

Sober topologies on a set

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Abstract

The collection of all topologies on a set X forms a complete lattice with respect to the inclusion order, which have been investigated by many researchers. Sobriety is one of the core and extensively studied properties in non-Hausdorff topology. This property plays a crucial role in characterizing the spectral spaces of commutative rings and topological spaces determined by their lattices of open sets. In this paper, we investigate the statute of sober topologies in the complete lattice of all topologies on a given set. The main results to be proved include: (1) every T_1 topology is the join of some sober topologies; (2) every topology is the meet of some sober topologies; (3) the set of all sober topologies is directed complete; (4) every Alexanderoff - discrete topology is the meet of some sober Alexanderoff - discrete topologies; (5) the minimal sober topologies are exactly the Scott topologies of sup-complete chains; (6) an example will be constructed to show that the intersection of a decreasing sequence of Hausdorff topologies need not be sober.

Keywords: sober space, lattice of topologies, sup-complete chain, Scott topology, minimal sober topology.

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In mathematics, Lattice Theory often shows to be useful in the study of the set of objects of a given type. Once the considered objects form a (complete) lattice \mathcal{L} , one immediately have some natural problems about a sub class $\mathcal{B} \subseteq \mathcal{L}$: (i) Is \mathcal{B} closed under finite (arbitrary) joins or meets? (ii) which objects in \mathcal{L} are the joins (meets, resp.) of objects in \mathcal{B} ? (iii) what are the maximal (minimal, resp.) objects in \mathcal{B} , etc. Garrett Birkhoff [5] first considered the lattice of all topologies on a set with respect to the inclusion order. Such lattices have then been extensively studied by people from different point of views. Some classic results include (i) the compact Hausdorff topologies are minimal Hausdorff; (ii) the lattice of all topologies on a set is complemented; (iii) the co-finite topology is the unique minimal T_1 topology on a set, etc.

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In their paper [18], Larson and Andima presented a fairly complete survey on the properties of $\Lambda(X)$ of all topologies on a fixed set X satisfying a property p for various p . See also [7] for a survey on maximal topologies and [1][4][6][8][15][20][23] for additional results on maximal and minimal topologies. For other work on this topic, see [2][5][24][25].

A topological space X is sober if its every irreducible closed set is the closure of a unique singleton set. Sober topologies appeared naturally in several parts of mathematics. The well known result by M. Hochster states that a topological space is homeomorphic to the spectral space of a commutative ring if and only if (i) X is sober, (ii) the compact open sets form a base of X , (iii) the intersection of two compact open sets is compact and (iv) X is compact [13]. The sobriety was also used in characterizing the topological spaces which are determined by their open set lattices [10]. Moreover, the category of all sober spaces is reflective in the category of all T_0 space. With the emerging and development of domain theory, sobriety has become one of the most extensively studied non-Hausdorff properties, in particular for the Scott spaces of directed complete posets [21][22][26][27].

In this paper, we study the class of all sober topologies in the lattice of all topologies on a fixed set. The main results to be proved include: (1) every T_1 topology is the join (supremum) of some sober topologies; (2) every topology is the meet of some sober topologies; (3) the set of all sober topologies is directed complete; (4) the minimal sober topologies are exactly the Scott topologies of sup-complete chains; (5) every Alexanderoff - discrete topology is the meet of some sober Alexanderoff - discrete topologies; (6) the meet of a decreasing sequence of Hausdorff topologies need not be a sober topology.

1. Preliminary

Given a set X , the set $T(X)$ of all topologies on X , equipped with the set inclusion order \subseteq , is a complete lattice. For any $\mathcal{A} \subseteq T(X)$,

$$\bigwedge \mathcal{A} = \inf \mathcal{A} = \bigcap \mathcal{A},$$

and

$$\bigvee \mathcal{A}$$

equals the topology with $\bigcup \mathcal{A}$ as a subbase. Usually, $\bigvee \mathcal{A}$ is called the topology generated by $\bigcup \mathcal{A}$.

In the following, for a subset A in a topological space (X, τ) , we shall use $cl(A)$ or \bar{A} to denote the closure of A . We may also use $cl_\tau(A)$ to denote the closure of A if we wish to indicate explicitly the topology τ . Also $\Gamma(X, \tau)$ (or $\Gamma_\tau(X)$) will be used to denote the set of all closed sets of (X, τ) , called the *co-topology* of (X, τ) .

A non-empty subset A in a topological space X is *irreducible* if for any closed sets F, G in X , $A \subseteq F \cup G$ implies either $A \subseteq F$ or $A \subseteq G$.

We shall use $\text{Irr}(X, \tau)$ (or just $\text{Irr}(X)$) to denote the set of all irreducible sets of space (X, τ) , and use $\text{Irr}_c(X, \tau)$ (or just $\text{Irr}_c(X)$) to denote the set of all closed irreducible sets in (X, τ) .

Remark 1.1. (1) If A is an irreducible set, then so is $cl(A)$.

- (2) A is irreducible if and only if for any open sets U and V , $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$ imply $A \cap U \cap V \neq \emptyset$.

- (3) If $f : X \longrightarrow Y$ is a continuous function between two topological spaces and $A \subseteq X$ is irreducible, then $f(A)$ is an irreducible set in Y .
- (4) Every singleton set $\{x\}$ is irreducible. Thus every $cl(\{x\})$ is irreducible.
- (5) If τ_1 and τ_2 are two topologies on a set X such that $\tau_1 \subseteq \tau_2$, then

$$\text{Irr}(X, \tau_2) \subseteq \text{Irr}(X, \tau_1).$$

Definition 1.2. A topological space (X, τ) is *sober* if for every closed irreducible set A in X , there is a unique $x \in X$ such that $A = cl(\{x\})$.

In this case, τ is also called a *sober* topology on X .

By the uniqueness of x in the above definition, it follows that every sober space is T_0 .

If X is a sober space, then the function $f : X \longrightarrow \text{Irr}_c(X)$, defined by $f(x) = cl(\{x\})$, is a bijection, hence $|X| = |\text{Irr}_c(X)|$.

Lemma 1.3. Let $\{\tau_i : i \in I\} \subseteq \mathcal{T}(X)$ and $\tau = \bigvee \{\tau_i : i \in I\}$. Then for any $A \in \text{Irr}(X, \tau)$,

$$cl_\tau(A) = \bigcap \{cl_{\tau_i}(A) : i \in I\}.$$

In particular, for each $A \in \text{Irr}_c(X, \tau)$,

$$A = \bigcap \{cl_{\tau_i}(A) : i \in I\}.$$

Proof. Let $B = \bigcap \{cl_{\tau_i}(A) : i \in I\}$. Since $\tau_i \subseteq \tau$ for each $i \in I$, $cl_\tau(A) \subseteq cl_{\tau_i}(A)$ for each $i \in I$. Hence, $cl_\tau(A) \subseteq B$.

For any $x \in X - cl_\tau(A)$, as $X - cl_\tau(A) \in \tau$ and τ has $\bigcup \{\tau_i : i \in I\}$ as a subbase, there are $U_k \in \tau_{i_k} (k = 1, 2, \dots, m)$ such that $x \in \bigcap \{U_k : k = 1, 2, \dots, m\} \subseteq X - cl_\tau(A)$. Then $A \subseteq cl_\tau(A) \subseteq U_1^c \cup U_2^c \cup \dots \cup U_m^c$, where U_k^c is the complement of U_k . Note that for each k , $U_k \in \tau$, thus U_k^c is a closed set in (X, τ) . By the irreducibility of A , $A \subseteq U_{k'}^c$ for some k' . Then, as $x \in U_{k'} \in \tau_{i_{k'}}$ and $A \cap U_{k'} = \emptyset$, we have that $x \notin cl_{\tau_{i_{k'}}}(A)$, implying $x \notin B$. Thus $B \subseteq cl_\tau(A)$, therefore $cl_\tau(A) = B$ as desired. \square

Note that $cl_\tau(A) = \bigcap \{cl_{\tau_i}(A) : i \in I\}$ may fail to be true if A is not irreducible.

Remark 1.4. (1) Let X be a finite T_0 space. Then $cl(\{x\}) \neq cl(\{y\})$ for any $x, y \in X$ if $x \neq y$. For any $F \in \text{Irr}_c(X)$, $F = \bigcup \{cl(\{x\}) : x \in F\}$. Since $F \subseteq X$ is a finite set, we have $F = cl(\{x\})$ for some $x \in F$. It follows that X is sober.

(2) Every T_2 topology is sober.

For any T_0 space (X, τ) , the *specialization order* \leq_τ , which is a partial order on X , is defined by

$$x \leq_\tau y \text{ if and only if } x \in cl(\{y\}).$$

A nonempty subset D of a poset P is *directed* if for any two elements $x, y \in D$, there is a $z \in D$ such that $x \leq z, y \leq z$. A poset P is *directed complete* if for any directed subset $D \subseteq P$, $\bigvee D$ exists in P . A directed complete poset will also be called a dcpo.

A subset A of a poset P is an upper (lower, resp.) set if $A = \uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}$ ($A = \downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$, resp.)

A subset U of a poset P is Scott open if (i) U is an upper set, and (ii) for any directed set D with $\sup D$ existing, $\sup D \in U$ implies $D \cap U \neq \emptyset$. All Scott open sets of a poset P form a topology on P , called the Scott topology of P and is denoted by $\sigma(P)$.

For any poset (P, \leq) , the specialization order \leq_σ of $(P, \sigma(P))$ coincides with the partial order \leq on P .

A subset A of a space (X, τ) is saturated if it equals the intersection of all open sets containing A . A T_0 space X is well-filtered if for any open set U and filter \mathcal{F} of saturated compact subsets of X ,

$$\bigcap \mathcal{F} \subseteq U \text{ implies } F \subseteq U \text{ for some } F \in \mathcal{F}.$$

A T_0 space (X, τ) is called a d-space (or monotone convergence space) if (X, \leq_τ) is a dcpo and every $U \in \tau$ is a Scott open set of (X, \leq_τ) (that is, $\tau \subseteq \sigma(X, \leq_\tau)$). By [19], (X, τ) is a d-space if and only if for any directed set D of (X, \leq_τ) and open set U ,

$$\bigcap \{\uparrow d : d \in D\} \subseteq U \text{ implies } d \in U \text{ for some } d \in D,$$

here $\uparrow d = \uparrow \{d\}$.

