Stars and Planets Problem Set6

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Exercise VI.1 Synodical timescale

(a)

The t_{syn} is the least common multiple of P_1 and P_2:

$$t_{syn} = [P_1, P_2]$$

(b)

how to calculate?

Exercise VI.2 Epicycle approximation

(a)

Assuming that $e << 1$, from the Kelper equation $M = E - e \sin E$ we can have:

$$\begin{aligned}
\sin E &= \sin(M + e\sin E) = \sin M \cos(e \sin E) + \cos M \sin(e \sin E) \\
&= \sin M(1 + \frac{1}{2}(e\sin E)^2 + \cdots) + \cos M(e \sin E + \cdots) \\
&= \sin M + \mathcal{O}(e)
\end{aligned}$$

and:

$$\begin{aligned}
\cos E &= \cos(M + e\sin E) = \cos M \cos(e \sin E) - \sin M \sin(e \sin E) \\
&= \cos M(1 + \frac{1}{2}(e\sin E)^2 + \cdots) - \sin M(e \sin E + \cdots) \\
&= \cos M - e \sin M \sin E + \mathcal{O}(e^2) = \cos M - e\sin^2 M + \mathcal{O}(e^2) \\
&= \cos M + \mathcal{O}(e)
\end{aligned}$$

Firstly consider r:

$$\cos E = e + \frac{r}{a}\cos\nu = \frac{e+\cos\nu}{1+e\cos\nu}$$
$$\cos\nu = \frac{\cos E - e}{1 - e\cos E}$$
$$r = \frac{a(\cos E - e)}{\cos\nu} = a(1 - e\cos E)$$

Plug in $\cos E = \cos M + \mathcal{O}(e)$ and we have:

$$r \approx a(1 - e\cos M) + \mathcal{O}(e^2)$$

Secondly consider ν:

$$\cos\nu = \frac{\cos E - e}{1 - e\cos E} = (\cos E - e)(1 + e\cos E) = \cos E - e\sin^2 E + \mathcal{O}(e^2)$$

Plug in $\sin E = \sin M + \mathcal{O}(e)$ and $\cos E = \cos M - e\sin^2 M + \mathcal{O}(e^2)$, and we have:

$$\cos\nu = \cos M - 2e\sin^2 M + \mathcal{O}(e^2)$$

If we assume that $\nu \approx M + 2e\sin M + \mathcal{O}(e^2)$, we can have:

$$\begin{aligned}\cos\nu &= \cos(M + 2e\sin M + \mathcal{O}(e^2))\\ &= \cos M\cos(2e\sin M) - \sin M\sin(2e\sin M) + \mathcal{O}(e^2)\\ &= \cos M - 2e\sin^2 M + \mathcal{O}(e^2)\end{aligned}$$

Therefore, we can conclude that $\nu \approx M + 2e\sin M + \mathcal{O}(e^2)$.

(b)

how to prove?

(c)

$$\phi'_{\text{eff}}(r_o) = Anr_o^{n-1} - \frac{l_z^2}{2}\frac{2}{r_o^3} = 0$$
$$r_o = (\frac{l_z^2}{An})^{\frac{1}{n+2}}$$

(d)

Expanding the potential around $r = r_o$:

$$\phi_{\text{eff}} = \phi_{\text{eff}}(r_o) + \phi'_{\text{eff}}(r_o)x + \frac{1}{2}\phi''_{\text{eff}}(r_o)x^2$$

where $x = r - r_o$ and $\phi'_{\text{eff}}(r_o) = 0$. Therefore the equation of motion (3) becomes:

$$\ddot{x} = -x\phi''_{\text{eff}}(r_o) = -x(An(n-1)r_o^{n-2} + 3l_z^2 r_o^{-4}) = -(n+2)l_z^2 r_o^{-4}x$$

Compared to $\ddot{x} = -\kappa^2 x$, we have:

$$\kappa = \frac{\sqrt{n+2}l_z}{r_o^2} = \sqrt{n+2}(\frac{l_z^{\frac{2-n}{2}}}{An})^{-\frac{2}{n+2}}$$

In a Keplerian potential (n=-1):

$$\kappa = A^2/l_z^3 = \Omega$$

where Ω is the orbital frequency. why the orbital frequency

(e)

For which values of n does the circular orbit solution become unstable? What is the physical reason?

