

Disappointment Aversion and Expectiles

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Abstract

This paper recasts Gul (1991)’s theory of disappointment aversion in a Savage framework, with general outcomes, new explicit axioms of disappointment aversion, and novel explicit representations. These permit broader applications of the theory and better understanding of its decision-theoretic foundations. Our results exploit an unexpected connection of Gul’s theory and the econometric framework of Newey and Powell (1987) of asymmetric least square estimation.

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1 Introduction

The *theory of disappointment aversion* of Gul (1991) is one of the most influential models in decision-making under risk. This model posits that individuals evaluate the outcomes of any risky action relative to an endogenous reference point: the ex-ante value of the action itself. Such a reference point balances the negative emotional responses to outcomes falling short of expectations—that are *disappointing*—against the positive emotional responses to outcomes exceeding expectations—that are *elating*.

The model captures two core emotions, sadness (from disappointment) and joy (from elation). It has proven valuable in explaining economic behaviors beyond the reach of expected utility theory, it performs well in experimental settings, and it aligns closely with recent neuroscientific findings on reference-dependent decision-making. Yet, despite its success, the original framework, including its axiomatic formulation and functional representation, has some known limitations. The present paper addresses these comprehensively, while keeping the core theory intact.

Gul’s model is characterized by two parameters: a *utility function* $u : \mathbb{R} \rightarrow \mathbb{R}$, evaluating monetary outcomes, and a *disappointment aversion coefficient* $\beta \in (-1, \infty)$, measuring how much a decision maker (DM) discounts elating outcomes relative to disappointing ones. A positive $\beta > 0$ corresponds to disappointment aversion, $\beta = 0$ aligns with standard expected utility preferences, and a negative $\beta < 0$ indicates elation seeking behavior (rather than being discounted, elating outcomes carry additional value).

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Motivated by the Allais paradox, Gul (1991) shows that a weakening of the von Neumann-Morgenstern Independence Axiom characterizes evaluation of a random variable X as the solution v_X to the equation:

$$v = \mathbb{E}[k_v(X)] \quad (1)$$

where $k_v : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$k_v(x) = \begin{cases} \frac{u(x) + \beta v}{1 + \beta} & \text{if } u(x) \geq v \\ u(x) & \text{if } u(x) < v \end{cases}$$

for all $v \in \mathbb{R}$. The DM prefers X over Y if, and only if, $v_X \geq v_Y$.

Our research is motivated by the following challenges:

1. The original formulation above is restricted to monetary outcomes, even though disappointment over non-monetary outcomes, such as the results of personal endeavors or medical treatments, seem to be equally, if not more, significant.
2. The axiomatic characterization does not include explicit axioms that capture disappointment aversion or elation seeking, and, as a result, much of the theory's interpretation depends on its functional representation.
3. The representation itself is implicit, the solution of equation (1).

The insightful work of Cerreia-Vioglio, Dillenberger, and Ortoleva (2020) addresses the last point by providing an explicit representation for the monetary certainty equivalents of Gul's model in terms of cautious expected utility,¹ using solely the parameters u and β of the original implicit representation.

The present paper tackles all the challenges outlined above and offers the following contributions:

1. An extension of the theory to *general outcomes* in a framework à la Savage (1954).
2. A behavioral foundation based on *explicit axioms* of disappointment aversion/elation seeking, which are easy to interpret and determine the sign of β . The magnitude of β is then captured via comparative statics.
3. An explicit representation in terms of *maxmin expected utility* à la Gilboa and Schmeidler (1989), and another one in terms of *asymmetric least squares estimation* of the random payoff $u(X)$ that the DM is facing:

$$v_X = \arg \min_{v \in \mathbb{R}} \mathbb{E}[\ell_\beta(u(X) - v)] \quad (2)$$

where the deviation penalty $\ell_\beta(s) = s^2 + \beta s^2 1_{(-\infty, 0]}(s)$ captures the different emotional impact of elating and disappointing outcomes.

Relative to the last point, similarly to Cerreia-Vioglio et al. (2020), both the explicit representations that we obtain use solely the parameters of equation (1) and yield exactly the same solution v_X .

Altogether, our results provide a novel framework, a new set of axioms, and explicit representations for Gul's theory, offering the following benefits:

¹Cerreia-Vioglio, Dillenberger, and Ortoleva (2015).

1. The extended domain (general outcomes) broadens the applicability of the model, allowing it to capture disappointment and elation beyond monetary rewards.
2. The new axiomatization opens up avenues for empirical testing on the descriptive side, while offering a solid decision-theoretic foundation and deeper insights on the prescriptive side.
3. The explicit representations enhance the interpretability of the model, facilitate comparison with alternative theories, improve its computational tractability, and relate decision theory with econometric modelling.

Conceptually and mathematically, these contributions are enabled by an unexpected connection between Gul’s (1991) theory of disappointment aversion and the asymmetric least squares estimation framework of Newey and Powell (1987), as foreshadowed by representation (2) above and further detailed in the paper outline below.

1.1 Outline of the paper

1.1.1 Beyond monetary outcomes

The need to extend the original theory of Gul beyond monetary outcomes is clearly illustrated by personal endeavors and medical treatments. Imagine a novelist’s frustration when her manuscript is rejected, a student’s reaction to being denied admission to his dream college, an athlete’s discouragement when chronic pain persists despite strict adherence to recovery training, an actor’s dismay after cosmetic surgery fails to match the envisioned transformation. In all these cases, outcomes are judged against internal standards —of success, performance, fitness, appearance— highlighting how elation and disappointment are shaped by subjective expectations.

Extending the original arguments of Gul (1991), where the DM compares lotteries supported in an interval $[w, b]$ of monetary outcomes, is a nonobvious task. Moreover, working in a von Neumann–Morgenstern framework with lotteries prevents the separation of statistical modelling and decision modelling, something made possible by the Savage framework. For these reasons, in Section 2, we introduce our framework where (deterministic) *outcomes* x belong to a generic measurable space \mathcal{X} and a DM is comparing random outcomes $X : \Omega \rightarrow \mathcal{X}$, defined on a probability space (Ω, \mathcal{F}, P) and called *acts*. In the same section, we further discuss the choice of modelling risky actions as acts (random outcomes) rather than as lotteries (probability distributions over outcomes).

1.1.2 New formulas and interpretations

Section 3 opens with a simple observation (Proposition 1): in our general framework, v_X is a solution of equation (1) if, and only if, it is a solution of²

$$\underbrace{\mathbb{E}[(u(X) - v)^+]}_{\text{expected elation}} = (1 + \beta) \underbrace{\mathbb{E}[(v - u(X))^+]}_{\text{expected disappointment}}. \quad (3)$$

Conceptually, this means that expected elation $\mathbb{E}[(u(X) - v_X)^+]$ must overcompensate expected disappointment by a factor $1 + \beta$. Mathematically, setting $U = u(X)$ and $\alpha = (2 + \beta)^{-1}$, equation (3) shows that v_X is a solution of

$$\alpha \mathbb{E}[(U - v)^+] = (1 - \alpha) \mathbb{E}[(v - U)^+]. \quad (4)$$

²We tacitly assume suitable integrability in the Introduction.

This establishes the anticipated link between Gul’s theory (3) and asymmetric least squares estimation. In fact, Newey and Powell (1987) introduce the solutions of (4), denoted by $\mathbb{E}_\beta[U]$, as asymmetric least squares estimators. Under the name *expectiles*, these estimators became a popular tool in econometrics due to their robustness properties and statistical elicibility (e.g. Gneiting, 2011), and subsequently gained prominence as coherent measures of risk in finance and actuarial science (e.g. Bellini, Klar, Müller, Rosazza-Gianin, 2014, and Ziegel, 2016).

The original results of Newey and Powell guarantee that equation (3) always has a unique solution

$$v_X = \mathbb{E}_\beta[u(X)]. \quad (5)$$

In turn, our Proposition 1 shows that $\mathbb{E}_\beta[u(X)]$ is the only solution of Gul’s equation (1). Then we call $\mathbb{E}_\beta[u(X)]$ the *expectiled utility* of X (expectile of the utility), in analogy with the *expected utility* of X (expectation of the utility) which corresponds to $\beta = 0$.

Another direct application of the results of Newey and Powell leads to the representation

$$\mathbb{E}_\beta[u(X)] = \arg \min_{v \in \mathbb{R}} \mathbb{E} [\ell_\beta(u(X) - v)] \quad (6)$$

where $\ell_\beta(s) = s^2 + \beta s^2 1_{(-\infty, 0]}(s)$ is an asymmetric quadratic loss function. This representation recasts the theory of Gul in terms of (internal) utility estimation on part of a DM who anticipates asymmetric emotional responses to positive and negative deviations. Newey and Powell also show that $\mathbb{E}_\beta[\cdot]$ is positively homogeneous and constant-additive; furthermore, $\mathbb{E}_\beta[\cdot]$ is monotone and, for $\beta \geq 0$, superadditive, two important properties used in the mathematical finance literature.³ To the decision theorist, this says that expectiled utility is a specification of Gilboa and Schmeidler’s (1989) maxmin expected utility in a Savage framework.⁴ Our Theorem 3 characterizes expectiled utility within the class of these utilities:

$$\mathbb{E}_\beta[u(X)] = \min_{Q \in \mathcal{Q}_\beta} \mathbb{E}^Q[u(X)] \quad (7)$$

where

$$\mathcal{Q}_\beta = \left\{ Q : \frac{dQ}{dP} = \frac{1_{D^c} + (1 + \beta)1_D}{1 + \beta P(D)} \text{ for some } D \in \mathcal{F} \right\} \quad (8)$$

and an optimum is attained at Q^* given by

$$\frac{dQ^*}{dP} = \frac{1_{D_X^c} + (1 + \beta)1_{D_X}}{1 + \beta P(D_X)} \quad (9)$$

where $D_X = \{\omega : u(X(\omega)) < \mathbb{E}_\beta[u(X)]\}$.

The representation (7)–(9) admits a clear interpretation in terms of a (zero-sum) game against Nature.⁵ A disappointment-averse DM envisages an adversarial Nature who is able to distort the reference probability P within a set \mathcal{Q}_β of alternative distributions to minimize his expected utility. Anticipating this, the value of act X to the DM is given by (7). The set \mathcal{Q}_β , given by (8), explicitly captures these adversarial distributions, with Nature being able to increase the reference probability of any single event D by a factor of $(1 + \beta)$, up to a normalization constraint. Finally, the “optimal sabotage” on part of Nature, Q^* given by (9), precisely embodies the fears of the DM by making salient the *disappointing states* for act X , that is, the states ω in which the ex post utility $u(X(\omega))$ is inferior to the ex-ante value $\mathbb{E}_\beta[u(X)]$. Fearing disappointment, the DM constructs a *psychological armor* of weight β —represented by the set \mathcal{Q}_β —and consistently adopts decisions that are robust to this adversarial scenario.

³See, for instance, Bellini et al. (2014).

⁴Casadesus-Masanell, Klbanoff, and Ozdenoren (2000), Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2001, 2003), and Alon and Schmeidler (2014).

⁵See the classical Milnor (1954) and Luce and Raiffa (1957, p. 279), as well as Gilboa and Schmeidler (1989).

1.1.3 New axiomatizations

In the light of the above representations, the preferences \succsim of a DM adhering to Gul's theory have to satisfy some standard axioms on the set of all simple acts.⁶ First, these preferences are *probabilistically sophisticated*:

$$P(\omega : X(\omega) \succsim x) \geq P(\omega : Y(\omega) \succsim x) \text{ for all } x \in \mathcal{X} \text{ implies } X \succsim Y$$

with strict preference if the inequality is strict for some x (Machina and Schmeidler, 1992, p. 754). Second, these preferences are *invariant biseparable* in the sense of Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2001, henceforth GMMS).⁷ In particular, there exist a nonconstant and continuous function $u : \mathcal{X} \rightarrow \mathbb{R}$, and a monotone, positively homogeneous, and constant-additive functional I on the set of simple random variables such that, for all simple acts X and Y ,

$$X \succsim Y \iff I(u(X)) \geq I(u(Y)). \quad (10)$$

GMMS provide axioms characterizing this representation. Under the same axioms, for all outcomes x and y , it is possible to elicit (up to indifference) a *preference midpoint*

$$\frac{1}{2}x \oplus \frac{1}{2}y \text{ in } \mathcal{X} \text{ such that } u\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) = \frac{1}{2}u(x) + \frac{1}{2}u(y).$$

Moreover, GMMS show that the additional axiom needed to characterize *maxmin expected utility* within the class of invariant biseparable ones is:

$$X \sim Y \implies X \precsim \frac{1}{2}X \oplus \frac{1}{2}Y \quad (11)$$

called *ambiguity hedging*. The interpretation of (11) is literally the one of Schmeidler (1989, p. 582): "smoothing" or averaging utility distributions makes the DM better off.

In Section 4, we propose a new axiom, called *disappointment hedging*, that, together with probabilistic sophistication and invariant biseparability characterizes expected utility:

$$X \sim Y \implies \frac{1}{2}W \oplus \frac{1}{2}X \precsim \frac{1}{2}W \oplus \frac{1}{2}Y \quad (12)$$

for all simple acts W that have the same disappointment states as X , that is, $\{\omega : W(\omega) \prec W\} = \{\omega : X(\omega) \prec X\}$. This novel axiom admits a clear interpretation in terms of disappointment aversion. If W has the same disappointment states as X , then mixing X with W offers no protection: both acts disappoint in the same states. By contrast, mixing Y with W may offer some hedging benefits, thus making the mixture of Y with W preferable. Also observe that considering $W = X$ in (12) shows that disappointment hedging is a stronger axiom than ambiguity hedging.

In the GMMS framework, our Expected Utility Theorem (Theorem 4) shows that a binary relation \succsim between simple acts is probabilistically sophisticated, invariant biseparable, and disappointment hedging if, and only if, there exists a nonconstant and continuous function $u : \mathcal{X} \rightarrow \mathbb{R}$ and a number $\beta \geq 0$ such that

$$X \succsim Y \iff \mathbb{E}_\beta[u(X)] \geq \mathbb{E}_\beta[u(Y)]. \quad (13)$$

In this case, u is cardinally unique and $\beta \geq 0$ is unique. When the preference in (12) is reversed, we have *elation speculating* preferences and the representation (13) with $-1 < \beta \leq 0$ is characterized.

This result is the main and technically most demanding result of the paper, and it requires the creation of novel techniques to analyze attitudes toward different dependence structures among random variables.

⁶As customary in axiomatic decision theory, preferences are represented by a binary relation \succsim on the set of all acts that take a finite number of values in a connected, metric, and separable space \mathcal{X} , called *simple acts*. Moreover, in the tradition of Savage (1954), (Ω, \mathcal{F}, P) is assumed to be nonatomic, so that simple acts generate all simple lotteries.

⁷Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003) is the abridged published version.

