# Non-stationary wormholes with the presence of scalar fields and modified gravity

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Some novel regular spacetimes are considered that show a non-stationary wormhole structure. A Simpson-Visser-like procedure is applied to reconstruct these regular spacetimes, free of time-like and space-like singularities. Such a procedure is also applied to describe a regular cosmological expansion, where the universe reaches a minimum scale and then rebounds. This type of regular spacetime is achieved by considering some scalar fields as sources, with the appropriate kinetic term and scalar potential. We show that all these sources turn out to be ghosts due to the wrong sign of the kinetic term. These ghosts can, however, be eliminated by constraints. Nevertheless, the same procedure is also explored in the framework of modified gravities, particularly within the so-called $f(R)$ gravity, where a new wormhole spacetime is also obtained that does not require a ghost scalar field.

## I. INTRODUCTION

Over the last few years, General Relativity (GR) and Cosmology have gained significant attention. This is due to the present era of high-precision measurements of black holes and cosmological effects, which allowed the direct measurement of gravitational waves by LIGO, VIRGO and KAGRA, the image of supermassive black holes by the Event Horizon Telescope (EHT), precise of the cosmological parameters and recent works that investigate detection methods for wormholes in both general relativity and alternative theories of gravity [1–5]. In GR (and other theories of gravity), wormholes are hypothetical solutions that describe regular spacetimes, featuring a tunnel-like structure that connects different regions/universes. Most of these types of spacetime known in the literature are static spherically symmetric solutions of Einstein's field equations with the presence of additional fields that generally do not satisfy the energy conditions [6, 7]. Wormholes function as shortcuts that connect two different regions of the spacetime or even different spacetimes, since, unlike black holes, wormholes do not have event horizons.

The first wormhole-like solution was the so-called Schwarzschild wormhole, also known as Einstein-Rosen bridge [8], which is based on the Schwarzschild metric, where the maximal extension of such a spacetime leads to two separate regions/universes that might be connected through a tunnel. However, these wormholes are highly unstable. Small disturbances in spacetime would cause the structure to collapse instantly, preventing any safe passage [9, 10]. Another type of wormhole is the Morris-Thorne wormhole [11], which, unlike the Schwarzschild wormhole, is traversable. However, for this wormhole to remain open and stable, it would require the presence of exotic matter, a type of theoretical matter with negative energy density. These theoretical objects have been extensively studied in the literature [12–14]. Recent studies have focused largely on the metrics of black bounces [15–17, 21], which are spacetimes that might transit from regular black holes into traversable wormholes, depending on a free parameter that characterises the behaviour of the spacetime. One of the procedures for reconstructing black bounce solutions was proposed by the Simpson-Visser [15] and then implemented in a plethora of different spacetimes [16–21]. These interesting objects have gained attention as their properties and dynamics have been explored in depth, also in some extensions of GR [22–24].

One of the most studied exact solutions for these structures is the Ellis-Bronnikov wormhole [25, 26]. Unlike black holes, the Ellis-Bronnikov wormhole does not have a central singularity, which makes it an interesting, regular, and theoretical solution for the study of hypothetical non-stationary objects. Such a spacetime is a specific solution of Einstein's field equations in the presence of an exotic type of matter, often called phantom matter with negative energy density, which violates the zero energy condition and enables the stability of the structure. This type of exotic matter can be described

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by a free scalar field with negative kinetic energy. Nevertheless, other wormhole solutions can be reconstructed with several scalar fields that might avoid the presence of ghosts and, consequently, no instabilities arise [27].

In this work, a new non-stationary solution is found based on the Ellis-Bronnikov wormhole by exploring the types of sources responsible for this solution. To do so, firstly, a time regularisation is considered for Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime in order to reconstruct cosmologies free of time-like singularities, where a Simpson-Visser-like procedure is applied [28], leading to bouncing cosmological models [29–31]. Then, in order to construct a FLRW cosmology free of singularities, a particular scalar field model, called cuscuton, is considered [32]. The cuscuton field has been widely analysed previously in the literature, since it lacks any dynamical degree of freedom, and it might reproduce not only the late-time accelerating expansion of the universe [33], but also the inflationary phase [34, 35]. Here, the presence of the cuscuton field transforms singular cosmologies into regular ones. The procedure is applied to construct a non-stationary Ellis-Bronnikov wormhole that is consequently free of any type of singularities, and its throat expands/contracts with time, reaching a minimum size of the throat and then bouncing. Nevertheless, we show that in order to obtain such regular solutions, the scalar field, including the cuscuton field, has a negative kinetic term, turning into a ghost field and consequently, the model might not be physically acceptable. However, this reconstruction procedure might be applied to any other type of spacetime to make it regular. Furthermore, we show that the ghosts can be eliminated by constraints. Finally, such analysis is extended to $f(R)$ gravities [30]. To do so, the same regular solutions are attempted to be held in $f(R)$ gravities. Cosmological bouncing solutions are well known in modified gravities, where a regular cosmological expansion can be easily reconstructed [29]. Moreover, while some wormhole solutions are known to exist in the framework of $f(R)$ gravity (for a recent review, see [36]), we have revisited the Ellis-Bronnikov wormhole in the framework of $f(R)$ gravity [37] and found that the solution is not satisfied when considering the same scalar field as in GR, but other types of sources are necessary. In addition, a new wormhole spacetime is obtained in the framework of $f(R)$ gravity, which has similar properties to the Ellis-Bronnikov wormhole, but does not require the presence of a ghost field.

This paper is organised as follows: in Section II, a brief review of FLRW cosmologies is followed, and the cuscuton field is introduced as a source for the field equations. In Section III, the time regularisation procedure is followed to reconstruct FLRW regular solutions with the presence of the cuscuton field. Then, Section IV reviews the Ellis-Bronnikov wormhole, whereas Section V introduces a new non-stationary and regular wormhole solution. We use a scalar field besides the cuscuton scalar field to construct the above solutions, but we find that these scalar fields are ghosts. In Section VI, we show that the ghosts can be eliminated by constraints. In Section VII, these regular solutions are analysed in the framework of $f(R)$ gravity and a new solution is found. Finally, Section VIII summarises the main results of the paper.

## II.   FLRW COSMOLOGIES

The Friedmann-Lemaître-Robertson-Walker (FLRW) model describes an isotropic and homogeneous expanding universe that characterises the geometry of the universe. In comoving coordinates, the metric can be written as follows

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right], \qquad (1)$$

where $a(t)$ is the scale factor that describes how the universe expands over time and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$, which expresses the metric of a two-dimensional sphere with a unit radius. The $K$ defines the spatial curvature, which can be normalised to be $+1$, $-1$, or $0$ corresponding to a closed, open, or flat universe, respectively. The dynamics of this model is described by the Friedmann equations, which can be obtained directly from Einstein's field equations, leading to

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \qquad (2)$$

$$\frac{\ddot{a}}{a} = \frac{\kappa^2}{6}(\rho + 3p) + \frac{\Lambda}{3}, \qquad (3)$$

where $\kappa^2 = 8\pi G$, $\rho$ is the energy density, $p$ is the pressure and $\Lambda$ is a cosmological constant.

In the context of the FLRW model, these quantities play an important role in describing the dynamics and evolution of the universe. The energy density and pressure can come from different species present in the universe: dust matter (both ordinary and dark matter), where $\rho_{\mathrm{m}}$ is the energy density and $p_{\mathrm{m}} \approx 0$, radiation, which has energy density $\rho_{\mathrm{r}}$ and $p_{\mathrm{r}} = \frac{1}{3}\rho_{\mathrm{r}}$, and dark energy (often modeled as a cosmological constant $\Lambda$), which has energy density $p_{\mathrm{DE}} = \omega_{\mathrm{DE}}\rho_{\mathrm{DE}}$ with $\omega_{\mathrm{DE}} < -1/3$.

