

Bohr-Sommerfeld Quantization Rules for 1-D Semiclassical Pseudo-Differential Operator: the Method of Microlocal Wronskian and Gram Matrix

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Abstract

In this paper, we revisit the well known Bohr-Sommerfeld quantization rule (BS) of order 2 for a self-adjoint 1-D semiclassical pseudo-differential operator, within the algebraic and microlocal framework of B. Helffer and J. Sjöstrand. BS holds precisely when the Gram matrix consisting of scalar products of certain WKB solutions with respect to the "flux norm" is not invertible. This condition is obtained using the microlocal Wronskian and does not rely on traditional matching techniques. It is simplified by using action-angle variables. The interest of this procedure lies in its possible generalization to matrixvalued Hamiltonians, like BdG Hamiltonian.

1 Introduction

Bohr-Sommerfeld quantization rule, in its first formulation, allows to compute the energy levels E of a particle in a one-dimensional potential well, its dynamics being described by the semi-classical Schrödinger operator

$$P(x, hD_x) = (hD_x)^2 + V(x), \quad \text{with } D_x = -i\frac{d}{dx}.$$

It is given at first order in h by the well known formula

$$\frac{1}{2\pi h} \oint_{\gamma_E} \xi(x) dx = n + \frac{1}{2}$$

Here $\xi(x) = \sqrt{E - V(x)}$ denotes the momentum of the particle on its orbit $\gamma_E \subset T^*\mathbb{R}$ above the potential well $\{V(x) \leq E\}$, n is an integer and the integral is computed over γ_E in the phase space $T^*\mathbb{R}$.

By the implicit function theorem, we then find $E = E_n(h)$. In other words, the number of wavelengths (associated with the particle along γ_E by de Broglie correspondence) must be an integer plus $1/2$, called *Maslov correction*. Let $p(x, \xi; h)$ be a smooth real classical Hamiltonian on $T^*\mathbb{R}$, admitting a semiclassical expansion

$$p(x, \xi; h) \sim p_0(x, \xi) + h p_1(x, \xi) + h^2 p_2(x, \xi) + \cdots, \quad h \rightarrow 0,$$

where p_0 is the principal symbol, and p_1 the sub-principal symbol. We assume that $p \in S^0(m)$ for some order function m , and that $p + i$ is elliptic. Here, $S^0(m)$ denotes the class of symbols satisfying

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi; h)| \leq C_{\alpha, \beta} m(x, \xi), \quad \forall \alpha, \beta \in \mathbb{N}, (x, \xi) \in T^*\mathbb{R},$$

for some order function $m(x, \xi) \geq 1$, uniformly in $h \in (0, h_0]$. This allows to take the Weyl quantization of p , namely

$$P(x, hD_x; h)u(x; h) = (Op^W(p)(u))(x) = (2\pi h)^{-1} \int \int_{\mathbb{R} \times \mathbb{R}} e^{\frac{i}{h}(x-y)\eta} p\left(\frac{x+y}{2}, \eta\right) u(y) dy d\eta. \quad (1.1)$$

We make the geometric assumption (H) of [7]: fix a compact interval $I = [E_-, E_+]$, and assume that there exists a topological ring $\mathcal{A} \subset p_0^{-1}(I)$, such that $\partial\mathcal{A} = A_- \cup A_+$, with A_\pm connected components of $p_0^{-1}(E_\pm)$, and that p_0 has no critical point in \mathcal{A} . Moreover, A_- is included in the disk bounded by A_+ . (if it is not the case, we can always change p to $-p$). These conditions guarantee that the spectrum of P in I is discrete and can be described semiclassically.

We define the microlocal well W as the disk bounded by A_+ . For each $E \in I$, let $\gamma_E \subset W$ be a periodic orbit on the energy surface $\{p_0(x, \xi) = E\}$, so that γ_E is an embedded Lagrangian manifold.

Let $\mathcal{K}_h^N(E)$ be the microlocal kernel of $P - E$ of order N , i.e. the space of local solutions of $(P - E)u = \mathcal{O}(h^{N+1})$ in the distributional sense, microlocalized on γ_E . This is a smooth complex vector bundle over $\pi_x(\gamma_E)$. Here we address the problem of finding the set of $E = E(h)$ such that $\mathcal{K}_h^N(E)$ contains a global section, i.e. of constructing a sequence of quasi-modes $(u_n(h), E_n(h))$ of a given order N . As usual we denote by $\mathcal{K}_h(E)$ the microlocal kernel of $P - E \bmod \mathcal{O}(h^\infty)$; since the distinction between $\mathcal{K}_h^N(E)$ and $\mathcal{K}_h(E)$ plays no important role here, we shall content to write $\mathcal{K}_h(E)$.

Then if $E_+ < E_0 = \liminf_{|x, \xi| \rightarrow +\infty} p_0(x, \xi)$, all eigenvalues of P in I are indeed by *Bohr-Sommerfeld quantization condition* (BS) $\mathcal{S}_h(E_n(h)) = 2\pi nh$, where the semiclassical action $\mathcal{S}_h(E)$ has the asymptotics

$$\mathcal{S}_h(E) \sim S_0(E) + hS_1(E) + h^2S_2(E) + \dots$$

We determine BS at any accuracy by computing quasi-modes. There are a lot of ways to derive BS: the method of matching of WKB solutions [4], known also as Liouville-Green method [29], which has received many improvements (see [36]), the method of the monodromy operator (see [16] and references therein), the method of quantization deformation based on Functional Calculus and Trace Formulas [22], [7], [30], [14], [1].

Note that the latter one already assumes BS, it only gives a very convenient way to derive it. In the real analytic case, BS rule, and also tunneling expansions, can be obtained using the so-called "exact WKB method" see e.g. [13], [10], [11] when $P(x, hD_x) = (hD_x)^2 + V(x)$ is Schrödinger operator.

Here we present another way to construct quasi-modes of order 2, based on [32], [15]. We stress that our method in the present scalar case, when carried to second order, is a bit more intricate than [22], [7] and its refinements [14]; it is most useful for matrix valued operators with double characteristics such as Bogoliubov-de Gennes Hamiltonian ([21], [26], [12]), or Born-Oppenheimer type Hamiltonians ([2], [31]).

Example 1.1. (BS quantization rule for the Harmonic Oscillator on \mathbb{R})

We consider the one-dimensional quantum harmonic oscillator given by the semiclassical operator:

$$P_0(x, hD_x) = \frac{1}{2} (x^2 + (hD_x)^2)$$

where h is the semiclassical parameter. The associated classical Hamiltonian (principal symbol) is:

$$p_0(x, \xi) = \frac{1}{2} (x^2 + \xi^2)$$

In the phase space (x, ξ) , the energy surface $\{p_0(x, \xi) = E\}$ is a circle of radius $\sqrt{2E}$. The classical motion along this orbit is described by:

$$x(t) = \sqrt{2E} \cos(t), \quad \xi(t) = -\sqrt{2E} \sin(t),$$

which is a closed trajectory with period $T = 2\pi$. To apply the Bohr-Sommerfeld quantization rule, we compute the classical action $S_0(E)$, which is the integral of the momentum along the closed orbit γ_E at energy E :

$$S_0(E) = \oint_{\gamma_E} \xi \, dx.$$

Using the parametrization of the orbit, we compute:

$$\begin{aligned} S_0(E) &= \int_0^{2\pi} \xi(t) \dot{x}(t) \, dt \\ &= \int_0^{2\pi} \left(-\sqrt{2E} \sin t \right) \cdot \left(-\sqrt{2E} \sin t \right) \, dt \\ &= 2E \int_0^{2\pi} \sin^2 t \, dt \\ &= 2E \cdot \pi = 2\pi E. \end{aligned}$$

The Bohr-Sommerfeld quantization condition, including the Maslov correction, reads:

$$S_0(E) + hS_1(E) = 2\pi h n, \quad n \in \mathbb{N}.$$

For the harmonic oscillator, the Maslov index is $\mu = 2$, which gives:

$$S_1(E) = -\frac{\pi}{2}.$$

This leads to the equation:

$$2\pi E - \pi h = 2\pi h n \quad \Rightarrow \quad E = h \left(n + \frac{1}{2} \right).$$

Therefore, the quantized energy levels of the harmonic oscillator are:

$$E_n = h \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots$$

These are exactly the energy levels of the quantum harmonic oscillator.

2 The microlocal Wronskian

The best algebraic and microlocal framework for computing 1-D quantization rules in the self-adjoint case, developed in the fundamental works of [32], [15], is based on Fredholm theory and the classical *positive commutator method*, which involves conservation of a quantity called *quantum flux*.

Bohr-Sommerfeld quantization rules are derived by constructing quasi-modes using the WKB approximation along a closed Lagrangian manifold $\Lambda_E \subset \{p_0 = E\}$, i.e. a periodic orbit of the Hamiltonian vector field H_p with energy E . This construction is local and depends on the rank of the projection $\Lambda_E \rightarrow \mathbb{R}_x$.

Thus, the set $K_h(E)$ of microlocal solutions to $(P - E)u = 0$ along Λ_E can be seen as a bundle over \mathbb{R} with a compact base, corresponding to the classically allowed region at energy E . The eigenvalues $E_n(h)$ are then determined by the condition that the global quasi-mode obtained by gluing local WKB solutions along Λ_E is singlevalued, i.e. that $K_h(E)$ has trivial holonomy.

Assuming Λ_E is smoothly embedded in $T^*\mathbb{R}$, it can always be parametrized by a non-degenerate phase function. Of particular interest are the *focal points*, i.e. critical points of the phase functions, which are responsible for the change in Maslov index. A point $a_E = (x_E, \xi_E) \in \Lambda_E$ is a focal point if Λ_E "turns vertical" at a_E , meaning that the tangent space $T_{a_E}\Lambda_E$ is no longer transverse to the fiber $x = \text{const.}$ in $T^*\mathbb{R}$.

In any case however, Λ_E can locally be parametrized either by a phase function $S(x)$ (spatial representation) or $\tilde{S}(\xi)$ (Fourier representation). We fix an orientation on Λ_E and for any point $a \in \Lambda_E$ (not necessarily a focal point), we denote by $\rho = \pm 1$ the oriented segments near a . Let $\chi^a \in C_0^\infty(\mathbb{R}^2)$ be a smooth cut-off function equal to 1 near a , and ω_ρ^a a small neighborhood of $\text{supp}[P, \chi^a] \cap \Lambda_E$ near ρ . Here, χ^a holds for $\chi^a(x, hD_x)$ as in (1.1), and we shall equally write $P(x, hD_x)$ in spatial representation, or $P(-hD_\xi, \xi)$ in Fourier representation.

Definition 2.1. Let P be self-adjoint and $u_a, v_a \in K_h(E)$ be microlocal solutions supported on Λ_E . We define the microlocal Wronskian of $(u^a, \overline{v^a})$ near a in ω_ρ^a as

$$\mathcal{W}_\rho^a(u^a, \overline{v^a}) = \left(\frac{i}{h} [P, \chi^a]_\rho u^a | v^a\right) \quad (2.1)$$

where $\frac{i}{h} [P, \chi^a]_\rho$ denotes the part of the commutator supported microlocally on ω_ρ^a .

To clarify the meaning of this definition, consider the Schrödinger operator $P(x, hD_x) = (hD_x)^2 + V(x)$, with $x_E = 0$, and take χ to be the Heaviside step function $\chi(x)$. Then, in the distributional sense,

$$\frac{i}{h} [P, \chi] = -ih\chi'' + 2\chi' hD_x = -ih\delta' + 2\delta hD_x,$$

so that $\left(\frac{i}{h} [P, \chi] u | u\right) = -ih(u'(0)\overline{u(0)} - u(0)\overline{u'(0)})$ which is the usual Wronskian of (u, \bar{u}) .

Proposition 2.1. Let $u^a, v^a \in K_h(E)$, and denote by \hat{u} the h -Fourier (unitary) transform of u . Then:

$$\mathcal{W} = \left(\frac{i}{h} [P, \chi^a] u^a | v^a\right) = \left(\frac{i}{h} [P, \chi^a] \hat{u}^a | \hat{v}^a\right) = 0 \quad (2.2)$$

and

$$\mathcal{W}_+^a(u^a, \overline{v^a}) = -\mathcal{W}_-^a(u^a, \overline{v^a}) \quad (2.3)$$

(all equalities being understood mod $\mathcal{O}(h^\infty)$, resp $\mathcal{O}(h^{N+1})$ when considering $u^a, v^a \in K_h(E)$). Moreover, $\mathcal{W}_\rho^a(u^a, \overline{v^a})$ does not depend modulo $\mathcal{O}(h^\infty)$ (resp. $\mathcal{O}(h^{N+1})$) on the choice of χ^a above.

Proof. Since $u^a, v^a \in K_h(E)$ are distributions in L^2 , the first equality (2.2) follows from the Plancherel formula and the regularity of microlocal solutions in L^2 , $p + i$ being elliptic. If a is not a focal point, u^a, v^a are smooth WKB solutions near a , so we can expand the commutator in $\mathcal{W} = \left(\frac{i}{h} [P, \chi^a] u^a | v^a\right)$ and use that P is self-adjoint to show that $\mathcal{W} = \mathcal{O}(h^\infty)$. If a is a focal point, u^a, v^a are smooth WKB solutions in Fourier representation, so again $\mathcal{W} = \mathcal{O}(h^\infty)$. Then (2.3) follows from Definition 2.1. \square

3 Second-order BS quantization for a self-adjoint 1-D h -PDO

We apply the method of the microlocal Wronskian and Gram matrix, to derive BS quantization conditions at order 2 for a h -PDO of the type (1.1). To simplify, we assume that the principal symbol p_0 contains only two focal points along the classical orbit γ_E , but it is clear that by matching together microlocal solutions, the result does not depend on this simplification.

In fact, BS depend on the geometry of γ_E only through its Maslov index, which equals 2 when γ_E is a smooth embedded Lagrangian submanifold. The case where γ_E is not a submanifold (for example, homeomorphic to the figure-eight) and has Maslov index 0, is not considered here (see [32] or [8]). (Recall that in dimension 1, the Maslov index, defined modulo 4, is an even number, hence either 0 or 2.)

Our main result is the following:

Theorem 3.1. Let $P(x, hD_x; h)$ be a self-adjoint h -PDO, given as the Weyl quantization of a real classical symbol

$$p(x, \xi; h) \sim p_0(x, \xi) + h p_1(x, \xi) + h^2 p_2(x, \xi) + \dots$$

Assume that the geometry of p_0 satisfies the hypothesis (H) of Section 1, and that $E_+ < E_0 = \liminf_{|x, \xi| \rightarrow +\infty} p_0(x, \xi)$. Then the spectrum of P in a fixed energy interval $I \subset \mathbb{R}$ is discrete, and given by the BS quantization condition:

$$S_h(E) := S_0(E) + h S_1(E) + h^2 S_2(E) + \dots = 2\pi n h, \quad n \in \mathbb{Z}$$

where:

- $S_0(E) = \oint_{\gamma_E} \xi(x) dx = \int \int_{\{p_0 \leq E\} \cap W} d\xi \wedge dx$ is the classical action along the closed orbit $\gamma_E \subset \{p_0 = E\}$;
- $S_1(E) = -\pi - \int_{\gamma_E} p_1(x(t), \xi(t)) dt$ is the first-order correction, including the Maslov index and the integral of the subprincipal 1-form $p_1 dt$;
- $S_2(E)$ is the second-order correction, given by

$$S_2(E) = \frac{1}{24} \frac{d}{dE} \int_{\gamma_E} \Delta dt - \int_{\gamma_E} p_2 dt - \frac{1}{2} \frac{d}{dE} \int_{\gamma_E} p_1^2 dt$$

with

$$\Delta(x, \xi) = \frac{\partial^2 p_0}{\partial x^2} \frac{\partial^2 p_0}{\partial \xi^2} - \left(\frac{\partial^2 p_0}{\partial x \partial \xi} \right)^2,$$

and $(x(t), \xi(t))$ is a parametrization of γ_E by the Hamiltonian flow.

Let us note that the deformation quantization method (see [7]), easily recovers this result, as well as higher-order terms, in particular $S_4(E)$ (see [30] and [14] for a diagrammatic approach). Recall that all odd-order terms $S_j(E)$ with $j \geq 3$ vanish.

