

ON FIXED POINT THEOREMS IN BIPOLAR METRIC SPACES INVOLVING POLYNOMIAL-TYPE CONTRACTIONS

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ABSTRACT. In this paper, we investigate the existence and uniqueness of fixed points for self-mappings defined on bipolar metric spaces using a new class of contractive conditions, namely polynomial-type contractions. Our main results establish sufficient conditions under which a mapping on a complete bipolar metric space admits a UFP. Several illustrative examples are provided to demonstrate the applicability of our theorems, and we further show how our results generalize and improve upon existing fixed point theorems in both standard and generalized metric settings.

1. INTRODUCTION

The most commonly used techniques for establishing the existence and uniqueness of solutions to nonlinear problems—such as differential equations, integral equations, evolution equations—typically and fractional differential equations involve reducing the problem to an equation of the form $\check{F}e = e$ and $e \in \check{E}$ is the unknown solution. This is referred to as a FP of \check{F} .

The Banach Fixed Point Theorem (FPT) stands as one of the most fundamental and widely recognized results in FP theory [2], which states that, if (\check{E}, ϑ) is a complete \mathcal{MS} and $\check{F}: \check{E} \rightarrow \check{E}$ is a mapping satisfies

$$\vartheta(\check{F}e, \check{F}f) \leq \pi \vartheta(e, f)$$

for all $e, f \in \check{E}$ and $\pi \in (0, 1)$ is a constant, then

(M_1) \check{F} has a UFP;

(M_2) For all $e_0 \in \check{E}$, the sequence $\{e_\kappa\}$ given by $e_{\kappa+1} = \check{F}e_\kappa$, approaches to FP.

Agrawal et. al [1] has established FPT in \mathcal{MS} . Banach [2] has presented banach contraction principle. Berinde [4] has established general constructive FPTs in Ćirić-type almost contractions in \mathcal{MS} s. Berinde [5] has proved FPT in banach spaces by using a retraction-displacement condition. Berinde et. al [6] has proved FPTs in almost contractions. Berinde et. al [7] has proved FPs of enriched contractions in banach spaces. Boyd et. al [8] has proved FPT in metrically complete \mathcal{MS} .

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The concept of FPs has been a central theme in nonlinear analysis and has led to significant developments in various branches of mathematics. In 2000, Branciari [11] introduced a FPT of Banach-Caccioppoli type in a class of generalized \mathcal{MS} s, which extends the classical Banach contraction principle. Earlier, Chatterjea [12] presented a FP result based on a distinct type of contraction condition, now known as the Chatterjea contraction. In a further generalization, Ćirić [13] proposed a broader class of contractive mappings that unify and extend several known FPTs. The structure of generalized \mathcal{MS} s was further enriched by Czerwik [14], who explored contraction mappings in $b\text{-}\mathcal{MS}$ s, thereby opening new avenues for FP theory. Additionally, Dhage [15] introduced and analyzed mappings in generalized \mathcal{MS} s, contributing valuable FP results applicable in more abstract settings. These foundational works collectively form the basis for many contemporary studies in FP theory and its applications.

FPT continues to evolve with the development of novel contraction principles and generalizations applicable to diverse mathematical structures. Petrov [16] introduced a geometric approach by studying mappings that contract the perimeters of triangles, offering a unique perspective within metric frameworks. Petruşel and Rus [17] expanded FPT by incorporating both metric and order structures, enabling broader applicability in ordered \mathcal{MS} s. Popescu and Păcurar [18] recently proposed FP results for generalized Chatterjea-type mappings, contributing to ongoing efforts to unify and generalize contraction conditions. Classical results by Rakotch [19] and Reich [20] laid essential groundwork in the theory of contractive functions, which continues to inspire modern generalizations.

The idea of weakly Picard mappings was explored by Rus [21], leading to a series of studies on Picard operators [22, 23, 24] that examined their properties, equivalences, and applications. These contributions underscore the importance of iterative behavior and convergence in FP theory. Recent advances in bipolar metric spaces have extended classical fixed point results to accommodate asymmetric structures. Gaba et al. [9] established Banach-type fixed point theorems in this context, while Aphane, and et al. [10] introduced (α, BK) -contractions, broadening the range of applicable contractive conditions. Zhang and Song [25] further enriched the field by introducing the framework of generalized $(\psi - \phi)$ -weak contractions, which accommodate a wider class of nonlinear mappings. Together, these works highlight the depth and versatility of FP theory in both classical and modern mathematical analysis. Gunaseelan mani et. al [26] has presented FPT in F -contraction in bipolar \mathcal{MS} . Gunaseelan mani et. al [27] has established FPT in controlled bipolar \mathcal{MS} . Gunaseelan mani et. al. [28] has given common FPT in bipolar orthogonal \mathcal{MS} . Further more details see [[29, 30, 31, 32, 33]]. Throughout in this paper \mathcal{MS} means metric space and PC means polynomial contraction.

2. PRELIMINARIES

Definition 2.1. Let \check{E} and \check{P} be a two non-void set and $\vartheta: \check{E} \times \check{P} \rightarrow [0, +\infty)$ be a mapping implies that

- (1) $e = f$ iff $\vartheta(e, f) = 0$;
- (2) $\vartheta(e, f) = \vartheta(f, e)$ if $e, f \in \check{E} \cap \check{P}$;
- (3) $\vartheta(e, f) \leq \vartheta(e, \mathfrak{z}) + \vartheta(\mathfrak{r}, \mathfrak{z}) + \vartheta(\mathfrak{r}, f)$, $\forall e, \mathfrak{r} \in \check{E}$ and $\mathfrak{z}, f \in \check{P}$.

Then, the mapping ϑ is called a bipolar \mathcal{MS} on the pair (\check{E}, \check{P}) and the triple $(\check{E}, \check{P}, \vartheta)$ is called bipolar \mathcal{MS} .

Definition 2.2. Let (\check{E}, ϑ) be a \mathcal{MS} . A mapping $\check{F}: \check{E} \rightarrow \check{E}$ is called an almost contraction, if there exist $\pi \in (0, 1)$ and $\rho > 0$ implies that

$$\vartheta(\check{F}e, \check{F}f) \leq \pi\vartheta(e, f) + \rho\vartheta(f, \check{F}e), \forall e, f \in \check{E}.$$

The following FPT for the above class of mappings was proven by Berinde [3].

Theorem 2.1. Let (\check{E}, ϑ) be a complete \mathcal{MS} and the mapping $\check{F}: \check{E} \rightarrow \check{E}$ be an almost contraction. Then

- (i) \check{F} admits at least one FP;
- (ii) For all $\{g_\kappa\}$ defined by $g_{\kappa+1} = \check{F}g_\kappa$, converges to a FP of \check{F} .

