Problem Set II – Astrophysics of Stellar Matter

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Hand in HW-problems before the announced date

Hand in before the announced deadline, either before the start of class or by uploading your solutions to the WebLearning system (pdf format only!). Write down your answers in English and state any additional assumptions that you have made, e.g., in case when you think that the phrasing is unclear. Collaboration is allowed, but you should hand in your own solution and be able to reproduce your answer independently if asked to (e.g, in class!). Students whom typeset their answers (with, e.g., LaTeX) or present an excellent layout will be awarded a 10% bonus, if all layout standards have been met. Late returns are penalized at a rate of 10% per day.

Exercise I.1 Radiation Pressure

donnot use stelfen-boltzman constant

(a) Following the lecture, derive an expression for the radiation pressure $P_{\rm rad}$ using dimensional analysis only (and not something else). Motivate your choice for the physical quantities and fundamental physical constants you use and do not use the Stefan-Boltzmann constant as it is not a fundamental physical constant.

(b) Now derive the precise expression. First, convert the expression of the energy density u_ν (see slides in Module 1) into the distribution function for the (photon) momenta $f(p)$. Then use the pressure integral to obtain the expression for $P_{\rm rad}$. Finally, relate $P_{\rm rad}$ to the radiation energy density $u_{\rm rad}$. (Hint: you may encounter this integral:

$$\int_0^\infty \frac{x^3}{(e^{ax}-1)}dx = \frac{\pi^4}{15a^4}\Bigg) \tag{1}$$

Exercise I.2 Physical structure of protoplanetary disks

Consider a cylindrical coordinate system (r, ϕ, z) with the midplane of the disk at $z = 0$. Ignore the (self-)gravity of the disk.

(a) The gas in the protoplanetary disk is in *molecular* form. Assuming standard astrophysical "cosmic" abundances for the elements, what is the mean molecular weight μ?

(b) Assuming that $z \ll r$ show that for a point z above the midplane the *vertical component* of the stellar gravity reads $g_{z,\star} \simeq -\Omega_K^2 z$ where Ω_K is the Keplerian frequency corresponding to radius r.

(c) Assume an isothermal equation of state (constant temperature) and the ideal gas law. Apply hydrostatic balance to find $\rho(z)$. Your solution should feature a constant of integration C.

(d) What is the standard deviation σ of the density distribution? This is more often referred to as the scaleheight H.

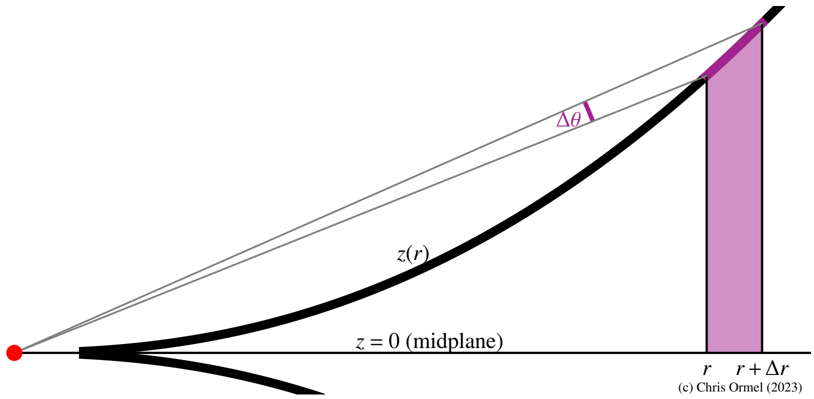

Figure 1: Geometry for a flared disk, illustrating the relationship between $\Delta\theta$ and Δr. The vertical dimension and the amount of flaring have been greatly exaggerated.

(e) Let the *surface density* Σ be defined:

$$\Sigma = \int_{-\infty}^{\infty} \rho(z)\,\mathrm{d}z \tag{2}$$

write down $\rho(z)$ in terms of Σ and H (eliminate the integration constant C).

(f) Assume that the disk is *flared*, H/r increases with cylindrical radius r, allowing the disk to intercept stellar photons. Assume that interception occurs at a height $z = \beta H$ with β constant. The column of material within $r, r + \Delta r$ is heated by stellar radiation intercepted at angle $\Delta\theta$ (see Figure 1). Show that

$$\frac{d\theta}{dr} = \frac{d}{dr}\left(\frac{z}{r}\right) \tag{3}$$

(where we may have assumed that $z/r \ll 1$.) What is the fraction of the stellar luminosity $L_\star$ that is being intercepted by the disk at radius interval $[r, r + \Delta r]$?

(g) Assume that half of the intercepted radiation can be used to heat the disk column in the vertical direction (the other half being lost to empty space). Then assume that this radiation is in turn intercepted at the disk midplane ($z = 0$), from which it re-radiates freely back into space at (effective) temperature T. By balancing heating and cooling, find an expression for the disk temperature as function of r, $T(r)$. If you feel you need to make additional assumption, please state them.

