Stars and Planets Problem Set4

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Exercise IV.1 Photon diffusion

(a)

The mean free path of the distribution $\bar{s}$ is:

$$\bar{s} = \frac{\int_0^{+\infty} s l^{-1} e^{-s/l} \mathrm{d}s}{\int_0^{+\infty} l^{-1} e^{-s/l} \mathrm{d}s} = \frac{l \int_0^{+\infty} x e^{-x} \mathrm{d}x}{\int_0^{+\infty} e^{-x} \mathrm{d}x} = l \text{ (where } x = s/l)$$

(b)

The motion of the photon is:

$$\vec{S} = \vec{s_1} + \vec{s_2} + \vec{s_3} + \cdots$$

The average of the squared length:

$$< S^2 > = (\vec{s_1} + \vec{s_2} + \vec{s_3} + \cdots)(\vec{s_1} + \vec{s_2} + \vec{s_3} + \cdots) = \sum_{i=1}^{N} \vec{s_i} \cdot \vec{s_i} + \sum_{i=1}^{N} \sum_{i \neq j}^{N} \vec{s_i} \vec{s_j}$$

The second equals 0 because the symmetry of the random walk, so we have:

$$< S^2 > = R^2 = \sum_{i=1}^{N} \vec{s_i} \cdot \vec{s_i} = N < s^2 >$$

$$< s^2 > = \int_0^{+\infty} s^2 l^{-1} e^{-s/l} \mathrm{d}s = 2l^2$$

$$N = \frac{R^2}{2l^2}$$

Therefore, the escape time will be shorter:

$$t_{\text{escp}} = \frac{Nl}{c} = \frac{R^2}{2cl}$$

(c)

I think the escape time will become shorter. If there exists density gradient, diffusion will come into the picture. Diffusion will help photons to move from dense core to sparse surface, making the escape time shorter.

Exercise IV.2 Ledoux stability criterion

(a)

The expression for the two gradients are:

$$\nabla = \frac{\mathrm{d}\log T}{\mathrm{d}\log P}$$

$$\nabla_\mu = \frac{\mathrm{d}\log \mu}{\mathrm{d}\log P}$$

Divide both sides of Equation (3) by $\mathrm{d}P/p = \mathrm{d}\log P$:

$$1 = \chi_\rho \frac{\mathrm{d}\log \rho}{\mathrm{d}\log P} + \chi_T \frac{\mathrm{d}\log T}{\mathrm{d}\log P} + \chi_\mu \frac{\mathrm{d}\log \mu}{\mathrm{d}\log P}$$

$$1 = \chi_\rho \frac{\mathrm{d}\log \rho}{\mathrm{d}\log P} + \chi_T \nabla + \chi_\mu \nabla_\mu$$

(b)

Since for an adiabatic gas the composition is the same, $\nabla_\mu = 0$. For constant entropy process, $\nabla = \nabla_{\mathrm{ad}}$ and $P \propto \rho^{\gamma_{\mathrm{ad}}}, \mathrm{d}\log P/\mathrm{d}\log \rho = \gamma_{\mathrm{ad}}$. Equation(3) becomes:

$$1 = \chi_\rho/\gamma_{\mathrm{ad}} + \chi_T \nabla_{\mathrm{ad}}$$

$$\nabla_{\mathrm{ad}} = (1 - \chi_\rho/\gamma_{\mathrm{ad}})/\chi_T$$

(c)

For an ideal mono-atomic gas, the equation of state is $P = \frac{\rho k T}{\mu m_u}$ and $\gamma_{\mathrm{ad}} = 5/3$. Besides, $\chi_\rho = 1$ and $\chi_T = 1$.

$$\nabla_{\mathrm{ad}} = (1 - \chi_\rho/\gamma_{\mathrm{ad}})/\chi_T = 2/5$$

(d)

From Equation(4) we can get:

$$\frac{\mathrm{d}\log \rho}{\mathrm{d}\log P} = (1 - \chi_T \nabla - \chi_\mu \nabla_\mu)/\chi_\rho$$

From section (b) we can get:

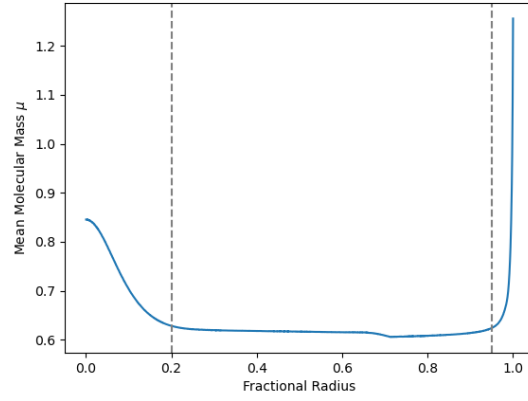$$\frac{1}{\gamma_{\mathrm{ad}}} = (1 - \chi_T \nabla_{\mathrm{ad}})/\chi_\rho$$

Th e instability criterion can lead to the condition for convection:

$$\frac{\mathrm{d}\log \rho}{\mathrm{d}\log P} < \frac{1}{\gamma_{\mathrm{ad}}}$$

$$(1 - \chi_T \nabla - \chi_\mu \nabla_\mu)/\chi_\rho < (1 - \chi_T \nabla_{\mathrm{ad}})/\chi_\rho$$

$$\nabla > \nabla_{\mathrm{ad}} - \frac{\chi_\mu}{\chi_T}\nabla_\mu$$

Exercise IV.3 Structure of the Sun

(a)

The relation between the mean molecular mass μ and the fractional radius is shown as below.

