Problem Set I - Introduction

Zhecheng Hu

2023-02-27

1 Sun's efficiency

(a) Compute the energy output (power) of the Sun per unit mass. Estimate the same quantity for a human being.

Use the quantity of the Sun's mass and luminosity, we can derive the energy output of the Sun per unit mass. The luminosity of the Sun is $L_\odot = 3.828 \times 10^{33}$ erg/s, and the mass of the Sun is $M_\odot = 1.989 \times 10^{33}$ g.

$$\frac{L_\odot}{M_\odot} = \frac{3.828 \times 10^{33} \text{ erg/s}}{1.989 \times 10^{33} \text{ g}} = 1.92 \text{ erg/s/g} \tag{1}$$

A human need to eat 2000 kilo-calories per day to keep the weight, which is $2000 \text{ kcal} = 8.4 \times 10^{13}$ erg. At this case, we can consider the energy is conserved, which means that all the energy is consumed. Note that the if we just lie on a bed all day, we still feel hungry, so we can infer that most energy will finally be radiated. The mass of a human is 7×10^4 g, so the power of a human being per unit mass approximately:

$$\frac{8.4 \times 10^{13} \text{ erg}}{24 \times 3600 \text{ s} \times 7 \times 10^4 \text{ g}} = 1.4 \times 10^4 \text{ erg/s/g} \tag{2}$$

I guess the reason why the power per unit mass of the human is much larger than the that of the Sun is because that the nucluear fusion, which is the source of the energy of the sun, only happens in the core of the Sun, which only occupies a small part. So the average value will be much smaller.

2 Color-Magnitude diagram

(a) Plot the color-magnitude diagram

The CMD is shown in Figure 1.

(b) Why does the B-V color index reach the same value, irrespective of whether magnitudes are given as apparent or absolute?

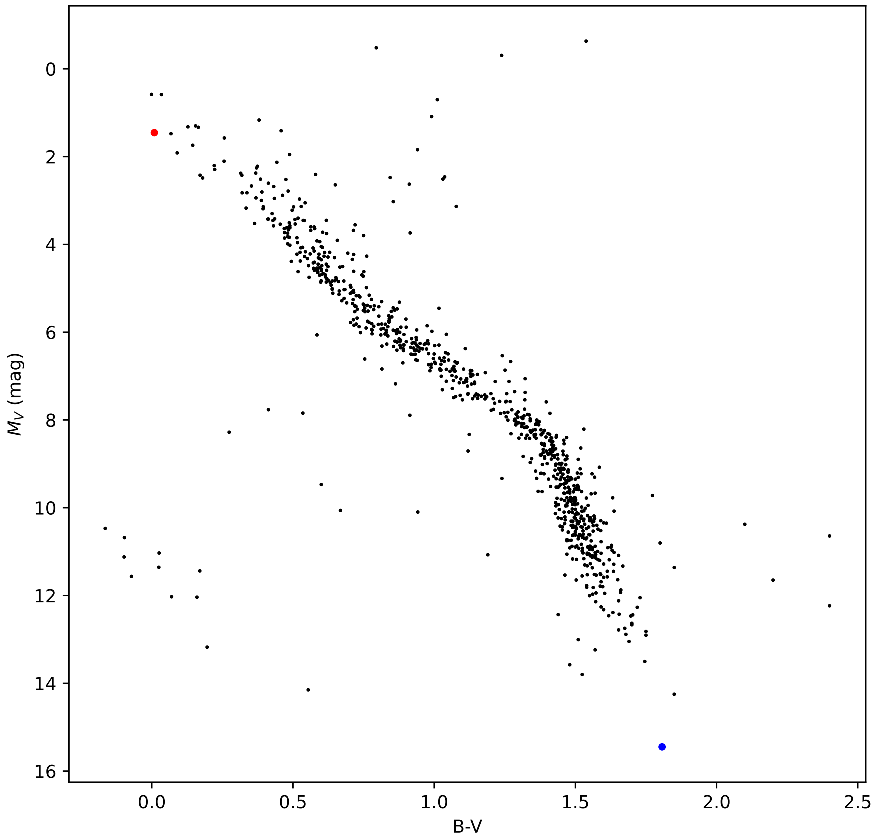

Figure 1: The color-magnitude diagram of the stars in the Hipparcos with parallax exceeding 50 mas. The x-axis is the color index $B-V$, and the y-axis is the absolute magnitude M_V. The red dot is the brightest star in the sky and the blue dot is the nearest star, i.e. the star that has the greatest parallax.