(h) Evaluate this expression for a $1\,L_\odot$, $1\,M_\odot$ star at a distance of 1 au. Take $\beta = 3$. If you failed to derive the expression in the previous questions, or you don't believe your answer, then please tell what $T(r)$ would be *without* a disk.

(i) In the above we assumed that the radiation intercepted at height z can freely reach the midplane. How would your answer change if the photons instead are absorbed along the way. Would the midplane temperature become:

(a) higher
(b) lower
(c) stay the same

Explain your answer.

Exercise I.3 A solar-mass White Dwarf

Consider a White Dwarf of mass $1\, M_\odot$.

(a) Calculate the radius using the expression for a degenerate non-relativistic electron gas. Express you answer in Earth radii.

(b) Calculate the density at the center of the White Dwarf.

(c) Are the electrons at the center approaching relativistic motions? Explain your answer.

Exercise I.4 Mass-radius relationship for solid planets

We will apply the Lane-Emden equation to "rocky" planets. As we are interested in analytical solutions, we take $n = 1$. Let the outer outer boundary conditions, furthermore, be given by

$$\rho(r = R) = \rho_0 \tag{4}$$

(this is different from the book!) reflecting that the density of solids at zero pressure is finite.

(a) Show that the planet mass can be written as

$$M = 4\pi\lambda_1^3\rho_0 \frac{\xi_R}{\sin\xi_R} (\sin\xi_R - \xi_R \cos\xi_R) \tag{5}$$

where $\xi_R = R/\lambda_1$. (You don't need to re-derive the Lane-Emden equation. Just take the $n = 1$ solution for the density and integrate for the mass.)

(b) Show that for $\xi_R \ll 1$, this equation reduces to the familiar expression for the mass of a homogeneous sphere.

(c) Assume Equation (5) holds for the Earth and that the uncompressed value for the density is $\rho_0 = 3\,\mathrm{g\,cm^{-3}}$. What is the value of λ_1 (express your answer in terms of Earth radii and with an accuracy of two digits. Note that you need to find the root numerically)?

(d) Under this (rather unrealistic) scenario, what is the maximum radius rocky planets can obtain?

(e) Rocky exoplanets have been found with radii above $1.5\, R_\oplus$. Do you think that (in the case planets can be approximated as polytropes) the polytropic index n will be higher or lower than 1. Motivate your answer.

Exercise I.5 The Gamow peak

In class, we derived the following expression for the reaction rate

$$r_{12} = n_1 n_2 \sqrt{\frac{8}{\pi m_\mu}} (kT)^{-3/2} \int S(E) e^{f(E)} dE; \qquad f(E) = -\pi \left(\frac{E_*}{E}\right)^{1/2} - \frac{E}{kT}; \qquad E_* = \frac{8\pi^2 Z_1^2 Z_2^2 e^4 m_\mu}{h^2} \tag{6}$$

We will be investigating the temperature dependence of the reaction rate r_{12}.

(a) Show that $f(E)$ has a peak at an energy

$$E_0 = \left[\frac{1}{4}\pi^2 E_* (kT)^2\right]^{1/3} \tag{7}$$

To evaluate the integral r_{12}, assume that $S(E)$ is a slowly varying function with E, approximate $S(E) \approx S(E_0)$ and pull this factor outside the integral

$$r_{12} = ...S(E_0) \int \exp\left[f(E)\right] dE \tag{8}$$

Now write $f(E)$ as a Taylor-series around $E = E_0$

$$f(E) \approx f(E_0) + \frac{1}{2} f''(E_0)(E - E_0)^2 \tag{9}$$

(b) Why is there no term with $(E - E_0)$ in Equation (9)?

(c) Write down expression for the terms $A = f(E_0)$ and $B = \frac{1}{2} f''(E_0)$ in terms of E_* and T. Then perform the integration to solve for r_{12}. It is OK to Give your answer in terms of A and B (i.e., it's not necessary to insert these factors in your answer for r_{12}.)

(d) Using this answer, derive the following expresion for the temperature power-law index of the reaction rate (as in $r_{12} \propto T^{\nu}$):

$$\nu = -\frac{2}{3} + \left(\frac{\pi^2 E_*}{4kT}\right)^{1/3} \tag{10}$$

(Hint:

$$\nu = \frac{d \log r_{12}}{d \log T}\Bigg) \tag{11}$$

(e) Calculate ν for a temperature of $T = 10^7$ K for the

- pp-chain (consider ${}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{H} + e^+ + \nu_e$)
- CNO cycle (consider ${}^1_1\mathrm{H} + {}^{14}_7\mathrm{N} \rightarrow {}^{15}_8\mathrm{O} + \gamma$)