

LOCAL SPECTRAL THEORY FOR SUBORDINATED OPERATORS: THE CESÀRO OPERATOR AND BEYOND

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ABSTRACT. We study local spectral properties for subordinated operators arising from C_0 -semigroups. Specifically, if $\mathcal{T} = (T_t)_{t \geq 0}$ is a C_0 -semigroup acting boundedly on a complex Banach space and

$$\mathcal{H}_\nu = \int_0^\infty T_t \, d\nu(t)$$

is the subordinated operator associated to \mathcal{T} , where ν is a sufficiently regular complex Borel measure supported on $[0, \infty)$, it is shown that \mathcal{H}_ν does not enjoy the *Single Valued Extension Property* (SVEP) and has dense *glocal spectral subspaces* in terms of the spectrum of the generator of \mathcal{T} . Likewise, the adjoint \mathcal{H}_ν^* has trivial spectral subspaces and enjoys the Dunford property. As an application, for the classical Cesàro operator \mathcal{C} acting on the Hardy spaces H^p ($1 < p < \infty$), it follows that the local spectrum of \mathcal{C} at any non-zero H^p -function or the spectrum of the restriction of \mathcal{C} to any of its non-trivial closed invariant subspaces coincides with the spectrum of \mathcal{C} . Finally, we characterize the local spectral properties of subordinated operators arising from hyperbolic semigroups of composition operators acting on H^p ($1 < p < \infty$), which will depend only on the geometry of the associated Koenigs domain.

1. INTRODUCTION

Our starting point in this work is the remarkable theorem of Kriete and Trutt [23] which states that the classical Cesàro operator \mathcal{C} is *subnormal* in the Hardy space H^2 , as well as its counterpart in the Banach space setting H^p by Miller, Miller and Smith [26] for $1 < p < \infty$ and Persson [28] for $p = 1$, showing that \mathcal{C} is *subdecomposable*. More precisely, in H^2 the Cesàro operator \mathcal{C} is similar to the restriction of a normal operator to one of its closed invariant subspaces and, analogously, in H^p it is similar to the restriction of a decomposable operator. The decomposable operators were introduced by Foias [17] in the sixties as a generalization of spectral operators in the sense of Dunford and despite the fact that many spectral operators in Hilbert spaces as unitary operators, self-adjoint operators or more generally, normal operators are decomposable, there are still many questions about their restrictions to closed invariant subspaces unsettled. We refer to the monograph of Laursen and Neumann [24] for more on the subject.

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A classical result of Hardy [22] yields that the Cesàro operator \mathcal{C} , which takes a complex sequence $\mathbf{a} = (a_0, a_1, a_2, \dots)$ to that with n -th entry:

$$(\mathcal{C} \mathbf{a})_n = \frac{1}{n+1} \sum_{k=0}^n a_k, \quad (n \geq 0),$$

is bounded in ℓ^2 . Upon identifying sequences with Taylor coefficients of power series, \mathcal{C} acts *formally* on $f(z) = \sum_{k=0}^{\infty} a_k z^k$ as

$$(1.1) \quad \mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n,$$

and turns out to be a bounded operator on the classical Hardy spaces on H^p for $1 \leq p < \infty$ (see [30] and [31], for instance).

In this setting, the theory of C_0 -semigroups turned out to be a powerful tool for studying the Cesàro operator, as shown in Siskakis's paper [30] or Cowen's [12], which gives an alternative proof that \mathcal{C} is subnormal. More recently, in [19] the authors made use that the adjoint of the Cesàro operator \mathcal{C} appears as the resolvent operator of the infinitesimal generator of the semigroup of composition operators $(C_{\varphi_t})_{t \geq 0}$ acting on H^p , given by

$$\varphi_t(z) = e^{-t}z + (1 - e^{-t}), \quad (z \in \mathbb{D}, t \geq 0).$$

That is,

$$(\mathcal{C}^* f)(z) = - \int_0^{\infty} e^{-t} f(e^{-t}z + (1 - e^{-t})) dt,$$

for $f \in H^p$ and $z \in \mathbb{D}$. This representation of \mathcal{C}^* turns out to be a particular instance of what is called a *subordinated operator* associated to a semigroup. In general, if $\mathcal{T} = (T_t)_{t \geq 0}$ is a C_0 -semigroup of bounded operators acting on a Banach space X , and ν is a 'sufficiently regular' complex Borel measure supported on $[0, \infty)$, the operator

$$\mathcal{H}_{\nu} = \int_0^{\infty} T_t d\nu(t)$$

is said to be a subordinated operator to \mathcal{T} (see [21, Remark 3.3.3], for instance).

Recently, the authors of [18] undertook a study of the Local Spectral Properties in H^p -spaces of composition operators induced by linear fractional maps, characterizing in particular the associated local spectra and the local spectral subspaces, among others (see Section 2 or [24] for definitions). Our main goal in this work is to study the local spectral properties of subordinated operators associated to C_0 -semigroups. Having in mind that the spectrum of the Cesàro operator \mathcal{C} in H^p is the closed disc

$$\sigma(\mathcal{C}; H^p) = \left\{ z \in \mathbb{C} : \left| z - \frac{2}{p} \right| \leq \frac{2}{p} \right\},$$

(see [30]), as a particular instance, we derive the following result for the local spectra and the local spectral subspaces of \mathcal{C} :

Theorem. *Let $1 < p < \infty$ and \mathcal{C} be the Cesàro operator acting on the Hardy space H^p . Then:*

- (1) *The local spectrum of \mathcal{C} at any non-zero H^p -function coincides with the spectrum of \mathcal{C} .*

- (2) *The local spectral subspace associated to any closed proper subset F of $\sigma(\mathcal{C}; H^p)$ is trivial, namely, it is the zero subspace.*
- (3) *If M is a non-trivial closed invariant subspace of \mathcal{C} , then $\sigma(\mathcal{C}|_M; H^p) = \sigma(\mathcal{C}; H^p)$.*

It is worthy to point out the increasing interest in studying (local) spectral properties of the so-called *generalized Cesàro operators*

$$f \in H^p \mapsto \begin{cases} \frac{1}{z} \int_0^z f(s)g'(s)ds, & z \in \mathbb{D} \setminus \{0\}, \\ f(0) & z = 0; \end{cases},$$

or the non-normalized version, namely *integration operators*

$$f \in H^p \mapsto \int_0^z f(s)g'(s)ds, \quad (z \in \mathbb{D}),$$

where g is a fixed analytic function in the unit disc. While these operators turn out to be bounded on H^p whenever $g' \in BMOA$ (see [2], for instance), regarding spectral properties or subdecomposability, the most satisfying results hold assuming that g' is a ‘small’ perturbation of a rational function, and the general questions remain open (see [3, 4, 25, 36], for instance).

Subordinated operators in H^p associated to semigroups (or semiflows) of holomorphic functions can also be understood as generalizations of Cesàro operators. In this sense, the stated theorem for the Cesàro operator will hold equivalently for a wider class of operators that arise as the adjoints of operators of the form

$$(\mathcal{H}_{\nu_\rho} f)(z) = \int_z^1 h'(s)\rho(h(s) - h(z))f(s)ds, \quad (z \in \mathbb{D} \text{ and } f \in H^p),$$

where $d\nu_\rho(t) = \rho(t)dt$ is an absolutely continuous measure respect to the Lebesgue measure with $\rho : [0, \infty) \rightarrow \mathbb{C}$ a sufficiently regular density function and h is the Koenigs function associated to the semigroup (or semiflow) of holomorphic functions $(\varphi_t)_{t \geq 0}$ in \mathbb{D} (see Subsection 4.3).

We conclude this introduction with an outline of the rest of the manuscript. In Section 2 we recall some of the preliminaries needed throughout the rest of the text, such as C_0 -semigroups and their subordinated operators, as well as some background on Local Spectral Theory.

In Section 3 we prove Theorem 3.1, which will allow us to transfer some of the (local) spectral properties of the generator of a C_0 -semigroup to associated subordinated operators. Indeed, if a weak spectral mapping theorem holds for the C_0 -semigroup, such properties can be also transferred to the operators of the semigroup (see Remark 3.2).

In Section 4 we introduce some of the basic notions of semigroups (or semiflows) of holomorphic functions in \mathbb{D} and derive a result that assures the density of the eigenfunctions of infinitesimal generators of hyperbolic semigroups in terms of the geometric properties of the associated Koenigs domain (Proposition 4.3) based on a recent work of Bracci, Gallardo-Gutiérrez and Yakubovich [8]. As a consequence, we prove a result that provides local spectral properties for subordinated operators depending only on the geometry of the Koenigs domain associated to the hyperbolic semigroup (Theorem 4.4). Finally, we exhibit particular instances of subordinated operators for which the result applies (Subsection 4.3) and discuss further applications to non-hyperbolic semigroups of composition operators.

2. PRELIMINARIES

Throughout the rest of the text, X will stand for an infinite-dimensional complex Banach space and $\mathcal{L}(X)$ for the algebra of linear bounded operators acting on X .