The B-V color is essentially depend on the ratio of flux of the blue band and the that of the visual band. Because the absolute magnitude is the apparent magnitude corrected for the distance, the ratio of the fluxes is the same for the apparent and absolute magnitudes.

$$M_B - M_V = -2.5\log\left(\frac{L_B/4\pi(10\ \mathrm{pc})^2}{L_V/4\pi(10\ \mathrm{pc})^2}\right) = -2.5\log\left(\frac{L_B/4\pi d^2}{L_V/4\pi d^2}\right) = m_B - m_V \tag{3}$$

(c) Explain qualitatively that the difference between the magnitude in the "blue" band (B) and the magnitude in the "visual" (V) band reflect the stellar effective temperature, such that stars of higher B-V have lower T_{eff}.

Here we use the mid value of the band wavelength and the bandwidth to estimate the flux of the band.

$$F_{v_B} = \int_S I_{v_B}\cos\theta \mathrm{d}\Omega = \frac{4\pi h\nu_B^3}{c^2}\frac{1}{e^{h\nu_B/kT}-1} \tag{4}$$

$$F_{v_V} = \int_S I_{v_V}\cos\theta \mathrm{d}\Omega = \frac{4\pi h\nu_V^3}{c^2}\frac{1}{e^{h\nu_V/kT}-1} \tag{5}$$

So the ratio of the fluxes is

$$\frac{F_{v_B}}{F_{v_V}} = \frac{\nu_B^3(e^{h\nu_V/kT}-1)}{\nu_V^3(e^{h\nu_B/kT}-1)} \tag{6}$$

Now we calculate the derivative of the ratio with respect to the temperature T.

$$\frac{\mathrm{d}\frac{F_{v_B}}{F_{v_V}}}{\mathrm{d}T} = \frac{\nu_B^3}{\nu_V^3}\frac{h}{\left(-1+e^{\frac{h\nu_B}{kT}}\right)^2 kT^2}\left(-e^{\frac{h\nu_B}{kT}}\nu_B + e^{\frac{h(\nu_B+\nu_V)}{kT}}(\nu_B-\nu_V) + e^{\frac{h\nu_V}{kT}}\nu_V\right) \tag{7}$$

The first two terms are positive, so we need to decide whether the third term is positive or negative. In order to use $B-V$ to decide the T_{eff}, this derivative must be monotonic. For simplicity, here I rewrite the third term as

$$-e^{\frac{h(\nu+\Delta\nu)}{kT}}(\nu+\Delta\nu) + e^{\frac{h(2\nu+\Delta\nu)}{kT}}\Delta\nu + e^{\frac{h\nu}{kT}}\nu \tag{8}$$

by applying $\nu_B = \nu + \Delta\nu$ and $\nu_V = \nu$. Note that both ν and $\Delta\nu$ are positive. If the following equation has no solution, the derivative is monotonic

$$\begin{aligned}
-e^{\frac{h(\nu+\Delta\nu)}{kT}}(\nu+\Delta\nu) + e^{\frac{h(2\nu+\Delta\nu)}{kT}}\Delta\nu + e^{\frac{h\nu}{kT}}\nu = 0\\
-e^{\frac{h\Delta\nu}{kT}}(\nu+\Delta\nu) + e^{\frac{h(\nu+\Delta\nu)}{kT}}\Delta\nu + \nu = 0\\
\Delta\nu e^{\frac{h\Delta\nu}{kT}}(e^{\frac{h\nu}{kT}}-1) = \nu(e^{\frac{h\Delta\nu}{kT}}-1)\\
\frac{\Delta\nu e^{\frac{h\Delta\nu}{kT}}}{e^{\frac{h\Delta\nu}{kT}}-1} = \frac{\nu}{e^{\frac{h\nu}{kT}}-1}
\end{aligned} \tag{9}$$

With some algebra, we can show that the left hand side is monotonically increasing with $\Delta\nu$, and the right hand side is monotonically decreasing with ν. Besides, only when $\Delta\nu = \nu = 0$, the two sides are equal. So there is no solution for the equation above, and the derivative of the flux ratio, i.e. the Expression 7, is always negative, which means that the $B-V$ is monotonically decreasing with the temperature.

