Problem Set III – Birth

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Hand in HW-problems before the announced date

Hand in before the announced deadline, either before the start of class or by uploading your solutions to the WebLearning system (pdf format only!). Write down your answers in English and state any additional assumptions that you have made, e.g., in case when you think that the phrasing is unclear. Collaboration is allowed, but you should hand in your own solution and be able to reproduce your answer independently if asked to (e.g, in class!). Students whom typeset their answers (with, e.g., LaTeX) or present an excellent layout will be awarded a 10% bonus, if all layout standards have been met. Late returns are penalized at a rate of 10% per day.

Exercise III.1 Energetics of collapsing clouds

The free-fall timescale is the time a particle at the edge of the cloud collapses to $r = 0$. It reads

$$t_{\rm ff} = \sqrt{\frac{3\pi}{32G\rho}} \tag{1}$$

for a homogeneous sphere of mass M and (initial) radius R. There are (at least) two ways to derive this expression. One is to solve the equation of motion

$$\ddot{r} = -\frac{GM}{r^2}. \tag{2}$$

You are welcome to find the solution $r(t)$ in this way (but you do not get any credits). Alternatively, you obtain Equation (1) by considering the solution to a Kepler orbit.

(a) Which are the eccentricity, semi-major axis and orbital period of a particle that starts from $r = R$ and collapses (along a line as we ignore rotation) towards the center? Then, obtain the free-fall time.

Consider the free-fall collapse of a homogeneous cloud. Assume constant temperature T such that the cloud is fragmenting, i.e., $M = M_J$ is decreasing with time.

(b) Show that the radius corresponding to the fragment mass M_J is

$$R_{\rm J} = \frac{G\mu m_{\rm m}}{5kT} M_J \tag{3}$$

(c) Argue that the fragment radiates away energy at rate

$$\frac{dE}{dt} \sim \frac{GM^2}{Rt_{\rm ff}} \tag{4}$$

and work out this expression, eliminating ρ and R, for a fragment that has $M = M_J$ and $R = R_J$.

(d) What is the expression for the maximum rate at which objects can cool by black body radiation? Write down this expression in terms of T and M.

(e) Now plot the previous two expressions you have derived as function of mass M for temperatures of 100 and 1,000 K in a single plot (so four lines in total). Mass M should range between 10^{-3} and 100 solar mass. For the y-axis choose units that are "astrophysical".

(f) You will see (hopefully) that the lines will intersect at a certain mass. Why does fragmentation stop at this point?

Exercise III.2 The Messier 80 globular cluster

Messier 80 is a globular cluster in the constellation Scorpius. Its apparent magitude is $m_V = +7.87$. We will use the virial theorem to estimate the distance d to Messier 80. Its "core radius" is observed to be 8 arcsec.

(a) Given that the star Vega has magnitude $m_V = 0.026$, luminosity $40\,L_\odot$ and lies at a distance of 7.7 pc, give an expression for L as function of the (unknown) distance d to Messier 80. Write your expression as:

$$\frac{L_{\mathrm{M81}}}{L_\odot} = A_1 \left(\frac{d}{\mathrm{pc}}\right)^2 \tag{5}$$

and give a value of A_1. If you get stuck here, take $A_1 = 10^{-4}$.

Assume that the stellar density follows an $n = 5$ polytrope with scaling parameter $R_s = \sqrt{3}\lambda_5$. This is referred to as a Plummer model.

(b) The core radius R_c is defined as the distance from the center of the cluster where the surface brightness (or stellar surface density) is half of that at the center. Show that $R_c = 0.64 R_s$.

(c) Incidentally, do you know why $M/L \gg M_\odot/L_\odot$ for most globular clusters?

It can be shown that the gravitational energy of an $n = 5$ polytropic mass distribution is

$$W = -\frac{3\pi}{32}\frac{GM^2}{R_s} \tag{6}$$

(which is not hard to derive)

(d) The line-of-sight (1D) velocity dispersion of the stars in M80 is measured to be $\sigma = 10\,\mathrm{km\,s^{-1}}$. Also, assume that the luminosity-to-mass ratio of M80 is $0.2\,L_\odot/M_\odot$. Now apply the virial theorem together with the relations derived above to obtain the distance d to M80.

Exercise III.3 The initial mass function (IMF)

We consider the Chabrier IMF

$$\frac{dn}{dm} = \xi(m) \propto \begin{cases} m^{-1} \exp\left[-\dfrac{(\log_{10} m - \log_{10} 0.22)^2}{0.65}\right] & (m < 1) \\ m^{-2.35} \exp\left[-\dfrac{(-\log_{10} 0.22)^2}{0.65}\right] & (m \geq 1) \end{cases} \tag{7}$$

Here m is expressed in units of solar mass.

(a) Let Brown Dwarfs be stars of mass less than 0.075 $M_\odot$. What is the number fraction of Brown Dwarfs? And what is the *mass* fraction of Brown Dwarfs. It is most useful to solve for these numbers numerically.

Consider stars above solar mass and let's for simplicity omit the exponential dependence, such that $\xi(m) \propto m^{-2.35}$. For main-sequence stars, let their luminosity scale with mass as $L \propto m^\eta$ and let their lifetime be $t \propto m/L$.

(b) Suppose you are in the Gobi desert on a clear night and that you can see all stars out to a certain (apparent) magnitude. How does the distance d out to which you see stars scale with the stellar mass?

(c) Let $\eta = 3.5$. Are most stars that you see light or massive?

