Application of Portfolio Optimization

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1 Introduction

Allocating investments in assets such as stocks and bonds provides investors with both short-term stability and long-term potential. Therefore, portfolio optimization is a critical topic in finance and investments. The risks and returns of assets varies a lot across the markets. In general, risky assets have higher returns than risk-less assets, so it would be beneficial to diversify among assets with different risk levels.

In our project, we will consider an investor who has an amount of money to be invested in a number of different securities (stocks, bonds, money markets, etc.). He has some expectations on the returns and he also can accept some level of risk. The investor now has several options: Maximizing the expected portfolio return, Minimizing the portfolio risk, Maximizing the utility function, and finding a balance point between these problems that aligns with the investor's goals.

Suppose there are n assets in the portfolio and their respective weights are $w_1, w_2, w_3, ..., w_n$. Weights can be negative when the investor chooses to short sell the assets. Their rates of return are denoted as $R_1, R_2, R_3, ..., R_n$. Each asset has variance of σ_i^2 and the correlation between two randomly selected assets is denoted as $\rho_{i,j}$. The covariance matrix is denoted as Q.

2 Efficient Frontier

An optimal portfolio is efficient if no other portfolio can give a higher return from the same amount of risk or give a lower risk from the same amount of return.

In our project, we use the data set in Figure 1 to plot the efficient frontier and the capital market line, which is the tangent line at the portfolio with the highest Sharpe ratio. The slope of the capital market line represents the best expected return per unit of risk. We analyze the closing prices of the stocks of 22 companies in the data set, including the S&P 500 index, for each month from January 2012 to September 2014 to obtain the average monthly returns for each stock. We also included a risk-free asset with monthly return of 0.165%.

We obtain a covariance table(see Figure 3 in Appendix) by finding the monthly price change of each stock and the average monthly return of each stock, subtracting it, and multiplying it with the change of other stocks. The covariance between stock returns is an essential factor in portfolio optimization because the value of diversification comes from assets that are not perfectly correlated; the more significant the covariance, the more effectively we can diversify to reduce portfolio variability.

We find two efficient portfolios, A and B, by placing constraints of 7% and 8% on the standard deviations of the each portfolio. The total weights of assets in each portfolio sum to 100%. We plot the efficient frontier by assigning weights to portfolios A and B to obtain different means and standard deviations. The green line in Figure 2 contains all the efficient portfolios. The blue line in Figure 2 contains all the inefficient portfolios, which are inferior investment choices. Next, we analyze the two portfolios, A and B, to find their maximum Sharpe ratio to plot the tangent line; see the gray line in Figure 2. At the same time, we add each stock to Figure 2 based on its average monthly return and standard deviation. As a result, we obtain the whole efficient frontier below. By combining the optimized portfolio with risk-free assets, we can achieve expected returns above the efficient frontier at the respective standard deviations.