(d) On the plot, identify the following features (warning: depending on the sample you choose, they may not be there!):

- The Main Sequence
- RGB stars
- White Dwarfs

The labeled CMD is shown as Figure 2.

(e) Convert the CMD into Hertzsprung-Russell diagram.

The Hertzsprung-Russell diagram is shown as Figure 3.

3 Transits

(a) The probability that a planet transits a star

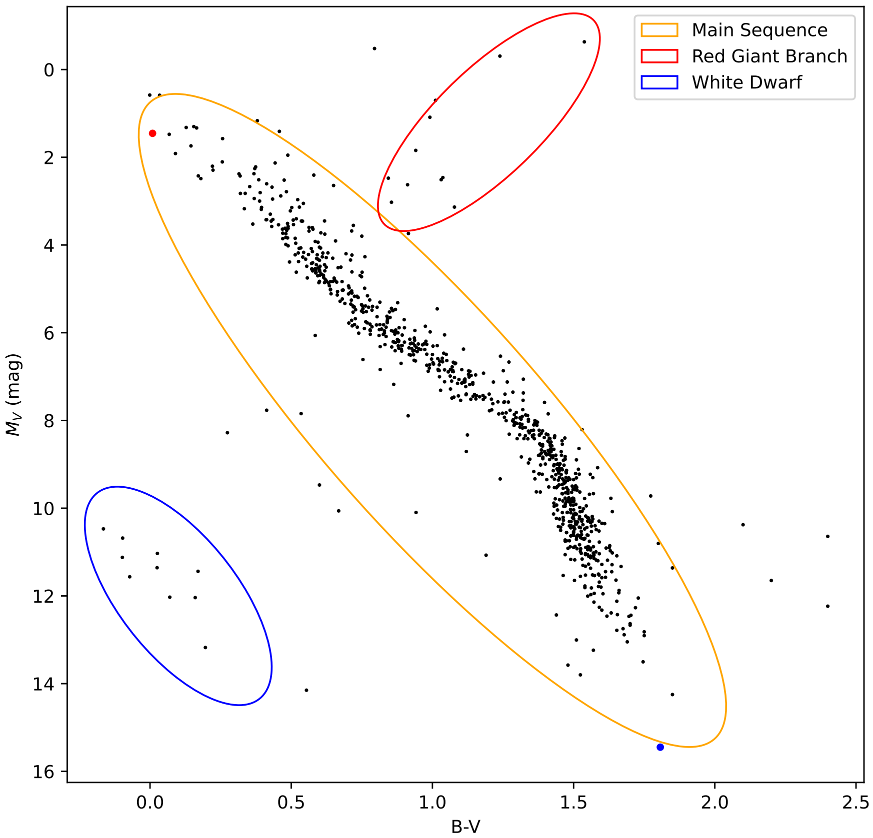

Figure 2: The labeled CMD as Figure 1. The main sequence is labeled with the orange ellipse. The RGB stars are labeled with the red ellipse. The white dwarfs are labeled with the blue ellipse.

Transit is caused by the eclipsing of a star by a planet. So only when

$$\frac{R_p + R_\star}{a} > \sin(\frac{\pi}{2} - i) \tag{10}$$

the planet can transit the star, which gives us limits of orbital inclination. If we assume identical possibility for the angular momentum of the planet point into an unit solid angle, the probability distribution of the inclination angle i is

$$\rho(i) = \frac{1}{2}\sin i \qquad \text{for} \quad 0 \leq i \leq \pi \tag{11}$$

Now integrate this distribution over the range mentioned above to get the probability of transit,

$$\int_{\arccos\left(\frac{R_p+R_\star}{a}\right)}^{\pi-\arccos\left(\frac{R_p+R_\star}{a}\right)} \frac{1}{2}\sin i \, \mathrm{d}i = \frac{1}{2}\left(\cos\arccos\left(\frac{R_p+R_\star}{a}\right) + \cos\arccos\left(\frac{R_p+R_\star}{a}\right)\right) \tag{12}$$