The causal structure of the FLRW model is an important aspect of understanding how different regions of the universe can influence each other over time and is strongly influenced by the evolution of the scale factor $a(t)$. This scale factor governs the expansion of the universe and defines the limits of the cosmological horizons, both the future and the past light cones, such that they delineate the regions of the universe that are "observable" from a given point in spacetime and consequently regions that are causally connected in the past and the future. Nevertheless, this causal structure generally has a singularity at the beginning of the universe, where we choose $t = 0$. For the standard model of cosmology, solutions to the equations (2) and (3) make the density $\rho$

diverge as $t \to 0$, leading to the Big Bang singularity, where the scale factor $a(t)$ turns out to be zero, and the temperature becomes infinite. This is the case when the universe is filled with any kind of matter that satisfies the energy conditions. For instance, in the presence of dust matter or radiation, the FLRW equation (2) can be easily solved to lead to:

$$\text{For dust:} \, a(t) \sim t^{2/3} \, ,$$
$$\text{For radiation:} \, a(t) \sim t^{1/2} \, . \tag{4}$$

Such a singularity represents a significant challenge for physics, as the extreme conditions of density and temperature demand a theory that unifies general relativity with quantum mechanics. General relativity, which is excellent for describing objects on large scales, such as stars and galaxies, fails to characterise phenomena on the quantum scale, such as those that would occur near singularities. This is because the curvature of spacetime becomes infinite, and Einstein's equations are no longer valid or cannot be solved accurately. Various approaches have been proposed to address or mitigate the singularity at $t = 0$, such as modified theories of gravity or classical approaches such as regularisations.

In order to find a source that regularises the FLRW spacetime, a cuscuton-type scalar field, $\phi_1$, is considered. The action for this scalar field is given by [33]

$$S_{\phi_1} = \int d^4x \sqrt{-g} \left[ \mu \sqrt{2X} - U(\phi_1) \right] \, , \tag{5}$$

where $\mu$ is a constant that might be either positive or negative, whereas,

$$X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi_1 \partial_\nu \phi_1 \, . \tag{6}$$

Here we assume $X > 0$, that is, the vector $(\partial_\mu \phi_1)$ is time-like. The equation of motion can be obtained by varying the action (5) with respect to the scalar field $\phi$, where one obtains

$$\frac{1}{\sqrt{-g}} \partial_\gamma \left[ \frac{\sqrt{-g} \partial^\gamma \phi_1}{\sqrt{-g^{\alpha\beta} \partial_\alpha \phi_1 \partial_\beta \phi_1}} \right] - \frac{1}{\mu} \frac{dU}{d\phi_1} = 0 \, . \tag{7}$$

In addition, the variation of the action $S_{\phi_1}$ over the metric $g_{\mu\nu}$ provides the energy-momentum tensor for the cuscuton field:

$$T^\mu_{\ \nu} = \frac{\mu g^{\mu\gamma} \partial_\gamma \phi_1 \partial_\nu \phi_1}{\sqrt{-g^{\alpha\beta} \partial_\alpha \phi_1 \partial_\beta \phi_1}}$$
$$+ \delta^\mu_\nu \left[ \mu \sqrt{-g^{\gamma\alpha} \partial_\gamma \phi_1 \partial_\alpha \phi_1} - U(\phi_1) \right] \, . \tag{8}$$

Then, the components of the energy-momentum tensor are

$$T^t_{\ t} = -U(\phi_1) \, , \tag{9}$$
$$T^r_{\ r} = T^\theta_{\ \theta} = T^\varphi_{\ \varphi} = \mu \left| \dot{\phi}_1 \right| - U(\phi_1) \, . \tag{10}$$

While the field equation (7) for the scalar field reduces to:

$$\frac{dU}{d\phi_1} = -3\mu H \frac{\dot{\phi}_1}{\sqrt{\dot{\phi}_1^2}} \, . \tag{11}$$

One should note that the term $\dot{\phi}_1/\sqrt{\dot{\phi}_1^2}$ in the r.h.s. of the equation provides a sign. Thus, with no presence of any other type of matter and a null cosmological constant, the FLRW equations are

$$3H^2 = U(\phi_1) \, , \tag{12}$$
$$-2\dot{H} - 3H^2 = \mu \left| \dot{\phi}_1 \right| - U(\phi_1) \, , \tag{13}$$

and thus,

$$\left| \dot{\phi}_1 \right| = -\frac{2}{\mu} \dot{H} = -\frac{2}{\mu} \left( \frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a(t)^2} \right) \, . \tag{14}$$

By (12), the scalar field equation (11) can be expressed in the form:

$$\left( \frac{dU}{d\phi_1} \right)^2 - 3\mu^2 U(\phi_1) = 0 \, . \tag{15}$$

Here, one of the main features of the cuscuton arises. Since the action for the cuscuton field is linear in the velocities, the scalar field equation (11) does not provide any information on the dynamics of the scalar field, and consequently, by an arbitrary potential $U(\phi_1)$, the only possible solutions are provided by a constant scalar field $\phi_1 = \phi_0$ that are the roots of the equation (15), and it plays the role of an effective cosmological constant. Nevertheless, that is not the case if one considers (15) as a differential equation for the potential with respect to the cuscuton field. In such a case, by solving the equation (15), the following form for the potential is obtained

$$U(\phi_1) = \frac{3}{4} \mu^2 (\phi_1 - \phi_0)^2 \, , \tag{16}$$

where $\phi_0$ is an integration constant. It is straightforward to show that for this potential, the system of differential equations (12)-(13) is undetermined and the model does not provide any consistent and unique solution. However, as we will show in the section below, the cuscuton field in the presence of other types of matter might provide a way to regularise the cosmological expansion.

## III. TIME REGULARIZATION OF THE FLRW COSMOLOGIES

As we mentioned in Section II, the FLRW metric in isotropic coordinates (1) has a temporal singularity when $a(t) \to 0$. This singularity is known as the Big Bang (or Big Crunch, depending on the evolution of

the universe). In the Big Bang model, this singularity occurs at $t = 0$, marking the beginning of physical time. At $t = 0$, the spatial distances between any two points collapse to zero, since all spatial distances are multiplied by $a(t)$. The density $\rho$ also diverges, as we can see in (2). Furthermore, the curvature of spacetime diverges, since the curvature scalars tend to infinity. For example, the Ricci scalar is given by

$$R = \frac{6\left(a(t)\ddot{a}(t) + \dot{a}(t)^2\right)}{a(t)^2}, \tag{17}$$

and as we can see, as $a \to 0$ it tends to infinity. The singularity at $t = 0$ suggests that General Relativity is not sufficient to describe this region of spacetime, indicating the need for a quantum theory of gravity or an alternative method.

To obtain nonsingular cosmological solutions, the dependence of the scale factor on time is forced to undergo a Simpson-Visser-like regularisation:

$$\bar{t}^2 = t^2 + b^2 \rightarrow \bar{t} = \pm\sqrt{t^2 + b^2}. \tag{18}$$

So the line element for this case is

$$ds^2 = -dt^2 + a^2\left(\sqrt{t^2 + b^2}\right)\left[dr^2 + r^2 d\Omega^2\right], \tag{19}$$

where $b \neq 0$ is a constant, such that the scale factor reaches a minimum $a(t = 0) = a_{\min} \neq 0$ and the Big Bang singularity is avoided while the universe undergoes a bounce. Let us consider a model where the cuscuton field is part of the universe. The Friedmann equations now are

$$3H^2 = \kappa^2 \rho + U\left(\phi_1\right),$$
$$-2\dot{H} = \kappa^2\left(\rho + p\right) + \mu\left|\dot{\phi}_1\right|. \tag{20}$$

In addition, the continuity equation for the fluid reads

$$\dot{\rho} + 3H(1 + w)\rho = 0, \tag{21}$$

where the equation of state is $p = w\rho$. Then, the perfect fluid in terms of the scale factor evolves as

$$\rho = \rho_0 \left(\frac{1}{a}\right)^{3(1+w)}. \tag{22}$$

Hence, for a given cosmological expansion $H(t)$, the corresponding potential and scalar field evolution is obtained by the FLRW equations (20)

$$U\left(\phi_1\right) = 3H^2 - \kappa^2\rho_0 a^{-3(1+w)},$$
$$\mu\left|\dot{\phi}_1\right| = -2\dot{H} - \kappa^2\rho_0(1+w)a^{-3(1+w)}. \tag{23}$$