3.1 Quasi-modes mod $\mathcal{O}(h^2)$ in Fourier representation

We first recall Hörmander's asymptotic stationary phase theorem (see e.g. [17], Theorem 7.7.5):

Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}$ be a function such that $\text{Im}(\varphi(x)) \geq 0$, and suppose that φ has a non-degenerate critical point at x_0 . Then, we have the asymptotic expansion:

$$\int_{\mathbb{R}^d} e^{\frac{i}{h} \varphi(x)} u(x) dx \sim e^{\frac{i}{h} \varphi(x_0)} \left(\det \left(\frac{\varphi''(x_0)}{2i\pi h} \right) \right)^{-\frac{1}{2}} \sum_j h^j L_j u(x_0) \quad (3.1)$$

where L_j are linear differential operators, with $L_0 u(x_0) = u(x_0)$ and in particular:

$$L_1 u(x_0) = \sum_{n=0}^2 \frac{2^{-(n+1)}}{in! (n+1)!} \langle (\varphi''(x_0))^{-1} D_x, D_x \rangle^{n+1} (\Phi_{x_0}^n u)(x_0) \quad (3.2)$$

with:

$$\Phi_{x_0}(x) = \varphi(x) - \varphi(x_0) - \frac{1}{2} \langle \varphi''(x_0), (x - x_0), x - x_0 \rangle \quad (3.3)$$

We note that Φ_{x_0} vanishes to order 3 at x_0 (i.e., $\Phi_{x_0}(x_0) = 0$, $\Phi'_{x_0}(x_0) = 0$, $\Phi''_{x_0}(x_0) = 0$).

In the sequel, we present formulas with accuracy up to the second order in h . It is helpful to start building the quasi-modes from a focal point, because this gives both outgoing and incoming approximate solutions at the same time.

For $E \in I$, let $a_E = (x_E, \xi_E) \in \gamma_E$ be such that $\left(\frac{\partial p_0}{\partial \xi}\right)(a_E) = 0$ (i.e., a_E is a focal point). Since $\left(\frac{\partial p_0}{\partial x}\right)(a_E) \neq 0$, the orbit γ_E can be locally parametrized near a_E using a phase function $\psi(\xi) = \psi(\xi; E)$, which satisfies the Hamilton-Jacobi equation:

$$p_0(-\psi'(\xi), \xi) = E \quad (3.4)$$

and is normalized by $\psi(\xi_E) = 0$. We then look for an asymptotic solution of $(P(x, hD_x; h) - E)u(x; h) = 0$ of the form

$$u(x; h) = (2\pi h)^{-1/2} \int e^{\frac{i}{h}x\xi} \hat{u}(\xi; h) d\xi = (2\pi h)^{-1/2} \int e^{\frac{i}{h}(x\xi + \psi(\xi))} b(\xi; h) d\xi \quad (3.5)$$

where ψ, b depend also on E . We aim to compute:

$$Pu(x; h) = (2\pi h)^{-\frac{3}{2}} \int \int e^{\frac{i}{h}((x-y)\eta + y\xi + \psi(\xi))} p\left(\frac{x+y}{2}, \eta; h\right) b(\xi; h) d\xi dy d\eta \quad (3.6)$$

In (3.6), we integrate with respect to the variables (y, η) . For fixed ξ , the phase is:

$$(y, \eta) \mapsto (x-y)\eta + y\xi$$

with critical point $(y_c, \eta_c) = (x, \xi)$. Let us set the change of variables:

$$y - x = 2y', \quad \eta - \xi = \eta'$$

The Jacobian of this transformation is 2, so:

$$Pu(x; h) = 2(2\pi h)^{-\frac{3}{2}} \int \int \int e^{\frac{i}{h}(-2y'\eta' + x\xi + \psi(\xi))} p(x + y', \xi + \eta'; h) b(\xi; h) d\xi dy' d\eta' \quad (3.7)$$

Now, for fixed ξ , applying the stationary phase formula to the variables (y', η') gives:

$$\begin{aligned} \int \int e^{-\frac{i}{h}2y'\eta'} p(x + y', \xi + \eta'; h) dy' d\eta' &\sim \pi h \sum_{j=0}^{N-1} \frac{1}{j!(2/h)^{jij}} (\partial_{y'}^j \partial_{\eta'}^j p(x + y', \xi + \eta'; h))_{(y', \eta')=(0,0)} \\ &\sim \pi h (p(x, \xi; h) + \frac{h}{2i} \frac{\partial^2 p}{\partial x \partial \xi}(x, \xi; h) - \frac{h^2}{8} \frac{\partial^4 p}{\partial x^2 \partial \xi^2}(x, \xi; h) + \mathcal{O}(h^3)) \end{aligned}$$

Hence:

$$\begin{aligned} Pu(x; h) &= (2\pi h)^{-\frac{1}{2}} \int e^{\frac{i}{h}(x\xi + \psi(\xi))} b(\xi; h) (p(x, \xi; h) + \frac{h}{2i} \frac{\partial^2 p}{\partial x \partial \xi}(x, \xi; h) - \frac{h^2}{8} \frac{\partial^4 p}{\partial x^2 \partial \xi^2}(x, \xi; h) + \mathcal{O}(h^3)) d\xi \\ &= (2\pi h)^{-\frac{1}{2}} \int e^{\frac{i}{h}(x\xi + \psi(\xi))} b(\xi; h) (p_0(x, \xi) + h\tilde{p}_1(x, \xi) + h^2\tilde{p}_2(x, \xi) + \mathcal{O}(h^3)) d\xi \end{aligned}$$

where

$$\tilde{p}_1(x, \xi) = p_1(x, \xi) + \frac{1}{2i} \frac{\partial^2 p_0}{\partial x \partial \xi}(x, \xi) \quad (3.8)$$

and

$$\tilde{p}_2(x, \xi) = p_2(x, \xi) + \frac{1}{2i} \frac{\partial^2 p_1}{\partial x \partial \xi}(x, \xi) - \frac{1}{8} \frac{\partial^4 p_0}{\partial x^2 \partial \xi^2}(x, \xi) \quad (3.9)$$

Following ([6], the Maslov Ansatz and Theorem 43), we look for $b(\xi; h) \sim b_0(\xi) + hb_1(\xi) + \dots$, a classical elliptic symbol, with $b_0(0) \neq 0$, such that there exists a symbol $a(x, \xi; h) \sim a_0(x, \xi) + ha_1(x, \xi) + \dots$ satisfying:

$$hD_\xi(e^{\frac{i}{h}(x\xi + \psi(\xi))} a(x, \xi; h)) = e^{\frac{i}{h}(x\xi + \psi(\xi))} b(\xi; h) (p_0(x, \xi) - E + h\tilde{p}_1(x, \xi) + h^2\tilde{p}_2(x, \xi) + \mathcal{O}(h^3)) \quad (3.10)$$

or more explicitly

$$(x + \psi'(\xi)) a(x, \xi; h) + h D_\xi a(x, \xi; h) = b(\xi; h) (p_0(x, \xi) - E + h \tilde{p}_1(x, \xi) + h^2 \tilde{p}_2(x, \xi) + \mathcal{O}(h^3)) \quad (3.11)$$

In order to solve (3.11) for ξ near ξ_E , x near x_E , E near E_0 , it is sufficient to solve the sequence of equations,

$$(x + \psi'(\xi)) a_0(x, \xi) = (p_0(x, \xi) - E) b_0(\xi) \quad (3.12)$$

$$(x + \psi'(\xi)) a_1(x, \xi) + D_\xi a_0(x, \xi) = (p_0(x, \xi) - E) b_1(\xi) + \tilde{p}_1(x, \xi) b_0(\xi) \quad (3.13)$$

$$(x + \psi'(\xi)) a_2(x, \xi) + D_\xi a_1(x, \xi) = (p_0(x, \xi) - E) b_2(\xi) + \tilde{p}_1(x, \xi) b_1(\xi) + \tilde{p}_2(x, \xi) b_0(\xi) \quad (3.14)$$

Here, we have grouped the terms according to the powers of h in equation (3.11); equation (3.12) is obtained by annihilating the term in h^0 , equation (3.13) by annihilating the term in h^1 , and equation (3.14) by annihilating the term in h^2 . We define the function $\lambda(x, \xi)$ by:

$$\lambda(x, \xi) := \frac{p_0(x, \xi) - E}{x + \psi'(\xi)} \quad (3.15)$$

From equation (3.15), we deduce that:

$$\lambda(-\psi'(\xi), \xi) = (\partial_x p_0)(-\psi'(\xi), \xi) := \alpha(\xi) \quad (3.16)$$

Differentiating both sides of equation (3.4) with respect to ξ gives:

$$\psi''(\xi) = \frac{(\partial_\xi p_0)(-\psi'(\xi), \xi)}{\alpha(\xi)} \quad (3.17)$$

which vanishes at ξ_E . For a given b_0 , the unique solution of (3.12) is $a_0(x, \xi) = \lambda(x, \xi) b_0(\xi)$. In order to solve equation (3.13), it is necessary and sufficient that

$$(D_\xi a_0)(-\psi'(\xi), \xi) = \tilde{p}_1(-\psi'(\xi), \xi) b_0(\xi) \quad (3.18)$$

This is equivalent to:

$$(\partial_\xi \lambda)(-\psi'(\xi), \xi) b_0(\xi) + \alpha(\xi) b_0'(\xi) = \left(i p_1(-\psi'(\xi), \xi) + \frac{1}{2} \left(\frac{\partial^2 p_0}{\partial x \partial \xi} \right)(-\psi'(\xi), \xi) \right) b_0(\xi) \quad (3.19)$$

A direct computation from (3.15) shows that:

$$(\partial_x \lambda)(-\psi'(\xi), \xi) = \frac{1}{2} \left(\frac{\partial^2 p_0}{\partial x^2} \right)(-\psi'(\xi), \xi) \quad (3.20)$$

Differentiating both sides of equation (3.16) with respect to ξ gives:

$$(\partial_\xi \lambda)(-\psi'(\xi), \xi) = -\frac{\psi''(\xi)}{2} \left(\frac{\partial^2 p_0}{\partial x^2} \right)(-\psi'(\xi), \xi) + \left(\frac{\partial^2 p_0}{\partial x \partial \xi} \right)(-\psi'(\xi), \xi) \quad (3.21)$$

Substituting this expression into the equation (3.19) gives the differential equation for b_0 :

$$\alpha(\xi) b_0'(\xi) + \left(\frac{1}{2} \alpha'(\xi) - i p_1(-\psi'(\xi), \xi) \right) b_0(\xi) = 0 \quad (3.22)$$

whose general solution is:

$$b_0(\xi) = C_0 |\alpha(\xi)|^{-\frac{1}{2}} \exp \left(i \int \frac{p_1(-\psi'(\xi), \xi)}{\alpha(\xi)} d\xi \right) \quad (3.23)$$

In order to solve (3.14) it is necessary and sufficient that

$$(D_\xi a_1)(-\psi'(\xi), \xi) = \tilde{p}_1(-\psi'(\xi), \xi) b_1(\xi) + \tilde{p}_2(-\psi'(\xi), \xi) b_0(\xi) \quad (3.24)$$

From equation (3.13), we get:

$$a_1(x, \xi) = \lambda(x, \xi) b_1(\xi) + \lambda_0(x, \xi) \quad (3.25)$$

where

$$\lambda_0(x, \xi) = \frac{\tilde{p}_1(x, \xi) b_0(\xi) + i \partial_\xi a_0(x, \xi)}{x + \psi'(\xi)} \quad (3.26)$$

Before continuing, we state a lemma we will use.

Lemma 3.1.

$$\lambda_0(-\psi'(\xi), \xi) = b_0(\xi) \left(\partial_x p_1 - \frac{p_1}{2\alpha} \frac{\partial^2 p_0}{\partial x^2} \right)_{x=-\psi'(\xi)} - i b_0(\xi) \left(\frac{\psi''(\xi)}{6} \frac{\partial^3 p_0}{\partial x^3} + \frac{\alpha'}{4\alpha} \frac{\partial^2 p_0}{\partial x^2} \right)_{x=-\psi'(\xi)} \quad (3.27)$$

$$\left(\frac{\partial \lambda_0}{\partial x} \right)(-\psi'(\xi), \xi) = \frac{b_0(\xi)}{2} \left(\frac{\partial^2 p_1}{\partial x^2} - \frac{p_1}{3\alpha} \frac{\partial^3 p_0}{\partial x^3} \right)_{x=-\psi'(\xi)} - i \frac{b_0(\xi)}{12} \left(\frac{\partial^4 p_0}{\partial x^3 \partial \xi} + \frac{\psi''(\xi)}{2} \frac{\partial^4 p_0}{\partial x^4} + \frac{\alpha'}{\alpha} \frac{\partial^3 p_0}{\partial x^3} \right)_{x=-\psi'(\xi)} \quad (3.28)$$

$$\left(\frac{\partial^n \lambda}{\partial x^n} \right)(-\psi'(\xi), \xi) = \frac{1}{n+1} \left(\frac{\partial^{n+1} p_0}{\partial x^{n+1}} \right)(-\psi'(\xi), \xi); \forall n \in \mathbb{N} \quad (3.29)$$

Differentiating both sides of equation (3.25) with respect to ξ and evaluating at $x = -\psi'(\xi)$ gives

$$(D_\xi a_1)(-\psi'(\xi), \xi) = \frac{1}{i} (\partial_\xi \lambda)(-\psi'(\xi), \xi) b_1(\xi) + \frac{1}{i} \alpha(\xi) b_1'(\xi) + \frac{1}{i} (\partial_\xi \lambda_0)(-\psi'(\xi), \xi) \quad (3.30)$$

Then, comparing this with equation (3.24), we see that b_1 must satisfy the differential equation

$$\alpha(\xi) b_1'(\xi) + \left(\frac{1}{2} \alpha'(\xi) - i p_1(-\psi'(\xi), \xi) \right) b_1(\xi) = i \tilde{p}_2(-\psi'(\xi), \xi) b_0(\xi) - (\partial_\xi \lambda_0)(-\psi'(\xi), \xi) \quad (3.31)$$

The homogeneous part of this equation is the same as in (3.22); therefore, we seek a particular solution of the form

$$D_1(\xi) |\alpha(\xi)|^{-\frac{1}{2}} \exp \left(i \int_{\xi_E}^{\xi} \frac{p_1(-\psi'(\zeta), \zeta)}{\alpha(\zeta)} d\zeta \right) \quad (3.32)$$

Using variation of constants, we find

$$D_1(\xi) = \text{sgn}(\alpha(\xi_E)) \int_{\xi_E}^{\xi} |\alpha(\zeta)|^{-\frac{1}{2}} \left(i b_0(\zeta) \tilde{p}_2(-\psi'(\zeta), \zeta) - (\partial_\zeta \lambda_0)(-\psi'(\zeta), \zeta) \right) \exp \left(-i \int_{\xi_E}^{\zeta} \frac{p_1(-\psi'(s), s)}{\alpha(s)} ds \right) d\zeta \quad (3.33)$$

We normalize by setting

$$D_1(\xi_E) = 0 \quad (3.34)$$

So the general solution of the equation is:

$$b_1(\xi) = (C_1 + D_1(\xi)) |\alpha(\xi)|^{-\frac{1}{2}} \exp \left(i \int_{\xi_E}^{\xi} \frac{p_1(-\psi'(\zeta), \zeta)}{\alpha(\zeta)} d\zeta \right) \quad (3.35)$$

It follows that:

$$b_0(\xi) + h b_1(\xi) = (C_0 + h C_1 + h D_1(\xi)) |\alpha(\xi)|^{-\frac{1}{2}} \exp \left(i \int_{\xi_E}^{\xi} \frac{p_1(-\psi'(\zeta), \zeta)}{\alpha(\zeta)} d\zeta \right) \quad (3.36)$$

The integration constants C_0 and $C_1 = C_1(a_E)$ will be determined by normalizing the microlocal Wronskians as follows

3.2 Normalisation

We compute the microlocal Wronskian of $(u^a, \overline{u^a}) = (u, \overline{u})$ in ω_ρ^a . Our goal is to normalize the microlocal solution

$$\hat{u}(\xi; h) = e^{\frac{i}{h}\psi(\xi)} b(\xi; h)$$

using the microlocal Wronskian. That is, we seek constants C_0 and $C_1 = C_1(a_E)$ such that

$$\mathcal{W}^a(\hat{u}, \overline{\hat{u}}) = 1 + \mathcal{O}(h^2).$$

In the Fourier representation, we write:

$$\mathcal{W}_\rho^a(\hat{u}, \overline{\hat{u}}) = \left(\frac{i}{h} [P, \chi^a]_\rho \hat{u} | \hat{u}\right)$$

where $\chi^a \in C_0^\infty(\mathbb{R}^2)$ is a smooth cut-off equal to 1 near the focal point $a = a_E$. Without loss of generality, we can take $\chi^a(x, \xi) = \chi_1(x) \chi_2(\xi)$, with $\chi_2 \equiv 1$ on small neighborhoods ω_\pm^a , of $\text{supp}(\frac{i}{h} [P, \chi^a]) \cap \{p_0(x, \xi) = E\}$ in the region $\pm(\xi - \xi_E) > 0$. Therefore, it is sufficient to consider variations of the function $\chi_1(x)$ only.