1.1.4 Synopsis and additional results

The three main sections of this paper that we summarized above show that the theory of disappointment aversion of Gul (1991) can be “remastered” in a Savage framework with general outcomes, new explicit representations, and an axiomatization that directly appeals to the idea of robustifying decisions to take protection against disappointment (or exploit elation opportunities).

The final Section 5 shows that:

- as it happens with expected utility (a theorem of de Finetti, 1931), the axiomatization of expected utility can be simplified and made even more expressive when acts are directly represented in utils;
- comparative statics descends naturally, with greater β ’s corresponding to a stronger preference for constant acts (those that by definition cannot disappoint);
- computation of expected utility is made very efficient by applying our results;
- the axiom of disappointment hedging leads to Gul’s evaluations also beyond the realm of invariant biseparable preferences.

2 Framework

We adopt a Savage framework augmented with a reference probability P . Specifically:

- (Ω, \mathcal{F}, P) is a probability space of *states*;
- \mathcal{X} is a measurable space of (deterministic) *outcomes*;
- *acts* are random outcomes (measurable mappings) $X : \Omega \rightarrow \mathcal{X}$.

An act X represents a risky action the outcome of which is $X(\omega)$ if state ω occurs. Each act X induces a distribution $P_X = P \circ X^{-1}$ of outcomes, called *lottery* in the decision theory jargon. Beyond the advantages discussed in the introduction, representing risky actions as acts rather than lotteries allows for a natural description of dependence structures across different actions. When considering two acts simultaneously, the Savage framework provides a joint specification of outcomes across states, enabling the modelling of patterns that would be lost if acts were reduced to separate lotteries (their marginal distributions). This capability is crucial for our axiomatization since it allows us to specify when two acts disappoint the DM in exactly the same states.⁸ Furthermore, as in Machina and Schmeidler (1992), the probability measure P in the present framework can be interpreted as a subjective one.

Translated into this Savage framework, Gul’s theory describes a DM who evaluates act $X : \Omega \rightarrow \mathcal{X}$ through a solution v_X of

$$v = \mathbb{E} [k_v(X)] \tag{14}$$

where, $u : \mathcal{X} \rightarrow \mathbb{R}$ is a (measurable) *utility function*, $\beta \in (-1, \infty)$ a *disappointment aversion coefficient*, and $k_v : \mathbb{R} \rightarrow \mathbb{R}$ is the continuous transformation of u given by

$$k_v(x) = \begin{cases} \frac{u(x) + \beta v}{1 + \beta} & \text{if } u(x) \geq v \\ u(x) & \text{if } u(x) < v \end{cases}$$

⁸See the *disappointment hedging* axiom below.

for all $v \in \mathbb{R}$.

Note that, for $\beta \geq 0$,

$$u(x) \geq v \implies \beta v \leq \beta u(x) \implies \frac{u(x) + \beta v}{1 + \beta} \leq \frac{u(x) + \beta u(x)}{1 + \beta} = u(x).$$

Therefore, when $v = v_X$, the utility of disappointing outcomes, for which $u(x) < v_X$, is unaffected, while that of elating ones, for which $u(x) \geq v_X$, is discounted. Thus Gul's implicit representation captures the asymmetry between positive and negative deviations from the endogenously determined reference point v_X . This describes the full emotional impact of disappointment and the deflated one of elation. The coefficient β governs the strength of this asymmetry.

Nota Bene. For brevity, we focus mostly on the case of disappointment aversion, corresponding to $\beta \geq 0$. The case $-1 < \beta \leq 0$, which reflects elation seeking, admits analogous formal results and interpretive reversals. We address it only when the duality between disappointment aversion and elation seeking yields additional conceptual insight.

3 Expected utility representations

Our first result shows that, also in the framework that we introduced in the previous section, Gul's equation (14) always has a unique solution v_X , an internal equilibrium between elation and disappointment.⁹

Proposition 1 *If $\mathbb{E}[u(X)]$ exists finite, then the following conditions are equivalent for $v \in \mathbb{R}$:*

- v is a solution to Gul's equation (14);
- v is a solution of

$$\mathbb{E}[(u(X) - v)^+] = (1 + \beta) \mathbb{E}[(v - u(X))^+]. \quad (15)$$

In particular, v exists and is unique, denoted $\mathbb{E}_\beta[u(X)]$.

This formulation provides an alternative take on Gul's theory. Rather than working with a piecewise-transform k_v , Proposition 1 characterizes the DM's evaluation $\mathbb{E}_\beta[u(X)]$ of act X as the sure utility level v that balances expected elation, $\mathbb{E}[(u(X) - v)^+]$, and expected disappointment, $\mathbb{E}[(v - u(X))^+]$. The factor $(1 + \beta)$ explicitly measures the degree of emotional compensation required: expected elation must outweigh expected disappointment by such a factor to achieve internal equilibrium.

As anticipated in the introduction, $\mathbb{E}_\beta[u(X)]$ is the α -*expectile* of $u(X)$ in the sense of Newey and Powell (1987) for $\alpha = (2 + \beta)^{-1}$. Therefore, in analogy with the *expected utility* terminology for $\mathbb{E}[u(X)]$, expectation of the utility of X , we call $\mathbb{E}_\beta[u(X)]$ *expected utility* of X . The econometric connection that we have just established presents us with another representation that bears a conceptual nuance of subjective estimation.

Theorem 2 *If $\mathbb{E}[u(X)^2]$ exists finite, then*

$$\mathbb{E}_\beta[u(X)] = \arg \min_{v \in \mathbb{R}} \mathbb{E}[\ell_\beta(u(X) - v)],$$

where $\ell_\beta : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\ell_\beta(s) = \begin{cases} s^2 & \text{if } s \geq 0, \\ (1 + \beta)s^2 & \text{if } s < 0. \end{cases}$$

⁹In reading it, recall that $s^+ = \max\{s, 0\}$ denotes the positive part of a real number s .

Here the agent looks for the sure utility level that best approximates the random utility levels yielded by X . So doing, he acknowledges that ex-post utility gains and shortfalls will have different psychological “distance” from the ex-ante evaluation of the act. This formulation highlights the optimization nature of value formation. The DM behaves like an asymmetric least squares estimator: minimizing the average squared deviation of payoffs from a reference value, but doing so with a bias that mirrors emotional asymmetry. This perspective reframes the DM’s valuation of risk not as a fixed point solution, but as the result of a deliberate, internal optimization based on emotional awareness.

Proposition 1 and Theorem 2 describe the DM as an active mitigator of emotional distress aligning with neuroscientific evidence on nonlinear loss responses (e.g. Tom, Fox, Trepel, and Poldrack, 2007).

We conclude with another explicit representation, this time with a robustness flavor.

Theorem 3 *If $\mathbb{E}[u(X)]$ exists finite and $\beta \geq 0$, then*

$$\mathbb{E}_\beta[u(X)] = \min_{Q \in \mathcal{Q}_\beta} \mathbb{E}^Q[u(X)],$$

where

$$\mathcal{Q}_\beta = \left\{ Q : \frac{dQ}{dP} = \frac{1_{D^c} + (1 + \beta)1_D}{1 + \beta P(D)} \text{ for some } D \in \mathcal{F} \right\}.$$

and an optimum is attained at Q^* given by

$$\frac{dQ^*}{dP} = \frac{1_{D_X^c} + (1 + \beta)1_{D_X}}{1 + \beta P(D_X)} \quad (16)$$

where $D_X = \{\omega : u(X(\omega)) < \mathbb{E}_\beta[u(X)]\}$.

As anticipated (see Section 1.1.2), Theorem 3 reveals disappointment aversion as a *robust approach against fictitious adversarial scenarios*. This maxmin representation features:

- A game-theoretic interpretation. The DM views the decision problem as a game against a malevolent Nature that is able to distort probabilities, overweighting disappointment states, to minimize his utility.
- A “Psychological armor.” The coefficient β quantifies defensive preparedness. Higher β implies greater anticipated sabotage of favorable outcomes, but higher protection comes at an higher opportunity cost. Formally, this corresponds to the fact that $\mathbb{E}_0[u(X)] = \mathbb{E}[u(X)]$, $\mathbb{E}_\beta[u(X)]$ decreases as β increases, and the limit $\mathbb{E}_\infty[u(X)]$ is the essential infimum of $u(X)$ on Ω as $\beta \rightarrow \infty$.
- An explicit minimizer. The optimal sabotage Q^* magnifies the likelihood of states ω in which $u(X(\omega)) < \mathbb{E}_\beta[u(X)]$, making disappointment salient through probability inflation.

Denoting by \succsim the preference relation between acts X that is represented by $\mathbb{E}_\beta[u(X)]$, Theorem 3 provides an entirely new lens through which Gul’s theory can be understood. For each act X , the event

$$\{\omega : u(X(\omega)) < \mathbb{E}_\beta[u(X)]\} = \{\omega : X(\omega) \prec X\}$$

is the one in which the DM is disappointed. The emotional salience of this event induces the DM to overweight the probability of this set and to evaluate the expected utility of X with respect to the distorted probability Q^* given by (16) that inflates the likelihood of this event and correspondingly deflates that of its complement.

Theorem 3 provides the general perspective behind the fundamental intuition of Ghirardo and Marinacci (2001) regarding the behavior of a disappointment-averse DM who confronts bets (binary acts). Consider two outcomes $x \succ y$ and an act xEy that delivers x if E occurs and y otherwise, so that the elating states are the ones in E and the disappointing states the ones in $D = E^c$. Ghirardo and Marinacci show that

$$\mathbb{E}_\beta [u(xEy)] = \frac{P(E)}{1 + \beta P(D)} u(x) + (1 + \beta) \frac{P(D)}{1 + \beta P(D)} u(y) \quad (17)$$

highlighting how the probability of disappointing states is augmented and that of elating states reduced. The formula

$$\mathbb{E}_\beta [u(X)] = \mathbb{E}^{Q^*} [u(X)]$$

implied by Theorem 3 is the generalization of (17) to nonbinary acts.

4 Expected utility axioms

In this section, we consider a preference relation \succsim over the set \mathbb{X} of all acts that take a finite number of values in a connected, metric, and separable space \mathcal{X} , called *simple acts*. Moreover, in the tradition of Savage (1954), (Ω, \mathcal{F}, P) is assumed to be adequate, that is, either nonatomic or such that \mathcal{F} is generated by a finite partition over which P is uniform. The set of all simple random variables, which are \mathcal{F} -measurable real-valued functions that take finitely many values, is denoted by $B_0(\Omega, \mathcal{F})$. In this framework we provide an axiomatic characterization of the preferences on \mathbb{X} that admit an expected utility representation.

The result of the previous sections guarantee that expected utility preferences are probabilistically sophisticated and invariant biseparable. The first property, probabilistic sophistication (Machina and Schmeidler, 1992), is preferential form of first order stochastic dominance. It says that acts that deliver better outcomes with higher probability are preferred. Formally:

$$P(\omega : X(\omega) \succsim x) \geq P(\omega : Y(\omega) \succsim x) \text{ for all } x \in \mathcal{X} \text{ implies } X \succsim Y$$

with strict preference if the inequality is strict for some x .

The second property, *invariant biseparability*, requires the possibility of separating a cardinal utility from attitudes toward uncertainty (Ghirardo, Maccheroni, and Marinacci, 2005). Formally, this means that there exists a continuous nonconstant function $u : \mathcal{X} \rightarrow \mathbb{R}$ and a monotone, positively homogeneous, and constant-additive functional $I : B_0(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ such that, for all acts X and Y ,

$$X \succsim Y \iff I(u(X)) \geq I(u(Y)) \quad (18)$$

with $I(1_E) \in (0, 1)$ for some $E \in \mathcal{F}$. GMMS provide an axiomatization of invariant biseparable preferences in the framework of this section. The utility function u that they obtain is cardinally unique and I is unique. Recently, Castagnoli, Cattelan, Maccheroni, Tebaldi, and Wang (2022) and Chandrasekhar, Frick, Iijima, and Le Yaouanq (2022) provided some concrete representation of the functional I appearing in (18). When preferences are invariant biseparable, for all outcomes x and y , it is possible to elicit from betting behavior (that is from the restriction of \succsim to bets) a *preference midpoint*

$$\frac{1}{2}x \oplus \frac{1}{2}y \text{ in } \mathcal{X} \text{ such that } u\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) = \frac{1}{2}u(x) + \frac{1}{2}u(y),$$

which is unique up to indifference.

Remark. This was first shown by GMMS (Lemma 3), subsequently the theoretical and the experimental literatures presented alternative ways of obtaining $(1/2)x \oplus (1/2)y$ from the preference \succsim , and put this technology into action. We refer the reader to Köbberling and Wakker (2003), Abdellaoui, Bleichrodt, Paraschiv (2007), Baillon, Driesen, and Wakker (2012), Dean and Ortoleva (2017), Ghirardato and Pennesi (2020), Chateauneuf, Maccheroni, and Zank (2025).

The additional axiom required to characterize maxmin expected utility preferences within invariant biseparable ones is:

$$X \sim Y \implies X \precsim \frac{1}{2}X \oplus \frac{1}{2}Y \quad (19)$$

called *ambiguity hedging* (GMMS, Proposition 10). Our novel key axiom, *disappointment hedging*, is indeed a stronger version of it:

$$X \sim Y \implies \frac{1}{2}W \oplus \frac{1}{2}X \precsim \frac{1}{2}W \oplus \frac{1}{2}Y \quad (20)$$

for all simple acts W that have the same disappointment states as X , that is, $\{\omega : W(\omega) \prec W\} = \{\omega : X(\omega) \prec X\}$. We already discussed its interpretation in the introduction: if W has the same disappointment states as X , then mixing X with W offers no protection since both acts disappoint in the same states. By contrast, mixing Y with W may offer some hedging benefits, thus making the mixture of Y with W preferable. By reverting the preference in (20) we obtain a dual axiom, *elation speculating*, with the inverse meaning.

Theorem 4 (Expected Utility) *Let (Ω, \mathcal{F}, P) be an adequate probability space and \mathcal{X} be a connected and separable metric space. The following conditions are equivalent for a binary relation \succsim on \mathbb{X} :*

- (i) \succsim is probabilistically sophisticated, invariant biseparable, and disappointment hedging (resp. elation speculating);
- (ii) there exists a continuous and nonconstant function $u : \mathcal{X} \rightarrow \mathbb{R}$ and a number $\beta \geq 0$ (resp. $-1 < \beta \leq 0$) such that

$$X \succsim Y \iff \mathbb{E}_\beta[u(X)] \geq \mathbb{E}_\beta[u(Y)].$$

In this case, u is cardinally unique and β is unique.

This is the main theorem of this paper providing a novel axiomatic foundation of Gul's theory: in a general Savage framework, with transparent axioms that reveal the asymmetry between disappointment and elation, and supported by the explicit representations of the previous section.