Recall that a one-parameter family $\mathcal{T} = (T_t)_{t \geq 0} \subset \mathcal{L}(X)$ is a C_0 -semigroup if:

- (1) $T_0 = Id_X$, the identity operator;
- (2) $T_{t+s} = T_t T_s$ for every $t, s \geq 0$;
- (3) $T_t \rightarrow Id_X$ when $t \rightarrow 0$ in the strong operator topology.

Every C_0 -semigroup satisfies

$$(2.1) \quad \|T_t\| \leq M e^{\omega t} \quad \text{for } t \geq 0,$$

for some $M \geq 0$ and $\omega \in \mathbb{R}$. The infimum of those ω such that (2.1) holds for some $M \geq 0$ is called the *type* of the C_0 -semigroup (see [16, p. 250]) and can be computed as

$$\omega_0 := \inf_{t > 0} \frac{\log(\|T_t\|)}{t}.$$

Given a C_0 -semigroup $(T_t)_{t \geq 0}$ on a Banach space X , recall that its generator is the closed and densely defined linear operator Δ defined by

$$\Delta x = \lim_{t \rightarrow 0^+} \frac{T_t x - x}{t}$$

with domain $\text{Dom}(\Delta) = \{x \in X : \lim_{t \rightarrow 0^+} \frac{T_t x - x}{t} \text{ exists}\}$. The semigroup is determined uniquely by its generator.

It is a classical fact that the spectrum of Δ in X , namely, those complex numbers z such that $\Delta - zI$ is not invertible in X , satisfies

$$\sigma(\Delta) \subseteq \{z \in \mathbb{C} : \text{Re}(z) \leq \omega_0\}$$

(see [16, Proposition 2.2], for instance).

A word about notation. For simplicity, the spectrum of an operator T on X is denoted by $\sigma(T)$ unless it is necessary to specify the underlying space, when it is denoted by $\sigma(T; X)$.

For $z \in \mathbb{C}$ with $\text{Re}(z) > \omega_0$, it is possible to express the resolvent operator

$$R(z, \Delta) = -(\Delta - zI)^{-1}$$

in terms of the Laplace transform of $(T_t)_{t \geq 0}$, namely

$$(2.2) \quad R(z, \Delta)x = \int_0^\infty e^{-zt} T_t x \, dt \quad (x \in X),$$

(see [16, p. 231]). Likewise, the spectral properties of the resolvent operators $R(z, \Delta)$ as well as the operators T_t in the semigroup can be related with the ones of the infinitesimal generator by means of *spectral mapping theorems*. In particular, if $\sigma_p(T)$ denotes the point spectrum of an operator T , namely, the set consisting of eigenvalues of T , the following holds:

Proposition 2.1. *Let X be an infinite-dimensional complex Banach space, $(T_t)_{t \geq 0} \subset \mathcal{L}(X)$ a C_0 -semigroup and Δ its infinitesimal generator. Then,*

- (1) $\{e^{zt} : z \in \sigma(\Delta)\} \subseteq \sigma(T_t)$ for every $t \geq 0$;
- (2) $\{e^{zt} : z \in \sigma_p(\Delta)\} = \sigma_p(T_t) \setminus \{0\}$ for every $t \geq 0$.

Moreover, for every complex number λ in the resolvent of Δ , it holds that

- (3) $\sigma(R(\lambda, \Delta)) = \{0\} \cup \{(\lambda - z)^{-1} : z \in \sigma(\Delta)\}$;
- (4) $\{(\lambda - z)^{-1} : z \in \sigma_p(\Delta)\} \subseteq \sigma_p(R(\lambda, \Delta))$.

We refer to [16, Chapter IV, Sections 3.6 and 3.7] and [20, Theorem 6.4 and Corollary 6.6], for instance.

The equation (2.2) provides particular instances of subordinated operators associated to a semigroup according to the following:

Definition 2.2. Let ν be a finite complex Borel measure in $[0, +\infty)$ such that $\int_0^\infty e^{(\omega_0 + \delta)t} d|\nu|(t) < \infty$ for some $\delta > 0$ and $\mathcal{T} = (T_t)_{t \geq 0} \subset \mathcal{L}(X)$ a C_0 -semigroup. The subordinated operator \mathcal{H}_ν associated to \mathcal{T} is given by

$$(2.3) \quad \mathcal{H}_\nu x = \int_0^\infty T_t x \, d\nu(t), \quad (x \in X).$$

Before going further, let us point out that (2.3) defines a linear bounded operator in X because the integral is Bochner-convergent. Clearly, the absolute continuous measure $d\nu_\lambda(t) = e^{-\lambda t} dt$ where $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > \omega_0$ yields that $\mathcal{H}_{\nu_\lambda} = R(\lambda, \Delta)$.

2.1. Sectorial operators. In order to study the local spectral properties of subordinated operators, we recall some spectral mapping theorems in the lines of Proposition 2.1. For such a purpose, we introduce a *toolkit* regarding functional calculus for sectorial operators and refer the reader to the monograph [21].

Given ν a finite complex Borel measure in $[0, \infty)$ of bounded variation, the *Laplace transform* of ν is given by

$$\mathcal{L}(\nu)(z) = \int_0^\infty e^{-zt} \, d\nu(t),$$

where $z \in \mathbb{C}$ with $\operatorname{Re}(z) \geq 0$. It is a classical fact that $\mathcal{L}(\nu)$ is a holomorphic function in the right-half plane $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ and continuous in its closure.

Given $0 \leq \omega \leq \pi$, let S_ω be the sector in the complex plane given by

$$S_\omega = \begin{cases} \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\} & \text{if } \omega \in (0, \pi) \\ \{z \in \mathbb{C} \setminus \{0\} : \operatorname{Im}(z) = 0, \operatorname{Re}(z) > 0\} & \text{if } \omega = 0. \end{cases}$$

Recall that a (possibly unbounded operator) Δ acting on X is called *sectorial of angle* $\omega \in [0, \pi)$, denoted by $\Delta \in \operatorname{Sect}(\omega)$ if it satisfies

- (i) $\sigma(\Delta) \subset \overline{S_\omega}$ and
- (ii) $M(A, \omega') := \sup\{|\lambda R(\lambda, \Delta)| : \lambda \in \mathbb{C} \setminus \overline{S_{\omega'}}\} < \infty$ for all $\omega' \in (\omega, \pi)$.

It is a standard fact that if Δ is the infinitesimal generator of a uniformly bounded semigroup of operators, that is,

$$\sup_{t \geq 0} \|T_t\| < \infty,$$

then $-\Delta$ is a sectorial operator of angle $\pi/2$. In particular, if $(T_t)_{t \geq 0} \subset \mathcal{L}(X)$ is a C_0 -semigroup of type $\omega_0 \in \mathbb{R}$ and Δ is its infinitesimal generator, the semigroup

$$(e^{-(\omega_0 + \varepsilon)t} T_t)_{t \geq 0}$$

is a uniformly bounded semigroup for every $\varepsilon > 0$, so $(\omega_0 + \varepsilon)I - \Delta$ is sectorial of angle $\pi/2$.

Now, we introduce a functional calculus for sectorial operators of angle $\pi/2$. First, we recall the following:

Definition 2.3. Let A be a sectorial operator of angle $\pi/2$ in X and $\omega \in (\pi/2, \pi)$. The domain of the functional calculus of A , denoted by $\mathcal{E}(A)$, consists of holomorphic functions $f : S_\omega \rightarrow \mathbb{C}$ such that the limits

$$\lim_{z \rightarrow 0} f(z) := d_0 \quad \lim_{z \rightarrow \infty} f(z) := d_\infty$$

are finite and satisfy that for some $r, R > 0$, that the integrals

$$\int_{\partial(S_{\omega'} \cap \{|z| < r\})} \left| \frac{f(z) - d_0}{z} \right| |dz| < \infty, \quad \int_{\partial(S_{\omega'} \cap \{|z| > R\})} \left| \frac{f(z) - d_\infty}{z} \right| |dz| < \infty$$

converges for every $\omega' \in (\pi/2, \omega)$.

Here $\partial(S_{\omega'} \cap \{|z| < r\})$ denotes the boundary of the set $S_{\omega'} \cap \{|z| < r\}$.

Given A a sectorial operator of angle $\pi/2$ in X and $f \in \mathcal{E}(A)$, the operator $f(A)$ may be defined as

$$f(A) = d_\infty + d_0(I + A)^{-1} + \int_{\partial S_{\omega'}} (f(z) - d_\infty - \frac{d_0}{z+1})(zI - A)^{-1} dz,$$

where $\omega' \in (\pi/2, \pi)$. Note that this definition does not depend on the selected ω' .

In particular, if $\mathcal{T} = (T_t)_{t \geq 0}$ is a C_0 -semigroup with infinitesimal generator Δ of type $w_0 \in \mathbb{R}$, the operator $(\omega_0 + \varepsilon)I - \Delta$ is a sectorial operator of angle $\pi/2$, so the functional calculus defined applies.

