Stars and Planets Problem Set6

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Exercise VI.1 Synodical timescale

(a)

Assuming $P_1 < P_2$, the synodical timescale satisfies:

$$(\omega_1 - \omega_2)t_{\rm syn} = 2\pi$$
$$(2\pi/P_1 - 2\pi/P_2)t_{\rm syn} = 2\pi$$

Therefore:

$$t_{\rm syn} = \frac{1}{1/P_1 - 1/P_2} = \frac{P_1 P_2}{P_2 - P_1}$$

(b)

According to the third Kelpler law:

$$\frac{a_1^3}{P_1^2} = \frac{a_2^3}{P_2^2} = \frac{Gm}{4\pi^2}$$

Expand P_1/P_2 in terms of $b/a_2 << 1$ ($b = a_2 - a_1, b/a_1 << 1$):

$$\frac{P_1}{P_2} = (\frac{a_1}{a_2})^{3/2} = (1 - \frac{b}{a_2})^{3/2} = 1 - \frac{3}{2}\frac{b}{a_2}$$

Plug into $t_{\rm syn}$ and we have:

$$t_{\rm syn} = P_1 \frac{1}{1 - P_1/P_2} = \frac{2}{3}\frac{a_2}{b}P_1$$

Exercise VI.2 Epicycle approximation

(a)

Assuming that $e << 1$, from the Kelper equation $M = E - e\sin E$ we can have:

$$\begin{aligned}\sin E &= \sin(M + e\sin E) = \sin M\cos(e\sin E) + \cos M\sin(e\sin E)\\ &= \sin M(1 + \frac{1}{2}(e\sin E)^2 + \cdots) + \cos M(e\sin E + \cdots)\\ &= \sin M + \mathcal{O}(e)\end{aligned}$$

and:

$$\begin{aligned}\cos E &= \cos(M + e\sin E) = \cos M\cos(e\sin E) - \sin M\sin(e\sin E)\\ &= \cos M(1 + \frac{1}{2}(e\sin E)^2 + \cdots) - \sin M(e\sin E + \cdots)\\ &= \cos M - e\sin M\sin E + \mathcal{O}(e^2) = \cos M - e\sin^2 M + \mathcal{O}(e^2)\\ &= \cos M + \mathcal{O}(e)\end{aligned}$$

Firstly consider r:

$$\begin{aligned}\cos E &= e + \frac{r}{a}\cos\nu = \frac{e + \cos\nu}{1 + e\cos\nu}\\ \cos\nu &= \frac{\cos E - e}{1 - e\cos E}\\ r &= \frac{a(\cos E - e)}{\cos\nu} = a(1 - e\cos E)\end{aligned}$$

Plug in $\cos E = \cos M + \mathcal{O}(e)$ and we have:

$$r \approx a(1 - e\cos M) + \mathcal{O}(e^2)$$

Secondly consider ν:

$$\cos\nu = \frac{\cos E - e}{1 - e\cos E} = (\cos E - e)(1 + e\cos E) = \cos E - e\sin^2 E + \mathcal{O}(e^2)$$

Plug in $\sin E = \sin M + \mathcal{O}(e)$ and $\cos E = \cos M - e\sin^2 M + \mathcal{O}(e^2)$, and we have:

$$\cos\nu = \cos M - 2e\sin^2 M + \mathcal{O}(e^2)$$

If we assume that $\nu \approx M + 2e\sin M + \mathcal{O}(e^2)$, we can have:

$$\begin{aligned}\cos\nu &= \cos(M + 2e\sin M + \mathcal{O}(e^2))\\ &= \cos M\cos(2e\sin M) - \sin M\sin(2e\sin M) + \mathcal{O}(e^2)\\ &= \cos M - 2e\sin^2 M + \mathcal{O}(e^2)\end{aligned}$$

Therefore, we can conclude that $\nu \approx M + 2e\sin M + \mathcal{O}(e^2)$.