Then, one might try to find whether a particular cosmological solution that owns a Big Bang singularity might be regularised in the presence of the cuscuton field. One should note that according to the equation (23), the first

derivative $\dot{\phi}_1$ must be a monotonically increasing or decreasing function in order to keep the equations consistent. Let us consider the case of dust matter $w_{\mathrm{m}} = 0$. In the absence of the cuscuton field, the scale factor goes as (4), which clearly contains a singularity at $t = 0$. Nevertheless, one might consider the following bouncing solution:

$$a(t) = a_0 \left(t^2 + b^2\right)^{\frac{1}{3}},$$
$$\rightarrow \quad H(t) = \frac{2t}{3\left(t^2 + b^2\right)}. \tag{24}$$

For $t \gg b$, the solution is basically the same as the one for a matter-dominated universe. However, for $t \sim 0$, the universe reaches a bounce and the Big Bang singularity is avoided, as shown in Fig. 1, where the cosmological evolution experiences an accelerated phase around $t = 0$ and then the matter-dominated universe is recovered. For this simple model, the equations (23) can be solved and the scalar field and its potential lead to

$$\mu\left(\phi_1(t) - \phi_0\right) = -\frac{4}{3}\left[\frac{t}{t^2 + b^2} + \frac{\Omega_{\mathrm{m}}\arctan\left(\frac{t}{b}\right)}{b}\right],$$
$$U\left(\phi_1\right) = \frac{4t^2 - \Omega_{\mathrm{m}}\left(t^2 + b^2\right)}{3\left(t^2 + b^2\right)^2}, \tag{25}$$

where $\Omega_{\mathrm{m}} = \frac{\rho_0}{\frac{3}{\kappa^2}H_0^2}$ and $\phi_0$ is a constant of integration. In Fig. 2, the potential for the cuscuton field is depicted. As shown, the smaller the values of the parameter $b$ are, the deeper the potential arises. This is natural, as $b$ is the central parameter that regularises the solution, and as goes to zero, the scalar field in (25) diverges and the singularity at $t = 0$ is recovered. Moreover, in order to check the consistency of the second equation in (23), the behaviour of the scalar field is depicted in Fig. 3. As shown, one has to consider $\mu < 0$ in order to keep $\mu\left|\dot{\phi}_1\right| < 0$ around the bounce, located at $t = 0$, whereas away from the bounce $\mu\left|\dot{\phi}_1\right| > 0$. Since this is not possible as $\mu$ is a constant, this might be interpreted as the cuscuton field playing a central role just around the bounce at $t = 0$, where the solution is regularised. While away from $t = 0$, the cuscuton field is not allowed and the matter-dominated universe is recovered. In addition, $\mu < 0$ implies that the cuscuton field owns a negative kinetic energy and becomes a ghost, violating the energy conditions. The elimination of the ghost is discussed in Section VI.

In the sections below, the same technique is applied to construct a non-stationary solution for the Ellis-Bronnikov wormhole.

## IV. ELLIS-BRONNIKOV WORMHOLE

As previously mentioned, the Ellis-Bronnikov wormhole is a solution to Einstein's field equations in the
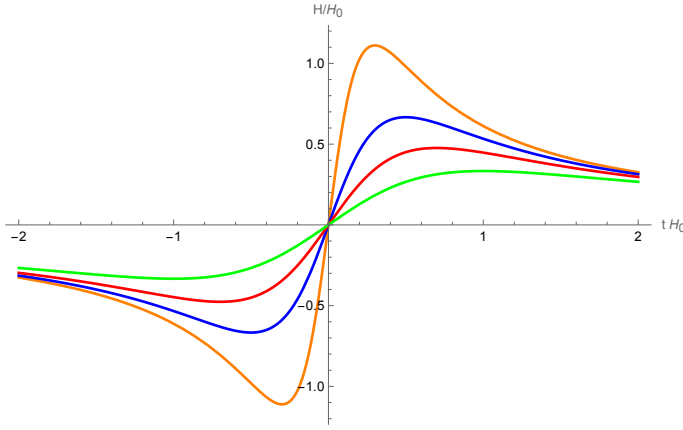
FIG. 1: Hubble parameter for $b = 0.3H_0$ (orange), $b = 0.5H_0$ (blue), $b = 0.7H_0$ (red) and $b = 1.0H_0$ (green), and $\Omega_{\rm m} = 0.3$.
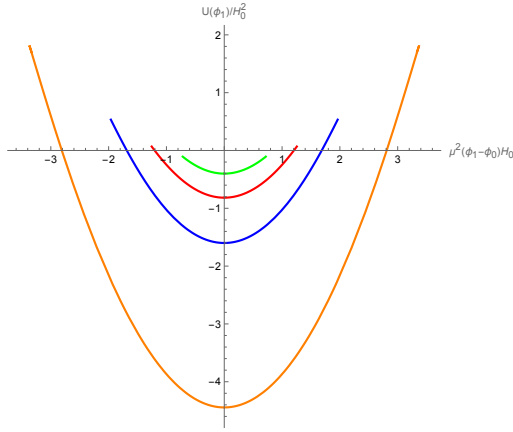


FIG. 2: Potential of the cuscuton field $U(\phi_1)$ for $b = 0.3H_0$ (orange), $b = 0.5H_0$ (blue), $b = 0.7H_0$ (red) and $b = 1.0H_0$ (green), and $\Omega_{\rm m} = 0.3$.
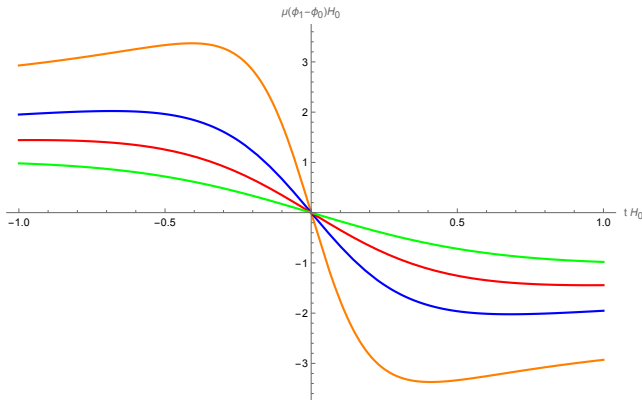


FIG. 3: Evolution of the cuscuton field $\phi_1$ for $b = 0.3H_0$ (orange), $b = 0.5H_0$ (blue), $b = 0.7H_0$ (red) and $b = 1.0H_0$ (green), and $\Omega_{\rm m} = 0.3$.

presence of a massless scalar field with negative kinetic energy acting as the source of curvature. The spheri-

cally symmetric line element is typically expressed as

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \dfrac{h(r)}{r}} + r^2 d\Omega^2 \,, \qquad (26)$$

where $h(r)$ is the shape function that governs the behaviour of the wormhole. For the Ellis-Bronnikov wormhole, the shape function takes the particular form:

$$h(r) = \frac{b^2}{r} \,, \qquad (27)$$

where $b$ is a constant. The metric (26) might be expressed more naturally through the coordinate transformation $r^2 = b^2 + \bar{r}^2$, such that the line element (26) becomes

$$ds^2 = -dt^2 + d\bar{r}^2 + \Sigma^2 d\Omega^2 \,, \quad \Sigma(\bar{r}) = \sqrt{\bar{r}^2 + b^2} \,. \quad (28)$$

Here, we can see clearly that the metric is regular everywhere, and the radial coordinate $-\infty < \bar{r} < \infty$ covers two universes/regions connected by a throat located at $\bar{r} = 0$, with a radius $\bar{R} = b$. Thus, the non-vanishing components of the Einstein tensor are

$$G^t{}_t = \frac{b^2}{\left(b^2 + \bar{r}^2\right)^2} \,,$$

$$G^r{}_r = -\frac{b^2}{\left(b^2 + \bar{r}^2\right)^2} \,,$$

$$G^\theta{}_\theta = G^\varphi{}_\varphi = \frac{b^2}{\left(b^2 + \bar{r}^2\right)^2} \,. \qquad (29)$$