5 Additional results

5.1 Expected value

The axiomatization of expected utility à la Savage (1954) descends from that of *expected value* of de Finetti (1931). Moreover, the latter builds on a set of even more transparent axioms. In this section we show that the same is true for expected utility (axiomatized in the previous section) and *expected value* (studied here). In this case the acts that the DM is facing are simple random variables, thus \mathbb{X} is $B_0(\Omega, \mathcal{F})$, constant acts are real numbers, and subjective mixtures (with \oplus) are replaced by usual averages (with $+$).

The following properties are considered for a binary relation \succsim on \mathbb{X} , where typical elements are denoted by U , V , and Z .

- *monotonicity*: given any $x, y \in \mathbb{R}$, $x \succsim y \iff x \geq y$;
- *continuity*: the upper and lower level sets of \succsim are closed in the L^∞ -norm;
- *disappointment aversion*: given any $U, V \in \mathbb{X}$,

$$U \sim V \implies Z + U \succsim Z + W$$

for all $Z \in \mathbb{X}$ that have the same disappointment states as U .

The latter is the only new axiom since the others are well known and well studied in the literature. Its interpretation is very clear, the DM shies the accumulation of disappointment. If $\{\omega : Z(\omega) \prec Z\} = \{\omega : U(\omega) \prec U\}$, and the DM holding Z acquires U then he runs the risk of being simultaneously disappointed by both his investments, V instead may allow some elating compensation when Z disappoints. An even more precise name for this axiom would be *disappointment stacking aversion*, we opted for the shorter disappointment aversion for brevity. The obvious dual axiom is called *elation (stacking) seeking*.

Theorem 5 (Expected Value) *Let (Ω, \mathcal{F}, P) be an adequate probability space. The following conditions are equivalent for a binary relation \succsim on $\mathbb{X} = B_0(\Omega, \mathcal{F})$:*

- (i) \succsim is monotone, continuous, probabilistically sophisticated, and disappointment averse (resp. elation seeking);
- (ii) there exists a number $\beta \geq 0$ (resp. $1 < \beta \leq 0$) such that
$$U \succsim V \iff \mathbb{E}_\beta[U] \geq \mathbb{E}_\beta[V];$$
- (iii) \succsim is monotone, continuous, probabilistically sophisticated, and disappointment hedging (resp. elation speculating).¹⁰

In this case, β is unique.

5.2 Comparative statics

The next theorem, a variation of Gul (1991, p. 676) that we present for completeness, shows that, for expected utility preferences, comparative statics à la Yaari (1969) and Ghirardato and Marinacci (2002) amounts to comparison of the magnitudes of the disappointment aversion parameters.

Theorem 6 *For two expected utility preferences \succsim_A and \succsim_B , the following conditions are equivalent:*

- (i) given any act X and any constant act y ,

$$X \succsim_A y \implies X \succsim_B y \tag{21}$$

and the two preferences share the same preference midpoints;

- (ii) $\beta_A \geq \beta_B$ and u_A is a strictly increasing affine transformation of u_B .

In particular, under the assumptions of Theorem 5, condition (21) is equivalent to $\beta_A \geq \beta_B$, because the other involved conditions are automatically satisfied. Classically, condition (21) has been interpreted as capturing greater uncertainty aversion, or greater risk aversion in the lottery setting. By requiring the two preferences have the same preference midpoints, that the utilities to be affinely related, we shut down the classical risk aversion component, and the remaining component of greater uncertainty aversion is fully captured by the different magnitudes of disappointment aversion, with the interpretation that a more disappointment-averse DM is more attached to constant acts because they cannot disappoint.

¹⁰With $\oplus = +$.

5.3 Iterative computation of expected utility

Newey and Powell (1987) mentioned that expectiles can be conveniently computed by an iterative reweighting procedure. First, the mean is computed. Then, the probability masses of the realizations falling short of the mean are reweighted by a factor $1 + \beta$. Finally, the total sum of probabilities is normalized to 1, and the mean is recalculated.

Theorem 7 *Let $X \in \mathbb{X}$ be a simple act that takes values $x_1, x_2, \dots, x_n \in \mathcal{X}$, with corresponding utilities $u_i = u(x_i) \in \mathbb{R}$, and probabilities $p_i > 0$, with $\sum_{i=1}^n p_i = 1$. For each $\beta \in (-1, \infty)$, the sequence defined by*

$$\left\{ \begin{array}{l} v^{(0)} = \sum_{i=1}^n p_i u_i = \mathbb{E}[u(X)], \\ v^{(k)} = \sum_{i=1}^n \tilde{p}_i^{(k)} u_i \quad \text{for } k = 1, 2, \dots, \\ \text{where} \\ w_i^{(k)} = \begin{cases} p_i & \text{if } u_i \geq v^{(k-1)}, \\ (1 + \beta)p_i & \text{if } u_i < v^{(k-1)}, \end{cases} \\ \tilde{p}_i^{(k)} = \frac{w_i^{(k)}}{\sum_{j=1}^n w_j^{(k)}} \end{array} \right.$$

converges monotonically, in at most $n - 1$ steps, to the expected utility $\mathbb{E}_\beta[u(X)]$ of X .

Note that each iteration reweights disappointing events (those where $u_i < v^{(k-1)}$) more heavily, reflecting the emotional asymmetry encoded by β . Since there are only finitely many distinct utility values u_i , the sequence stabilizes when the set of disappointing outcomes stops changing, thus leading to finite convergence.

The above procedure illustrates that the game against Nature that the DM is playing as discussed after Theorem 3 can also be thought of as a sequential game: every round, the DM computes the expected value of the utility as a benchmark, and Nature reweights all disappointing states by $1 + \beta$. The DM computes the expectation again, and this procedure yields $\mathbb{E}_\beta[u(X)]$ in at most $n - 1$ steps. It is immediate to see that, for bets, the procedure yields formula (17) of Ghirardato and Marinacci (2001) in one single step. Therefore, like Theorem 3, also Theorem 7 can be seen as an extension of (17), now with constructive content.

5.4 Expected utility under weaker assumptions

In Section 4, our characterization of expected utility builds on the assumption of invariant biseparability, which requires that the preference relation admits a representation of the form $X \mapsto I(u(X))$, where $u : \mathcal{X} \rightarrow \mathbb{R}$ is cardinally unique, continuous, and nonconstant, and $I : B_0(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ is monotone, positively homogeneous, and constant-additive (*a fortiori*, continuous). To highlight the scope our disappointment hedging axiom, we next show that a characterization of expected utility can be obtained within the broader class of preference relations that admit a representation of the form $X \mapsto I(u(X))$, where $u : \mathcal{X} \rightarrow \mathbb{R}$ is cardinally unique, continuous, and nonconstant, and $I : B_0(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ is only required to be monotone, continuous, and normalized.¹¹ These *separable preferences* include most of the preferences relations studied in decision theory under uncertainty.

Theorem 8 *Let (Ω, \mathcal{F}, P) be an adequate probability space and \mathcal{X} be a connected and separable metric space. The following conditions are equivalent for a separable preference \succsim on \mathbb{X} with representation $X \mapsto I(u(X))$:*

¹¹That is, such that $I(v) = v$ for all $v \in \mathbb{R}$.

(i) \succsim is probabilistically sophisticated and disappointment hedging (resp. elation speculating) with midpoints defined by

$$u\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) = \frac{1}{2}u(x) + \frac{1}{2}u(y) \quad (22)$$

for all $x, y \in \mathcal{X}$;

(ii) there exists a number $\beta \geq 0$ (resp. $-1 < \beta \leq 0$) such that $I = \mathbb{E}_\beta$.

This final result shows that the substantive axioms that deliver Gul's theory are indeed disappointment aversion and probabilistic sophistication. The latter assumption is relaxed in the companion paper Bellini, Mao, Wang, and Wu (2024).

Final remark. Preference midpoints represented as in (22) naturally emerge in measurement theories. For instance, in social welfare analysis and medical decision making it is important to measure how much better or worse one option is than another for the DM. In these fields, a pair of outcomes $(x, y) \in \mathcal{X} \times \mathcal{X}$ describes the DM's option of replacing outcome y by outcome x ; and, in addition to the preference relation $x \succsim y$ on \mathcal{X} , a preorder

$$(x, y) \succcurlyeq (z, w)$$

on $\mathcal{X} \times \mathcal{X}$ represents that replacing y by x is at least as good as that of replacing w by z . After the foundational work of Krantz, Luce, Suppes, and Tversky (1971), Shapley (1975) provides a beautiful axiomatic characterization of pairs (\succsim, \succcurlyeq) jointly represented by a continuous and nonconstant utility function $u : \mathcal{X} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} x \succsim y &\iff u(x) \geq u(y), \\ (x, y) \succcurlyeq (z, w) &\iff u(x) - u(y) \geq u(z) - u(w). \end{aligned}$$

In this case, u is cardinally unique, and the preference midpoints $z = (1/2)x \oplus (1/2)y$ of x and y are characterized by

$$(x, z) \approx (z, y)$$

with simple algebra and obvious interpretation.

A Properties of expectiles

In this section, we provide several properties of expectiles that will be used in the proofs of our results.

Lemma 9 *Let $X, Y : \Omega \rightarrow \mathbb{R}$ be two random variables, and $\beta > -1$. We have the following properties of the expectile \mathbb{E}_β .*

- (i) *Law invariance: If $P_X = P_Y$, then $\mathbb{E}_\beta[X] = \mathbb{E}_\beta[Y]$;*
- (ii) *Strong monotonicity: If $P(X \geq t) \geq P(Y \geq t)$ for all $t \in \mathbb{R}$, then $\mathbb{E}_\beta[X] \geq \mathbb{E}_\beta[Y]$, with strict inequality if the inequality is strict for some t ;*
- (iii) *Constant additivity: For any $m \in \mathbb{R}$, $\mathbb{E}_\beta[X + m] = \mathbb{E}_\beta[X] + m$;*

- (iv) *Positive homogeneity*: For any $\lambda \geq 0$, $\mathbb{E}_\beta[\lambda X] = \lambda \mathbb{E}_\beta[X]$;
- (v) *Superadditivity (resp. subadditivity)*: If $\beta \geq 0$ (resp. $\beta \in (-1, 0]$), then $\mathbb{E}_\beta[X + Y] \geq \mathbb{E}_\beta[X] + \mathbb{E}_\beta[Y]$ (resp. $\mathbb{E}_\beta[X + Y] \leq \mathbb{E}_\beta[X] + \mathbb{E}_\beta[Y]$);
- (vi) *Bounded a.s. continuity*: If a sequence $\{X_n\}_{n \in \mathbb{N}}$ converges to X bounded a.s., then $\mathbb{E}_\beta[X_n] \rightarrow \mathbb{E}_\beta[X]$;
- (vii) *Additivity in concordant sums*: If $\{\omega : X(\omega) < \mathbb{E}_\beta[X]\} = \{\omega : Y(\omega) < \mathbb{E}_\beta[Y]\}$, then $\mathbb{E}_\beta[X + Y] = \mathbb{E}_\beta[X] + \mathbb{E}_\beta[Y]$;
- (viii) *Dual representation*: We have the following representation:

$$\mathbb{E}_\beta[X] = \begin{cases} \min_{\varphi \in \mathcal{M}_\beta} \mathbb{E}[\varphi X] & \text{if } \beta \geq 0, \\ \max_{\varphi \in \mathcal{M}_\beta} \mathbb{E}[\varphi X] & \text{if } \beta \in (-1, 0], \end{cases}$$

where

$$\mathcal{M}_\beta = \left\{ \varphi \in L^\infty : \varphi > 0 \text{ a.s., } \mathbb{E}[\varphi] = 1, \frac{\text{ess sup } \varphi}{\text{ess inf } \varphi} \leq \gamma(\beta) \right\},$$

with $\gamma(\beta) = \max\{1 + \beta, 1/(1 + \beta)\}$. Moreover, an optimal scenario $\bar{\varphi}$ is given by

$$\bar{\varphi} := \frac{1_{\{X > \mathbb{E}_\beta[X]\}} + (1 + \beta)1_{\{X \leq \mathbb{E}_\beta[X]\}}}{\mathbb{E}\left[1_{\{X > \mathbb{E}_\beta[X]\}} + (1 + \beta)1_{\{X \leq \mathbb{E}_\beta[X]\}}\right]}.$$

Proof. The properties (i) and (iii)-(v) hold as expectile is a coherent risk measure (see e.g. Corollary 4.6 of Ziegel, 2016). The properties (vi), (vii) and (viii) follow from Theorem 10 of Bellini et al. (2014), Theorem 3 of Bellini et al. (2021) and Proposition 8 of Bellini et al. (2014), respectively. To see the property (ii), if $P(X \geq t) \geq P(Y \geq t)$ for all $t \in \mathbb{R}$, then, for $v = \mathbb{E}_\beta[X]$ and $v' > v$, we have

$$\begin{aligned} \mathbb{E}[(Y - v')^+] &\leq \mathbb{E}[(Y - v)^+] \leq \mathbb{E}[(X - v)^+] \\ &= (1 + \beta)\mathbb{E}[(v - X)^+] \\ &\leq (1 + \beta)\mathbb{E}[(v - Y)^+] < (1 + \beta)\mathbb{E}[(v' - Y)^+], \end{aligned}$$

where the last strict equality follows from $P(Y < v') \geq P(Y \leq v) \geq P(X \leq v) > 0$. This shows that $\mathbb{E}_\beta[Y] > v$ cannot hold. Therefore, $\mathbb{E}_\beta[X] \geq \mathbb{E}_\beta[Y]$. To see the strict inequality under strict dominance, note that if $\mathbb{E}_\beta[Y] = v$, then at least one of the two inequalities in

$$\begin{aligned} (1 + \beta)\mathbb{E}[(v - Y)^+] &= \mathbb{E}[(Y - v)^+] \leq \mathbb{E}[(X - v)^+] \\ &= (1 + \beta)\mathbb{E}[(v - X)^+] \leq (1 + \beta)\mathbb{E}[(v - Y)^+] \end{aligned}$$

is strict, a contradiction. □

B Characterize expectiles within functionals

Recall the properties of expectiles presented in Lemma 9. In this section, we establish a characterization theorem showing that expectiles are the only class of functionals that satisfy strong monotonicity, constant additivity, positive homogeneity, and additivity in concordant sums.

Theorem 10 *A functional $I : B_0(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ is strongly monotone, constant-additive, positively homogeneous, and additive in concordant sums if and only if there exists $\beta > -1$ such that $I = \mathbb{E}_\beta$.*

Theorem 10 does not constrain the sign of the parameter β , as additivity in concordant sums has an additive structure. To capture the effect of concordance, we introduce the two properties of monotonicity in concordant sums. A functional $I : B_0(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ is *decreasing* (resp. *increasing*) in concordant sums if, for all $X, Y, W \in B_0(\Omega, \mathcal{F})$ such that $I(X) = I(Y)$ and $\{\omega : X(\omega) < I(X)\} = \{\omega : W(\omega) < I(W)\}$, it holds that $I(W + X) \leq I(W + Y)$ (resp. $I(W + X) \geq I(W + Y)$).