In general, if P and Q are two h -pseudodifferential operators, whose Weyl symbols admit the expansions

$$\sigma^W(P)(x, \xi; h) = p_0(x, \xi) + h p_1(x, \xi) + h^2 p_2(x, \xi) + \dots$$

$$\sigma^W(Q)(x, \xi; h) = q_0(x, \xi) + h q_1(x, \xi) + h^2 q_2(x, \xi) + \dots$$

then we have

$$\sigma^W\left(\frac{i}{h} [P, Q]\right) = \{\sigma^W(P), \sigma^W(Q)\} + \mathcal{O}(h^2)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. In particular,

$$\sigma^W\left(\frac{i}{h} [P, \chi^a]\right)(x, \xi; h) := c(x, \xi; h) = (\partial_\xi p_0(x, \xi) + h \partial_\xi p_1(x, \xi)) \chi_1'(x) + \mathcal{O}(h^2)$$

Using the Weyl calculus, the operator acts in Fourier representation as

$$c^w(-hD_\xi, \xi; h) v(\xi; h) = (2\pi h)^{-1} \int \int e^{-\frac{i}{h}(\xi - \eta)y} c\left(y, \frac{\xi + \eta}{2}; h\right) v(\eta; h) dy d\eta$$

Hence, applying to \hat{u} , we get

$$\frac{i}{h} [P, \chi^a] \hat{u}(\xi; h) = (2\pi h)^{-1} \int \int e^{\frac{i}{h}(\psi(\eta) - (\xi - \eta)y)} c\left(y, \frac{\xi + \eta}{2}; h\right) b(\eta; h) dy d\eta$$

For fixed ξ , the phase function corresponding to the oscillatory integral defining $\frac{i}{h} [P, \chi^a] \hat{u}$ is given by

$$\varphi_\xi(y, \eta) = \psi(\eta) - (\xi - \eta)y$$

The critical points of φ_ξ are

$$(y_c(\xi), \eta_c(\xi)) = (-\psi'(\xi), \xi),$$

and therefore, the corresponding critical values of φ_ξ are

$$\varphi_\xi(y_c(\xi), \eta_c(\xi)) = \varphi_\xi(-\psi'(\xi), \xi) = \psi(\xi)$$

A direct computation shows that

$$(\text{Hess } \varphi_\xi)(y_c(\xi), \eta_c(\xi)) = \begin{pmatrix} 0 & 1 \\ 1 & \psi''(\xi) \end{pmatrix}$$

Let $c_j(y, \eta) := \partial_\eta p_j(y, \eta) \chi_1'(y)$, for all $j \in \{0, 1\}$. By the stationary phase theorem (3.1), we obtain:

$$\frac{i}{h} [P, \chi^a] \hat{u}(\xi; h) = e^{\frac{i}{h} \psi(\xi)} (d_0(\xi) + h d_1(\xi) + \mathcal{O}(h^2))$$

with

$$d_0(\xi) = c_0(-\psi'(\xi), \xi) b_0(\xi)$$

and

$$d_1(\xi) = c_0(-\psi'(\xi), \xi) b_1(\xi) + c_1(-\psi'(\xi), \xi) b_0(\xi) + \frac{i}{2} J(\xi)$$

where

$$J(\xi) = e_0'(\xi) b_0(\xi) + 2 e_0(\xi) b_0'(\xi)$$

and where we have set

$$e_0(\xi) = \partial_x c_0(-\psi'(\xi), \xi)$$

It follows that

$$\begin{aligned} \mathcal{W}_+^a(\hat{u}, \bar{u}) &= \int_{\xi_E}^{+\infty} d_0(\xi) \overline{b_0(\xi)} d\xi + h \int_{\xi_E}^{+\infty} (d_0(\xi) \overline{b_1(\xi)} + d_1(\xi) \overline{b_0(\xi)}) d\xi + \mathcal{O}(h^2) \\ &= \mathcal{M}_0^+ + h \mathcal{M}_1^+ + \mathcal{O}(h^2) \end{aligned}$$

First, we have

$$\begin{aligned} \mathcal{M}_0^+ &= \int_{\xi_E}^{+\infty} \partial_\xi p_0(-\psi'(\xi), \xi) \chi_1'(-\psi'(\xi)) |b_0(\xi)|^2 d\xi \\ &= |C_0|^2 \int_{\xi_E}^{+\infty} \frac{\alpha(\xi)}{|\alpha(\xi)|} \psi''(\xi) \chi_1'(-\psi'(\xi)) d\xi \\ &= -|C_0|^2 \int_{\xi_E}^{+\infty} \operatorname{sgn}(\alpha(\xi)) \frac{d}{d\xi} (\chi_1(-\psi'(\xi))) d\xi \\ &= |C_0|^2 \operatorname{sgn}(\alpha(\xi_E)) \end{aligned}$$

The next step is to compute

$$\mathcal{M}_1^+ := \int_{\xi_E}^{+\infty} (d_0(\xi) \overline{b_1(\xi)} + d_1(\xi) \overline{b_0(\xi)}) d\xi \quad (3.37)$$

A few lines of calculations show that

$$\begin{aligned} d_0 \overline{b_1} + d_1 \overline{b_0} &= -2 \operatorname{Re}(\overline{C_0} C_1) \frac{d}{d\xi} (\chi_1) \operatorname{sgn}(\alpha) - 2 \operatorname{Re}(\overline{C_0} D_1) \frac{d}{d\xi} (\chi_1) \operatorname{sgn}(\alpha) \\ &\quad + \frac{|C_0|^2}{|\alpha|} (\partial_\xi p_1 \chi_1' - s_0 p_1) + \frac{i}{2} |C_0|^2 \operatorname{sgn}(\alpha) s_0' \end{aligned}$$

where we set

$$s_0(\xi) = \frac{e_0(\xi)}{\alpha(\xi)}$$

and therefore

$$\begin{aligned} \mathcal{M}_1^+ &= -2 \operatorname{Re}(\overline{C_0} C_1) \int_{\xi_E}^{+\infty} \operatorname{sgn}(\alpha) \frac{d}{d\xi} (\chi_1) d\xi - 2 \operatorname{Re}(\overline{C_0} D_1) \int_{\xi_E}^{+\infty} \operatorname{sgn}(\alpha) D_1(\xi) \frac{d}{d\xi} (\chi_1) d\xi \\ &\quad + |C_0|^2 \int_{\xi_E}^{+\infty} \frac{1}{|\alpha|} (\partial_\xi p_1 \chi_1' - s_0 p_1) d\xi + \frac{i}{2} |C_0|^2 \int_{\xi_E}^{+\infty} \operatorname{sgn}(\alpha) s_0' d\xi \end{aligned}$$

Note that

$$\partial_x c_0(x, \xi) = \frac{\partial^2 p_0(x, \xi)}{\partial x \partial \xi} \chi_1'(x) + \partial_\xi p_0(x, \xi) \chi_1''(x)$$

and

$$\chi_1'(x_E) = 0, \chi_1''(x_E) = 0, \lim_{\xi \rightarrow +\infty} \chi_1'(-\psi'(\xi)) = 0, \lim_{\xi \rightarrow +\infty} \chi_1''(-\psi'(\xi)) = 0, \alpha(\xi_E) \neq 0$$

This implies that

$$\begin{aligned} \int_{\xi_E}^{+\infty} \operatorname{sgn}(\alpha(\xi)) s_0'(\xi) d\xi &= \int_{\xi_E}^{+\infty} \operatorname{sgn}(\alpha(\xi)) \frac{d}{d\xi} \left(\frac{\tilde{s}_0(\xi)}{\alpha(\xi)} \right) d\xi \\ &= \int_{\xi_E}^{+\infty} \frac{d}{d\xi} \left(\frac{\tilde{s}_0(\xi)}{|\alpha(\xi)|} \right) d\xi \\ &= \left[\frac{\tilde{s}_0(\xi)}{|\alpha(\xi)|} \right]_{\xi_E}^{+\infty} = \left[\frac{\partial_x c_0(-\psi'(\xi), \xi)}{|\alpha(\xi)|} \right]_{\xi_E}^{+\infty} = 0 \end{aligned}$$

and

$$\int_{\xi_E}^{+\infty} \operatorname{sgn}(\alpha(\xi)) \frac{d}{d\xi} \left(\chi_1(-\psi'(\xi)) \right) d\xi = \left[\operatorname{sgn}(\alpha(\xi)) \chi_1(-\psi'(\xi)) \right]_{\xi_E}^{+\infty} = -\operatorname{sgn}(\alpha(\xi_E)) \chi_1(x_E) = -\operatorname{sgn}(\alpha(\xi_E))$$

So equation (3.37) becomes:

$$\mathcal{M}_1^+ = 2 \operatorname{sgn}(\alpha(\xi_E)) \operatorname{Re}(\overline{C_0} C_1) - 2I_1 + I_2 \quad (3.38)$$

where we set

$$I_1 = \operatorname{Re} \left(\overline{C_0} \int_{\xi_E}^{+\infty} \operatorname{sgn}(\alpha(\xi)) D_1(\xi) \frac{d}{d\xi} (\chi_1(-\psi'(\xi))) d\xi \right) \quad (3.39)$$

and

$$I_2 = |C_0|^2 \int_{\xi_E}^{+\infty} \frac{1}{|\alpha(\xi)|} \left(\partial_\xi p_1(-\psi'(\xi), \xi) \chi_1'(-\psi'(\xi)) - s_0(\xi) p_1(-\psi'(\xi), \xi) \right) d\xi \quad (3.40)$$

After a few integrations by parts, we obtain that

$$I_1 = -\frac{|C_0|^2}{2} \operatorname{sgn}(\alpha(\xi_E)) \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (a_E) + \frac{|C_0|^2}{2} \int_{\xi_E}^{+\infty} \frac{\psi'' \chi_1'}{|\alpha|} \left(\partial_x p_1 - \frac{p_1}{\alpha} \frac{\partial^2 p_0}{\partial x^2} \right) d\xi \quad (3.41)$$

and

$$I_2 = |C_0|^2 \int_{\xi_E}^{+\infty} \frac{\psi'' \chi_1'}{|\alpha|} \left(\partial_x p_1 - \frac{p_1}{\alpha} \frac{\partial^2 p_0}{\partial x^2} \right) d\xi \quad (3.42)$$

Finally

$$\mathcal{M}_1^+ = 2 \operatorname{sgn}(\alpha(\xi_E)) \operatorname{Re}(\overline{C_0} C_1) + |C_0|^2 \operatorname{sgn}(\alpha(\xi_E)) \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (a_E)$$

We thus have modulo $\mathcal{O}(h^2)$

$$\mathcal{W}_+^a(\hat{u}, \bar{\hat{u}}) = |C_0|^2 \operatorname{sgn}(\alpha(\xi_E)) + h \operatorname{sgn}(\alpha(\xi_E)) \left(2 \operatorname{Re}(\overline{C_0} C_1) + |C_0|^2 \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (a_E) \right) \quad (3.43)$$

Similarly, one shows that modulo $\mathcal{O}(h^2)$

$$\mathcal{W}_-^a(\hat{u}, \bar{\hat{u}}) = -|C_0|^2 \operatorname{sgn}(\alpha(\xi_E)) - h \operatorname{sgn}(\alpha(\xi_E)) \left(2 \operatorname{Re}(\overline{C_0} C_1) + |C_0|^2 \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (a_E) \right) \quad (3.44)$$

which allows us to conclude that modulo $\mathcal{O}(h^2)$

$$\mathcal{W}^a(\hat{u}, \bar{\hat{u}}) := \mathcal{W}_+^a(\hat{u}, \bar{\hat{u}}) - \mathcal{W}_-^a(\hat{u}, \bar{\hat{u}}) = 2|C_0|^2 \operatorname{sgn}(\alpha(\xi_E)) + 2h \operatorname{sgn}(\alpha(\xi_E)) \left(2 \operatorname{Re}(\overline{C_0} C_1) + |C_0|^2 \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (a_E) \right) \quad (3.45)$$

Assuming $\alpha(\xi_E) > 0$, $C_0 > 0$, and $C_1 \in \mathbb{R}$, it follows that

$$C_0 = 2^{-1/2} \quad (3.46)$$

and

$$C_1 := C_1(a_E) = -2^{-3/2} \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (a_E) \quad (3.47)$$

For the values of C_0 and C_1 found above, we indeed have

$$\mathcal{W}^a(\hat{u}, \bar{\hat{u}}) = 1 + \mathcal{O}(h^2) \quad (3.48)$$

We say that u^a is well-normalized mod $\mathcal{O}(h^2)$. This can be formalized by considering $\{a_E\}$ as a Poincaré section (see Section 4), and *Poisson operator* the operator that assigns, in a unique way, to the initial condition C_0 on $\{a_E\}$ the well normalized (forward) solution u^a to $(P - E)u^a = 0$: namely, $C_1(E)$ and $D_1(\xi)$, hence also \hat{u}^a , depend linearly on C_0 .

Remark 3.1. *So far, under the assumption $\alpha(\xi_E) > 0$, we have obtained the following expression*

$$\hat{u}^a(\xi; h) = (C_0 + hC_1(a_E) + hD_1(\xi) + \mathcal{O}(h^2)) |\alpha(\xi)|^{-\frac{1}{2}} \exp \left[\frac{i}{h} \left(\psi(\xi) + h \int_{\xi_E}^{\xi} \frac{p_1(-\psi'(\zeta), \zeta)}{\alpha(\zeta)} d\zeta \right) \right] \quad (3.49)$$

Thanks to the identity

$$C_0 + hC_1(a_E) + hD_1(\xi) = \left(C_0 + hC_1 + h\text{Re}(D_1(\xi)) \right) \exp \left[\frac{ih}{C_0} \text{Im}(D_1(\xi)) \right] + \mathcal{O}(h^2) \quad (3.50)$$

we can refine both the phase and the half-density up to the next order as follows

$$\hat{u}^a(\xi; h) = \left(C_0 + hC_1(a_E) + h\text{Re}(D_1(\xi)) \right) |\alpha(\xi)|^{-\frac{1}{2}} \exp \left[\frac{i}{h} \tilde{S}(\xi, \xi_E; h) \right] (1 + \mathcal{O}(h^2)) \quad (3.51)$$

where the improved phase \tilde{S} is defined by

$$\tilde{S}(\xi, \xi_E; h) = \psi(\xi) + h \int_{\xi_E}^{\xi} \frac{p_1(-\psi'(\zeta), \zeta)}{\alpha(\zeta)} d\zeta + \frac{h^2}{C_0} \text{Im}(D_1(\xi)) \quad (3.52)$$

3.3 The homology class of the generalized action: Fourier representation

Here we identify the various term in (3.51). First on a γ_E (i.e. Λ_E) we have

$$\psi(\xi) = \int -x(\xi) d\xi + \text{Const}$$

and

$$\varphi(x) = \int \xi(x) dx + \text{Const}$$

By Hamilton equations

$$\begin{cases} \dot{\xi}(t) &= -\partial_x p_0(x(t), \xi(t)) \\ \dot{x}(t) &= \partial_\xi p_0(x(t), \xi(t)) \end{cases}$$

so

$$\int \frac{p_1(-\psi'(\xi), \xi)}{\alpha(\xi)} d\xi = - \int_{\gamma_E} p_1(x(t), \xi(t)) dt$$

where t is the parametrization of γ_E by the time evolution. The form $p_1 dt$ is called sub-principal 1-form. Next we consider $D_1(\xi)$ as the integral over γ_E of the 1-form $\Omega_1(\xi)$, defined near ξ_E in Fourier representation by the

expression (3.33). Using WKB construction, $\Omega_1(\xi)$ can also be extended to the spatial representation. Since γ_E is Lagrangian, $\Omega_1(\xi)$ is clearly closed on γ_E ; our goal is to compute it modulo exact forms. Using integration by parts in (3.33), together with the condition $D_1(\xi_E) = 0$, we derive the following relations

$$\begin{aligned}
\sqrt{2} \operatorname{Re}(D_1(\xi)) &= \frac{1}{2} \left[\frac{p_1(-\psi'(\xi), \xi) \left(\frac{\partial^2 p_0}{\partial x^2} \right) (-\psi'(\xi), \xi) - \alpha(\xi) (\partial_x p_1) (-\psi'(\xi), \xi)}{\alpha^2(\xi)} \right]_{\xi_E}^{\xi} \\
&= -\frac{1}{2} \left[\partial_x \left(\frac{p_1}{\partial_x p_0} \right) (-\psi'(\xi), \xi) \right]_{\xi_E}^{\xi} \\
&= -\frac{1}{2} \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (-\psi'(\xi), \xi) + \frac{1}{2} \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (a_E) \\
&= -\frac{1}{2} \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (-\psi'(\xi), \xi) - \sqrt{2} C_1(a_E)
\end{aligned} \tag{3.53}$$

and

$$\sqrt{2} \operatorname{Im}(D_1(\xi)) = \int_{\xi_E}^{\xi} T_1(\zeta) d\zeta + \left[\frac{\psi''}{6\alpha} \frac{\partial^3 p_0}{\partial x^3} + \frac{\alpha'}{4\alpha^2} \frac{\partial^2 p_0}{\partial x^2} \right]_{\xi_E}^{\xi} \tag{3.54}$$

with

$$T_1(\zeta) = \frac{1}{\alpha} \left(p_2 - \frac{1}{8} \frac{\partial^4 p_0}{\partial x^2 \partial \zeta^2} + \frac{\psi''}{12} \frac{\partial^4 p_0}{\partial x^3 \partial \zeta} + \frac{(\psi'')^2}{24} \frac{\partial^4 p_0}{\partial x^4} \right) + \frac{1}{8} \frac{(\alpha')^2}{\alpha^3} \frac{\partial^2 p_0}{\partial x^2} + \frac{1}{6} \psi'' \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^3} - \frac{p_1}{\alpha^2} \left(\partial_x p_1 - \frac{p_1}{2\alpha} \frac{\partial^2 p_0}{\partial x^2} \right) \tag{3.55}$$

There follows:

Lemma 3.2. *Modulo the integral of an exact form in \mathcal{A} , with T_1 as in (3.55) we have:*

$$\begin{aligned}
\operatorname{Re}(D_1(\xi)) &\equiv 0 \\
\sqrt{2} \operatorname{Im}(D_1(\xi)) &\equiv \int_{\xi_E}^{\xi} T_1(\zeta) d\zeta
\end{aligned} \tag{3.56}$$

Passing from Fourier to spatial representation, we can carry the integration in x -variable between the focal points a_E and a'_E , and in ξ -variable again near a'_E . Since γ_E is smoothly embedded, the microlocal solution \hat{u}^a extends uniquely along γ_E .