Let us now consider a scalar field described by the action:

$$S_{\phi_2} = \int d^4x \sqrt{-g} \left[-\varepsilon g^{\mu\nu} \partial_\mu \phi_2 \partial_\nu \phi_2 - V(\phi_2)\right] \,, \quad (30)$$

where $\varepsilon$ is a constant that can be either positive or negative. If $\varepsilon$ is negative, the scalar field $\phi_2$ becomes a ghost, and therefore the model turns out to be physically inconsistent. Since the metric is static and spherically symmetric, the scalar field by the field equations depends just on the radial coordinate, $\phi_2 = \phi_2(r)$. Then, the expression of the energy-momentum tensor for the scalar field is given by

$$T^\mu{}_\nu = 2\varepsilon g^{\mu\gamma} \partial_\gamma \phi_2 \partial_\nu \phi_2 - \delta^\mu_\nu \left(\varepsilon g^{\alpha\beta} \partial_\alpha \phi_2 \partial_\beta \phi_2 - V(\phi)\right) \,, \quad (31)$$

where its non-vanishing components are

$$T^t{}_t = -\varepsilon \phi_2'^2 + V(\phi_2) \,,$$
$$T^r{}_r = \varepsilon \phi_2'^2 + V(\phi_2) \,,$$
$$T^\theta{}_\theta = T^\varphi{}_\varphi = -\varepsilon \phi_2'^2 + V(\phi_2) \,.$$

Here, the primes denote differentiation with respect to the radial coordinate. By solving the field equations, one gets the solution:

$$V(\phi_2) = 0 \,, \quad \varepsilon = -1 \,. \qquad (32)$$

As $\varepsilon = -1$, the scalar field is a ghost and the model might not be physically unacceptable. While a first integral for the scalar field equation yields

$$\phi_2' = \frac{|b|}{(b^2 + \bar{r}^2)} , \qquad (33)$$

whose solution is given by

$$\phi_2 = \phi_0 + \arctan\left(\frac{\bar{r}}{b}\right) , \qquad (34)$$

where $\phi_0$ is an integration constant. In Fig. 4, the shape of the dynamics of the scalar field is depicted. As shown, the scalar field reaches a minimum at the throat of the wormhole, located at $\bar{r} = 0$. Therefore, the source for this wormhole is a free phantom scalar field. As far as $b \to 0$, the metric (28) turns out to be Minkowski spacetime and the scalar field becomes a constant that does not contribute to the equations, as expected.
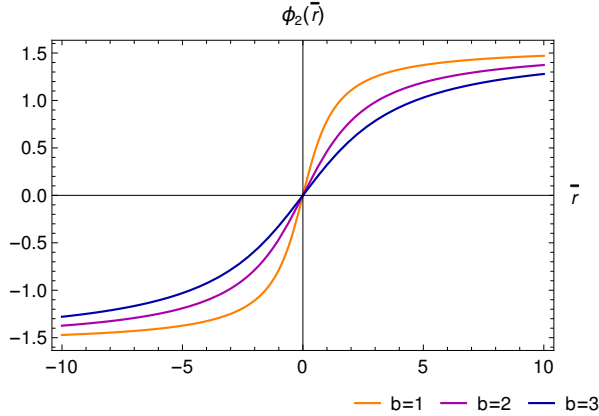


$\phi_2(\bar{r})$

FIG. 4: Motion of the scalar field $\phi_2(\bar{r})$ for different values of $b$. Here, $\phi_0 = 0$ is assumed.

## V. NON-STATIONARY ELLIS-BRONNIKOV WORMHOLE

Let us now reconstruct a non-stationary and regular wormhole, based on the Ellis-Bronnikov solution and the regularisation of FLRW spacetime followed above. To do so, we start with the Minkowski metric, which in spherical coordinates is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 . \qquad (35)$$

Since the FLRW metric is conformal to Minkowski, it can be obtained just by multiplying the spatial part of Minkowski by the scale factor,

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega^2\right) . \qquad (36)$$

On the other hand, by replacing the radius $r$ of the two-dimensional sphere $\mathcal{S}^2$ by $\Sigma = \sqrt{\bar{r}^2 + b^2}$ of (28) in (35),

which is a kind of the Simpson-Visser regularisation procedure, one obtains

$$ds^2 = -dt^2 + d\bar{r}^2 + \Sigma^2 d\Omega^2 , \qquad (37)$$

which is the stationary Ellis-Bronnikov wormhole. The question now arises: how can we define a non-stationary wormhole? There are many possibilities. For example, if we apply the Simpson-Visser procedure directly on $\bar{r}$ in Eq. (36), the spacetime metric becomes

$$ds^2 = -dt^2 + a^2(t)\left(d\bar{r}^2 + \Sigma^2 d\Omega^2\right) . \qquad (38)$$

On the other hand, if the last term of Eq. (37) is multiplied by $a(t)$, then the spacetime turns out (38) again. Hence, both procedures are equivalent, and this metric describes a non-stationary wormhole connecting two regions/universes $-\infty < \bar{r} < \infty$, whose throat, with a topology of two-dimensional sphere $\mathcal{S}^2$, is not constant but evolves with time as $R(t) = a(t)b$. For this metric, the Einstein tensor is given by

$$G^t_{\ t} = \frac{b^2}{a(t)^2 \left(b^2 + \bar{r}^2\right)^2} - \frac{3\dot{a}^2(t)}{a(t)^2} , \qquad (39)$$

$$G^{\bar{r}}_{\ \bar{r}} = -\frac{b^2}{a(t)^2 \left(b^2 + \bar{r}^2\right)^2} - \frac{2\ddot{a}(t)}{a(t)} - \frac{\dot{a}(t)^2}{a(t)^2} , \qquad (40)$$

$$G^\theta_{\ \theta} = G^\varphi_{\ \varphi} = \frac{b^2}{a(t)^2 \left(b^2 + \bar{r}^2\right)^2} - \frac{2\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a(t)^2} . \qquad (41)$$

In order to reconstruct the corresponding matter Lagrangian that reproduces this solution, two scalar fields, as the ones analysed in previous sections, are considered. Then, the action is given by

$$S = \int d^4x \sqrt{-g}\left[\varepsilon g^{\mu\nu}\partial_\mu\phi_2\partial_\nu\phi_2 - V\left(\phi_2\right) \right.$$
$$\left. + \mu\sqrt{-g^{\mu\nu}\partial_\mu\phi_1\partial_\nu\phi_1} - U\left(\phi_1\right)\right] . \qquad (42)$$

That is, the above wormhole will be described exactly by the sum of these two scalar fields. The components of the energy-momentum tensor are shown below.

For the cuscuton field $\phi_1 = \phi_1(t)$, the energy-momentum tensor is given by (8), whose components do not change,

$$T^t_{\ t} = -U\left(\phi_1\right) ,$$
$$T^{\bar{r}}_{\ \bar{r}} = T^\theta_{\ \theta} = T^\varphi_{\ \varphi} = \mu\left|\dot{\phi}_1\right| - U\left(\phi_1\right) . \qquad (43)$$

Whereas the second scalar field $\phi_2 = \phi_2\left(\bar{r}\right)$ is described by the energy-momentum tensor (31) with $\varepsilon = -1$, that is, $\phi_2$ is a physically inconsistent ghost.

$$T^t_{\ t} = \frac{\phi_2'^2}{a(t)^2} + V\left(\phi_2\right) ,$$

$$T^{\bar{r}}_{\ \bar{r}} = -\frac{\phi_2'^2}{a(t)^2} + V\left(\phi_2\right) ,$$

$$T^\theta_{\ \theta} = T^\varphi_{\ \varphi} = \frac{\phi_2'^2}{a(t)^2} + V\left(\phi_2\right) . \qquad (44)$$

The field equation for $\phi_2$ yields:

$$\frac{\phi_2''}{a^2} + \frac{2\bar{r}}{a^2\left(\bar{r}^2 + b^2\right)}\phi_2' + \frac{dV}{d\phi_2} = 0 \, . \qquad (45)$$

For $V\left(\phi_2\right) = 0$, the scale factor $a(t)$ can be removed from (45) and the solution of the equation reproduces the same one as in the stationary Ellis-Bronnikov wormhole, i.e.