The next result, fundamental to the proofs of Theorems 4 and 5, provides an alternative characterization of expectiles in which decrease or increase in concordant sums serves as a key axiom, alongside two standard conditions: strong monotonicity and L^∞ -norm continuity. Specifically, L^∞ -norm continuity refers to the property that $I(X_n) \rightarrow I(X)$ if $X_n \rightarrow X$ in L^∞ , and it can be deduced from strong monotonicity and constant additivity (see e.g. Lemma 4.3 in Föllmer and Schied, 2016).

Theorem 11 *Let $I : B_0(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ satisfy $I(v) = v$ for all $v \in \mathbb{R}$. It is strongly monotone, L^∞ -norm continuous, and decreasing (resp. increasing) in concordant sums if and only if there exists $\beta \geq 0$ (resp. $\beta \in (-1, 0]$) such that $I = \mathbb{E}_\beta$.*

Theorem 11 can be extended to the case where I is defined on the set of acts supported on a nonempty interval $A \subseteq \mathbb{R}$ with 0 in its interior. In this setting, the properties of decrease and increase in concordant sums need to be slightly modified, since an interval is generally not a linear space. Specifically, denote by $B_0(\Omega, \mathcal{F}, A)$ the set of all simple random variables taking values in A . A functional $I : B_0(\Omega, \mathcal{F}, A) \rightarrow \mathbb{R}$ is *decreasing* (resp. *increasing*) in concordant mixtures if, for all $X, Y, W \in B_0(\Omega, \mathcal{F}, A)$ such that $I(X) = I(Y)$ and $\{\omega : X(\omega) < I(X)\} = \{\omega : W(\omega) < I(W)\}$, it holds that

$$I\left(\frac{W + X}{2}\right) \leq I\left(\frac{W + Y}{2}\right) \quad \left(\text{resp. } I\left(\frac{W + X}{2}\right) \geq I\left(\frac{W + Y}{2}\right)\right).$$

Theorem 12 *Let $A \subseteq \mathbb{R}$ be an interval with 0 in its interior. Suppose that $I : B_0(\Omega, \mathcal{F}, A) \rightarrow \mathbb{R}$ satisfies $I(v) = v$ for all $v \in A$. It is strongly monotone, L^∞ -norm continuous, and decreasing (resp. increasing) in concordant mixtures if and only if there exists $\beta \geq 0$ (resp. $\beta \in (-1, 0]$) such that $I = \mathbb{E}_\beta$.*

Below, we present the complete proofs of Theorems 10, 11 and 12. For an act X and a functional I , we define

$$D_I(X) = \{\omega : X(\omega) < I(X)\}. \quad (23)$$

In the proofs of Theorems 11 and 12, we focus on the case of decrease in concordant counterpart, as the increasing case can be treated analogously.

Proof of Theorem 10. The sufficiency is established in Lemma 9. We now turn to the necessity and address it in two parts, by considering first the finite case and then the infinite case of Ω .

Proof for the case of finite Ω . Assume that $\Omega = \{\omega_1, \dots, \omega_n\}$ and $P(\omega_i) = 1/n$ for all $i \in [n]$. Since I satisfies constant additivity and positive homogeneity, it is natural to focus on the study of the following set:

$$\mathcal{X}^0 = \{X \in B_0(\Omega, \mathcal{F}) : I(X) = 0\}.$$

To make use of additivity in concordant sums, we aim to construct acts X and Y such that $\{\omega : X(\omega) \geq I(X)\} = \{\omega : Y(\omega) \geq I(Y)\}$. This motivates to introduce the following subset:

$$\mathcal{X}_S = \{X \in \mathcal{X}^0 : \{X \geq 0\} = S\}, \quad S \in \mathcal{F}. \quad (24)$$

It is clear that $\mathcal{X}^0 = \bigcup_{S \in \mathcal{F}} \mathcal{X}_S$. Note that positive homogeneity implies that $I(0) = 0$, and thus, combining with strong monotonicity yields $\mathcal{X}_\Omega = \{0\}$, $\mathcal{X}_\emptyset = \emptyset$, and $\mathcal{X}_S \neq \emptyset$ for all $S \in \mathcal{F} \setminus \{\Omega, \emptyset\}$. We proceed the rest proof in three steps.

- (a) For any $S \in \mathcal{F}$, there exist measures P_S and Q_S on (Ω, \mathcal{F}) such that $P_S(S)Q_S(S^c) > 0$ for all $S \in \mathcal{F}$ and $\mathbb{E}^{P_S}[X_+] = \mathbb{E}^{Q_S}[X_-]$ for all $X \in \mathcal{X}_S$. This step is the most challenging.
- (b) Note that strong monotonicity imply law invariance. We use law invariance to show that $P_S(\omega) = \lambda_S$ for all $\omega \in S$ and $Q_S(\omega) = \eta_S$ for all $\omega \in S^c$, where λ_S, η_S are two constants. Moreover, $\lambda_{S_1}/\eta_{S_1} = \lambda_{S_2}/\eta_{S_2}$ if $|S_1| = |S_2|$, where $|S|$ is the cardinality of S .
- (c) Note that strong monotonicity and constant additivity together imply the continuity of I with respect to uniform convergence (see, e.g. Lemma 4.3 of Föllmer and Schied, 2016), that is, $\text{ess sup } |X_n - X| \rightarrow 0$ implies $I(X_n) \rightarrow I(X)$. We apply this continuity property to deduce that λ_S/η_S must be constant for any $S \in \mathcal{F}$. This observation uniquely determines the form of I , implying that $I = \mathbb{E}_\beta$ for some $\beta > -1$.

(a) The cases that $S = \Omega$ and $S = \emptyset$ are trivial. For $S \in \mathcal{F} \setminus \{\Omega, \emptyset\}$, denote by $a_S = I(1_S)$, and strong monotonicity of I yields $a_S \in (0, 1)$. Based on this, we define a nonempty set as

$$\mathcal{X}_S^+ = \{X \in B_0(\Omega, \mathcal{F}) : 0 < \max X < \min X/a_S\},$$

and a functional on \mathcal{X}_S^+ as

$$\phi_S(X) := I(X1_S), \quad X \in \mathcal{X}_S^+.$$

Since $\max(X + Y) \leq \max X + \max Y$ and $\min(X + Y) \geq \min X + \min Y$, it is straightforward to verify that \mathcal{X}_S^+ is a linear space. We also have that $\phi_S(\lambda) = \lambda a_S$ for all $\lambda \geq 0$ by the positive homogeneity of I . For any $X \in \mathcal{X}_S^+$, we have

$$0 < I(X1_S) \leq I((\max X)1_S) = \phi_S(\max X) = (\max X)a_S < \min X,$$

where the first and the second inequalities follow from the strong monotonicity of I . This gives

$$D_I(X1_S) = S^c, \quad X \in \mathcal{X}_S^+. \quad (25)$$

Therefore,

$$\phi_S(X + Y) = I(X1_S + Y1_S) = I(X1_S) + I(Y1_S) = \phi_S(X) + \phi_S(Y), \quad \forall X, Y \in \mathcal{X}_S^+,$$

where we have used additivity in concordant sums in the second equality. Hence, ϕ_S is additive on \mathcal{X}_S^+ . Note that for any $X \in B_0(\Omega, \mathcal{F})$, $X + m \in \mathcal{X}_S^+$ for large enough $m > 0$. This guarantees to extend ϕ_S to $B_0(\Omega, \mathcal{F})$ by defining

$$\widehat{\phi}_S(X) = \phi_S(X + m_X + 1) - (m_X + 1)a_S, \quad \text{with } m_X = \inf\{m : X + m \in \mathcal{X}_S^+\},$$

for $X \in B_0(\Omega, \mathcal{F})$. We clarify that $X + m_X$ may not belong to \mathcal{X}_S^+ when \mathcal{X}_S^+ is an open set. Therefore, we use $X + m_X + 1$ to ensure it lies within the domain of ϕ_S . We assert that

$$\widehat{\phi}_S(X) = \phi_S(X + m + 1) - (m + 1)a_S \quad \text{for any } m > m_X. \quad (26)$$

To see this, for $m > m_X$,

$$\begin{aligned} \phi_S(X + m + 1) - (m + 1)a_S &= \phi_S(X + m_X + 1 + m - m_X) - (m + 1)a_S \\ &= \phi_S(X + m_X + 1) + \phi_S(m - m_X) - (m + 1)a_S \\ &= \phi_S(X + m_X + 1) + (m - m_X)a_S - (m + 1)a_S \\ &= \phi_S(X + m_X + 1) - (m_X + 1)a_S = \widehat{\phi}_S(X), \end{aligned}$$

where the second equality follows from the additivity of ϕ_S . For any $X, Y \in B_0(\Omega, \mathcal{F})$, it holds that

$$X + m_X + 1, Y + m_Y + 1, X + Y + m_X + m_Y + 2 \in \mathcal{X}_S^+.$$

Hence,

$$\begin{aligned}\widehat{\phi}_S(X + Y) &= \phi_S(X + Y + m_X + m_Y + 2) - (m_X + m_Y + 2)a_S \\ &= \phi_S(X + m_X + 1) + \phi_S(Y + m_Y + 1) - (m_X + m_Y + 2)a_S \\ &= \widehat{\phi}_S(X) + \widehat{\phi}_S(Y),\end{aligned}$$

where the second equality follows from the additivity of ϕ_S . This implies that $\widehat{\phi}_S : B_0(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ is additive. Note that $\phi_S : \mathcal{X}_S^+ \rightarrow \mathbb{R}$ is monotone as I is strongly monotone. By the representation of $\widehat{\phi}_S$ in (26), we have that $\widehat{\phi}_S$ is also monotone. Hence, there exists a measure P_S on (Ω, \mathcal{F}) such that $\widehat{\phi}_S(X) = \mathbb{E}^{P_S}[X]$ for $X \in B_0(\Omega, \mathcal{F})$, and we have

$$\phi_S(X) = I(X1_S) = \mathbb{E}^{P_S}[X], \quad \forall X \in \mathcal{X}_S^+.$$

It is straightforward to verify that $P_S(S^c) = 0$ as $X1_S = 0$ on S^c for any $X \in \mathcal{X}_S^+$. Then, we claim that $P_S(S) > 0$. Let $X = (2 + a_S)1_S + 3a_S1_{S^c}$ and $Y = (1 + 2a_S)1_S + 3a_S1_{S^c}$. It holds that $X, Y \in \mathcal{X}_S^+$ and

$$\begin{aligned}\phi_S(X) &= I(X1_S) = \mathbb{E}^{P_S}[X] = P_S(S)(2 + a_S); \\ \phi_S(Y) &= I(Y1_S) = \mathbb{E}^{P_S}[Y] = P_S(S)(1 + 2a_S).\end{aligned}$$

Since $a_S \in (0, 1)$, one can verify that $X1_S \geq Y1_S$, and $X1_S > Y1_S$ on S . Strong monotonicity yields $I(X1_S) > I(Y1_S)$, and thus, $P_S(S)(2 + a_S) > P_S(S)(1 + 2a_S)$, which in turn implies that $P_S(S) > 0$.

Next, we turn to the negative counterpart, which follows similarly from the preceding arguments. Hence, we provide a brief proof for this case. Recall that $a_S = I(1_S) \in (0, 1)$. Define

$$\mathcal{X}_S^- = \{X \in B_0(\Omega, \mathcal{F}) : \max X / (1 - a_S) < \min X < 0\}$$

and

$$\psi_S(X) := I(X1_{S^c}), \quad X \in \mathcal{X}_S^-.$$

We aim to show that $\psi_S(X) = \mathbb{E}^{Q_S}[X]$ for $X \in \mathcal{X}_S^-$ with some measure Q_S on (Ω, \mathcal{F}) . It follows from constant additivity of I that $I(1_S - 1) = I(1_S) - 1 = a_S - 1$. Hence, we have

$$\psi_S(m) = I((-m)(1_S - 1)) = (-m)I(1_S - 1) = (1 - a_S)m, \quad m < 0,$$

where the third equality follows from the positive homogeneity of I . For any $X \in \mathcal{X}_S^-$, we have

$$0 > I(X1_{S^c}) \geq I((\min X)(1 - 1_S)) = \psi(\min X) = (1 - a_S) \min X > \max X.$$

where the first step and the second inequalities follow from the strong monotonicity of I . This implies that

$$D_I(X1_{S^c}) = S^c, \quad X \in \mathcal{X}_S^-. \tag{27}$$

By the similar arguments in the previous, we can extend ψ_S to $B_0(\Omega, \mathcal{F})$ by defining

$$\widehat{\psi}_S(X) = \phi_S(X - m_X - 1) + (m_X + 1)a_S, \quad \text{with } m_X = \inf\{m : X - m \in \mathcal{X}_S^-\},$$

for $X \in B_0(\Omega, \mathcal{F})$. Similarly, one can check that $\widehat{\psi}_S$ is monotone and additive. Hence, there exists a measure Q_S on (Ω, \mathcal{F}) such that $\widehat{\psi}_S = \mathbb{E}^{Q_S}$, and this implies

$$\psi_S(X) = I(X1_{S^c}) = \mathbb{E}^{Q_S}[X], \quad \forall X \in \mathcal{X}_S^-.$$

We can also conclude that $Q_S(S) = 0$ and $Q_S(S^c) > 0$ for $S \in \mathcal{F}$.

We now assume that $X \in \mathcal{X}_S$. Choose large enough $\eta > 0$ such that $X - \eta \in \mathcal{X}_S^-$ and $X + \eta \in \mathcal{X}_S^+$. By the definition of \mathcal{X}_S in (24), and combining with (25) and (27), we have that $S^c = D_I(Z)$ for all Z to be the following acts:

$$X, \quad -\eta 1_{S^c}, \quad \eta 1_S, \quad (X - \eta)1_{S^c}, \quad (X + \eta)1_S, \quad -\eta 1_{S^c} + \eta 1_S.$$

Hence, the following equality chains holds:

$$\begin{aligned} -\mathbb{E}^{Q_S}[\eta] + \mathbb{E}^{P_S}[\eta] &= I(X) + \psi_S(-\eta) + \phi_S(\eta) \\ &= I(X) + I(-\eta 1_S) + I(\eta 1_S) = I((X - \eta)1_{S^c} + (X + \eta)1_S) \\ &= I((X - \eta)1_{S^c}) + I((X + \eta)1_S) = \psi_S(X - \eta) + \phi_S(X + \eta) \\ &= \mathbb{E}^{Q_S}[X] + \mathbb{E}^{P_S}[X] - \mathbb{E}^{Q_S}[\eta] + \mathbb{E}^{P_S}[\eta] \\ &= -\mathbb{E}^{Q_S}[X_-] + \mathbb{E}^{P_S}[X_+] - \mathbb{E}^{Q_S}[\eta] + \mathbb{E}^{P_S}[\eta], \end{aligned}$$

where the first equality follows from $I(X) = 0$ as $X \in \mathcal{X}_S$, and we have used additivity in concordant sums of I in the third and fourth equalities, and the last equality holds because $P_S(S^c) = Q_S(S) = 0$. This completes the step (a).