3. THE CLASS OF POLYNOMIAL CONTRACTIONS(PC)

Definition 3.1. Let (\check{E}, ϑ) be a bipolar \mathcal{MS} and $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ be a mapping. It is stated that \check{F} is a PC, if there exist $\pi \in (0, 1)$, $\sigma \geq 1$ and a mappings $q_v: \check{E} \times \check{P} \rightarrow [0, \infty)$, $v = 0, \dots, \sigma$ implies that

$$\sum_{v=0}^{\sigma} q_v(\check{F}e, \check{F}f)\vartheta^v(\check{F}e, \check{F}f) \leq \pi \sum_{v=0}^{\sigma} q_v(e, f)\vartheta^v(e, f)$$

for all $e \in \check{E}, f \in \check{P}$.

In this section, we focus on the study of FPs within bipolar \mathcal{MS} involving the class of polynomial functions.

Theorem 3.1. Let $(\check{E}, \check{P}, \vartheta)$ be a complete bipolar \mathcal{MS} and the covariant mapping $\check{F}: \check{E} \cup \check{P} \rightrightarrows \check{E} \cup \check{P}$ be a PC such that

- (i) \check{F} is continuous;
- (ii) We can find that $\varrho \in \{1, \dots, \sigma\}$ and $\check{Q}_\varrho > 0$ implies that

$$q_\varrho(e, f) \geq \check{Q}_\varrho, e \in \check{E}, f \in \check{P}.$$

Then, \check{F} admits a UFP. Moreover for every, $g_0 \in \check{E}$, the picard sequence $\{g_\kappa\} \subset \check{E}$ by $g_{\kappa+1} = \check{F}g_\kappa$ and $h_0 \in \check{P}$, the picard sequence $\{h_\kappa\} \subset \check{P}$ by $h_{\kappa+1} = \check{F}h_\kappa, \forall \kappa \geq 0$.

Proof. Initially, we show that the set of FPs of \check{F} is non-empty. Let $g_0 \in \check{E}$ and $h_0 \in \check{P}$ be FPs and $\{g_\kappa\} \subset \check{E}$ and $\{h_\kappa\} \subset \check{P}$ is defined by

$$g_{\kappa+1} = \check{F}g_\kappa \quad \text{and} \quad h_{\kappa+1} = \check{F}h_\kappa, \quad \kappa \geq 0.$$

By definition of PC, with $(e, f) = (g_0, h_0)$, we get

$$\sum_{v=0}^{\sigma} q_v(\check{F}g_0, \check{F}h_0)\vartheta^v(\check{F}g_0, \check{F}h_0) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_0)\vartheta^v(g_0, h_0),$$

i.e.,

$$\sum_{v=0}^{\sigma} q_v(g_1, h_1)\vartheta^v(g_1, h_1) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_0)\vartheta^v(g_0, h_0).$$

Generally we can write,

$$\vartheta^{\varrho}(g_{\kappa}, h_{\kappa}) \leq \pi^{\kappa} \sum_{v=0}^{\sigma} q_v(g_0, h_0) \vartheta^v(g_0, h_0), \quad \kappa \geq 0.$$

By definition of PC, with $(e, f) = (g_0, h_1)$, we get

$$\sum_{v=0}^{\sigma} q_v(\check{F}g_0, \check{F}h_1) \vartheta^v(\check{F}g_0, \check{F}h_1) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1),$$

i.e.,

$$\sum_{v=0}^{\sigma} q_v(g_1, h_2) \vartheta^v(g_1, h_2) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1). \quad (1)$$

Again by definition of PC with $(e, f) = (g_1, h_2)$

$$\sum_{v=0}^{\sigma} q_v(\check{F}g_1, \check{F}h_2) \vartheta^v(\check{F}g_1, \check{F}h_2) \leq \pi \sum_{v=0}^{\sigma} q_v(g_1, h_2) \vartheta^v(g_1, h_2),$$

i.e.,

$$\sum_{v=0}^{\sigma} q_v(g_2, h_3) \vartheta^v(g_2, h_3) \leq \pi \sum_{v=0}^{\sigma} q_v(g_1, h_2) \vartheta^v(g_1, h_2),$$

which implies

$$\sum_{v=0}^{\sigma} q_v(g_2, h_3) \vartheta^v(g_2, h_3) \leq \pi^2 \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1).$$

Continuing in the same way, we get

$$\sum_{v=0}^{\sigma} q_v(g_{\kappa}, h_{\kappa+1}) \vartheta^v(g_{\kappa}, h_{\kappa+1}) \leq \pi^{\kappa} \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1). \quad (2)$$

Since

$$q_{\varrho}(g_{\kappa}, h_{\kappa+1}) \vartheta^{\varrho}(g_{\kappa}, h_{\kappa+1}) \leq \sum_{v=0}^{\sigma} q_v(g_{\kappa}, h_{\kappa+1}) \vartheta^v(g_{\kappa}, h_{\kappa+1}).$$

We obtain by (ii) that

$$\check{Q}_{\varrho} \vartheta^{\varrho}(g_{\kappa}, h_{\kappa+1}) \leq \sum_{v=0}^{\sigma} q_v(g_{\kappa}, h_{\kappa+1}) \vartheta^v(g_{\kappa}, h_{\kappa+1}),$$

which implies by (2), that

$$\vartheta^{\varrho}(g_{\kappa}, h_{\kappa+1}) \leq \pi^{\kappa} \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1), \quad \kappa \geq 0. \quad (3)$$

Similarly we have,

$$\vartheta^{\varrho}(g_{\kappa}, h_{\kappa}) \leq \pi^{\kappa} \sum_{v=0}^{\sigma} q_v(g_0, h_0) \vartheta^v(g_0, h_0), \quad \kappa \geq 0,$$

where,

$$\mathcal{M} = \check{Q}_\varrho^{-1} \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1) + \check{Q}_\varrho^{-1} \sum_{v=0}^{\sigma} q_v(g_0, h_0) \vartheta^v(g_0, h_0). \quad (4)$$