Every sober space is well-filtered and every well-filtered space is a d-space. For any d-space (X, τ) , (X, \leq_τ) is a dcpo.

See [11] and [12] for more about Scott topologies, sober spaces, well-filtered spaces and d-spaces.

Let x, y are elements in a poset P . We say that x is *way-below* y , in symbols $x \ll y$, if for every directed subset $D \subseteq P$ for which $\sup D$ exists, $y \leq \sup D$ implies the existence of a $d \in D$ with $x \leq d$. For each $x \in P$, let $\downarrow x = \{y \in P : y \ll x\}$.

A dcpo P is called a *domain* if for each $x \in P$, $\downarrow x$ is a directed set and $x = \bigvee \downarrow x$.

For every domain P , $(P, \sigma(P))$ is sober [11] and [12].

2. Joins of sober topologies

In this section, we consider the following problem: Which topologies $\tau \in \mathbf{T}(X)$ are the join of sober topologies? The main result is that every T_1 topology is such a topology. There is a T_0 topology which is not the join of sober topologies.

For any set X , we shall use $\mathbf{T}_{sob}(X)$ to denote the set of all sober topologies on X .

Proposition 2.1. *If $\tau \in \mathbf{T}(X)$ is the join of a finite number of sober topologies, then $|X| = |\text{Irr}_c(X, \tau)|$, here $|X|$ is the cardinality of X .*

Proof. Let $\tau = \bigvee \{\tau_i : i \in D\}$ with $D = \{1, 2, \dots, m\}$ and each τ_i sober.

Since τ is finer than the sober topology τ_1 , which is T_0 , τ is a T_0 topology.

If X is a finite set, then by Remark 1.4, (X, τ) is sober, thus $|X| = |\{cl(\{x\}) : x \in X\}| = |\text{Irr}_c(X, \tau)|$.

Next, we assume that X is an infinite set.

Since X is T_0 , for any $x, y \in X$, $x = y$ if and only if $cl(\{x\}) = cl(\{y\})$. Hence, $|X| = |\{cl(\{x\}) : x \in X\}| \leq |\text{Irr}_c(X, \tau)|$ because $cl(\{x\}) \in \text{Irr}_c(X, \tau)$ for each $x \in X$.

For each $A \in \text{Irr}_c(X, \tau)$, by Lemma 1.3,

$$A = \bigcap \{cl_{\tau_i}(A) : i \in D\}.$$

Also $cl_{\tau_i}(A) \in \text{Irr}_c(X, \tau_i)$ for each $i \in D$, by Remark 1.1(4). Thus there is an injective $\phi : \text{Irr}_c(X, \tau) \longrightarrow \prod_{i \in D} \text{Irr}_c(X, \tau_i)$ defined by

$$\phi(A) = (cl_{\tau_1}(A), cl_{\tau_2}(A), \dots, cl_{\tau_m}(A)),$$

for each $A \in \text{Irr}_c(X, \tau)$.

So $|X| \leq |\text{Irr}_c(X, \tau)| \leq |\prod_{i \in D} \text{Irr}_c(X, \tau_i)| = |\text{Irr}_c(X, \tau_1)| \times |\text{Irr}_c(X, \tau_2)| \times \dots \times |\text{Irr}_c(X, \tau_m)|$.

Since each (X, τ_i) is sober, $|X| = |\text{Irr}_c(X, \tau_i)|$. Therefore, $|\text{Irr}_c(X, \tau_1)| \times |\text{Irr}_c(X, \tau_2)| \times \dots \times |\text{Irr}_c(X, \tau_m)| = |X|^m = |X|$, because $|X|$ is infinite. All these together deduce that

$$|X| = |\text{Irr}_c(X, \tau)|.$$

□

Example 2.2. Let τ_{cof} be the co-finite topology on the set \mathbb{N} of all positive integers ($U \in \tau_{cof}$ if and only if either $U = \emptyset$ or $\mathbb{N} - U$ is a finite set).

Then (\mathbb{N}, τ_{cof}) is a T_1 space. As $\mathbb{N} \in \text{Irr}_c(X, \tau_{cof})$ and $\mathbb{N} \neq cl(\{x\})$ for any x , (\mathbb{N}, τ_{cof}) is not sober.

Let τ_1 be the topology on \mathbb{N} such that $U \in \tau_1$ iff either $U = \emptyset$ or $1 \in U$ and $\mathbb{N} - U$ is a finite set.

Let τ_2 be the topology on \mathbb{N} such that $U \in \tau_2$ iff either $U = \emptyset$ or $2 \in U$ and $\mathbb{N} - U$ is a finite set.

Then we can verify that both τ_1 and τ_2 are sober topologies and $\tau_{cof} = \tau_1 \vee \tau_2$. Thus the join of finite numbers of sober topologies need not be sober.

The example below shows that not every topology is the join of sober topologies.

Example 2.3. Let \mathbb{R} be the set of all real numbers and $\tau = \{(r, +\infty) : r \in \mathbb{R}\} \cup \{\mathbb{R}\}$ be the upper topology on \mathbb{R} . Then τ is a minimal T_0 topology on \mathbb{R} by a characterization of minimal T_0 topologies given in [17]. Hence, any topology strictly smaller than τ is not T_0 , thus not sober. It follows that this T_0 topology τ is not the join of sober topologies.

In general, any non-sober minimal T_0 topology is not the join of sober topologies.

However, if τ is a T_1 topology, then it is the join of sober topologies.

Theorem 2.4. For any T_1 topology $\tau \in \mathbf{T}(X)$, there is a set $\mathcal{A} \subseteq \mathbf{T}_{sob}(X)$ such that

$$\tau = \bigvee \mathcal{A}.$$

Proof. We only need to consider the non-trivial case where (X, τ) is not sober, thus X must be an infinite set.

(1) For each closed proper nonempty subset $A \subseteq X$, choose two points $x_A \in A$ and $y_A \notin A$. Let

$$\tau_A = \{U \subseteq X : x_A, y_A \in U \text{ and } U^c \text{ is finite}\} \cup \{A^c \cap V : V \subseteq X \text{ with } x_A, y_A \in V \text{ and } V^c \text{ is finite}\}.$$

One easily verify that τ_A is indeed a topology on X .

The closed sets of τ_A are

$$\{F : F \subseteq X - \{x_A, y_A\} \text{ and } F \text{ is finite}\} \cup \{A \cup F : F \subseteq X - \{x_A, y_A\} \text{ and } F \text{ is finite}\} \cup \{X\}.$$

In particular, $A = A \cup \emptyset$ is closed in (X, τ_A) , that is, $A^c \in \tau_A$.

It is easily seen that τ_A is T_0 and

$$\text{Irr}_c(X, \tau_A) = \{\{x\} : x \notin \{x_A, y_A\}\} \cup \{A, X\}.$$

If $x \notin \{x_A, y_A\}$, then $\{x\} = cl_{\tau_A}(\{x\})$. Also $A = cl_{\tau_A}(\{x_A\})$ and $X = cl_{\tau_A}(\{y_A\})$. Therefore, τ_A is a sober topology.

Also, as (X, τ) is T_1 , $\tau_A \subseteq \tau$ holds, therefore $\bigvee \{\tau_A : A^c \in \tau, A \neq X, A \neq \emptyset\} \subseteq \tau$.

(2) Now let $U \in \tau$.

Case 1: $U = X$ or \emptyset . Then $U \in \tau_A$ for each τ_A .

Case 2: $U \neq X$ and $U \neq \emptyset$. Let $A = U^c$. Then A is a proper, nonempty closed set of (X, τ_A) . By (1), $U = A^c \in \tau_A$.

Hence $\tau \subseteq \bigvee \{\tau_A : A \text{ is a nonempty proper closed set of } (X, \tau)\}$.

All these together then show that τ equals the join of these sober topologies τ'_A s.

The theorem is proved. \square

Remark 2.5. (1) The reader may wonder whether the above theorem can be strengthened to that every T_1 topology is the join of T_1 sober topologies. Consider the co-finite topology τ_{cof} on the set \mathbb{N} of all positive integers. Then τ_{cof} is the coarsest T_1 topology on \mathbb{N} . Thus if τ is a T_1 sober topology on \mathbb{N} , then $\tau_{cof} \subseteq \tau$ and $\tau_{cof} \neq \tau$. Hence τ_{cof} is not the join of T_1 sober topologies.

(2) For each of the topology τ_A constructed in the proof of Theorem 2.4, τ_A actually possesses several other properties, such as (i) τ_A is connected and locally connected; (ii) every subset of X is compact.

An element a of a complete lattice L is *strongly irreducible* if for any $C \subseteq L, C \neq \emptyset$ and $a = \bigvee C$ imply $a = c$ for some $c \in C$ [10][28].

By Theorem 2.4, we easily deduce the following.

Corollary 2.6. *If $\tau \in \mathbf{T}(X)$ is a T_1 topology and a strongly irreducible element of $\mathbf{T}(X)$, then τ is sober.*

A subset C of a poset (P, \leq) is an upper set, if $x \leq y$ and $x \in C$ imply $y \in C$, equivalently, if $C = \uparrow\{y \in P : x \leq y \text{ for some } x \in C\}$.

The following result shows that the subset of $\mathbf{T}(X)$ consisting of all T_1 sober topologies is an upper set of $\mathbf{T}(X)$. In addition, all T_D sober topologies is also an upper set of $\mathbf{T}(X)$.

Recall that a space (X, τ) is T_D if for each $x \in X$, $cl(\{x\}) - \{x\}$ is a closed set. Trivially, every T_1 spaces is T_D .

The following lemma should have been proved by other people. For reader's convenience, we give a brief proof.

Lemma 2.7. *Let (X, τ) be a topological space. Then the following statements are equivalent.*

- (1) (X, τ) is a T_D space.
- (2) For any $x \in X$ and $A \subseteq X$, $cl(A) = cl(\{x\})$ implies $x \in A$.

Proof. Let (X, τ) be T_D . Assume that $cl(\{x\}) = cl(A)$ for some $x \in X$ and $A \subseteq X$. If $x \notin A$, then $A \subseteq cl(\{x\}) - \{x\}$. Hence $cl(A) \subseteq cl(cl(\{x\}) - \{x\}) = cl(\{x\}) - \{x\} \subsetneq cl(\{x\})$, a contradiction.