$$= \frac{R_p + R_\star}{a} \tag{13}$$

(b) The probability that a planet is transiting a star currently

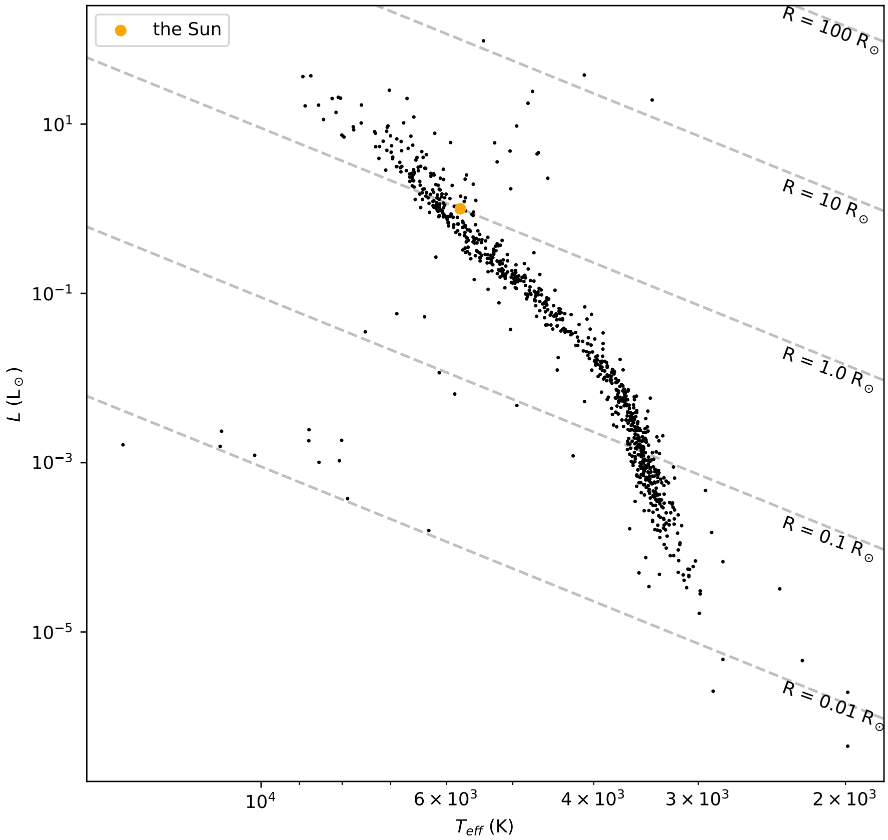

Figure 3: The Hertzsprung-Russell diagram.

Assuming that the prbability of the planet at any phase in the orbit is equal. Note that the phase coverage of the transit is not the same with different orbital inclination, which is due to the distance for the planet to transit is different, which is

$$(R_p + R_\star)\sqrt{1 - \left(\frac{a \cos i}{R_p + R_\star}\right)^2} \tag{14}$$

And thus the phase coverage is

$$\Delta\phi = 2 \arcsin\left(\frac{(R_p + R_\star)}{a}\sqrt{1 - \left(\frac{a \cos i}{R_p + R_\star}\right)^2}\right) \approx 2\left(\frac{(R_p + R_\star)}{a}\sqrt{1 - \left(\frac{a \cos i}{R_p + R_\star}\right)^2}\right) \tag{15}$$

The approximation is the small angle approximation. Given the total phase is 2π, the probability of the planet transiting the star is

$$\int_{\arccos\left(\frac{R_p+R_\star}{a}\right)}^{\pi - \arccos\left(\frac{R_p+R_\star}{a}\right)} \frac{\Delta\phi}{2\pi}\frac{1}{2}\sin i \,\mathrm{d}i = -\frac{1}{4\pi}\int_{\beta}^{-\beta} \Delta\phi \,\mathrm{d}\cos i \tag{16}$$

$$= \frac{1}{4}\left(\frac{R_p + R_\star}{a}\right)^2 \tag{17}$$

while there is a π in the result of interation from $\arctan\infty$, which can be canceled out by the π in the denominator.

(c) Assuming that all of the 100,000 stars harbor such an Earth-like planet, how many will (eventually) show a transit?

By substituting the corresponding value in solar system and multiply it by the total number of the stars, we can get the number of transit is

$$N_\star \frac{R_\oplus + R_\odot}{\mathrm{AU}} \approx 469 \tag{18}$$

(d) What is the (maximum) percentage change in brightness of an Earth-like transit? What is the drop in magnitude? State the numerical values.