Let $u(x, \xi)$ and $v(x, \xi)$ be two smooth functions on \mathcal{A} , and define the 1-form $\Omega(x, \xi) = u(x, \xi) dx + v(x, \xi) d\xi$. By Stokes' formula, we have:

$$\int_{\gamma_E} \Omega(x, \xi) = \int \int_{\{p_0 \leq E\}} (\partial_x v - \partial_\xi u) dx \wedge d\xi$$

According to [7], we can extend p_0 inside the disk bounded by A_- (which, without loss of generality, may be assumed to contain the origin), so that it coincides with a harmonic oscillator in a neighborhood of a point inside, say $p_0(0, 0) = 0$. Making the symplectic change of coordinates $(x, \xi) \mapsto (t, E)$ in $T^*\mathbb{R}$

$$\int \int_{\{p_0 \leq E\}} (\partial_x v - \partial_\xi u) dx \wedge d\xi = \int_0^E \int_0^{T(E')} (\partial_x v - \partial_\xi u) dt \wedge dE' \tag{3.57}$$

where $T(E')$ is the period of the flow of Hamilton vector field H_{p_0} at energy E' ($T(E')$ being a constant near $(0, 0)$). Taking derivative with respect to E , we find:

$$\frac{d}{dE} \int_{\gamma_E} \Omega(x, \xi) = \int_0^{T(E)} (\partial_x v - \partial_\xi u) dt \tag{3.58}$$

We compute $\int_{\xi_E}^{\xi} T_1(\zeta) d\zeta$ with T_1 as in (3.55), and start to simplify $J_1 = \int \omega_1$, where the 1-form ω_1 is expressed in terms of the variable ξ as follows:

$$\omega_1(-\psi'(\xi), \xi) = \frac{p_1(-\psi'(\xi), \xi)}{\alpha^2(\xi)} \left(\partial_x p_1(-\psi'(\xi), \xi) - \frac{p_1(-\psi'(\xi), \xi)}{2\alpha(\xi)} \frac{\partial^2 p_0}{\partial x^2}(-\psi'(\xi), \xi) \right) d\xi$$

Let

$$f_1(x, \xi) := \frac{p_1^2(x, \xi)}{\partial_x p_0(x, \xi)},$$

then its partial derivative with respect to x is given by:

$$\partial_x f_1(x, \xi) = \frac{2p_1(x, \xi)}{\partial_x p_0(x, \xi)} \left(\partial_x p_1(x, \xi) - \frac{p_1(x, \xi)}{2\partial_x p_0(x, \xi)} \frac{\partial^2 p_0(x, \xi)}{\partial x^2} \right)$$

By (3.58) we get

$$\begin{aligned} J_1 &= \frac{1}{2} \int_{\gamma_E} \frac{\partial_x f_1(x, \xi)}{\partial_x p_0(x, \xi)} d\xi = -\frac{1}{2} \int_0^{T(E)} \partial_x f_1(x(t), \xi(t)) dt \\ &= -\frac{1}{2} \frac{d}{dE} \int_{\gamma_E} f_1(x, \xi) d\xi = -\frac{1}{2} \frac{d}{dE} \int_{\gamma_E} \frac{p_1^2(x, \xi)}{\partial_x p_0(x, \xi)} d\xi \\ &= \frac{1}{2} \frac{d}{dE} \int_0^{T(E)} p_1^2(x(t), \xi(t)) dt \end{aligned} \quad (3.59)$$

which is the contribution of p_1 to the second term S_2 of generalized action in ([7], Thm2). Here $T(E)$ is the period on γ_E . We also have

$$\int_{\xi_E}^{\xi} \frac{1}{\alpha(\zeta)} p_2(-\psi'(\zeta), \zeta) d\zeta = \int_{\gamma_E} \frac{p_2(x, \xi)}{\partial_x p_0(x, \xi)} d\xi = -\int_0^{T(E)} p_2(x(t), \xi(t)) dt \quad (3.60)$$

In order to compute T_1 modulo exact forms, it remains to simplify in equation (3.55) the expression

$$\begin{aligned} J_2 &= \int_{\xi_E}^{\xi} \frac{1}{\alpha} \left(-\frac{1}{8} \frac{\partial^4 p_0}{\partial x^2 \partial \zeta^2} + \frac{\psi''}{12} \frac{\partial^4 p_0}{\partial x^3 \partial \zeta} + \frac{(\psi'')^2}{24} \frac{\partial^4 p_0}{\partial x^4} \right) d\zeta + \frac{1}{8} \int_{\xi_E}^{\xi} \frac{(\alpha')^2}{\alpha^3} \frac{\partial^2 p_0}{\partial x^2} d\zeta \\ &\quad + \frac{1}{6} \int_{\xi_E}^{\xi} \psi'' \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^3} d\zeta + \left[\frac{\psi''}{6\alpha} \frac{\partial^3 p_0}{\partial x^3} + \frac{\alpha'}{4\alpha^2} \frac{\partial^2 p_0}{\partial x^2} \right]_{\xi_E}^{\xi} \end{aligned}$$

Let

$$f_0(x, \xi) := \frac{\Delta(x, \xi)}{\partial_x p_0(x, \xi)},$$

where we have set according to [7]

$$\Delta(x, \xi) = \frac{\partial^2 p_0}{\partial x^2} \frac{\partial^2 p_0}{\partial \xi^2} - \left(\frac{\partial^2 p_0}{\partial x \partial \xi} \right)^2$$

From the eikonal equation (3.4), we deduce

$$\begin{aligned} \left(\frac{\partial^2 p_0}{\partial \xi^2} \right) (-\psi'(\xi), \xi) &= \psi'''(\xi) \alpha(\xi) + \psi''(\xi) \alpha'(\xi) + \psi''(\xi) \left(\frac{\partial^2 p_0}{\partial x \partial \xi} \right) (-\psi'(\xi), \xi) \\ &= \psi'''(\xi) \alpha(\xi) + 2\psi''(\xi) \alpha'(\xi) + (\psi''(\xi))^2 \left(\frac{\partial^2 p_0}{\partial x^2} \right) (-\psi'(\xi), \xi) \end{aligned}$$

Consequently,

$$\Delta(-\psi'(\xi), \xi) = -(\alpha'(\xi))^2 + \psi'''(\xi) \alpha(\xi) \left(\frac{\partial^2 p_0}{\partial x^2} \right) (-\psi'(\xi), \xi)$$

and

$$(\partial_x \Delta)(-\psi'(\zeta), \zeta) = \frac{\partial^3 p_0}{\partial x^3} (\psi''' \alpha + 2 \psi'' \alpha') + \frac{\partial^2 p_0}{\partial x^2} (\alpha'' + \psi''' \frac{\partial^2 p_0}{\partial x^2}) - 2 \alpha' \frac{\partial^3 p_0}{\partial x^2 \partial \zeta}$$

which implies

$$\frac{(\partial_x f_0)(-\psi'(\zeta), \zeta)}{\alpha(\zeta)} = \frac{\psi'''}{\alpha} \frac{\partial^3 p_0}{\partial x^3} + 2 \psi'' \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^3} + \frac{\alpha''}{\alpha^2} \frac{\partial^2 p_0}{\partial x^2} - 2 \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^2 \partial \zeta} + \frac{(\alpha')^2}{\alpha^3} \frac{\partial^2 p_0}{\partial x^2}$$

By integration by parts, we obtain:

$$\begin{aligned} \int_{\xi_E}^{\xi} \frac{\psi'''}{\alpha} \frac{\partial^3 p_0}{\partial x^3} d\zeta &= \left[\frac{\psi''}{\alpha} \frac{\partial^3 p_0}{\partial x^3} \right]_{\xi_E}^{\xi} + \int_{\xi_E}^{\xi} \psi'' \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^3} d\zeta + \int_{\xi_E}^{\xi} \frac{(\psi'')^2}{\alpha} \frac{\partial^4 p_0}{\partial x^4} d\zeta - \int_{\xi_E}^{\xi} \frac{\psi''}{\alpha} \frac{\partial^4 p_0}{\partial x^3 \partial \zeta} d\zeta \\ \int_{\xi_E}^{\xi} \frac{\alpha''}{\alpha^2} \frac{\partial^2 p_0}{\partial x^2} d\zeta &= \left[\frac{\alpha'}{\alpha^2} \frac{\partial^2 p_0}{\partial x^2} \right]_{\xi_E}^{\xi} + 2 \int_{\xi_E}^{\xi} \frac{(\alpha')^2}{\alpha^3} \frac{\partial^2 p_0}{\partial x^2} d\zeta + \int_{\xi_E}^{\xi} \psi'' \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^3} d\zeta - \int_{\xi_E}^{\xi} \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^2 \partial \zeta} d\zeta \\ \int_{\xi_E}^{\xi} \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^2 \partial \zeta} d\zeta &= - \left[\frac{1}{\alpha} \frac{\partial^3 p_0}{\partial x^2 \partial \zeta} \right]_{\xi_E}^{\xi} - \int_{\xi_E}^{\xi} \frac{\psi''}{\alpha} \frac{\partial^4 p_0}{\partial x^3 \partial \zeta} d\zeta + \int_{\xi_E}^{\xi} \frac{1}{\alpha} \frac{\partial^4 p_0}{\partial x^2 \partial \zeta^2} d\zeta \end{aligned}$$

and

$$\begin{aligned} \int_{\xi_E}^{\xi} \frac{\alpha''}{\alpha^2} \frac{\partial^2 p_0}{\partial x^2} d\zeta &= \left[\frac{\alpha'}{\alpha^2} \frac{\partial^2 p_0}{\partial x^2} \right]_{\xi_E}^{\xi} + 2 \int_{\xi_E}^{\xi} \frac{(\alpha')^2}{\alpha^3} \frac{\partial^2 p_0}{\partial x^2} d\zeta + \int_{\xi_E}^{\xi} \psi'' \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^3} d\zeta + \left[\frac{1}{\alpha} \frac{\partial^3 p_0}{\partial x^2 \partial \zeta} \right]_{\xi_E}^{\xi} \\ &\quad + \int_{\xi_E}^{\xi} \frac{\psi''}{\alpha} \frac{\partial^4 p_0}{\partial x^3 \partial \zeta} d\zeta - \int_{\xi_E}^{\xi} \frac{1}{\alpha} \frac{\partial^4 p_0}{\partial x^2 \partial \zeta^2} d\zeta \end{aligned}$$

It immediately follows that

$$\begin{aligned} \int_{\xi_E}^{\xi} \frac{(\partial_x f_0)(-\psi'(\zeta), \zeta)}{\alpha(\zeta)} d\zeta &= -3 \int_{\xi_E}^{\xi} \frac{1}{\alpha} \frac{\partial^4 p_0}{\partial x^2 \partial \zeta^2} d\zeta + 2 \int_{\xi_E}^{\xi} \frac{\psi''}{\alpha} \frac{\partial^4 p_0}{\partial x^3 \partial \zeta} d\zeta + \int_{\xi_E}^{\xi} \frac{(\psi'')^2}{\alpha} \frac{\partial^4 p_0}{\partial x^4} d\zeta \\ &\quad + 3 \int_{\xi_E}^{\xi} \frac{(\alpha')^2}{\alpha^3} \frac{\partial^2 p_0}{\partial x^2} d\zeta + 4 \int_{\xi_E}^{\xi} \psi'' \frac{\alpha'}{\alpha^2} \frac{\partial^3 p_0}{\partial x^3} d\zeta + \left[\frac{\psi''}{\alpha} \frac{\partial^3 p_0}{\partial x^3} \right]_{\xi_E}^{\xi} \\ &\quad + \left[\frac{\alpha'}{\alpha^2} \frac{\partial^2 p_0}{\partial x^2} \right]_{\xi_E}^{\xi} + 3 \left[\frac{1}{\alpha} \frac{\partial^3 p_0}{\partial x^2 \partial \zeta} \right]_{\xi_E}^{\xi} \end{aligned}$$

hence

$$24 J_2 = \int_{\xi_E}^{\xi} \frac{(\partial_x f_0)(-\psi'(\zeta), \zeta)}{\alpha(\zeta)} d\zeta + 3 \left[\frac{\psi''}{\alpha} \frac{\partial^3 p_0}{\partial x^3} \right]_{\xi_E}^{\xi} + 5 \left[\frac{\alpha'}{\alpha^2} \frac{\partial^2 p_0}{\partial x^2} \right]_{\xi_E}^{\xi} - 3 \left[\frac{1}{\alpha} \frac{\partial^3 p_0}{\partial x^2 \partial \zeta} \right]_{\xi_E}^{\xi} \quad (3.61)$$

and modulo the integral of an exact form in \mathcal{A}

$$\begin{aligned} J_2 &\equiv \frac{1}{24} \int_{\xi_E}^{\xi} \frac{(\partial_x f_0)(-\psi'(\zeta), \zeta)}{\alpha(\zeta)} d\zeta = \frac{1}{24} \int_{\gamma_E} \frac{\partial_x f_0(x, \xi)}{\partial_x p_0(x, \xi)} d\xi \\ &= -\frac{1}{24} \int_0^{T(E)} \partial_x f_0(x(t), \xi(t)) dt = -\frac{1}{24} \frac{d}{dE} \int_{\gamma_E} f_0(x, \xi) d\xi \\ &= -\frac{1}{24} \frac{d}{dE} \int_{\gamma_E} \frac{\Delta(x, \xi)}{\partial_x p_0(x, \xi)} d\xi = \frac{1}{24} \frac{d}{dE} \int_0^{T(E)} \Delta(x(t), \xi(t)) dt \end{aligned}$$

Using these expressions, we recover the well known action integrals (see e.g. [7]): We know that

$$\frac{d}{dE} \int_0^{T(E)} \Gamma(x(t), \xi(t)) dt = 2 \int_0^{T(E)} \Delta(x(t), \xi(t)) dt$$

where Γdt is the restriction to γ_E of the 1-form ω_0 in \mathbb{R}^2 , defined by

$$\omega_0(x, \xi) = \left(\frac{\partial^2 p_0}{\partial x^2} \frac{\partial p_0}{\partial \xi} - \frac{\partial^2 p_0}{\partial x \partial \xi} \frac{\partial p_0}{\partial x} \right) dx + \left(\frac{\partial^2 p_0}{\partial x \partial \xi} \frac{\partial p_0}{\partial \xi} - \frac{\partial^2 p_0}{\partial \xi^2} \frac{\partial p_0}{\partial x} \right) d\xi$$

By writing

$$\sqrt{2} D_1(\xi) = \int_{\xi_E}^{\xi} \Omega_1(\zeta)$$

we find that

$$\begin{aligned} \operatorname{Im} \oint_{\gamma_E} \Omega_1 &= \frac{1}{24} \frac{d}{dE} \int_{\gamma_E} \Delta dt - \int_{\gamma_E} p_2 dt - \frac{1}{2} \frac{d}{dE} \int_{\gamma_E} p_1^2 dt \\ &= \frac{1}{48} \left(\frac{d}{dE} \right)^2 \int_{\gamma_E} \Gamma dt - \int_{\gamma_E} p_2 dt - \frac{1}{2} \frac{d}{dE} \int_{\gamma_E} p_1^2 dt \end{aligned} \quad (3.62)$$

Using relation (3.53), we conclude that

$$\operatorname{Re} \oint_{\gamma_E} \Omega_1 = 0 \quad (3.63)$$

3.4 Well normalized QM mod $\mathcal{O}(h^2)$ in the spatial representation

The next task consists in extending the solutions away from $a_E = (x_E, \xi_E)$ in the spatial representation.