Exercise III.4 Toomre-Q and disk instability

A physically-intuitive way to obtain Q_T approximately is to compare the total internal, rotational, and gravitational energies. When

$$\left|\frac{E_{\rm therm}}{E_{\rm grav}}\right| \times \left|\frac{E_{\rm rot}}{E_{\rm grav}}\right| < 1 \tag{8}$$

the gravitational energy dominates over the combined rotational and thermal energies, leading to instability.

(a) Write down expressions for $E_{\rm therm}$, $E_{\rm grav}$, and $E_{\rm rot}$ as function of scale λ. Work in two dimensions. For example, the mass corresponding to scale λ is $\sim\Sigma\lambda^2$ (you can forget numerical factors) with Σ the surface density. Then show that the above estimate results in Q_T, barring a factor of unity.

(b) Take a disk with a solar-mass star and temperature $T = 200\,{\rm K} \times r_{\rm au}^{-1/2}$ and surface density $\Sigma = 2 \times 10^3$ g cm$^{-2} \times r_{\rm au}^{-1}$ where $r_{\rm au}$ is the orbital radius expressed in astronomical units. According to the Toomre-Q criterion, where does the disk become unstable?

(c) What is the mass associated with the critical wavelength λ_c?

Exercise III.5 Radial drift of solids in disks

Protoplanetary disks are partially supported by pressure. It can be shown that if the pressure gradient force is

$$\Delta g = -\frac{1}{\rho}\frac{dP}{dr} \equiv 2\eta\Omega_K^2 r \tag{9}$$

where Ω_K is the Keplerian orbital frequency and η is a dimensionless quantity $\ll 1$, then the disk rotates at a velocity that is slightly below Keplerian, $v_{\phi,\rm gas} = (1-\eta)v_K$, where $v_K = \Omega_K r$ is the Keplerian orbital velocity.

The sub-Keplerian velocity imparts a gas drag force on particles, which causes them to spiral inwards. In this exercise we will work with the "stopping time" $t_{\rm stop}$ instead of the gas drag force:

$$F_{\rm drag} = \frac{m\Delta v}{t_{\rm stop}} \tag{10}$$

where Δv is the relative velocity between the solid particle and the gas.

(a) Do you have any idea why t_{stop} is called "stopping time"?

(b) Consider the limit of very small particles. They closely follow the gas and hence we have that their velocity $v_\phi \simeq v_{\phi,\mathrm{gas}}$. However, solid particles do not experience a pressure gradient force. Therefore, their (sub-Keplerian) centrifugal motion does not make up for the gravitational force, causing them to drift radially. Show by a force balance that the radial velocity is

$$v_r \simeq -2\eta v_K \Omega t_{\mathrm{stop}}; \qquad \Omega t_{\mathrm{stop}} \ll 1 \tag{11}$$

(c) Now consider the limit of very large particles. They are hardly perturbed by gas drag and very closely obey Keplerian motion, $v_\phi \simeq v_K$. Therefore, they move faster than the gas and experience a headwind of ηv_K. Calculate the torque T that these bodies experience and subsequently the radial velocity due to angular momentum loss. Hint: the torque T and rate of angular momentum loss can be cast as:

$$T = m\frac{dl}{dt} = m\frac{dl}{dr}\frac{dr}{dt} \tag{12}$$

and l is the specific angular momentum corresponding to a Keplerian circular orbit.

(d) Can you think of a formula that combines these two limiting expressions? (An expression that is continuous and has the correct limiting behavior.)

(e) Calculate the radial drift timescale $t_{\mathrm{drift}} = r/|v_r|$ for particles that obey $t_{\mathrm{stop}}\Omega = 1$ at (i) 1 au; (ii) 10 au; and (iii) 100 au. Take $\eta = 10^{-3}$ and consider a solar-mass star. Give your answer in years. (If you failed to solve the previous answer, use Equation (11)).

Exercise III.6 Planet formation

Consider a test body of mass M immersed in a sea of smaller bodies of mass m. Assume that the M-body is on a circular orbit at semi-major axis a and orbital frequency Ω_K, while the m-bodies are in Kepler orbits with eccentricity $e \ll 1$ and inclinations $i = \frac{1}{2}e$.

(a) Ignoring (for the moment) gravitational focusing, show that the growth timescale of the M-body is:

$$t_{\mathrm{growth}} \equiv \frac{M}{dM/dt} \sim \frac{R\rho_\bullet}{\Sigma_m \Omega_K} \qquad \text{(no focusing)} \tag{13}$$

where R is the radius corresponding to M, $\rho_\bullet$ the internal density of the material and Σ_m the surface density in m-bodies. Assume that the relative motion between mass M and m is due the eccentricity of the latter. (*Hint:* With these assumptions provide simple expressions for Δv and the scaleheight of the m-particles, then perform an $n - \sigma - \Delta v$ calculation. If necessary, make other reasonable assumptions.)

(b) Take a surface density profile for planetesimals of $\Sigma_{\mathrm{pltsm}} = 10\ \mathrm{g\ cm^{-2}}\,(a/\mathrm{au})^{-1}$. How long would it take to form an Earth-mass planet (around a solar-mass star) at (i) 0.1, (ii) 1, and (iii) 10 au without gravitational focusing? Comment on the timescales that you have obtained – too short, or too long to explain the solar system's planets? (say why)

(c) Take $e = 0.01$. How do the growth timescales change when we account for gravitational focusing?