

# Trichotomy for the HRT Conjecture for mixed integer configuration

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## Abstract

Let  $\Lambda$  consist of  $N - 1$  lattice points in  $\mathbb{Z}^d \times \mathbb{Z}^d$  together with a single off-lattice point  $(\alpha, \beta)$ . We prove that the finite Gabor system  $\{M_y T_x f : (x, y) \in \Lambda\}$  is linearly independent for every non-zero window  $f \in \mathcal{S}(\mathbb{R}^d)$ . A supposed dependence gives rise, via the Zak transform, to a system of modulus and phase equations driven by the torus translation  $z \mapsto z + (-\alpha, \beta)$ . Classifying the resulting orbits as dense, finite, or infinite-non-dense yields a sharp trichotomy: (i) dense orbits force  $f = 0$ , (ii) finite orbits reduce to Linnell's lattice theorem, and (iii) a new rigidity argument excludes the infinite non-dense case.

*Dedicated to the memory of Jean-Pierre Gabardo*

## 1 Introduction

Gabor theory is a branch of harmonic analysis that primarily deals with building basis-like signal systems through discrete time-frequency shifts. These systems typically emerge by discretizing a single orbit produced through the action of a Schrödinger representation [17, 18]. The groups underlying these representations belong to a family of noncommutative, nilpotent Lie groups known as Heisenberg Lie groups. Such groups act irreducibly on the Hilbert space  $L^2(\mathbb{R}^d)$  via translation and modulation. The systems of vectors obtained by sampling a single orbit in  $L^2(\mathbb{R}^d)$  along countable subsets of the Heisenberg group modulo its center have been studied extensively in the literature. Among the wealth of results available, the following stand out: Density Theorem, Wexler-Raz Biorthogonality Relations, Ron-Shen Duality Principle, Janssen Representation of Frame Operator and the Balian-Low Theorem, just to name a few [17].

During his distinguished career, the late Jean-Pierre Gabardo made a substantial and impactful contribution to Gabor and frame theory. Many of Gabardo's works either substantially extend classical results or provide deep connections with these findings (see [13, 8, 4, 16, 5, 9, 10, 6, 14, 11, 7, 15] .) He has reshaped many facets of Gabor theory by showing that several cornerstone results, first proved for the full Hilbert space  $L^2(\mathbb{R}^d)$ , persist when one restricts to (cyclic) closed subspaces that are invariant under time–frequency shifts. For instance, Gabardo and Han [13] developed an operator–algebraic framework for Weyl–Heisenberg (Gabor) frames restricted to closed, time–frequency–invariant subspaces of  $L^2(\mathbb{R})$ . Interpreting a Gabor system as the orbit of a projective unitary representation, they prove that the subspace frame  $\{E_{m\alpha}T_{n\beta}g\}_{m,n\in\mathbb{Z}}$  possesses a *unique* Weyl–Heisenberg dual (within the same subspace) precisely in the following situations:

1. **Integer density:**  $\alpha\beta = k \in \mathbb{N}$  (hence  $\alpha\beta \geq 1$ )  $\implies$  uniqueness always holds;
2. **Irrational density:**  $\alpha\beta \notin \mathbb{Q}$   $\implies$  uniqueness holds *iff* the system is a Riesz sequence;
3. **Over–complete regime:**  $\alpha\beta < 1$   $\implies$  the Weyl–Heisenberg dual is never unique.

Consequences include a subspace version of the Balian–Low uncertainty principle and a unified operator–algebraic perspective on density, completeness, and duality for Gabor frames.

I first became acquainted with Jean-Pierre Gabardo through his work on Gabor and frame theory. As an abstract harmonic analyst, what particularly drew me to Jean-Pierre's work was his remarkable versatility. He possessed the unique ability to engage both the abstract and applied aspects of harmonic analysis, and this dual strength demonstrated through his publications underscores his legacy as a remarkable mathematician [13, 8, 12]. Over the years, I had the great privilege of meeting Jean-Pierre at several conferences. A particularly memorable event was the **7th International Conference on Computational Harmonic Analysis**, held at Vanderbilt University in Nashville, Tennessee, in 2018 (co-organized by Akram Aldroubi.) At this gathering, I spent meaningful time with Jean-Pierre Gabardo and Chun-Kit Lai. This period coincides with my active engagement with the Discretization Problem [26], addressing conditions for discretizing unitary representations of solvable Lie groups to construct frames with specific desired properties [27, 23, 24, 25]. In my work with solvable Lie groups, certain group actions could partially be represented through pointwise multiplication by complex exponentials, involving nonlinear parameters. Recognizing potential intersections between my research interests and Jean-Pierre's extensive work on exponential frames, I shared some preliminary ideas with him during a session break at the conference. Right on the spot, Jean-Pierre was able to extend my ideas in a positive direction. Together

with Chun-Kit Lai, we formulated a question of shared interest, eventually resulting in our collaborative article published in the Journal of Fourier Analysis and Applications [15]. This work stands as a testament to Jean-Pierre's generosity not only with his ideas but also with his time.

Unfortunately, subsequent opportunities for continued collaboration were halted by the global COVID-19 pandemic and compounded by the personal demands of teaching and family life, preventing further advancement of this research trajectory. Since that 2018 conference, circumstances sadly prevented me from seeing Jean-Pierre again, making the news of his passing, conveyed by Deguang Han in fall 2024, profoundly saddening. It is with deep respect and honor that I present this contribution on the HRT Conjecture, dedicated to the memory of Jean-Pierre Gabardo, who would undoubtedly have appreciated the present work.

The HRT Conjecture can be stated via the language of representation theory as follows.