Recall that in the Fourier representation and for ξ near ξ_E , the microlocal solution \hat{u}^a of the eigenvalue equation $(P(-hD_\xi, \xi; h) - E) \hat{u}^a(\xi; h) = 0$ is given by:

$$\hat{u}^a(\xi; h) = e^{\frac{i}{h} \psi(\xi)} (b_0(\xi) + h b_1(\xi) + \mathcal{O}(h^2)) \quad (3.64)$$

Next, applying the inverse semi-classical Fourier transform to \hat{u}^a , we obtain:

$$u^a(x; h) = (2\pi h)^{-1/2} \int e^{\frac{i}{h} (x\xi + \psi(\xi))} (b_0(\xi) + h b_1(\xi) + \mathcal{O}(h^2)) d\xi \quad (3.65)$$

The phase of the oscillatory integral defining u^a has two critical points, $\xi_+(x) > \xi_E$ and $\xi_-(x) < \xi_E$. The critical values of the phase are given by:

$$\varphi_\pm(x) = x \xi_\pm(x) + \psi(\xi_\pm(x))$$

From the relation $x + \psi'(\xi_\pm(x)) = 0$, it follows that:

$$\partial_x \varphi_\pm(x) = \xi_\pm(x)$$

Since

$$\varphi_\pm(x_E) = x_E \xi_E + \psi(\xi_E) = x_E \xi_E$$

we deduce that:

$$\varphi_\pm(x) := \varphi_\pm(x_E, x) = x_E \xi_E + \int_{x_E}^x \xi_\pm(y) dy$$

Because the phase has two critical points on the support of $b(\xi; h)$, the contributions from each critical point must be summed. By the stationary phase theorem (3.1), we obtain:

$$\int e^{\frac{i}{h} (x\xi + \psi(\xi))} b(\xi; h) d\xi = \sum_{\pm} e^{\frac{i}{h} \varphi_\pm(x)} \left(\frac{\psi''(\xi_\pm(x))}{2i\pi h} \right)^{-\frac{1}{2}} \left(b_0(\xi_\pm(x)) + h b_1(\xi_\pm(x)) + h L_1 b_0(\xi_\pm(x)) + \mathcal{O}(h^2) \right) \quad (3.66)$$

where

$$L_1 b_0(\xi_\pm(x)) = \sum_{n=0}^2 \frac{2^{-(n+1)}}{in!(n+1)!} \left\langle \left(\psi''(\xi_\pm(x)) \right)^{-1} D_\xi, D_\xi \right\rangle^{n+1} (\phi_x^n b_0)(\xi_\pm(x)) \quad (3.67)$$

and

$$\phi_x(\xi) = x(\xi - \xi_{\pm}(x)) + \psi(\xi) - \psi(\xi_{\pm}(x)) - \frac{1}{2} \psi''(\xi_{\pm}(x)) (\xi - \xi_{\pm}(x))^2 = \mathcal{O}\left((\xi - \xi_{\pm}(x))^3\right) \quad (3.68)$$

A straightforward calculation shows that:

$$\begin{aligned} \langle (\psi''(\xi_{\pm}(x)))^{-1} D_{\xi}, D_{\xi} \rangle b_0(\xi_{\pm}(x)) &= \left[-(\psi''(\xi))^{-1} b_0''(\xi) \right]_{\xi=\xi_{\pm}(x)} \\ \langle (\psi''(\xi_{\pm}(x)))^{-1} D_{\xi}, D_{\xi} \rangle^2 (\phi_x b_0)(\xi_{\pm}(x)) &= \left[(\psi''(\xi))^{-2} (\psi^{(4)}(\xi) b_0(\xi) + 4 \psi^{(3)}(\xi) b_0'(\xi)) \right]_{\xi=\xi_{\pm}(x)} \end{aligned}$$

and

$$\langle (\psi''(\xi_{\pm}(x)))^{-1} D_{\xi}, D_{\xi} \rangle^3 (\phi_x^2 b_0)(\xi_{\pm}(x)) = -20 \left[(\psi''(\xi))^{-3} (\psi'''(\xi))^2 b_0(\xi) \right]_{\xi=\xi_{\pm}(x)}$$

In a neighborhood of the focal point a_E and for $x < x_E$, the microlocal solution of $(P(x, hD_x) - E)u(x; h) = 0$ is given mod $\mathcal{O}(h^2)$ by:

$$\begin{aligned} u_{\pm}^a(x; h) &= \sum_{\pm} u_{\pm}^a(x; h) \\ &= 2^{-1/2} \sum_{\pm} e^{\pm i \frac{\pi}{4}} \left(\pm \partial_{\xi} p_0(x, \xi_{\pm}(x)) \right)^{-\frac{1}{2}} \exp \left[\frac{i}{h} \left(\phi_{\pm}(x) - h \int_{x_E}^x \frac{p_1(y, \xi_{\pm}(y))}{\partial_{\xi} p_0(y, \xi_{\pm}(y))} dy \right) \right] \left(1 + h \frac{b_1(\xi_{\pm}(x))}{b_0(\xi_{\pm}(x))} + h D_2(\xi_{\pm}(x)) \right), \end{aligned} \quad (3.69)$$

with $\pm \partial_{\xi} p_0(x, \xi_{\pm}(x)) > 0$, and where we define:

$$D_2(\xi) = -\frac{1}{2i} (\psi''(\xi))^{-1} \frac{b_0''(\xi)}{b_0(\xi)} + \frac{1}{8i} (\psi''(\xi))^{-2} \left(\psi^{(4)}(\xi) + 4 \psi^{(3)}(\xi) \frac{b_0'(\xi)}{b_0(\xi)} \right) - \frac{5}{24i} (\psi''(\xi))^{-3} (\psi^{(3)}(\xi))^2 \quad (3.70)$$

and $\psi^{(j)}$, $j \geq 3$, denotes the j -th derivative of ψ . It is also easy to see that

$$\frac{b_1(\xi)}{b_0(\xi)} = \sqrt{2} (C_1(E) + D_1(\xi)) = -\frac{1}{2} \partial_x \left(\frac{p_1}{\partial_x p_0} \right) (-\psi'(\xi), \xi) + i \sqrt{2} \operatorname{Im}(D_1(\xi)) \quad (3.71)$$

We also have

$$\frac{b_0'(\xi)}{b_0(\xi)} = -\frac{\alpha'(\xi)}{2\alpha(\xi)} + \frac{i p_1(-\psi'(\xi), \xi)}{\alpha(\xi)}$$

and

$$\frac{b_0''(\xi)}{b_0(\xi)} = \left(-\frac{\alpha'(\xi)}{2\alpha(\xi)} + \frac{i p_1(-\psi'(\xi), \xi)}{\alpha(\xi)} \right)^2 + \frac{d}{d\xi} \left(-\frac{\alpha'(\xi)}{2\alpha(\xi)} + \frac{i p_1(-\psi'(\xi), \xi)}{\alpha(\xi)} \right)$$

First, we observe that $D_2(\xi_{\pm}(x))$ does not contribute to the homology class of the semi-classical forms defining the action, as it contains no integral term. Thus, the phase in (3.69) can be replaced, modulo $\mathcal{O}(h^3)$ by

$$\begin{aligned} S_{\pm}(x_E, x; h) &= x_E \xi_E + \int_{x_E}^x \xi_{\pm}(y) dy - h \int_{x_E}^x \frac{p_1(y, \xi_{\pm}(y))}{\partial_{\xi} p_0(y, \xi_{\pm}(y))} dy + \sqrt{2} h^2 \operatorname{Im}(D_1(\xi_{\pm}(x))) \\ &= x_E \xi_E + \int_{x_E}^x \xi_{\pm}(y) dy - h \int_{x_E}^x \frac{p_1(y, \xi_{\pm}(y))}{\partial_{\xi} p_0(y, \xi_{\pm}(y))} dy + h^2 \int_{x_E}^x T_1(\xi_{\pm}(y)) \xi'_{\pm}(y) dy \end{aligned} \quad (3.72)$$

Proposition 3.1. *In the spatial representation, the microlocal Wronskian near a focal point a_E is given by*

$$\mathcal{W}^a(u^a, \bar{u}^a) = \mathcal{W}_+^a(u^a, \bar{u}^a) - \mathcal{W}_-^a(u^a, \bar{u}^a) = 1 + \mathcal{O}(h^2) \quad (3.73)$$

Proof. Let $\chi^a \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function as defined in Subsection 3.2. The Weyl symbol of the commutator $\frac{i}{h}[P, \chi^a]$ is given by:

$$c(x, \xi; h) = (\partial_\xi p_0(x, \xi) + h \partial_\xi p_1(x, \xi)) \chi_1'(x) + \mathcal{O}(h^2) = c_0(x, \xi) + h c_1(x, \xi) + \mathcal{O}(h^2)$$

Let:

$$B_\pm^a := \frac{i}{h} [P, \chi^a]_\pm u_\pm^a \quad (3.74)$$

so:

$$B_\pm^a(x; h) = \frac{1}{2\pi h} \int \int e^{\frac{i}{h}(x-y)\eta} c\left(\frac{x+y}{2}, \eta; h\right) u_\pm^a(y; h) dy d\eta$$

For fixed x , the phase of the oscillatory integral defining $F_\pm^a(x; h)$ is:

$$\phi_x^\pm(y, \eta) = (x-y)\eta + \varphi_\pm(y).$$

Its critical points are:

$$(y_c(x), \eta_c^\pm(x)) = (x, \xi_\pm(x)),$$

and the corresponding critical values are:

$$\phi_x^\pm(y_c(x), \eta_c^\pm(x)) = \varphi_\pm(x).$$

A direct calculation shows that the Hessian matrix is:

$$(\text{Hess } \phi_x^\pm)(x, \xi_\pm(x)) = \begin{pmatrix} \xi_\pm'(x) & -1 \\ -1 & 0 \end{pmatrix}$$

We define:

$$u_x^\pm(y, \eta; h) = c\left(\frac{x+y}{2}, \eta; h\right) (\pm \partial_\xi p_0(y, \xi_\pm(y)))^{-\frac{1}{2}} \exp\left[-i \int_{x_E}^y \frac{p_1(z, \xi_\pm(z))}{\partial_\xi p_0(z, \xi_\pm(z))} dz\right] (1 + hZ(\xi_\pm(y)) + \mathcal{O}(h^2))$$

where:

$$Z(\xi_\pm(y)) = -\frac{1}{2} \partial_x \left(\frac{p_1}{\partial_x p_0} \right)(y, \xi_\pm(y)) + i\sqrt{2} \text{Im}(D_1(\xi_\pm(y))) + D_2(\xi_\pm(y))$$

Thus, the leading term of $u_x^\pm(y, \eta; h)$ is:

$$u_x^{(0, \pm)}(y, \eta) = c_0\left(\frac{x+y}{2}, \eta\right) (\pm \partial_\xi p_0(y, \xi_\pm(y)))^{-\frac{1}{2}} \exp\left[-i \int_{x_E}^y \frac{p_1(z, \xi_\pm(z))}{\partial_\xi p_0(z, \xi_\pm(z))} dz\right] = c_0\left(\frac{x+y}{2}, \eta\right) v_\pm(y).$$

By the stationary phase theorem (3.1), we obtain:

$$B_\pm^a(x; h) = \frac{1}{\sqrt{2}} e^{\pm i \frac{\pi}{4}} e^{\frac{i}{h} \varphi_\pm(x)} \left(u_x^\pm(x, \xi_\pm(x); h) + h L_1 u_x^{(0, \pm)}(x, \xi_\pm(x)) + \mathcal{O}(h^2) \right)$$

where:

$$L_1 u_x^{(0, \pm)}(x, \xi_\pm(x)) = \sum_{n=0}^2 \frac{2^{-(n+1)}}{in!(n+1)!} \left(2 \frac{\partial^2}{\partial y \partial \eta} + \xi_\pm'(x) \frac{\partial^2}{\partial \eta^2} \right)^{n+1} (\psi_x^n u_x^{(0, \pm)})(x, \xi_\pm(x)) \quad (3.75)$$

and

$$\psi_x(y, \eta) = (x-y) \xi_\pm(x) + \varphi_\pm(y) - \varphi_\pm(x) - \frac{1}{2} \xi_\pm'(x) (y-x)^2 = \mathcal{O}((y-x)^3) \quad (3.76)$$

A few calculations show that:

$$\left(2 \frac{\partial^2}{\partial y \partial \eta} + \xi_\pm'(x) \frac{\partial^2}{\partial \eta^2} \right) u_x^{(0, \pm)}(x, \xi_\pm(x)) = 2v_\pm'(x) s_\pm(x) + v_\pm(x) s_\pm'(x),$$

where:

$$s_{\pm}(x) = \left(\frac{\partial c_0}{\partial \xi}\right)(x, \xi_{\pm}(x)).$$

Moreover:

$$\left(2 \frac{\partial^2}{\partial y \partial \eta} + \xi'_{\pm}(x) \frac{\partial^2}{\partial \eta^2}\right)^{n+1} (\psi_x^n u_x^{(0, \pm)})(x, \xi_{\pm}(x)) = 0, \quad \forall n \in \{1, 2\}.$$

It is easy to see that:

$$v'_{\pm}(x) = \theta_{\pm}(x) v_{\pm}(x),$$

where:

$$\theta_{\pm}(x) = -\frac{1}{\psi''(\xi_{\pm}(x)) \alpha(\xi_{\pm}(x))} \left(i p_1(x, \xi_{\pm}(x)) - \frac{\psi'''(\xi_{\pm}(x)) \alpha(\xi_{\pm}(x)) + \psi''(\xi_{\pm}(x)) \alpha'(\xi_{\pm}(x))}{2 \psi''(\xi_{\pm}(x))} \right)$$

and:

$$c_0(x, \xi_{\pm}(x)) (\pm \partial_{\xi} p_0(x, \xi_{\pm}(x)))^{-\frac{1}{2}} = \pm (\pm \partial_{\xi} p_0(x, \xi_{\pm}(x)))^{\frac{1}{2}} \chi'_1(x)$$

Consequently:

$$\begin{aligned} B_{\pm}^a(x; h) &= \pm 2^{-1/2} e^{\pm i \frac{\pi}{4}} \exp \left[\frac{i}{h} \left(\varphi_{\pm}(x) - h \int_{x_E}^x \frac{p_1(y, \xi_{\pm}(y))}{\partial_{\xi} p_0(y, \xi_{\pm}(y))} dy \right) \right] (\pm \partial_{\xi} p_0(x, \xi_{\pm}(x)))^{\frac{1}{2}} \chi'_1(x) \\ &\times \left(1 + h Z(\xi_{\pm}(x)) + h \frac{c_1(x, \xi_{\pm}(x))}{c_0(x, \xi_{\pm}(x))} + \left(\frac{2 s_{\pm}(x) \theta_{\pm}(x) + s'_{\pm}(x)}{2 i c_0(x, \xi_{\pm}(x))} \right) h + \mathcal{O}(h^2) \right). \end{aligned}$$

Next, observing that:

$$s_{\pm}(x) = \left(\frac{\partial^2 p_0}{\partial \xi^2}\right)(x, \xi_{\pm}(x)) \chi'_1(x) = \omega_{\pm}(x) \chi'_1(x),$$

and that:

$$\partial_{\xi} p_0(x, \xi_{\pm}(x)) = \psi''(\xi_{\pm}(x)) \alpha(\xi_{\pm}(x)),$$

we obtain:

$$\begin{aligned} B_{\pm}^a(x; h) &= \pm 2^{-1/2} e^{\pm i \frac{\pi}{4}} \exp \left[\frac{i}{h} \left(\varphi_{\pm}(x) - h \int_{x_E}^x \frac{p_1(y, \xi_{\pm}(y))}{\partial_{\xi} p_0(y, \xi_{\pm}(y))} dy \right) \right] (\pm \partial_{\xi} p_0(x, \xi_{\pm}(x)))^{\frac{1}{2}} \chi'_1(x) \\ &\times \left(1 + h Z(\xi_{\pm}(x)) + h \frac{\partial_{\xi} p_1(x, \xi_{\pm}(x))}{\partial_{\xi} p_0(x, \xi_{\pm}(x))} - \frac{i h \omega_{\pm}(x) \theta_{\pm}(x)}{\partial_{\xi} p_0(x, \xi_{\pm}(x))} - \frac{i h}{2} \frac{\frac{d}{dx}(\omega_{\pm}(x) \chi'_1(x))}{\partial_{\xi} p_0(x, \xi_{\pm}(x)) \chi'_1(x)} + \mathcal{O}(h^2) \right). \end{aligned} \quad (3.77)$$