(b) From the step (a), we have concluded that

$$\mathbb{E}^{P_S}[X_+] = \mathbb{E}^{Q_S}[X_-], \quad X \in \mathcal{X}_S, \quad S \in \mathcal{F}, \quad (28)$$

where $P_S(S) > 0$ and $Q_S(S^c) > 0$ for $S \in \mathcal{F}$. Fix $S \in \mathcal{F}$ with $|S| \geq 2$, let $\theta_1, \theta_2 \in S$ with $\theta_1 \neq \theta_2$, and we aim to verify $P_S(\theta_1) = P_S(\theta_2)$. Let $X \in \mathcal{X}_S$ satisfying $X(\theta_1) \neq X(\theta_2)$. Define $Y = X1_{\Omega \setminus \{\theta_1, \theta_2\}} + X(\theta_1)1_{\theta_2} + X(\theta_2)1_{\theta_1}$. Note that strong monotonicity implies law invariance, which further yields $I(Y) = I(X) = 0$ as $P_Y = P_X$. By the definition of \mathcal{X}_S and noting that $\theta_1, \theta_2 \in S$, we have $X(\theta_1), X(\theta_2) \geq 0$, and hence, $\{Y \geq 0\} = \{X \geq 0\} = S$. Therefore, we have concluded that $Y \in \mathcal{X}_S$. Substituting both X and Y into (28), we have

$$\begin{aligned} \sum_{\omega \in S \setminus \{\theta_1, \theta_2\}} X(\omega)P_S(\omega) + P_S(\theta_1)X(\theta_1) + P_S(\theta_2)X(\theta_2) &= \sum_{\omega \in S^c} |X(\omega)|Q_S(\omega); \\ \sum_{\omega \in S \setminus \{\theta_1, \theta_2\}} X(\omega)P_S(\omega) + P_S(\theta_1)X(\theta_2) + P_S(\theta_2)X(\theta_1) &= \sum_{\omega \in S^c} |X(\omega)|Q_S(\omega). \end{aligned}$$

This yields

$$P_S(\theta_1)(X(\theta_1) - X(\theta_2)) = P_S(\theta_2)(X(\theta_1) - X(\theta_2)),$$

and thus, $P_S(\theta_1) = P_S(\theta_2)$. Therefore, there exists $\lambda_S > 0$ such that $P_S(\omega) = \lambda_S$ for all $\omega \in S$. Following similar arguments, we can also verify that there exists $\eta_S > 0$ such that $Q_S(\omega) = \eta_S$ for all $\omega \in S^c$. Suppose now that $|S_1| = |S_2|$. Let $X = a1_{S_1} - b1_{S_1^c}$ satisfying $I(X) = 0$ with some $a, b > 0$. It is immediate to get $X \in \mathcal{X}_{S_1}$. Define $Y = a1_{S_2} - b1_{S_2^c}$. By law invariance of I , we have $I(Y) = I(X) = 0$, and thus, $Y \in \mathcal{X}_{S_2}$. Substituting X and Y into (28), we have

$$\begin{aligned} a|S_1|\lambda_{S_1} &= aP_{S_1}(S_1) = bQ_{S_1}(S_1^c) = b|S_1^c|\eta_{S_1}; \\ a|S_2|\lambda_{S_2} &= aP_{S_2}(S_2) = bQ_{S_2}(S_2^c) = b|S_2^c|\eta_{S_2}. \end{aligned}$$

Combining with $|S_1| = |S_2|$ yields $\lambda_{S_1}/\eta_{S_1} = \lambda_{S_2}/\eta_{S_2}$, and this completes the proof of the step (b).

(c) From step (b), we have shown that $P_S(\omega) = \lambda_S$ for all $\omega \in S$ and $Q_{S^c}(\omega) = \eta_S$ for all $\omega \in S^c$, and $\overline{S} \mapsto \lambda_S/\eta_S$ is a constant mapping on the sets $\{S \in \mathcal{F} : |S| = k\}$ for each $k \in [n]$. In this step, we aim to verify that $S \mapsto \lambda_S/\eta_S$ is a constant mapping on \mathcal{F} . To see this, it suffices to check that $\lambda_{S_1}/\eta_{S_1} = \lambda_{S_2}/\eta_{S_2}$ for $S_1 = \{\omega_1, \dots, \omega_s\}$ and $S_2 = \{\omega_1, \dots, \omega_t\}$ with $s < t$. Define

$$X_\epsilon = 1_{S_1} - \epsilon 1_{S_2 \setminus S_1} - f(\epsilon) 1_{S_2^c}, \quad \epsilon \geq 0,$$

where $f(\epsilon)$ is a function such that $I(X_\epsilon) = 0$. Indeed, if $\epsilon = 0$, then $X_\epsilon \in \mathcal{X}_{S_2}$, and applying (28) yields

$$f(0) = P_{S_2}(S_1)/Q_{S_2}(S_2^c). \quad (29)$$

For $0 < \epsilon < P_{S_1}(S_1)/Q_{S_1}(S_2 \setminus S_1)$, we have $X_\epsilon \in \mathcal{X}_{S_1}$, and using (28) yields

$$f(\epsilon) = \frac{P_{S_1}(S_1) - \epsilon Q_{S_1}(S_2 \setminus S_1)}{Q_{S_1}(S_2^c)} > 0. \quad (30)$$

Since I satisfies strong monotonicity, $f(\epsilon)$ is strictly decreasing. We claim that $\lim_{\epsilon \downarrow 0} f(\epsilon) = f(0)$. Indeed,

$$I(1_{S_1} - f(0) 1_{S_2^c}) = I(X_0) = 0 = \lim_{\epsilon \downarrow 0} I(X_\epsilon) = I\left(1_{S_1} - \lim_{\epsilon \downarrow 0} f(\epsilon) 1_{S_2^c}\right), \quad (31)$$

where the last equality holds because strong monotonicity and constant additivity together imply L^∞ -norm continuity of I , and $X_\epsilon \rightarrow 1_{S_1} - \lim_{\epsilon \downarrow 0} f(\epsilon) 1_{S_2^c}$ uniformly. Combining (31) with strong monotonicity of I yields $\lim_{\epsilon \downarrow 0} f(\epsilon) = f(0)$. Therefore,

$$\frac{s\lambda_{S_2}}{(n-t)\eta_{S_2}} = \frac{P_{S_2}(S_1)}{Q_{S_2}(S_2^c)} = f(0) = \lim_{\epsilon \downarrow 0} f(\epsilon) = \frac{P_{S_1}(S_1)}{Q_{S_1}(S_2^c)} = \frac{s\lambda_{S_1}}{(n-t)\eta_{S_1}},$$

where the second and the fourth equalities follow from (29) and (30), respectively. This concludes that $S \mapsto \lambda_S/\eta_S$ is a constant on \mathcal{F} , which is denoted as $1/(1+\beta)$. Since it is positive, we have $\beta > -1$. For any $X \in \mathcal{X}^0$, using the representation (28) implies that

$$\mathbb{E}[X^+] = (1+\beta)\mathbb{E}[X^-].$$

For any $X \in B_0(\Omega, \mathcal{F})$, note that $X - I(X) \in \mathcal{X}^0$ by translation invariance of I , and thus,

$$\mathbb{E}[(X - I(X))^+] = (1+\beta)\mathbb{E}[(I(X) - X)^+],$$

which means that $I = \mathbb{E}_\beta$ with $\beta > -1$. This completes the proof of the case of finite Ω .

Proof for the case of infinite Ω . We now assume that (Ω, \mathcal{F}, P) is a nonatomic probability space. The proof builds upon the result established for the finite case and proceeds by applying standard convergence arguments.

We begin by introducing some notation from Maccheroni et al. (2025), which will be used in the proof. Let V be a random variable with uniform distribution on $(0, 1)$, i.e., $P(V \leq x) = x$ for $x \in [0, 1]$. Denote by q_X the quantile function of X under P , i.e., $q_X(\alpha) = \inf\{x \in \mathbb{R} : P(X \leq x) \geq \alpha\}$ for $\alpha \in (0, 1]$. Also write $q_X(0) = \inf\{x \in \mathbb{R} : P(X \leq x) > 0\}$. Define

$$\Psi_k = \left\{ \left(\frac{0}{2^k}, \frac{1}{2^k} \right], \left(\frac{1}{2^k}, \frac{2}{2^k} \right], \dots, \left(\frac{2^k - 1}{2^k}, \frac{2^k}{2^k} \right] \right\}, \quad k \in \mathbb{N}$$

as the partition of $(0, 1]$ into segments of equal length 2^{-k} . Further define

$$\Pi_k = V^{-1}(\Psi_k), \quad k \in \mathbb{N}$$

as a partition of Ω in \mathcal{F} such that $P(E) = 1/2^k$ for all $E \in \Pi_k$. By setting $\mathcal{F}_k = \sigma(\Pi_k) = V^{-1}(\sigma(\Psi_k))$ for all $k \in \mathbb{N}$, we have a filtration $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ in \mathcal{F} . Denote by $P|_{\mathcal{F}_k}$ the restriction of P on \mathcal{F}_k .

Let $B_0(\Omega, \mathcal{F}_k)$ denote the set of all \mathcal{F}_k -measurable simple random variables, and let $I|_{\mathcal{F}_k}$ represent the restriction of the functional I to the domain $B_0(\Omega, \mathcal{F}_k)$. It is straightforward to verify that $I|_{\mathcal{F}_k}$ satisfies strong monotonicity, constant additivity, positive homogeneity and additivity in concordant sums on $B_0(\Omega, \mathcal{F}_k)$. By the result for finite case, we have $I|_{\mathcal{F}_k} = \mathbb{E}_{\beta_k}$ with $\beta_k > -1$ for all $k \in \mathbb{N}$. Since $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ is a filtration, we have

$$\mathbb{E}_{\beta_k}[X] = I|_{\mathcal{F}_k}(X) = I|_{\mathcal{F}_1}(X) = \mathbb{E}_{\beta_1}[X]$$

for any $X \in B_0(\Omega, \mathcal{F}_1)$. We claim that $\beta_k = \beta_1$ for all $k \in \mathbb{N}$. Denote $\Pi_1 = \{E_1, E_2\}$ with $P(E_1) = P(E_2) = 1/2$. Let $X = 1_{E_1}$, and denote $t = \mathbb{E}_{\beta_1}[X] = \mathbb{E}_{\beta_k}[X]$. It holds that $t \in (0, 1)$ by strong monotonicity, and

$$1 - t = (1 + \beta_1)t \quad \text{and} \quad 1 - t = (1 + \beta_k)t$$

implying $\beta_k = \beta_1$. Therefore, the following representation holds:

$$I(X) = \mathbb{E}_{\beta}[X], \quad X \in \bigcup_{k \in \mathbb{N}} B_0(\Omega, \mathcal{F}_k), \quad (32)$$

where we denote $\beta = \beta_k$ for $k \in \mathbb{N}$. Below, we aim to verify that the representation holds for all acts in $B_0(\Omega, \mathcal{F})$.

We first recall that strong monotonicity implies law invariance. Let $\underline{V}_k = 2^{-k}[2^k V - 1]$ and $\overline{V}_k = 2^{-k}[2^k V]$ for $k \in \mathbb{N}$, where $[x]$ represents the least integer not less than x ; that is, $(\underline{V}_k, \overline{V}_k]$ is the interval in Ψ_k that contains V . For $X \in B_0(\Omega, \mathcal{F})$, denote by $\underline{X}_k = q_X(\underline{V}_k)$ and $\overline{X}_k = q_X(\overline{V}_k)$. It is straightforward to check that $\underline{X}_k, \overline{X}_k \in \bigcup_{k \in \mathbb{N}} B_0(\Omega, \mathcal{F}_k)$ and $\underline{X}_k \leq X \leq \overline{X}_k$. By strong monotonicity, we have

$$I(\underline{X}_k) \leq I(X) \leq I(\overline{X}_k),$$

and combining with (32) implies

$$\mathbb{E}_{\beta}[\underline{X}_k] \leq I(X) \leq \mathbb{E}_{\beta}[\overline{X}_k]. \quad (33)$$

Since \underline{X}_k and \overline{X}_k both converge to $q_X(V)$ bounded a.s., we have

$$\mathbb{E}_{\beta}[\underline{X}_k], \mathbb{E}_{\beta}[\overline{X}_k] \rightarrow \mathbb{E}_{\beta}[q_V(X)] = \mathbb{E}_{\beta}[X],$$

where the convergence follows from the continuity property in Lemma 9, and the equality is due to law invariance. Combining with (33) yields the representation of \mathbb{E}_{β} on $B_0(\Omega, \mathcal{F})$. This completes the proof. \square

Proof of Theorem 11. Sufficiency. Suppose that $I = \mathbb{E}_{\beta}$ with $\beta \geq 0$. By Lemma 9, strong monotonicity and L^{∞} -norm continuity hold. To see decrease in concordant sums, let $X, Y, W \in B_0(\Omega, \mathcal{F})$ be such that $\mathbb{E}_{\beta}[X] = \mathbb{E}_{\beta}[Y]$ and $D_{\mathbb{E}_{\beta}}(X) = D_{\mathbb{E}_{\beta}}(W)$, where $D_{\mathbb{E}_{\beta}}(X)$ is defined by (23) with the form:

$$D_{\mathbb{E}_{\beta}}(X) = \{\omega : X(\omega) < \mathbb{E}_{\beta}[X]\}.$$

It holds that

$$\mathbb{E}_\beta[W + X] = \mathbb{E}_\beta[W] + \mathbb{E}_\beta[X] = \mathbb{E}_\beta[W] + \mathbb{E}_\beta[Y] \leq \mathbb{E}_\beta[W + Y],$$

where the first equality and the inequality follow from additivity in concordant sums and superadditivity in Lemma 9, respectively.

Necessity. Suppose that I satisfies strong monotonicity, L^∞ -norm continuity, and decrease in concordant sums with $I(m) = m$ for all $m \in \mathbb{R}$. We aim to show that these properties together imply constant additivity, superadditivity, additivity in concordant sums, and positive homogeneity. Consequently, by Theorem 10, we conclude that $I = \mathbb{E}_\beta$ for some $\beta > -1$. Moreover, superadditivity implies that $\beta \geq 0$, as established in Lemma 9. Below, we proceed to verify each of the required properties in turn.