Now assume that (X, τ) satisfies (2). Let $x \in X$ and $A = cl(\{x\}) - \{x\}$. If $cl(A) \neq A$, then, as $A \subseteq cl(A) \subseteq cl(\{x\}) = A \cup \{x\}$, it follows that $x \in cl(A)$. By (2), $x \in A = cl(\{x\}) - \{x\}$, which is not possible. Hence $cl(A) = A = cl(\{x\}) - \{x\}$, showing that $cl(\{x\}) - \{x\}$ is closed. Hence (X, τ) is T_D . \square

Proposition 2.8. (1) The set of all T_D topologies on X is an upper set of $T(X)$.

- (2) If τ is a T_D and sober topology on X , then for any $\mu \in T(X)$, $\tau \subseteq \mu$ implies μ is T_D and sober.

Proof. (1) Let τ be a T_D topology and $\tau \subseteq \mu \in T(X)$. For any $x \in X$, $cl_\mu(\{x\}) - \{x\} = cl_\mu(\{x\}) \cap (cl_\tau(\{x\}) - \{x\})$ by $cl_\mu(\{x\}) \subseteq cl_\tau(\{x\})$. Now $cl_\tau(\{x\}) - \{x\} \in \Gamma(X, \tau) \subseteq \Gamma(X, \mu)$, thus $cl_\mu(\{x\}) - \{x\} = cl_\mu(\{x\}) \cap (cl_\tau(\{x\}) - \{x\}) \in \Gamma(X, \mu)$. Hence (X, μ) is T_D .

(2) By (1), μ is T_D . Assume $C \in Irr_c(X, \mu)$. Then $C \in Irr(X, \tau)$ and so $cl_\tau(C) = cl_\tau(\{x\})$ for some $x \in X$. By Lemma 2.7(2), $x \in C$. We now prove that $C = cl_\mu(\{x\})$. Assume that $C \neq cl_\mu(\{x\})$, so $C - cl_\mu(\{x\}) \neq \emptyset$. Then $C \subseteq cl_\tau(\{x\}) \subseteq (cl_\tau(\{x\}) - \{x\}) \cup cl_\mu(\{x\})$, and $C \not\subseteq (cl_\tau(\{x\}) - \{x\})$ and $C \not\subseteq cl_\mu(X)$. Note that $cl_\tau(\{x\}) - \{x\} \in \Gamma(X, \tau) \subseteq \Gamma(X, \mu)$ and $cl_\mu(\{x\}) \in \Gamma(X, \mu)$. These contradict to that $C \in Irr_c(X, \mu)$. Thus $C = cl_\mu(\{x\})$. Hence (X, μ) is sober. \square

Corollary 2.9. *If τ is a T_1 and sober topology on X , then for any $\mu \in T(X)$, $\tau \subseteq \mu$ implies μ is T_1 and sober.*

Proof. The topology μ is T_1 as it is finer than a T_1 topology. Since τ is T_1 , it is T_D . By Proposition 2.8, μ is also sober. \square

Corollary 2.10. *For any collection $\{\tau_i : i \in I\}$ of T_1 and sober topologies on a set X ,*

$$\bigvee \{\tau_i : i \in I\}$$

is T_1 and sober.

The following propositions can be found in [16, Lemma 13.3] and [9, Lemma 72], respectively.

Proposition 2.11. *If τ_0 is a quasi-polish topology on X and $\{\tau_i : i \in \mathbb{N}\}$ is a set of quasi-polish topologies on X such that $\tau_0 \subseteq \tau_i$ for each $i \in \mathbb{N}$, then $\tau = \bigvee \{\tau_i : i \in \mathbb{N}\}$ is quasi-polish.*

Proposition 2.12. *If τ_0 is a quasi-polish topology on X and $\{\tau_i : i \in \mathbb{N}\}$ is a set of polish topologies on X such that $\tau_0 \subseteq \tau_i$ for each $i \in \mathbb{N}$, then $\tau = \bigvee \{\tau_i : i \in \mathbb{N}\}$ is polish.*

For sober topologies, we have a similar result where the countable index set \mathbb{N} can be replaced by any set.

Lemma 2.13. *Let τ_1, τ_2 be two sober topologies on a set X with $\tau_1 \subseteq \tau_2$. Then we have the following.*

- (1) $\text{Irr}_c(X, \tau_1) = \{cl_{\tau_1}(A) : A \in \text{Irr}_c(X, \tau_2)\}$.
- (2) For any $a, b \in X$, $cl_{\tau_1}(cl_{\tau_2}(\{a\})) = cl_{\tau_1}(\{b\})$ implies $a = b$.

Proof. (1) By Remark 1.1 (1) (4), $\{cl_{\tau_1}(A) : A \in \text{Irr}_c(X, \tau_2)\} \subseteq \text{Irr}_c(X, \tau_1)$. Now let $F \in \text{Irr}_c(X, \tau_1)$. Thus $F = cl_{\tau_1}(\{x_0\})$ for a unique $x_0 \in X$. Now $cl_{\tau_2}(\{x_0\}) \in \text{Irr}_c(X, \tau_2)$ because $\{x_0\}$ is an irreducible set in (X, τ_2) . So, $F \subseteq cl_{\tau_1}(cl_{\tau_2}(\{x_0\}))$ because $\{x_0\} \subseteq cl_{\tau_2}(\{x_0\})$. Also $cl_{\tau_2}(\{x_0\}) \subseteq cl_{\tau_1}(\{x_0\})$, it follows that $F \subseteq cl_{\tau_1}(cl_{\tau_2}(\{x_0\})) \subseteq cl_{\tau_1}(cl_{\tau_1}(\{x_0\})) = cl_{\tau_1}(\{x_0\}) = F$. Hence,

$$F = cl_{\tau_1}(K),$$

where $K = cl_{\tau_2}(\{x_0\}) \in \text{Irr}_c(X, \tau_2)$. So the equation holds.

(2) If $cl_{\tau_1}(cl_{\tau_2}(\{a\})) = cl_{\tau_1}(\{b\})$ with $a, b \in X$, then, similar to the proof of (1), we deduce that $cl_{\tau_1}(\{a\}) = cl_{\tau_1}(cl_{\tau_2}(\{a\})) = cl_{\tau_1}(\{b\})$. Thus $a = b$ because (X, τ_1) is T_0 . \square

Proposition 2.14. *If τ_0 is a sober topology on X and $\{\tau_i : i \in I\}$ is a set of sober topologies on X such that $\tau_0 \subseteq \tau_i$ for each $i \in I$, then $\tau = \bigvee \{\tau_i : i \in I\}$ is sober.*

Proof. Firstly, (X, τ) is T_0 because τ is finer than the T_0 topologies $\tau_i (i \in I)$.

Let $A \in \text{Irr}_c(X, \tau)$. Then, again, by Remark 1.1 (1)(4), $cl_{\tau_j}(A) \in \text{Irr}_c(X, \tau_j)$ for each $j \in I \cup \{0\}$ and $A = \bigcap \{cl_{\tau_i}(A) : i \in I\}$, by Proposition 1.3. Since τ_j is sober for each $j \in I \cup \{0\}$, $cl_{\tau_j}(A) = cl_{\tau_j}(\{x_j\})$ for some $x_j \in X$. Now $cl_{\tau_0}(A) = cl_{\tau_0}(\{x_0\}) \subseteq cl_{\tau_0}(cl_{\tau_i}(\{x_i\})) \subseteq cl_{\tau_0}(cl_{\tau_0}(\{x_0\})) = cl_{\tau_0}(\{x_0\})$, implying $cl_{\tau_0}(cl_{\tau_i}(\{x_i\})) = cl_{\tau_0}(\{x_0\})$. By Lemma 2.13(2), $x_0 = x_i$ for each $i \in I$.

Then $A = \bigcap \{cl_{\tau_i}(\{x_0\}) : i \in I\}$. Note that the set $\{x_0\}$ is irreducible in (X, τ) , by Lemma 1.3, we also have

$$cl_{\tau}(\{x_0\}) = \bigcap \{cl_{\tau_i}(\{x_0\}) : i \in I\}.$$

At last, we have $A = cl_{\tau}(\{x_0\})$. The uniqueness of x_0 follows from that τ is T_0 . Therefore, (X, τ) is sober. \square

If D is a directed subset of poset P and $d_0 \in D$, then $D_{d_0} = \{d \in D : d_0 \leq d\}$ is also a directed set and $\bigvee D = \bigvee D_{d_0}$ if $\bigvee D$ exists in P .

Let $\mathcal{B} = \{\tau_i : i \in I\} \subseteq T(X)$ be a directed set consisting of sober topologies. Fixed one τ_{i_0} and let $\mathcal{B}_{i_0} = \{\tau_i : \tau_{i_0} \subseteq \tau_i\}$. Then by the previous remark, \mathcal{B}_{i_0} is also a directed set and $\bigvee \mathcal{B} = \bigvee \mathcal{B}_{i_0}$, here the suprema are taken in $T(X)$. Since \mathcal{B}_{i_0} has a bottom element τ_{i_0} , by Proposition 2.14, $\bigvee \mathcal{B}_{i_0}$ is sober. Hence $\bigvee \mathcal{B} = \bigvee \mathcal{B}_{i_0}$ is sober.

Corollary 2.15. *The poset $(T_{sob}(X), \subseteq)$ of all sober topologies on a set X is a dcpo.*

For any $\tau \in T(X)$, define

$$\omega_{sob}(\tau) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq T_{sob}(X) \text{ and } \tau = \bigvee \mathcal{A}\},$$

where $|\mathcal{A}|$ is the cardinality of the set \mathcal{A} .

Then $\omega_{sob}(\tau) = 1$ if and only if τ is sober.

By Example 2.2, $\omega_{sob}(\tau_{cof}) = 2$, where τ_{cof} is the co-finite topology on \mathbb{N} .

Lemma 2.16. *Let $\{\tau_i : i \in I\} \subseteq T_{sob}(X)$ and $\tau = \bigvee \{\tau_i : i \in I\}$. Then*

$$|Irr_c(X, \tau)| \leq |X|^{|I|}.$$

Proof. Define the mapping $\phi : Irr_c(X, \tau) \longrightarrow \prod_{i \in I} Irr_c(X, \tau_i)$ by

$$\phi(A) = (cl_{\tau_i}(A))_{i \in I}, A \in Irr_c(X, \tau).$$

By Lemma 1.3, ϕ is well defined and injective. For each $i \in I$, since (X, τ_i) is sober, we have $Irr_c(X, \tau_i) = \{cl_{\tau_i}\{x\} : x \in X\}$. Thus $|Irr_c(X, \tau_i)| = |X|$. Therefore, $|Irr_c(X, \tau)| \leq |\prod_{i \in I} Irr_c(X, \tau_i)| = |X|^{|I|}$, as desired. \square

Now we give a topology τ such that $\omega_{sob}(\tau)$ is not finite.