$$\phi_2(r) = \phi_0 + \arctan\left(\frac{\bar{r}}{b}\right) \, . \qquad (46)$$

Then, by the Einstein field equations,

$$G^t_{\ t} = T^t_{\ t} \, , \quad G^{\bar{r}}_{\ \bar{r}} = T^{\bar{r}}_{\ \bar{r}} \, ,$$
$$G^\theta_{\ \theta} = T^\theta_{\ \theta} \, , \quad G^\varphi_{\ \varphi} = T^\varphi_{\ \varphi} \, , \qquad (47)$$

the general solution can be obtained, since the equations (47) are separable in the radial and temporal parts, and the radial part fulfils the solution for $\phi_2$ given in (46). Nevertheless, the temporal side of the equations, which implies the cuscuton field, reduces to the same equations as in the cosmological case analysed above, just with the presence of the cuscuton, which are given by:

$$3H^2 = U\left(\phi_1\right) \, , \quad -2\dot{H} - 3H^2 = \mu\left|\dot{\phi}_1\right| - U\left(\phi_1\right) \, . \qquad (48)$$

As pointed out in Section III, the only possible solution for an arbitrary potential is given by the roots $\phi_1 = \phi_0 = \text{constant}$ of the equation:

$$\left(\frac{dU}{d\phi}\right)^2 - 3\mu^2 U(\phi) = 0 \, . \qquad (49)$$

For such a case, the scale factor grows exponentially,

$$a(t) \propto e^{\sqrt{\frac{U(\phi_0)}{3}}t} \, . \qquad (50)$$

For this case, the metric (38) is free of timelike singularities and describes a wormhole whose throat grows exponentially with time, as shown in Fig. 5, where the structure of the wormhole is depicted at the equatorial plane, $\theta = \pi/2$. This is the only possible regular solution with no presence of any other kind of fields. Of course, as far as one includes additional species in the matter Lagrangian, as dust, for instance, the solution might be the one found in Section III with the presence of $\rho_\mathrm{m}$, where the scale factor evolutes as $a(t) \propto (t^2 + \tilde{b}^2)^{1/3}$. Note also that one might construct a non-stationary wormhole in the absence of the cuscuton field but with the presence of dust matter or radiation. However, in such cases, the spacetime metric would contain a time-like singularity, since the scale factor becomes null at $t = 0$, as shown in (4). This feature can be seen in Fig. 6b, as $\tilde{b}$ gets smaller, the throat of the wormhole is shrinking at $t = 0$.

## VI. ELIMINATING GHOSTS

We have found that the cuscuton-type scalar field $\phi_1$ in (5) is a ghost because $\mu$ is negative and another scalar field $\phi_2$ in (30) is also a ghost because $\varepsilon$ is negative. In [27], it has been shown that such ghosts can be eliminated by using the constraints. In this section, we discuss how the ghost is eliminated.

We assume the solutions of the model as $\phi_1 = \phi_1(t) = \Phi_1(t)$ as in (25), $\phi_2 = \phi_2\left(\bar{r}\right) = \Phi_2\left(\bar{r}\right)$ as in (34) or (46). Here $\Phi_1(t)$ and $\Phi_2\left(\bar{r}\right)$ are not field variables but given functions of $t$ and $\bar{r}$, respectively. If $\phi_1$ is a constant as in (49), we can just drop the kinetic term of $\phi_1$ and therefore the cuscuton-type scalar field $\phi_1$ becomes non-dynamical and $\phi_1$ is not a ghost.

For the FLRW universe, we add the following action to the action (5)

$$S_{\mathrm{cnstrnt}\,\phi_1} = \int d^4x \sqrt{-g}\lambda_1\left(g^{\mu\nu}\partial_\mu\hat{\phi}_1\partial_\nu\hat{\phi}_1 + 1\right) \, . \quad (51)$$

Here $\lambda_1$ is a Lagrange multiplier field and $\hat{\phi}_1$ is defined by $\phi_1 = \Phi_1\left(\hat{\phi}_1\right)$. The variation of the action (51) with rspect to $\lambda_1$, we obtain the following constraint for $\hat{\phi}_1$ and therefore $\phi_1$,

$$0 = g^{\mu\nu}\partial_\mu\hat{\phi}_1\partial_\nu\hat{\phi}_1 + 1 \, . \qquad (52)$$

Due to the constraint (52), $\hat{\phi}_1$ and therefore $\phi_1$ become dynamical, that is, they do not propagate and therefore $\phi_1$ is not a ghost. The solution of the constraint (52) is given by $\hat{\phi}_1 = t$, which gives $\phi_1 = \Phi_1(t)$, consistently.

For the Ellis-Bronnikov wormhole (28), we add the following action to the action (30),

$$S_{\mathrm{cnstrnt}\,\phi_2} = \int d^4x \sqrt{-g}\lambda_2\left(g^{\mu\nu}\partial_\mu\hat{\phi}_2\partial_\nu\hat{\phi}_2 - 1\right) \, . \quad (53)$$

Here $\lambda_2$ is also a Lagrange multiplier field and $\hat{\phi}_2$ is defined by $\phi_2 = \Phi_2\left(\hat{\phi}_2\right)$. The variation of the action (53) with rspect to $\lambda_2$, we obtain the following constraint for $\hat{\phi}_2$ and therefore $\phi_2$,

$$0 = g^{\mu\nu}\partial_\mu\hat{\phi}_2\partial_\nu\hat{\phi}_2 - 1 \, . \qquad (54)$$

Due to the constraint (54), $\hat{\phi}_2$ and therefore $\phi_2$ also become dynamical, that is, they do not propagate, and therefore $\phi_2$ is not a ghost. The solution of the constraint (54) is given by $\hat{\phi}_2 = \bar{r}$, which gives $\phi_2 = \Phi_2\left(\bar{r}\right)$, consistently.

For the time-dependent Simpson-Visser space time (38), we need to be technical a little bit, Because we are considering the solution that $\phi_1$ is a constant, we may eliminate the kinetic term of $\phi_1$ so that the cuscuton-type scalar field $\phi_1$ becomes non-dynamical and $\phi_1$ is not a ghost. Although $\phi_1$ is a constant, in order to include the field corresponding to the cosmological time
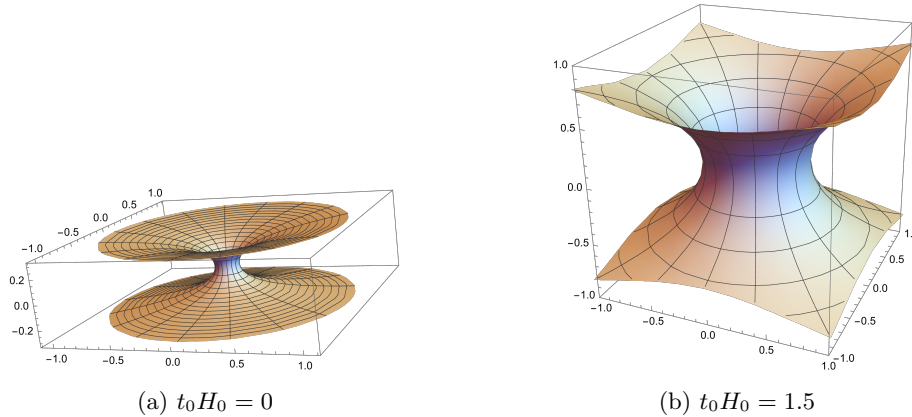
(a) $t_0 H_0 = 0$

(b) $t_0 H_0 = 1.5$

FIG. 5: Evolution of the Ellis-Bronnikov wormhole for a de Sitter-like expansion $a(t) \propto \mathrm{e}^{tH_0}$ and $b = 0.1$.



(a) $\tilde{b} = 0.1$
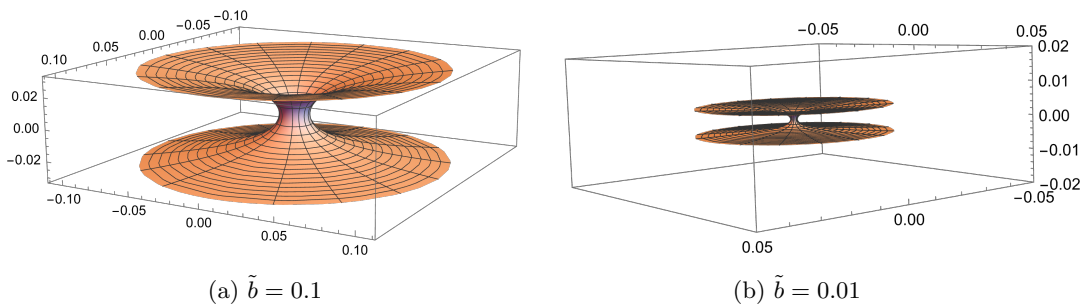
(b) $\tilde{b} = 0.01$

FIG. 6: Ellis-Bronnikov wormhole with $b = 0.1$ for a dust-like scale factor $a(t) \propto (t^2 + \tilde{b}^2)^{1/3}$ at $t = 0$.