(b)

In the polar coordinate, the acceleration is:

$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

We consider the $\ddot{r}$:

$$\ddot{r} = \ddot{\boldsymbol{r}} \cdot \hat{r} + r\dot{\theta}^2$$

We have:

$$\begin{aligned}
\ddot{\boldsymbol{r}} \cdot \hat{r} &= -\nabla\phi \cdot \hat{r} = -\frac{\partial \phi}{\partial r} = -Anr^{n-1} \\
r\dot{\theta}^2 &= (r^2\dot{\theta})^2/r^3 = l_z^2/r^3 \\
\ddot{r} &= -Anr^{n-1} + l_z^2/r^3 = -\frac{\partial}{\partial r}(Ar^n + \frac{l_z^2}{2r^2}) = -\frac{\partial \phi_{\text{eff}}}{\partial r}
\end{aligned}$$

where $\phi_{\text{eff}} = \phi(r) + \frac{l_z^2}{2r^2}$ is the effective potential.

(c)

$$\begin{aligned}
\phi'_{\text{eff}}(r_o) &= Anr_o^{n-1} - \frac{l_z^2}{2}\frac{2}{r_o^3} = 0 \\
r_o &= (\frac{l_z^2}{An})^{\frac{1}{n+2}}
\end{aligned}$$

(d)

Expanding the potential around $r = r_o$:

$$\phi_{\text{eff}} = \phi_{\text{eff}}(r_o) + \phi'_{\text{eff}}(r_o)x + \frac{1}{2}\phi''_{\text{eff}}(r_o)x^2$$

where $x = r - r_o$ and $\phi'_{\text{eff}}(r_o) = 0$. Therefore the equation of motion (3) becomes:

$$\ddot{x} = -x\phi''_{\text{eff}}(r_o) = -x(An(n-1)r_o^{n-2} + 3l_z^2 r_o^{-4}) = -(n+2)l_z^2 r_o^{-4}x$$

Compared to $\ddot{x} = -\kappa^2 x$, we have:

$$\kappa = \frac{\sqrt{n+2}l_z}{r_o^2} = \sqrt{n+2}(\frac{l_z^{\frac{2-n}{2}}}{An})^{-\frac{2}{n+2}}$$

In a Keplerian potential (n=-1):

$$\kappa = A^2/l_z^3 = \frac{(GM)^2}{(\sqrt{GMa})^3} = \sqrt{\frac{GM}{a^3}} = \Omega$$

where Ω is the keplerian orbital frequency.

(e)

For $n < -2$ the circular orbit solution becomes unstable.

If $n < -2$ the equation of motion will become:

$$\ddot{x} = -(n+2)l_z^2 r_o^{-4} x = \kappa^2 x$$
$$\kappa > 0$$

The general solution for this differential equation is either $x(t) = x_0 e^{kt}$ or $x(t) = x_0 e^{-kt}$, which indicates that the second object will be scattered to infinity or collide into the primary object.

Exercise VI.3 The Trojans

(a)

From the law of cos, we have:

$$r_1^2 = m^2 + 2mr\cos\theta + r^2$$
$$r_2^2 = (1-m)^2 - 2(1-m)r\cos\theta + r^2$$

Because $m << 1$, we can expand r_1^{-1} as:

$$\begin{aligned} r_1^{-1} &= (m^2 + 2mr\cos\theta + r^2)^{-1/2} \\ &\approx (1 + 2\Delta + \Delta^2 + 2m\cos\theta)^{-1/2} \\ &\approx 1 - 1/2(2\Delta + \Delta^2 + 2m\cos\theta) + 3/8(2\Delta + \Delta^2 + 2m\cos\theta)^2 \\ &\approx 1 - \Delta + \Delta^2 - m\cos\theta \end{aligned}$$

We can expand r_2^{-1} as:

$$\begin{aligned} r_2^{-1} &= ((1-m)^2 - 2(1-m)r\cos\theta + r^2)^{-1/2} \\ &\approx (1 - 2\cos\theta + 1)^{-1/2} \\ &= \frac{1}{\sqrt{2(1-\cos\theta)}} \end{aligned}$$