Then, by (3) and triangle inequality, we get

$$\begin{aligned} \vartheta^\varrho(g_{\kappa+\varpi}, h_\kappa) &\leq \vartheta^\varrho(g_{\kappa+\varpi}, h_{\kappa+1}) + \vartheta^\varrho(g_\kappa, h_{\kappa+1}) + \vartheta^\varrho(g_\kappa, h_\kappa) \\ &\leq \vartheta^\varrho(g_{\kappa+\varpi}, h_{\kappa+1}) + \pi^\kappa \mathcal{M} \\ &\leq \vartheta^\varrho(g_{\kappa+\varpi}, h_{\kappa+2}) + \vartheta^\varrho(g_{\kappa+1}, h_{\kappa+2}) + \vartheta^\varrho(g_{\kappa+1}, h_{\kappa+1}) + \pi^\kappa \mathcal{M} \\ &\leq \vartheta^\varrho(g_{\kappa+\varpi}, h_{\kappa+2}) + (\pi^{\kappa+1} + \pi^\kappa) \mathcal{M} \\ &\vdots \\ &\leq \vartheta^\varrho(g_{\kappa+\varpi}, h_{\kappa+\varpi}) + (\pi^{\kappa+\varpi-1} + \dots + \pi^{\kappa+1} + \pi^\kappa) \mathcal{M} \\ &\leq (\pi^{\kappa+\varpi} + \dots + \pi^{\kappa+1} + \pi^\kappa) \mathcal{M} \\ &\leq \mathcal{M} \frac{\pi^\kappa}{1 - \pi}, \end{aligned}$$

which implies,

$$\vartheta(g_{\kappa+\varpi}, h_\kappa) \leq \left(\frac{\mathcal{M}}{1 - \pi} \right)^{\frac{1}{\varrho}} \left(\pi^{\frac{1}{\varrho}} \right)^\kappa \rightarrow 0 \quad \text{as } \kappa, \varpi \rightarrow \infty.$$

Similarly,

$$\vartheta(g_\kappa, h_{\kappa+\varpi}) \leq \left(\frac{\mathcal{M}}{1 - \pi} \right)^{\frac{1}{\varrho}} \left(\pi^{\frac{1}{\varrho}} \right)^\kappa \rightarrow 0 \quad \text{as } \kappa, \varpi \rightarrow \infty.$$

This shows that $\{g_\kappa\}$ and $\{h_\kappa\}$ is a Cauchy bisequence. Since $(\check{E}, \check{P}, \vartheta)$ is complete, there exist $g \in \check{E}$ such that

$$\lim_{\kappa \rightarrow \infty} \vartheta(g_\kappa, g) = 0,$$

due to the continuity of \check{F} that

$$\lim_{\kappa \rightarrow \infty} \vartheta(g_{\kappa+1}, \check{F}g) = \lim_{\kappa \rightarrow \infty} \vartheta(\check{F}g_\kappa, \check{F}g) = 0.$$

By the uniqueness of the limit, we conclude that $\check{F}g = g$ i.e., g is a FP of \check{F} .

Now, we show that g is the UFP of \check{F} . Indeed, if $g^* \in \check{E}$ is another FP of \check{F} i.e., $\check{F}g^* = g^*$ and $\vartheta(g, g^*) > 0$ then by definition of PC with $(e, f) = (g, g^*)$

$$\sum_{v=0}^{\sigma} q_v(\check{F}g, \check{F}g^*) \vartheta^v(\check{F}g, \check{F}g^*) \leq \pi \sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*),$$

i.e.,

$$\sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*) \leq \pi \sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*). \quad (5)$$

On the other hand, from (ii), we get

$$\begin{aligned} \sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*) &\geq q_{\varrho}(g, g^*) \vartheta^v(g, g^*) \\ &\geq \check{Q}_{\varrho} \vartheta^v(g, g^*). \end{aligned}$$

Since $\check{Q}_{\varrho} > 0$ and $\vartheta(g, g^*) > 0$, we deduce that

$$\sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*) > 0.$$

Then, dividing by (5), we get $\sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*)$, we reach a contraction with $\pi \in (0, 1)$. Consequently, g is a UFP of \check{F} . Hence its completes. \square

Theorem 3.2. *Let $(\check{E}, \check{P}, \vartheta)$ be a complete bipolar \mathcal{MS} and the contravariant mapping $\check{F}: \check{E} \cup \check{P} \rightrightarrows \check{E} \cup \check{P}$ be a PC such that*

- (i) \check{F} is continuous;
- (ii) We can find that $\varrho \in \{1, \dots, \sigma\}$ and $\check{Q}_{\varrho} > 0$ implies that

$$q_{\varrho}(e, f) \geq \check{Q}_{\varrho}, e \in \check{E}, f \in \check{P}.$$

Then, \check{F} admits a UFP. Moreover for every, $g_0 \in \check{E}$, the picard sequence $\{g_{\kappa}\} \subset \check{E}$ by $g_{\kappa+1} = \check{F}g_{\kappa}$ and $h_0 \in \check{P}$, the picard sequence $\{h_{\kappa}\} \subset \check{P}$ by $h_{\kappa+1} = \check{F}h_{\kappa}, \forall \kappa \geq 0$.

Proof. Initially, we show that the set of FPs of \check{F} is non-empty. Let $g_0 \in \check{E}$ and $h_0 \in \check{P}$ be FPs and $\{g_{\kappa}\} \subset \check{E}$ and $\{h_{\kappa}\} \subset \check{P}$ is defined by

$$h_{\kappa} = \check{F}g_{\kappa} \quad \text{and} \quad g_{\kappa+1} = \check{F}h_{\kappa}, \quad \kappa \geq 0.$$

By definition of PC, with $(e, f) = (g_0, h_0)$, we get

$$\sum_{v=0}^{\sigma} q_v(\check{F}g_0, \check{F}h_0) \vartheta^v(\check{F}g_0, \check{F}h_0) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_0) \vartheta^v(g_0, h_0),$$

i.e.,

$$\sum_{v=0}^{\sigma} q_v(h_0, g_1) \vartheta^v(h_0, g_1) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_0) \vartheta^v(g_0, h_0). \quad (6)$$

Generally we can write,

$$\vartheta^{\varrho}(g_{\kappa}, h_{\kappa}) \leq \pi^{2\kappa} \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1), \quad \kappa \geq 0.$$

Again by definition of PC with $(e, f) = (g_0, h_1)$

$$\sum_{v=0}^{\sigma} q_v(\check{F}g_0, \check{F}h_1) \vartheta^v(\check{F}g_0, \check{F}h_1) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1),$$

i.e.,

$$\sum_{v=0}^{\sigma} q_v(h_0, g_2) \vartheta^v(h_0, g_2) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1),$$

which implies

$$\sum_{v=0}^{\sigma} q_v(g_2, h_0) \vartheta^v(g_2, h_0) \leq \pi^2 \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1).$$