In order to simplify the notation, and following that of [1, p. 27], we write $f \in \mathcal{E}(-\Delta)$ if $f_{\omega_0 + \varepsilon} \in \mathcal{E}((\omega_0 + \varepsilon)I - \Delta)$, where $f_{\omega_0 + \varepsilon}(z) = f(z - \omega_0 - \varepsilon)$. In such a case, we define

$$f(-\Delta) = f_{\omega_0 + \varepsilon}((\omega_0 + \varepsilon)I - \Delta).$$

With this notation, we can state the desired Spectral Mapping Theorem for our purposes:

Theorem 2.4. Let $\mathcal{T} = (T_t)_{t \geq 0} \subset \mathcal{L}(X)$ be a C_0 -semigroup of operators with infinitesimal generator Δ of type $w_0 \in \mathbb{R}$. Let ν be a finite complex Borel measure in $[0, \infty)$ such that $\int_0^\infty e^{(\omega_0 + \delta)t} d|\nu|(t) < \infty$ for some $\delta > 0$. Let \mathcal{H}_ν be the subordinated operator associated to \mathcal{T} and assume that $\mathcal{L}(\nu) \in \mathcal{E}(-\Delta)$. Then the following assertions hold:

- (1) The spectrum of \mathcal{H}_ν is

$$\sigma(\mathcal{H}_\nu) = \mathcal{L}(\nu)(\tilde{\sigma}(-\Delta))$$

where $\tilde{\sigma}(-\Delta) = \sigma(-\Delta) \cup \{\infty\}$.

- (2) The point spectrum of \mathcal{H}_ν satisfies

$$\mathcal{L}(\nu)(\sigma_p(-\Delta)) \subseteq \sigma_p(\mathcal{H}_\nu) \subseteq \mathcal{L}(\nu)(\sigma_p(-\Delta)) \cup \mathcal{L}(\nu)(M),$$

where $M = \{0, \infty\} \cap \tilde{\sigma}(-\Delta)$.

Moreover, $\ker(-\Delta - \lambda I) \subseteq \ker(\mathcal{H}_\nu - \mathcal{L}(\nu)(\lambda) I)$ for every $\lambda \in \sigma_p(-\Delta)$.

Proof. An argument along the lines of [1, Corollary 5.3] yields that $\mathcal{L}(\nu) \in \mathcal{E}(-\Delta)$. Moreover, $\mathcal{H}_\nu = \mathcal{L}(\nu)(-\Delta)$ by [21, Proposition 3.3.2], so both statements (1) and (2) in Theorem 2.4 follows by [20, Theorem 6.4 and Corollary 6.6].

Finally, let $\lambda \in \sigma_p(-\Delta)$ and $x \in \ker(-\Delta - \lambda I)$. By Proposition 2.1, $x \in \ker(T_t - e^{-\lambda t})$. Since,

$$\mathcal{H}_\nu x = \int_0^\infty T_t x d\nu(t) = x \int_0^\infty e^{-\lambda t} d\nu(t) = \mathcal{L}(\nu)x,$$

the last statement of Theorem 2.4 holds and the proof follows. \square

It is worthy to remark that as a consequence of [1, Lemma 5.2 and Corollary 5.3], next proposition yields a relatively easy way to provide measures ν satisfying the hypothesis of the previous result.

Proposition 2.5. *Let ν be a finite complex Borel measure on $[0, \infty)$ such that $\int_0^\infty e^{(\omega_0 + \delta)t} |d\nu|(t) < \infty$ for some $\delta > 0$. Assume $d\nu(t) = h(t)dt$, where h can be extended holomorphically to a sector S_ω with $\omega \in (0, \pi/2]$. Suppose further that there exist $\eta \in (0, 1]$ and $\xi \in (0, 1)$ such that*

$$\sup_{z \in S_\xi \cap \{|z| \leq 1\}} |z^{1-\eta} h(z)| < \infty \quad \sup_{z \in S_\xi \cap \{|z| \geq 1\}} |z^{1+\xi} e^{(\omega_0 + \delta)z} h(z)| < \infty$$

for all $0 < \varepsilon < \omega$. Then, $\mathcal{L}(\nu) \in \mathcal{E}(-\Delta)$.

We conclude this subsection with a basic fact about adjoints of C_0 -semigroups of operators. For such a task, recall that if Z is a closed operator with dense domain \mathcal{D} in a Banach space X , the *adjoint* Z^* with domain $\mathcal{D}^* \subseteq X^*$ is defined as follows: $\phi \in \mathcal{D}^*$ if and only if the linear functional $f \mapsto \langle Zf, \phi \rangle$ is norm bounded on \mathcal{D} ; if $\phi \in \mathcal{D}^*$, then $Z^*\phi$ is the unique bounded extension of that linear functional to X . Hence

$$\langle Zf, \phi \rangle = \langle f, Z^*\phi \rangle$$

for all $f \in \mathcal{D}$ and $\phi \in \mathcal{D}^*$ (see [13]).

Theorem 2.6. [13, Theorem 1.34] *Let X be a reflexive Banach space and $(T_t)_{t \geq 0} \subseteq \mathcal{L}(X)$ a C_0 -semigroup with infinitesimal generator Δ . Then, $(T_t^*)_{t \geq 0} \subseteq \mathcal{L}(X^*)$ is also a C_0 -semigroup whose infinitesimal generator is Δ^* .*

2.2. Local Spectral Theory. In this subsection we will recall some of the basic features of Local Spectral Theory and refer the reader to the monograph by Laursen and Neumann [24].

Given $T \in \mathcal{L}(X)$ and $x \in X$, the *local resolvent* of T at x , denoted as $\rho_T(x)$, is defined as the union of all open sets $U \subset \mathbb{C}$ such that there exists an analytic function $f_x : U \rightarrow X$ satisfying

$$(2.4) \quad (T - zI)f_x(z) = x \quad (z \in U).$$

In particular, $\rho_T(x)$ is an open set that contains the resolvent of the operator $\rho(T)$. An operator $T \in \mathcal{L}(X)$ is said to have the *single-valued extension property* (SVEP) if for every open set U and every analytic function $f_x : U \rightarrow X$ the equality

$$(T - zI)f_x(z) = 0 \quad (z \in U)$$

implies that $f_x \equiv 0$. This property assures that the analytic function solving (2.4) is unique, so one can define the *local resolvent function* of T at x as the unique analytic map $f_x : \rho_T(x) \rightarrow X$ such that (2.4) holds.

The *local spectrum* of $T \in \mathcal{L}(X)$ at $x \in X$ is defined as $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. This set is always a (possibly empty) compact subset of the spectrum $\sigma(T)$. Having in mind the spectral radius formula

$$r(T) = \limsup_{n \rightarrow \infty} \|T^n\|^{1/n},$$

the *local spectral radius* of T at x is defined by

$$r_T(x) := \limsup_{n \rightarrow \infty} \|T^n x\|^{1/n}.$$

If T has the SVEP, a local spectral radius formula satisfies:

$$(2.5) \quad r_T(x) = \max\{|\lambda| : \lambda \in \sigma_T(x)\}, \quad (x \in X \setminus \{0\}),$$

(see [24, Proposition 3.3.13]).

The definition of the local spectrum allows us to introduce the local spectral subspaces, or simply, the spectral subspaces associated to T .

For any set $A \subset \mathbb{C}$, the *spectral subspace* of T associated to A , denoted as $X_T(A)$, is defined as

$$X_T(A) = \{x \in X : \sigma_T(x) \subseteq A\}.$$

In other words, $X_T(A)$ consists of those vectors $x \in X$ for which $(T - zI)$ is locally invertible for $z \in \mathbb{C} \setminus A$. We note that $X_T(A) = X_T(A \cap \sigma(T))$ for every $A \subset \mathbb{C}$.

The spectral subspaces turn out to be (non-necessarily closed) subspaces which are hyperinvariant by T , namely, invariant under every operator in the commutant of T . If all the spectral subspaces of an operator T associated to closed sets are closed, then T is said to enjoy the *Dunford property (C)*. Clearly, the Dunford property implies the SVEP.

We also recall the *glocal spectral subspaces* $\mathcal{X}_T(F)$, which are a variant of spectral subspaces that in general are better suited for operators lacking the SVEP. If $F \subset \mathbb{C}$ is a closed set, the glocal spectral subspace $\mathcal{X}_T(F)$ consists of all $x \in X$ such that there exists an analytic function $f_x : \mathbb{C} \setminus F \rightarrow X$ satisfying

$$(T - zI)f_x(z) = x, \quad (z \in \mathbb{C} \setminus F).$$

If T has the SVEP, then $X_T(F) = \mathcal{X}_T(F)$ for every closed set $F \subset \mathbb{C}$. In general, one has the inclusion $\mathcal{X}_T(F) \subseteq X_T(F)$.

Glocal spectral subspaces are also of interest when dealing with adjoint of operators. Namely, if X is a complex Banach space and $T \in \mathcal{L}(X)$, for every disjoint closed sets $F_1, F_2 \subseteq \mathbb{C}$, it follows:

$$(2.6) \quad \mathcal{X}_{T^*}^*(F_1) \subseteq \mathcal{X}_T(F_2)^\perp,$$

(see [24, Proposition 2.5.1]).