We can expand r^2 as:

$$r^2 = (1+\Delta)^2 = 1 + 2\Delta + \Delta^2$$

Plug the three above relations into the effective potential:

$$\phi_{\text{eff}} = -\frac{1-m}{r_1} - \frac{m}{r_2} - \frac{1}{2}r^2$$
$$\phi_{\text{eff}} = m(\cos\theta - \frac{1}{\sqrt{2(1-\cos\theta)}}) - \frac{3}{2}\Delta^2 + m - 3/2$$

Ignore the constants in the potential and we get:

$$\phi_{\text{eff}} = m(\cos\theta - \frac{1}{\sqrt{2(1-\cos\theta)}}) - \frac{3}{2}\Delta^2$$

(b)

The vector form of the equation of motion is:

$$\ddot{\boldsymbol{r}} + 2(\boldsymbol{\omega} \times \dot{\boldsymbol{r}}) = -\nabla\phi_{\text{eff}}$$

The radial component of the equation of motion is:

$$\ddot{r} - r\dot{\theta}^2 - 2r\dot{\theta} = -\frac{\partial\phi_{\text{eff}}}{\partial r}$$

Plug in the expression of the effective potential and we have:

$$\ddot{\Delta} - (1+\Delta)\dot{\theta}^2 - 2(1+\Delta)\dot{\theta} = 3\Delta$$

(c)

Assume $\Delta << 1$ and we get:

$$\ddot{\Delta} - \dot{\theta}^2 - 2\dot{\theta} = 3\Delta$$

And $\dot{\theta}^2$ must be the higher order small value compared to $\dot{\theta}$. Therefore, the above equation can be reduced to:

$$\ddot{\Delta} - 2\dot{\theta} = 3\Delta$$

If the objects are confined in a small range of radius, then the motion along the radial direction can not be too large, so $\ddot{\Delta}$ is also a higher order small value. Equation (8) can be reduced to:

$$2\dot{\theta} + 3\Delta = 0$$

(d)

The azimuthal component of the equation of motion is:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r} = -\frac{1}{r}\frac{\partial\phi_{\text{eff}}}{\partial\theta}$$

Plug in $r = 1 + \Delta$ and we have:

$$(1+\Delta)\ddot{\theta} + 2\dot{\Delta}\dot{\theta} + 2\dot{\Delta} = -\frac{1}{1+\Delta}\frac{\partial\phi_{\text{eff}}}{\partial\theta}$$

Ignore the higher order small value:

$$\ddot{\theta} + 2\dot{\Delta} = -\frac{\partial\phi_{\text{eff}}}{\partial\theta}$$

(e)

The effective potential can be rewritten as:

$$\phi_{\text{eff}} = -\frac{m}{2}(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2}) - \frac{3}{2}\Delta^2 + m$$

From the radial component of the equation of motion we have $\Delta = -2/3\dot{\theta}$. Plug the above two expressions into the azimuthal component and we can have:

$$-\frac{1}{3}\ddot{\theta} = \frac{m}{2}\frac{\partial}{\partial\theta}(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2}) + 3\Delta\frac{\partial\Delta}{\partial\theta}$$

Ignore the higher order small value $3\Delta\frac{\partial\Delta}{\partial\theta}$ and times $\dot{\theta}$ on both sides:

$$\begin{aligned} -\dot{\theta}\ddot{\theta} &= \frac{3}{2}m\frac{\partial}{\partial\theta}(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2})\dot{\theta} \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\dot{\theta}^2 + \frac{3}{2}m(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2})) \end{aligned}$$

The integral of motion under this approximation gives a conserved quantity I:

$$I = \frac{1}{2}\dot{\theta}^2 + \frac{3}{2}m(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2})$$

(f)