	SPX	AXP	BA	CAT	CSCO	CVX	DO	DIS	GE	GS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE
14-Sep	1972.3	87.28	127	98.28	25.17	118.22	71.28	87.91	25.38	182.99	91.3	188.5	34.59	105.9	59.84	42.37	93.98	140.92	58.84	46.07	88.94	29.32
14-Aug	2003.4	89.28	126	108.3	24.8	128.26	65.67	88.75	25.52	178.54	92.59	191	34.69	103.1	59.05	41.13	92.9	143.23	59.23	45.15	78.32	29.14
14-Jul	1930.7	87.74	119	99.99	25.04	126.97	63.42	84.8	24.71	171.79	80.06	189.3	33.45	98.77	57.28	38.74	92.93	139.31	55.91	42.62	76.68	28.45
14-Jun	1960.2	94.33	126	107.2	24.48	128.26	64.54	84.66	25.82	166.39	80.17	179	30.5	103.2	56.84	41.76	99	141.63	57	41.18	77.09	29.17
14-May	1923.6	90.98	134	100.8	24.25	120.64	68.36	82.96	26.1	158.81	78.99	182	26.96	100.1	54.81	40.03	99.68	140.95	56.58	40.43	76.46	29.12
14-Apr	1884	86.93	127	104	22.76	122.27	65.96	78.34	26.2	158.28	78.28	192.9	26.12	99.27	55.22	39.92	98.83	136.69	57.27	39.62	72.29	30.47
14-Mar	1872.3	89.29	123	97.43	21.89	115.84	65.74	79.07	25.23	162.27	77.9	189	25.25	96.27	59.51	37.83	95.57	133.32	55.52	40.2	73.2	31.29
14-Feb	1859.5	90.53	127	95.07	21.29	112.35	65.27	79.8	24.82	164.84	80.3	181.8	24.23	90.28	55.7	37.09	92.76	132.4	55.3	37.57	77.6	31.28
14-Jan	1782.6	84.32	122	92.07	21.4	107.79	59.35	71.7	24.27	161.99	75.23	172.5	23.79	86.08	54.27	36.72	91.03	125.15	51.4	36.83	71.97	29.37
13-Dec	1848.4	89.75	133	88.45	21.74	120.61	63.21	75.44	27.07	174.96	80.6	183.1	25.16	89.12	56.95	40.11	93.8	136.93	48.57	36.41	77.69	29.59
13-Nov	1805.8	84.88	131	82.41	20.59	118.22	59.71	68.81	25.54	166.75	78.58	175.4	23.11	92.11	55.72	39.02	94.12	130.35	47.92	38.11	77.94	30.65
13-Oct	1756.5	80.92	127	81.2	21.86	114.87	59.11	66.9	25.05	158.25	75.87	174	23.5	89.49	50.19	38.15	92.53	122.27	43.36	34.21	74.61	29.42
13-Sep	1681.6	74.48	114	80.67	22.54	116.34	56.56	62.9	22.89	155.65	73.88	179.8	22.01	83.77	49.97	36.52	92.24	116.01	45.78	32.15	71.54	27.54
13-Aug	1633	70.92	101	79.84	22.43	115.32	54.68	59.33	22	149.66	72.18	177	21.11	83.5	48.85	36.55	90.46	110.35	45.07	32.27	61.87	27.04
13-Jul	1685.7	72.75	102	80.2	24.62	119.56	55.3	63.06	23.17	160.85	76.58	188.5	22.2	89.67	53.88	38.39	93.27	113.46	45.91	30.54	61.76	28.02
13-Jun	1606.3	73.51	99.3	79.24	23.25	112.39	50.32	61.6	22.05	148.31	75.07	184.7	23.04	82.34	50.66	38.02	94.15	105.66	44.27	33.13	62.51	26.63
13-May	1630.7	74.44	96	82.42	23.04	116.57	53.48	61.53	22	158.93	75.84	201	23.09	80.73	52.39	40.24	91.84	106.54	44.1	33.48	60.53	25.89
13-Apr	1597.6	67.26	82.8	81.33	19.99	114.94	51.83	61.29	21.03	142.78	70.72	194.8	22.58	81.13	47.04	38.44	96.38	100.6	44.33	31.53	62.22	27.41
13-Mar	1569.2	66.13	74.2	83.01	19.81	111.94	46.74	55.4	21.81	143.84	67.28	205.1	20.58	77.61	45.27	36.54	94.07	102.14	41.74	27.25	57.73	27.21
13-Feb	1514.7	60.93	70.8	88.16	19.77	110.37	45.54	53.25	21.9	146.39	65.69	193.2	19.67	72.45	46.66	35.15	90.49	99.92	39.97	26.48	53.28	25.81
13-Jan	1498.1	57.15	72.2	93.91	19.5	107.64	44.71	52.55	20.85	144.04	64.17	194.5	19.61	69.81	44.88	34.21	89.2	96.01	40.45	25.93	52.68	25.72
12-Dec	1426.2	56.15	72.2	85.53	18.62	101.09	42.38	48.57	19.64	124.27	59.31	183.5	19.22	66.2	41.65	34.21	82.57	88.66	38.29	25.23	50.29	23.44
12-Nov	1416.2	54.61	71.2	80.89	17.92	98.8	40.65	47.71	19.59	114.75	62.4	182	18.24	66.85	38.91	35.79	81.47	86.84	41.04	25.15	47.3	23.99
12-Oct	1412.2	54.68	67.1	80.48	16.13	102.17	41.53	47.19	19.53	118.73	58.59	185.5	19.69	66.29	39.48	34.85	80.53	83.09	42.28	26.74	44.34	23.04
12-Sep	1440.7	55.35	66.3	81.16	17.84	108.03	46.89	50.22	21.06	110.28	57.63	197.8	20.91	64.5	38.06	35.55	85.12	87.66	41.78	27.89	46.05	23.02
12-Aug	1406.6	56.75	68	80.49	17.82	103.95	46.41	47.52	19.06	102.56	54.17	185.8	22.91	63.12	34.92	34.82	83.02	87.84	39.51	28.88	47.24	22.11
12-Jul	1379.3	56.18	70	79.44	14.9	100.75	46	47.21	19.09	97.45	49.56	186.1	23.5	64.21	33.85	37.61	82.25	85.99	40.54	27.43	45.13	22.07
12-Jun	1326.2	56.48	70.3	79.59	15.96	97	46.78	46.59	19.18	92.58	50.33	185.7	24.37	62.67	33.32	36.4	81.49	84.45	38.32	28.48	42.43	21.11
12-May	1310.3	54.17	65.9	82.13	15.18	90.39	44.64	43.91	17.37	92.43	46.86	183.2	23.63	57.91	30.91	34.55	82.24	79.56	34.12	27.17	52.3	20.08
12-Apr	1397.9	58.42	72.3	96.33	18.74	97.1	49.04	41.41	17.81	110.69	48.9	195.8	25.79	59.81	40.08	35.29	89	83.65	35.62	29.61	53.9	20.82
12-Mar	1408.5	55.34	70	99.42	19.59	97.7	48.53	42.06	18.26	119.55	47.5	197.3	25.53	60.6	42.6	34.22	89.6	83.51	34.86	29.83	52.25	20.59
12-Feb	1365.7	51.14	70.5	106.6	18.41	99.44	46.65	40.34	17.33	110.68	44.64	186	24.41	59.79	36.36	32.07	90.68	82	34.27	29.35	51.82	19.21
12-Jan	1312.4	48.48	69.4	101.9	18.2	93.25	46.31	37.37	16.87	106.83	41.65	181.4	23.8	60.02	34.56	31	89.82	80.62	34.36	27.13	49.94	19.25

