There are collections of standard Fibonacci and Lucas formulas online at *MathWorld* [3] and at *Fibonacci Numbers, the Golden Section and the Golden String* [33], denoted respectively as '*Eric*' and '*Ron*'. Eric's collection is the more sophisticated.

Any candidate relation can be checked either by induction or with the Binet formulas[1] eq. (2.8) and eq. (3.13) that give $f_k$ and $\ell_k$ in terms of $\lambda_\pm$ — *eg.* as in Sec. A.6. It's more satisfying, however, to find an explicit construction, and the $Q$-matrix makes this possible. We illustrate with some formulas in Ron's list — using his scheme of reference to Dunlap [34], Vadja [4], *etc.*

**Linear and bilinear formulas.** Generally, the relations that are at most bilinear come as special cases of the index-reduction formula (IRF) eq. (3.23):

$$g_a h_b - g_c h_d = (-1)^r (g_{a-r} h_{b-r} - g_{c-r} h_{d-r})$$

— *ie*, with $gh$ as $ff$, $f\ell$ or $\ell\ell$, and with suitable choices for $(a, b, c, d, r)$. However some $ff$-plus-$\ell\ell$ results come more naturally from the mixed IRF in eq. (3.10). Note that eq. (2.4) is the $ff$ version of eq. (3.23).

- Of the 'order-two' Fibonacci relationships, Catalan's identity [35] is eq. (2.4) with

$$(a, b, c, d, r) \rightarrow (n, n, n+r, n-r, n-r)$$

  where Cassini/Simson in eq. (1.2) is a special case. Eq. (3.7) gives the corresponding Lucas identity: $\quad \ell_{k+r}\ell_{k-r} - \ell_k^2 = (-1)^k[(-1)^r\ell_r^2 - 4].$

---

[1]The manipulations use $\lambda_\pm^2 = \lambda_\pm + 1$, implying that $\lambda_+ + \lambda_- = 1$, $\lambda_+\lambda_- = -1$ and $\lambda_+ - \lambda_- = \sqrt{5}$. Note that Eric uses $\alpha$ and $\beta$ for $\lambda_\pm$, but also denotes the golden ratio $\tau \equiv \lambda_+$ by $\phi$, while Ron consistently uses $\Phi \equiv \lambda_+$ and $\phi \equiv -\lambda_-$.

- Equally, for d'Ocagne's identity [36] use eq. (2.4) with

$$(a,\, b,\, c,\, d,\, r) \rightarrow (m,\, n+1,\, m+1,\, n,\, n)\,.$$

For others listed, take values of $(a,\, b,\, c,\, d,\, r)$ in eq. (2.4) as follows —

- Lucas' duplication formula:  $(2,\, 2n,\, n+1,\, n+1,\, 2)$,

- Vadja-11, Dunlap-7:  $(n+1,\, -n-1,\, n,\, -n,\, n)$,

- Vadja-20a:  $(n+i,\, n+k,\, n,\, n+i+k,\, n)$,

and it's pleasant exercise to deal with the rest likewise ... however ...

- Vadja-12 (and its analogue with $f \rightarrow \ell$) comes most naturally direct from eq. (1.1) — multiply  $f_{n-1} = f_{n+1} - f_n$  by  $f_{n+2} = f_{n+1} + f_n$.

- Dunlap-10 is just eq. (2.2) with $(k,\, l) \rightarrow (m,\, n-m)$.

To illustrate the Lucas formulas, take $(a,\, b,\, c,\, d,\, r)$ in the $\ell\ell$ version of eq. (3.23) as follows — remembering that $(\ell_0,\, \ell_1) = (2,\, 1)$ —

- 'Cassini' (see eq. (3.8)):  $(n+1,\, n-1,\, n,\, n,\, n+1)$,

- Vadja-17a:  $(n+m,\, 0,\, n,\, m,\, m)$,

- Vadja-17c (duplication):  $(2n,\, 0,\, n,\, n,\, n)$.

Among the Fibonacci and Lucas results —

- Vadja-13 comes from eq. (2.4) with

$$(a,\, b,\, c,\, d,\, r) \rightarrow (2n,\, 1,\, n,\, n+1,\, n)\,,$$

  using $(f_0,\, f_1) = (0,\, 1)$ plus eq. (2.1) and eq. (3.1). See also Sec. 3.3.

- Vadja-15a follows from the $f\ell$ version of eq. (3.23) with

$$(a,\, b,\, c,\, d,\, r) \rightarrow (n,\, m,\, 0,\, m+n,\, m)\,,$$

  plus eq. (3.2).

Otherwise from the mixed formula eq. (3.10) we have *eg.*

- Vadja-24:  $(a,\, b,\, c,\, d,\, r) \rightarrow (n,\, n,\, n,\, n,\, n+1)$,

- Vadja-25:  $(a,\, b,\, c,\, d,\, r) \rightarrow (n,\, n,\, n,\, n,\, -1)$.

The remaining 'order-two' (bilinear) identities are straightforward.

**Higher-order Fibonacci and Lucas.**

- Vadja-32: from eq. (2.2) we have $f_{m+n} = f_{m+1}f_n + f_m f_{n-1}$ when $n \to -n$ gives also $(-1)^{n-1}f_{m-n} = f_{m+1}f_n - f_m f_{n+1}$. Multiply these two, and use eq. (1.1) twice.

- the Gelin-Cesàro identity [37] is (Cassini)$^2$ plus the basic recurrence eq. (1.1) several times. For the Lucasian equivalent, square eq. (3.8).

**Fibonacci and Lucas summations.** The finite sums are either special cases of the results in Sec. 2.8, Sec. 2.9 and Sec. 3.2 or they are telescoping series — *ie*, replacing each summand by a difference.

For instance, if $g_{k+1} = g_k + g_{k-1}$ then

$$\sum_{p}^{q} g_r = g_{q+2} - g_{p+1} \tag{B.1}$$

when $g_{r+2} - g_{r+1}$ is inserted in place of $g_r$. This generalises the familiar results

$$\sum_{0}^{n} f_r = f_{n+2} - 1 \quad \text{and} \quad \sum_{0}^{n} \ell_r = \ell_{n+2} - 1.$$

Also, from the Vadja-12 difference formula add to the list

$$f_{n+2}^2 = 1 + \sum_{1}^{n} f_r f_{r+3},$$

which holds also for $f \to \ell$.

There are endless extra results from eq. (2.23) and eq. (3.12) — *eg.*

$$\sum_{1}^{n} f_{3r} = \tfrac{1}{4}(f_{3n+3} + f_{3n} - 2)$$

as well as

$$\sum_{1}^{n} (-1)^r f_{3r} = \tfrac{1}{4}[(-1)^n (f_{3n+3} - f_{3n}) - 2].$$

Others are left as an exercise.

The infinite sums are specific $\xi$-values either in the generating-function expansions eq. (2.14) and eq. (3.19) or in a derivative with respect to $\xi$.

To illustrate with eq. (2.16) — putting *eg.* $k = 2$, $\xi = \tfrac{1}{4}$:

$$\sum_{1}^{\infty} 2^{-2r} f_{2r} = \tfrac{4}{5}.$$

These examples show how to tackle most of Ron's remaining sum formulas, given eq. (3.22) plus *eg.* the results of Sec. 2.9. Your next target is Eric's list.