$t$, we need to include the field $\hat{\phi}_1$ and the action (51). Furthermore, the action (53) is modified to be

$$S^{(2)}_{\mathrm{cnstrnt}\,\phi_2} = \int d^4 x \sqrt{-g}\lambda_2 \left( \frac{1}{a\left(\hat{\phi}_1\right)} g^{\mu\nu} \partial_\mu \hat{\phi}_2 \partial_\nu \hat{\phi}_2 - 1 \right),$$

(55)

which gives the following constraint

$$0 = \frac{1}{a\left(\hat{\phi}_1\right)} g^{\mu\nu} \partial_\mu \hat{\phi}_2 \partial_\nu \hat{\phi}_2 - 1 .$$

(56)

Then by using $\phi_1 = t$, which is a solution of (52), and the metric in (38), we find that $\hat{\phi}_2 = \bar{r}$ is a solution.

We should note that we can consider the solution where $\lambda_1 = \lambda_2 = 0$. Then the solutions in the previous sections are solutions even if we include the actions (51), and (53) or (54). Because both $\phi_1$ and $\phi_2$ become non-dynamical, the ghosts are excluded.

## VII. THE CASE IN $f(R)$ GRAVITY

We might consider now the cases analysed above within $f(R)$ gravity, whose general action is given by

$$S_G = \int d^4 x \sqrt{-g} f(R) ,$$

(57)

where $f(R)$ is an arbitrary non-linear function of the Ricci scalar. The general form of the field equations is obtained by varying the action (57) with respect to the metric, which, including the matter fields, leads to

$$f_R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) f_R = \kappa^2 T^{\mathrm{matter}}_{\mu\nu} .$$

(58)

Here $f_R = \frac{df}{dR}$. This type of modified gravity is known to carry an extra scalar mode, such that the action might be rewritten as,

$$S_G = \int d^4 x \sqrt{-g} \left( \psi R - V(\psi) \right) .$$

(59)

Then, by varying this action and comparing (57) and (59), it yields

$$\psi = f_R , \quad V(\psi) = f_R R - f(R) .$$

(60)

The field equations might be expressed now in an Einstein-like form as follows

$$G_{\mu\nu} = \frac{1}{\psi} \left( T^{(m)}_{\mu\nu} + T^{\psi}_{\mu\nu} \right) ,$$

(61)

where

$$T^{\psi}_{\mu\nu} = \nabla_\mu \nabla_\nu \psi - g_{\mu\nu}\Box\psi - \frac{1}{2} g_{\mu\nu} V(\psi) .$$

(62)

We shall now find whether the above regular solutions, both cosmological as the Ellis-Bronnikov wormhole, can be reproduced in this type of theories. For the FLRW metric (1), the only two independent equations are the $tt-$ and $rr-$ components in (61), which are given by

$$3H^2 + 3H\frac{\dot{\psi}}{\psi} - \frac{1}{2}\frac{V(\psi)}{\psi} = -\kappa^2 T^{(m)t}{}_t,$$

$$-2\frac{\ddot{a}}{a}\psi - H^2 - 2H\dot{\psi} + \frac{1}{2}\frac{V(\psi)}{\psi} = \kappa^2 T^{(m)r}{}_r. \quad (63)$$

For the sake and with no loss of generality, we can assume here $T^m_{\mu\nu} = 0$, i.e., no presence of any other fields but just vacuum solutions. At the end, for a given cosmological solution $a(t)$, equations (63) are differential equations with respect to the auxiliary field $\psi$ and its potential $V(\psi)$, whereas matter terms provide some particular solutions. We might consider the previous case of

a scalar factor that mimics a dust-dominated universe, $a(t) \sim t^{2/3}$. Equations (63) can be solved, which yield [38, 39],

$$\psi(t) = c_1 t^{\frac{5+\sqrt{73}}{6}} + c_2 t^{\frac{5-\sqrt{73}}{6}},$$

$$V(t) = c_+ t^{\frac{-7+\sqrt{73}}{6}} + c_- t^{\frac{-7-\sqrt{73}}{6}}, \quad (64)$$

where $c_\pm = \frac{2(9\pm\sqrt{73})}{3}c_{1,2}$. By inverting the expressions (60), one finally gets the following $f(R)$ gravity:

$$f(R) = \tilde{c}_1 R^{n_+} + \tilde{c}_2 R^{n_-}. \quad (65)$$

Here $n_\pm = \frac{7\pm\sqrt{73}}{12}$ and $\tilde{c}_{1,2}$ are just redefinitions of the integration constants. One might consider the same cosmological solution in the presence of dust matter and the cuscuton field,

$$\psi = t^{\frac{5+\sqrt{73}}{6}}\left[c_1 - \frac{3}{\sqrt{73}}\int\left(\kappa^2\rho_0 t^{-\frac{11+\sqrt{73}}{6}} + \mu^2 t^{\frac{1-\sqrt{73}}{6}}\dot{\phi}\right)dt\right] + t^{\frac{5-\sqrt{73}}{6}}\left[c_2 + \frac{3}{\sqrt{73}}\int\left(\kappa^2\rho_0 t^{\frac{-11+\sqrt{73}}{6}} + \mu^2 t^{\frac{1+\sqrt{73}}{6}}\dot{\phi}\right)dt\right].$$

$$(66)$$

Then, with the combination of the cuscuton equation (11) and the cosmological equations (63), the corresponding $f(R)$ action can be reconstructed. Nevertheless, one should note that the system of equations is undetermined, since there are three independent equations and four variables. In any case, the solution $a(t) \sim t^{2/3}$ can be well reproduced in $f(R)$ gravity, as shown in previous literature. We might try now to find out whether the regularised version of this cosmological solution, $a(t) \sim (t^2 + b^2)^{1/3}$, is a solution of $f(R)$ gravity, as it is in the case in GR with the presence of the cuscuton field. For such a case, the equations (63) in vacuum provide the following solution for the auxiliary scalar field,

$$\psi(t) = (t^2 + b^2)^{2/3}\left[c_1\mathcal{P}_j^k(it/b) + c_2\mathcal{Q}_j^k(it/b)\right]. \quad (67)$$

Here, $\mathcal{P}_j^k$ and $\mathcal{Q}_j^k$ are the associated Legendre polynomials of the first and second kind, respectively, whereas $j = \frac{-3+\sqrt{73}}{6}$ and $k = \frac{2\sqrt{10}}{3}$. This is not a real solution, and consequently, the gravitational action $f(R)$ turns out complex, and it cannot be considered a consistent theory of gravity. Moreover, even by considering the presence of the cuscuton field and/or dust matter, the only real-valued gravitational action that can be obtained is the Hilbert-Einstein action. Indeed, the cuscuton and any other matter fields just provide particular solutions to (67), and the general solution becomes complex as well. In addition, the same result is found for any regular solution of the type $a(t) \sim (t^2 + b^2)^n$. Hence, one can conclude that this type of regular solution is ex-

clusive of General Relativity (with the presence of the appropriate matter fields).

For the stationary wormhole (28), one reaches a similar conclusion in the cosmological case. The effective tensor (62) yields:

$$T^{(\psi)t}{}_t = -\frac{1}{\psi}\left(\psi'' + \frac{2r\psi'}{r^2+b^2} + \frac{1}{2}V(\psi)\right),$$

$$T^{(\psi)r}{}_r = -\frac{1}{\psi}\left(\frac{2r\psi'}{r^2+b^2} + \frac{1}{2}V(\psi)\right).$$

$$T^{(\psi)\theta}{}_\theta = T^{(\psi)\varphi}{}_\varphi = -\frac{1}{\psi}\left(\psi'' + \frac{r\psi'}{r^2+b^2} + \frac{1}{2}V(\psi)\right). \quad (68)$$

We might or might not consider the scalar field given by (31), but the result turns out to be the same. By combining the $tt-$ and $\varphi\varphi-$ field equations, it gives

$$\frac{r}{r^2+b^2}\frac{\psi'}{\psi} = 0. \quad (69)$$

The only possible solution for this equation is basically $\psi =$ constant. Since the Ricci scalar is not constant, $R = -\frac{2b^2}{(r^2+b^2)^2}$ and $\psi = df/dR$, the only gravitational action that admits the Ellis-Bronnikov wormhole is again the Hilbert-Einstein action. Nevertheless, one might consider additional sources of matter to reproduce the Ellis-Bronnikov wormhole in $f(R)$ gravity, but at the price of increasing the number of degrees of freedom.

