Problem Set VI – Dynamics

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Hand in HW-problems before the announced date

Hand in before the announced deadline, either before the start of class or by uploading your solutions to the WebLearning system (pdf format only!). Write down your answers in English and state any additional assumptions that you have made, e.g., in case when you think that the phrasing is unclear. Collaboration is allowed, but you should hand in your own solution and be able to reproduce your answer independently if asked to (e.g, in class!). Students whom typeset their answers (with, e.g., LaTeX) or present an excellent layout will be awarded a 10% bonus, if all layout standards have been met. Late returns are penalized at a rate of 10% per day.

Exercise VI.1 Synodical timescale

Two planets are in conjunction when they line up and are on the same side with respect to the star. (If they are on different sides, it is called opposition). The time interval between two conjunctions is called the *synodical timescale*.

(a) Give an expression for t_{syn} in terms of the period of the two planets P_1 and P_2. You can assume that the planets are on circular orbits and ignore their mutual interaction.

(b) Let the difference between the planets' semi-major axes be $b = a_2 - a_1$. Express t_{syn} in terms of b and the period of P_1 in the limit where $b/a_1 \ll 1$.

Exercise VI.2 Epicycle approximation

(a) From the expressions derived in the course for the mean (M) and true (ν) anomaly, prove the *guiding center approximation*:

$$r \simeq a(1 - e\cos M) + O(e^2) \tag{1}$$

$$\nu \simeq M + 2e\sin M + O(e^2) \tag{2}$$

hint: you obviously need to assume that $e \ll 1$.

We see that to first order in eccentricity, the motion is a superposition of uniform circular motion around the guiding center ($r = a$, $\nu = M$) and uniform motion of an ellipse with aspect ratio 2:1.

Let us now generalize the guiding center approximation to any radial potential, $\phi(r) \equiv Ar^n$, where A is a constant.

(b) Show that the equation of motion, $\ddot{\boldsymbol{r}} = -\nabla\phi$ (in vector form), can be written

$$\ddot{r} = -\frac{\partial \phi_{\mathrm{eff}}}{\partial r} \tag{3}$$

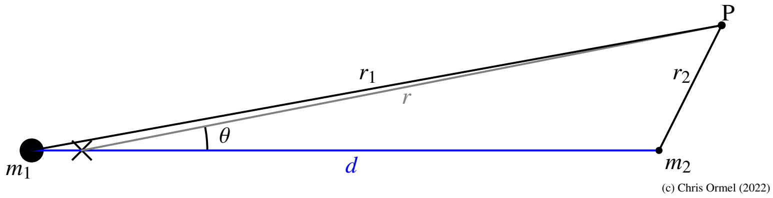


Figure 1: Definition of θ in this problem. In this problem you will assume that $r_1 \approx r \approx d$.

(in scalar form) where

$$\phi_{\rm eff} = \phi + \frac{l_z^2}{2r^2} \tag{4}$$

is the effective potential. *P.S.: it's OK to consider the planar dimensions only in your answer ($z = 0$).*

(c) Show that circular orbits correspond to an extremum of the effective potential, $\phi'_{\rm eff}(r) = 0$. Give an expression for the corresponding radius r_o (in terms of A, n ...).

(d) Expanding the potential around $r = r_o$, show that

$$\ddot{x} \simeq -\kappa^2 x \tag{5}$$

where $x = r - r_o$ and κ the epicyclic frequency. Give an expression for κ (in terms of A, n...) and verify that for a Keplerian potential ($n = -1$), $\kappa = \Omega$ where Ω is the orbital frequency.

(e) For which values of n does the circular orbit solution become <u>unstable</u>? What is the physical reason?

Exercise VI.3 The Trojans

Consider the circularly-restricted three body problem with effective potential

$$\phi_{\rm eff} = -\frac{1-m}{r_1} - \frac{m}{r_2} - \frac{1}{2}r^2 \tag{6}$$

where we have defined $m_1 + m_2 = G = d = \omega = 1$ and renamed $m_2 = m$ for convenience. In class we have derived an approximation in the limit where $r_2 \ll 1$.