Continuing in the same way, we get

$$\sum_{v=0}^{\sigma} q_v(g_{\kappa+1}, h_{\kappa}) \vartheta^v(g_{\kappa+1}, h_{\kappa}) \leq \pi^{2\kappa+1} \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1). \quad (7)$$

Since

$$q_{\varrho}(g_{\kappa+1}, h_{\kappa}) \vartheta^{\varrho}(g_{\kappa+1}, h_{\kappa}) \leq \sum_{v=0}^{\sigma} q_v(g_{\kappa+1}, h_{\kappa}) \vartheta^v(g_{\kappa+1}, h_{\kappa}).$$

We obtain by (ii) that

$$\check{Q}_{\varrho} \vartheta^{\varrho}(g_{\kappa+1}, h_{\kappa}) \leq \sum_{v=0}^{\sigma} q_v(g_{\kappa+1}, h_{\kappa}) \vartheta^v(g_{\kappa+1}, h_{\kappa}),$$

which implies by (7), that

$$\vartheta^{\varrho}(g_{\kappa+1}, h_{\kappa}) \leq \pi^{2\kappa+1} \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1), \quad \kappa \geq 0, \quad (8)$$

similarly we have,

$$\vartheta^{\varrho}(g_{\kappa}, h_{\kappa}) \leq \pi^{2\kappa} \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1), \quad \kappa \geq 0,$$

where,

$$\mathcal{M} = \check{Q}_{\varrho}^{-1} \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1) + \check{Q}_{\varrho}^{-1} \sum_{v=0}^{\sigma} q_v(g_0, h_1) \vartheta^v(g_0, h_1). \quad (9)$$

Then, by (8) and triangle inequality, we get

$$\begin{aligned}
\vartheta^{\varrho}(g_{\kappa+\varpi}, h_{\kappa}) &\leq \vartheta^{\varrho}(g_{\kappa+\varpi}, h_{\kappa+1}) + \vartheta^{\varrho}(g_{\kappa+1}, h_{\kappa+1}) + \vartheta^{\varrho}(g_{\kappa+1}, h_{\kappa}) \\
&\leq \vartheta^{\varrho}(g_{\kappa+\varpi}, h_{\kappa+1}) + (\pi^{2\kappa+2} + \pi^{2\kappa+1})\mathcal{M} \\
&\leq \vartheta^{\varrho}(g_{\kappa+\varpi}, h_{\kappa+2}) + \vartheta^{\varrho}(g_{\kappa+2}, h_{\kappa+2}) + \vartheta^{\varrho}(g_{\kappa+2}, h_{\kappa+1}) \\
&\quad + (\pi^{2\kappa+2} + \pi^{2\kappa+1})\mathcal{M} \\
&\vdots \\
&\leq \vartheta^{\varrho}(g_{\kappa+\varpi}, h_{\kappa+\varpi-1}) + (\pi^{2\kappa+2\varpi-2} + \dots + \pi^{2\kappa+1} + \pi^{\kappa})\mathcal{M} \\
&\leq (\pi^{2\kappa+2\varpi-1} + \pi^{2\kappa+2\varpi-2} + \pi^{2\kappa+2\varpi-3} + \dots + \pi^{2\kappa+1})\mathcal{M} \\
&\leq \mathcal{M} \frac{\pi^{2\kappa}}{1-\pi},
\end{aligned}$$

which implies,

$$\vartheta(g_{\kappa+\varpi}, h_{\kappa}) \leq \left(\frac{\mathcal{M}}{1-\pi} \right)^{\frac{1}{\varrho}} \left(\pi^{\frac{1}{\varrho}} \right)^{2\kappa} \rightarrow 0 \quad \text{as } \kappa, \varpi \rightarrow \infty,$$

similarly,

$$\vartheta(g_{\kappa}, h_{\kappa+\varpi}) \leq \left(\frac{\mathcal{M}}{1-\pi} \right)^{\frac{1}{\varrho}} \left(\pi^{\frac{1}{\varrho}} \right)^{2\kappa} \rightarrow 0 \quad \text{as } \kappa, \varpi \rightarrow \infty.$$

This shows that $\{g_{\kappa}\}$ and $\{h_{\kappa}\}$ is a Cauchy bisequence. Since $(\check{E}, \check{P}, \vartheta)$ is complete, there exist $g \in \check{E}$ such that

$$\lim_{\kappa \rightarrow \infty} \vartheta(g_{\kappa}, g) = 0,$$

due to the continuity of \check{F} that

$$\lim_{\kappa \rightarrow \infty} \vartheta(g_{\kappa+1}, \check{F}g) = \lim_{\kappa \rightarrow \infty} \vartheta(\check{F}g_{\kappa}, \check{F}g) = 0.$$

By the uniqueness of the limit, we conclude that $\check{F}g = g$ i.e., g is a FP of \check{F} .

Now, we show that g is the UFP of \check{F} . Indeed, if $g^* \in \check{E}$ is another FP of \check{F} i.e., $\check{F}g^* = g^*$ and $\vartheta(g, g^*) > 0$ then by definition of PC with $(e, f) = (g, g^*)$

$$\sum_{v=0}^{\sigma} q_v(\check{F}g, \check{F}g^*) \vartheta^v(\check{F}g, \check{F}g^*) \leq \pi \sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*),$$

i.e.,

$$\sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*) \leq \pi \sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*). \quad (10)$$

On the other hand, from (ii), we get

$$\begin{aligned}
\sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*) &\geq q_{\varrho}(g, g^*) \vartheta^v(g, g^*) \\
&\geq \check{Q}_{\varrho} \vartheta^v(g, g^*).
\end{aligned}$$

Since $\check{Q}_\varrho > 0$ and $\vartheta(g, g^*) > 0$, we deduce that

$$\sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*) > 0.$$

Then, dividing by (10), we get $\sum_{v=0}^{\sigma} q_v(g, g^*) \vartheta^v(g, g^*)$, we reach a contraction with $\pi \in (0, 1)$. Consequently, g is a UFP of \check{F} . Hence its completes. \square

Proposition 3.3. *Let $(\check{E}, \check{P}, \vartheta)$ be a bipolar \mathcal{MS} and the mapping $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ be a PC such that*

- (i) $q_0 \equiv 0$, i.e., $q_0(e, f) = 0, \forall e \in \check{E}, f \in \check{P}$;
- (ii) For all, $v \in \{1, \dots, \sigma\}$, there exist $\check{W}_v > 0$ such that
$$q_v(e, f) \leq \check{W}_v, e \in \check{E}, f \in \check{P}$$
- (iii) There exist $\varrho \in \{1, \dots, \sigma\}$ and $\check{Q}_\varrho > 0$ such that
$$q_\varrho(e, f) \geq \check{Q}_\varrho, e \in \check{E}, f \in \check{P}.$$

Then \check{F} is continuous.