**Conjecture 1.1.** *Let  $\pi$  be a continuous, irreducible, and infinite-dimensional representation of the Heisenberg group of dimension  $2d + 1$  acting on  $L^2(\mathbb{R}^d)$ . Fix a nonzero vector  $f \in L^2(\mathbb{R}^d)$ , and let  $\Lambda$  be a finite subset of the Heisenberg group modulo its center. Then the set  $\pi(\Lambda)f = \{\pi(\lambda)f : \lambda \in \Lambda\}$  forms a basis for its span.*

Simply put, define

$$\Lambda_{(N,d)} = \{(x_k, y_k) : 1 \leq k \leq N\} \subseteq \mathbb{R}^d \times \mathbb{R}^d$$

with  $N \in \mathbb{N}$ , as a finite collection of points in the time-frequency plane. The HRT Conjecture states that the system of vectors (Gabor systems)

$$C_{(f,N,d)} = \{t \mapsto e^{-2\pi i \langle y_k, t \rangle} f(t - x_k) : (x_k, y_k) \in \Lambda_{(N,d)}\}$$

is linearly independent.

This conjecture was initially stated by Heil, Ramanathan and Topiwala in 1996. In the original work [19] in which the conjecture was stated, the authors proved that if  $\Lambda_{(N,d)}$  is a finite collection of regularly spaced colinear points in the time-frequency plane with one additional point then  $C_{(f,N,1)}$  is linearly independent. This is also true if  $\Lambda_{(N,1)}$  has three points or fewer. Also, with a straightforward application of the Zak transform, they also show that when  $\Lambda_{(N,d)}$  is contained in an integer lattice, the conjecture is true as well.

A couple of years after the conjecture was stated, in 1998, Linnell proved it to be true when  $\Lambda_{(N,d)}$  lies in a full-rank lattice of the time-frequency plane or contains at most three points [20]. Since the first case for which the result of Linnell is not applicable is the arbitrary case where  $N = 4$ , much effort has been devoted to this case. For instance, if  $\Lambda_{(4,1)}$  represents

the vertices of any planar trapezoid then the conjecture is also corroborated. This result is due to Demeter and Zaharescu [3].

There are also a cluster of results establishing that the HRT Conjecture holds true when the window function has various decay properties. For instance, in 2012-2013, Bownik and Speegle [2] establish linear independence of time-frequency translates for functions with sub-exponential decays, including one-sided logarithmic-exponential decay ( $\lim_{x \rightarrow \infty} |f(x)|e^{cx \log x} = 0$  for all  $c > 0$ ) and faster-than-exponential decay ( $\lim_{x \rightarrow \infty} |f(x)|e^{cx} = 0$  for all  $c > 0$ ) under mild conditions like quasi-monotonicity. They introduce a technical condition on sets where trigonometric polynomials dominate sums of shifts. Furthermore, Benedetto and Bourouhiya [1] proved the HRT Conjecture for every finite Gabor system whose generator fell into one of several broad asymptotic classes. They covered all square-integrable windows that are ultimately analytic and whose germs lay in a Hardy field closed under translations. This contains every logarithmico-exponential and Pfaffian function. Their results remain valid if such a generator is perturbed by a term that decayed strictly faster than the base window, or when the window admits finite ratio limits for every positive translate provided a mild non-degeneracy condition on the modulation frequencies was met. They further established independence for functions decaying faster than any exponential (i.e. satisfying  $\lim_{|x| \rightarrow \infty} e^{t|x|}|g(x)| = 0$  for all  $t > 0$ ), with extensions from  $L^2$  to  $L^p$  settings. They also addressed the case for ultimately positive windows by showing that linear independence holds whenever the modulation frequencies are linearly independent over  $\mathbb{Q}$ , and within four-point systems, whenever both  $g(x)$  and  $g(-x)$  are ultimately decreasing.

In [21], Okoudjou introduced an inductive approach to the HRT conjecture by investigating to what extent it is possible to extend the known results for  $N$  points to  $N + 1$  points. With his method, he was able to corroborate the conjecture for a class of configurations. More recently, together with Okoudjou [22], we focused on the particular case of a mixed-integer configuration, in which  $(N - 1)$  time-frequency shifts are integer lattice points and one is not. Note that this partly covers the  $\Lambda_{(4,1)}$ -case. For instance, the specific configuration of points:  $\Lambda_{(4,1)} = \{(0, 0), (1, 0), (0, 1), (\sqrt{2}, \sqrt{2})\}$  or  $\Lambda_{(4,1)} = \{(0, 0), (1, 0), (0, 1), (\sqrt{2}, \sqrt{3})\}$  fall under the class studied in [22]. Unsurprisingly, the Zak transform played an important role in the following ways: for a square-integrable window with continuous Zak transform, the zero set of the Zak transform:  $\text{Zero}(Zf)$  is non-empty and is invariant under the toral translation generated by an off-integer shift. This invariance leads to three distinct orbit types:

- **Case 1: (Dense orbits)** If a dense orbit meets  $\text{Zero}(Zf)$ , the zero set of the Zak transform, continuity forces it to be the whole torus and this is possible only if the window function is trivial.

- **Case 2: (Infinite but non-dense orbits)** Surprisingly, this case is significantly harder than the first and in our initial note [22], we were only able to observe that  $\text{Zero}(Zf)$  cannot be finite. <sup>1</sup>
- **Case 3: (Finite orbits)** This corresponds to the off-integer point being rational in all of its coordinates (a case for which the HRT Conjecture is already known to be true [20].)