The potential component is:

$$U = \frac{3}{2}m(4\sin^2\frac{\theta}{2} + \frac{1}{\sin\theta/2})$$

Take the first derivative of the potential and we can get the Lagrange points:

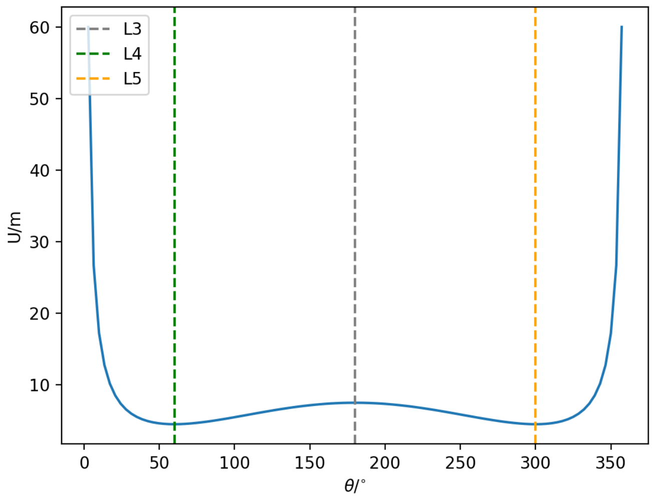
$$\begin{aligned} U'/m &= 3/2(4\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{1}{2}\frac{\cos\theta/2}{\sin^2\theta/2}) \\ \cos\frac{\theta}{2} &= 0 \text{ or } \sin\frac{\theta}{2} = 1/2 \\ \theta &= \pi/3, \pi/2 \text{ or } 5\pi/3 \end{aligned}$$

The potential and the Lagrange points are shown as below.

(g)

The widest possible Trojan orbit extend from L4 to near L3. By solving the inequality $U < U(L_3)$ we can have:

$$\theta \in [2\arcsin\frac{\sqrt{2}-1}{2}, \pi]$$



L4 is a local minimum so the asteroids can stay motionless around L4 $\theta(\dot{L}_4) = 0$. From the conserved quantity I we can have:

$$\frac{1}{2}\theta(\dot{L}_3)^2 + U(\theta(L_3)) = \frac{1}{2}\theta(\dot{L}_4)^2 + U(\theta(L_4))$$

$$\theta(\dot{L}_3) = \sqrt{2(U(\theta(L_4)) - U(\theta(L_3)))} = \sqrt{3m}$$

Terefore:

$$2\dot{\theta} + 3\Delta = 0$$

$$\Delta \in [-2\sqrt{\frac{m}{3}}, 2\frac{m}{3}]$$

the total radial width of these Trojans is:

$$b - a = 4\sqrt{\frac{m}{3}} = 4 \times \sqrt{\frac{9.5 \times 10^{-4}}{3}} = 0.37 \text{ au}$$

(h)

Expand the potential to the second order at L4 $\theta = \pi/3$:

$$U''/m = 3/2(2\cos\theta + \frac{1}{2}\sin^{-3}\frac{\theta}{2}\cos^2\frac{\theta}{2} + \frac{1}{4}\sin^{-1}\frac{\theta}{2})$$

$$\omega_{\text{lib}} = \sqrt{k/m} = \sqrt{U''/m} = \sqrt{4.5} = \frac{3\sqrt{3}}{2}\,\omega$$

$$t_{\text{lib}} = \frac{2\pi}{\omega_{\text{lib}}} = \frac{1}{2.12}\frac{2\pi}{\omega} = 0.47t$$

The orbital period for Jupiter is $t = 12$ years, so the orbital period for Jupiter's Trojans around L4 is $t_{\text{lib}} = 5.6$ years. not correct

Exercise VI.4 Tides

(a)

For the Earth-Moon system, the value for n (the mean motion) is $n = \frac{2\pi}{28\text{days}} = 0.22\text{day}^{-1}$.