(a) Let us instead assume that $r = 1 + \Delta$ with $\Delta \ll 1$ to expand $1/r_1$ and r^2 similar to what has done in class. Show that in the limit $m \ll 1$ and $\Delta \ll 1$:

$$\phi_{\rm eff} = m\left(\cos\theta - \frac{1}{\sqrt{2(1-\cos\theta)}}\right) - \frac{3}{2}\Delta^2 \tag{7}$$

where θ is the azimuthal angle seen from the center of mass (different from the figure shown in class, where it was defined with respect to m_2!).[1]
Hint: you can forget about terms $O(m^2)$, $O(m\Delta)$, $O(\Delta^3)$ and any higher order terms. Please, intelligently use this information in your expansion of the r_1^{-1} and r_2^{-1} terms.

[1] or formally: the angle subtended by $\boldsymbol{r}_2$ from the center-of-mass.

(b) Decompose the equation of motion into its radial (r) and azimuthal (θ) components. Focusing on the radial component, show that:

$$\ddot{\Delta} - (1+\Delta)\dot{\theta}^2 - 2(1+\Delta)\dot{\theta} = 3\Delta \tag{8}$$

Hint: you should know that:

$$\ddot{\boldsymbol{r}} = \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{pmatrix} \tag{9}$$

in polar coordinates

(c) We will assume that Equation (8) can be reduced to

$$2\dot{\theta} + 3\Delta = 0. \tag{10}$$

i.e., the other terms appearing in Equation (8) will be ignored. Can you give a motivation why, or a condition when, it is justified to ignore these terms? Why is it justified to ignore terms as $\Delta\dot{\theta}$; and under which conditions can $\ddot{\Delta}$ be ignored?

(d) In a similar vain, show that the azimuthal equation of motion reduces to

$$\ddot{\theta} + 2\dot{\Delta} = -\frac{\partial}{\partial\theta}\phi_{\rm eff} \tag{11}$$

(e) Then show that

$$I = \frac{1}{2}\dot{\theta}^2 + \frac{3}{2}m\left(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2}\right) \tag{12}$$

is an integral of motion under this approximation.

(f) Writing $I = \frac{1}{2}\dot{\theta}^2 + U$ we identify U as the "potential" component to the conserved quantity I. Plot U/m vs θ for θ in the range $[0, 2\pi]$. Identify the L3, L4, and L5 Lagrange points in this figure.

Trojans are bodies that orbit either L4 or L5.

(g) What is the extent of the widest possible Trojan orbit? Give the range in θ and Δ (like in $\Delta = [a, b]$). Then give the numerical value for the total *radial* width ($b - a$) of these Trojans in units of au for Jupiter.

(h) Assume a Trojan orbits close to L4. What is the orbital period $t_{\rm lib}$ about L4 (in dimensionless units)? What is this $t_{\rm lib}$ for Jupiter's Trojan's in years?

Exercise VI.4 Tides

In this question we are going to calculate several timescales related to tidal interactions. The expressions were already derived in class (use them as well as the tidal constants!).

(a) For the Earth-Moon system what is the value for n (the mean motion) that must be used?

(b) Calculate the timescale for (i) the Moon to spin-down the Earth; and (ii) for the Sun to spin down the Earth.

(c) Suppose the Earth has been tidally locked with the Sun (just ignore its spin altogether). How long will it take for the Moon to "crash" into the Earth.

(d) One suggestion for the peanut shape of the KBO Arrokoth is that it is a product of a merger between two separate KBOs[2] after the two bodies became tidally locked to each other. In this scenario, do you think the collision velocity would have been

A) much larger than;

B) similar to;

C) much smaller than

their mutual surface escape velocity $v_{\rm esc} \sim \sqrt{2G(m_1+m_2)/(R_1+R_2)}$? Motivate your answer.

Exercise VI.5 Hot Jupiter migration by tides

Hot-Jupiters are planets orbiting at distances ~0.05 au from their host star. A proposed scenario for this close distance is that they started much further out, but experienced a violent interaction ("scattering") with a third body.