Example 2.17. Let $X = \mathbb{Q} \cap [0, 1]$ be the set of all rational numbers in $[0, 1]$. Let τ be the topology on X whose family of closed sets equals

$$\{F \subset X : F \text{ is finite}\} \cup \{([0, x] \cap \mathbb{Q}) \cup F : x \in [0, 1], F \subseteq X \text{ is finite}\}.$$

Then we can verify that

$$Irr_c(X, \tau) = \{\{x\} : x \in X\} \cup \{[0, x] \cap \mathbb{Q} : x \in [0, 1]\}.$$

Now $|Irr_c(X, \tau)| = |X| + |[0, 1]| = \aleph_1$. If $\tau = \bigvee \{\tau_i : i \in H\}$ with H a finite set, then by Lemma 2.16, $\aleph_1 \leq |X|^{|H|} = \aleph_0^{|H|} = \aleph_0$, a contradiction. It follows that $\omega_{sob}(\tau) \geq \aleph_0$.

In fact, $\Gamma(X, \tau)$ has a subbase $\{\{x\} : x \in X\} \cup \{[0, q] : q \in \mathbb{Q}\} \cup \{\emptyset\}$. And we have $\omega_{sob}(\tau) \leq \aleph_0$, by the proof of Theorem 2.4.

Hence, $\omega_{sob}(\tau) = \aleph_0$.

3. Meets of sober topologies

In this section, we study the meets of sober topologies in $T(X)$. The main result is that every topology is the meet of a collection of sober topologies. Thus, the set $T_{sob}(X)$ is meet dense in $T(X)$.

Note that for any $\mathcal{A} \subseteq T(X)$, $\inf \mathcal{A} = \bigwedge \mathcal{A} = \bigcap \mathcal{A}$.

First, we show that every topology is the meet of some T_0 topologies. This result seems having been known for quite some time (we saw it in some online forum without proof), but we could not find a reliable source for it. For reader's convenience, we give a proof here.

Let (M, μ) be a topological space and $M \subseteq X$. Let τ_M be the topology on X such that

$$\Gamma(X, \tau_M) = \{A \cup B : A \cap M = \emptyset \text{ and } B \in \Gamma(M, \mu)\}.$$

By the definition of τ_M , (M, μ) is a closed subspace of (X, τ_M) .

Lemma 3.1. *With the above notations, we have the following statements.*

- (1) (M, μ) is T_0 if and only if (X, τ_M) is T_0 .
- (2) (M, μ) is T_1 if and only if (X, τ_M) is T_1 .
- (3) (M, μ) is sober if and only if (X, τ_M) is sober.

The space (X, τ_M) is actually the direct sum $(X - M) \oplus M$ of the discrete space $X - M$ and the space (M, μ) . Hence the three statements hold.

Given a topology $\tau \in \mathbf{T}(X)$, let \sim_τ be the equivalence relation on X defined by

$$x \sim_\tau y \text{ if and only if } cl_\tau(\{x\}) = cl_\tau(\{y\}).$$

Let $X/\sim_\tau = \{[x_i] : i \in I\}$ be the collection of all distinct equivalence classes determined by \sim_τ . Let $P = \{f \in X^I : f(i) \in [x_i] \text{ for each } i \in I\}$.

For each $f \in P$, let $F_f : X \rightarrow X$ be the mapping such that for each $x \in [x_i]_{\sim_\tau}$, $F_f(x) = f(i)$.

For each $f \in P$, let τ_f be the topology on X such that

$$\Gamma(X, \tau) \cup \{F \subseteq X : F \text{ is finite and } f(i) \in F \subseteq [x_i]_{\sim_\tau} \text{ for some } i \in I\}$$

is a sub base of the co-topology of τ_f .

Trivially, $\tau \subseteq \tau_f$. Also for any $a \in [x_i]_{\sim_\tau}$, $\{a, f(i)\}$ is the smallest closed set in (X, τ_f) containing a , hence $cl_{\tau_f}(\{a\}) = \{a, f(i)\}$ when $a \neq f(i)$, and $cl_{\tau_f}(\{a\}) = \{a\}$ if $a = f(i)$.

Note that a topology $\mu \in \mathbf{T}(X, \tau)$ is T_0 iff for any $a, b \in X$, $a \neq b$ implies $cl_\mu(\{a\}) \neq cl_\mu(\{b\})$.

Lemma 3.2. *For each $f \in P$, τ_f is a T_0 topology.*

Proof. Let $a, b \in X$ with $a \neq b$.

Case 1: $a \sim_\tau b$.

Then $\{a, b\} \subseteq [x_i]_{\sim_\tau}$ for some $i \in I$.

If $a = f(i)$ (or $b = f(i)$), then $cl_{\tau_f}(\{a\}) = \{a\}$ ($cl_{\tau_f}(\{b\}) = \{b\}$, resp.), thus $cl_{\tau_f}(\{a\}) \neq cl_{\tau_f}(\{b\})$.

If $a \neq f(i)$ and $b \neq f(i)$, then one can verify that $cl_{\tau_f}(\{a\}) = \{f(i), a\}$, and as $b \notin \{f(i), a\}$, again $cl_{\tau_f}(\{a\}) \neq cl_{\tau_f}(\{b\})$.

Case 2: $a \sim_\tau b$ does not hold.

Then either $a \notin cl_\tau(\{b\})$ or $b \notin cl_\tau(\{a\})$. As $cl_{\tau_f}(\{b\}) \subseteq cl_\tau(\{b\})$, $cl_{\tau_f}(\{a\}) \subseteq cl_\tau(\{a\})$, we have that $a \notin cl_{\tau_f}(\{b\})$ or $b \notin cl_{\tau_f}(\{a\})$, implying $cl_{\tau_f}(\{a\}) \neq cl_{\tau_f}(\{b\})$.

All these together show that (X, τ_f) is T_0 . □

Remark 3.3. For each subset A of a space (X, τ) , let $\tilde{A} = \bigcup \{cl_\tau(\{x\}) : x \in A\}$. Then

- (1) $A \subseteq \tilde{A} \subseteq cl_\tau(A)$;
- (2) $(A \tilde{\cup} B) = \tilde{A} \cup \tilde{B}$ for any $A, B \subseteq X$;

$$(3) \quad \tilde{A} = \tilde{A}.$$

Hence $A \rightarrow \tilde{A}$ defines a closure operator on $\mathcal{P}(X)$.

Lemma 3.4. *For any $\tau \in \mathbf{T}(X)$, there is a collection $\mathcal{A} \subseteq \mathbf{T}(X)$ of T_0 topologies such that*

$$\tau = \bigcap \mathcal{A}.$$

Proof. As before, let $X/\sim_\tau = \{[x_i]_{\sim_\tau} : i \in I\}$ and $P = \{f \in X^I : f(i) \in [x_i] \text{ for each } i \in I\}$.

Let $\lambda = \bigcap \{\mu \in \mathbf{T}(X) : \tau \subseteq \mu, \mu \text{ is } T_0\}$. Then $\tau \subseteq \lambda$.

Assume that $A \subseteq X$ with $A \notin \Gamma(X, \tau)$, we show that $A \notin \Gamma(X, \lambda)$.

Case 1: $A \neq \tilde{A} = \bigcup \{cl_\tau(\{x\}) : x \in A\}$.

Then there exist $a \in A$ and $y \in cl_\tau(\{a\}) - A$, thus $y \sim_\tau a$ and so $y, a \in [x_i]_{\sim_\tau}$ for some $i \in I$.

Choose an $f \in P$ such that $y = f(i)$. Then $cl_{\tau_f}(\{a\}) = \{a, f(i)\} = \{a, y\}$. It follows that $A \notin \Gamma(X, \tau_f)$, otherwise $y \in cl_{\tau_f}(\{a\}) \subseteq cl_{\tau_f}(A) = A$, contradicting $y \notin A$.

Note that τ_f is T_0 and $\tau \subseteq \tau_f$, thus $\lambda \subseteq \tau_f$. Hence, $A \notin \Gamma(X, \lambda)$.

Case 2: $A = \tilde{A}$.

Then for any $x \in A$, $y \sim_\tau x$ implies $y \in A$.

Since $A \notin \Gamma(X, \tau)$, there exists $b \in cl_\tau(A) - A$.

Choose any $f \in P$ and let $M = F_f(X) = \{f(x) : x \in X\}$. Take (M, ν) be the subspace of (X, τ) with underlying set M . Then, as different element of M are not \sim_τ equivalent, (M, ν) is T_0 , thus (X, τ_M) is T_0 by Lemma 3.1.

For any $F \in \Gamma(X, \tau)$, $F = (F - M) \cup (F \cap M)$, and $F - M \subseteq X - M$, $F \cap M \in \Gamma(M, \nu)$, hence, $F \in \Gamma(X, \tau_M)$. It follows that $\tau \subseteq \tau_M$.

We now show that $A \notin \Gamma(X, \tau_M)$. Assume, on the contrary, that $A \in \Gamma(X, \tau_M)$. Then $A = C \cup (E \cap M)$, where $C \cap M = \emptyset$ and $E \in \Gamma(X, \tau)$.

If $F_f(b) \in E \cap M \subseteq A$, then $b \in cl_\tau(\{F_f(b)\}) \in \tilde{A} = A$, which contradicts $b \in cl_\tau(A) - A$. Hence, $F_f(b) \notin E \cap M$.

Also note that $F_f(b) = f(i) \in M$, where $b \sim_\tau f(i)$. Thus $F_f(b) \in X - E \in \tau$. Since $F_f(b) \in cl_\tau(\{F_f(b)\}) = cl_\tau(\{b\})$, then $b \in X - E$. Hence, $A \cap (X - E) \neq \emptyset$ because $b \in cl_\tau(A) \cap (X - E)$ and $X - E \in \tau$.

Choose $a \in A \cap (X - E)$. Then, as $cl_\tau(\{a\}) = cl_\tau(\{F_f(a)\})$, we have that $F_f(a) \in X - E$ and $F_f(a) \in A$. However, $F_f(a) \in M$ and $F_f(a) \in A = C \cup (E \cap M)$ (note that $C \subseteq X - M$). So $F_f(a) \in E \cap M$, implying $F_f(a) \in E$. This contradicts $F_f(a) \in X - E$.