Constant additivity. The case that $X \in B_0(\Omega, \mathcal{F})$ is a constant is trivial as $I(m) = m$ for all $m \in \mathbb{R}$. For a nonconstant $X \in B_0(\Omega, \mathcal{F})$, we have

$$I(X + m) \geq I(I(X) + m) = I(X) + m,$$

where the inequality follows from decrease in concordant sums and the fact that $I(I(X)) = I(X)$ and $D_I(I(X)) = D_I(m) = \emptyset$. On the other hand, for $\epsilon > 0$, define $Y_\epsilon = m1_{D_I(X)} + (m + \epsilon)1_{D_I(X)^c}$. Strong monotonicity implies $I(Y_\epsilon) \in (I(m), I(m + \epsilon)) = (m, m + \epsilon)$, and thus, $D_I(Y_\epsilon) = D_I(X)$. Decrease in concordant sums yields

$$I(X + Y_\epsilon) \leq I(I(X) + Y_\epsilon) \quad \forall \epsilon > 0.$$

Note that $\text{ess sup } |Y_\epsilon - m| \rightarrow 0$. Letting $\epsilon \rightarrow 0$ in the above equation, and using L^∞ -norm continuity of I implies

$$I(X + m) \leq I(I(X) + m) = I(X) + m.$$

Hence, we have concluded that $I(X + m) = I(X) + m$ for all $m \in \mathbb{R}$, and constant additivity holds.

Superadditivity. For $X, Y \in B_0(\Omega, \mathcal{F})$, if either X or Y is constant, then constant additivity immediately yields $I(X + Y) = I(X) + I(Y)$. Now, consider the case where both X and Y are nonconstant. Define $X_\epsilon = \epsilon 1_{D_I(X)^c}$ for $\epsilon > 0$. Strong monotonicity implies $D_I(X_\epsilon) = D_I(X)$. Note that

$$I(X - I(X) + I(X_\epsilon)) = I(X) - I(X) + I(X_\epsilon) = I(X_\epsilon),$$

where we have used constant additivity in the first equality. Therefore,

$$I(X_\epsilon + Y) \leq I(X - I(X) + I(X_\epsilon) + Y) = I(X + Y) + I(X_\epsilon) - I(X),$$

where the inequality follows from decrease in concordant sums and the equality is due to constant additivity. Letting $\epsilon \downarrow 0$ in the above equation, and using L^∞ -norm continuity of I yields

$$I(Y) \leq I(X + Y) - I(X).$$

This gives superadditivity.

Additivity in concordant sums. Suppose that $X, Y \in B_0(\Omega, \mathcal{F})$ satisfy $D_I(X) = D_I(Y)$. By decrease in concordant sums and constant additivity, we have $I(X + Y) \leq I(I(X) + Y) = I(X) + I(Y)$. Combining with superadditivity yields additivity in concordant sums.

Positive homogeneity. For $X \in B_0(\Omega, \mathcal{F})$, $D_I(X) = D_I(X)$ holds trivially. By additivity in concordant sums, we have $I(2X) = 2I(X)$, which implies $D_I(X) = D_I(2X)$. Applying additivity in concordant sums again, we obtain $I(3X) = 3I(X)$. By iterating this argument, it follows that

$I(\lambda X) = \lambda I(X)$ for all rational $\lambda > 0$. Since I satisfies L^∞ -norm continuity, this identity can be extended to all real $\lambda \geq 0$ via a standard convergence argument. \square

Proof of Theorem 12. The proof of sufficiency is similar to that of Theorem 11, and follows directly from Lemma 9. We now consider the necessity. Suppose that $I : B_0(\Omega, \mathcal{F}, A) \rightarrow \mathbb{R}$ satisfies strong monotonicity, L^∞ -norm continuity, and decrease in concordant mixtures with $I(m) = m$ for all $m \in A$. The proof can be proceeded by verifying the following claims:

- (a) $I\left(\frac{X+m}{2}\right) = \frac{1}{2}(I(X) + m)$ for all $X \in B_0(\Omega, \mathcal{F}, A)$ and $m \in A$.
- (b) $I(\lambda X) = \lambda I(X)$ for all $\lambda \in [0, 1]$ and $X \in B_0(\Omega, \mathcal{F}, A)$. Because 0 in the interior of A , $\lambda \in [0, 1]$ and $X \in B_0(\Omega, \mathcal{F}, A)$ imply $\lambda X \in B_0(\Omega, \mathcal{F}, A)$.
- (c) The functional I can be extended to $B_0(\Omega, \mathcal{F})$ in such a way that the extension satisfies strong monotonicity, L^∞ -norm continuity, and decrease in concordant sums, thereby allowing the desired result to follow from Theorem 11.

(a) For $X \in B_0(\Omega, \mathcal{F}, A)$ and $m \in A$, it is trivial for the case that X is a constant. Suppose now that X is nonconstant. It holds that

$$\frac{I(X) + m}{2} = I\left(\frac{I(X) + m}{2}\right) \leq I\left(\frac{X + m}{2}\right), \quad (34)$$

where the inequality follows from decrease in concordant mixtures by noting that $I(I(X)) = I(X)$ and $D_I(I(X)) = D_I(m) = \emptyset$, where we recall that D_I is defined by (23) with the form:

$$D_I(Z) = \{\omega : Z(\omega) < I(Z)\}, \quad Z \in B_0(\Omega, \mathcal{F}, A).$$

On the other hand, strong monotonicity implies $I(X) \in (\text{ess inf } X, \text{ess sup } X)$, and thus, $P(D_I(X)) > 0$. Define $Y_\epsilon = m - \epsilon 1_{D_I(X)}$ if m is the right endpoint of A , and $Y_\epsilon = m + \epsilon 1_{D_I^c(X)}$ otherwise. Applying strong monotonicity again, it is straightforward to verify that $D_I(Y_\epsilon) = D_I(X)$ for all $\epsilon > 0$ such that $Y_\epsilon \in B_0(\Omega, \mathcal{F}, A)$. Hence, we have

$$I\left(\frac{X + m}{2}\right) \xleftarrow{\epsilon \downarrow 0} I\left(\frac{X + Y_\epsilon}{2}\right) \leq I\left(\frac{I(X) + Y_\epsilon}{2}\right) \xrightarrow{\epsilon \downarrow 0} I\left(\frac{I(X) + m}{2}\right) = \frac{I(X) + m}{2}, \quad (35)$$

where the convergences are due to the L^∞ -norm continuity of I and the inequality follows from decrease in concordant mixtures. Hence, statement (a) holds by combining (34) and (35).

(b) We first consider a result as follows:

$$I\left(\frac{k}{2^n}X\right) = \frac{k}{2^n}I(X) \quad \text{for all } X \in B_0(\Omega, \mathcal{F}, A), \quad n \in \mathbb{N}, \quad k \in [2^n]. \quad (36)$$

We use induction to prove this conclusion. We only consider the nontrivial case that X is not a constant. The cases of $(n, k) \in \{(0, 1), (1, 2)\}$ holds directly. The case of $(n, k) = (1, 1)$ follows immediately from statement (a) with $m = 0$. Assume now that (36) holds for $(n - 1, k)$ with $k \in [2^{n-1}]$ and (n, k) with $k \in [2^n]$. Let us consider the situations that $(n + 1, k)$ with $k \in [2^{n+1}]$. If $k \in [2^{n+1}]$ is even, then we have

$$I\left(\frac{k}{2^{n+1}}X\right) = I\left(\frac{k/2}{2^n}X\right) = \frac{k/2}{2^n}I(X) = \frac{k}{2^{n+1}}I(X),$$

where we have used the inductive assumption in the second equality. For odd $k \in [2^{n+1}]$, let $s = (k - 1)/2 \in \mathbb{N}$, and thus, $k = 2s + 1$. Denote $\lambda_1 = s/2^n$ and $\lambda_2 = (s + 1)/2^n$. Since $\lambda_1, \lambda_2 \in [0, 1]$, we

have $\lambda_1 X, \lambda_2 X \in B_0(\Omega, \mathcal{F}, A)$ because 0 belongs to the interior of A . It follows from the assumption that $I(\lambda_i X) = \lambda_i I(X)$ for $i = 1, 2$, which implies

$$D_I(\lambda_1 X) = D_I(\lambda_2 X) = D_I(X). \quad (37)$$

Therefore,

$$\begin{aligned} I\left(\frac{k}{2^{n+1}}X\right) &= I\left(\frac{\lambda_1 X + \lambda_2 X}{2}\right) \\ &\leq I\left(\frac{I(\lambda_1 X) + \lambda_2 X}{2}\right) \end{aligned} \quad (38)$$

$$= \frac{I(\lambda_1 X) + I(\lambda_2 X)}{2} \quad (39)$$

$$= \frac{\lambda_1 I(X) + \lambda_2 I(X)}{2} = \frac{k}{2^{n+1}} I(X), \quad (40)$$

where the inequality follows from decrease in concordant mixtures, the second equality is due to statement (a), and we have used the inductive assumption in the third equality. On the other hand, define $Y_\epsilon = I(\lambda_1 X) + \epsilon 1_{D_I^c(X)}$ for $\epsilon > 0$. Since X is nonconstant, strong monotonicity implies

$$I(\lambda_1 X) < \text{ess sup}(\lambda_1 X) \leq \text{ess sup } X.$$

Note that $I(\lambda_1 X), \text{ess sup } X \in A$, and we have $Y_\epsilon \in B_0(\Omega, \mathcal{F}, A)$ for any $\epsilon \in (0, \text{ess sup } X - I(\lambda_1 X))$. Moreover, it follows from strong monotonicity and (37) that

$$D_I(Y_\epsilon) = D_I(\lambda_1 X) = D_I(\lambda_2 X) = D_I(X).$$

Define $\eta_\epsilon \in \mathbb{R}$ as the number satisfying $I(\lambda_1 X + \eta_\epsilon) = I(Y_\epsilon)$. For $\epsilon > 0$, strong monotonicity implies that $I(Y_\epsilon) > I(\lambda_1 X)$, and $I(Y_\epsilon)$ is increasing in ϵ . Hence, η_ϵ is positive and increasing in ϵ . Moreover, the L^∞ -norm continuity implies $\lim_{\epsilon \downarrow 0} \eta_\epsilon = 0$. Therefore,

$$\begin{aligned} \frac{k}{2^{n+1}} I(X) &= I\left(\frac{I(\lambda_1 X) + \lambda_2 X}{2}\right) \xleftarrow{\epsilon \downarrow 0} I\left(\frac{Y_\epsilon + \lambda_2 X}{2}\right) \\ &\leq I\left(\frac{\lambda_1 X + \eta_\epsilon + \lambda_2 X}{2}\right) \\ &\xrightarrow{\epsilon \downarrow 0} I\left(\frac{\lambda_1 X + \lambda_2 X}{2}\right) = I\left(\frac{k}{2^{n+1}}X\right), \end{aligned}$$

where the first equality has been verified in (38)-(40), the convergences are due to the L^∞ -norm continuity of I , and the inequality follows from decrease in concordant mixtures by noting that $I(Y_\epsilon) = I(\lambda_1 X + \eta_\epsilon)$ and $D_I(Y_\epsilon) = D_I(\lambda_2 X)$. Hence, we have concluded that the equation in (36) holds for $(n+1, k)$ with $k \in [2^{n+1}]$. This completes the proof of (36). Since $\{k/2^n : n \in \mathbb{N}, k \in [2^n]\}$ is a dense subset of $[0, 1]$, this combined with the L^∞ -norm continuity of I yields $I(\lambda X) = \lambda I(X)$ for all $X \in B_0(\Omega, \mathcal{F}, A)$ and $\lambda \in [0, 1]$.

(c) Define

$$\tilde{I}(X) = \frac{1}{\lambda_X} I(\lambda_X X) \text{ for } X \in B_0(\Omega, \mathcal{F}),$$

where

$$\lambda_X := \frac{1}{2} \sup\{\lambda \in [0, 1] : \lambda X \in B_0(\Omega, \mathcal{F}, A)\}.$$

We clarify that λ_X is defined as half the value of the supremum problem above, rather than the full value, in order to avoid situations where $\lambda_X X \notin B_0(\Omega, \mathcal{F}, A)$ when A is an open set. We aim to

prove that \tilde{I} is an extension of I on $B_0(\Omega, \mathcal{F})$ satisfying strong monotonicity, L^∞ -norm continuity and decrease in concordant sums with $\tilde{I}(m) = m$ for all $m \in \mathbb{R}$. If $X \in B_0(\Omega, \mathcal{F}, A)$, it is clear that $\lambda_X = 1/2$, and statement (b) implies $\tilde{I}(X) = I(X)$, which means that \tilde{I} is an extension of I . Let $X \in B_0(\Omega, \mathcal{F})$, and we claim that

$$\tilde{I}(X) = \frac{1}{\lambda} I(\lambda X) \text{ whenever } \lambda \in (0, 1] \text{ and } \lambda X \in B_0(\Omega, \mathcal{F}, A). \quad (41)$$

If $\lambda \geq \lambda_X$, then denote by $\theta = \lambda_X/\lambda \in [0, 1]$, and we have

$$\frac{1}{\lambda} I(\lambda X) = \frac{1}{\lambda} \left(\frac{1}{\theta} I(\theta(\lambda X)) \right) = \frac{1}{\lambda_X} I(\lambda_X X) = \tilde{I}(X).$$

If $\lambda \leq \lambda_X$, a similar argument holds with λ and λ_X interchanged. Hence, (41) holds. For $m \in \mathbb{R}$, let $\lambda \in (0, 1]$ be such that $\lambda m \in A$, and we have $\tilde{I}(m) = I(\lambda m)/\lambda = m$. For $X, Y \in B_0(\Omega, \mathcal{F}, A)$ with $P(X \geq t) \geq P(Y \geq t)$ for all $t \in \mathbb{R}$, let $\lambda \in (0, 1]$ be such that $\lambda X, \lambda Y \in B_0(\Omega, \mathcal{F}, A)$, and we have

$$\tilde{I}(X) = \frac{I(\lambda X)}{\lambda} \geq \frac{I(\lambda Y)}{\lambda} = \tilde{I}(Y),$$

where the inequality follows from strong monotonicity of I and $P(\lambda X \geq t) \geq P(\lambda Y \geq t)$ for all $t \in \mathbb{R}$. Hence, \tilde{I} satisfies strong monotonicity. For $\{X_n\}_{n \in \mathbb{N}} \subseteq B_0(\Omega, \mathcal{F})$ and $X \in B_0(\Omega, \mathcal{F}, A)$ with $X_n \rightarrow X$ in L^∞ , there exists $\lambda \in (0, 1]$ such that $\lambda X \in B_0(\Omega, \mathcal{F}, A)$ and $\lambda X_n \in B_0(\Omega, \mathcal{F}, A)$ for all $n \in \mathbb{N}$, and hence,

$$\tilde{I}(X_n) = \frac{I(\lambda X_n)}{\lambda} \rightarrow \frac{I(\lambda X)}{\lambda} = \tilde{I}(X),$$

where the convergence follows from the L^∞ -norm continuity of I . This yields the L^∞ -norm continuity of \tilde{I} . Finally, let $W, X, Y \in B_0(\Omega, \mathcal{F})$ with $\tilde{I}(X) = \tilde{I}(Y)$ and $D_{\tilde{I}}(W) = D_{\tilde{I}}(X)$. There exists $\lambda \in (0, 1]$ such that $\lambda W, \lambda X, \lambda Y, \lambda(W + X), \lambda(W + Y) \in B_0(\Omega, \mathcal{F}, A)$, and we have

$$\frac{I(\lambda X)}{\lambda} = \tilde{I}(X) = \tilde{I}(Y) = \frac{I(\lambda Y)}{\lambda} \quad (42)$$

and

$$\begin{aligned} D_I(\lambda W) &= \left\{ \omega : W(\omega) < \frac{I(\lambda W)}{\lambda} \right\} = \left\{ \omega : W(\omega) < \tilde{I}(W) \right\} = D_{\tilde{I}}(W) \\ &= D_{\tilde{I}}(X) = \left\{ \omega : X(\omega) < \tilde{I}(X) \right\} = \left\{ \omega : X(\omega) < \frac{I(\lambda X)}{\lambda} \right\} = D_I(\lambda X). \end{aligned} \quad (43)$$