3. TRANSFERRING THE (LOCAL) SPECTRAL PROPERTIES OF INFINITESIMAL GENERATORS

In this section we prove Theorem 3.1 which will allow us to transfer some of the (local) spectral properties of the infinitesimal generator Δ of a C_0 -semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ to subordinated operators \mathcal{H}_ν associated to \mathcal{T} , as well as for their adjoints. The result extends the conclusions of Theorem 2.1 in [18] to the context of a C_0 -semigroups and reads as follows:

Theorem 3.1. *Let X be a complex Banach space and $\mathcal{T} = (T_t)_{t \geq 0} \subset \mathcal{L}(X)$ a C_0 -semigroup of type $w_0 \in \mathbb{R}$ and infinitesimal generator Δ . Assume that $\overline{\text{int}(\sigma(-\Delta))} = \sigma(-\Delta)$ is connected and that there exists a non-vanishing analytic map $f_\Delta : \text{int}(\sigma(-\Delta)) \rightarrow \text{Dom}(-\Delta)$ such that*

$$(-\Delta - zI)f_\Delta(z) = 0 \quad \text{for } z \in \text{int}(\sigma(-\Delta)).$$

Assume further that the linear span

$$\text{span} \{f_\Delta(\lambda) : \lambda \in \text{int}(\sigma(-\Delta))\}$$

is dense in X . Let ν be a finite complex Borel measure supported in $[0, \infty)$ such that $\int_0^\infty e^{(\omega_0 + \delta)t} d|\nu|(t) < \infty$ for some $\delta > 0$ and $\mathcal{L}(\nu)$ is non-constant in $\mathcal{E}(-\Delta)$. Then the subordinated operator \mathcal{H}_ν to \mathcal{T} satisfies the following properties:

- (1) \mathcal{H}_ν does not enjoy the SVEP.
- (2) For every relatively open subset non-empty $U \subset \sigma(\mathcal{H}_\nu)$, $\mathcal{X}_{\mathcal{H}_\nu}(\overline{U})$ is dense in X .
- (3) $\sigma_p(\mathcal{H}_\nu^*) = \emptyset$.
- (4) $X_{\mathcal{H}_\nu^*}(F) = \{0\}$ for every closed subset $F \subsetneq \sigma(\mathcal{H}_\nu^*)$.
- (5) \mathcal{H}_ν^* has the Dunford property (C).
- (6) $\sigma_{\mathcal{H}_\nu^*}(x) = \sigma(\mathcal{H}_\nu^*)$ for every $x \in X^* \setminus \{0\}$.
- (7) $r_{\mathcal{H}_\nu^*}(x) = r(\mathcal{H}_\nu^*)$ for every $x \in X^* \setminus \{0\}$.
- (8) If M is a non-trivial closed invariant subspace for \mathcal{H}_ν^* , then $\sigma(\mathcal{H}_\nu^*) \subseteq \sigma(\mathcal{H}_\nu^*|_M) \subset \eta(\sigma(\mathcal{H}_\nu^*))$, where $\eta(\sigma(\mathcal{H}_\nu^*))$ denotes the full spectrum of $\sigma(\mathcal{H}_\nu^*)$, namely, the union of $\sigma(\mathcal{H}_\nu^*)$ and all the bounded components of the resolvent set $\rho(\mathcal{H}_\nu^*)$.

Proof. We begin by noting that the existence of the analytic function f_Δ implies that $\text{int}(\sigma(-\Delta)) \subseteq \sigma_p(-\Delta)$. In particular, by Theorem 2.4, it follows that

$$\sigma(\mathcal{H}_\nu) = \mathcal{L}(\nu)(\tilde{\sigma}(-\Delta)) \text{ and } \mathcal{L}(\nu)(\sigma_p(-\Delta)) \subseteq \sigma_p(\mathcal{H}_\nu).$$

Let us start with the proof of (1). Since $\mathcal{L}(\nu)$ is holomorphic in the translated sector $-(\omega_0 + \delta) + S_\omega$, which contains $\sigma(-\Delta)$, the set $\mathcal{L}(\nu)(\text{int}(\sigma(-\Delta)))$ is open and

$$(3.1) \quad \mathcal{L}(\nu)(\text{int}(\sigma(-\Delta))) \subset \sigma(\mathcal{H}_\nu).$$

By hypothesis, the function $\mathcal{L}(\nu)'$ is not zero, so there exists an open subset $G \subseteq \text{int}(\sigma(-\Delta)) \subseteq \sigma_p(-\Delta)$ such that

$$\mathcal{L}(\nu) : G \rightarrow \mathcal{L}(\nu)(G)$$

is biholomorphic. The analytic map $f_\Delta \circ \mathcal{L}(\nu)^{-1} : \mathcal{L}(\nu)(G) \rightarrow X$ satisfies that

$$(\mathcal{H}_\nu - zI)f_\Delta \circ \mathcal{L}(\nu)^{-1}(z) = 0$$

for every $z \in \mathcal{L}(\nu)(G)$. Accordingly, \mathcal{H}_ν does not enjoy the SVEP and (1) follows.

In order to prove (2), we note that for every non-empty open set $\Omega \subset \text{int}(\sigma(-\Delta))$,

$$\text{span} \{f_\Delta(\lambda) : \lambda \in \Omega\}$$

is dense in X since the map $f_\Delta : \text{int}(\sigma(-\Delta)) \rightarrow \text{Dom}(-\Delta)$ is holomorphic. Likewise, the hypotheses on the spectrum $\sigma(-\Delta)$ yield that it has no isolated points. So, if $U \subset \sigma(\mathcal{H}_\nu)$ is a relatively open non-empty set, having in mind that $\sigma(\mathcal{H}_\nu) = \mathcal{L}(\nu)(\tilde{\sigma}(-\Delta))$, $\text{int}(\sigma(-\Delta)) \subset \sigma_p(-\Delta)$ and $\mathcal{L}(\nu)(\sigma_p(-\Delta)) \subseteq \sigma_p(\mathcal{H}_\nu)$, there exists an open subset $U_1 \subseteq U \cap \sigma_p(\mathcal{H}_\nu)$. Now, by Theorem 2.4, $U_1 \subseteq U \cap \sigma_p(\mathcal{H}_\nu) =$

$U \cap \mathcal{L}(\nu)(\sigma_p(-\Delta)) \setminus \mathcal{L}(\nu)(M)$, so there exists a non-empty open subset $U_2 \subseteq \mathcal{L}(\nu)(\text{int}(\sigma_p(-\Delta))) \cap U_1$, since $\mathcal{L}(\nu)(M)$ consists, at most, of two points. Accordingly, there exists a non-empty open subset $G \subseteq \text{int}(\sigma_p(-\Delta)) \subseteq \text{int}(\sigma(-\Delta))$ such that $\mathcal{L}(\nu)(G)$ is contained in U_2 .

Now,

$$\text{span } \{f_\Delta(\lambda) : \lambda \in G\}$$

is dense in X and observe that

$$\mathcal{H}_\nu f_\Delta(\lambda) = \mathcal{L}(\nu)(\lambda) f_\Delta(\lambda), \quad (\lambda \in G);$$

so $\mathcal{L}(\nu)(\lambda)$ is an eigenvalue for \mathcal{H}_ν for every $\lambda \in G$. But $\mathcal{L}(\nu)(G) \subset U$, so by [24, Proposition 3.3.1] and the inclusion properties of global spectral subspaces, it yields that $\text{span } \{f(\lambda) : \lambda \in G\} \subseteq \mathcal{X}_{\mathcal{H}_\nu}(\overline{U})$ and (2) follows.

In order to show (3), recall that $\sigma(-\Delta)$ is not a singleton by hypothesis, so neither is $\sigma(\mathcal{H}_\nu)$. Fix $\lambda_0 \in \sigma(\mathcal{H}_\nu^*) = \sigma(\mathcal{H}_\nu)$. By [24, Proposition 3.3.1], we have that

$$\ker(\mathcal{H}_\nu^* - \lambda I) \subset \mathcal{X}_{\mathcal{H}_\nu^*}(\{\lambda_0\}).$$

Now, let $U \subset \sigma(\mathcal{H}_\nu)$ be a non-empty relatively open set not containing λ_0 in its closure, which can be achieved since the spectrum $\sigma(\mathcal{H}_\nu)$ is not a singleton. Then, having in mind (2.6), we deduce that

$$\mathcal{X}_{\mathcal{H}_\nu^*}(\{\lambda_0\}) \subseteq \mathcal{X}_{\mathcal{H}_\nu}(\overline{U})^\perp = \{0\}.$$

Hence, $\lambda_0 \notin \sigma_p(\mathcal{H}_\nu)$, which concludes the proof. Observe that, in particular, \mathcal{H}_ν^* has the SVEP, and $X_{\mathcal{H}_\nu^*}^*(F) = \mathcal{X}_{\mathcal{H}_\nu^*}^*(F)$ for every closed set $F \subset \mathbb{C}$.