Proof. Let $\{g_\kappa\} \subset \check{E}$ and $\{h_\kappa\} \subset \check{P}$ be a bisequence such that

$$\lim_{\kappa \rightarrow \infty} \vartheta(g_\kappa, h_\kappa) = 0, \quad (11)$$

for some $g \in \check{E}$ and $h \in \check{P}$. Using (i) and by definition of PC with $(e, f) = (g_\kappa, h_\kappa)$, we get

$$\sum_{v=0}^{\sigma} q_v(\check{F}g_\kappa, \check{F}h_\kappa) \vartheta^v(\check{F}g_\kappa, \check{F}h_\kappa) \leq \pi \sum_{v=0}^{\sigma} q_v(g_\kappa, h_\kappa) \vartheta^v(g_\kappa, h_\kappa), \kappa \geq 0,$$

by (ii) and (iii) implies that

$$\check{Q}_\varrho \vartheta^\varrho(\check{F}g_\kappa, \check{F}h_\kappa) \leq \pi \sum_{v=1}^{\sigma} \check{W}_v \vartheta^v(g_\kappa, h_\kappa), \kappa \geq 0. \quad (12)$$

Then, by (11) and taking the limit as $\kappa \rightarrow \infty$ in (12), we obtain

$$\lim_{\kappa \rightarrow \infty} \vartheta^\varrho(\check{F}g_\kappa, \check{F}h_\kappa) = 0,$$

which is equivalent to,

$$\lim_{\kappa \rightarrow \infty} \vartheta(\check{F}g_\kappa, \check{F}h_\kappa) = 0.$$

Thus \check{F} is a continuous mapping. \square

The following result is derived from Theorem 3.1 and Proposition 3.3.

Corollary 3.4. *Let $(\check{E}, \check{P}, \vartheta)$ be a complete bipolar \mathcal{MS} and $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ be a polynomial contraction such that*

- (i) $q_0 \equiv 0$;
- (ii) For all $v \in \{1, \dots, \sigma\}$, there exists $\check{W}_v > 0$ such that

$$q_v(e, f) \leq \check{W}_v, e \in \check{E}, f \in \check{P};$$

(iii) There exist $\varrho \in \{1, \dots, \sigma\}$ and $\check{Q}_\varrho > 0$ such that

$$q_\varrho(e, f) \leq \check{Q}_\varrho, e \in \check{E}, f \in \check{P};$$

Then \check{F} admits a UFP.

The following result follows directly from the above corollary.

Corollary 3.5. Let $(\check{E}, \check{P}, \vartheta)$ be a complete bipolar \mathcal{MS} and $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ be a mapping. Consider we can find that $\pi \in (0, 1)$, $\sigma \geq 1$ and a sequence $\{q_v\}_{v=1}^\sigma \subset (0, \infty)$ implies that

$$\sum_{v=1}^\sigma q_v \vartheta^v(\check{F}e, \check{F}f) \leq \pi \sum_{v=1}^\sigma q_v \vartheta^v(e, f), \forall e \in \check{E}, f \in \check{P}$$

Then \check{F} admits a UFP.

Example 3.6. Let $\check{E} = \{e_1, e_2, e_3\}$ and $f = \{e_1, e_5\}$ and the mapping be $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ is defined by

$$\begin{aligned} \check{F}e_1 &= e_1, \check{F}e_2 = e_3, \check{F}e_3 = e_4, \check{F}e_4 = e_2, \check{F}e_5 = e_4, \\ \check{F}f_1 &= f_1, \check{F}f_2 = f_3, \check{F}f_3 = f_4, \check{F}f_4 = f_2, \check{F}f_5 = f_4 \end{aligned}$$

Let ϑ be discrete metric on $\check{E} \cup \check{P}$ i.e.,

$$\vartheta(e_v, f_\varrho) = \begin{cases} 1, & \text{if } v \neq \varrho \\ 0, & \text{if } v = \varrho \end{cases}.$$

Consider the mapping $q_0: \check{E} \times \check{P} \rightarrow [0, \infty)$ defined by

$$\begin{aligned} q_0(e_v, f_\varrho) &= q_0(f_\varrho, e_v) \\ q_0(e_v, e_v) &= 0 \\ q_0(e_1, f_2) &= q_0(e_2, f_3) = 4, \\ q_0(e_1, f_3) &= q_0(e_3, f_4) = 3, \\ q_0(e_2, f_5) &= q_0(e_3, f_5) = 5, \\ q_0(e_1, f_4) &= 2, \\ q_0(e_1, f_5) &= 1, \\ q_0(e_2, f_4) &= 7, \\ q_0(e_4, f_5) &= 8. \end{aligned}$$

We claim that

$$q_0(\check{F}e, \check{F}f) + \vartheta(\check{F}e, \check{F}f) \leq \frac{1}{2} q_0(e, f) + \vartheta(e, f), \quad (13)$$

for every $e \in \check{E}, f \in \check{P}$ i.e., \check{F} is a PC in the sense of definition of PC with $\sigma = 1, q_1 \equiv 1$ and $\pi = \frac{1}{2}$. If $e = f$ (or) $(e, f) = (e_1, e_5)$ then equation (13) is obvious. We have to show that equation (13) holds for all $e_v \in \check{E}, f_v \in \check{P}$ with $1 \leq v < \varrho \leq 5$ and $(v, \varrho) \neq (1, 5)$. Then all condition as follows in table below:

(v, ϱ)	$q_0(\check{F}e_v, \check{F}f_\varrho) + \vartheta(\check{F}e_v, \check{F}f_\varrho)$	$q_0(e_v, f_\varrho) + \vartheta(e_v, f_\varrho)$
$(1,2)$	4	5
$(1,3)$	3	4
$(1,4)$	5	3
$(1,5)$	3	2
$(2,3)$	4	5
$(2,4)$	5	8
$(2,5)$	4	6
$(3,4)$	8	4
$(3,5)$	1	6
$(4,5)$	8	9

of Theorem 3.1 are fulfilled ((ii) is satisfied with $\check{Q} = 1$).

Hence \check{F} admits a UFP.

Definition 3.2. Let $(\check{E}, \check{P}, \vartheta)$ be a bipolar \mathcal{MS} . The mapping $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ is reffered as picard-continuous, if for all $g \in \check{E}, h \in \check{P}$, we have

$$\lim_{\kappa \rightarrow \infty} \vartheta(\check{F}^\kappa g, h) = 0 \rightarrow \lim_{\kappa \rightarrow \infty} \vartheta(\check{F}(\check{F}^\kappa g), \check{F}h) = 0$$

where $\check{F}^0 g = g$ and $\check{F}^{\kappa+1} g = \check{F}(\check{F}^\kappa g), \forall \kappa \geq 0$.