Figure 1: Data Set

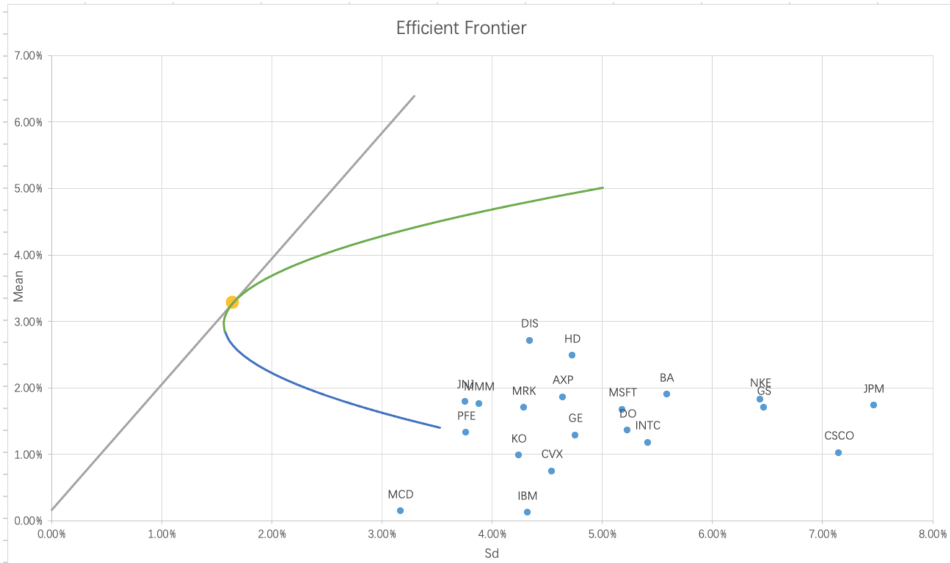

Figure 2: Efficient Frontier

3 Maximize the expected portfolio return

The return of portfolio is the weighted summation of return of each asset, and the goal is to maximize the portfolio return with constrained variance within the portfolio. The weights of the portfolio should have a sum of 1 and each weight should be at least 0. We can summarize maximization problem and constraints into the standard form as follows:

Return of portfolio:

$$R_p = w_1 R_1 + w_2 R_2 + w_3 R_3 + \cdots + w_k R_n = \textstyle\sum_{i=1}^{n} w_i R_i$$

$$\begin{aligned} \text{Maximize} \quad & \textstyle\sum_{i=1}^{n} w_i R_i \\ \text{Subject to} \quad & w^T Q w \le \gamma^2 \\ & \textstyle\sum_{i=1}^{n} w_i = 1 \end{aligned}$$

The risk limit γ^2 is set to be 0.001. We used the Python package numpy to solve this problem and defined the Fusion model. The maximum of $\sum_{i=1}^{n} w_i R_i$ is 0.0455 and the portfolio composition is:

$$w_i = \begin{bmatrix} -0.7286; & 0.2205; & -0.3354; & 0.2834; & 0.4946; & -0.1270; & -0.4303; & 0.4417; \cdots \\ 0.7102; & 0.1154; & -0.3780; & -0.1673; & -0.1311; & 0.9453; & -0.2787; & -0.1259; \cdots \\ -1.6865; & 0.1201; & 0.8094; & 0.4673; & 0.7778; & 0.0035 & & \end{bmatrix}$$

The complete codes and outcome are attached in the appendix.

4 Minimize the Portfolio Risk

The second option is to minimize the portfolio risk, which depends on the standard deviation of each asset and correlations between the assets in the portfolio. Simply finding the minimum of the variance might lead to a result of a lower return than expected by the investor, since we will have to put heavy weights on risk-free assets and risk-free assets usually have low returns. Therefore the portfolio return should be bounded from below according to the investor's request. We can summarize the minimization problem and constraints into the standard form as follows:

Variance of portfolio:

$$Var[P] = \sum_{i,j} \rho_{ij}\sigma_i\sigma_j w_i w_j = w^T Q w$$

$$\begin{aligned} \text{Maximize} \quad & \frac{1}{2} w^T Q w \\ \text{Subject to} \quad & \sum_{i=1}^{n} w_i R_i \geq R, \\ & \sum_{i=1}^{n} w_i = 1 \end{aligned}$$

We use the same data set as in the efficient frontier section and we use Langrange Multiplier to solve this problem. We assume the minimum acceptable expected return of the portfolio is 0.015 per month.

$$\begin{aligned} & L = \frac{1}{2}\sum_{i,j=1}^{n} Q_{ij}w_i w_j - \lambda_1\Big(\sum_{j=1}^{n} w_j - 1\Big) - \lambda_2\Big(\sum_{j=1}^{n} w_j R_j - R\Big) \\ & \frac{\partial L}{\partial w_1} = \sum_{j=1}^{n} Q_{1j}w_j - \lambda_1 - \lambda_2 R_1 = 0 \\ & \vdots \\ & \frac{\partial L}{\partial w_n} = \sum_{j=1}^{n} Q_{nj}w_j - \lambda_1 - \lambda_2 R_n = 0 \\ & \frac{\partial L}{\partial \lambda_1} = -\Big(\sum_{j=1}^{n} w_j - 1\Big) = 0 \\ & \frac{\partial L}{\partial \lambda_2} = -\Big(\sum_{j=1}^{n} w_j R_j - R\Big) = 0 \end{aligned}$$

Let Q represent the covariance matrix of the 22 stocks in our data set. By using Lagrange multiplier, we obtain the unique solution of weights, λ_1, and λ_2.

$$\begin{bmatrix} Q_{11} & \dots & Q_{1n} & -1 & -R_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_{n1} & \dots & Q_{nn} & -1 & -R_n \\ 1 & \dots & 1 & 0 & 0 \\ R_1 & \dots & R_n & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ R \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ \vdots \\ w_n \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ R \end{bmatrix} = \begin{bmatrix} 0.3723; & 0.0142; & -0.0437; & 0.072; & 0.2589; & -0.1509; & -0.2708; & 0.0858; \cdots \\ 0.2802; & 0.1371; & -0.2923; & 0.0966; & -0.0250; & 0.1670; & -0.3244; & 0.1140; \cdots \\ -0.3539; & -0.0863; & 0.3269; & 0.2900; & 0.1949; & 0.1373; & 0.0003; & -0.0010 \end{bmatrix}$$