Let us prove (4). Let F be a proper closed subset of $\sigma(\mathcal{H}_\nu^*)$. Since F is proper, there exists a relatively open subset $U \subset \sigma(\mathcal{H}_\nu^*)$ such that $\overline{U} \cap F = \emptyset$. Then, as in the proof of (3), by (2.6) and the density of $\mathcal{X}_{\mathcal{H}_\nu}(\overline{U})$, we deduce that $X_{\mathcal{H}_\nu^*}^*(F) = \{0\}$, as we wished to prove.

As a consequence of (4), the proof of (5) follows directly: if $F \subsetneq \sigma(\mathcal{H}_\nu^*)$, then $X_{\mathcal{H}_\nu^*}^*(F) = \{0\}$, and if $F = \sigma(T)$, it is trivial to check that $X_{\mathcal{H}_\nu^*}^*(F) = X^*$. In particular, $X_{\mathcal{H}_\nu^*}^*(F)$ is a closed linear manifold for every closed set $F \subset \sigma(\mathcal{H}_\nu^*)$, which yields the statement.

In order to prove (6) and (7), we observe that statement (4) directly implies that (6) holds, and such identity together with (2.5) yields (7).

To conclude the proof of Theorem 3.1, we show (8). Let $M \subset X^*$ be a non-trivial closed invariant subspace for \mathcal{H}_ν^* . Then, by [24, Proposition 1.2.16 (e)] it follows that $M \subset X_{\mathcal{H}_\nu^*}(\sigma(\mathcal{H}_\nu^*|_M))$. But $M \neq \{0\}$, so by (4) we deduce that $\sigma(\mathcal{H}_\nu^*) \subseteq \sigma(\mathcal{H}_\nu^*|_M)$. For the second containment, it is enough to apply [29, Theorem 0.8]. \square

Remark 3.2. Under the hypothesis of Theorem 3.1, if a weak spectral mapping formula of the form

$$(3.2) \quad \sigma(T_t) = \overline{\{e^{\lambda t} : \lambda \in \sigma(\Delta)\}} \quad (t \geq 0)$$

holds, then the statements (1) – (8) in Theorem 3.1 are also satisfied for each of the operators T_t in the semigroup \mathcal{T} besides the subordinated operator \mathcal{H}_ν associated to it.

It is worth mentioning that the equality (3.2) does not hold for every C_0 -semigroup acting on a complex Banach space X , see for instance [16, Chapter IV, Section 3.3]. Nevertheless, (3.2), even in

a stronger version, does hold for a large variety of C_0 -semigroups, such as eventually compact semigroups, analytic semigroups, uniformly continuous semigroups or C_0 -semigroups of normal operators on Hilbert spaces [16, Chapter IV, Corollaries 3.12 and 3.14].

4. SUBORDINATED OPERATORS ARISING FROM SEMIGROUPS OF COMPOSITION OPERATORS

Our main goal in this section is applying Theorem 3.1 to a large collection of semigroups of composition operators acting on the Hardy spaces H^p of the unit disc \mathbb{D} , $1 < p < \infty$, which will allow us to get local spectral properties of the subordinated operators associated to such semigroups as well as of their adjoints. A particular instance is the classical Cesàro operator, mentioned in the introduction. The subordinated operators that emerge turn out to be *averaging operators* introduced by Siskakis in 1993 [32].

4.1. Semigroups of analytic functions in \mathbb{D} . Recall that a family $(\varphi_t)_{t \geq 0}$ of holomorphic functions on \mathbb{D} is said to be a *semigroup (or semiflow)* in \mathbb{D} if the map $t \in [0, +\infty) \mapsto \mathcal{H}(\mathbb{D})$ is a continuous semigroup homomorphism, where $[0, +\infty)$ is endowed with the Euclidean topology and $\mathcal{H}(\mathbb{D})$ with the topology of uniform convergence on compact subsets of \mathbb{D} . In particular, the following properties yield:

- (1) $\varphi_0(z) = z$ for every $z \in \mathbb{D}$.
- (2) $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for every $s, t \geq 0$.
- (3) $\lim_{t \rightarrow s} \varphi_t = \varphi_s$ uniformly on compact subsets of \mathbb{D} for $s, t \geq 0$.

In such a case, the function $t \mapsto \varphi_t(z)$ is differentiable for each $z \in \mathbb{D}$, and there exists a unique function $G : \mathbb{D} \rightarrow \mathbb{C}$ such that

$$\frac{\partial \varphi_t(z)}{\partial t} \Big|_{t=t_0} = G(\varphi_{t_0}(z)), \quad (z \in \mathbb{D}, t_0 \geq 0).$$

This function G is the *infinitesimal generator* of the semigroup $(\varphi_t)_{t \geq 0}$.

The semigroups in \mathbb{D} can be classified with respect to the configuration of the fixed points of the functions of the semigroups. In this respect, a semigroup $(\varphi_t)_{t \geq 0}$ is *elliptic* if there exists $z_0 \in \mathbb{D}$ such that $\varphi_{t_0}(z_0) = z_0$ for some $t_0 > 0$. Otherwise, we will say that the semigroup is *non-elliptic*.

In this latter case, there exists a unique point $\tau \in \mathbb{T}$, called the *Denjoy-Wolff point* such that φ_t converges uniformly on compact subsets of \mathbb{D} to the constant function $z \mapsto \tau$ when $t \rightarrow +\infty$. Moreover, $\varphi_t(z)$ has non-tangential limits $e^{t\alpha}$ at the Denjoy-Wolff point, with some $\alpha \leq 0$. If $\alpha < 0$, the semigroup $(\varphi_t)_{t \geq 0}$ is said to be *hyperbolic*, and it is said to be *parabolic* when $\alpha = 0$.

For non-elliptic semigroups, there exists a unique representation

$$(4.1) \quad \varphi_t(z) = h^{-1}(h(z) + t), \quad (z \in \mathbb{D}, t \geq 0),$$

where $h : \mathbb{D} \mapsto \mathbb{C}$ is a univalent function with $h(0) = 0$. h is called the *Koenigs function* of the semigroup $(\varphi_t)_{t \geq 0}$ and it is unique up to the normalization $h(0) = 0$. The infinitesimal generator G of the semigroup $(\varphi_t)_{t \geq 0}$ and the Koenigs function are related by means of

$$(4.2) \quad G(z)h'(z) = 1 \quad (z \in \mathbb{D}).$$

The range of the Koenigs function $\Omega = h(\mathbb{D})$ is the associated *Koenigs domain* of $(\varphi_t)_{t \geq 0}$, which is *convex in the positive direction*, that is, if $w \in \Omega$ then $w + t \in \Omega$ for every $t \geq 0$. The geometry of

Koenigs domain plays a significant role in understanding the semigroup. In particular, a non-elliptic semigroup $(\varphi_t)_{t \geq 0}$ is hyperbolic if and only if its Koenigs domain Ω is contained in a horizontal strip (we refer to the recent monograph [7] for this and further results and references).

4.2. Semigroups of composition operators on H^p . It is a well known result that every semigroup $(\varphi_t)_{t \geq 0}$ in \mathbb{D} induces a C_0 -semigroup of composition operators $(C_{\varphi_t})_{t \geq 0}$ acting on the Hardy space H^p of the unit disc when $1 \leq p < \infty$. In such a case, the infinitesimal generator Δ of the semigroup $(C_{\varphi_t})_{t \geq 0}$ is given by

$$\Delta f = Gf', \quad (f \in \text{Dom}(\Delta)),$$

where G is the infinitesimal generator of $(\varphi_t)_{t \geq 0}$ and $\text{Dom}(\Delta) = \{f \in H^p : Gf' \in H^p\}$.

Assume that $(\varphi_t)_{t \geq 0}$ is a non-elliptic semigroup. It is straightforward that $\lambda \in \sigma_p(\Delta)$ if and only if there exists $f \in \text{Dom}(\Delta)$ such that $\lambda f = Gf'$. This differential equation has a unique solution up to a constant $e^{\lambda h} \in \mathcal{H}(\mathbb{D})$ (see [34]), where h is the Koenigs function of the semigroup $(\varphi_t)_{t \geq 0}$. Thus, $\lambda \in \sigma_p(\Delta)$ if and only if $e^{\lambda h} \in H^p$, and in such a case $\ker(\Delta - \lambda I) = \text{span}\{e^{\lambda h}\}$.