## 2 Main Result and its proof

**Theorem 2.1** (Mixed–Integer HRT). *Let*

$$\Lambda = \left\{ (x_k, y_k) \in \mathbb{Z}^d \times \mathbb{Z}^d : 1 \leq k \leq N-1 \right\} \cup \{(\alpha, \beta)\}, \quad (\alpha, \beta) \notin \mathbb{Z}^d \times \mathbb{Z}^d.$$

*For every dimension  $d \geq 1$  and every non-zero window  $f \in \mathcal{S}(\mathbb{R}^d)$  of Schwartz class, the finite Gabor system associated with  $\Lambda$  is linearly independent.*

### 2.1 The trichotomy and orbit Rigidity for the mixed-integer case

#### 2.1.1 Application of Zak transform

Let us suppose that there exists a nonzero function  $f \in L^2(\mathbb{R}^d)$  such that  $C_{(f, N, d)}$  is linearly dependent and its Zak transform is continuous. By suitably rescaling, there exist nonzero scalars  $c_1, \dots, c_{N-1} \in \mathbb{C}$  such that

$$\sum_{k=1}^{N-1} c_k e^{-2\pi i \langle y_k, t \rangle} f(t - x_k) = e^{-2\pi i \langle \beta, t \rangle} f(t - \alpha). \quad (1)$$

We recall that the Zak transform [17] is a unitary map defined as  $Z : L^2(\mathbb{R}^d) \rightarrow L^2([0, 1)^{2d})$  such that

$$Zf(t, \omega) = \sum_{\kappa \in \mathbb{Z}^d} e^{-2\pi i \langle \omega, \kappa \rangle} f(t + \kappa).$$

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<sup>1</sup>This left a gap in the literature which we shall settle in this work (see Theorem 2.1.)

By applying the Zak transform to the left hand side of the Equation (1) we obtain:

$$\begin{aligned} & \sum_{\kappa \in \mathbb{Z}^d} \sum_{k=1}^{N-1} c_k e^{-2\pi i \langle \omega, \kappa \rangle} e^{-2\pi i \langle y_k, t + \kappa \rangle} f(t + \kappa - x_k) \\ &= \left( \sum_{k=1}^{N-1} c_k e^{-2\pi i \langle y_k, t \rangle} e^{-2\pi i \langle \omega, x_k \rangle} \right) \cdot Zf(t, \omega). \end{aligned}$$

On the other hand, if we apply the Zak transform on the right-hand side of the equation, we obtain:  $e^{-2\pi i \langle t, \beta \rangle} \cdot Zf(t - \alpha, \omega + \beta)$ . As a result,

$$\left( \sum_{k=1}^{N-1} c_k e^{-2\pi i \langle y_k, t \rangle} e^{-2\pi i \langle \omega, x_k \rangle} \right) \cdot Zf(t, \omega) = e^{-2\pi i \langle t, \beta \rangle} \cdot Zf(t - \alpha, \omega + \beta).$$

Letting

$$p(t, \omega) = \sum_{k=1}^{N-1} c_k e^{-2\pi i \langle y_k, t \rangle} e^{-2\pi i \langle \omega, x_k \rangle}$$

be the trigonometric polynomial in the equation above, we derive the following functional equation: For all  $(t, \omega) \in \mathbb{R}^{2d}$ ,

$$p(t, \omega) Zf(t, \omega) = e^{-2\pi i \langle t, \beta \rangle} Zf(t - \alpha, \omega + \beta).$$

### 2.1.2 From Functional to Ergodic Equation

Put  $z = (t, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$  and  $\gamma = (-\alpha, \beta)$ . Then,

$$p(z) Zf(z) = e^{-2\pi i \langle z, (\beta, 0) \rangle} Zf(z + \gamma)$$

and  $|p(z)| \cdot |Zf(z)| = |Zf(z + \gamma)|$ . Put  $F = |Zf|$  and  $q = |p|$ . By appealing to the quasi-periodicity (periodic up to a phase) of the Zak transform [17, Page 150], it is clear that  $F$  is periodic and the above may be written as follows:

$$F(z + \gamma) = q(z) F(z). \tag{2}$$

Next by iterating the equation above, we derive that

$$F(z + n\gamma) = \left( \prod_{j=0}^{n-1} q(z + j\gamma) \right) \cdot F(z)$$

for every natural number  $n$ .

### 2.1.3 Orbit analysis

Assume that the Zak transform of  $f$  is continuous. According to [17, Lemma 8.4.2], it must have at least one zero in its fundamental domain. Fix  $\lambda \in \text{Zero}(F)$ . Then for every  $n \in \mathbb{N}$ ,  $F(\lambda + n\gamma) = 0$ . Define

$$\tau: [0, 1)^{2d} \rightarrow [0, 1)^{2d}, \quad \tau(z) = (z + \gamma) \bmod \mathbb{Z}^{2d}.$$

Since  $F(\tau(z)) = q(z)F(z)$ , the zero-set of  $F$  is invariant under  $\tau$ . In particular, the forward orbit

$$\mathcal{O}(\lambda) = \{\tau^n(\lambda) : n \in \mathbb{N}\}$$

satisfies  $\mathcal{O}(\lambda) \subseteq \text{Zero}(F)$ .

**Case 1: Dense orbit.** Assume that the additive subgroup generated by  $\gamma$  is dense in the torus  $\mathbb{T}^{2d} = [0, 1)^{2d}$ . Consequently,  $\lambda$  in the zero set of  $F$ , fixing the orbit  $\mathcal{O}(\lambda) = \{\lambda + n\gamma : n \in \mathbb{Z}\}$  is dense, and the zero set of the continuous function  $F$  is also dense. Density plus continuity gives  $F \equiv 0$ , but the Zak transform is unitary, so  $F \equiv 0$  would force  $f = 0$ , contradicting our assumption  $f \neq 0$ . Hence, the dense-orbit alternative cannot occur.