Therefore,

$$\tilde{I}(W + X) = \frac{I(\lambda W/2 + \lambda X/2)}{\lambda/2} \leq \frac{I(\lambda W/2 + \lambda Y/2)}{\lambda/2} = \tilde{I}(W + Y),$$

where the inequality follows from the property of decrease in concordant mixtures, along with the relations in (42) and (43). This completes the proof. \square

C Proofs of Section 3

Proof of Proposition 1. The solution to (15) is an expectile and is unique; see, e.g. Bellini et al. (2014). Next, we aim to verify that a solution to Gul's equation (1) is the solution of (15). Suppose that v is a solution of (1), and thus,

$$v = \mathbb{E} \left[\frac{u(X) + \beta v}{1 + \beta} 1_{\{u(X) \geq v\}} + u(X) 1_{\{u(X) < v\}} \right].$$

It is straightforward to verify that the two terms inside the bracket in the above equation can be reformulated as follows:

$$\frac{u(X) + \beta v}{1 + \beta} 1_{\{u(X) \geq v\}} = \frac{1}{1 + \beta} ((u(X) - v)^+ + (1 + \beta)v 1_{\{u(X) \geq v\}})$$

and

$$u(X) 1_{\{u(X) < v\}} = -(v - u(X))^+ + v 1_{\{u(X) < v\}}.$$

Therefore,

$$\begin{aligned} v &= \mathbb{E} \left[\frac{1}{1 + \beta} ((u(X) - v)^+ + (1 + \beta)v 1_{\{u(X) \geq v\}}) - (v - u(X))^+ + v 1_{\{u(X) < v\}} \right] \\ &= \frac{1}{1 + \beta} \mathbb{E}[(u(X) - v)^+] - \mathbb{E}[(v - u(X))^+] + v, \end{aligned}$$

which implies that v is a solution of (15). This completes the proof. \square

Proof of Theorem 2. The result follows directly from the foundational works of Newey and Powell (1987) and Bellini et al. (2014), where expectiles are characterized as solutions to asymmetric least squares minimization problems and as coherent risk measures under suitable conditions. \square

Proof of Theorem 3. For notational simplicity, we denote $U = u(X)$. Define \mathcal{P} as the set of all probability measures. Note that $\beta \geq 0$. By the dual representation of \mathbb{E}_β in Lemma 9, we have

$$\mathbb{E}_\beta[U] = \min_{Q \in \mathcal{P}_\beta} \mathbb{E}^Q[U],$$

where

$$\mathcal{P}_\beta = \left\{ Q \in \mathcal{P} : \frac{\text{ess sup } dQ/dP}{\text{ess inf } dQ/dP} \leq 1 + \beta \right\},$$

and the optimal probability measure Q^* is attained at

$$dQ^* = \frac{1_{D_{u(X)}^c} + (1 + \beta)1_{D_{u(X)}}}{1 + \beta P(D_{u(X)})} dP.$$

Therefore, it suffices to verify that $\mathcal{Q}_\beta \subseteq \mathcal{P}_\beta$. This follows directly from the fact that, for any $Q \in \mathcal{Q}_\beta$, dQ/dP is supported on two points, and the ratio of their values is bounded above by $1 + \beta$. \square

D Proofs and related axioms in Section 4

In this section, we begin by presenting the detailed axioms underlying invariant biseparable preferences and some related result as introduced in GMMS.

Axiom D.1 (Weak Order) (a) For all $X, Y \in \mathbb{X}$, $X \succsim Y$ or $Y \succsim X$. (b) For all $X, Y, Z \in \mathbb{X}$, if $X \succsim Y$ and $Y \succsim Z$, then $X \succsim Z$.

Axiom D.2 (Dominance) For every $X, Y \in \mathbb{X}$, if $X(\omega) \succsim Y(\omega)$ for all $\omega \in \Omega$, then $X \succsim Y$.

Axiom D.3 (Essentiality) There exists an event $E \in \mathcal{F}$ such that $x \succ xEy \succ y$ for some consequences $x, y \in \mathcal{X}$. Such an event is called essential.

Given E , we denote by $\sigma(E)$ the algebra generated by E . We use the following terminology: An event $A \in \mathcal{F}$ is null (resp. universal) if $y \sim xAy$ (resp. $x \sim xAy$) for every $x \succ y$. It follows from Axiom D.1 that an event can be only one of null, essential, or universal.

Axiom D.4 (E -Monotonicity) For every non-null $A \in \sigma(E)$ and every $x, y \succsim z \in \mathcal{X}$,

$$x \succ y \implies xAz \succ yAz.$$

For every non-universal $A \in \sigma(E)$ and every $x, y \succsim z \in X$,

$$x \succ y \implies zAx \succ zAy.$$

Axiom D.5 (E -Continuity) Let $\{X_n\}_{n \in \mathbb{N}} \subseteq \mathbb{X}$ be a sequence of $\sigma(E)$ -measurable acts that point-wise converges to X . For every $Y \in \mathbb{X}$, if $X_n \succsim Y$ (resp. $Y \succsim X_n$) for all $n \in \mathbb{N}$, then $X \succsim Y$ (resp. $Y \succsim X$).

It is straightforward to show (see e.g. Lemma 12 of GMMS) that any binary relation satisfying axioms D.1–D.3 and D.5 has certainty equivalents. That is, for every $X \in \mathbb{X}$, there exists $x \in \mathcal{X}$ such that $x \sim X$. Granted this, we henceforth denote by c_X an arbitrarily chosen certainty equivalent of $X \in \mathbb{X}$.

The next axiom imposes a behavioral restriction. We write $x \succsim \{z', z''\}$ (resp. $\{z', z''\} \succsim y$) if $x \succsim z'$ and $x \succsim z''$ (resp. $z' \succsim y$ and $z'' \succsim y$).

Axiom D.6 (E -Substitution) For all $x, y, z', z'' \in X$ and $A, B \in \sigma(E)$. Suppose that $x \succsim \{z', z''\} \succsim y$. Then

$$c_{xAz'}Bc_{z''Ay} \sim c_{xBz''}Ac_{z'B y}$$

The final axiom of GMMS uses the notion of mixture thus derived to impose a very weak and natural property of separability of preferences. Before stating the axiom, we first introduce some necessary preliminaries.

Definition 1 (Preference Average) Given $x, y \in \mathcal{X}$ such that $x \succsim y$ (resp. $y \succsim x$), the preference average of x, y given the event E , denote by $(1/2)x \oplus (1/2)y$, is a consequence $z \in \mathcal{X}$ such that $x \succsim z \succsim y$ (resp. $y \succsim z \succsim x$) and

$$xEy \sim c_{xEz}Ec_{zEy} \quad (\text{resp. } yEx \sim c_{yEz}Ec_{zEx}).$$

We note that the term preference average used in the above definition corresponds to what is referred to as the preference midpoint in the main text of this paper. Based on Axioms D.1–D.6, Lemma 1 of GMMS establishes that the DM's preferences admit a canonical representation over the set of all $\sigma(E)$ -measurable acts, characterized by a continuous and nonconstant canonical utility function u .

Lemma 13 (Lemma 1 of GMMS) The binary relation \succsim satisfies Axioms D.1–D.6 if and only if there is a continuous nonconstant utility index $u : \mathcal{X} \rightarrow \mathbb{R}$ and a capacity $\rho_E : \sigma(E) \rightarrow [0, 1]$, with $\rho_E(E) \in (0, 1)$, such that the functional $V : \mathcal{F} \rightarrow \mathbb{R}$ defined by $V(f) \equiv u(c_f)$ for any $f \in \mathcal{F}$ represents \succsim , it is monotone and it satisfies, for all $x \succsim y$ and all $A \in \sigma(E)$,

$$V(xAy) = u(x)\rho_E(A) + u(y)(1 - \rho_E(A)).$$

Moreover, such u and V are unique up to a positive affine transformation and ρ_E is unique.

Furthermore, Lemma 3 of GMMS shows that this utility function is additive with respect to the preference average operator. We present the corresponding result below.

Lemma 14 (Lemma 3 of GMMS) *Suppose that the binary relation \succsim satisfies Axioms D.1–D.6. For any $x, y \in \mathcal{X}$, there exists $z = (1/2)x \oplus (1/2)y$. If u is the cardinal utility that represents \succsim by Lemma 13, then*

$$u\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) = \frac{1}{2}u(x) + \frac{1}{2}u(y).$$

Let w be a preference average of x and y , and z is a preference average of x and w if and only if $u(z) = (3/4)u(x) + (1/4)u(y)$; that is, z is a $(3/4) : (1/4)$ utility mixture of x and y . This allows us to identify $(3/4)x \oplus (1/4)y$ behaviorally. Proceeding along these lines and using the continuity axiom (Axiom D.5), it is possible to identify behaviorally the $\alpha : 1 - \alpha$ utility mixtures of x and y , for any $\alpha \in [0, 1]$. Specifically, Lemma 13 of GMMS shows that

$$u(\alpha x \oplus (1 - \alpha)y) = \alpha u(x) + (1 - \alpha)u(y), \quad \forall x, y \in \mathcal{X}, \alpha \in [0, 1].$$

Subjective mixtures of acts may then be defined pointwise, as usual. That is, given $X, Y \in \mathbb{X}$ and $\alpha \in [0, 1]$, $\alpha X \oplus (1 - \alpha)Y$ is the act $Z \in \mathbb{X}$ defined by $Z(\omega) = \alpha X(\omega) \oplus (1 - \alpha)Y(\omega)$ for any $\omega \in \Omega$.

Axiom D.7 (Weak Certainty Independence) *For all $X, Y \in \mathbb{X}$, $x \in \mathcal{X}$ and $\alpha \in [0, 1]$,*

$$X \sim Y \implies \alpha X \oplus (1 - \alpha)x \sim \alpha Y \oplus (1 - \alpha)x.$$

So far, we have presented all the relevant axioms from GMMS. A binary relation that satisfies Axioms D.1–D.7 is referred to as *invariant biseparable*. Next, we present the representation result for invariant biseparable preference given by GMMS.

Lemma 15 (Theorem 5 of GMMS) *Let \succsim be a binary relation on \mathbb{X} . Then \succsim is invariant biseparable if and only if there exist a continuous nonconstant function $u : \mathcal{X} \rightarrow \mathbb{R}$ and a monotone, constant-additive and positively homogeneous functional $I : B_0(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ such that for all $X, Y \in \mathbb{X}$,*

$$X \succsim Y \iff I(u(X)) \geq I(u(Y))$$

and such that $I(1_E) \notin \{0, 1\}$ for some $E \in \mathcal{F}$. Moreover, u is unique up to a positive affine transformation and I is unique.

We are now ready to present the complete proof of our main characterization result—Theorem 4.

Proof of Theorem 4. In this proof, we focus on the case of disappointment hedging, as the elation-speculating case can be treated analogously.

(ii) \Rightarrow (i). By Lemma 9, \mathbb{E}_β satisfies monotonicity, constant additivity and positively homogeneity. Combining with Lemma 15 implies that \succsim is invariant biseparable. To see probabilistically sophistication, let $X, Y \in \mathbb{X}$ be such that

$$P(\omega : X(\omega) \succsim x) \geq P(\omega : Y(\omega) \succsim x) \text{ for all } x \in \mathcal{X}. \quad (44)$$

This implies

$$P(u(X) \geq u(x)) \geq P(u(Y) \geq u(x)) \text{ for all } x \in \mathcal{X},$$

which is equivalent to

$$P(u(X) \geq t) \geq P(u(Y) \geq t) \text{ for all } t \in \mathbb{R}. \quad (45)$$

By strong monotonicity of \mathbb{E}_β in Lemma 9, we have $\mathbb{E}_\beta[u(X)] \geq \mathbb{E}_\beta[u(Y)]$, and thus, $X \succsim Y$. If the inequality in (44) is strict for some $x \in \mathcal{X}$, then the inequality in (44) is strict for $t = u(x)$, and using strong monotonicity again yields $X \succ Y$. It remains to verify disappointment hedging. Let $X, Y, W \in \mathbb{X}$ be such that $X \sim Y$ and $D_X = D_W$. It holds that

$$\mathbb{E}_\beta[u(X)] = \mathbb{E}_\beta[u(Y)] \quad \text{and} \quad \{\omega : u(X(\omega)) < \mathbb{E}_\beta[u(X)]\} = \{\omega : u(W(\omega)) < \mathbb{E}_\beta[u(W)]\}.$$

Therefore,

$$\begin{aligned} \mathbb{E}_\beta \left[u \left(\frac{1}{2}W \oplus \frac{1}{2}X \right) \right] &= \mathbb{E}_\beta \left[\frac{1}{2}u(W) + \frac{1}{2}u(X) \right] \\ &= \frac{1}{2}\mathbb{E}_\beta[u(W) + u(X)] \\ &= \frac{1}{2}(\mathbb{E}_\beta[u(W)] + \mathbb{E}_\beta[u(X)]) \\ &= \frac{1}{2}(\mathbb{E}_\beta[u(W)] + \mathbb{E}_\beta[u(Y)]) \\ &\leq \frac{1}{2}\mathbb{E}_\beta[u(W) + u(Y)] \\ &= \mathbb{E}_\beta \left[\frac{1}{2}u(W) + \frac{1}{2}u(Y) \right] = \mathbb{E}_\beta \left[u \left(\frac{1}{2}W \oplus \frac{1}{2}Y \right) \right], \end{aligned}$$

where the first and the last equalities follow from Lemma 14, the second and the fifth equalities are due to positive homogeneity of \mathbb{E}_β , and the third equality and the inequality come from additivity in concordant sums and superadditivity of \mathbb{E}_β in Lemma 9, respectively. Thus, we have verified disappointment hedging.