The point spectrum $\sigma_p(\Delta)$ was characterized by Betsakos in [5] for hyperbolic semigroups. We present a slight reformulation of his results that will be easier to handle for our purposes:

Theorem 4.1. [5] *Let $(\varphi_t)_{t \geq 0}$ be a hyperbolic semigroup of holomorphic functions on \mathbb{D} and assume without loss of generality that its Denjoy-Wolff point is $\tau = 1$. Let Ω be the associated Koenigs domain of the semigroup and γ be the width of the smallest horizontal strip containing Ω . Let $(C_{\varphi_t})_{t \geq 0}$ be the induced semigroup of composition operators on H^p with $1 \leq p < \infty$, and Δ its infinitesimal generator.*

(a) *Suppose that Ω does not contain any horizontal strip. Then*

$$\{\lambda \in \mathbb{C} : -\infty < \text{Re}(\lambda) < \frac{\pi}{p\gamma}\} \subseteq \sigma_p(\Delta; H^p) \subseteq \{\lambda \in \mathbb{C} : -\infty < \text{Re}(\lambda) \leq \frac{\pi}{p\gamma}\}.$$

(b) *Suppose that Ω contains a horizontal strip. Then there exists a horizontal strip contained in Ω with width*

$$\beta_{\max} = \max\{\beta > 0 : \Omega \text{ contains a horizontal strip of width } \beta\}.$$

Moreover,

$$\{\lambda \in \mathbb{C} : -\frac{\pi}{p\beta_{\max}} < \text{Re}(\lambda) < \frac{\pi}{p\gamma}\} \subseteq \sigma_p(\Delta; H^p) \subseteq \{\lambda \in \mathbb{C} : -\frac{\pi}{p\beta_{\max}} < \text{Re}(\lambda) \leq \frac{\pi}{p\gamma}\}.$$

With this result at hand, we state the following proposition regarding hyperbolic semigroups. Recall that a horizontal strip is a domain defined as

$$\{z + it_0 \in \mathbb{C} : |\text{Im}(z)| < t_1\} \subset \mathbb{C},$$

where $t_0 \in \mathbb{R}$ and $t_1 > 0$.

Theorem 4.2. *Let $(\varphi_t)_{t \geq 0}$ be a hyperbolic semigroup of holomorphic functions on \mathbb{D} and assume without loss of generality that its Denjoy-Wolff point is $\tau = 1$. Let Ω be the associated Koenigs domain of the semigroup and γ be the width of the smallest horizontal strip containing Ω . Let $(C_{\varphi_t})_{t \geq 0}$ be the induced semigroup of composition operators on H^p , $1 \leq p < \infty$, and Δ its infinitesimal generator.*

(a) Assume that Ω does not contain any horizontal strip. Then,

$$\sigma(\Delta; H^p) = \{\lambda \in \mathbb{C} : -\infty < \operatorname{Re}(\lambda) \leq \frac{\pi}{p\gamma}\}.$$

(b) Assume that Ω contains a horizontal strip, and let β_{\max} be as in Theorem 4.1.

(i) If Ω is a horizontal strip, then C_{φ_t} is invertible for every $t \geq 0$ and $\beta_{\max} = \gamma$ and

$$\sigma(\Delta; H^p) = \{\lambda \in \mathbb{C} : |\operatorname{Re}(\lambda)| \leq \frac{\pi}{p\gamma}\}.$$

(ii) If Ω is not a horizontal strip, then C_{φ_t} is not invertible for any $t > 0$.

Proof. We start proving (a). If Ω does not contain any horizontal strip, by Theorem 4.1, it follows that

$$\{\lambda \in \mathbb{C} : -\infty < \operatorname{Re}(\lambda) \leq \frac{\pi}{p\gamma}\} \subseteq \sigma(\Delta; H^p).$$

For the reverse containment, recall that the spectral radius of C_{φ_t} in H^p is given by $r(C_{\varphi_t}; H^p) = \varphi'_t(1)^{-1/p}$ (see [33, Corollary 3.2], for instance). Moreover, $\varphi'_t(1) = e^{-\frac{\pi t}{\gamma}}$ (see [10, Theorem 2.1], for instance), so

$$r(C_{\varphi_t}; H^p) = e^{\frac{\pi t}{\gamma p}}.$$

Upon applying Proposition 2.1, it follows that for every $\lambda \in \sigma(\Delta)$ the inequality $\operatorname{Re}(\lambda) \leq \frac{\pi}{p\gamma}$ holds and (a) is proved.

Now, we deal with (b). Assume that Ω contains a horizontal strip and consider the following two cases:

(i) Assume Ω is a horizontal strip, and let

$$g(z) = \log \left(\frac{1+z}{1-z} \right), \quad (z \in \mathbb{D}),$$

where we are considering the principal branch of the logarithm. Note that g is a univalent function with $g(0) = 0$ that maps the disc \mathbb{D} onto the horizontal strip symmetric with respect to the real axis and with width π . It is clear that there exists $\alpha_1, \alpha_2 > 0$ such that $\tilde{g}(z) := \alpha_1 g(z) + \alpha_2$ maps the unit disc onto Ω . Now, if h is the Koenigs function of the semigroup, it is clear that the map

$$\Gamma : z \in \mathbb{D} \mapsto h^{-1} \circ \tilde{g}(z) \in \mathbb{D}$$

is a biholomorphic map of \mathbb{D} . Moreover, it is easy to check that $C_\Gamma C_{\varphi_t} = C_{\psi_t} C_\Gamma$, where $\psi_t(z) = \tilde{g}^{-1}(\tilde{g}(z) + t)$. Then, the operators C_{φ_t} and C_{ψ_t} are similar for all $t \geq 0$, so it suffices to show the result for C_{ψ_t} . But, in this case,

$$\psi_t(z) = \frac{(e^{t/\alpha_1} + 1)z + (e^{t/\alpha_1} - 1)}{(e^{t/\alpha_1} - 1)z + (e^{t/\alpha_1} + 1)} \quad (z \in \mathbb{D}),$$

which turns out to be a hyperbolic automorphism of \mathbb{D} and, hence, C_{ψ_t} is invertible.

Accordingly, C_{φ_t} is invertible for every $t \geq 0$ and the assertion about the spectrum follows from [27, Theorem 6], Theorem 4.1 and Proposition 2.1.

- (ii) Assume Ω contains a horizontal strip but it is not one. We will argue by contradiction. Let us suppose that there exists $t_0 > 0$ such that $C_{\varphi_{t_0}}$ is invertible. Then, φ_{t_0} is an automorphism of the unit disc which is hyperbolic. Now, by [6, Theorem 3.2], every member of the semigroup is a linear fractional map, and since they all commute, by [11, Theorem 8], it follows that every φ_t with $t > 0$ has to be a hyperbolic automorphism of the disc having the same fixed points. Finally, arguing as in the proof of (i), it is easy to see that the Koenigs function of the semigroup $(\varphi_t)_{t \geq 0}$ (which can be extended to a group) maps the disc \mathbb{D} onto a horizontal strip, which contradicts our assumption. Then, we have shown that C_{φ_t} is not invertible for any $t > 0$, and the proof is done. \square

At this point, we can discuss the density of the eigenfunctions of infinitesimal generators of hyperbolic semigroups, which will be carried out by applying a result of [8]:

Proposition 4.3. *Let $(\varphi_t)_{t \geq 0}$ be a hyperbolic semigroup of holomorphic functions in \mathbb{D} and let h and Ω denote its Koenigs function and Koenigs domain, respectively. Assume, without loss of generality, that $\tau = 1$. Let $(C_{\varphi_t})_{t \geq 0}$ be the induced semigroup of composition operators acting on H^p ($1 \leq p < \infty$) and Δ its infinitesimal generator. Assume that*

- (i) *Either Ω does not contain any horizontal strip, $\text{int}(\overline{\Omega}) = \Omega$ and $\mathbb{C} \setminus \overline{\Omega}$ has at most two connected components,*
- (ii) *or Ω is a horizontal strip.*

Then, the linear manifold

$$\text{span} \{e^{\lambda h} \in H^p : \lambda \in \text{int}(\sigma(\Delta; H^p))\}$$

is dense in H^p for every $1 \leq p < \infty$.

The proof is based on the ideas developed in [8], and we include it for the sake of completeness.

Proof. For $1 \leq p < \infty$, let $H^p(\Omega)$ be the Hardy space associated to the Koenigs domain Ω , that is, the Banach space consisting of holomorphic functions $f : \Omega \rightarrow \mathbb{C}$ such that there exists a harmonic function $u : \Omega \rightarrow \mathbb{R}$ such that $|f(z)|^p \leq u(z)$ for every $z \in \Omega$ (see Duren's book [15, Chapter 10] for more on these spaces). It turns out that the map $f \in H^p(\Omega) \mapsto f \circ h \in H^p$ is a surjective isometry, so it suffices to show that the linear manifold $\text{span} \{e^{\lambda z} : \lambda \in \text{int}(\sigma(\Delta; H^p))\}$ is dense in $H^p(\Omega)$.

