Problem Set IV – Evolution

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Hand in HW-problems before the announced date

Hand in before the announced deadline, either before the start of class or by uploading your solutions to the WebLearning system (pdf format only!). Write down your answers in English and state any additional assumptions that you have made, e.g., in case when you think that the phrasing is unclear. Collaboration is allowed, but you should hand in your own solution and be able to reproduce your answer independently if asked to (e.g, in class!). Students whom typeset their answers (with, e.g., LaTeX) or present an excellent layout will be awarded a 10% bonus, if all layout standards have been met. Late returns are penalized at a rate of 10% per day.

Exercise IV.1 Photon diffusion

In class we derived the photon escape time assuming that the step size in the random walk has the same length, equal to the mean free path (ℓ). Let's instead assume a distribution:

$$P(s) = \ell^{-1} e^{-s/\ell} \tag{1}$$

with $P(s)ds$ the probability that the step size s falls between $s, s + ds$.

(a) Verify that the mean free path of the distribution in Equation (1) is ℓ.

(b) How does the distribution change the escape time (Longer, shorter, the same)? Motivate your answer.

(c) We also assumed that ℓ is constant throughout the star, i.e., we assumed the average density to calculate ℓ. More realistically, there is a density gradient. How would this affect the escape time (longer, shorter, the same; and motivate).

Exercise IV.2 Ledoux stability criterion

In class, we derived the condition for convection:

$$\frac{d \log \rho}{d \log P} < \frac{1}{\gamma_{\mathrm{ad}}} \tag{2}$$

where $\gamma_{\mathrm{ad}} = (\partial \log P / \partial \log \rho)_s$ is the adiabatic index (evaluated at constant entropy s). Second, we introduced the general equation of state:

$$\frac{dP}{P} = \chi_\rho \frac{d\rho}{\rho} + \chi_T \frac{dT}{T} + \chi_\mu \frac{d\mu}{\mu}. \tag{3}$$

where P is pressure, T temperature, and μ the molecular weight. The χ_i are numerical prefactors.

(a) Show that Equation (3) can equivalently be written as:

$$1 = \chi_\rho \frac{d \log \rho}{d \log P} + \chi_T \nabla + \chi_\mu \nabla_\mu \tag{4}$$

where we introduced the logarithmic temperature gradient ∇ (see lecture notes) as well as ∇_μ. Give the expression for ∇_μ.

(b) Evaluate Equation (3) at constant entropy. In that case, the logarithmic temperature gradient $\nabla = \nabla_{\rm ad}$ per definition. Give an expression for $\nabla_{\rm ad}$ in terms of $\gamma_{\rm ad}$. Hint: for an adiabatic gas, the composition is the same.

(c) What is $\nabla_{\rm ad}$ for an ideal mono-atomic gas?

(d) Go back to the instability criterion, Equation (2). Use Equation (4) to find the condition for convection:

$$\nabla > \nabla_{\rm ad} - \frac{\chi_\mu}{\chi_T} \nabla_\mu \tag{5}$$

Exercise IV.3 Structure of the Sun

Download the file `solar_standard_model.txt`, which contains a table on the physical and chemical properties of the Sun. It contains columns for fractional mass, fractional radius, temperature (Kelvin), density, pressure (all cgs units), etc. You can use your favorite programming tool to process and plot these data (I recommend `python` and `matplotlib`).

(a) Using the ideal gas law, compute the mean molecular mass μ as function of radius. Plot μ as function of fractional radius.

(b) Crudely, according to $\mu(r)$ the Sun can be divided into three regions: (a) the central 20%, (b) the interior 20%–95%, and (c) the outer 5%. Explain the behavior of $\mu(r)$ in each of these three regions.

(c) Now compute (numerically!) the logarithmic temperature gradient

$$\nabla \equiv \frac{\mathrm{d} \log T}{\mathrm{d} \log P} \tag{6}$$

and also plot this quantity as function of (fractional) radius. (Hint: you should try to avoid numerical errors or at least identify them!)

(d) Which region of the Sun is convective?

Exercise IV.4 Mass-radius relationships for stars

In class we derived the following homology relationships:

$$\rho \propto M R^{-3}; \quad T \propto M R^{-1} \mu; \quad L \propto M^3 \mu^4 \tag{7}$$

(a) Now apply homology to the 4$^{\rm th}$ stellar structure equation, assuming that the nuclear energy generation rate obeys:

$$\frac{dL}{dm} = \epsilon = \epsilon_0 \rho T^\nu \tag{8}$$

where ν is the power-law index for energy generation by nuclear fusion. You should give another expression for L ($L \propto \ldots$).

(b) Show that this results in the following expression for the radius

$$R \propto M^{\frac{\nu-1}{\nu+3}} \mu^{\frac{\nu-4}{\nu+3}} \tag{9}$$

(c) In previous exercises in this class you have derived:

- The central temperature of the Sun, $\approx 2 \times 10^7$ K.
- The required temperature for nuclear fusion, $T = 10^7$ K.
- $\nu = 4$ for the pp-chain (low mass stars)

What is the minimum mass stars should have to fuse hydrogen?

(d) Is this answer correct you think? Motivate your answer.

(e) In reality one of the "ingredients" used to derive the minimum stellar mass will break down for low-mass stars. Can you identify which?

Exercise IV.5 Critical core mass for planets

We will calculate the structure for the envelope of a protoplanet. Assume the following:

- the opacity κ and luminosity L are constant
- the gravitational mass interior to r, GM_r can be approximated by the *total* mass of the planet core *and* planet envelope, $GM_{\rm tot}$
- the envelope is radiatively supported, $\nabla = \nabla_{\rm rad}$
- the ideal equation of state

(a) Under these assumptions, show that the temperature transport equation

$$\frac{d\log T}{d\log P} = \nabla_{\rm rad} \tag{10}$$

gives the following relation between pressure and temperature:

$$P = \frac{GM_{\rm tot}T^4}{4W} \tag{11}$$

where $W = 3\kappa L/64\pi\sigma_{\rm sb}$. Note that P is not anchored at the surface (we have ignored a boundary condition). This solution is applicable to the interior.

(b) Using the ideal gas law and hydrostatic balance, solve for the temperature as function of radius, $T(r)$. Again do not bother about applying a boundary condition.

(c) Similarly, write down the expression for the density profile $\rho(r)$ and integrate this density profile to find the mass of the envelope. Answer:

$$M_{\rm env} = \frac{4\pi}{W}\left(\frac{GM\mu m_{\rm u}}{4k}\right)^4 \Lambda \tag{12}$$

where $\Lambda = \log(r_{\rm out}/r_c)$ and $r_{\rm out}$ and $r_{\rm in}$ are the inner and outer radius of the envelope, respectively. By virtue of the logarithmic dependence, Λ can be regarded as constant.

(d) For $L = 10^{-8}$, 10^{-7} and $10^{-6}\,L_\odot$, plot the core mass $M_{\rm tot} - M_{\rm env}$ (y-axis) vs. $M_{\rm tot}$ (x-axis) in units of Earth masses (three curves in one plot). Take $\Lambda = 7$, $\kappa = 1\,{\rm cm^2\,g^{-1}}$ and assume a composition typical for a mixture of molecular hydrogen and helium. Present the plot in an intelligent way (see next question) such that the main features show up.

(e) Your plot should feature that the curves peak at some point. This maximum corresponds to the *critical core mass*. Explain why (Why are there no solutions for core masses beyond the critical point?) What happens for higher core masses?

(f) Explain the behavior of the three curves (the L dependence). Why does a higher L results in higher critical core masses?