Hence $A \notin \Gamma(X, \tau_M)$.

In summary, for any $A \notin \Gamma(X, \tau)$, there is a T_0 topology $\mu \in \mathbf{T}(X)$ such that $\tau \subseteq \mu$ and $A \notin \Gamma(X, \mu)$.

Hence, $\lambda = \bigcap \{\mu \in \mathbf{T}(X) : \tau \subseteq \mu, \mu \text{ is } T_0\} \subseteq \tau$. At last, we have

$$\bigcap \{\mu \in \mathbf{T}(X) : \tau \subseteq \mu, \mu \text{ is } T_0\} = \tau.$$

□

Lemma 3.5. *Let (X, τ) be a T_0 space. Then there is a collection \mathcal{A} of sober topologies on X such that*

$$\tau = \bigcap \mathcal{A}.$$

Proof. It is enough to prove that $\nu = \tau$, where $\nu = \bigcap \{\mu \in T_{sob}(X) : \tau \subseteq \mu\}$.

As the discrete topology on X is sober and contains τ , the family $\{\lambda \in T_{sob}(X) : \tau \subseteq \lambda\}$ is nonempty and clearly $\tau \subseteq \nu$.

It remains to show that if $A \notin \Gamma(X, \tau)$, then there is a sober topology μ containing τ and $A \notin \Gamma(X, \mu)$.

Case 1: $A \neq \tilde{A}$.

Then there exists $b \in \bigcup \{cl_\tau(\{x\}) : x \in A\} - A$, that is, $b \in cl_\tau(\{a\}) - A$ for some $a \in A$. Let $M = \{a, b\}$ with the subspace topology $\tau_{a,b}$, which equals $\{\emptyset, \{a, b\}, \{a\}\}$. Then clearly $(M, \tau_{a,b})$ is sober (it is homeomorphic to the Sierpinski space). By Lemma 3.1, (X, τ_M) is sober. Also, similar to the conclusion in the case 2 of the proof of Lemma 3.4, $\tau \subseteq \tau_M$. If $A \in \Gamma(X, \tau_M)$, then $A = C \cup (D \cap M)$ where $C \subseteq X - M, D \in \Gamma(X, \tau)$. As $a \in M, a \notin C$, we have $a \in D$. Then $b \in cl_\tau(\{a\}) \subseteq D \subseteq A$, which contradicts the assumption that $b \notin A$.

Hence $A \notin \Gamma(X, \tau_M)$.

Case 2: $A = \tilde{A}$.

Since $A \neq cl_\tau(A)$, there exists $b \in cl_\tau(A) - A$. Let $M = A \cup \{b\}$ and λ_M be the topology on M such that $\{(C \cap M) - \tilde{B} : C \in \Gamma(X, \tau), B \subseteq A\}$ is a subbase of $\Gamma(M, \lambda_M)$.

(i) (M, λ_M) is sober.

For any $z \in M$, as (X, τ) is T_0 , $\{z\} = cl_\tau(\{z\}) - \tilde{B}_z = (cl_\tau(\{z\}) \cap M) - \tilde{B}_z$, where $B_z = cl_\tau(\{z\}) - \{z\}$. In addition $b \notin B_z$ (if $z = b$, then clearly $b \notin B_z$; if $b \neq z$, then $z \in A$, so $B_z \subseteq cl_\tau(\{z\}) \subseteq \tilde{A} = A$, then $b \notin B_z$ as $b \notin A$).

By the definition of λ_M , if $z \neq b$, then $z \in A$, $(cl_\tau(\{z\}) \cap M) - \tilde{B}_z \in \Gamma(M, \lambda_M)$, hence $\{z\}$ is closed in (M, λ_M) . For $b \in M$, $\{b\} = (cl_\tau(\{b\}) \cap M) - A = (cl_\tau(\{b\}) \cap M) - \tilde{A}$ is also closed in (M, λ_M) . It follows that (M, λ_M) is T_1 .

We now show that (M, λ_M) is sober. Let $D \in \Gamma(M, \lambda_M)$ and D contains two distinct elements a, c . If one of a and c equals b , say $a = b$, then $c \in A$ and $a \notin cl_\tau(\{c\})$. Otherwise both a and c are in A and we can assume that $a \notin cl_\tau(c)$.

Let

$$D = \bigcap_{j \in J} \left(\bigcup_{k \in K(j)} (C_k - \tilde{B}_k) \right),$$

where C_k is a closed subset in the subspace M of (X, τ) , $K(j)$ is a finite set and $B_k \subseteq A$.

Then

$$D \subseteq (D - cl_\tau(\{c\})) \cup (cl_\tau(\{c\}) \cap M).$$

Also $D - cl_\tau(\{c\}) = \bigcap_{j \in J} (\bigcup_{k \in K(j)} (C_k - (B_k \cup \{c\}))) \in \Gamma(M, \lambda_M)$ (note that $B_k \cup \{c\} \subseteq A$ as we assume $c \in A$) and $cl_\tau(\{c\}) \cap M \in \Gamma(M, \lambda_M)$. But $D \not\subseteq D - cl_\tau(\{c\})$ and $D \not\subseteq cl_\tau(\{c\}) \cap M$ (note $a \in D$ and $a \notin cl_\tau(\{c\})$). Thus D is not irreducible. So every irreducible closed set of (M, τ_M) is a singleton, hence (M, λ_M) is sober.

By Lemma 3.1, the space (X, τ_M) is also sober.

For any $F \in \Gamma(X, \tau)$, $F = (F \cap M) \cup (F - M) \in \Gamma(X, \tau_M)$, thus $\Gamma(X, \tau) \subseteq \Gamma(X, \tau_M)$.

We now show that $A \notin \Gamma(X, \tau_M)$.

For any $D \in \Gamma(X, \tau_M)$, $D = E \cup F$ for some $E \in \Gamma(M, \tau_M)$ and $F \subseteq X - M$. We show that $A \neq D$ by considering the following cases:

(i) $F \neq \emptyset$.

Then $D \not\subseteq M$, so $D \neq A$ because $A \subseteq M$.

(ii) $F = \emptyset$ and $b \in D$.

Then, as $b \notin A$, $D \neq A$.

(iii) $F = \emptyset$ and $b \notin D$.

Assume that $D = \bigcap_{j \in J} (\bigcup_{k \in K(j)} (C_k - \tilde{B}_k))$, where C_k is a closed subset in the subspace M of (X, τ) , $B_k \subseteq A$ (thus $\tilde{B}_k \subseteq A$) and $K(j)$ is a finite set.

Then, as $b \notin D$, there is a $j_0 \in J$ such that $b \notin \bigcup_{k \in K(j_0)} (C_k - \tilde{B}_k)$.

Since $b \notin \tilde{B}_k$ for each k , $b \notin \bigcup_{k \in K(j_0)} C_k$, thus $\bigcup_{k \in K(j_0)} C_k \subseteq A$.

We now show that $\bigcup_{k \in K(j_0)} C_k = A$ does not hold. If not, let $C_k = G_k \cap M$ with $G_k \in \Gamma(X, \tau)$ for each $k \in K(j_0)$. Then $X - \bigcup_{k \in K(j_0)} G_k \in \tau$ and $b \in X - \bigcup_{k \in K(j_0)} G_k$. This contradicts $b \in cl_\tau(A)$ and $A \cap (X - \bigcup_{k \in K(j_0)} G_k) \subseteq A \cap (X - \bigcup_{k \in K(j_0)} C_k) = A \cap (X - A) = \emptyset$. Therefore, $\bigcup_{k \in K(j_0)} C_k \neq A$. Then, as $D \subseteq \bigcup_{k \in K(j_0)} C_k \subsetneq A$, we have $D \neq A$.

In summary, for any $A \notin \Gamma(X, \tau)$, there is a sober topology $\mu \in T(X)$ such that $\tau \subseteq \mu$ and $A \notin \Gamma(X, \mu)$.

Hence, $\tau = \bigcap \{\mu \in T_{sob}(X) : \tau \subseteq \mu\}$, as desired. The proof is completed. \square

The combination of Lemma 3.4 and Lemma 3.5 deduces the main result in this section.

Theorem 3.6. *Every topology on a set X is the meet of some sober topologies on X .*

Thus $T_{sob}(X)$ is meet dense in $T(X)$.

The following result follows from Theorem 3.6 straightforwardly.

Corollary 3.7. *A topology $\tau \in T(X)$ is T_1 if and only if it is the meet of some T_1 sober topologies.*

Proposition 2.15 says that the join of a directed family of sober topologies on a set is also sober. It is then nature to ask whether the meet of a filtered family of sober topologies is sober. The following example shows that the answer is no, even each topology is T_2 .

To verify the counterexample below, we need to make use of the Chinese Remainder Theorem shown below. Let \mathbb{N}_0 be the set of all nonnegative integers and \mathbb{N} the set of all positive integers. For each $n, x \in \mathbb{N}_0$, let

$$[x]_n = \{y \in \mathbb{N}_0 : y \equiv x \pmod{n}\} = \{kn + x : k \in \mathbb{N}_0\}.$$

Lemma 3.8. *(Chinese Remainder Theorem) Let n_1, n_2, \dots, n_k be pairwise coprime integers with $n_i > 1$ for each i . Assume that for each $1 \leq i \leq k$, a_i is an integer such that $0 \leq a_i < n_i$. Put $N = n_1 n_2 \cdots n_k$ be the product of n_i 's. Then there is a unique integer x such that $0 \leq x < N$ and $x \equiv a_i \pmod{n_i}$ for each $1 \leq i \leq k$.*

In particular, if $\{p_1, p_2, \dots, p_m\}$ is a finite set of distinct prime numbers and for each $1 \leq i \leq m$, b_i is an integer such that $0 \leq b_i < p_i$, then

$$\bigcap \{[b_i]_{p_i} : 1 \leq i \leq m\} \neq \emptyset.$$

Example 3.9. Let $X = \mathbb{N}_0$ and $P = \{p_k : k \in \mathbb{N}\}$ be the set of all prime numbers, where we assume that $p_k < p_{k+1}$ for all k .

For each $n \in \mathbb{N}$, let τ_n be the topology on X of which $\mathcal{S}_n = \{[k]_{p_t} : n \leq t, 0 \leq k < p_t\}$ is a subbase.