Exercise VI.3 The Trojans

(a)

From the law of cos, we have:

$$r_1^2 = m^2 + 2mr\cos\theta + r^2$$
$$r_2^2 = (1-m)^2 - 2(1-m)r\cos\theta + r^2$$

Because $m << 1$, we can expand r_1^{-1} as:

$$\begin{aligned}
r_1^{-1} &= (m^2 + 2mr\cos\theta + r^2)^{-1/2} \\
&\approx (1 + 2\Delta + \Delta^2 + 2m\cos\theta)^{-1/2} \\
&\approx 1 - 1/2(2\Delta + \Delta^2 + 2m\cos\theta) + 3/8(2\Delta + \Delta^2 + 2m\cos\theta)^2 \\
&\approx 1 - \Delta + \Delta^2 - m\cos\theta
\end{aligned}$$

We can expand r_2^{-1} as:

$$
\begin{aligned}
r_2^{-1} &= ((1-m)^2 - 2(1-m)r\cos\theta + r^2)^{-1/2} \\
&\approx (1 - 2\cos\theta + 1)^{-1/2} \\
&= \frac{1}{\sqrt{2(1-\cos\theta)}}
\end{aligned}
$$

We can expand r^2 as:

$$
r^2 = (1+\Delta)^2 = 1 + 2\Delta + \Delta^2
$$

Plug the three above relations into the effective potential:

$$
\begin{aligned}
\phi_{\text{eff}} &= -\frac{1-m}{r_1} - \frac{m}{r_2} - \frac{1}{2}r^2 \\
\phi_{\text{eff}} &= m(\cos\theta - \frac{1}{\sqrt{2(1-\cos\theta)}}) - \frac{3}{2}\Delta^2 + m - 3/2
\end{aligned}
$$

Ignore the constants in the potential and we get:

$$
\phi_{\text{eff}} = m(\cos\theta - \frac{1}{\sqrt{2(1-\cos\theta)}}) - \frac{3}{2}\Delta^2
$$

(b)

The vector form of the equation of motion is:

$$
\ddot{\boldsymbol{r}} + 2(\boldsymbol{\omega} \times \dot{\boldsymbol{r}}) = -\nabla\phi_{\text{eff}}
$$

The radial component of the equation of motion is:

$$
\ddot{r} - r\dot{\theta}^2 - 2r\dot{\theta} = -\frac{\partial\phi_{\text{eff}}}{\partial r}
$$

Plug in the expression of the effective potential and we have:

$$
\ddot{\Delta} - (1+\Delta)\dot{\theta}^2 - 2(1+\Delta)\dot{\theta} = 3\Delta
$$

(c)

Assume $\Delta << 1$ and we get:

$$
\ddot{\Delta} - \dot{\theta}^2 - 2\dot{\theta} = 3\Delta
$$

And $\dot{\theta}^2$ must be the higher order small value compared to $\dot{\theta}$. Therefore, the above equation can be reduced to:

$$
\ddot{\Delta} - 2\dot{\theta} = 3\Delta
$$

If the objects are confined in a small range of radius, then the motion along the radial direction can not be too large, so $\ddot{\Delta}$ is also a higher order small value. Equation (8) can be reduced to:

$$2\dot{\theta} + 3\Delta = 0$$

(d)

The azimuthal component of the equation of motion is:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r} = -\frac{1}{r}\frac{\partial \phi_{\text{eff}}}{\partial \theta}$$

Plug in $r = 1 + \Delta$ and we have:

$$(1+\Delta)\ddot{\theta} + 2\dot{\Delta}\dot{\theta} + 2\dot{\Delta} = -\frac{1}{1+\Delta}\frac{\partial \phi_{\text{eff}}}{\partial \theta}$$