(b)

The spin-down timescale is:

$$t_{\text{de- spin, p}}^{-1} = \frac{\dot{\Omega}_p}{\Omega_p} = -\frac{\Gamma}{\Omega_p I_p} = -\frac{3k_{2p}}{2QC_I}\frac{m_s^2}{(m_s+m_p)\,m_p}\left(\frac{R_p}{d}\right)^3\frac{n}{\Omega_p}n$$

For the Moon to spin-down the Earth, the love number $k_{2p} =$, the Quality factor $Q =$, the inertia factor $C_I =$, $m_s = 7.3477\times10^{25}$ g, $m_p = 5.974\times10^{27}$ g, $R_p = 6.378\times10^8$ cm, $d = 384,401\text{km}$, $n = \frac{2\pi}{28\text{days}}$, and $\Omega_P = \frac{2\pi}{1\text{ day}}$. Therefore the timescale for the Moon to spin down the Earth is 4.4 billion years .

For the Sun to spin-down the Earth, the love number $k_{2p} = 0.3$, the Quality factor $Q = 12$, the inertia factor $C_I = 0.33$, $m_s = 1.99\times10^{33}$ g, $m_p = 5.974\times10^{27}$ g, $R_p = 6.378\times10^8$ cm, $d = 1\ au$, $n = \frac{2\pi}{365\text{days}}$, and $\Omega_P = \frac{2\pi}{1\text{ day}}$. Therefore the timescale for the Sun to spin down the Earth is 19.9 billion years.

(c)

The time taken for the Moon to "crash" into the Earth is:

$$t_{\text{orbit}}^{-1} = \frac{9k}{2Q}\frac{m_s}{m_p}(\frac{R_p}{d})^5 n$$

Plug in the values from (b) and we can get $t_{\text{orbit}} = 7$ Gyr.

(d)

The collision velocity would have been similar to their mutual surface escape velocity $v_{\text{esc}} = \sqrt{2G(m_1+m_2)/(R_1+R_2)}$. The typical physical properties that are included in the tidal effects are just $m_{1,2}$ and $R_{1,2}$, so from analysis of the dimension, the collision velocity can only be in the form of $\sim \sqrt{G(m_1+m_2)/(R_1+R_2)}$, which can only be different from the mutual surface escape velocity of a constant (that is, they are of the same magnitude).

Exercise VI.5 Hot Jupiter migration by tides

(a)

The total energy of the original orbit is:

$$E_0 = -\frac{Gm}{2a_0}$$

where the m is the total mass of the star and the hot jupiter. The kinetic energy of the original orbit is:

$$K_0 = \frac{Gm}{2a_0}$$

The gravitational potential energy of the orbit is:

$$U_0 = -\frac{Gm}{a_0}$$

When the magnitude of the orbit velocity is suddenly changed by a factor of f, the kinetic energy changes to:

$$K_1 = f^2 K_0 = \frac{f^2}{2}\frac{Gm}{a_0}$$

The gravitational energy remains the same:

$$U_1 = U_0 = -\frac{Gm}{a_0}$$

Therefore, the total energy changes to:

$$E_1 = U_1 + K_1 = (\frac{f^2}{2} - 1)\frac{Gm}{a_0}$$

Compared to the energy expression of the Kelperian orbit $E_1 = -\frac{Gm}{2a_1}$ we can get:

$$a_1 = \frac{a_0}{2 - f^2}$$

When the magnitude of the orbit velocity is suddenly changed by a factor of f, the total angular momentum changes to:

$$l_1 = f l_0 = f\sqrt{Gma_0(1 - e_0^2)}$$

Compared to the angular momentum expression of the Kelperian orbit $l_1 = \sqrt{Gma_1(1 - e_1^2)}$ we can get:

$$e_1 = 1 - f^2$$

The pericenter r_{p1} is:

$$r_{p1} = a_1(1 - e_1) = \frac{f^2}{2 - f^2}a_0$$

(b)

During this tidal dissipation step, the hot Jupiter's angular momentum is roughly conserved as it circularizes to a final, close-in semimajor axis $e_2 = 0$:

$$l_1 = \sqrt{Gma_1(1 - e_1^2)} = l_2 = \sqrt{Gma_2}$$
$$a_2 = a_1(1 - e_1^2) = a_1 f^2(2 - f^2) = a_0 f^2$$

If $a_0 = 5$ au and $a_2 = 0.05$ au, then $f = 0.1$.