Let $H^\infty(\Omega)$ denote the space of bounded holomorphic functions in Ω . By Theorems 4.1 and 4.2 it follows that $\text{int}(\sigma(\Delta; H^p)) \subseteq \sigma_p(\Delta; H^p)$, so for every $\lambda \in \text{int}(\sigma(\Delta|_{H^p}))$ the kernel $\ker(\Delta - \lambda I)$ has dimension one and it is spanned by $e^{\lambda h}$. Observe that, by Theorem 4.2 and the convexity in the positive direction of Ω ,

$$\{e^{\lambda z} \in H^\infty(\Omega) : \lambda \in \mathbb{C}\} \subset \{e^{\lambda h} \in H^p : \lambda \in \text{int}(\sigma(\Delta; H^p))\}$$

for all $1 \leq p < \infty$. Now, [8, Theorem 1.1] yields that the linear manifold

$$\text{span} \{e^{\lambda z} \in H^\infty(\Omega) : \lambda \in \mathbb{C}\}$$

is weak-star dense in $H^\infty(\Omega)$. As a consequence, if $1 < p < \infty$, such linear manifold is weakly dense in $H^p(\Omega)$, and by Mazur's Lemma ([14, Corollary 3, Chapter 2]), it is dense on $H^p(\Omega)$. Finally, the continuity of the inclusion $H^p(\Omega) \hookrightarrow H^1(\Omega)$ and the density of $H^p(\Omega)$ in $H^1(\Omega)$ yields the density for $p = 1$, which completes the proof. \square

The next theorem is the counterpart of Theorem 3.1 in the setting of hyperbolic semigroups of composition operators:

Theorem 4.4. *Let $(\varphi_t)_{t \geq 0}$ be a hyperbolic semigroup of holomorphic functions in \mathbb{D} and Ω its Koenigs domain. Assume, without loss of generality, that $\tau = 1$. Let $(C_{\varphi_t})_{t \geq 0}$ be the induced semigroup of composition operators in $X = H^p$, $1 \leq p < \infty$, and Δ its infinitesimal generator of type $\omega_0 \in \mathbb{R}$. Assume that*

- (i) *Either Ω does not contain any horizontal strip, $\text{int}(\overline{\Omega}) = \Omega$ and $\mathbb{C} \setminus \overline{\Omega}$ has at most two connected components,*
- (ii) *or Ω is a horizontal strip.*

Let ν be a finite complex Borel measure supported in $[0, \infty)$ such that $\int_0^\infty e^{(w_0 + \delta)t} d\nu(t) < \infty$ for some $\delta > 0$ and $\mathcal{L}(\nu)$ is a non-constant function in $\mathcal{E}(-\Delta)$. If \mathcal{H}_ν is the subordinated operator associated to $(C_{\varphi_t})_{t \geq 0}$ the following holds:

- (1) *\mathcal{H}_ν does not have the SVEP.*
- (2) *For every relatively open subset $U \subset \sigma(\mathcal{H}_\nu)$, $\mathcal{K}_{\mathcal{H}_\nu}(\overline{U})$ is dense in X .*
- (3) *$\sigma_p(\mathcal{H}_\nu^*) = \emptyset$.*
- (4) *$X_{\mathcal{H}_\nu^*}(F) = \{0\}$ for every closed subset $F \subsetneq \sigma(\mathcal{H}_\nu^*)$.*
- (5) *\mathcal{H}_ν^* has the Dunford property (C).*
- (6) *$\sigma_{\mathcal{H}_\nu^*}(f) = \sigma(\mathcal{H}_\nu^*)$ for every $f \in X^* \setminus \{0\}$.*
- (7) *$r_{\mathcal{H}_\nu^*}(f) = r(\mathcal{H}_\nu^*)$ for every $f \in X^* \setminus \{0\}$.*
- (8) *If M is a non-trivial closed invariant subspace for \mathcal{H}_ν^* , then $\sigma(\mathcal{H}_\nu^*) \subseteq \sigma(\mathcal{H}_\nu^*|_M) \subset \eta(\sigma(\mathcal{H}_\nu^*))$.*

The proof follows as an application of Theorem 3.1. Note that by Theorem 4.2, $\overline{\text{int}(\sigma(-\Delta))} = \sigma(-\Delta)$, so the map

$$f_\Delta : z \in \text{int}(\sigma(-\Delta)) \mapsto e^{-zh} \in \text{Dom}(-\Delta)$$

is well defined and for every $-z \in \text{int}(\sigma(\Delta))$

$$(-\Delta - zI)f_\Delta(z) = -(\Delta - (-z)I)e^{-zh} = 0,$$

since $\text{int}(\sigma(\Delta)) \subset \sigma_p(\Delta)$ and $\text{int}(\sigma(-\Delta)) = -\text{int}(\sigma(\Delta))$ holds. Finally, by Proposition 4.3,

$$\{f_\Delta(\lambda) : \lambda \in \text{int}(\sigma(-\Delta))\}$$

is dense in H^p , so the conditions of Theorem 3.1 are therefore fulfilled.

In particular, as it was discussed in Remark 3.2, it is possible to derive the following consequence for each operator in the hyperbolic composition semigroup $(C_{\varphi_t})_{t \geq 0}$:

Corollary 4.5. *Let $(\varphi_t)_{t \geq 0}$ be a hyperbolic semigroup of holomorphic functions in \mathbb{D} and Ω its Koenigs domain. Assume, without loss of generality, that $\tau = 1$. Let $(C_{\varphi_t})_{t \geq 0}$ be the induced semigroup of composition operators in $X = H^p$, $1 \leq p < \infty$, and Δ its infinitesimal generator. Assume that*

- (i) *Either Ω does not contain any horizontal strip, $\text{int}(\overline{\Omega}) = \Omega$ and $\mathbb{C} \setminus \overline{\Omega}$ has at most two connected components;*
- (ii) *or Ω is a horizontal strip.*

For every $t > 0$ the following holds:

- (1) C_{φ_t} does not have the SVEP.
- (2) For every relatively open subset $U \subset \sigma(C_{\varphi_t})$, $\mathcal{X}_{C_{\varphi_t}}(\overline{U})$ is dense in X .
- (3) $\sigma_p(C_{\varphi_t}^*) = \emptyset$.
- (4) $X_{C_{\varphi_t}^*}(F) = \{0\}$ for every closed subset $F \subsetneq \sigma(C_{\varphi_t}^*)$.
- (5) $C_{\varphi_t}^*$ has the Dunford property (C).
- (6) $\sigma_{C_{\varphi_t}^*}(f) = \sigma(C_{\varphi_t}^*)$ for every $f \in X^* \setminus \{0\}$.
- (7) $r_{C_{\varphi_t}^*}(f) = r(C_{\varphi_t}^*)$ for every $f \in X^* \setminus \{0\}$.
- (8) If M is a non-trivial closed invariant subspace for T^* , then $\sigma(C_{\varphi_t}^*) \subseteq \sigma(C_{\varphi_t}^*|_M) \subset \eta(\sigma(C_{\varphi_t}^*))$.

Proof. Assume that Ω does not contain any horizontal strip, the proof of Theorem 4.2 yields that

$$(4.3) \quad \sigma(C_{\varphi_t}; H^p) = \overline{\{e^{\lambda t} : \lambda \in \sigma(\Delta)\}}$$

for every $t > 0$ having in mind Proposition 2.1 and $r(C_{\varphi_t}; H^p) = e^{\frac{\pi t}{\gamma p}}$.

On the other hand, if Ω is a horizontal strip, the semigroup is similar to a semigroup of hyperbolic automorphisms and in such a case Theorem 4.1 along with [27, Theorem 6] yields (4.3). \square

Remark 4.6. It is important to note that properties (1) – (8) were already proved for composition operators induced by hyperbolic linear fractional maps in [18]. Corollary 4.5 extends these results to every composition operator that can be embedded in a hyperbolic semigroup such that its Koenigs domain meets the geometrical conditions (i) or (ii).

4.3. Examples of subordinated operators. In this subsection, we provide particular instances of hyperbolic semigroups of holomorphic functions in \mathbb{D} which yield examples of subordinated operators for which Theorem 4.4 applies. Of particular interest is the classical Cesàro operator.

- (1) For each $t \geq 0$, let

$$\varphi_t(z) = e^{-t}z + (1 - e^{-t}), \quad (z \in \mathbb{D}).$$

Each φ_t is an affine map, in particular a hyperbolic non-automorphism linear fractional map in \mathbb{D} which fixes 1 and ∞ . Clearly $(\varphi_t)_{t \geq 0}$ is a semigroup of holomorphic functions in \mathbb{D} and its Koenigs function is given by $h(z) = -\log(1 - z)$, for $z \in \mathbb{D}$, where the principal branch of the logarithm is considered. The Koenigs domain Ω is contained in

$$\{z \in \mathbb{C} : \operatorname{Re}(z) \geq -\log(2), -\pi/2 < \operatorname{Im}(z) < \pi/2\}$$

and the semigroup is hyperbolic since it is contained in a horizontal strip. Indeed, Ω does not contain any horizontal strip and $\operatorname{int}(\overline{\Omega}) = \Omega$, so Theorem 4.4 clearly applies to the semigroup $(C_{\varphi_t})_{t \geq 0}$ acting on H^p .