Remark 3.7. Remark that, if $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ is continuous, then \check{F} is picard-continuous, the converse part is not true.

Example 3.8. Let $\check{E} = [q, w], \check{P} = [\mu, \vartheta]$, where $q, w, \mu, \vartheta \in \mathbb{R}$ and $q < w, \mu < \vartheta$. The mapping $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ defined by

$$\check{F}e = \begin{cases} q, & \text{if } q \leq e < w, \\ \frac{q+w}{3}, & \text{if } e = w \end{cases}$$

$$\check{F}f = \begin{cases} \mu, & \text{if } \mu \leq f < \vartheta, \\ \frac{\mu+\vartheta}{3}, & \text{if } f = \vartheta. \end{cases}.$$

Let ϑ be the standard metric on \check{E} , i.e., $\vartheta(e, f) = |e - f|$ for all $e, f \in \check{E} \cup \check{P}$. Although the mapping \check{F} is not continuous at w and ϑ , it is Picard-continuous as per the preceding definition.

$$\check{F}^\kappa g = q, \quad \forall \kappa \geq 3.$$

So, if for some $g \in \check{E}, h \in \check{P}$, we have

$$\lim_{\kappa \rightarrow \infty} \vartheta(\check{F}^\kappa g, h) = 0,$$

then $h = q$.

$$\lim_{\kappa \rightarrow \infty} \vartheta(\check{F}(\check{F}^\kappa g), h) = \lim_{\kappa \rightarrow \infty} \vartheta(\check{F}q, h) = \vartheta(\check{F}q, h) = 0.$$

Hence \check{F} is picard-continuous.

Theorem 3.9. Let $(\check{E}, \check{P}, \vartheta)$ be a complete bipolar \mathcal{MS} and the mapping $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ be a PC such that

- (i) \check{F} is picard-continuous,
- (ii) We can find that $\varrho \in \{1, \dots, \sigma\}$ and $\check{Q}_\varrho > 0$ implies that

$$q_\varrho(e, f) \geq \check{Q}_\varrho, \quad e \in \check{E}, f \in \check{P}.$$

Then \check{F} admits a UFP.

Proof. Let \check{F} be a non-empty set. Assume $g_0 \in \check{E}, h_0 \in \check{P}$ and $\{g_\kappa\} \subset \check{E}$ and $\{h_\kappa\} \subset \check{P}$ be the picard bisequence defined by

$$g_{\kappa+1} = \check{F}g_\kappa \quad \text{and} \quad h_{\kappa+1} = \check{F}h_\kappa, \quad \forall \kappa \geq 0,$$

i.e.,

$$g_\kappa = \check{F}^\kappa g_0 \quad \text{and} \quad h_\kappa = \check{F}^\kappa h_0, \quad \forall \kappa \geq 0.$$

By using the proof of Theorem 3.1, w.k.t $\{g_\kappa\}$ and $\{h_\kappa\}$ is a Cauchy bisequence, by completeness of $(\check{E}, \check{P}, \vartheta)$ that we can find that $g \in \check{E}$ and $h \in \check{P}$ implies that

$$\lim_{\kappa \rightarrow \infty} \vartheta(\check{F}^\kappa g_0, g) = 0 \quad \text{and} \quad \lim_{\kappa \rightarrow \infty} \vartheta(\check{F}^\kappa h_0, h) = 0.$$

Using the picard continuity of \check{F} , it hold that

$$\lim_{\kappa \rightarrow \infty} \vartheta(\check{F}^{\kappa+1} g_0, \check{F}g) = \lim_{\kappa \rightarrow \infty} \vartheta(\check{F}(\check{F}^\kappa g_0), \check{F}g) = 0,$$

and

$$\lim_{\kappa \rightarrow \infty} \vartheta(\check{F}^{\kappa+1} h_0, \check{F}h) = \lim_{\kappa \rightarrow \infty} \vartheta(\check{F}(\check{F}^\kappa h_0), \check{F}h) = 0,$$

using the uniqueness of limit, then g is a FP of \check{F} . □

4. THE CLASS OF ALMOST POLYNOMIAL CONTRACTION

Definition 4.1. Let $(\check{E}, \check{P}, \vartheta)$ be a bipolar \mathcal{MS} and the mapping $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$. We say that \check{F} is an almost PC, if we can find that $\pi \in (0, 1)$, $\sigma \geq 1$, a sequence $\{\check{H}_v\}_{v=0}^\sigma \subset (0, \infty)$ and a mapping $q_v: \check{E} \times \check{P} \rightarrow [0, \infty)$, $v = 0, \dots, \sigma$ implies that

$$\sum_{v=0}^\sigma q_v(\check{F}e, \check{F}f) \vartheta^v(\check{F}e, \check{F}f) \leq \pi \sum_{v=0}^\sigma q_v(e, f) [\vartheta^v(e, f) + \check{H}_v \vartheta^v(f, \check{F}e)], \quad e \in \check{E}, f \in \check{P}.$$

Definition 4.2. Let $(\check{E}, \check{P}, \vartheta)$ be a bipolar \mathcal{MS} and the mapping be $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$. We reffered as \check{F} is a weakly picard operator, if

- (i) All FPs of \check{F} is non-empty,
- (ii) For all $g_0 \in \check{E}, h_0 \in \check{P}$, the picard sequence $\{\check{F}^\kappa g_0\}$ and $\{\check{F}^\kappa h_0\}$ is biconvergent and its limit belongs to the set of FPs of \check{F} .

Theorem 4.1. Let $(\check{E}, \check{P}, \vartheta)$ be a complete bipolar \mathcal{MS} and the covariant mapping $\check{F}: \check{E} \cup \check{P} \rightrightarrows \check{E} \cup \check{P}$ be an atmost PC such that

- (i) \check{F} is picard-continuous,

(ii) we can find that $\varrho \in \{1, \dots, \sigma\}$ and $\check{Q}_\varrho > 0$ implies that

$$q_\varrho(e, f) \geq \check{Q}_\varrho, \quad e \in \check{E}, f \in \check{P}.$$

Then \check{F} is a weakly picard operator.

