Stars and Planets Problem Set2

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Exercise I.1 Radiation Pressure

(a)

The dimension of P_{rad} is $[E][L]^{-3}$. The physical quantities that are related to pressure from the derivation of state equations are:

$$kT \sim [E]$$
$$h \sim [E][T]$$

However, only with the two quantities above we cannot cancel out T and add in L. Given the pressure is produced by radiation, we can introduce the speed of light $c \sim [L][T]^{-1}$ to help us. After some calculations, we can get that the dimension of radiation pressure is:

$$P_{\text{rad}} \sim \frac{(kT)^4}{h^3c^3}$$

(b)

From the energy density u_ν, we can get the distribution function for the energy $f(\nu)$:

$$f(\nu) = u_\nu/(h\nu) = \frac{8\pi\nu^2}{c^3}\frac{1}{e^{h\nu/kT}-1}$$

And the relation between the distribution function for the energy and the momenta is:

$$f(\nu)\mathrm{d}\nu = f(p)\mathrm{d}p$$
$$p = \frac{h}{\lambda} = \frac{h\nu}{c}$$
$$\mathrm{d}p = \frac{h}{c}\mathrm{d}\nu$$

Therefore, the distribution function for the momenta is:

$$f(p) = \frac{8\pi}{h^3}p^2\frac{1}{e^{cp/kT}-1}$$

The radiation pressure is:

$$P_{\text{rad}} = \int_0^{+\infty} cpf(p)\mathrm{d}p = \frac{8\pi^5}{45}\frac{(kT)^4}{h^3c^3}$$

The relation between radiation energy density and radiation pressure is:

$$P_{\text{rad}} = \frac{u_{\text{rad}}}{3}$$

Exercise I.2 Physical structure of protoplanetary disks

(a)

The standard astrophysical "cosmic" abundances for the elements gives:

$$X = 0.75,\ Y = 0.25$$

Since the hydrogen and helium gas are in molecular form, the relative mean molecular weight is:

$$\mu_{\mathrm{H_2}} = 2$$
$$\mu_{\mathrm{He}} = 4$$

The mean molecular weight μ of the gas is:

$$\frac{1}{\mu} = \frac{X}{\mu_{\mathrm{H_2}}} + \frac{Y}{\mu_{\mathrm{He}}}$$
$$\mu = 2.29$$

(b)

Assume the mass of the star is M, the gravitational constant is G and the positive direction of z axis is pointing upward. Write down Newton's second law in the vertical direction:

$$g_{\mathrm{z},\star} = -\frac{GM}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}} = -\frac{GM}{r^3}\frac{z}{(1+(\frac{z}{r})^2)^{\frac{3}{2}}}$$

For $z << r$:

$$\frac{z}{(1+(\frac{z}{r})^2)^{\frac{3}{2}}} = z(1-\frac{3}{2}(\frac{z}{r})^2) \sim z$$

Therefore:

$$g_{\mathrm{z},\star} \approx -\frac{GM}{r^3}z = -\Omega_{\mathrm{K}}^2 z$$

where

$$\Omega_{\mathrm{K}} = \sqrt{\frac{GM}{r^3}}$$

(c)

The equation of state for the gas in the disk is:

$$P = nkT = \frac{\rho}{\mu m_u}kT$$

Since the $g_{\mathrm{z},\star}$ has included the minus sign, the hydrostatic balance is:

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \rho g_{\mathrm{z},\star}$$

Combine the two equations and we get the ordinary differential equation for $\rho(z)$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\frac{\Omega_{\mathrm{K}}^2\mu m_u}{kT}z\rho$$

The solution is:

$$\rho(z) = Ce^{-\frac{\Omega_{\mathrm{K}}^2\mu m_u}{2kT}z^2}$$

(d)

The standard deviation σ or the scaleheight H is:

$$\sigma = H = \sqrt{\frac{kT}{\mu m_u \Omega_{\mathrm{K}}^2}} = \sqrt{\frac{kT}{\mu m_u}} = \sqrt{\frac{kT}{\mu m_u GM}} r^{\frac{3}{2}}$$

And the $\rho(z)$ can be written as:

$$\rho(z) = Ce^{-\frac{z^2}{2H^2}}$$

(e)

The surface density is:

$$\Sigma = \int_{-\infty}^{+\infty} \rho(z)\mathrm{d}z = \int_{-\infty}^{+\infty} Ce^{-\frac{z^2}{2H^2}}\mathrm{d}z$$

$$\Sigma = CH\sqrt{2\pi}$$

$$C = \frac{\Sigma}{H\sqrt{2\pi}}$$

Therefore, $\rho(z)$ can be rewritten as:

$$\rho(z) = \frac{\Sigma}{H\sqrt{2\pi}} e^{-\frac{z^2}{2H^2}}$$

(f)

When $z << r$, use the small angle approximation:

$$\tan\theta = \frac{z}{r} \approx \theta$$

Since z only depends on r, take derivatives on both sides:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}(\frac{z}{r})$$

The fraction ϵ of the stellar luminosity $L_\star$ that is being intercepted by the disk at radius interval $[r, r + \Delta r]$ is:

$$\epsilon = \frac{1}{2\pi}\int \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \int_{\theta}^{\theta+\mathrm{d}\theta} \sin\theta \mathrm{d}\theta = \int_{r}^{r+\mathrm{d}r} \frac{z}{r}\frac{\mathrm{d}}{\mathrm{d}r}(\frac{z}{r})\mathrm{d}r$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}(\frac{z}{r})^2 \Delta r = \frac{\beta^2}{2}\frac{\mathrm{d}}{\mathrm{d}r}(\frac{H}{r})^2\Delta r = \beta^2 \frac{H}{r}(\frac{H}{r})'\Delta r$$

Pulge in $\frac{H}{r} = \sqrt{\frac{kT}{\mu m_u GM}} r^{\frac{1}{2}}$ and denote $\alpha = \sqrt{\frac{k}{\mu m_u GM}}$:

$$\epsilon = \frac{1}{2}\beta^2\alpha^2 T\Delta r$$

(g)

The radiation is in turn intercepted at the disk midplane($z = 0$). For the disk at radius interval $[r, r + \Delta r]$, the radiation area is $2\pi r \Delta r$. The heating and cooling radiation equation reads:

$$\frac{1}{2}L_\star \epsilon = 2\pi r \Delta r \sigma T^4$$

$$T(r) = (\frac{L_\star \beta^2 \alpha^2}{8\pi\sigma}\frac{1}{r})^{\frac{1}{3}}$$

(h)

Given $L_\star = 1\ L_\odot$, $M = 1\ M_\odot$, $r = 1\ au$ and $\beta = 3$:

$$T(r) = (\frac{L_\star \beta^2 \alpha^2}{8\pi\sigma}\frac{1}{r})^{\frac{1}{3}} = 76.64\ K$$

(i)

If the photons instead are absorbed along the way, the midplane temperature would become lower. If the photons are absorbed along the way, less radiation energy will reach the midplane. According to the heating and cooling radiation balance, less energy will be emittted by the midplane, so the midplane temperature will be lower.