**Case 2: the orbit  $\mathcal{O}(\lambda)$  is finite.** Because the forward orbit is finite, every coordinate of  $\gamma = (-\alpha, \beta)$  is rational. Appealing to the result of Linnell (see Lemma B.3) we immediately conclude that  $f = 0$ .

**Case 3:  $\mathcal{O}(\lambda)$  is infinite but not dense in  $[0, 1)^{2d}$ .**<sup>2</sup> As a starting point, fix  $z$  in  $S = \{w \in [0, 1)^{2d} \mid F(w) > 0\}$ . Since  $F$  is continuous,  $S$  must be an open subset of  $[0, 1)^{2d}$ . Moreover, note that at the points where  $q$  vanishes,  $\ln(q)$  exhibits singularities and  $F$  vanishes as well. However, since  $q$  is the absolute value of a trigonometric polynomial, its zero set has Lebesgue zero measure on the torus. In fact,  $\ln(q)$  is real-analytic on an open and dense subset of the  $2d$ -dimensional torus. Applying the natural log to each side of the following equation:

$$F(z + n\gamma) = \left( \prod_{j=0}^{n-1} q(z + j\gamma) \right) \cdot F(z) \quad (n \in \mathbb{N})$$

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<sup>2</sup>In this trichotomy, this case presents us with the most challenge.

we obtain that

$$\ln(F(z + n\gamma)) - \ln(F(z)) = \sum_{j=0}^{n-1} \ln(q(z + j\gamma))$$

almost everywhere on  $S$ . Consequently,

$$\lim_{n \rightarrow \infty} \frac{(\ln(F(z + n\gamma)) - \ln(F(z)))}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln q(z + j\gamma)$$

almost everywhere on  $S$ .

**Proposition 2.2** (Asymptotic log-growth along  $H$ ). *Let  $H \subset [0, 1]^{2d}$  be the compact subgroup generated by  $\gamma = (-\alpha, \beta)$ , and let  $m_H$  be its normalized Haar measure. Choose a Borel transversal  $T$  for the quotient  $[0, 1]^{2d}/H$  and set*

$$U = \{\lambda \in T : p(\lambda + h) \neq 0 \text{ for all } h \in H\}, \quad S = \{z : F(z) \neq 0\}.$$

(i)  $U$  is open and non-empty in  $T$ .

(ii) For every  $\lambda \in S \cap U$  the limit

$$\Theta(\lambda) := \lim_{n \rightarrow \infty} \frac{\ln F(\lambda + n\gamma) - \ln F(\lambda)}{n}$$

exists and equals the continuous function

$$\Theta(\lambda) = \int_H \ln q(\lambda + h) dm_H(h).$$

(iii)  $\Theta$  is real-analytic on each connected component of  $S \cap U$ .

*Proof.* (i) *Openness.* For  $\lambda \in T$  put  $g(\lambda) = \min_{h \in H} |p(\lambda + h)|$ . Because  $p$  is continuous and  $H$  is compact,  $g$  is continuous; hence  $U = \{\lambda : g(\lambda) > 0\}$  is open. If  $U$  were empty,  $p$  would vanish on every coset  $\lambda + H$  and, by the cocycle  $F(z + \gamma) = q(z)F(z)$ , we would get  $F \equiv 0$ , contradicting  $f \neq 0$ .

(ii) *Existence of the limit.* The translation  $h \mapsto h + \gamma$  acts ergodically on  $H$ , so by Birkhoff's theorem

$$A_n(\lambda) := \frac{1}{n} \sum_{j=0}^{n-1} \ln q(\lambda + j\gamma) \xrightarrow{n \rightarrow \infty} \int_H \ln q(\lambda + h) dm_H(h) \quad \text{for a.e. } \lambda \in S \cap U.$$

Because  $\ln q$  is continuous and  $H$  is compact,  $(\lambda, h) \mapsto \ln q(\lambda + h)$  is *uniformly* continuous



on any compact  $K \subset S \cap U$ . Fix  $\lambda \in S \cap U$  and choose a sequence  $\lambda_k \rightarrow \lambda$  inside the full-measure set where the above convergence already holds. Uniform continuity gives

$$\|A_n(\lambda_k) - A_n(\lambda)\|_\infty \xrightarrow{k \rightarrow \infty} 0 \quad \text{for every fixed } n,$$

so taking  $n \rightarrow \infty$  first (for each  $\lambda_k$ ) and then  $k \rightarrow \infty$  yields the desired limit at  $\lambda$  itself. Hence the formula for  $\Theta(\lambda)$  is valid *for all*  $\lambda \in S \cap U$ .

(iii) *Analyticity.* On  $S \cap U$  the function  $\ln q$  is real-analytic (because  $p$  is a non-vanishing trigonometric polynomial there). Lemma B.2 shows that convolution with the Haar measure of a compact subgroup preserves real-analyticity, so  $\Theta(\lambda) = \int_H \ln q(\lambda + h) dm_H(h)$  is real-analytic on each connected component of  $S \cap U$ .  $\square$

### Real-analytic dichotomy on $S \cap U$

Let

$$Z_{S \cap U} := \{\lambda \in S \cap U : \Theta(\lambda) = 0\}.$$

Then two mutually exclusive cases present themselves:

**Case 3 (A)**  $Z_{S \cap U}$  is a null set

**Case 3 (B)**  $Z_{S \cap U}$  has positive measure.

**Case 3 (A)** Choose  $\lambda_0 \in (S \cap U) \setminus Z_{S \cap U}$ . Writing the cocycle average along the orbit  $\lambda_0 + H$  gives

$$\lim_{n \rightarrow \infty} \frac{\ln F(\lambda_0 + n\gamma) - \ln F(\lambda_0)}{n} = \Theta(\lambda_0) \neq 0$$

and the following is immediate:

**Proposition 2.3.** *If  $\Theta(\lambda_0) \neq 0$  for some  $\lambda_0 \in (S \cap U) \setminus Z_{S \cap U}$ , then  $F \equiv 0$  on the entire coset  $\lambda_0 + H$ .*

*Proof.* Fix  $\lambda_0 \in (S \cap U) \setminus Z_{S \cap U}$  with  $\Theta(\lambda_0) \neq 0$ . By the cocycle identity, we have

$$F(\lambda_0 + n\gamma) = \left( \prod_{k=0}^{n-1} |p(\lambda_0 + k\gamma)| \right) F(\lambda_0).$$

Taking logarithms and dividing by  $n$ , we obtain

$$\frac{1}{n} \log F(\lambda_0 + n\gamma) = \frac{1}{n} \log F(\lambda_0) + \frac{1}{n} \sum_{k=0}^{n-1} \log |p(\lambda_0 + k\gamma)|.$$

By ergodic averaging along the orbit of  $\gamma$ , which is dense in  $H$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |p(\lambda_0 + k\gamma)| = \Theta(\lambda_0).$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log F(\lambda_0 + n\gamma) = \Theta(\lambda_0).$$

**Case 3 (A)-1:** If  $\Theta(\lambda_0) > 0$ , then for all large  $n$ ,

$$F(\lambda_0 + n\gamma) \geq F(\lambda_0) \cdot e^{\frac{n}{2}\Theta(\lambda_0)} \rightarrow \infty,$$

which contradicts the continuity and boundedness of  $F$  on the compact torus. Thus,  $F(\lambda_0) = 0$ .

**Case 3 (A)-2:** If  $\Theta(\lambda_0) < 0$ , then

$$F(\lambda_0 + n\gamma) \leq F(\lambda_0) \cdot e^{\frac{n}{2}\Theta(\lambda_0)} \rightarrow 0.$$

Since  $\lambda_0 + H$  is a compact set, there exists a subsequence of natural numbers  $(n_j)_{j \in \mathbb{N}}$  such that  $(\lambda_0 + n_j\gamma)_{j \in \mathbb{N}}$  converges to  $\lambda_0$ . Consequently,

$$\lim_{j \rightarrow \infty} F(\lambda_0 + n_j\gamma) = F(\lambda_0) = 0,$$

and this violates the assumption that  $F$  does not vanish on the coset  $\lambda_0 + H$ .

In both cases, we conclude  $F(\lambda_0) = 0$ , and by the cocycle relation, it follows that  $F(\lambda_0 + h) = 0$  for all  $h \in H$ . Therefore,  $F \equiv 0$  on  $\lambda_0 + H$ . Since  $\lambda_0$  was an arbitrary point in  $(S \cap U) \setminus Z_{S \cap U}$  with  $\Theta(\lambda_0) \neq 0$ , it follows that

$$F \equiv 0 \quad \text{on} \quad \bigcup_{\lambda \in (S \cap U) \setminus Z_{S \cap U}} (\lambda + H)$$

and therefore  $F$  vanishes everywhere. □

### Case 3 (B)

**Remark 2.4.** *The following examples illustrate the shortcomings of the methods provided by the analysis of modulus equation and indicate that the modulus arguments alone cannot settle Case 3. This makes Theorem 2.5 establishing the rigidity of phase indispensable.*

- First, consider the trigonometric polynomial

$$p(t, w) = 1 + e^{-2\pi i t} - e^{-2\pi i w}, \quad (t, w) \in \mathbb{T}^2,$$

and set  $\gamma = (0, \sqrt{2})$ . Elementary calculations show that  $\Theta(t) := \int_0^1 \log |p(t, w)| dw$  is a nonzero piecewise real-analytic function that vanishes on the interval  $t \in [\frac{1}{3}, \frac{2}{3}]$ . Moreover, whenever  $t \in [\frac{1}{3}, \frac{2}{3}]$ , we cannot conclude that  $F$  vanishes on the cosets  $t + H$ .

- Secondly, consider the trigonometric polynomial

$$p(t, w) := 1 + \frac{1}{4} e^{2\pi i(t+w)} + \frac{1}{4} e^{4\pi i(2t-w)}, \quad (t, w) \in \mathbb{T}^2.$$

It satisfies the following:  $|p(t, w)| \geq 1 - (\frac{1}{4} + \frac{1}{4}) = \frac{1}{2} > 0$ , so  $p(t, w)$  has no zeros on the torus and  $\Theta(w) := \int_0^1 \log |p(t, w)| dt$  vanishes identically for all parameters  $w$ . This example is (optimistically arguing in favor of the fact that the HRT Conjecture might be true) worse than the first, as nothing can be said about  $F$  based on the modulus equation.

In this set up, the only place where it might be possible to find a counter example for the HRT Conjecture for our configuration is precisely when

$$\lambda \mapsto \Theta(\lambda) = \int_H \ln q(\lambda + h) dh$$

vanishes on a set of positive measures on  $S \cap U$ . However, even if  $\Theta$  vanishes everywhere, this only points to the direction that it is possible to find a non-trivial solution to the modulus equation  $F(z + \gamma) = F(z)q(z)$ . The existence of such a function does not provide a counterexample to the HRT Conjecture since the information provided by the modulus equation is only partial. But Theorem 2.5 shows that in this situation no non-lattice  $(\alpha, \beta)$  can exist if the window function is Schwartz and, as such, no counterexample arises.

