Stars and Planets Problem Set1

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Exercise I.1 Radiation Pressure

(a)

The dimension of $P_{\rm rad}$ is $[E][L]^{-3}$. The physical quantities that are related to pressure from the derivation of state equations are:

$$kT \sim [E]$$
$$h \sim [E][T]$$

However, only with the two quantities above we cannot cancel out T and add in L. Given the pressure is produced by radiation, we can introduce the speed of light $c \sim [L][T]^{-1}$ to help us. After some calculations, we can get that the dimension of radiation pressure is:

$$P_{\rm rad} \sim \frac{(kT)^4}{h^3 c^3}$$

(b)

From the energy density u_ν, we can get the distribution function for the energy $f(\nu)$:

$$f(\nu) = u_\nu/(h\nu) = \frac{8\pi\nu^2}{c^3}\frac{1}{e^{h\nu/kT} - 1}$$

And the relation between the distribution function for the energy and the momenta is:

$$f(\nu)\mathrm{d}\nu = f(p)\mathrm{d}p$$
$$p = \frac{h}{\lambda} = \frac{h\nu}{c}$$
$$\mathrm{d}p = \frac{h}{c}\mathrm{d}\nu$$

Therefore, the distribution function for the momenta is:

$$f(p) = \frac{8\pi}{h^3}p^2\frac{1}{e^{cp/kT} - 1}$$

The radiation pressure is:

$$P_{\rm rad} = \int_0^{+\infty} cpf(p)\mathrm{d}p = \frac{8\pi^5}{45}\frac{(kT)^4}{h^3c^3}$$

The relation between radiation energy density and radiation pressure is:

$$P_{\rm rad} = \frac{u_{\rm rad}}{3}$$

Exercise I.2 Physical structure of protoplanetary disks

(a)

The standard astrophysical "cosmic" abundances for the elements gives:

$$X = 0.75,\ Y = 0.25$$

Since the hydrogen and helium gas are in molecular form, the relative mean molecular weight is:

$$\mu_{\mathrm{H_2}} = 2$$
$$\mu_{\mathrm{He}} = 4$$

The mean molecular weight μ of the gas is:

$$\frac{1}{\mu} = \frac{X}{\mu_{\mathrm{H_2}}} + \frac{Y}{\mu_{\mathrm{He}}}$$
$$\mu = 2.29$$

(b)

Assume the mass of the star is M, the gravitational constant is G and the positive direction of z axis is pointing upward. Write down Newton's second law in the vertical direction:

$$g_{\mathrm{z},\star} = -\frac{GM}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}} = -\frac{GM}{r^2}\frac{z}{(1+(\frac{z}{r})^2)^{\frac{3}{2}}}$$

For $z << r$:

$$\frac{z}{(1+(\frac{z}{r})^2)^{\frac{3}{2}}} = z(1-\frac{3}{2}(\frac{z}{r})^2) \sim z$$

Therefore:

$$g_{\mathrm{z},\star} = -\frac{GM}{r^2}z \approx -\Omega_{\mathrm{K}}^2 z$$

(c)

The equation of state for the gas in the disk is:

$$P = nkT = \frac{\rho}{\mu m_u}kT$$

Since the $g_{\mathrm{z},\star}$ has included the minus sign, the hydrostatic balance is:

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \rho g_{\mathrm{z},\star}$$

Combine the two equations and we get the ordinary differential equation for $\rho(z)$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\frac{\Omega_{\mathrm{K}}^2\mu m_u}{kT}z\rho$$

The solution is:

$$\rho(z) = Ce^{-\frac{\Omega_{\mathrm{K}}^2\mu m_u}{2kT}z^2}$$

(d)

The standard deviation σ or the scaleheight H is:

$$\sigma = H = \sqrt{\frac{kT}{\mu m_u \Omega_{\mathrm{K}}^2}}$$

And the $\rho(z)$ can be written as:

$$\rho(z) = Ce^{-\frac{z^2}{2H^2}}$$

(e)

The surface density is:

$$\begin{aligned}
\Sigma &= \int_{-\infty}^{+\infty} \rho(z)\mathrm{d}z = \int_{-\infty}^{+\infty} Ce^{-\frac{z^2}{2H^2}}\mathrm{d}z \\
\Sigma &= CH\sqrt{2\pi} \\
C &= \frac{\Sigma}{H\sqrt{2\pi}}
\end{aligned}$$

Therefore, $\rho(z)$ can be rewritten as:

$$\rho(z) = \frac{\Sigma}{H\sqrt{2\pi}} e^{-\frac{z^2}{2H^2}}$$

(f)

When $z << r$, use the small angle approximation:

$$\tan\theta = \frac{z}{r} \approx \theta$$

Since z only depends on r, take derivatives on both sides:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}(\frac{z}{r})$$

The fraction ϵ of the stellar luminosity $L_\star$ that is being intercepted by the disk at radius interval $[r, r+\Delta r]$ is:

$$\begin{aligned}
\epsilon &= \frac{1}{2\pi}\int \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \int_{\theta}^{\theta+\mathrm{d}\theta} \sin\theta \mathrm{d}\theta = \int_{r}^{r+\mathrm{d}r} \frac{z}{r}\frac{\mathrm{d}}{\mathrm{d}r}(\frac{z}{r})\mathrm{d}r \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}(\frac{z}{r})^2\Delta r = \frac{\beta^2}{2}\frac{\mathrm{d}}{\mathrm{d}r}(\frac{H}{r})^2\Delta r = \beta^2\frac{H}{r}(\frac{H}{r})'\Delta r
\end{aligned}$$