This gives:

$$\begin{aligned} (u_+^a | B_+^a) &= \frac{1}{2} \int_{x_E}^{+\infty} \chi'_1(x) dx + \frac{h}{2} \int_{x_E}^{+\infty} \left(2 \operatorname{Re}(Z(\xi_+(x))) + \frac{\partial_{\xi} p_1(x, \xi_+(x))}{\psi''(\xi_+(x)) \alpha(\xi_+(x))} + \frac{i \omega_+(x) \overline{\theta_+(x)}}{\psi''(\xi_+(x)) \alpha(\xi_+(x))} \right) \chi'_1(x) dx \\ &+ \frac{i h}{4} \int_{x_E}^{+\infty} \frac{1}{\psi''(\xi_+(x)) \alpha(\xi_+(x))} \frac{d}{dx} (\omega_+(x) \chi'_1(x)) dx + \mathcal{O}(h^2) \\ &= \frac{1}{2} + \frac{h}{2} K_1 + \frac{i h}{4} K_2 + \mathcal{O}(h^2). \end{aligned}$$

A simple calculation shows that:

$$2 \operatorname{Re}(Z(\xi_+(x))) + \frac{\partial_{\xi} p_1(x, \xi_+(x))}{\psi''(\xi_+(x)) \alpha(\xi_+(x))} + \frac{i \omega_+(x) \overline{\theta_+(x)}}{\psi''(\xi_+(x)) \alpha(\xi_+(x))} = \frac{\omega_+(x)}{\psi''(\xi_+(x)) \alpha(\xi_+(x))} \left(i \overline{\theta_+(x)} + \frac{p_1(x, \xi_+(x))}{\psi''(\xi_+(x)) \alpha(\xi_+(x))} \right)$$

$$= \frac{i \omega_+(x)}{2 (\psi''(\xi_+(x)))^3 (\alpha(\xi_+(x)))^2} \left(\psi'''(\xi_+(x)) \alpha(\xi_+(x)) + \psi''(\xi_+(x)) \alpha'(\xi_+(x)) \right).$$

Hence:

$$K_1 = \frac{i}{2} \int_{x_E}^{+\infty} \frac{\omega_+(x)}{(\psi''(\xi_+(x)))^3 (\alpha(\xi_+(x)))^2} \left(\psi'''(\xi_+(x)) \alpha(\xi_+(x)) + \psi''(\xi_+(x)) \alpha'(\xi_+(x)) \right) \chi_1'(x) dx.$$

Here, we used the fact that:

$$\omega_+(x) := \left(\frac{\partial^2 p_0}{\partial \xi^2} \right)(x, \xi_+(x)) = \psi'''(\xi_+(x)) \alpha(\xi_+(x)) + 2 \psi''(\xi_+(x)) \alpha'(\xi_+(x)) + (\psi''(\xi_+(x)))^2 \left(\frac{\partial^2 p_0}{\partial x^2} \right)(x, \xi_+(x)).$$

Integrating by parts gives:

$$\begin{aligned} K_2 &= \left[\frac{\omega_+(x) \chi_1'(x)}{\psi''(\xi_+(x)) \alpha(\xi_+(x))} \right]_{x_E}^{+\infty} - \int_{x_E}^{+\infty} \frac{d}{dx} \left(\frac{1}{\psi''(\xi_+(x)) \alpha(\xi_+(x))} \right) \omega_+(x) \chi_1'(x) dx \\ &= - \int_{x_E}^{+\infty} \frac{\omega_+(x)}{(\psi''(\xi_+(x)))^3 (\alpha(\xi_+(x)))^2} \left(\psi'''(\xi_+(x)) \alpha(\xi_+(x)) + \psi''(\xi_+(x)) \alpha'(\xi_+(x)) \right) \chi_1'(x) dx \\ &= 2i K_1 \end{aligned}$$

Thus, we have:

$$(u_+^a | B_+^a) = \frac{1}{2} + \mathcal{O}(h^2),$$

and similarly:

$$(u_-^a | B_-^a) = -\frac{1}{2} + \mathcal{O}(h^2),$$

Consequently:

$$(u^a | B_+^a - B_-^a) = 1 + \mathcal{O}(h^2).$$

Note that the mixed terms $(u_\pm^a | B_\mp^a)$ are $\mathcal{O}(h^\infty)$ because the phase is non-stationary. \square

3.5 WKB solutions mod $\mathcal{O}(h^2)$ in the spatial representation

We begin by constructing the WKB solutions $u_\rho^a(x; h) = u_\pm^a(x; h)$ starting from the focal point $a = a_E$. These solutions are uniformly valid with respect to h for x in any interval $I \subset \subset]x_E', x_E[$. The solutions take the form:

$$u_\rho^a(x; h) = a_\rho(x; h) e^{\frac{i}{h} \varphi_\rho(x)}, \quad (3.78)$$

where $a_\rho(x; h)$ is a formal series in h , which we shall compute with h^2 accuracy

$$a_\rho(x; h) = a_{\rho,0}(x) + h a_{\rho,1}(x) + h^2 a_{\rho,2}(x) + \dots$$

The phase $\varphi_\rho(x)$ is a real smooth function that satisfies the eikonal equation

$$p_0(x, \varphi_\rho'(x)) = E. \quad (3.79)$$

For simplicity we shall omit indices $\rho = \pm$ whenever no confusion may occur. Let $Q(x, hD_x; h) = e^{-\frac{i}{h} \varphi(x)} P(x, hD_x) e^{\frac{i}{h} \varphi(x)}$, which is an h -pseudo-differential operator. Its action on $a(x; h)$ is given by:

$$(Q - E)a(x; h) = (2\pi h)^{-1} \int \int e^{\frac{i}{h}(x-y)\theta} p\left(\frac{x+y}{2}, \theta + F(x, y); h\right) a(y; h) dy d\theta,$$

where $F(x, y) = \int_0^1 \varphi'(x + t(y - x)) dt$. Applying stationary phase theorem (5.10) at order 2 (see Appendix), we find modulo $\mathcal{O}(h^3)$:

$$\begin{aligned} (Q(x, hD_x; h) - E)a(x; h) &= \left(p(x, \varphi'(x); h) - E \right) a(x; h) + \frac{h}{i} \left(\beta(x; h) \partial_x a(x; h) + \frac{1}{2} \partial_x \beta(x; h) a(x; h) \right) \\ &- h^2 \left(\frac{1}{8} \partial_x r(x; h) a(x; h) + \frac{1}{8} \varphi''(x) \partial_x \theta(x; h) a(x; h) + \frac{1}{2} \partial_x \gamma(x; h) \partial_x a(x; h) + \frac{1}{2} \gamma(x; h) \frac{\partial^2 a(x; h)}{\partial x^2} + \frac{1}{6} \varphi'''(x) \theta(x; h) a(x; h) \right). \end{aligned} \quad (3.80)$$

Suppose now that $p(x, \xi; h)$ is real, $p_0(x_E, \xi_E) = E$, $(\frac{\partial p_0}{\partial \xi})(x_E, \xi_E) \neq 0$. We look for formal solutions (i.e in the sense of formal classical symbols) of

$$(P(x, hD_x; h) - E)(a(x; h) e^{\frac{i}{h} \varphi(x)}) = 0 \Leftrightarrow (Q(x, hD_x; h) - E)a(x; h) = 0. \quad (3.81)$$

Once the eikonal equation (3.79) is satisfied, the first transport equation is obtained by setting the $\mathcal{O}(h)$ term in (3.80) to zero:

$$\beta_0(x) a'_0(x) + \left(i p_1(x, \varphi'(x)) + \frac{1}{2} \beta'_0(x) \right) a_0(x) = 0. \quad (3.82)$$

Its solutions are of the form:

$$a_0(x) = \widetilde{C}_0 |\beta_0(x)|^{-\frac{1}{2}} \exp \left(-i \int_{x_E}^x \frac{p_1(y, \varphi'(y))}{\beta_0(y)} dy \right), \quad (3.83)$$

\widetilde{C}_0 being so far an arbitrary constant.

Next, setting the $\mathcal{O}(h^2)$ term in (3.80) to zero yields a differential equation for $a_1(x)$:

$$\begin{aligned} \beta_0(x) a'_1(x) + \left(i p_1(x, \varphi'(x)) + \frac{1}{2} \beta'_0(x) \right) a_1(x) &= -\beta_1(x) a'_0(x) - \left(i p_2(x, \varphi'(x)) + \frac{1}{2} \beta'_1(x) \right) a_0(x) \\ &+ i \left(\frac{1}{8} r'_0(x) a_0(x) + \frac{1}{8} \varphi''(x) \theta'_0(x) a_0(x) + \frac{1}{2} \gamma'_0(x) a'_0(x) + \frac{1}{2} \gamma_0(x) a''_0(x) + \frac{1}{6} \varphi'''(x) \theta_0(x) a_0(x) \right). \end{aligned} \quad (3.84)$$

Here, we have introduced the notations:

$$\beta_0(x) = \left(\frac{\partial p_0}{\partial \xi} \right)(x, \varphi'(x)), \quad r_0(x) = \left(\frac{\partial^3 p_0}{\partial x \partial \xi^2} \right)(x, \varphi'(x)), \quad \gamma_0(x) = \left(\frac{\partial^2 p_0}{\partial \xi^2} \right)(x, \varphi'(x)), \quad \theta_0(x) = \left(\frac{\partial^3 p_0}{\partial \xi^3} \right)(x, \varphi'(x)).$$

The homogeneous equation associated with (3.84) is the same as (3.82); so we are looking for a particular solution of (3.84), integrating from x_E , of the form

$$a_1(x) = \widetilde{D}_1(x) |\beta_0(x)|^{-\frac{1}{2}} \exp \left(-i \int_{x_E}^x \frac{p_1(y, \varphi'(y))}{\beta_0(y)} dy \right). \quad (3.85)$$

Alternatively, we could integrate (3.84) from x'_E instead of x_E . So our main task will consist in computing $\widetilde{D}_1(x)$ as a multivalued function, due to the presence of the turning points, in the same way we have determined $D_1(\xi)$ in [23] (Formula (3.5)), using Fourier representation. We solve (3.84) by the method of variation of constants, and find

$$(\widetilde{C}_0)^{-1} \operatorname{Re}(\widetilde{D}_1(x)) = -\frac{1}{2} \left[\partial_\xi \left(\frac{p_1}{\partial_\xi p_0} \right)(y, \varphi'(y)) \right]_{x_E}^x, \quad (3.86)$$

$$\begin{aligned} (\widetilde{C}_0)^{-1} \operatorname{Im}(\widetilde{D}_1(x)) &= \int_{x_E}^x \frac{1}{\beta_0} \left(-p_2 + \frac{1}{8} \frac{\partial^4 p_0}{\partial y^2 \partial \xi^2} + \frac{\varphi''}{12} \frac{\partial^4 p_0}{\partial y \partial \xi^3} - \frac{(\varphi'')^2}{24} \frac{\partial^4 p_0}{\partial \xi^4} \right) dy - \frac{1}{8} \int_{x_E}^x \frac{(\beta'_0)^2}{\beta_0^3} \frac{\partial^2 p_0}{\partial \xi^2} dy \\ &+ \frac{1}{6} \int_{x_E}^x \varphi'' \frac{\beta'_0}{\beta_0^2} \frac{\partial^3 p_0}{\partial \xi^3} dy + \int_{x_E}^x \frac{p_1}{\beta_0^2} \left(\partial_\xi p_1 - \frac{p_1}{2\beta_0} \frac{\partial^2 p_0}{\partial \xi^2} \right) dy + \left[\frac{\varphi''}{6\beta_0} \frac{\partial^3 p_0}{\partial \xi^3} - \frac{\beta'_0}{4\beta_0^2} \frac{\partial^2 p_0}{\partial \xi^2} \right]_{x_E}^x. \end{aligned} \quad (3.87)$$

Function $\widetilde{D}_1(x)$ can be normalized by

$$\widetilde{D}_1(x_E) = 0$$

The general solution of (3.84) is then:

$$a_1(x) = (\widetilde{C}_1 + \widetilde{D}_1(x)) |\beta_0(x)|^{-\frac{1}{2}} \exp \left(-i \int_{x_E}^x \frac{p_1(y, \varphi'(y))}{\beta_0(y)} dy \right). \quad (3.88)$$

Consequently,

$$a(x; h) = \left(\widetilde{C}_0 + h(\widetilde{C}_1 + \widetilde{D}_1(x)) + \mathcal{O}(h^2) \right) |\beta_0(x)|^{-\frac{1}{2}} \exp \left(-i \int_{x_E}^x \frac{p_1(y, \varphi'(y))}{\beta_0(y)} dy \right).$$

Repeating this construction for the other branch ($\rho = -1$) yields the two branches of WKB solutions:

$$u_{\pm}^a(x; h) = |\beta_0^{\pm}(x)|^{-\frac{1}{2}} e^{\frac{i}{h} S_{\pm}(x_E, x; h)} \left(\widetilde{C}_0 + h(\widetilde{C}_1 + \widetilde{D}_1^{\pm}(x)) + \mathcal{O}(h^2) \right), \quad (3.89)$$

where

$$\begin{aligned} S_{\pm}(x_E, x; h) &= \varphi_{\pm}(x_E) + \int_{x_E}^x \xi_{\pm}(y) dy - h \int_{x_E}^x \frac{p_1(y, \varphi'_{\pm}(y))}{\beta_0^{\pm}(y)} dy, \\ \beta_0^{\pm}(x) &= (\partial_{\xi} p_0)(x, \varphi'_{\pm}(x)). \end{aligned} \quad (3.90)$$

Here we have used that $\varphi_{\pm}(x) = \varphi_{\pm}(x_E) + \int_{x_E}^x \xi_{\pm}(y) dy$, with $p_0(x, \xi_{\pm}(x)) = E$.

Normalization with respect to the "flux norm" consists as above in computing $B_{\pm}^a = \frac{i}{h} [P, \chi^a]_{\pm} u^a$ by stationary phase theorem (3.1) modulo $\mathcal{O}(h^2)$. Assuming already $\widetilde{C}_0, \widetilde{C}_1$ to be real, a simple calculation using integration by parts yields $\widetilde{C}_0 = C_0 = 2^{-1/2}$, and

$$\widetilde{C}_1 = \widetilde{C}_1(a_E) = -2^{-3/2} \partial_{\xi} \left(\frac{p_1}{\partial_{\xi} p_0} \right)(a_E).$$

As a result, outside any neighborhood of x_E , we have

$$u_{\pm}(x; h) = |\beta_0^{\pm}(x)|^{-1/2} e^{\frac{i}{h} S_{\pm}(x_E, x; h)} \left(\widetilde{C}_0 + h \widetilde{C}_1 + h \widetilde{D}_1^{\pm}(x) + \mathcal{O}(h^2) \right), \quad (3.91)$$

with

$$S_{\pm}(x_E, x; h) = \varphi_{\pm}(x_E) + \int_{x_E}^x \xi_{\pm}(y) dy - h \int_{x_E}^x \frac{p_1(y, \varphi'_{\pm}(y))}{\beta_0^{\pm}(y)} dy$$

From (3.91) we can recover the homology class of generalized action, considering the superposition $u(x; h) = e^{i\pi/4} u_+(x; h) + e^{-i\pi/4} u_-(x; h)$ near a_E . The argument is then similar to that of [20], formula (1).