In addition, the case of non-stationary wormholes described by the metric (38) does not provide any new

insights, but in such a case, equations just become much messier and results turn out the same, i.e., the Hilbert-Einstein action is the only one that admits this type of regular solutions unless additional matter sources are considered.

However, according to previous literature, some wormholes have been found in the framework of $f(R)$ gravity [40, 41]. Then, we might explore the existence of similar spacetimes in comparison to the Ellis-Bronnikov wormhole to test in $f(R)$ gravity. To do so, let us start with the following generic spacetime metric,

$$ds^2 = -e^{2\nu(r,t)}dt^2 + e^{2\lambda(r,t)}dr^2 + r^2 d\Omega^2\,. \tag{70}$$

We might now consider the simplest wormhole geometry given by,

$$e^{2\nu} = 1\,, \quad e^{2\lambda} = \frac{1}{1 - \frac{r_0}{r}}\,, \tag{71}$$

where $r_0$ is the minimum radius of the wormhole and $r \geq r_0$. When $r \sim r_0$, we find $e^{2\lambda} \sim \frac{r_0}{r - r_0}$, which looks singular at $r = r_0$. To avoid the singularity, if $r_0$ is a constant, by defining a new coordinate $l$, defined as follows,

$$r = \sqrt{r_0{}^2 + l^2}\,, \quad dr = \frac{l\,dl}{\sqrt{r_0{}^2 + l^2}}\,, \tag{72}$$

and then we obtain the following metric,

$$ds^2 = -dt^2 + \frac{l^2}{r_0{}^2 + l^2 - r_0\sqrt{r_0{}^2 + l^2}}dl^2 + \left(r_0{}^2 + l^2\right)d\Omega^2\,. \tag{73}$$

When $r \to r_0$, we find $l \to 0$, where $\frac{l^2}{r_0{}^2 + l^2 - r_0\sqrt{r_0{}^2 + l^2}} \to$ 2, and therefore the spacetime is regular $r = r_0$, that is, $l = 0$. We analytically continue the value of $l$ in the region where $l$ is negative. If the region with positive $l$ corresponds to our Universe, the region with negative $l$ corresponds to another Universe. The two Universes are connected by the wormhole, whose minimum radius is $r_0$. We should also note that when $|l| \to \infty$, we find $|l| \to r$ and $\frac{l^2}{r_0{}^2 + l^2 - r_0\sqrt{r_0{}^2 + l^2}} \to 1$ and therefore the spacetime is asymptotically flat. In Fig. 7, the structure of the wormhole is depicted at the equatorial plane, $\theta = \pi/2$, which shows a similar shape in comparison to the Ellis-Bronnikov wormhole.

For the geometry in (71), we find $R = 0$, and the equations in (A6) have the following forms,

$$-\frac{1}{2}f(0) = -\frac{\kappa^2}{2}\rho\,, \quad \frac{1}{2}f(0) + \frac{r_0}{r^3}f'(0) = -\frac{\kappa^2}{2}p_{\rm rad}\,,$$

$$\frac{1}{2}f(0) - \frac{r_0}{2r^3}f'(0) = -\frac{\kappa^2}{2}p_{\rm ang}\,, \quad 0 = -\frac{\kappa^2}{2}j\,. \tag{74}$$

This tells that if $f(0) = f'(0) = 0$, there appears the wormhole solution corresponding to (74) even in the vacuum, where $\rho = p_{\rm rad} = p_{\rm ang} = 0$. Because

the square of the effective gravitational coupling is proportional to the inverse power of $f'$, the effective coupling diverges in this case where $f'(0) = 0$ and therefore, the situation could not be realistic. We can consider, however, the expansion with respect to $f'(0)$ when $f'(0)$ does not vanish exactly, which could be regarded as a strong coupling expansion. Then, by the expansion, the geometry (74) can be deformed, but in the leading order, the geometry remains a solution.

This wormhole solution might also be extended to the non-stationary case by introducing a scale factor in front of the spatial part of the metric (73). Nevertheless, in such a case, the Ricci scalar turns out to be $R = 6\left(2H^2 + \dot{H}\right)$, which depends solely on the time coordinate and consequently $f(R)$ so it does, whereas the field equations do not. Hence, one has to consider additional sources that will depend on both the time coordinate and the radial coordinate. This is an interesting open issue that might be explored in the future.
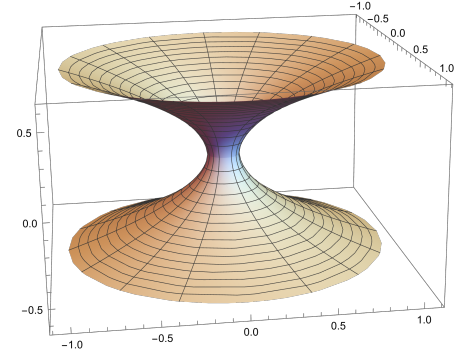


FIG. 7: Wormhole within the framework of $f(R)$ gravity, described by the metric (73) for $r_0 = 0.1$.

## VIII. CONCLUSIONS

In this work, we have explored the possibility of reconstructing non-stationary regular wormhole spacetimes by the use of two scalar fields, a usual phantom one that sources the wormhole and a cuscuton field that regularises the time expansion of the wormhole. To do so, we have focused on a particular case, the so-called Ellis-Bronnikov wormhole, highlighting its theoretical significance and potential implications. Firstly, the Simpson-Visser procedure is applied to regularise, in a general way, the FLRW metric, by introducing a bounce on the cosmological evolution that removes the Big Bang singularity at $t = 0$, but does not cause significant changes in the sources. In particular, by applying such regularisation on a matter-dominated universe, the solution describes an eternal universe that is contracting till it reaches a minimum of the scale factor and then expands again. In such a process, an accelerating phase arises that might be identified with inflation [34], providing

a regular cosmological model that might be well worth testing with the last observational data, which imposes severe constraints on the inflationary models. Such a regularisation is possible due to the presence of the cuscuton field, whose main feature, in comparison to the usual scalar field dynamics, is the linearity of its velocities in the action. While such a field with no presence of any other type of matter just describes de Sitter cosmological expansions, here we have shown that it can be useful to implement regular cosmological solutions.

Nevertheless, the aim of the paper has lain in the reconstruction of regular non-stationary wormholes, particularly the Ellis Bronnikov wormhole, where the cosmological bouncing solution sourced by the cuscuton field is applied to do so. Then, a generalisation of Ellis Bronnikov wormhole is proposed, where the spatial part of the metric evolves with time, by expanding and/or contracting. In order to avoid time-like singularities on the expansion of the spatial slices, the cuscuton field is assumed together with a phantom scalar field that sources the wormhole. The result is a non-stationary wormhole that, with the appropriate potential for the cuscuton field, it provides a completely regular spacetime, free of time-like and space-like singularities. Two particular cases are explored, one with an exponential expansion where the cuscuton field just acts as an effective cosmological constant and on the other hand, a dust matter-like expansion with the presence also of dust, where the cuscuton field plays an active role by regularising the time evolution of the scale factor, which reaches a minimum value and then bounces. Several cases are depicted to show the time evolution and the features of this generalised non-stationary wormhole solution. Nevertheless, we have shown that when this type of regularisation is followed in the framework of GR using scalar fields, their kinetic terms turn out negative and consequently, these scalar fields are ghosts that might not be considered acceptable physically. We have shown that the ghost-free models can be obtained by using the constraints.