The brightness change due to transit is caused by the eclipsing of the star by the planet. So the percentage of brightness change is the ratio of the area of the planet to the area of the star.

$$\frac{\Delta F}{F} = \frac{A_p}{A_\star} = \frac{R_\oplus^2}{R_\odot^2} = 8.4 \times 10^{-5} = 0.0084\% \tag{19}$$

And the derivative of the magnitude is

$$m - m_0 = -2.5 \log(\frac{f}{f_0}) \tag{20}$$

$$\mathrm{d}m = -\frac{2.5}{\ln 10} \frac{\mathrm{d}f}{f} \tag{21}$$

Then the drop (rise) in magnitude is

$$\Delta m = 9.1 \times 10^{-5} \tag{22}$$

(e) Angie argues that those White Dwarfs that are young and as bright as their host star (same effective temperature) do not show a transit signal. Bernie thinks they will. Who is right and why?

Bernie is right. When the white dwarf passing through the front of the host star or behind the host star, the brightness of the system will drop, which is a result of the declination of the total observed area. The drop is deeper in the latter case, because the white dwarf has a higher effective temperature than the host star, while the area of eclipse is the same.

(f) Calculate the radial-velocity amplitude generated by (i) the Earth-like planet; (ii) the White Dwarf. Still assume a distance of 1 au and a solar-mass host star.

According to the Kepler's third law, the orbital period of a Earth-like planet satisfies:

$$\frac{P_\oplus^2}{\mathrm{year}^2} \frac{\mathrm{AU}^3}{a^3} = \frac{M_\odot}{M_\odot + M_\oplus} \tag{23}$$

$$P_\oplus = \sqrt{\frac{(a/\mathrm{AU})^3}{(M_\odot + M_\oplus)/(M_\odot)}} \approx 1.0\mathrm{year} \tag{24}$$

So the radial velocity amplitude cause by a Earth-like planet is

$$K_\oplus = \frac{2\pi a}{P_\oplus} \frac{M_\oplus}{M_\odot + M_\oplus} \approx 8.9 \times 10^{-5} \mathrm{km/s} \tag{25}$$

With the same equation above and substitute the $M_\oplus$ with $M_\odot$, which is approximately the mass of a white dwarf, we can get the radial velocity amplitude cause by a white dwarf

$$P_{\mathrm{WD}} = \sqrt{\frac{(a/\mathrm{AU})^3}{(M_\odot + M_{\mathrm{WD}})/(M_\odot)}} \approx 0.7\mathrm{year} \tag{26}$$

$$K_{\mathrm{WD}} = \frac{2\pi a}{P_{\mathrm{WD}}} \frac{M_{\mathrm{WD}}}{M_\odot + M_{\mathrm{WD}}} \approx 21.3\mathrm{km/s} \tag{27}$$

(g) Is an RV-measurement useful to rule out that the transiting body is a White dwarf?

(h) Is an RV-measurement useful to ensure that the transiting body is an Earth-mass planet?

The accuracy limit of the RV-measurement is now approximately $n \times 10^{-1}\mathrm{m/s}$, which is limited by the stellar activity, especially the oscillations on star surface. We can see that this limit is smaller than the radial velocity amplitude caused by a white dwarf but is still larger than that caused by a Earth-like planet. So the RV-measurement is helpful to rule out that the transiting body is a white dwarf. From a high-precision transit light curve, we can infer the size of the transiting body. And if the RV follow up shows a none detection, I think in principal we can infer that the transiting body is an Earth-mass planet, or at least we can give an upper limit of the transting body, which is close to the mass of the Earth.

4 Microlensing

(a) find an expression for $v_\perp$.

By integrating the impulse on the direction perpendicular to the original velocity, we can get the momentum change on this direction.

$$\Delta p_\perp = \int_{-\infty}^{\infty} G\frac{Mm}{(vt)^2+b^2}\sqrt{\frac{b^2}{(vt)^2+b^2}}\,\mathrm{d}t = \frac{2GMm}{bv} \tag{28}$$

So the velocity change on this direction is

$$\Delta v_\perp = \frac{2GMm}{bv}\frac{1}{m} = \frac{2GM}{bv} \tag{29}$$

(b) From this, find the scattering angle ψ.

The scattering angle can be computed by campare the two velocities

$$\psi = \arctan\frac{v_\perp}{v_\parallel} = \arctan\frac{2GM}{bv^2} \approx \frac{2GM}{bv^2} \tag{30}$$

(c) Find the Einstein angle θ_E as defined in the class.