Clearly $\{\tau_n : n \in \mathbb{N}\}$ is a decreasing chain of topologies on \mathbb{N}_0 . We now verify that each τ_n is T_2 and their meet is not sober.

(1) For each $n \in \mathbb{N}$, τ_n is T_2 .

As a matter of fact, for any $a, b \in X$ with $a \neq b$, there is a prime number p_m such that $n \leq m$ and $a + b < p_m$. Then $a \in U = [a]_{p_m} \in \tau_n, b \in V = [b]_{p_m} \in \tau_n$. Also $U \cap V = [a]_{p_m} \cap [b]_{p_m} = \emptyset$ (because $a \neq b$ and $a, b < p_m$).

(2) Let $\tau = \bigwedge \{\tau_n : n \in \mathbb{N}\}$. We show that X is irreducible under τ . Assume that $U_1 \in \tau, U_2 \in \tau$ are nonempty open sets in τ (we show that $U_1 \cap U_2 \neq \emptyset$).

Choose $n_1 \in U_1$ and $n_2 \in U_2$. We just consider the nontrivial case $n_1 \neq n_2$.

Note that if $p \in P$ and $[k]_p \cap [h]_p \neq \emptyset$ and $0 \leq k < p, 0 \leq h < p$, then $k = h$.

Since $U_1 \in \tau \subseteq \tau_1$ and $U_1 \neq \emptyset$, it follows that there is a finite set $F_1 \subseteq \mathbb{N}$ such that $n_1 \in \bigcap \{[h_{p_i}]_{p_i} : i \in F_1\} \subseteq U_1$, where $0 \leq h_{p_i} < p_i$ for each $i \in F_1$. By the definition of elements in P , $p'_i s (i \in F_1)$ are different primes.

Let $m = \max\{i : i \in F_1\}$ be the largest member of F_1 . Again, since $n_2 \in U_2 \in \tau \subseteq \tau_{m+1}$, there is a finite set $F_2 \subseteq \mathbb{N}$ such that $n_2 \in \bigcap \{[g_{p_j}]_{p_j} : j \in F_2\}$, where $0 \leq g_{p_j} < p_j$ and $j \geq m + 1$ for each $j \in F_2$.

Now $F_1 \cap F_2 = \emptyset$, so $\{p_j : j \in F_1 \cup F_2\}$ is a set of distinct primes. By the Chinese Remainder Theorem,

$$\bigcap \{[h_{p_i}]_{p_i} : i \in F_1\} \cap \bigcap \{[g_{p_j}]_{p_j} : j \in F_2\} \neq \emptyset.$$

As $\bigcap \{[h_{p_i}]_{p_i} : i \in F_1\} \cap \bigcap \{[g_{p_j}]_{p_j} : j \in F_2\} \subseteq U_1 \cap U_2$, we have $U_1 \cap U_2 \neq \emptyset$. It follows that X is an irreducible closed set in (X, τ) .

As the intersection of T_1 topologies on a set is always T_1 , (X, τ) is T_1 . Since X is an infinite set, X is not the closure of any singleton set. Therefore (X, τ) is not sober.

Since every T_2 topology is sober, the above example also shows that the meet of a decreasing countable chain of T_2 topologies need not be T_2 .

Remark 3.10. It is well known that the co-finite topology τ_{cof} ($U \in \tau_{cof}$ if either $U = \emptyset$ or $X - U$ is a finite set) on each non-singleton set X is the smallest T_1 topology on X and is not sober.

It is natural to wonder whether the topology τ in Example 3.9 is actually the co-finite topology on \mathbb{N}_0 . We now show that it is not.

Let $A = \{0, p_1, p_1 p_2, \dots\} = \{0\} \cup \{\prod_{i=1}^n p_i : n \in \mathbb{N}\}$. We will show that $X - A \in \tau_n$ for each $n \in \mathbb{N}$, thus $X - A \in \tau$. Hence, τ is different from the τ_{cof} .

Fix $n \in \mathbb{N}$ and given $x \in (X - A)$.

Then there is a prime number p_m such that $n < m, x < p_m$. Thus $x \in \bigcap \{[x]_{p_i} : m \leq i \leq 2m\} \in \tau_n$. And we only need to prove that $\bigcap \{[x]_{p_i} : m \leq i \leq 2m\} \subseteq (X - A)$ (i.e., $\bigcap \{[x]_{p_i} : m \leq i \leq 2m\} \cap A = \emptyset$).

Let $A_1 = \{\prod_{i=1}^n p_i : 1 \leq n \leq m\}$ and $A_2 = \{\prod_{i=1}^n p_i : m < n\}$. Then $A = A_1 \cup A_2$.

First, we will prove that $A_1 \cap \bigcap \{[x]_{p_i} : m \leq i \leq 2m\} = \emptyset$. Clearly, $x \notin A_1$ because $x \notin A$. Note that $\max(A_1) = \prod_{i=1}^m p_i$ and $\min(\bigcap \{[x]_{p_i} : m \leq i \leq 2m\} - \{x\}) = x + \prod_{i=m+1}^{2m} p_i$. Hence $\min(\bigcap \{[x]_{p_i} : m \leq i \leq 2m\} - \{x\}) > \max(A_1)$, and it implies $\bigcap \{[x]_{p_i} : m \leq i \leq 2m\} \cap A_1 = \emptyset$.

Second, we will prove that $A_2 \cap \bigcap \{[x]_{p_i} : m \leq i \leq 2m\} = \emptyset$. Due to $x \notin A$, $x \neq 0$. Because $\bigcap \{[x]_{p_i} : m \leq i \leq 2m\} \subseteq [x]_{p_m}$, $A_2 \subseteq [0]_{p_m}$ and $[x]_{p_m} \cap [0]_{p_m} = \emptyset$, we have $\bigcap \{[x]_{p_i} : m \leq i \leq 2m\} \cap A_2 = \emptyset$.

Thus $\bigcap \{[x]_{p_i} : m \leq i \leq 2m\} \cap A = \emptyset$. It follows that $(X - A) \in \tau_n$ for each $n \in \mathbb{N}$. That is A is closed in $(X, \bigwedge_{n \in \mathbb{N}} \tau_n)$.

Actually, A is also compact. Given an open cover $\{U_j : j \in J\}$ of A . Then $0 \in U_{j_0}$ for some $j_0 \in J$. Due to $U_{j_0} \in \tau_1$, there is a finite set $F \subseteq \mathbb{N}$ with $a \geq 1$ for each $a \in F$, such that $0 \in \bigcap \{[0]_{p_i} : i \in F\} \subseteq U_{j_0}$. It follows that $\{\prod_{i=1}^n p_i : n > \max(F)\} \subseteq U_{j_0}$. And for each $1 \leq n \leq \max(F)$, there is $j_n \in J$ such that $\prod_{i=1}^n p_i \in U_{j_n}$. Thus $\{U_{j_n} : 0 \leq n \leq \max(F)\}$ is a finite subcover of A .

A space (X, τ) is Alexandroff-discrete if the intersection of any collection of open sets is open. In this case, we call τ an Alexandroff - discrete topology.

It then follows easily that a meet of any collection of Alexandroff-discrete topologies on a set X is also an Alexandroff - discrete topology.

Given any poset (P, \leq) , let $\Upsilon(P, \leq)$ (or just $\Upsilon(P)$) be the set of all upper sets of P . Then $(P, \Upsilon(P, \leq))$ is a T_0 Alexandroff - discrete space. The topology $\Upsilon(P, \leq)$ is called the Alexandroff topology on P [12][14].

Remark 3.11. (1) A space (X, τ) is T_0 Alexandroff-discrete if and only if there is a partial order \leq on X such that $\tau = \Upsilon(P, \leq)$ (see Exercise 4.2.13 of [12]).

- (2) A subset F of poset P is a closed irreducible set of $(P, \Upsilon(P, \leq))$ if and only if F is a directed and lower set of (P, \leq) .
- (3) For any $x \in P$, the closure of $\{x\}$ in $(P, \Upsilon(P, \leq))$ equals $\downarrow x = \{y \in P : y \leq x\}$. Now $\downarrow x - \{x\}$ is still a lower set hence a closed set in $(P, \Upsilon(P, \leq))$. Hence $(P, \Upsilon(P, \leq))$ is a T_D space.
- (4) If an Alexandroff - discrete topology τ is sober and μ is a topology on the same set such that $\tau \subseteq \mu$, then as τ is also T_D , by Proposition 2.8(2), μ is also sober.

Proposition 3.12. *Every Alexandroff-discrete topology is the meet of some sober Alexandroff - discrete topologies.*

Proof. Let (X, τ) be an Alexandroff-discrete space. We can assume that $\tau = \Upsilon(X, \leq)$, where \leq is a partial order on X .

For each pair of elements a, b in X with $a \leq b$ and $a \neq b$, let

$$\leq_{(a,b)} = \{(x, x) : x \in X\} \cup \{(a, b)\}.$$

Then $\leq_{(a,b)}$ is a partial order on X . A subset $F \subseteq X$ is a closed irreducible set in $(X, \Upsilon(X, \leq_{(a,b)}))$ if and only if either $F = \{x\}$ for some $x \in X$ with $x \neq b$, or $F = \{a, b\}$. In the first case, $F = cl(\{x\})$, while in the second case $F = cl(\{b\})$. Thus $(X, \Upsilon(X, \leq_{(a,b)}))$ is sober.

We now show that

$$\tau = \bigwedge \{\Upsilon(X, \leq_{(a,b)}) : a \neq b, a \leq b\}.$$

Clearly, $\tau \subseteq \bigwedge \{\Upsilon(X, \leq_{(a,b)}) : a \neq b, a \leq b\}$. Now let $U \in \bigwedge \{\Upsilon(X, \leq_{(a,b)}) : a \neq b, a \leq b\}$. Let $x \leq y$ and $x \in U$. If $y = x$, then immediately, $y \in U$. If $y \neq x$, then $x \leq_{(x,y)} y$. Since $U \in \Upsilon(X, \leq_{(x,y)})$ and $x \in U$, thus $y \in U$. It follows that $U \in \Upsilon(X, \leq) = \tau$. It follows that $\tau = \bigwedge \{\Upsilon(X, \leq_{(a,b)}) : a \neq b, a \leq b\}$. The proof is completed. \square

4. Minimal sober topologies

People have studied minimal members of several classes of topologies, such as minimal Hausdorff topologies, minimal regular topologies, minimal locally compact topologies and minimal normal topologies (see [3] [8]). In this section, we study minimal sober topologies. One necessary and sufficient conditions of such topologies is obtained.