Ignore the higher order small value:

$$\ddot{\theta} + 2\dot{\Delta} = -\frac{\partial \phi_{\text{eff}}}{\partial \theta}$$

(e)

The effective potential can be rewritten as:

$$\phi_{\text{eff}} = -\frac{m}{2}(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2}) - \frac{3}{2}\Delta^2 + m$$

From the radial component of the equation of motion we have $\Delta = -2/3\dot{\theta}$. Plug the above two expressions into the azimuthal component and we can have:

$$-\frac{1}{3}\ddot{\theta} = \frac{m}{2}\frac{\partial}{\partial \theta}(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2}) + 3\Delta\frac{\partial \Delta}{\partial \theta}$$

Ignore the higher order small value $3\Delta\frac{\partial\Delta}{\partial\theta}$ and times $\dot{\theta}$ on both sides:

$$\begin{aligned}\dot{\theta}\ddot{\theta} &= \frac{3}{2}m\frac{\partial}{\partial \theta}(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2})\dot{\theta} \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\dot{\theta}^2 + \frac{3}{2}m(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2}))\end{aligned}$$

The integral of motion under this approximation gives a conserved quantity I:

$$I = \frac{1}{2}\dot{\theta}^2 + \frac{3}{2}m(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2})$$

(f)

The potential component is:

$$U = \frac{3}{2}m(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2})$$

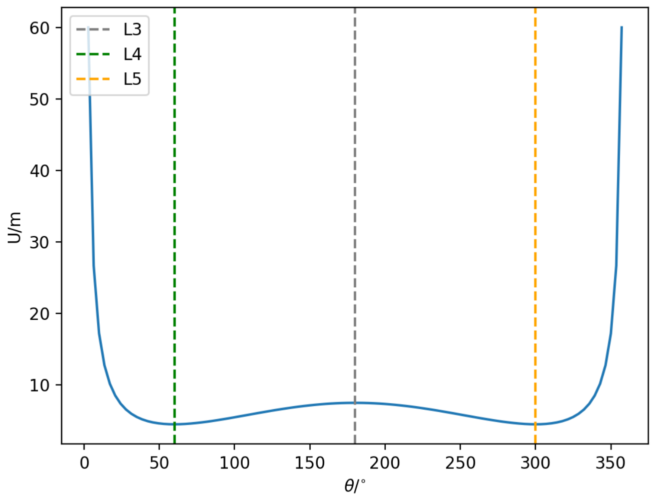
Take the first derivative of the potential and we can get the Lagrange points:

$$U'/m = 4\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{1}{2}\frac{\cos\theta/2}{\sin^2\theta/2}$$

$$\cos\frac{\theta}{2} = 0 \text{ or } \sin\frac{\theta}{2} = 1/2$$

$$\theta = \pi/3, \pi/2 \text{ or } 5\pi/3$$

The potential and the Lagrange points are shown as below.



(g)

the extent of the widest possible Trojan orbit?

(h)

Expand the potential to the second order at L4 $\theta = \pi/3$:

$$U''/m = 2\cos\theta + \frac{1}{2}\sin^{-3}\frac{\theta}{2}\cos^2\frac{\theta}{2} + \frac{1}{4}\sin^{-1}\frac{\theta}{2}$$

$$\omega_{\text{lib}} = \sqrt{k/m} = \sqrt{U''/m} = \sqrt{4.5} = 2.12\ \omega$$

$$t_{\text{lib}} = 3t$$

The orbital period for Jupiter is $t = 12$ years, so the orbital period for Jupiter's Trojans around L4 is $t_{\text{lib}} = 36$ years.

Exercise VI.4 Tides

(a)

For the Earth-Moon system, the value for n (the mean motion) is $n = \frac{2\pi}{28\text{days}} = 0.22\text{day}^{-1}$.