In addition, we have explored the possibility of reproducing this type of non-stationary wormhole in a class of modified gravities, the so-called $f(R)$ gravity. Results suggest that only within General Relativity with ghost scalar fields, the (non-)stationary Ellis-Bronnikov wormhole can be reconstructed, unless other types of anisotropic fluids for the matter sector are considered [37]. Moreover, we have reconstructed a novel wormhole solution in the framework of $f(R)$ gravity, similar to the Ellis-Bronnikov solution that does not require any ghost scalar field, which suggests that other time-dependent black bounce metrics can be described in these theories with no ghost fields.

These results also highlight the versatility of the Cuscuton field in modelling different gravitational regimes and suggest that it may play an important role in the formulation of new cosmological scenarios and the description of non-singular geometries. Further investigations might also point to a way for implementing a natural mechanism of formation for such a kind of compact objects, one of the Achilles' heel when dealing with wormholes. Hence, future analysis may explore the implications of these scenarios in broader contexts.

## Appendix A

For the spacetime, whose metric is given by,

$$ds^2 = -\mathrm{e}^{2\nu(r,t)}dt^2 + \mathrm{e}^{2\lambda(r,t)}dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{A1}$$

we find the following non-vanishing connections,

$$\Gamma^t_{tt} = \dot{\nu}, \quad \Gamma^r_{tt} = \mathrm{e}^{-2(\lambda-\nu)}\nu', \quad \Gamma^t_{tr} = \Gamma^t_{rt} = \nu',$$
$$\Gamma^t_{rr} = \mathrm{e}^{2\lambda-2\nu}\dot{\lambda}, \quad \Gamma^r_{tr} = \Gamma^r_{rt} = \dot{\lambda},$$
$$\Gamma^r_{rr} = \lambda', \Gamma^i_{jk} = \bar{\Gamma}^i_{jk}, \quad \Gamma^r_{ij} = -\mathrm{e}^{-2\lambda}r\bar{g}_{ij},$$
$$\Gamma^i_{rj} = \Gamma^i_{jr} = \frac{1}{r}\delta^i{}_j, \tag{A2}$$

where the metric $\bar{g}_{ij}$ is given by $\sum_{i,j=1}^{2}\bar{g}_{ij}dx^i dx^j = d\theta^2 + \sin^2\theta, d\phi^2$, with $\left(x^1 = \theta, \, x^2 = \phi\right)$, and $\bar{\Gamma}^i_{jk}$ represents the metric connection of $\bar{g}_{ij}$. The "dot" and "prime" symbols denote differentiation with respect to $t$ and $r$, respectively.

Non-vanishing components of the curvatures are given by,

$$R_{rtrt} = -\mathrm{e}^{2\lambda}\left\{\ddot{\lambda} + \left(\dot{\lambda} - \dot{\nu}\right)\dot{\lambda}\right\} + \mathrm{e}^{2\nu}\left\{\nu'' + \left(\nu' - \lambda'\right)\nu'\right\},$$
$$R_{titj} = r\nu'\mathrm{e}^{-2\lambda+2\nu}\bar{g}_{ij},$$

$$R_{rirj} = \lambda' r \bar{g}_{ij}\,, \quad R_{tirj} = \dot{\lambda} r \bar{g}_{ij}\,, \quad R_{ijkl} = \left(1 - \mathrm{e}^{-2\lambda}\right) r^2 \left(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}\right)\,,$$

$$R_{tt} = -\left\{\ddot{\lambda} + \left(\dot{\lambda} - \dot{\nu}\right)\dot{\lambda}\right\} + \mathrm{e}^{-2\lambda+2\nu}\left\{\nu'' + (\nu' - \lambda')\nu' + \frac{2\nu'}{r}\right\}\,,$$

$$R_{rr} = \mathrm{e}^{2\lambda-2\nu}\left\{\ddot{\lambda} + \left(\dot{\lambda} - \dot{\nu}\right)\dot{\lambda}\right\} - \left\{\nu'' + (\nu' - \lambda')\nu'\right\} + \frac{2\lambda'}{r}\,, \quad R_{tr} = R_{rt} = \frac{2\dot{\lambda}}{r}\,,$$

$$R_{ij} = \left\{1 + \left\{-1 - r\left(\nu' - \lambda'\right)\right\}\mathrm{e}^{-2\lambda}\right\}\bar{g}_{ij}\,,$$

$$R = 2\mathrm{e}^{-2\nu}\left\{\ddot{\lambda} + \left(\dot{\lambda} - \dot{\nu}\right)\dot{\lambda}\right\} + \mathrm{e}^{-2\lambda}\left\{-2\nu'' - 2\left(\nu' - \lambda'\right)\nu' - \frac{4\left(\nu' - \lambda'\right)}{r} + \frac{2\mathrm{e}^{2\lambda} - 2}{r^2}\right\}\,. \tag{A3}$$

By using the energy-momentum tensor $T_{\mu\nu}$, we now define the energy density $\rho$, the pressure in the radial direction $p_{\mathrm{rad}}$, the pressure in the angular direction $p_{\mathrm{ang}}$, and energy flow $j$ by,

$$T_{tt} = -\rho g_{tt}\,, \quad T_{rr} = p_{\parallel} g_{rr}\,, \quad T_{\theta\theta} = p_{\perp} g_{\theta\theta}\,,$$
$$T_{\phi\phi} = p_{\perp} g_{\phi\phi}\,, \quad T_{tr} = T_{rt} = j\,. \tag{A4}$$

The non-vanishing components of $f(R)$ equation,

$$\frac{1}{2}g_{\mu\nu}f(R) - R_{\mu\nu}f'(R) - g_{\mu\nu}\Box f'(R) + \nabla_\mu\nabla_\nu f'(R) = -\frac{\kappa^2}{2}T_{\mu\nu}\,, \tag{A5}$$

are given by

$$-\frac{\kappa^2 \mathrm{e}^{2\nu}}{2}\rho = -\frac{1}{2}\mathrm{e}^{2\nu}f(R) - \left[-\left\{\ddot{\lambda} + \left(\dot{\lambda} - \dot{\nu}\right)\dot{\lambda}\right\} + \mathrm{e}^{-2\lambda+2\nu}\left\{\nu'' + (\nu' - \lambda')\nu' + \frac{2\nu'}{r}\right\}\right]f'(R)$$
$$+ \mathrm{e}^{2\nu-2\lambda}\left\{\partial_r{}^2 - \mathrm{e}^{2\lambda-2\nu}\dot{\lambda}\partial_t - \lambda'\partial_r + \frac{2}{r}\partial_r\right\}f'(R)\,,$$

$$-\frac{\kappa^2 \mathrm{e}^{2\lambda}}{2}p_{\parallel} = \frac{1}{2}\mathrm{e}^{2\lambda}f(R) - \left[\mathrm{e}^{2\lambda-2\nu}\left\{\ddot{\lambda} + \left(\dot{\lambda} - \dot{\nu}\right)\dot{\lambda}\right\} - \left\{\nu'' + (\nu' - \lambda')\nu'\right\} + \frac{2\lambda'}{r}\right]f'(R)$$
$$- \left\{-\mathrm{e}^{2\lambda-2\nu}\left(\partial_t{}^2 - \dot{\nu}\partial_t - \mathrm{e}^{-2(\lambda-\nu)}\nu'\partial_r\right) + \frac{2}{r}\partial_r\right\}f'(R)\,,$$

$$-\frac{\kappa^2 r^2}{2}p_{\perp} = \frac{r^2}{2}f(R) - \left\{1 + \left\{-1 - r\left(\nu' - \lambda'\right)\right\}\mathrm{e}^{-2\lambda}\right\}f'(R)$$
$$- r^2\left\{-\mathrm{e}^{-2\nu}\left(\partial_t{}^2 - \dot{\nu}\partial_t - \mathrm{e}^{-2(\lambda-\nu)}\nu'\partial_r\right) + \mathrm{e}^{-2\lambda}\left(\partial_r{}^2 - \mathrm{e}^{2\lambda-2\nu}\dot{\lambda}\partial_t - \lambda'\partial_r\right)\right\}f'(R)\,,$$

$$-\frac{\kappa^2}{2}j = -\frac{2\dot{\lambda}}{r}f'(R) + \left(\partial_t\partial_r - \nu'\partial_t - \dot{\lambda}\partial_r\right)f'(R)\,. \tag{A6}$$

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