A topology τ on X is called a *minimal sober topology*, if it is sober and there is no sober topology on X strictly coarser than τ .

A topological space (X, τ) is called a *minimal sober space*, if τ is a minimal sober topology on X .

We first propose a construction method, that yields a necessary condition for minimal sober spaces.

Lemma 4.1. *Let (X, τ) be a T_0 space and $x, y \in X$ are noncomparable elements with respect to the specialization order \leq_τ (that is, $x \notin cl_\tau(\{y\})$ and $y \notin cl_\tau(\{x\})$).*

Define $\tau^ = \{U \in \tau : x \in U\} \cup \{U \in \tau : x \notin U, y \notin U\}$.*

If (X, τ) is a T_0 space, then (X, τ^) is also a T_0 space.*

Proof. It is easy to see that τ^* is a topology and $\tau^* \subseteq \tau$. The set of all closed sets of (X, τ^*) is

$$\Gamma(X, \tau^*) = \{C \in \Gamma(X, \tau) : x \notin C\} \cup \{C \in \Gamma(X), \{x, y\} \subseteq C\}.$$

Claim 1. For each $a \in X$, it holds that

$$cl_{\tau^*}(\{a\}) = \begin{cases} cl_\tau(\{a\}), & x \notin cl_\tau(\{a\}), \\ cl_\tau(\{a\}) \cup cl_\tau(\{y\}), & x \in cl_\tau(\{a\}). \end{cases}$$

As a matter of fact, if $x \notin cl_\tau(\{a\})$, then $cl_\tau(\{a\}) \in \Gamma(X, \tau^*)$ and $cl_\tau(\{a\}) \subseteq cl_{\tau^*}(\{a\})$, hence $cl_{\tau^*}(\{a\}) = cl_\tau(\{a\})$.

If $x \in cl_\tau(\{a\})$, then $a \in cl_\tau(\{a\}) \cup cl_\tau(\{y\}) \in \Gamma(X, \tau^*)$, showing that $cl_{\tau^*}(\{a\}) \subseteq cl_\tau(\{a\}) \cup cl_\tau(\{y\})$. In addition, $cl_{\tau^*}(\{a\}) \in \Gamma(X, \tau^*)$ and $x \in cl_\tau(\{a\}) \subseteq cl_{\tau^*}(\{a\})$, so $x \in cl_{\tau^*}(\{a\})$. Hence $y \in cl_{\tau^*}(\{a\})$. Therefore, $cl_\tau(\{a\}) \cup cl_\tau(\{y\}) \subseteq cl_{\tau^*}(\{a\})$. All these together show that $cl_{\tau^*}(\{a\}) = cl_\tau(\{a\}) \cup cl_\tau(\{y\})$.

Hence, $\leq_{\tau^*} = \leq_\tau \cup \{(u, v) : u \leq_\tau y \text{ and } x \leq_\tau v\}$.

Claim 2. (X, τ^*) is T_0 .

Let $a, b \in X$ with $a \neq b$. Since (X, τ) is T_0 , without loss of generality, we can assume that $a \notin cl_\tau(\{b\})$.

(i) If $x \notin cl_\tau(\{b\})$, by claim 1, $cl_\tau(\{b\}) = cl_{\tau^*}(\{b\})$. It implies that $a \notin cl_\tau(\{b\}) = cl_{\tau^*}(\{b\})$.

(ii) Now assume that $x \in cl_\tau(\{b\})$. Then by claim 1, $cl_{\tau^*}(\{b\}) = cl_\tau(\{b\}) \cup cl_\tau(\{y\})$.

If $a \notin cl_\tau(\{y\})$, then $a \notin cl_\tau(\{b\}) \cup cl_\tau(\{y\}) = cl_{\tau^*}(\{b\})$.

If $a \in cl_\tau(\{y\})$, then $x \notin cl_\tau(\{a\})$ because $x \notin cl_\tau(\{y\})$. By claim 1, $cl_{\tau^*}(\{a\}) = cl_\tau(\{a\})$.

Now as $x \in cl_\tau(\{b\})$ and $x \notin cl_\tau(\{a\})$, we have that $b \notin cl_\tau(\{a\}) = cl_{\tau^*}(\{a\})$.

All these together show that (X, τ^*) is T_0 . \square

Lemma 4.2. *Let τ be a topology on X and τ^* be the topology defined from τ and non-comparable elements $x, y \in X$ as in Lemma 4.1.*

(1) If (X, τ) is a sober space, then (X, τ^*) is a sober space.

(2) If (X, τ) is a well-filtered space, then (X, τ^*) is a well-filtered space.

(3) If (X, τ) is a d-space, then (X, τ^*) is a d-space.

Proof. First, note that for any $H \in \Gamma(X, \tau)$, $H \cup cl_\tau(\{y\}) \in \Gamma(X, \tau^*)$ (if $x \in H$, then $\{x, y\} \subseteq H \cup cl_\tau(\{y\})$; if $x \notin H$, then $x \notin H \cup cl_\tau(\{y\})$).

(1) Let (X, τ) be sober and $C \in Irr_c(X, \tau^*)$.

(i) Assume $x \notin C$.

If $C = C_1 \cup C_2$ for some $C_1, C_2 \in \Gamma(X, \tau)$, then $C_1, C_2 \in \Gamma(X, \tau^*)$ by $x \notin C$. Thus $C = C_1$ or $C = C_2$, as $C \in Irr_c(X, \tau^*)$. It follows that $C \in Irr_c(X, \tau)$. Since (X, τ) is sober, $C = cl_\tau(\{a\})$ for some $a \in X$. As $x \notin C$, so $x \notin cl_\tau(\{a\})$. By Lemma 4.1, $cl_{\tau^*}(\{a\}) = cl_\tau(\{a\})$. Hence $C = cl_{\tau^*}(\{a\})$.

(ii) Let $x \in C$. Then $cl_{\tau^*}(\{x\}) = cl_\tau(\{x\}) \cup cl_\tau(\{y\})$, which is contained in C as $C \in \Gamma(X, \tau^*)$. Hence $cl_\tau(\{y\}) \subseteq C$.

Let $\mathcal{A} = \{F : F \in \Gamma(X, \tau), C = F \cup cl_\tau(\{y\})\}$. By the above deduced fact, $C = C \cup cl_\tau(\{y\})$, hence $C \in \mathcal{A}$.

It follows that $C = (\bigcap_{F \in \mathcal{A}} F) \cup cl_\tau(\{y\})$.

Let $\hat{F} = \bigcap_{F \in \mathcal{A}} F$. Then $C = \hat{F} \cup cl_\tau(\{y\})$.

Assume $\hat{F} = D \cup E$ for some $D, E \in \Gamma(X, \tau)$.

Then $C = (D \cup cl_\tau(\{y\})) \cup (E \cup cl_\tau(\{y\}))$. Since C is irreducible in (X, τ^*) , hence $C = D \cup cl_\tau(\{y\})$ or $C = E \cup cl_\tau(\{y\})$, which further deduces that $\hat{F} = D$ or $\hat{F} = E$ (note that if $C = D \cup cl_\tau(\{y\})$, for example, then $D \in \mathcal{A}$, thus $D \subseteq \hat{F} \subseteq D$, implying $D = \hat{F}$).

That is $\hat{F} \in Irr_c(X, \tau)$. As (X, τ) is sober, $\hat{F} = cl_\tau(\{a\})$ for some $a \in X$. By $x \in C = cl_\tau(\{a\}) \cup cl_\tau(\{y\})$ and $x \notin cl_\tau(\{y\})$, we have $x \in cl_\tau(\{a\})$. Thus

$$C = \hat{F} \cup cl_\tau(\{y\}) = cl_\tau(\{a\}) \cup cl_\tau(\{y\}) = cl_{\tau^*}(\{a\}).$$

Since (X, τ^*) is T_0 , it is sober.

(2) Let (X, τ) be well-filtered. Assume that K is a compact saturated set of (X, τ^*) and $\{U_i \in \tau : i \in I\} \subseteq \tau$ be an open cover of K .

(i) Assume $x \notin K$.

Note that $cl_{\tau^*}(\{x\}) = cl_{\tau}(\{x\}) \cup cl_{\tau}(\{y\})$. As K is the intersection of all open sets $V \in \tau^*$ containing K , one has that $cl_{\tau^*}(\{x\}) \cap K = \emptyset$. Now $K = K - cl_{\tau^*}(\{x\}) \subseteq \bigcup \{U_i - cl_{\tau^*}(\{x\}) : i \in I\} = \bigcup \{U_i - (cl_{\tau}(\{x\}) \cup cl_{\tau}(\{y\})) : i \in I\}$. Note that each $U_i - (cl_{\tau}(\{x\}) \cup cl_{\tau}(\{y\})) \in \tau^*$. As K is compact in (X, τ^*) , $K \subseteq \bigcup \{U_i - (cl_{\tau}(\{x\}) \cup cl_{\tau}(\{y\})) : i \in F\} \subseteq \bigcup \{U_i : i \in F\}$ for some finite $F \subseteq I$.

(ii) Assume $x \in K$.

Then $x \in U_{i_0}$ for some $i_0 \in I$ and $K \subseteq \bigcup \{U_{i_0} \cup U_i : i \in I\}$. Because $U_{i_0} \cup U_i \in \tau^*$ for each $i \in I$, $K \subseteq \{U_{i_0} \cup U_i : i \in F\}$ for some finite $F \subseteq I$.

Thus K is also a compact set of (X, τ) . Furthermore, K is saturated in (X, τ) by $\tau^* \subseteq \tau$.

Then by the definition of well-filtered spaces and that $\tau^* \subseteq \tau$, (X, τ^*) is also well filtered.

(3) Assume that (X, τ) is a d-space.

We show that (X, τ^*) is also a d-space.

Let $U \in \tau^*$ and D be a directed set of (X, \leq_{τ^*}) such that $\bigcap \{\uparrow_{\tau^*} d : d \in D\} \subseteq U$. As $\tau^* \subseteq \tau$, D is also a directed subset of (X, \leq_{τ}) .

(i) Assume $y \in \uparrow_{\tau^*} d$ for each $d \in D$.