## 2.2 The phase equation

In this section, we will address the phase equation induced by linear dependence of time-frequency shifts viewed on the domain of the Zak transform. Our findings are summarized below.

**Theorem 2.5** (Rigidity of the mixed Zak-phase). *Let  $d \in \mathbb{N}$  and let  $\alpha, \beta \in \mathbb{R}^d$ . Let  $f \in L^2(\mathbb{R}^d)$  with Zak transform*

$$Zf : \mathbb{T}^{2d} := [0, 1)^{2d} \longrightarrow \mathbb{C}, \quad (t, \omega) \longmapsto Zf(t, \omega).$$

Assume the mixed functional equation

$$Zf(t - \alpha, \omega + \beta) = e^{2\pi i \langle t, \beta \rangle} p(t, \omega) Zf(t, \omega), \quad (t, \omega) \in \mathbb{T}^{2d} \quad (3)$$

for some Schwartz function  $f$ . Denote

$$W := Zf^{-1}(\mathbb{C} \setminus \{0\}) \cap p^{-1}(\mathbb{C} \setminus \{0\}) \subseteq \mathbb{T}^{2d},$$

so that  $W$  is open and invariant under the torus shift  $(t, \omega) \mapsto (t - \alpha, \omega + \beta)$ . Then no non-trivial time-frequency parameters  $(\alpha, \beta) \notin \mathbb{Z}^d \times \mathbb{Z}^d$  can satisfy (3).

*Proof.* The proof follows several steps:

**Step 1: From the mixed functional equation to the phase cocycle.** Let  $f \in L^2(\mathbb{R}^d)$  satisfy

$$Zf(t - \alpha, \omega + \beta) = e^{2\pi i \langle t, \beta \rangle} p(t, \omega) Zf(t, \omega), \quad (t, \omega) \in [0, 1)^{2d}, \quad (4)$$

where  $p$  is a trigonometric polynomial. Write the polar decompositions (see Lemma B.1)

$$Zf = |Zf| e^{2\pi i \theta}, \quad p = |p| e^{2\pi i \varphi},$$

obtaining the *phase equation*

$$e^{2\pi i \theta(t - \alpha, \omega + \beta)} = e^{2\pi i (\theta(t, \omega) + \varphi(t, \omega) + \langle t, \beta \rangle)}. \quad (5)$$

**Step 2: Iteration and normalization.** Iterating (5)  $n$  times yields:

$$e^{2\pi i \theta(t - n\alpha, \omega + n\beta)} = e^{2\pi i (\theta(t, \omega) + \sum_{j=0}^{n-1} \varphi(t - j\alpha, \omega + j\beta) + n\langle t, \beta \rangle - \frac{n(n-1)}{2} \langle \alpha, \beta \rangle)}. \quad (6)$$

Divide both sides of (6) by  $n$  in the exponent:

$$\exp\left(2\pi i \frac{\theta(t - n\alpha, \omega + n\beta)}{n}\right) = \exp\left(2\pi i \left(\frac{\theta(t, \omega)}{n} + \frac{1}{n} \sum_{j=0}^{n-1} \varphi(t - j\alpha, \omega + j\beta) + \langle t, \beta \rangle - \frac{n-1}{2} \langle \alpha, \beta \rangle\right)\right). \quad (7)$$

**Step 3. Another expression via quasi-periodicity.** With fractional part  $[u]$  and integer part  $\iota(u) = u - [u]$ , use quasi-periodicity of  $Zf$  to rewrite the left side of (7):

$$\begin{aligned} \exp\left(2\pi i \frac{\theta(t-n\alpha, \omega+n\beta)}{n}\right) &= \exp\left(2\pi i \frac{\theta([t-n\alpha] + \iota(t-n\alpha), [\omega+n\beta] + \iota(\omega+n\beta))}{n}\right) \\ &= \exp\left(2\pi i \frac{\theta([t-n\alpha], [\omega+n\beta])}{n}\right) \exp\left(2\pi i \frac{\langle t-n\alpha - [t-n\alpha], [\omega+n\beta] \rangle}{n}\right). \end{aligned}$$

**Step 4: Cluster points of the normalised phase sequence.** Set

$$\Sigma_{(t, \omega, \alpha, \beta)} := \left\{ \exp\left(\frac{2\pi i}{n} \theta(t - n\alpha, \omega + n\beta)\right) : n \in \mathbb{N} \right\}.$$

Fix a base point  $(t, \omega) \in W$ . For each  $n \geq 1$ , set

$$\zeta_n(t, \omega, \alpha, \beta) := \exp\left(\frac{2\pi i}{n} \theta(t - n\alpha, \omega + n\beta)\right), \quad \Sigma_{(t, \omega, \alpha, \beta)} := \{\zeta_n : n \in \mathbb{N}\}.$$

Iterating the phase cocycle and dividing by  $n$  gives

$$\zeta_n = e^{\frac{2\pi i}{n} \theta(t, \omega)} \exp\left(\frac{2\pi i}{n} \sum_{j=0}^{n-1} \varphi(t - j\alpha, \omega + j\beta)\right) e^{2\pi i \langle t, \beta \rangle} e^{-\pi i (n-1) \langle \alpha, \beta \rangle}.$$

Birkhoff's theorem yields the average

$$I(t, \omega) = \int_H \varphi((t, \omega) + h) dh.$$

Hence every cluster point of  $\Sigma_{(t, \omega, \alpha, \beta)}$  can be written as

$$z = e^{2\pi i I(t, \omega)} e^{2\pi i \langle t, \beta \rangle} \xi, \quad \xi \in \overline{\{e^{-\pi i (n-1) \langle \alpha, \beta \rangle} : n \in \mathbb{N}\}} \subset \mathbb{S}^1. \quad (\text{C}_1)$$