(b)

The spin-down timescale is:

$$t^{-1}_{\text{de- spin, p}} = \frac{\dot{\Omega}_p}{\Omega_p} = -\frac{\Gamma}{\Omega_p I_p} = -\frac{3k_{2p}}{2QC_I}\frac{m_s^2}{(m_s + m_p)\, m_p}\left(\frac{R_p}{d}\right)^3 \frac{n}{\Omega_p} n$$

For the Moon to spin-down the Earth, the love number $k_{2p} =$, the Quality factor $Q =$, the inertia factor $C_I =$, $m_s = 7.3477 \times 10^{25}$ g, $m_p = 5.974 \times 10^{27}$ g, $R_p = 6.378 \times 10^8$ cm, $d = 384,401$km, $n = \frac{2\pi}{28\text{days}}$, and $\Omega_P = \frac{2\pi}{1\text{ day}}$. Therefore the timescale for the Moon to spin down the Earth is 4.4 billion years .

For the Sun to spin-down the Earth, the love number $k_{2p} = 0.3$, the Quality factor $Q = 12$, the inertia factor $C_I = 0.33$, $m_s = 1.99 \times 10^{33}$ g, $m_p = 5.974 \times 10^{27}$ g, $R_p = 6.378 \times 10^8$ cm, $d = 1\ au$, $n = \frac{2\pi}{365\text{days}}$, and $\Omega_P = \frac{2\pi}{1\text{ day}}$. Therefore the timescale for the Sun to spin down the Earth is 19.9 billion years.

(c)

The time taken for the Moon to "crash" into the Earth is:

$$t^{-1}_{\text{orbit}} = \frac{9k}{2Q}\frac{m_s}{m_p}(\frac{R_p}{d})^5 n$$

Plug in the values from (b) and we can get $t_{\text{orbit}} = 7$ Gyr.

(d)

how to motivate?

Exercise VI.5 Hot Jupiter migration by tides

(a)

The total energy of the original orbit is:

$$E_0 = -\frac{Gm}{2a_0}$$

where the m is the total mass of the star and the hot jupiter. The kinetic energy of the original orbit is:

$$K_0 = \frac{Gm}{2a_0}$$

The gravitational potential energy of the orbit is:

$$U_0 = -\frac{Gm}{a_0}$$

When the magnitude of the orbit velocity is suddenly changed by a factor of f, the kinetic energy changes to:

$$K_1 = f^2 K_0 = \frac{f^2}{2}\frac{Gm}{a_0}$$

The gravitational energy remains the same:

$$U_1 = U_0 = -\frac{Gm}{a_0}$$

Therefore, the total energy changes to:

$$E_1 = U_1 + K_1 = (\frac{f^2}{2} - 1)\frac{Gm}{a_0}$$

Compared to the energy expression of the Kelperian orbit $E_1 = -\frac{Gm}{2a_1}$ we can get:

$$a_1 = \frac{a_0}{2 - f^2}$$

When the magnitude of the orbit velocity is suddenly changed by a factor of f, the total angular momentum changes to:

$$l_1 = f l_0 = f\sqrt{Gma_0(1 - e_0^2)}$$

Compared to the angular momentum expression of the Kelperian orbit $l_1 = \sqrt{Gma_1(1 - e_1^2)}$ we can get:

$$e_1 = 1 - f^2$$

The pericenter r_{p1} is:

$$r_{p1} = a_1(1 - e_1) = \frac{f^2}{2 - f^2}a_0$$

(b)

During this tidal dissipation step, the hot Jupiter's angular momentum is roughly conserved as it circularizes to a final, close-in semimajor axis $e_2 = 0$:

$$l_1 = \sqrt{Gma_1(1 - e_1^2)} = l_2 = \sqrt{Gma_2}$$
$$a_2 = a_1(1 - e_1^2) = a_1 f^2 (2 - f^2) = a_0 f^2$$

If $a_0 = 5$ au and $a_2 = 0.05$ au, then $f = 0.1$.

(c)

I don't think the existence of these planets can be explained by this mechanism. explain my answer

Exercise VI.6 Geometry of resonances

(a)

(b)

(c)

(d)

Exercise VI.7 Planet trapping

(a)

(b)

(c)

(d)