(b)

The two dashed lines in the above figure shows the 20% and 95% fractional radius. We all know that $\mu_H = 1/2$, $\mu_{He} = 4/3$ and $\mu_Z = 2$. In the central 20%, the core of the sun is mostly composed of Helium, so the mean molecular weight is between $1/2$ and $4/3$; the interior 20%-95% region is mostly composed of Hydrogen, so the mean molecular weight is near $1/2$; in the outer 5% region, metals have been brought into this shell due to convection, so the mean molecular weight increase sharply in the outer shell.

(c)

The numerical errors occur in regions near the center and the surface.
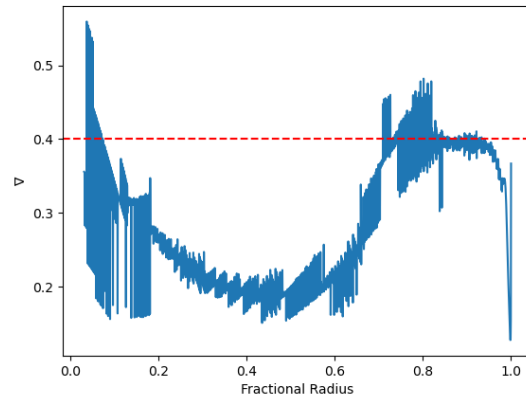
(d)

Assume the composition of the Sun is ideal mono-atomic gas. From Exercise2(c) we get that for an ideal mono-atomic gas $\nabla_{ad} = 2/5 = 0.4$. Therefore, we can roughly guess that the outer 30% region of the Sun is convective.

Exercise IV.4 Mass-radius relationships for stars

(a)

Time M to both sides of Equation(8) and use the homology:

$$\frac{\mathrm{d}L}{\mathrm{d}x} = \epsilon_0 \rho M T^\nu$$

$$\frac{\mathrm{d}L_1}{\mathrm{d}L_2} = \frac{\rho_1 M_1 T_1^\nu}{\rho_2 M_2 T_2^\nu}$$

$$L \propto M^{\nu+2} R^{-3-\nu} \mu^\nu$$

(b)

$$L \propto M^3 \nu^4$$

$$L \propto M^{\nu+2} R^{-3-\nu} \mu^\nu$$

$$M^{\nu+2} R^{-3-\nu} \mu^\nu \propto M^3 \nu^4$$

$$R \propto M^{\frac{\nu-1}{\nu+3}} \mu^{\frac{\nu-4}{3+\nu}}$$

(c)

From Equation(9) and the given equations we can have:

$$T \propto M R^{-1} \mu$$

$$R \propto M^{\frac{\nu-1}{\nu+3}} \mu^{\frac{\nu-4}{3+\nu}}$$

Therefore we can get:

$$T \propto M^{\frac{4}{3+\nu}} \mu^{\frac{7}{3+\nu}}$$

$$M \propto T \mu^{-7/4}$$

Assume the composition of the star are almost the same, so μ is a constant.

$$\frac{M_{\min}}{M_\odot} \propto \left(\frac{T_{\min}}{T_\odot}\right)^{\frac{3+\nu}{4}}$$

Plug in $\nu = 4$ and $T_\odot = 2 \times 10^7\ K, T_{\min} = 10^7\ K$, and $M_{\min} = 0.30 M_\odot$.

(d)

This answer is not correct. We have observed stars that burn hydrogen with masses smaller than $0.30 M_\odot$.

(e)

I think the relation $L \propto M^3 \mu^4$ will not hold for small mass stars. We derive this relation from radiative diffusion, but for small mass stars, convection is dominant, so the formula for radiative diffusion is not correct for small mass stars.

Exercise IV.5 Critical core mass for planets

(a)

For a radiatively supported envelope, assuming the opacity κ and luminosity L are constant:

$$\frac{L}{4\pi r^2} = F = -D_{phot}\nabla u_{rad} = -\frac{c}{3}\frac{1}{\kappa\rho}\frac{16\sigma}{c}4T^3\frac{dT}{dr}$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{64\pi r^2\sigma T^3}$$

From the hydrostatic balance, we also have:

$$\frac{dP}{dr} = -\frac{GM_{tot}}{r^2}\rho$$

Therefore, the radiation difussion gradient is:

$$\nabla_{rad} = \frac{d\log T}{d\log P} = \frac{P}{T}\frac{dT}{dr}\frac{dr}{dP} = \frac{3\kappa LP}{64\pi\sigma GM_{tot}T^4} = \frac{WP}{GM_{tot}T^4}$$

$$d(WP) = d(GM_{tot}T^4/4)$$

$$P = \frac{GM_{tot}T^4}{4W}$$

where $W = 3\kappa L/64\pi\sigma_{sb}$.

(b)

The ideal gas law and hydrostatic balance are:

$$P = \frac{\rho kT}{\mu m_u}$$

$$\frac{dP}{dr} = -\frac{GM_{tot}}{r^2}\rho$$

Plug in Equation(11) and we can get:

$$dT = -A\frac{dr}{r^2}$$

where $A = GM_{tot}\mu m_u/4k$. Integrate on both sides and we get:

$$T = A\frac{1}{r}$$

(c)

$$\rho = -\frac{r^2}{GM_{tot}}\frac{dP}{dT}\frac{dT}{dr} = \frac{A^4}{W}\frac{1}{r^3}$$

Integrate this density profile we can get:

$$M_{env} = \int_{r_{in}}^{r_{out}} \rho 4\pi r^2 dr = \frac{4\pi}{W}A^4\Lambda$$

where $\Lambda = \log(r_{out}/r_{in})$, $A = GM_{tot}\mu m_u/4k$.

(d)

(e)

(f)