Now $\uparrow_{\tau^*} d = \{a \in X : d \in cl_{\tau^*}(\{a\})\} = \{a \in X : d \in cl_{\tau^*}(\{a\}), x \notin cl_{\tau}(\{a\})\} \cup \{a \in X : d \in cl_{\tau^*}(\{a\}), x \in cl_{\tau}(\{a\})\}$. From this, we can deduce easily that $\uparrow_{\tau^*} d = \uparrow_{\tau} d \cup \uparrow_{\tau} x$.

It follows that $\bigcap \{\uparrow_{\tau^*} d : d \in D\} = \bigcap \{\uparrow_{\tau} d \cup \uparrow_{\tau} x : d \in D\} = \bigcap \{\uparrow_{\tau} d : d \in D\} \cup \uparrow_{\tau} x \subseteq U$. Thus $\bigcap \{\uparrow_{\tau} d : d \in D\}$, D is a directed set in (X, \leq_{τ}) and $U \subset \tau$. As (X, τ) is a d-space, $\uparrow_{\tau} d_0 \subseteq U$ holds for some $d_0 \in D$. Then $\uparrow_{\tau^*} d_0 = \uparrow_{\tau} d_0 \cup \uparrow_{\tau} x \subseteq U$.

(ii) Now assume $y \notin \uparrow_{\tau^*} d_0$ for some $d_0 \in D$.

Then $y \notin \uparrow_{\tau^*} d$ for all $d \in D$ with $d_0 \leq_{\tau^*} d$, that is $d \notin cl_{\tau^*}(\{y\}) = cl_{\tau}(\{y\})$. For each such d , $d \leq_{\tau^*} a$ if and only if $d \in cl_{\tau^*}(\{a\})$, if and only if $d \in cl_{\tau}(\{a\})$, if and only if $a \in \uparrow_{\tau} d$.

Hence $\bigcap \{\uparrow_{\tau^*} d : d \in D\} = \bigcap \{\uparrow_{\tau^*} d : d \in D, d_0 \leq_{\tau^*} d\} = \bigcap \{\uparrow_{\tau} d : d \in D, d_0 \leq_{\tau} d\} \subseteq U$. So, as (X, τ) is a d-space, there is $d \in D$ such that $d_0 \leq_{\tau^*} d$ and $\uparrow_{\tau^*} d = \uparrow_{\tau} d \subseteq U$.

All these together show that (X, τ^*) is a d-space using the characterization given in [19]. \square

As $cl_{\tau^*}(\{x\}) = cl_{\tau}(\{x\}) \cup cl_{\tau}(\{y\}) \neq cl_{\tau}(\{x\})$, the topology τ^* is strictly coarser than τ . Hence we have the following conclusions.

Corollary 4.3. *If (X, τ) is a minimal sober space (well-filtered space, d-space, respectively), then (X, \leq_{τ}) is a chain (for any $x, y \in X$, it holds that either $x \leq_{\tau} y$ or $y \leq_{\tau} x$).*

Recall that the upper topology $\nu(P)$ on a poset P is the topology of which $\{P - \downarrow x : x \in P\}$ is a subbase [11]. The following result should have been proved by other people already. For reader's convenience, we give a brief proof.

Lemma 4.4. *For any chain C , $\nu(C) = \sigma(C)$.*

Proof. Clearly, $\nu(C) \subseteq \sigma(C)$.

Let F be a proper closed set of $(C, \sigma(C))$ and A be the set of upper bound of F . Then $A \neq \emptyset$.

Since F is a lower set of a chain, for any $c \in C$, either $c \in F$ or $c \in A$.

For any $y \in \bigcap_{x \in A} \downarrow x$, if $y \notin F$, then $y \in A$ and y is the smallest element in A , thus $y = \sup F$. But F is Scott closed, and F is a directed set (every chain is a directed subset), thus $\sup F \in F$, so $y \in F$, a contradiction. So $y \in F$ must hold. It follows that $F = \bigcap_{x \in A} \downarrow x \in \Gamma(C, \nu(C))$. Therefore, $\nu(C) = \sigma(C)$. \square

One characterization of minimal T_0 spaces was given in [17, Theorem 1].

Proposition 4.5. *Let (X, τ) be a T_0 space, $\mathcal{A} = \{cl(\{x\}) : x \in X\}$ and \leq_τ be the specialization order of (X, τ) . Then the following conditions are equivalence.*

- (1) (X, τ) is a minimal T_0 space.
- (2) $\{X - A : A \in \mathcal{A}\}$ is a base of τ and $cl(F) \in \mathcal{A}$ for each $F \subseteq_{fin} X$.
- (3) (X, \leq_τ) is a chain and $\tau = \nu(X, \leq_\tau)$ where $\nu(X, \leq_\tau)$ is the upper topology on (X, \leq_τ) .

A poset P is sup-complete if for every nonempty subset $B \subseteq P$, $\bigvee B$ exists. Thus a chain P is sup-complete if and only if it is a depo.

Lemma 4.6. *Let (X, τ) be a T_0 space. If (X, \leq_τ) is a sup-complete chain and $\tau = \sigma(X, \leq_\tau)$, then (X, τ) is a minimal sober space (resp., well-filtered space, d-space).*

Proof. It is well-known that every sup-complete chain is a domain and the Scott space of each domain is sober [11][12]. Hence (X, τ) is sober.

By Lemma 4.4 and Proposition 4.5, (X, τ) is a minimal T_0 space. Thus there is no T_0 topology strictly coarser than τ . Hence (X, τ) is a minimal sober space. Recall that every sober space is well-filtered, every well-filtered space is a d-space and every d-space is T_0 [11][12], it follows that (X, τ) is also a minimal well-filtered and minimal d-space. \square

Now we have the main result of this section.

Theorem 4.7. *Let (X, τ) be a T_0 space. Then the following statements are equivalent.*

- (1) (X, τ) is a minimal sober space.
- (2) (X, τ) is a minimal well-filtered space.
- (3) (X, τ) is a minimal d-space.
- (4) (X, \leq_τ) is a sup-complete chain and $\tau = \sigma(X, \leq_\tau)$.

Proof. By Lemma 4.6, (4) \Rightarrow (1), (2), (3). And (1), (2), (3) \Rightarrow (4), by Lemmas 4.2, 4.4 and the fact that for any d-space (sober space, well-filtered space, respectively) (X, τ) , (X, \leq_τ) is a depo (thus (X, \leq_τ) is sup-complete if it is a chain). \square

It is natural to wonder whether for any sober topology τ on a set X , there is a minimal sober topology $\mu \subseteq \tau$. The answer is no, as shown by the following example.

Example 4.8. Let $X = \mathbb{R} \cup \{\top\}$ and $\tau = \{\emptyset\} \cup \{X - F : F \subseteq_{fin} \mathbb{R}\}$.

Obviously, (X, τ) is sober. Assume τ^* is a minimal sober topology with $\tau^* \subseteq \tau$. If $|cl_{\tau^*}(\{r\})|$ is infinite for some $r \in \mathbb{R}$, then $cl_{\tau^*}(\{r\}) = X$ by $\tau^* \subseteq \tau$. It contradicts $X = cl_{\tau}(\{\top\}) \subseteq cl_{\tau^*}(\{\top\})$ and $cl_{\tau^*}(\{r\}) \neq cl_{\tau^*}(\{\top\})$ (because τ^* is T_0). Hence $|cl_{\tau^*}(\{r\})|$ is a finite number for each $r \in \mathbb{R}$.

By Theorem 4.7, (X, \leq_{τ^*}) is a chain. Hence $cl_{\tau^*}(\{r\}) = \downarrow r = \{x \in X : x \leq_{\tau^*} r\}$.

It then follows that $|cl_{\tau^*}(\{r_1\})| = |cl_{\tau^*}(\{r_2\})|$ if and only if $r_1 = r_2$. Thus there is an injective map $f : \mathbb{R} \rightarrow \mathbb{N}$ such that $f(r) = |cl_{\tau^*}(r)|$. But it is impossible. Hence there is no minimal sober topology coarser than τ .

At the end of this section, we prove that the soberification of a minimal T_0 space is a minimal sober space.

For any T_0 space (X, τ) , the set $Irr_c(X, \tau)$ with the lower Vietoris topology $\tau^* = \{\diamond U : U \in \tau\}$ is a sober space called the soberification of (X, τ) . Here $\diamond U = \{C \in Irr_c(X, \tau) : C \cap U \neq \emptyset\}$. In addition, the specialization order \leq_{τ^*} on $Irr_c(X, \tau)$ coincides with the inclusion order \subseteq (see [11][12]).

Proposition 4.9. *If (X, τ) is a minimal T_0 space, then the soberification $(Irr_c(X, \tau), \tau^*)$ of (X, τ) is a minimal sober space.*

Proof. By Proposition 4.5, (X, \leq_{τ}) is a chain. Also every closed set of (X, τ) is a lower set of (X, \leq_{τ}) . In particular, every member of $Irr_c(X, \tau)$ is a lower set. As (X, \leq_{τ}) is a chain, every two lower sets are comparable, it follows that $(Irr_c(X, \tau), \leq_{\tau^*})$ is a chain. Also $(Irr_c(X, \tau), \leq_{\tau^*})$ is a dcpo, thus sup-complete.

Since τ^* is sober, we have

$$\nu(Irr_c(X, \tau), \leq_{\tau^*}) \subseteq \tau^* \subseteq \sigma(Irr_c(X, \tau), \leq_{\tau^*}).$$

Then $\tau^* = \sigma(Irr_c(X, \tau), \leq_{\tau^*})$, by Lemma 4.4.

By Theorem 4.7, $(Irr_c(X, \tau), \tau^*)$ is a minimal sober space. □

5. Summary and further work

In this paper we study the sober topologies in the lattice of all topologies on a set. The main results are (1) every T_1 topology is the join of some sober topologies; (2) every topology is the meet of some sober topologies; (3) the set of all sober topologies is directed complete; (4) the minimal sober spaces are precisely the sup-complete chains equipped with the Scott topology.

There are still some problems on sober topologies deserve to be considered.

There are T_0 topologies which are not the join of sober topologies. Thus we have the following problems:

- (1) Which T_0 topologies are the joins of sober topologies?
- (2) Which T_0 topologies are the joins of finite number of sober topologies?

In [24], Steiner proved that the lattice of topologies on each set is complemented. Thus we have the following problem.

- (3) Which topologies have a sober complement?

Given a set X with cardinality κ , people need to study the cardinality of all topologies on X of certain types (such as T_1 , Hausdorff, etc.)

On sobriety, we have the following problem.

(4) What is the cardinality of all sober topologies on a set X with $|X| = \kappa$?

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