The inner product  $\langle \alpha, \beta \rangle$  governs the shape of the set  $\overline{\{e^{-\pi i (n-1) \langle \alpha, \beta \rangle} : n \in \mathbb{N}\}}$ : it is a finite set of roots of unity when  $\langle \alpha, \beta \rangle \in \mathbb{Q}$  and is the whole unit circle if  $\langle \alpha, \beta \rangle \notin \mathbb{Q}$ . Using the quasi-periodicity of the Zak transform,  $\omega + n\beta = [\omega + n\beta] + \iota(\omega + n\beta)$ , one may eliminate the integer part and obtain the alternative description so that

$$z \in \overline{\{e^{-2\pi i \langle \alpha, [\omega + n\beta] \rangle} : n \in \mathbb{N}\}} \quad (\text{C}_2)$$

**Step 5: Equality of the two descriptions.** Take  $\xi_{\langle\alpha,\beta\rangle}, \mu_{\langle\alpha,\beta\rangle} \in [0, 1)$  such that a common cluster point (we have at our disposal two representations for the same set) reads

$$z = e^{2\pi i I(t,\omega)} e^{2\pi i \langle t,\beta \rangle} e^{2\pi i \xi_{\langle\alpha,\beta\rangle}} = e^{-2\pi i \langle \alpha, \mu_{\langle\alpha,\beta\rangle} \rangle}.$$

Hence there exists  $\kappa \in \mathbb{Z}$  with  $(t, \omega)$  taken on a connected component of  $W$  such that:

$$I(t, \omega) = -\langle t, \beta \rangle - \xi_{\langle\alpha,\beta\rangle} - \langle \alpha, \mu_{\langle\alpha,\beta\rangle} \rangle + \kappa. \quad (8)$$

**Step 6: Shifts in the  $t$ -variable.** Because  $\varphi$  is  $C^1$  and  $\mathbb{Z}$ -periodic on each component,

$$I(t + \ell, \omega) - I(t, \omega) = \int_H (\varphi((t + \ell, \omega) + h) - \varphi((t, \omega) + h)) dh = \varsigma(\ell), \quad \varsigma(\ell) \in \mathbb{Z}.$$

Insert (8) for  $t$  and  $t + \ell$ :

$$-\langle \ell, \beta \rangle = \varsigma(\ell) \in \mathbb{Z} \quad \forall \ell \in \mathbb{Z}^d \implies \beta \in \mathbb{Z}^d.$$

**Step 7: Return to the cluster set.** Assume from the previous deductions that  $\beta \in \mathbb{Z}^d$ . Then the fractional part is frozen:

$$[\omega + n\beta] = [\omega] \quad \text{for every } n \in \mathbb{N},$$

so the second characterization of the cluster set

$$\left\{ e^{-2\pi i \langle \alpha, [\omega + n\beta] \rangle} : n \in \mathbb{N} \right\}$$

reduces to the *single* point  $e^{-2\pi i \langle \alpha, [\omega] \rangle}$ . In the first characterization the only factor that can depend on  $n$  is the root of unity  $e^{-\pi i (n-1) \langle \alpha, \beta \rangle}$ . For the closure to consist of one point, we must have

$$e^{-\pi i \langle \alpha, \beta \rangle} = 1 \iff \langle \alpha, \beta \rangle \in 2\mathbb{Z},$$

i.e. the inner product of  $\alpha$  and  $\beta$  is an **even** integer. With this parity condition the full prefactor in the first description,  $e^{2\pi i I(t,\omega)} e^{2\pi i \langle t,\beta \rangle}$ , must equal the constant obtained in the second one, yielding the identity

$$e^{2\pi i I(t,\omega)} e^{2\pi i \langle t,\beta \rangle} = e^{-2\pi i \langle \alpha, [\omega] \rangle}.$$

This equality links the averaged phase  $I(t, \omega)$ , the integral shift  $\beta$ , and the fractional part  $[\omega]$  through a common phase on the unit circle.

**Step 8: Fourier duality and the final contradiction.** For  $g \in \mathcal{S}(\mathbb{R}^d)$ , we set

$$\widehat{g}(\xi) := \int_{\mathbb{R}^d} g(t) e^{-2\pi i \langle \xi, t \rangle} dt, \quad g(t) = \int_{\mathbb{R}^d} \widehat{g}(\xi) e^{2\pi i \langle \xi, t \rangle} d\xi.$$

Under this convention,

$$\widehat{g(\cdot - x)}(\xi) = e^{-2\pi i \langle \xi, x \rangle} \widehat{g}(\xi), \quad \widehat{e^{2\pi i \langle \eta, \cdot \rangle} g}(\xi) = \widehat{g}(\xi - \eta).$$

Assume the Zak-phase relation originates from the finite dependence

$$\sum_{k=1}^{N-1} c_k e^{-2\pi i \langle y_k, t \rangle} f(t - x_k) = e^{-2\pi i \langle \beta, t \rangle} f(t - \alpha), \quad (\star)$$

with  $(x_k, y_k) \in \mathbb{Z}^d \times \mathbb{Z}^d$  ( $1 \leq k \leq N-1$ ) and a single *non-lattice* parameter  $(\alpha, \beta) \notin \mathbb{Z}^d \times \mathbb{Z}^d$ . Applying  $\mathcal{F}$  and the rules above gives:

$$\sum_{k=1}^{N-1} d_k e^{-2\pi i \langle \xi, x_k \rangle} \widehat{f}(\xi + y_k) = e^{-2\pi i \langle \xi, \alpha \rangle} \widehat{f}(\xi + \beta). \quad (\widehat{\star})$$