It is well known that $\sigma(\Delta; H^p) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq \frac{1}{p}\}$. For each $\lambda \in \mathbb{C}$ such that $\operatorname{Re}(\lambda) > \frac{1}{p}$, let ν_λ the Borel measure

$$d\nu_\lambda(t) = e^{-\lambda t} dt$$

in $[0, \infty)$. Clearly ν_λ satisfies the hypothesis on the measure of Theorem 4.4 and by (2.2) $\mathcal{H}_{\nu_\lambda} = R(\lambda, \Delta)$. For every $f \in H^p$ we have

$$\begin{aligned} \mathcal{H}_{\nu_\lambda} f(z) &= \int_0^\infty e^{-\lambda t} (C_{\varphi_t} f)(z) dt = \int_0^\infty e^{-\lambda t} f(e^{-t} z + (1 - e^{-t})) dt \\ &= \frac{-1}{(z-1)^\lambda} \int_z^1 (s-1)^{\lambda-1} f(s) ds, \quad (z \in \mathbb{D}) \end{aligned}$$

so $\mathcal{H}_{\nu_\lambda}$ is an averaging operator. Theorem 4.4 assures that properties (1) – (8) are satisfied for such operators.

In particular, consider $1 < p < \infty$ and let $\lambda = 1$. In this case, we obtain the operator

$$\mathcal{H}_{\nu_1} f(z) = \frac{-1}{z-1} \int_z^1 f(s) ds, \quad (z \in \mathbb{D}).$$

By [26] $-\mathcal{H}_{\nu_1}$ is (similar to) the adjoint operator of the Cesàro operator \mathcal{C} acting on H^p , that is, the operator defined as

$$(\mathcal{C}f)(z) = \int_0^z \frac{f(s)}{1-s} ds, \quad (z \in \mathbb{D}).$$

Therefore, \mathcal{C}^* does not enjoy the SVEP and their global spectral subspaces associated to the closure of relatively open sets are dense. Moreover, the spectral subspaces associated to proper closed sets of the Cesàro operator are zero, so the Cesàro operator has the Dunford property, the local spectrum at non-zero functions coincides with the spectrum of \mathcal{C} , and $\sigma(\mathcal{C}|_M) = \sigma(\mathcal{C})$ for every non-trivial closed invariant subspace of \mathcal{C} .

It is worth pointing out that the Dunford property was already known for the Cesàro operator since Miller, Miller and Smith [26] proved that \mathcal{C} has the Bishop property which implies it. Nevertheless, the characterization of the spectral subspaces and the local spectra of \mathcal{C} had not been previously studied in the literature. Next result (already mentioned in the Introduction) states the new features regarding the local spectra of \mathcal{C} , and as a particular instance, yields that \mathcal{C} enjoys the Dunford property (C).

Theorem 4.7. *Let $1 < p < \infty$ and \mathcal{C} be the Cesàro operator acting on the Hardy space H^p . Then:*

- (a) $\sigma_{\mathcal{C}}(f) = \sigma(\mathcal{C}; H^p)$ for every $f \in H^p \setminus \{0\}$.
- (b) $r_{\mathcal{C}}(f) = r(\mathcal{C}; H^p)$ for every $f \in H^p \setminus \{0\}$.
- (c) $H_{\mathcal{C}}^p(F) = \{0\}$ for every closed subset $F \subsetneq \sigma(\mathcal{C}; H^p)$.
- (d) If M is a non-trivial closed invariant subspace of \mathcal{C} , then $\sigma(\mathcal{C}|_M; H^p) = \sigma(\mathcal{C}; H^p)$.

(2) For each $t \geq 0$, let φ_t be the hyperbolic automorphism of \mathbb{D} fixing ± 1 :

$$\varphi_t(z) = \frac{(e^t + 1)z + (e^t - 1)}{(e^t - 1)z + (e^t + 1)} \quad (z \in \mathbb{D}).$$

Clearly $(\varphi_t)_{t \geq 0}$ forms a semigroup of holomorphic functions in \mathbb{D} that can be extended to a group $(\varphi_t)_{t \in \mathbb{R}}$ with the obvious definition for $t < 0$. As it was discussed in the proof of Theorem 4.2, the Koenigs domain associated to such semigroup is a horizontal strip. In this

case, $\sigma(\Delta; H^p) = \{\lambda \in \mathbb{C} : |\operatorname{Re}(\lambda)| \leq \frac{1}{p}\}$. For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 1$ the resolvent operator

$$(\mathcal{H}_{\nu_\lambda} f)(z) = \int_0^\infty e^{-\lambda t} f(\varphi_t(z)) dt = \left(\frac{1+z}{1-z} \right)^\lambda \int_z^1 \left(\frac{1+s}{1-s} \right)^{-\lambda} \frac{2}{1-s^2} f(s) ds, \quad (z \in \mathbb{D}),$$

is an averaging operator in H^p satisfying the local spectral properties of Theorem 4.4.

- (3) In general, a formula for the resolvent operators arising from semigroups of composition operators can be obtained. If $(\varphi_t)_{t \geq 0}$ is a hyperbolic semigroup of holomorphic functions in \mathbb{D} with Denjoy-Wolff point $\tau = 1 \in \mathbb{T}$, h is the associated Koenigs function and $(C_{\varphi_t})_{t \geq 0}$ the associated semigroup of type $\omega_0 \in \mathbb{R}$, for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re}(\lambda) > \omega_0$ the resolvent at λ may be expressed as

$$(\mathcal{H}_{\nu_\lambda} f)(z) = e^{\lambda h(z)} \int_z^1 h'(s) e^{-\lambda h(s)} f(s) ds, \quad (f \in H^p, z \in \mathbb{D}).$$

Indeed, if the Laplace transform of the measure $d\nu_\rho(t) = \rho(t)dt$ belongs to $E(-\Delta)$, where $\rho : [0, \infty) \rightarrow \mathbb{C}$, it follows

$$(4.4) \quad (\mathcal{H}_{\nu_\rho} f)(z) = \int_z^1 h'(s) \rho(h(s) - h(z)) f(s) ds, \quad (f \in H^p, z \in \mathbb{D}).$$

Thus, if $(\varphi_t)_{t \geq 0}$ is a hyperbolic semigroup satisfying the hypothesis of Theorem 4.4, every operator of the form (4.4) has properties (1) – (8) from the aforementioned Theorem.

4.4. A remark on non-hyperbolic semigroups. To end up this section, we briefly comment the local spectral picture of composition operators semigroups induced by non-hyperbolic symbols, which looks quite different.

- (1) For $t \geq 0$, let φ_t be the holomorphic function

$$\varphi_t(z) = \frac{e^{-t}z}{(e^{-t} - 1)z + 1} \quad (z \in \mathbb{D}).$$

Clearly $\varphi_t(0) = 0$ and $(\varphi_t)_{t \geq 0}$ is an elliptic semigroup. It was shown in [18] that the composition operator C_{φ_t} acting on H^p , $1 \leq p < \infty$, satisfies the Dunford's property, so in particular it has the SVEP. Moreover, by considering $\psi_t(z) = 1 - (1 - z)e^{-t}$ for $t \geq 0$, which also forms an elliptic semigroup, the Cesàro operator appears as the resolvent at $\lambda = 1$ ([34, p. 19]):

$$\mathcal{C} = R(1, \Delta),$$

where Δ is the infinitesimal generator of $(C_{\psi_t})_{t \geq 0}$ acting on H^p . In this case, the Cesàro operator appears as a subordinated operator having the Bishop's property and \mathcal{C}^* not enjoying the SVEP (see [26]). This situation is drastically different to the one presented in Theorem 4.4 for subordinated operators induced by hyperbolic semigroups.

- (2) For $t \geq 0$, let φ_t be the parabolic automorphism of \mathbb{D} fixing 1:

$$\varphi_t(z) = \frac{(1 - it)z + it}{-it z + 1 + it}, \quad (z \in \mathbb{D}).$$

$(\varphi_t)_{t \geq 0}$ turns out to be a semigroup of holomorphic functions in \mathbb{D} . It is well known that $\sigma(C_{\varphi_t}; H^p) = \sigma_p(C_{\varphi_t}; H^p) = \mathbb{T}$ (see [27]). Moreover, if Δ is the infinitesimal generator of $(C_{\varphi_t})_{t \geq 0}$ acting on H^p , then

$$\sigma(\Delta; H^p) = \sigma_p(\Delta; H^p) = \{i\lambda : \lambda \leq 0\}$$

(see [34]). As a particular instance, the equality $\sigma(C_{\varphi_t}; H^p) = \{e^{\lambda t} : \lambda \in \sigma(\Delta)\}$ holds. Nevertheless, the operators C_{φ_t} are generalized scalar (see [35]), which in particular implies that C_{φ_t} and $C_{\varphi_t}^*$ have the Dunford's property. This fact follows since all generalized scalar operators and their adjoint are decomposable operators, and these operators have Dunford's property ([24, Section 1.8]). In particular, each spectral subspace of such operators associated to proper non-empty closed sets are non-trivial. Clearly, the local spectral picture of the operators of the semigroup differs noticeably from the one for hyperbolic semigroups described in Corollary 4.5.

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