3.6 Bohr-Sommerfeld quantization rule

Recall from (3.72) the modified phase function of the microlocal solutions u_{\pm}^a near the focal point a_E , accurate to $\mathcal{O}(h^2)$. Similarly, the phase for the asymptotic solution near the other focal point a'_E is given by:

$$S_{\pm}(x'_E, x; h) = x'_E \xi'_E + \int_{x'_E}^x \xi_{\pm}(y) dy - h \int_{x'_E}^x \frac{p_1(y, \xi_{\pm}(y))}{\partial_{\xi} p_0(y, \xi_{\pm}(y))} dy + h^2 \int_{x'_E}^x T_1(\xi_{\pm}(y)) \xi'_{\pm}(y) dy \quad (3.92)$$

Now, consider the function $B_{\pm}^a(x; h)$ with asymptotics (3.77), and similarly $B_{\pm}^{a'}(x; h)$. The normalized microlocal solutions u^a and $u^{a'}$, extended uniquely along γ_E are denoted u_1 and u_2 . It is then easy to show that

$$\begin{aligned} (u_1 | B_+^{a'} - B_-^{a'}) &= \frac{i}{2} \left(e^{\frac{i}{h} A_-(x_E, x'_E; h)} - e^{\frac{i}{h} A_+(x_E, x'_E; h)} \right) \\ (u_2 | B_+^a - B_-^a) &= \frac{i}{2} \left(e^{-\frac{i}{h} A_-(x_E, x'_E; h)} - e^{-\frac{i}{h} A_+(x_E, x'_E; h)} \right) \end{aligned} \quad (3.93)$$

modulo $\mathcal{O}(h^2)$. Here, the generalized actions are:

$$\begin{aligned} A_{\pm}(x_E, x'_E; h) &:= S_{\pm}(x_E, x; h) - S_{\pm}(x'_E, x; h) \\ &= x_E \xi_E - x'_E \xi'_E + \int_{x_E}^{x'_E} \xi_{\pm}(y) dy - h \int_{x_E}^{x'_E} \frac{p_1(y, \xi_{\pm}(y))}{\partial_{\xi} p_0(y, \xi_{\pm}(y))} dy + h^2 \int_{x_E}^{x'_E} T_1(\xi_{\pm}(y)) \xi'_{\pm}(y) dy \end{aligned}$$

The Gram matrix $G^{(a, a')}(E)$ of the solutions u_1, u_2 in the basis $(B_+^a - B_-^a, B_+^{a'} - B_-^{a'})$ is given by:

$$G^{(a, a')}(E) = \begin{pmatrix} 1 & \frac{i}{2} (e^{-\frac{i}{h} A_-(x_E, x'_E; h)} - e^{-\frac{i}{h} A_+(x_E, x'_E; h)}) \\ \frac{i}{2} (e^{\frac{i}{h} A_-(x_E, x'_E; h)} - e^{\frac{i}{h} A_+(x_E, x'_E; h)}) & -1 \end{pmatrix} \quad (3.94)$$

whose determinant is:

$$-\cos^2 \left(\frac{A_-(x_E, x'_E; h) - A_+(x_E, x'_E; h)}{2h} \right)$$

This determinant vanishes precisely at the eigenvalues of P in I , leading to the condition modulo $\mathcal{O}(h^3)$:

$$\begin{aligned} 2\pi n h &= \int_{x'_E}^{x_E} (\xi_+(y) - \xi_-(y)) dy - \pi h - h \int_{x'_E}^{x_E} \left(\frac{p_1(y, \xi_+(y))}{\partial_{\xi} p_0(y, \xi_+(y))} - \frac{p_1(y, \xi_-(y))}{\partial_{\xi} p_0(y, \xi_-(y))} \right) dy \\ &\quad + h^2 \int_{x'_E}^{x_E} (T_1(\xi_+(y)) \xi'_+(y) - T_1(\xi_-(y)) \xi'_-(y)) dy, \quad n \in \mathbb{Z} \end{aligned} \quad (3.95)$$

We now aim to simplify (3.95). First, observe that:

$$\int_{x'_E}^{x_E} (\xi_+(y) - \xi_-(y)) dy = \oint_{\gamma_E} \xi(y) dy$$

From the second Hamilton-Jacobi equation:

$$dy(t) = \partial_{\xi} p_0(y(t), \xi(t)) dt$$

we derive:

$$\begin{aligned} \int_{x'_E}^{x_E} \left(\frac{p_1(y, \xi_+(y))}{\partial_y p_0(y, \xi_+(y))} - \frac{p_1(y, \xi_-(y))}{\partial_y p_0(y, \xi_-(y))} \right) dy &= \int_{\gamma_E} p_1 dt \\ \int_{x'_E}^{x_E} (T_1(\xi_+(y)) \xi'_+(y) - T_1(\xi_-(y)) \xi'_-(y)) dy &= \oint_{\gamma_E} T_1(\xi(y)) \xi'(y) dy \\ &= \text{Im} \oint_{\gamma_E} \Omega_1(\xi(y)) \\ &= \frac{1}{24} \frac{d}{dE} \int_{\gamma_E} \Delta dt - \int_{\gamma_E} p_2 dt - \frac{1}{2} \frac{d}{dE} \int_{\gamma_E} p_1^2 dt \end{aligned}$$

In conclusion, the generalized Bohr-Sommerfeld quantization rule at second order for an h -pseudo-differential operator of the form (1.1) is given by:

$$\oint_{\gamma_E} \xi(x) dx + \left(-\pi - \int_{\gamma_E} p_1 dt \right) h + h^2 \left(\frac{1}{24} \frac{d}{dE} \int_{\gamma_E} \Delta dt - \int_{\gamma_E} p_2 dt - \frac{1}{2} \frac{d}{dE} \int_{\gamma_E} p_1^2 dt \right) = 2\pi n h + \mathcal{O}(h^3) \quad (3.96)$$

4 Bohr-Sommerfeld and Action-Angle Variables

Here, we present a simpler approach based on Birkhoff normal form and the monodromy operator, reminiscent of [16]. Let P be self-adjoint as in (1.1), with Weyl symbol $p \in S^0(m)$, and such that there exists a topological ring \mathcal{A} where p_0 satisfies the hypotheses (H_1) , (H_2) and (H_3) in the Introduction. Without loss of generality, we can assume that p_0 has a periodic orbit $\gamma_0 \subset \mathcal{A}$ with period 2π and energy $E = E_0$. From Hamilton-Jacobi theory (see [1]), there exists a smooth canonical transformation $(t, \tau) \mapsto \kappa(t, \tau) = (x, \xi)$, $t \in [0, 2\pi]$, defined in a neighborhood of γ_0 , and a smooth function $\tau \mapsto f_0(\tau)$, with $f_0(0) = 0$ and $f'_0(0) = 1$, such that

$$p_0 \circ \kappa(t, \tau) = f_0(\tau) \quad (4.1)$$

This transformation is given by generating function $S(\tau, x) = \int_{x_E}^x \xi(y) dy$, where $\xi(x) = \partial_x S(\tau, x)$, $\varphi = \partial_\tau S(\tau, x)$, and:

$$p_0(x, \partial_x S(\tau, x)) = f_0(\tau)$$

Energy E and momentum τ are related by the one-to-one transformation $E = f_0(\tau)$, with $f'_0(E_0) = 1$.

This map can be quantized semi-classically, known as the semi-classical Birkhoff normal form (BNF). Here, we take advantage of the fact (see [7], Proposition 2) that we can smoothly deform p in the interior of annulus \mathcal{A} , without changing its semi-classical spectrum in I , such that the "new" p_0 has a non-degenerate minimum at (x_0, ξ_0) , while all energies $E \in]0, E_+]$ remain regular. The BNF is achieved by introducing "harmonic oscillator" coordinates (y, η) , so that (4.1) becomes:

$$p_0 \circ \kappa(y, \eta) = f_0\left(\frac{1}{2}(\eta^2 + y^2)\right), \quad (4.2)$$

and $U^*PU = f\left(\frac{1}{2}((hD_y)^2 + y^2); h\right)$, has full symbol $f(\tau; h) = f_0(\tau) + hf_1(\tau) + h^2f_2(\tau) + \dots$. Here, f_1 includes the Maslov correction $1/2$, and U is a microlocally unitary h -FIO operator associated with κ (see [9], [15]). In \mathcal{A} , $\tau \neq 0$, so we can make the smooth symplectic change of coordinates $y = \sqrt{2\tau} \cos(t)$, $\eta = \sqrt{2\tau} \sin(t)$, and transform $\frac{1}{2}((hD_y)^2 + y^2)$ back to hD_t .

We do not provide explicit expressions for $f_j(\tau)$, $j \geq 1$, in terms of p_j , but note that f_j depends linearly on p_0, p_1, \dots, p_j and their derivatives. The BNF effectively eliminates focal points. The Poincaré section $t = 0$ in $\{f_0(\tau) = E\} = f_0^{-1}(E)$ reduces to a single point $\Sigma = \{a(E)\}$.

From ([24], [25]), the Poisson operator $\mathcal{K}(t, E)$ solves (globally near γ_0):

$$(f(hD_t; h) - E) \mathcal{K}(t, E) = 0, \quad (4.3)$$

and is given in the special 1-D case by the multiplication operator on $L^2(\Sigma) \approx \mathbb{C}^2$:

$$\mathcal{K}(t, E) = e^{\frac{i}{h}S(t, E)} a(t, E; h), \quad (4.4)$$

where $S(t, E)$ satisfies the eikonal equation $f_0(\partial_t S(t, E)) = E$, $S(0, E) = 0$, i.e. $S(t, E) = f_0^{-1}(E)t = \tau t$. Using formula (3.80), we have:

$$\begin{aligned} e^{-\frac{i}{h}S(t, E)} (f(hD_t; h) - E) (a(t, E; h) e^{\frac{i}{h}S(t, E)}) &= (f(\partial_t S; h) - E) a + \frac{h}{i} \left(\partial_\tau f \partial_t a + \frac{1}{2} \frac{\partial^2 S}{\partial t^2} \frac{\partial^2 f}{\partial \tau^2} a \right) \\ &\quad - h^2 \left(\frac{1}{8} \left(\frac{\partial^2 S}{\partial t^2} \right)^2 \frac{\partial^4 f}{\partial \tau^4} a + \frac{1}{2} \frac{\partial^2 S}{\partial t^2} \frac{\partial^3 f}{\partial \tau^3} \partial_t a + \frac{1}{2} \frac{\partial^2 f}{\partial \tau^2} \frac{\partial^2 a}{\partial t^2} + \frac{1}{6} \frac{\partial^3 S}{\partial t^3} \frac{\partial^3 f}{\partial \tau^3} a \right) + \mathcal{O}(h^3) \end{aligned} \quad (4.5)$$

where we have simplified the notation by setting:

$$S = S(t, E), \quad a = a(t, E; h), \quad \partial_\tau f = (\partial_\tau f)(\partial_t S(t, E); h), \quad \frac{\partial^2 f}{\partial \tau^2} = \frac{\partial^2 f}{\partial \tau^2}(\partial_t S(t, E); h), \quad \frac{\partial^3 f}{\partial \tau^3} = \frac{\partial^3 f}{\partial \tau^3}(\partial_t S(t, E); h), \dots$$

From relation

$$\frac{\partial^2 S(t, E)}{\partial t^2} = 0 = \frac{\partial^3 S(t, E)}{\partial t^3},$$

it follows that

$$\begin{aligned} e^{-\frac{i}{h}S(t, E)} (f(hD_t; h) - E) (a(t, E; h) e^{\frac{i}{h}S(t, E)}) &= (f(\tau; h) - E) a(t, E; h) + \frac{h}{i} \partial_\tau f(\tau; h) \partial_t a(t, E; h) \\ &\quad - \frac{h^2}{2} \frac{\partial^2 f(\tau; h)}{\partial \tau^2} \frac{\partial^2 a(t, E; h)}{\partial t^2} + \mathcal{O}(h^3) \end{aligned} \quad (4.6)$$

If the eikonal equation is satisfied, we obtain by eliminating the h term in (4.6) the first transport equation:

$$f'_0(\tau) \partial_t a_0(t, E) + i f_1(\tau) a_0(t, E) = 0 \quad (4.7)$$

whose solutions are the functions:

$$a_0(t, E) = C_0 e^{-it f_1(\tau)/f'_0(\tau)}, \quad C_0 \in \mathbb{R} \quad (4.8)$$

By eliminating the h^2 term in (4.6), we obtain the second transport equation:

$$f'_0(\tau) \partial_t a_1(t, E) + i f_1(\tau) a_1(t, E) = - (f'_1(\tau) \partial_t a_0(t, E) + i f_2(\tau) a_0(t, E)) + \frac{i}{2} f''_0(\tau) \frac{\partial^2 a_0(t, E)}{\partial t^2} \quad (4.9)$$

The homogeneous equation associated with (4.9) is the same as (4.7), whose solutions are the functions:

$$t \mapsto C_1 e^{-it f_1(\tau)/f'_0(\tau)}, \quad C_1 \in \mathbb{R}$$

Thus, we seek a particular solution to (4.6) of the form:

$$t \mapsto D_1(t, E) e^{-it f_1(\tau)/f'_0(\tau)}$$

Using the method of variation of constants, we find:

$$D_1(t, E) = i C_0 \tilde{S}_2(E) t; \quad \tilde{S}_2(E) = \frac{1}{f'_0(\tau)} \left(\frac{1}{2} \left(\frac{f_1^2}{f'_0} \right)'(\tau) - f_2(\tau) \right) \quad (4.10)$$

which we normalize by setting $D_1(0, E) = 0$, so that the general solution to (4.9) is:

$$a_1(t, E) = (C_1 + D_1(t, E)) e^{-it f_1(\tau)/f'_0(\tau)}$$

It follows that:

$$a(t, E; h) = (C_0 + h C_1 + i h C_0 \tilde{S}_2(E) t + \mathcal{O}(h^2)) e^{it \tilde{S}_1(E)}; \quad \tilde{S}_1(E) = - \left(\frac{f_1}{f'_0} \right)(\tau)$$

and thus:

$$\mathcal{K}(t, E) = e^{\frac{i}{h}(S(t, E) + h t \tilde{S}_1(E))} (C_0 + h C_1 + i h C_0 \tilde{S}_2(E) t + \mathcal{O}(h^2)) \quad (4.11)$$

We define the adjoint $\mathcal{K}^*(t, E)$ of $\mathcal{K}(t, E)$ by:

$$\mathcal{K}^*(t, E) = a^*(t, E; h) e^{-\frac{i}{h}S(t, E)} = \overline{a(t, E; h)} e^{-\frac{i}{h}S(t, E)}$$

et

$$\mathcal{K}^*(E) \cdot = \int \mathcal{K}^*(t, E) \cdot dt$$

The "flux norme" on \mathbb{C}^2 , is defined by:

$$(u|v)_\chi = \left(\frac{i}{h} [f(hD_t; h), \chi(t)] \mathcal{K}(t, E) u | \mathcal{K}(t, E) v \right) \quad (4.12)$$

with the scalar product of $L^2(\mathbb{R}_t)$ on the RHS, and $\chi \in C^\infty(\mathbb{R})$ is a smooth step-function, equal to 0 for $t \leq 0$ and 1 for $t \geq 2\pi$. To normalize the solution $\mathcal{K}(t, E)$, we start from:

$$\mathcal{K}^*(E) \frac{i}{h} [f(hD_t; h), \chi(t)] \mathcal{K}(t, E) = \text{Id}_{L^2(\mathbb{R})}$$

The Weyl symbol of $\frac{i}{h} [f(hD_t; h), \chi(t)]$ is

$$c(t, \tilde{\tau}; h) = \{f(\tilde{\tau}; h), \chi(t)\} = (f'_0(\tilde{\tau}) + h f'_1(\tilde{\tau})) \chi'(t) + \mathcal{O}(h^2) = Q(\tilde{\tau}; h) \chi'(t) + \mathcal{O}(h^2) \quad (\text{with } hD_t \text{ quantizing } \tilde{\tau})$$

where we have set:

$$Q(\tilde{\tau}; h) = f'_0(\tilde{\tau}) + h f'_1(\tilde{\tau})$$

We set:

$$I(t, E) := \frac{i}{h} [f(hD_t; h), \chi(t)] \mathcal{K}(t, E) = (2\pi h)^{-1} \int \int e^{\frac{i}{h} ((t-s)\tilde{\tau} + S(s, E))} Q(\tilde{\tau}; h) \chi'(\frac{t+s}{2}) a(s, E; h) ds d\tilde{\tau} \quad (4.13)$$

For fixed t , the phase corresponding to the oscillatory integral defining $I(t, E)$ is:

$$\varphi_t(s, \tilde{\tau}) = (t-s)\tilde{\tau} + S(s, E)$$

whose critical points are:

$$(s_c(t), \tilde{\tau}_c(t)) = (t, \partial_t S(t, E)) = (t, \tau)$$

and the corresponding critical values are:

$$\varphi_t(s_c(t), \tilde{\tau}_c(t)) = S(t, E)$$

A direct calculation shows that:

$$\text{Hess} \varphi_t(s, \tilde{\tau}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We set $u_t(s, \tilde{\tau}; h) = Q(\tilde{\tau}; h) \chi'(\frac{t+s}{2}) a(s, E; h)$. By the stationary phase theorem (3.1), we obtain:

$$I(t, E) = e^{\frac{i}{h} S(t, E)} (u_t(t, \tau; h) + h L_1 u_t(t, \tau; h) + \mathcal{O}(h^2)) \quad (4.14)$$

with:

$$L_1 u_t(t, \tau; h) = \sum_{n=0}^2 \frac{2^{-(n+1)}}{in!(n+1)!} \left\langle \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} D_s \\ D_{\tilde{\tau}} \end{pmatrix}, \begin{pmatrix} D_s \\ D_{\tilde{\tau}} \end{pmatrix} \right\rangle^{n+1} (\phi_t^n u_t)(t, \tau; h) \quad (4.15)$$

A quick calculation shows that:

$$\left\langle \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} D_s \\ D_{\tilde{\tau}} \end{pmatrix}, \begin{pmatrix} D_s \\ D_{\tilde{\tau}} \end{pmatrix} \right\rangle = 2 \frac{\partial^2}{\partial s \partial \tilde{\tau}}$$

and that:

$$\phi_t(s, \tilde{\tau}) = \varphi_t(s, \tilde{\tau}) - \varphi_t(t, \tau) - \frac{1}{2} \left\langle \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s-t \\ \tilde{\tau}-\tau \end{pmatrix}, \begin{pmatrix} s-t \\ \tilde{\tau}-\tau \end{pmatrix} \right\rangle = 0$$

It follows that:

$$L_1 u_t(t, \tau; h) = -i \left(\frac{\partial^2 u_t}{\partial s \partial \bar{\tau}} \right) (t, \tau; h)$$

Thus:

$$I(t, E) = e^{\frac{i}{h} S(t, E)} \left[Q(\tau; h) \chi'(t) a(t, E; h) - i \frac{h}{2} \partial_\tau Q(\tau; h) \chi''(t) a(t, E; h) - i h \partial_\tau Q(\tau; h) \chi'(t) \partial_t a(t, E; h) + \mathcal{O}(h^2) \right] \quad (4.16)$$

which gives:

$$\begin{aligned} (u|v)_\chi &= (I(t, E) u | \mathcal{K}(t, E) v) = u \bar{v} \int_0^{2\pi} I(t, E) \overline{\mathcal{K}(t, E)} dt \\ &= u \bar{v} \left(C_0^2 f_0'(\tau) + h (2C_0 C_1 f_0'(\tau) + C_0^2 f_1'(\tau) + C_0^2 \tilde{S}_1(E) f_0''(\tau)) + \mathcal{O}(h^2) \right) \end{aligned}$$

We should therefore have:

$$C_0^2 f_0'(\tau) = 1$$

and

$$2C_0 C_1 f_0'(\tau) + C_0^2 f_1'(\tau) + C_0^2 \tilde{S}_1(E) f_0''(\tau) = 0$$

If we choose $C_0 > 0$, we have:

$$C_0 = C_0(\tau) = (f_0'(\tau))^{-1/2} \quad (4.17)$$

and

$$C_1 = C_1(\tau) = -\frac{1}{2} (f_0'(\tau))^{-1/2} \left(\frac{f_1}{f_0} \right)'(\tau) \quad (4.18)$$

we end up with $(u|v)_\chi = u \bar{v} (1 + \mathcal{O}(h^2))$, which normalizes $\mathcal{K}(t, E)$ to order 2.