Proof. Let $g_0 \in \check{E}$ and $h_0 \in \check{P}$ be fixed and $\{g_\kappa\} \subset \check{E}$ and $\{h_\kappa\} \subset \check{P}$ be the picard bisequence defined by

$$g_{\kappa+1} = \check{F}g_\kappa \quad \text{and} \quad h_{\kappa+1} = \check{F}h_\kappa, \quad \forall \kappa \geq 0.$$

By the definition of PC with $(e, f) = (g_0, h_0)$

$$\sum_{v=0}^{\sigma} q_v(\check{F}g_0, \check{F}h_0) \vartheta^v(\check{F}g_0, \check{F}h_0) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_0) [\vartheta^v(g_0, h_0) + \check{H}_v \vartheta^v(h_0, \check{F}g_0)],$$

i.e.,

$$\sum_{v=0}^{\sigma} q_v(g_1, h_1) \vartheta^v(g_1, h_1) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_0) \vartheta^v(g_0, h_0).$$

Again by definition of PC with $(e, f) = (g_1, h_1)$, we get

$$\sum_{v=0}^{\sigma} q_v(\check{F}g_1, \check{F}h_1) \vartheta^v(\check{F}g_1, \check{F}h_1) \leq \pi \sum_{v=0}^{\sigma} q_v(g_1, h_1) [\vartheta^v(g_1, h_1) + \check{H}_v \vartheta^v(h_1, \check{F}g_1)],$$

i.e.,

$$\sum_{v=0}^{\sigma} q_v(g_2, h_2) \vartheta^v(g_2, h_2) \leq \pi \sum_{v=0}^{\sigma} q_v(g_1, h_1) \vartheta^v(g_1, h_1),$$

which implies

$$\sum_{v=0}^{\sigma} q_v(g_2, h_2) \vartheta^v(g_2, h_2) \leq \pi \sum_{v=0}^{\sigma} q_v(g_0, h_0) \vartheta^v(g_0, h_0),$$

continuing by induction process, we get

$$\sum_{v=0}^{\sigma} q_v(g_\kappa, h_\kappa) \vartheta^v(g_\kappa, h_\kappa) \leq \pi^\kappa \sum_{v=0}^{\sigma} q_v(g_0, h_0) \vartheta^v(g_0, h_0), \quad \kappa \geq 0,$$

which implies from (ii) that

$$\vartheta^0(g_\kappa, h_\kappa) \leq \pi^\kappa \mathcal{M}, \quad \kappa \geq 0,$$

where \mathcal{M} is given in (4). By the proof of Theorem 3.1. We obtain that $\{g_\kappa\}$ and $\{h_\kappa\}$ is a cauchy bisequence, by the completeness of $(\check{E}, \check{P}, \vartheta)$ the existence of $g \in \check{E}$ implies that

$$\lim_{\kappa \rightarrow \infty} \vartheta(g_\kappa, g) = 0.$$

In conclusion, considering that \check{F} is picard-continuous, we obtain

$$\lim_{\kappa \rightarrow \infty} \vartheta(g_{\kappa+1}, \check{F}g) = 0,$$

by the uniqueness of limit that $g = \check{F}g$. This completes the proof. \square

Proposition 4.2. Let $(\check{E}, \check{P}, \vartheta)$ be a bipolar \mathcal{MS} and the mapping $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$. Assume that there exist $\pi \in (0, 1)$, $\sigma \geq 1$ and two finite sequence $\{q_v\}_{v=1}^\sigma, \{\check{H}_v\}_{v=1}^\sigma \subset (0, \infty)$ such that

$$\sum_{v=1}^\sigma q_v \vartheta^v(\check{F}e, \check{F}f) \leq \pi \sum_{v=1}^\sigma q_v [\vartheta^v(e, f) + \check{H}_v \vartheta^v(f, \check{F}e)], \quad (14)$$

for every $e \in \check{E}, f \in \check{P}$. Then \check{F} is picard-continuous.

Proof. Let $g \in \check{E}, f \in \check{P}$ be such that

$$\lim_{\kappa \rightarrow \infty} \vartheta(\check{F}^\kappa g, h) = 0.$$

By equation (14) with $(e, f) = (\check{F}^\kappa g, h)$, we get

$$\sum_{v=1}^\sigma q_v \vartheta^v(\check{F}(\check{F}^\kappa g), \check{F}h) \leq \pi \sum_{v=1}^\sigma q_v [\vartheta^v(\check{F}^\kappa g, h) + \check{H}_v \vartheta^v(h, \check{F}(\check{F}^\kappa g))],$$

i.e.,

$$\sum_{v=1}^\sigma q_v \vartheta^v(\check{F}^{\kappa+1} g, \check{F}h) \leq \pi \sum_{v=1}^\sigma q_v [\vartheta^v(\check{F}^\kappa g, h) + \check{H}_v \vartheta^v(h, \check{F}^{\kappa+1} g)],$$

which implies that

$$\vartheta(\check{F}(\check{F}^\kappa g), \check{F}h) \leq \frac{\pi}{q_1} \sum_{v=1}^\sigma q_v [\vartheta^v(\check{F}^\kappa g, h) + \check{H}_v \vartheta^v(h, \check{F}^{\kappa+1} g)].$$

Then, passing to the limit as $\kappa \rightarrow \infty$ in the above inequality and by equation (9), we get

$$\lim_{\kappa \rightarrow \infty} \vartheta(\check{F}(\check{F}^\kappa g), \check{F}h) = 0.$$

Hence \check{F} is picard-continuous. \square

The following result is derived from the above theorem and proposition.

Corollary 4.3. Let $(\check{E}, \check{P}, \vartheta)$ be a complete bipolar \mathcal{MS} and the mapping $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$. Assume that there exist $\pi \in (0, 1)$, $\sigma \geq 1$, and two finite sequence $\{q_v\}_{v=1}^\sigma, \{\check{H}_v\}_{v=1}^\sigma \subset (0, \infty)$ such that equation (9) holds for every $e \in \check{E}, f \in \check{P}$. Then, \check{F} is a weakly picard continuous.

Proof. By equation (9) is a special case by definition of almost PC with $q_0 \equiv 0$ and q_v is constant, $\forall v \in \{1, \dots, \sigma\}$. Then by above proposition and theorem applies. \square

Remark 4.4. Taking $\sigma = 1, q_1 = 1$ and $\check{H}_1 = \frac{\rho}{\pi}$, where $\rho > 0$, equation (9) reduces to PC. Then, by above corollary, we recover Berinde's FPT.