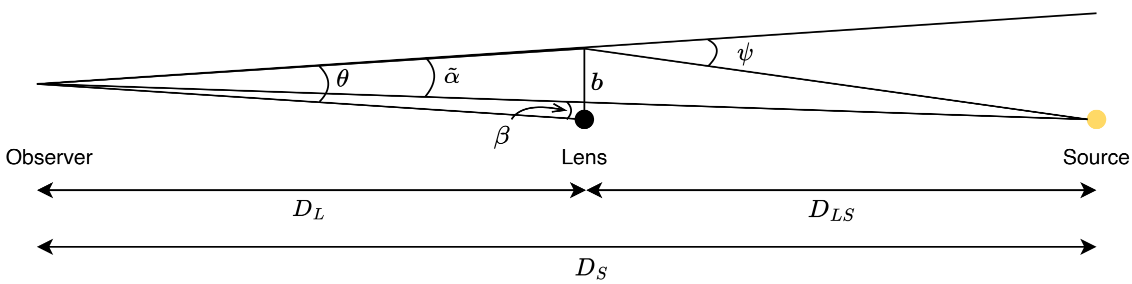

Figure 4: The geometry of the light deflection by micorlensing effect. Produced by Zhecheng Hu (2022).

The defintion of the Einstein radius θ_E is:

$$\frac{D_S - D_L}{D_S}\psi \equiv \frac{\theta_\mathrm{E}^2}{\theta} \tag{31}$$

while the definitions of the symbols are shown in Figure 4. So here the impact parameter, b, can be written into $a = \theta D_L$ with small angle approximation. If we substitute the impact parameter b into the Expression 30, i.e. the deflection angle and change v into c, we can get the deflection angle in microlensing,

$$\psi = \frac{2GM_L}{c^2 \theta D_L} \tag{32}$$

Combine this with the definition of the Einstein radius, i.e. Equation 31,

$$\theta_{\mathrm{E}} = \sqrt{\frac{2GM_L}{c^2} \frac{D_S - D_L}{D_S D_L}} \tag{33}$$

which is the Einstein radius.

(d) Estimate the typical value of θ_{E} for a bulge microlensing event.

$$\theta_{\mathrm{E}} \approx 0.71 \,\mathrm{mas} \left(\frac{D_{\mathrm{S}}}{8\mathrm{kpc}}\right)^{-1/2} \left(\frac{M}{0.5 M_{\odot}}\right)^{1/2} \tag{34}$$

$$r_{\mathrm{E}} = \theta_{\mathrm{E}} D_L \approx 2.8 \mathrm{AU} \tag{35}$$

where the r_{E} is the size of the Einstein ring in the lens plane.

(e) Compute the magnification.

From the module 2 slides, we know that the magnification A is

$$A = \theta^4 / \left(\theta^4 - \theta_E^4\right) \tag{36}$$

And with the lens euation,

$$\beta = \theta - \frac{\theta_E^2}{\theta} \tag{37}$$

one can rewrite the lens equation into the following form,

$$A = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}, \qquad u = \frac{\beta}{\theta_{\mathrm{E}}} \tag{38}$$

so if the source is on the Einstein ring, i.e. $\beta = \theta_{\mathrm{E}}$, the magnification is approximately $A = 3/\sqrt{5} \approx 1.342$. And the brightness change should be the same as the magnification if there is no light from other objects. The change in magnitude units is

$$\Delta m = -2.5 \log_{10} A = -0.31 \tag{39}$$

(f) Obtain the duration of the Einstein ring crossing.

The baseline magnitude is approximately $m_{\mathrm{b}} = 18.2$. With the assumption that there is no blending light and $\min(\beta)/\theta_{\mathrm{E}} \ll 1$, the duration of the whole Einstein ring crossing is approximately $t_{\mathrm{E,cross,star}} = 42.5$ day.

(g) Give an expression for the planet-to-star mass ratio $m_p/m_\star$ in terms of the Einstein crossing times. What is the planet mass (in units of Earth) if the star is 0.5 $M_{\odot}$?

The mass ratio is related with the Einstein crossing time,

$$\frac{t_{\mathrm{E,cross,planet}}}{t_{\mathrm{E,cross,star}}} = \frac{\theta_{\mathrm{E,planet}}}{\theta_{\mathrm{E,star}}} = \sqrt{m_p/m_\star} \tag{40}$$

So the planet mass is approximately $5.7M_\oplus$ assuming a host star mass of $0.5M_\odot$.