Here, each complex number  $d_k$  is a constant multiple of  $c_k$  due to the application of the Fourier transform. The time–frequency parameters in  $(\widehat{\star})$  are

$$\{(-y_k, x_k) : 1 \leq k \leq N-1\} \cup \{(-\beta, \alpha)\}.$$

Hence: the *lattice part* is still integral  $(-y_k, x_k) \in \mathbb{Z}^d \times \mathbb{Z}^d$  and the *single non-lattice point* is now  $(-\beta, \alpha)$ . In other words, the Fourier transform *preserves* the “mixed–integer” shape but swaps the roles of  $(\alpha, \beta)$ . Because  $\widehat{f}$  is again a Schwartz function, the whole Zak–phase machinery applies to  $(\widehat{\star})$  without any change. Replacing  $(\alpha, \beta)$  by  $(-\beta, \alpha)$  in Steps 1–7 one obtains

$$\beta \in \mathbb{Z}^d \implies \alpha \in \mathbb{Z}^d.$$

We already deduced  $\beta \in \mathbb{Z}^d$  in Step 6. The dual argument now forces  $\alpha \in \mathbb{Z}^d$  as well, contradicting the standing assumption that  $(\alpha, \beta) \notin \mathbb{Z}^d \times \mathbb{Z}^d$  (i.e. at least one of the two vectors has a non-integer coordinate). This contradiction shows that no non-lattice pair  $(\alpha, \beta)$  with an infinite but non-dense torus orbit can satisfy the functional equation (4). The rigidity theorem is proved.

□

## A Appendix

## B Measurable local phase

**Lemma B.1** (Measurable local phase). *Let  $g : \mathbb{T}^m \rightarrow \mathbb{C}$  be continuous and put  $W = g^{-1}(\mathbb{C} \setminus \{0\})$ . Define the principal-value argument*

$$\Theta_{\text{rad}}(z) = \begin{cases} \arctan\left(\frac{\text{Im } g(z)}{\text{Re } g(z)}\right) & \text{Re } g(z) > 0, \\ \arctan\left(\frac{\text{Im } g(z)}{\text{Re } g(z)}\right) + \pi & \text{Re } g(z) < 0, \\ \frac{\pi}{2} & \text{Re } g(z) = 0, \text{ Im } g(z) > 0, \\ \frac{3\pi}{2} & \text{Re } g(z) = 0, \text{ Im } g(z) < 0. \end{cases}$$

Put

$$\theta(z) = \frac{\Theta_{\text{rad}}(z)}{2\pi} \quad (z \in W).$$

Then  $\theta : W \rightarrow [0, 1)$  is Borel-measurable, locally bounded, and  $g(z) = |g(z)| e^{2\pi i \theta(z)}$  for all  $z \in W$ . If  $\theta_1, \theta_2$  are two such branches, their difference is integer-valued on  $W$ .

**Lemma B.2** (Analyticity preserved by Haar convolution). *Let  $H$  be a closed subgroup of the torus  $\mathbb{T}^m = [0, 1)^m$  with normalized Haar measure  $m_H$ . If  $g : \mathbb{T}^m \rightarrow \mathbb{C}$  is real-analytic, then the averaged function*

$$G(x) = \int_H g(x + h) dm_H(h)$$

*is also real-analytic. Moreover, its Fourier coefficients are*

$$\widehat{G}(\mu) = \widehat{g}(\mu) \cdot \mathbf{1}_{H^\perp}(\mu),$$

where  $H^\perp = \{\mu \in \mathbb{Z}^m : \langle \mu, h \rangle \in \mathbb{Z} \text{ for all } h \in H\}$ .

*Proof. Part (i): Analyticity of  $G$ .* Since  $g$  is real-analytic, its derivatives decay exponentially: there exist  $C, \rho > 0$  such that

$$|\partial^\nu g(x)| \leq C \rho^{-|\nu|} |\nu|!$$

for all multi-indices  $\nu = (\nu_1, \dots, \nu_m)$  and  $x \in \mathbb{T}^m$ . For any such  $\nu$ , differentiate under the



integral:

$$\partial^\nu G(x) = \int_H \partial^\nu g(x+h) dm_H(h).$$

Then

$$|\partial^\nu G(x)| \leq \int_H |\partial^\nu g(x+h)| dm_H(h) \leq C \rho^{-|\nu|} |\nu|,$$

so  $G$  satisfies the same derivative bounds and is real-analytic.

*Part (ii): Fourier coefficients.* Compute:

$$\begin{aligned} \widehat{G}(\mu) &= \int_{\mathbb{T}^m} G(x) e^{-2\pi i \langle \mu, x \rangle} dx \\ &= \int_{\mathbb{T}^m} \int_H g(x+h) dm_H(h) e^{-2\pi i \langle \mu, x \rangle} dx \\ &= \int_H \int_{\mathbb{T}^m} g(y) e^{-2\pi i \langle \mu, y-h \rangle} dy dm_H(h) \quad (y = x+h) \\ &= \widehat{g}(\mu) \int_H e^{2\pi i \langle \mu, h \rangle} dm_H(h). \end{aligned}$$

The last integral is 1 if  $\langle \mu, h \rangle \in \mathbb{Z}$  for all  $h \in H$  (i.e.,  $\mu \in H^\perp$ ), and 0 otherwise (by orthogonality of characters).  $\square$

**Lemma B.3** (Linnell's Theorem, convenient form, [20]). *Let  $\Lambda \subset \mathbb{R}^{2d}$  be contained in a full-rank lattice and  $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ . Then the associated finite Gabor system is linearly independent.*

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