We define $\mathcal{K}_0(t, E) = \mathcal{K}(t, E)$ (Poisson operator with data at $t = 0$), and $\mathcal{K}_{2\pi}(t, E) = \mathcal{K}_0(t - 2\pi, E)$ (Poisson operator with data at $t = 2\pi$).

The energy E is an eigenvalue of the operator $f(hD_t; h)$ if and only if 1 is an eigenvalue of the Monodromy operator

$$M(E) = \mathcal{K}_{2\pi}^*(E) I(t, E) = \int \mathcal{K}_0^*(t - 2\pi, E) I(t, E) dt$$

Note that in dimension 1, the monodromy operator $M(E)$ reduces to a multiplication operator.

A few lines of calculation then show that:

$$\begin{aligned} M(E) &= \exp(2i\pi\tau/h) \exp(2i\pi\tilde{S}_1(E)) (1 + 2i\pi h \tilde{S}_2(E) + \mathcal{O}(h^2)) \\ &= \exp \left[\frac{i}{h} (2\pi\tau + 2\pi h \tilde{S}_1(E) + 2\pi h^2 \tilde{S}_2(E)) \right] \end{aligned} \quad (4.19)$$

The Bohr-Sommerfeld quantization rule is written as:

$$f_0^{-1}(E) + h \tilde{S}_1(E) + h^2 \tilde{S}_2(E) + \mathcal{O}(h^3) = nh, \quad n \in \mathbb{Z}$$

Let $S_1(E) = 2\pi \tilde{S}_1(E)$, $S_2(E) = 2\pi \tilde{S}_2(E)$. Then, since $f_0^{-1}(E) = \tau(E) = \frac{1}{2\pi} \oint_{\gamma_E} \xi(x) dx = \frac{1}{2\pi} S_0(E)$, and we know that $S_3(E) = 0$, we obtain:

$$S_0(E) + h S_1(E) + h^2 S_2(E) + \mathcal{O}(h^4) = 2\pi n h, \quad n \in \mathbb{Z} \quad (4.20)$$

with:

$$S_1(E) = -2\pi \left(\frac{f_1}{f_0'} \right)(\tau), \quad S_2(E) = \frac{2\pi}{f_0'(\tau)} \left(\frac{1}{2} \left(\frac{f_1^2}{f_0'} \right)'(\tau) - f_2(\tau) \right) \quad (4.21)$$

5 The discrete spectrum of P in I

Here we recover the fact that BS determines asymptotically all eigenvalues of P in I . We adapt the argument of [33]. It is to think of $\{a_E\}$ and $\{a'_E\}$ as zero dimensional "Poincaré sections" of γ_E . Let $\mathcal{K}_h(E)$ be the operator (Poisson operator) that assigns to its "initial value" $C_0 \in L^2(\{a_E\}) \approx \mathbb{R}$ the well normalized solution $u(x; h) = \int e^{\frac{i}{h}(x\xi + \psi(\xi))} b(\xi; h) d\xi$ to $(P(x, hD_x) - E)u = 0$ near $\{a_E\}$. By construction, we have:

$$\pm(\mathcal{L}^a(E))^* \frac{i}{h} [P, \chi^\pm] \mathcal{L}^a(E) = \text{Id}_{a_E} = 1 \quad (5.1)$$

We define object "connecting" a to a' along γ_E as follows: let $\tilde{T} = \tilde{T}(E) > 0$ such that $\exp \tilde{T} H_{p_0}(a) = a'$ (in case p_0 is invariant by time reversal, i.e. $p_0(x, \xi) = p_0(x, -\xi)$ we take $\tilde{T}(E) = T(E)/2$).

Choose χ_f^a (f for "forward") be a cut-off function supported microlocally near γ_E , equal to 0 along $\exp t H_{p_0}(a)$ for $t \leq \varepsilon$, equal to 1 along γ_E for $t \in [2\varepsilon, \tilde{T} + \varepsilon]$, and back to 0 next to a' , e.g. for $t \geq \tilde{T} + 2\varepsilon$. Let similarly χ_b^a (b for "backward") be a cut-off function supported microlocally near γ_E , equal to 1 along $\exp t H_{p_0}(a)$ for $t \in [-\varepsilon, \tilde{T} - 2\varepsilon]$, and equal to 0 next to a' , e.g. for $t \geq \tilde{T} - \varepsilon$. By (5.1) we have

$$(\mathcal{L}^a(E))^* \frac{i}{h} [P, \chi_f^a] \mathcal{L}^a(E) = (\mathcal{L}^a(E))^* \frac{i}{h} [P, \chi_b^a] \mathcal{L}^a(E) = 1 \quad (5.2)$$

$$-(\mathcal{L}^a(E))^* \frac{i}{h} [P, \chi_f^a] \mathcal{L}^a(E) = -(\mathcal{L}^a(E))^* \frac{i}{h} [P, \chi_b^a] \mathcal{L}^a(E) = 1 \quad (5.3)$$

which define a left inverse $\mathcal{R}_+^a(E) = (\mathcal{L}^a(E))^* \frac{i}{h} [P, \chi_f^a]$ to $\mathcal{L}^a(E)$ and a right inverse $\mathcal{R}_-^a(E) = -\frac{i}{h} [P, \chi_b^a] \mathcal{L}^a(E)$ to $(\mathcal{L}^a(E))^*$. We define similar objects connecting a' to a , $\tilde{T}' = \tilde{T}'(E) > 0$ such that $\exp \tilde{T}' H_{p_0}(a) = a'$ ($\tilde{T} = \tilde{T}'$ if p_0 is invariant by time reversal), in particular a left inverse $\mathcal{R}_+^{a'}(E) = (\mathcal{L}^{a'}(E))^* \frac{i}{h} [P, \chi_f^{a'}]$ to $\mathcal{L}^{a'}(E)$ and a right inverse $\mathcal{R}_-^{a'}(E) = -\frac{i}{h} [P, \chi_b^{a'}] \mathcal{L}^{a'}(E)$ to $(\mathcal{L}^{a'}(E))^*$, with the additional requirement

$$\chi_b^a + \chi_b^{a'} = 1 \quad (5.4)$$

near γ_E . Define now the pair $\mathcal{R}_+(E)u = (\mathcal{R}_+^a(E)u, \mathcal{R}_+^{a'}(E)u)$, $u \in L^2(\mathbb{R})$ and $\mathcal{R}_-(E)$ by $\mathcal{R}_-(E)u_- = \mathcal{R}_-^a(E)u_-^a + \mathcal{R}_-^{a'}(E)u_-^{a'}$, $u_- = (u_-^a, u_-^{a'}) \in \mathbb{C}^2$, we call *Grushin operator* $\mathcal{P}(z)$ the operator defined by the linear system

$$\frac{i}{h}(\mathcal{P} - z)u + \mathcal{R}_-(z)u_- = v, \quad \mathcal{R}_+(z)u = v_+ \quad (5.5)$$

From [33], we know that the problem (5.5) is well posed, and

$$(\mathcal{P}(z))^{-1} = \begin{pmatrix} \mathcal{E}(z) & \mathcal{E}_+(z) \\ \mathcal{E}_-(z) & \mathcal{E}_{-+}(z) \end{pmatrix} \quad (5.6)$$

with $(P - z)^{-1} = \mathcal{E}(z) - \mathcal{E}_+(z) (\mathcal{E}_{-+}(z))^{-1} \mathcal{E}_-(z)$. Actually one can show that the effective Hamiltonian $\mathcal{E}_{-+}(z)$ is singular precisely when 1 belongs to the spectrum of the monodromy operator, or when the microlocal solutions $u_1, u_2 \in \mathcal{K}_h(E)$ computed in (3.93) are colinear, which amounts to say that Gram matrix (3.94) is singular. There follows that the spectrum of P in I is precisely the set of z we have determined by BS quantization rule.

Appendix. Weyl quantization and conjugation by Fourier integral operators

To fix the ideas we choose the real case (analytic or C^∞). To start with, we consider the simple case of conjugation by an elliptic factor. Let $p(x, \xi; h) \sim p_0(x, \xi) + h p_1(x, \xi) + \dots$ be a classical symbol of order 0, defined near $(x_0, \xi_0) \in \mathbb{R}^2$. Let $\varphi(x)$ be a real valued smooth function, defined near x_0 , such that $\varphi'(x_0) = \xi_0$. Let $P(x, hDx; h) = p^w(x, hDx; h)$ be the Weyl h -quantization of $p(x, \xi; h)$. We are then interested in the Weyl symbol of the pseudodifferential operator $Q = e^{-\frac{i}{h}\varphi(x)} \circ P \circ e^{\frac{i}{h}\varphi(x)}$, which is defined near $(x_0, \xi_0 - \xi_0)$. We proceed formally, by first writing the integral kernel of Q as,

$$K_Q(x, y) = \int e^{\frac{i}{h}((x-y)\theta + \varphi(y) - \varphi(x))} p\left(\frac{x+y}{2}, \theta; h\right) \widetilde{d\theta}, \quad \widetilde{d\theta} = (2\pi h)^{-1} d\theta \quad (5.7)$$

It is well known (Kuranishi's Trick) that

$$\varphi(x) - \varphi(y) = F(x, y)(x - y)$$

with

$$F(x, y) = \varphi'_x\left(\frac{x+y}{2}\right) + \mathcal{O}(x-y)$$

The change of variables $\theta \mapsto \theta - F(x, y)$ then allows us to write

$$K_Q(x, y) = \int e^{\frac{i}{h}(x-y)\theta} p\left(\frac{x+y}{2}, \theta + F(x, y); h\right) \widetilde{d\theta} \quad (5.8)$$

We have

$$p\left(\frac{x+y}{2}, \theta + F(x, y); h\right) = p\left(\frac{x+y}{2}, \theta + \varphi'_x\left(\frac{x+y}{2}\right); h\right) + r(x, y, \theta)(x-y)^2$$

where

$$r(x, y, \cdot) \in \mathcal{S}(\mathbb{R})$$

By integrating by parts, we obtain:

$$\int e^{\frac{i}{h}(x-y)\theta} r(x, y, \theta)(x-y)^2 \widetilde{d\theta} = \mathcal{O}(h^2)$$

Here the \mathcal{O} term in (5.8) contributes to the Weyl symbol of Q by $\mathcal{O}(h^2)$ (i.e., by a classical symbol of order -2). Therefore, if we denote by q the Weyl symbol of Q , we have:

$$q(x, \xi; h) = p(x, \xi + \varphi'(x); h) + \mathcal{O}(h^2)$$

If $a = a(x; h) \sim a_0(x) + h a_1(x) + \dots$ is a classical symbol defined near x_0 , then:

$$\begin{aligned} Qa(x; h) &= \left(e^{-\frac{i}{h}\varphi(x)} \circ P \circ e^{\frac{i}{h}\varphi(x)}\right)(a(x; h)) = e^{-\frac{i}{h}\varphi(x)} P\left(e^{\frac{i}{h}\varphi(x)} a(x; h)\right) \\ &= (2\pi h)^{-1} \int \int e^{\frac{i}{h}(x-y)\theta} p\left(\frac{x+y}{2}, \theta + F(x, y); h\right) a(y; h) dy d\theta \end{aligned} \quad (5.9)$$

Thus, the operator $e^{-\frac{i}{h}\varphi(x)} \circ P \circ e^{\frac{i}{h}\varphi(x)}$ is a semi-classical pseudo-differential operator.

Recall that in dimension 1, we have:

$$(2\pi h)^{-1} \int \int e^{-\frac{i}{h}z\theta} u(z, \theta) dz d\theta \sim \sum_{k=0}^2 \frac{h^k}{k! i^k} ((\partial_z \partial_\theta)^k u)(0, 0) + \mathcal{O}(h^3) \quad (5.10)$$

In (5.9), we perform the change of variables $(z, \theta) = (y - x, \theta)$. The Jacobian of this transformation is equal to 1.

Now, for a fixed x , we expand using the stationary phase formula (5.10), yielding:

$$Qa(x;h) = (2\pi h)^{-1} \int \int e^{-\frac{i}{h}z\theta} p\left(x + \frac{z}{2}, \theta + F(x, z+x); h\right) a(z+x;h) dz d\theta \quad (5.11)$$

Let

$$u(z, \theta) = p\left(x + \frac{z}{2}, \theta + F(x, z+x); h\right) a(z+x;h)$$

We then have:

$$Qa(x;h) \sim u(0,0) + \frac{h}{i} \left(\frac{\partial^2 u}{\partial z \partial \theta} \right)(0,0) - \frac{h^2}{2} \left(\frac{\partial^4 u}{\partial z^2 \partial \theta^2} \right)(0,0) + \mathcal{O}(h^3)$$

Moreover:

$$F(x, x) = \int_0^1 \varphi'(x) dt = \varphi'(x)$$

$$\partial_z (F(x, z+x))_{z=0} = \int_0^1 (1-t) \varphi''(x) dt = \frac{1}{2} \varphi''(x)$$

$$\frac{\partial^2}{\partial z^2} (F(x, z+x))_{z=0} = \int_0^1 (1-t)^2 \varphi'''(x) dt = \frac{1}{3} \varphi'''(x)$$

A straightforward calculation shows that:

$$u(0,0) = p(x, \varphi'(x); h) a(x;h)$$

$$\left(\frac{\partial^2 u}{\partial z \partial \theta} \right)(0,0) = \beta(x;h) \partial_x a(x;h) + \frac{1}{2} \partial_x \beta(x;h) a(x;h)$$

where we have defined

$$\beta(x;h) = \left(\frac{\partial p}{\partial \xi} \right)(x, \varphi'(x); h)$$

and

$$\left(\frac{\partial^4 u}{\partial z^2 \partial \theta^2} \right)(0,0) = \frac{1}{4} \partial_x r(x;h) a(x;h) + \frac{1}{4} \varphi''(x) \partial_x \theta(x;h) a(x;h) + \partial_x \gamma(x;h) \partial_x a(x;h) + \gamma(x;h) \frac{\partial^2 a(x;h)}{\partial x^2} + \frac{1}{3} \varphi'''(x) \theta(x;h) a(x;h)$$

where we have set

$$r(x;h) = \left(\frac{\partial^3 p}{\partial x \partial \xi^2} \right)(x, \varphi'(x)), \quad \gamma(x;h) = \left(\frac{\partial^2 p}{\partial \xi^2} \right)(x, \varphi'(x); h), \quad \theta(x;h) = \left(\frac{\partial^3 p}{\partial \xi^3} \right)(x, \varphi'(x); h)$$

Thus, we obtain modulo $\mathcal{O}(h^3)$:

$$\begin{aligned} (Q - E)(a(x;h)) &= \left(p(x, \varphi'(x); h) - E \right) a(x;h) + \frac{h}{i} \left(\beta(x;h) \partial_x a(x;h) + \frac{1}{2} \partial_x \beta(x;h) a(x;h) \right) \\ &\quad - h^2 \left(\frac{1}{8} \partial_x r(x;h) a(x;h) + \frac{1}{8} \varphi''(x) \partial_x \theta(x;h) a(x;h) + \frac{1}{2} \partial_x \gamma(x;h) \partial_x a(x;h) + \frac{1}{2} \gamma(x;h) \frac{\partial^2 a(x;h)}{\partial x^2} + \frac{1}{6} \varphi'''(x) \theta(x;h) a(x;h) \right) \end{aligned} \quad (5.12)$$

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