Example 4.5. Let $\check{E} = \{e_1, e_2\}, \check{P} = \{e_1, f_2\}$ and the mapping be $\check{F}: \check{E} \cup \check{P} \rightarrow \check{E} \cup \check{P}$ defined by

$$\check{F}e_1 = e_1, \check{F}e_2 = e_1, \check{F}f_1 = e_1, \check{F}f_2 = e_1.$$

Define the two metrics as

$$\vartheta_1(e_v, f_\varrho) = \begin{cases} 1 & e_v \neq f_\varrho \\ 0 & e_v = f_\varrho \end{cases}$$

$$\vartheta_2(e_v, f_\varrho) = \begin{cases} 2 & e_v \in \check{E}, f_\varrho \in \check{P} \\ 1 & e_v \neq f_\varrho \\ 0 & e_v = f_\varrho \end{cases}$$

(e_v, f_ϱ)	$\vartheta_1(e_v, f_\varrho)$	$\vartheta_2(e_v, f_\varrho)$	$\vartheta_1(\check{F}e_v, \check{F}f_\varrho)$	$\vartheta_2(\check{F}e_v, \check{F}f_\varrho)$
(e_1, e_2)	1	1	0	0
(e_1, f_1)	1	2	0	0
(e_2, f_2)	1	2	0	0
(f_1, f_2)	1	1	0	0

From the table:

$$\vartheta_1(\check{F}e_v, \check{F}f_\varrho) \leq \vartheta_1(e_v, e_\varrho), \vartheta_2(\check{F}e_v, \check{F}f_\varrho) \leq \vartheta_2(e_v, e_\varrho) \quad e \in \check{E}, f \in \check{P}.$$

Hence the mapping \check{F} is non-expansive with bipolar \mathcal{MS} .

5. APPLICATION TO FRACTIONAL CALCULUS

Let $(\check{V}[0, 1])$ be the set of all continuous function on $[0, 1]$ and the mapping be $\vartheta: \check{V}([0, 1]) \times \check{V}([0, 1]) \rightarrow \mathbb{R}$ defined by $\vartheta(e, f) = \|e - f\|_\infty = \sup_{\rho \in [0, 1]} |e(\rho) - f(\rho)|$. Define $e \in \check{E}$ and $f \in \check{P}$.

The Caputo fractional derivative of order β applied to a continuous function $\mathfrak{h}: [0, +\infty) \rightarrow \mathbb{R}$ is defined as

$$\check{V}\check{O}^\beta(\mathfrak{h}(\rho)) = \frac{1}{\Gamma(\varpi - \beta)} \int_0^\rho (\rho - \eta)^{\varpi - \beta - 1} \mathfrak{g}^{(\varpi)}(\eta) d\eta, \quad (\varpi - 1 < \beta < \kappa, \varpi = [\beta] + 1),$$

where Γ is the gamma function and $[\beta]$ denotes the integer part of a real number. Consider $\check{E} = (\check{V}[0, 1], [0, \infty) = \{\vartheta: [0, 1] \rightarrow [0, \infty) \text{ be a continuous}\}$ and $\check{P} = (\check{V}[0, 1], [0, \infty) = \{\vartheta: [0, 1] \rightarrow [-\infty, 0) \text{ be a continuous}\}$.

Let us see, the existence of the solution of non-linear fractional differential equation

$$\check{V}\check{O}^\beta(g(\rho)) + \omega(\rho, g(\rho)) = 0 \quad (0 \leq \rho \leq 1, \beta < 1), \quad (15)$$

with $g(0) = g(1) = 0$ and $\omega: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and the Green's function corresponding to problem (15) is defined as,

$$\check{A}(\rho, \eta) = \begin{cases} \rho(1 - \eta)^{q-1} - (\rho - \eta)^{q-1}, & \text{if } 0 \leq \rho \leq \eta \leq 1, \\ \frac{\rho(1 - \eta)^{q-1}}{\Gamma(q)}, & \text{if } 0 \leq \eta \leq \rho \leq 1. \end{cases}$$

Suppose the following conditions are satisfied:

- (1) There exist $\sigma \in (0, 1)$ such that $|\omega(\rho, e) - \omega(\rho, f)| \leq \sigma|e - f|$ for each $\rho \in [0, 1]$ and $e, f \in \mathbb{R}$,

$$(2) \sup_{\rho \in [0,1]} \left(\int_0^1 \check{A}(\rho, \eta) d\eta \right) \leq 1.$$

Next, we establish the existence of a solution to the fractional differential equation (15).

Theorem 5.1. *Assuming the condition holds ((1))-(2), from (15) has a unique solution.*

Proof. Define the mapping $\check{F}: \check{V}[0, 1] \rightarrow \check{V}[0, 1]$ is defined by

$$\check{F}(e(\rho)) = \int_0^1 \check{A}(\rho, \eta) \omega(\eta, e(\eta)) d\eta.$$

Under our assumption, e solves (15) precisely when $e \in \check{E}$ solves the integral equation.

$$g(\rho) = \int_0^1 \check{A}(\rho, \eta) \omega(\eta, g(\eta)) d\eta, \quad \forall \rho \in [0, 1].$$

Consider,

$$\begin{aligned} |\check{F}e(\rho) - \check{F}f(\rho)| &= \left| \int_0^1 \check{A}(\rho, \eta) \omega(\eta, e(\eta)) d\eta - \int_0^1 \check{A}(\rho, \eta) \omega(\eta, f(\eta)) d\eta \right| \\ &\leq \int_0^1 |\check{A}(\rho, \eta) (\omega(\eta, e(\eta)) - \omega(\eta, f(\eta)))| d\eta \\ &\leq \int_0^1 \check{A}(\rho, \eta) |(\omega(\eta, e(\eta)) - \omega(\eta, f(\eta)))| d\eta \\ &\leq \int_0^1 \check{A}(\rho, \eta) \sigma |e(\eta) - f(\eta)| d\eta. \end{aligned}$$

Taking supremum on both sides, we get

$$\vartheta(\check{F}e, \check{F}f) \leq \sigma \vartheta(e, f).$$

Therefore, all the hypothesis of theorem 3.1 and 4.1 are satisfied. Hence, \check{F} has a unique solution in $\check{V}[0, 1]$. \square

6. CONCLUSION

In this work, we have explored FP results within the framework of bipolar \mathcal{MS} s by introducing and analyzing polynomial-type contractions and almost polynomial contractions. The generalization of classical contraction principles under this extended metric structure offers a richer context for addressing FP problems. The results obtained not only unify and extend several existing theorems in the literature but also establish a novel approach applicable to fractional differential equations. The applicability of the developed FPT to problems in fractional calculus highlights the potential of bipolar \mathcal{MS} s as a powerful tool in nonlinear analysis. Future research may focus on extending these results to multivalued mappings, dynamic systems, and other abstract spaces, thereby broadening the theoretical and practical impact of the developed concepts.

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