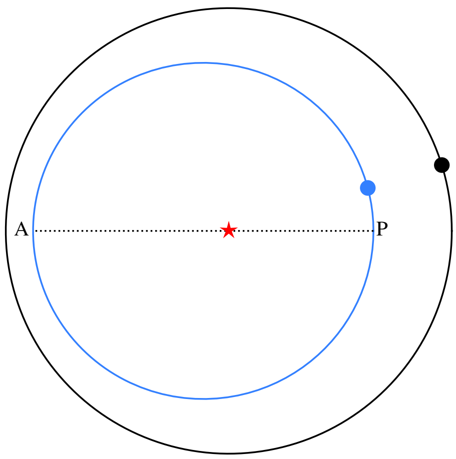
(a) Suppose that initially the Jupiter was on a circular orbit of semi-major axis a_0. The gravitational interaction *instantaneously* changed the *magnitude* (but not the direction) of its orbital velocity by a factor f with $0 < f < 1$. What is the eccentricity e_1, the pericenter r_{p1} and the semi-major axis a_1 of the new orbit? Give your (symbolic) expressions in terms of f and a_0.

(b) If r_{p1} is sufficiently small, tidal interactions with the star would have subsequently reduced the eccentricity to 0. What is the final semi-major axis a_2 in terms of a_0 and f? If $a_0 = 5$ au and $a_2 = 0.05$ au, what is f?

(c) Some hot-Jupiters have been found to orbit 1–2 Roche radii from their host star. Do you think the existence of these planets can be explained by this mechanism? Explain your answer.

Exercise VI.6 Geometry of resonances

In class, we considered conjunctions near periastron of planet 2, for an inner perturber (planet 1) at a circular orbit. We showed that (i) the pre-conjunction interaction is <u>stronger</u>, resulting in (ii) a <u>negative</u> net torque, which causes the conjunctions to (iii) <u>diverge</u> (=move away) from pericenter, that is, resonances with conjunctions at pericenter are <u>unstable</u>.



Now consider the case where the outer planet is the perturber on a circular orbit, with the inner (eccentric) planet experiencing conjunctions near pericenter (see figure). Choose an initial conjunction point slightly away from pericenter (like in the example in class). Subsequently (for each choice <u>motivate your answer</u>):

(a) interactions $\frac{\text{before}}{\text{after}}$ conjunction are stronger.

[2] KBO = Kuiper Belt Object

(b) This results in a net $\frac{\text{positive}}{\text{negative}}$ torque on planet 1

(c) ... which causes the next conjuction point to be $\frac{\text{closer to}}{\text{further from}}$ pericenter.

(d) Resonances near pericenter are therefore $\frac{\text{stable}}{\text{unstable}}$.

Exercise VI.7 Planet trapping

Consider the resonance forcing equations for $\dot{a}$, $\dot{e}$ and $\dot{\phi}_{\text{res}}$ as discussed in class for an interior perturber. In the planet-forming disk, the motion of bodies is damped by interactions with the gas, reducing the eccentricy and semi-major axis.[3] We therefore add a corresponding *damping* rate to the resonance forcing expressions, chararcterized by their timescales t_e and t_a. For example, the equation tor the semi-major axis becomes:

$$\frac{da}{dt} = 2(j+1)G_e^j q_1 n_1 ae \sin\phi_{\text{res}} - \frac{a}{t_a} \tag{13}$$

and similar for e and ϕ_{res}. Note again that the G_e^j are negative for an internal perturber. You can find the numerical value in the lecture slides.

(a) We will be looking for steady-state solutions ($\dot{a} = \dot{e} = \dot{\phi}_{\text{res}} = 0$). Combining the resonance forcing equation for semi-major axis and eccentricity, give the equilibrium expression for eccentricity e_{eq} and resonance angle ϕ_{eq}. The mathematics should be straightforward but choose the right, physically plausible solutions for the resonance angle! Specifically, in a plane where $x = \cos\phi_{\text{res}}$ and $y = \sin\phi_{\text{res}}$ indicate in which quadrant the solutions lie. (If there is any ambiguity in your answer you will not get credits.)

(b) What are the value of the equilibrium eccentricity and the equilibrium resonance angle, for a planet with $t_a = 10^4 t_e = 1\,\text{Myr}$ trapped in a 3:2 resonance? You can further assume that the perturber is an 10 Earth-mass planet orbiting a solar-mass star at 1 au.

(c) From the equation for $\dot{\phi}_{\text{res}}$, find the departure of the period ratio from exact resonance:

$$\Delta \equiv \frac{P_2}{P_1} - \frac{j+1}{j} \tag{14}$$

What is Δ for the above numbers?

(d) Below which $n_1 t_a$ does the equation for the resonance angle you derived above no longer has a solution? Give the analytical expression. What happens to the planet for lower t_a?

[3] For small bodies the damping is aerodynamical; for larger bodies it is gravitational.