Problem Set III – Birth. I

(c) Chris Ormel

Hand in HW-problems before the announced date

Hand in before the announced deadline, either before the start of class or by uploading your solutions to the WebLearning system (pdf format only!). Write down your answers in English and state any additional assumptions that you have made, e.g., in case when you think that the phrasing is unclear. Collaboration is allowed, but you should hand in your own solution and be able to reproduce your answer independently if asked to (e.g, in class!). Students whom typeset their answers (with, e.g., LaTeX) or present an excellent layout will be awarded a 10% bonus, if all layout standards have been met. Late returns are penalized at a rate of 10% per day.

Exercise III.1 Energetics of collapsing clouds

The free-fall timescale is the time a particle at the edge of the cloud collapses to $r = 0$. It reads

$$t_{\mathrm{ff}} = \sqrt{\frac{3\pi}{32G\rho}} \tag{1}$$

for a homogeneous sphere of mass M and (initial) radius R. There are (at least) two ways to derive this expression. One is to solve the equation of motion

$$\ddot{r} = -\frac{GM}{r^2}. \tag{2}$$

You are welcome to find the solution $r(t)$ in this way (but you do not get any credits). Alternatively, you obtain Equation (1) by considering the solution to a Kepler orbit.

(a) Which are the eccentricity, semi-major axis and orbital period of a particle that starts from $r = R$ and collapses (along a line as we ignore rotation) towards the center? Then, obtain the free-fall time.

Consider the free-fall collapse of a homogeneous cloud. Assume constant temperature T such that the cloud is fragmenting, i.e., $M = M_J$ is decreasing with time.

(b) Show that the radius corresponding to the fragment mass M_J is

$$R_{\mathrm{J}} = \frac{G\mu m_{\mathrm{m}}}{5kT} M_J \tag{3}$$

(c) Argue that the fragment radiates away energy at rate

$$\frac{dE}{dt} \sim \frac{GM^2}{R t_{\mathrm{ff}}} \tag{4}$$

and work out this expression, eliminating ρ and R, for a fragment that has $M = M_J$ and $R = R_J$.

(d) What is the expression for the maximum rate at which objects can cool by black body radiation? Write down this expression in terms of T and M.

(e) Now plot the previous two expressions you have derived as function of mass M for temperatures of 100 and 1,000 K in a single plot (so four lines in total). Mass M should range between 10^{-3} and 100 solar mass. For the y-axis choose units that are "astrophysical".

(f) You will see (hopefully) that the lines will intersect at a certain mass. Why does fragmentation stop at this point?

Exercise III.2 The Messier 80 globular cluster

Messier 80 is a globular cluster in the constellation Scorpius. Its apparent magitude is $m_V = +7.87$. We will use the virial theorem to estimate the distance d to Messier 80. Its "core radius" is observed to be 8 arcsec.

(a) Given that the star Vega has magnitude $m_V = 0.026$, luminosity $40\,L_\odot$ and lies at a distance of 7.7 pc, give an expression for L as function of the (unknown) distance d to Messier 80. Write your expression as:

$$\frac{L_{\mathrm{M81}}}{L_\odot} = A_1 \left(\frac{d}{\mathrm{pc}}\right)^2 \tag{5}$$

and give a value of A_1. If you get stuck here, take $A_1 = 10^{-4}$.

Assume that the stellar density follows an $n = 5$ polytrope with scaling parameter $R_s = \sqrt{3}\lambda_5$. This is referred to as a Plummer model.

(b) The core radius R_c is defined as the distance from the center of the cluster where the surface brightness (or stellar surface density) is half of that at the center. Show that $R_c = 0.64 R_s$.

(c) Incidentally, do you know why $M/L \gg M_\odot/L_\odot$ for most globular clusters?

It can be shown that the gravitational energy of an $n = 5$ polytropic mass distribution is

$$W = -\frac{3\pi}{32}\frac{GM^2}{R_s} \tag{6}$$

(which is not hard to derive)

(d) The line-of-sight (1D) velocity dispersion of the stars in M80 is measured to be $\sigma = 10\,\mathrm{km\,s^{-1}}$. Also, assume that the luminosity-to-mass ratio of M80 is $0.2\,L_\odot/M_\odot$. Now apply the virial theorem together with the relations derived above to obtain the distance d to M80.

Exercise III.3 The initial mass function (IMF)

We consider the Chabrier IMF

$$\frac{dn}{dm} = \xi(m) \propto \begin{cases} m^{-1} \exp\left[-\dfrac{(\log_{10} m - \log_{10} 0.22)^2}{0.65}\right] & (m < 1) \\ m^{-2.35} \exp\left[-\dfrac{(-\log_{10} 0.22)^2}{0.65}\right] & (m \geq 1) \end{cases} \tag{7}$$

Here m is expressed in units of solar mass.

(a) Let Brown Dwarfs be stars of mass less than 0.075 $M_\odot$. What is the number fraction of Brown Dwarfs? And what is the *mass* fraction of Brown Dwarfs. It is most useful to solve for these numbers numerically.

Consider stars above solar mass and let's for simplicity omit the exponential dependence, such that $\xi(m) \propto m^{-2.35}$. For main-sequence stars, let their luminosity scale with mass as $L \propto m^\eta$ and let their lifetime be $t \propto m/L$.

(b) Suppose you are in the Gobi desert on a clear night and that you can see all stars out to a certain (apparent) magnitude. How does the distance d out to which you see stars scale with the stellar mass?

(c) Let $\eta = 3.5$. Are most stars that you see light or massive?