(i) \Rightarrow (ii). Suppose now that \succsim is probabilistically sophisticated, invariant biseparable, and satisfies disappointment hedging. Then, there exist a continuous nonconstant function $u : \mathcal{X} \rightarrow \mathbb{R}$ and a monotone, positively homogeneous, and constant-additive $I : B_0(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ such that, for all acts $X, Y \in \mathbb{X}$,

$$X \succsim Y \iff I(u(X)) \geq I(u(Y)).$$

Below, we aim to establish that $I = \mathbb{E}_\beta$ for some $\beta \geq 0$. Note that positive homogeneity and constant additivity together imply L^∞ -norm continuity (see e.g. Lemma 4.3 of Föllmer and Schied, 2016). According to Theorem 11, it suffices to verify that I satisfies strong monotonicity and disappointment aversion. One can check that probabilistic sophistication implies strong monotonicity of I on $B_0(\Omega, \mathcal{F}, u(\mathcal{X}))$, that is, for $U, V \in B_0(\Omega, \mathcal{F}, u(\mathcal{X}))$ if $P(U \geq t) \geq P(V \geq t)$ for all $t \in \mathbb{R}$, then $I(U) \geq I(V)$, with strict inequality if the inequality is strict for some t . Moreover, for $X, Y, W \in \mathbb{X}$ with $X \sim Y$ and $D_W = D_X$,

$$\begin{aligned} \frac{1}{2}(I(u(W)) + I(u(X))) &= I \left(\frac{1}{2}(u(W) + u(X)) \right) \\ &= I \left(u \left(\frac{1}{2}W \oplus \frac{1}{2}X \right) \right) \\ &\leq I \left(u \left(\frac{1}{2}W \oplus \frac{1}{2}Y \right) \right) \\ &= I \left(\frac{1}{2}(u(W) + u(Y)) \right) = \frac{1}{2}(I(u(W)) + I(u(Y))), \end{aligned}$$

where the first and the last equalities follow from positive homogeneity of I , the second and the third equalities come from Lemma 14, and the inequality is due to disappointment hedging. This yields disappointment aversion of I restricted on $B_0(\Omega, \mathcal{F}, u(\mathcal{X}))$. Since u is continuous and nonconstant, and \mathcal{X} is connected, we have that $u(\mathcal{X}) \subseteq \mathbb{R}$ is a nonempty interval. Thus, it is straightforward to extend strong monotonicity and disappointment aversion from $B_0(\Omega, \mathcal{F}, u(\mathcal{X}))$ to $B_0(\Omega, \mathcal{F})$ by positive homogeneity and constant additivity of I . This completes the proof. \square

E Proofs in Section 5

Proof of Theorem 5. We only consider the disappointment averse case as the relation seeking case is similar.

First, it is an immediate and well-known consequence that monotonicity, probabilistic sophistication, together with continuity, implies the existence of a unique certainty equivalent. To see this, for $U \in B_0(\Omega, \mathcal{F})$, monotonicity implies that the sets $A := \{u \in \mathbb{R} : u \geq U\}$ and $B := \{u \in \mathbb{R} : U \geq u\}$ are both intervals. Continuity further yields $A \cap B \neq \emptyset$. Using probabilistic sophistication, we know that $A \cap B$ is a singleton, which can be defined as the unique certainty equivalent of \succsim . Second, we denote such a certainty equivalent as I . It is straightforward to verify that monotonicity, probabilistic sophistication, continuity, and disappointment aversion (resp. disappointment hedging) of \succsim are equivalent to the strong monotonicity, L^∞ -norm continuity, and decrease in concordant sums (resp. mixtures) of I . Hence, the equivalence between (i) and (ii) (resp. (ii) and (iii)) follows directly from Theorem 11 (resp. Theorem 12). \square

Proof of Theorem 6. (ii) \Rightarrow (i): This is a direct result from Theorem 5 of Gul (1991).

(i) \Rightarrow (ii): First, it follows from Theorem 5 of Gul (1991) that $\beta_A \geq \beta_B$.

Second, we verify that the condition that the two preferences share the same preference midpoints implies that u_A is an affine transformation of u_B . To see this, note that the condition that the two preferences share the same preference midpoints implies

$$u_A(z) = \frac{1}{2}u_A(x) + \frac{1}{2}u_A(y) \iff u_B(z) = \frac{1}{2}u_B(x) + \frac{1}{2}u_B(y), \quad \forall x, y, z \in \mathcal{X}.$$

Applying Lemma 13 of GMMS and noting that u_A and u_B are both continuous, we have

$$u_A(z) = \alpha u_A(x) + (1 - \alpha)u_A(y) \iff u_B(z) = \alpha u_B(x) + (1 - \alpha)u_B(y), \quad \forall x, y, z \in \mathcal{X}, \alpha \in [0, 1]. \quad (46)$$

Let $[a, b]$ be a subset of the interior of $u_A(\mathcal{X})$. Define

$$\mathcal{D}_A([a, b]) = \{z : u_A(z) \in [a, b]\}.$$

There exist $x_0, y_0 \in \mathcal{X}$ such that $u_A(x_0) = a$ and $u_A(y_0) = b$. For any $z \in \mathcal{D}_A([a, b])$, let $\alpha_z \in [0, 1]$ be such that

$$u_A(z) = \alpha_z u_A(x_0) + (1 - \alpha_z)u_A(y_0) = (a - b)\alpha_z + b. \quad (47)$$

Applying (46) yields

$$u_B(z) = \alpha_z u_B(x_0) + (1 - \alpha_z)u_B(y_0) = (u_B(x_0) - u_B(y_0))\alpha_z + u_B(y_0). \quad (48)$$

Note that u_B is not a constant on \mathcal{X} . Thus, it holds that u_B is not a constant on $\mathcal{D}_A([a, b])$ for large enough interval $[a, b]$. In this case, we have $u_B(x_0) - u_B(y_0) \neq 0$, and (47) and (48) imply u_A is an

affine transformation of u_B on $\mathcal{D}_A([a, b])$. Note that $[a, b]$ can be arbitrarily chosen as the subset of the interior of $u_A(\mathcal{X})$, and thus, u_A is an affine transformation of u_B on \mathcal{X} .

Finally, (21) in the statement (i) implies

$$u_A(x) \geq u_A(y) \implies u_B(x) \geq u_B(y), \quad \forall x, y \in \mathcal{X}.$$

This further guarantees that the affine transformation between u_A and u_B must be strictly increasing. Hence, we complete the proof. \square

Proof of Theorem 7. Assume that X is supported by a finite number of points. If $\beta = 0$ then $w_i^{(k)} = p_i$ and $v^{(k)} = \mathbb{E}[u(X)]$ for each $k \geq 1$. Assume that $\beta > 0$, the other case being similar. Notice first that, if $w_i^{(k)} = w_i^{(k+1)}$ for all $i = 1, \dots, n$, then

$$w_i^{(k)} = \begin{cases} p_i & \text{if } u_i \geq v^{(k)}, \\ (1 + \beta)p_i & \text{if } u_i < v^{(k)}, \end{cases}$$

and since $v^{(k)} = \sum_{i=1}^n \tilde{p}_i^{(k)} u_i$, it follows that $v^{(k)} = \mathbb{E}_\beta[u(X)]$. Let now

$$L_k := \{i \in [n] : u_i < v^{(k)}\}.$$

Under the assumption that $\beta > 0$, each reweighting transfers probability mass to the left, so it follows that $v^{(k+1)} \leq v^{(k)}$, implying that $L_{k+1} \subseteq L_k$. The previous observation guarantees that if $L_{k+1} = L_k$ then $v^{(k)} = \mathbb{E}_\beta[u(X)]$ and the procedure stops. If otherwise $L_{k+1} \subset L_k$ then, since L_k is finite, there is a $\bar{k} > k$ such that $L_{\bar{k}+1} = L_{\bar{k}}$, proving that the procedure stops in a finite number of steps. The case $\beta < 0$ is similar, the only difference being that now probability mass is transferred to the right so the sequence $v^{(k)}$ is increasing. The proof in the general case follows by similar arguments. \square

Proof of Theorem 8. We only provide the proof of the disappointment hedging case as the elation speculating case is similar.

(ii) \Rightarrow (i). The proof follows a similar argument to that of Theorem 4.

(i) \Rightarrow (ii). According to Theorem 12, it suffices to verify that I satisfies strong monotonicity, L^∞ -norm continuity and decrease in concordant mixtures on $B_0(\Omega, \mathcal{F}, u(\mathcal{X}))$. By probabilistic sophistication, it is straightforward to verify that I satisfies strong monotonicity. For $X, Y, W \in \mathbb{X}$ with $X \sim Y$ and $D_W = D_X$,

$$\begin{aligned} I\left(\frac{1}{2}(u(W) + u(X))\right) &= I\left(u\left(\frac{1}{2}W \oplus \frac{1}{2}X\right)\right) \\ &\leq I\left(u\left(\frac{1}{2}W \oplus \frac{1}{2}Y\right)\right) = I\left(\frac{1}{2}(u(W) + u(Y))\right), \end{aligned}$$

where the equalities follows from (22) in the statement (i), and the inequality is due to disappointment hedging. This yields decrease in concordant mixtures of I on $B_0(\Omega, \mathcal{F}, u(\mathcal{X}))$. Hence, we complete the proof. \square

References

- Abdellaoui, M., Bleichrodt, H., and Paraschiv, C. (2007). Loss aversion under prospect theory: A parameter-free measurement. *Management Science*, 53, 1659–1674.
- Alon, S. and Schmeidler, D. (2014). Purely subjective maxmin expected utility. *Journal of Economic Theory*, 152, 382–412.

- Baillon, A., Driesen, B., and Wakker, P. P. (2012). Relative concave utility for risk and ambiguity. *Games and Economic Behavior*, 75, 481–489.
- Bellini, F., Cesarone, F., Colombo, C. and Tardella, F. (2021). Risk parity with expectiles. *European Journal of Operational Research*, 291(3), 1149–1163.
- Bellini, F., Klar, B., Müller, A., and Rosazza Gianin, E. (2014). Generalized quantiles as risk measures. *Insurance: Mathematics and Economics*, 54, 41–48.
- Bellini, F., Mao, T., Wang, R., and Wu, Q. (2024). Disappointment concordance and duet expectiles. arXiv:2404.17751.
- Casadesus-Masanell, R., Klibanoff, P., and Ozdenoren, E. (2000). Maxmin expected utility over Savage acts with a set of priors. *Journal of Economic Theory*, 92(1), 35–65.
- Castagnoli, E., Cattelan, G., Maccheroni, F., Tebaldi, C., and Wang, R. (2022). Star-shaped risk measures. *Operations Research*, 70, 2637–2654.
- Cerreia-Vioglio, S., Dillenberger, D., and Ortoleva, P. (2015). Cautious expected utility and the certainty effect. *Econometrica*, 83, 693–728.
- Cerreia-Vioglio, S., Dillenberger, D., and Ortoleva, P. (2020). An explicit representation for disappointment aversion and other betweenness preferences. *Theoretical Economics*, 15, 1509–1546.
- Chandrasekhar, M., Frick, M., Iijima, R., and Le Yaouanq, Y. (2022). Dual-self representations of ambiguity preferences. *Econometrica*, 90, 1029–1061.
- Chateauneuf, A., Maccheroni, F., and Zank, H. (2025). A separation of utility and beliefs through betting consistency. *Management Science*, forthcoming.
- de Finetti, B. (1931). Sul concetto di media. *Giornale dell’Istituto Italiano degli Attuari*, 2(3), 369–396.
- Dean, M. and Ortoleva, P. (2017). Allais, Ellsberg, and preferences for hedging. *Theoretical Economics*, 12(1), 377–424.
- Föllmer, H. and Schied, A. (2016). *Stochastic Finance. An Introduction in Discrete Time*. Fourth Edition. Walter de Gruyter, Berlin.
- Ghirardato, P. and Marinacci, M. (2001). Risk, ambiguity, and the separation of utility and beliefs. *Mathematics of Operations Research*, 26(4), 864–890.
- Ghirardato, P., Maccheroni, F., Marinacci, M., and Siniscalchi, M. (2001). A subjective spin on roulette wheels (full version). *Caltech Social Science Working Paper*, 1127, August 2001. SSRN:278235.
- Ghirardato, P., Maccheroni, F., Marinacci, M., and Siniscalchi, M. (2003). A subjective spin on roulette wheels. *Econometrica*, 71(6), 1897–1908.
- Ghirardato, P., Maccheroni, F., and Marinacci, M. (2005). Certainty independence and the separation of utility and beliefs. *Journal of Economic Theory*, 120, 129–136.
- Ghirardato, P. and Pennesi, D. (2020). A general theory of subjective mixtures. *Journal of Economic Theory*, 188, 105056.

- Gilboa, I. and Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18, 141–153.
- Gneiting, T. (2011). Making and evaluating point forecasts. *Journal of the American Statistical Association*, 106, 746–762.
- Gul, F. (1991). A theory of disappointment aversion. *Econometrica*, 59, 667–686.
- Luce, R. D. and Raiffa, H. (1957). *Games and Decisions: Introduction and Critical Survey*. Wiley, New York.
- Köbberling, V. and Wakker, P. P. (2003). Preference foundations for nonexpected utility: A generalized and simplified technique. *Mathematics of Operations Research*, 28, 395–423.
- Krantz, D. H., Luce, R. D., Suppes, P., and Tversky, A. (1971). *Foundations of Measurement, Volume I: Additive and Polynomial Representations*. Academic Press, New York.
- Maccheroni, F., Marinacci, M., Wang, R. and Wu, Q. (2025). Risk aversion and insurance propensity. *American Economic Review*, 115(5), 1597–1649.
- Machina, M.J. and Schmeidler, D. (1992). A more robust definition of subjective probability. *Econometrica*, 60(4), 745–780.
- Milnor, J.W. (1954). Games against nature. In: Coombs, C.H. Davis, R.L., and Thrall, R.M. (Eds.), *Decision Processes*, 49–60. Wiley, New York.
- Newey, W.K. and Powell, J.L. (1987). Asymmetric least squares estimation and testing. *Econometrica*, 55, 819–847.
- Savage, L. (1954). *The Foundations of Statistics*. Wiley.
- Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, 57(3), 571–587.
- Shapley, L.S. (1975). Cardinal utility from intensity comparisons. *RAND Corporation Working Paper* (No. R-1683-PR).
- Tom, S.M., Fox, C.R., Trepel, C. and Poldrack, R.A. (2007). The neural basis of loss aversion in decision-making under risk. *Science*, 315(5811), 515–518.
- Yaari, M.E. (1969). Some Remarks on Measures of Risk Aversion and on Their Uses. *Journal of Economic Theory*, 1, 315–329.
- Ziegel, J.F. (2016). Coherence and elicibility. *Mathematical Finance*, 26, 901–918.