(c)

I don't think the existence of these planets can be explained by this mechanism.

A planet's initial periapse is approximately half of their final orbit radius if the palnet is perturbed onto a highly eccentric orbit $e >> 1$:

$$a_2 = a_1(1 - e_1^2) \approx 2a_1(1 - e_1) = 2r_{p1}$$

The periapse must be larger than 1 Roche radius of the planet to survive the tidal disruption of its host star. Therefore, we expect to see surviving planets beyond 2 Roche radius if the planet is formed through the high eccentricity tidal migration.

Exercise VI.6 Geometry of resonances

(a)

Interactions after the conjunction are stronger, because the two planets are closer after the conjunction.

(b)

This results in a net negative torque on planet 1, because planet 1 moves ahead of planet 2 and is dragged down by planet 2.

(c)

It causes the next conjunction point to be closer to pericenter, because the inner planet 1 loses angular momentum and the outer planet 2 gains angular momentum.

(d)

Resonances near pericenter are therefore stable.

Exercise VI.7 Planet trapping

(a)

Combine the steady-state solutions of the resonance forcing equation for semi-major axis and eccentricity:

$$\dot{a} = 2(j+\delta_1)G_e^j q_1 n_1 a e \sin\phi_{\text{eq}} - \frac{a}{t_a} = 0$$
$$\dot{e} = G_e^j q_1 n_1 \sin\phi_{\text{eq}} - \frac{e}{t_e} = 0$$

Solve the above two equations and we get:

$$\sin\phi_{\text{eq}} = \frac{1}{G_e^j q_1 n_1}\frac{1}{\sqrt{2(j+\delta_1)t_e t_a}}$$
$$e_{\text{eq}} = \frac{\sqrt{t_e}}{\sqrt{2(j+\delta_1)t_a}}$$

Since G_e^j are negative for an internal perturber, $\sin\phi_{\text{eq}}$ is negative according to the equation of e. And for an inner circular orbit and an outer eccentric orbit, the stable resonance is at apocenter, so $\cos\phi_{\text{eq}}$ should also be negative. The point (x, y) is in the third quadrant.

(b)

For the inner perturber, when in a 3:2 resonance $j = 2$, $G_e^j = -1.66$ and $\delta_1 = 1$. Assuming the inner perturber is an 10 Earth-mass planet orbiting a solar-mass star at 1 au, $q_1 = 10m_\oplus/m_\odot$ and $n_1 = 2\pi/1\text{yr}$. The equilibrium eccentricity is $e_{\text{eq}} = 0.004$ and the equilibrium resonance angle is $\phi_{\text{eq}} = -187.5°$.

(c)

The equation for $\dot{\phi_{\text{res}}}$ is:

$$\dot{\phi_{\text{res}}} = -\Delta n_{\text{res}} + G_e^j \frac{q_1 n_1 \cos\phi_{\text{res}}}{e}$$
$$\dot{\phi_{\text{res}}} = -jn_2\Delta + G_e^j \frac{q_1 n_1 \cos\phi_{\text{res}}}{e}$$

where $\Delta n_{\rm res} = jn_1 - (j+1)n_2$ and $\Delta \equiv \frac{P_2}{P_1} - \frac{j+1}{j}$.

Set $\dot{\phi_{\rm res}} = 0$ and assume that $n_2 = 2/3n_1$:

$$\Delta = G_e^j \frac{q_1 n_1 \cos\phi_{\rm res}}{ejn_2} = 0.01$$

(d)

what???