The values of w_i's are listed in the matrix in order from left to right and from top to bottom, and the last two numbers in the last row are the values of λ and μ respectively. These are the optimal weights that minimize the portfolio risk when the expected return of the portfolio is at least 0.015.

5 Maximize the Utility Function of the Investor

The quadratic utility function represents the trade-off between portfolio return and portfolio risk and δ is the risk-averse factor that can be used to adjust the preferred utility.

Utility function:

$$\sum_{i=1}^{n} w_i R_i - \frac{\delta}{2} w^T Q w$$

$$\begin{aligned} \text{Maximize} \quad & \sum_{i=1}^{n} w_i R_i - \frac{\delta}{2} w^T Q w \\ \text{Subject to} \quad & \sum_{i=1}^{n} w_i = 1, \end{aligned}$$

We use the same data set as in the efficient frontier section and we use Langrange Multiplier to solve this problem. We assume the risk-aversion coefficient δ is 100.

$$L = \sum_{i=1}^{n} w_i R_i - \frac{\delta}{2} \sum_{i,j=1}^{n} Q_{ij} w_i w_j - \lambda_1 (\sum_{i=1}^{n} w_i - 1)$$

$$\nabla_i L = R_i - \delta Q w_i - \lambda_1 \Rightarrow Q = Q, \vec{\boldsymbol{a}} = \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$$

$$h(x) = \sum_{i=1}^{n} w_i - 1 \Rightarrow A = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} = \vec{\boldsymbol{a}}, b = \begin{bmatrix} 1 \end{bmatrix}$$

Let Q represent the covariance matrix of the 22 stocks in our data set. By using Lagrange multiplier, we obtain the unique solution of weights, λ_1.

$$\begin{bmatrix} \delta Q_{11} & \dots & \delta Q_{1n} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta Q_{n1} & \dots & \delta Q_{nn} & 1 \\ 1 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{w}}_1 \\ \vdots \\ \vec{\boldsymbol{w}}_n \\ \vec{\boldsymbol{\lambda}}_1 \end{bmatrix} = \begin{bmatrix} R_1 \\ \vdots \\ R_n \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{\boldsymbol{w}}_1 \\ \vdots \\ \vec{\boldsymbol{w}}_n \\ \vec{\boldsymbol{\lambda}}_1 \end{bmatrix} = \begin{bmatrix} \delta Q_{11} & \dots & \delta Q_{1n} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta Q_{n1} & \dots & \delta Q_{nn} & 1 \\ 1 & \dots & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} R_1 \\ \vdots \\ R_n \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \vec{\boldsymbol{w}}_1 \\ \vdots \\ \vec{\boldsymbol{w}}_n \\ \vec{\boldsymbol{\lambda}}_1 \end{bmatrix} = \begin{bmatrix} -0.1257; & 0.1075; & -0.1757; & 0.1676; & 0.3655; & -0.1400; & -0.3430; & 0.2468; \cdots \\ 0.4748; & 0.1273; & -0.3311; & -0.0229; & -0.0730; & 0.5191; & -0.3037; & 0.0054; \cdots \\ -0.9568; & 0.0070; & 0.5452; & 0.3702; & 0.4586; & 0.0768; & -0.01557 & \end{bmatrix}$$

The values of w_i's are listed in the matrix in order from left to right and from top to bottom, and the last number in the last row is the value of λ. These are the optimal weights that maximize the utility function of the investor when risk-averse factor is 100.

6 Conclusion and Discussion

With sufficient information on the return of stocks of different companies, we are able to plot the efficient frontier to exclude the inefficient compositions of the investor's portfolio. Also, a risk-free asset can be added to the portfolio and it will be shown as the tangent line of the efficient frontier at the point which has the maximum Sharpe ratio. The investor can then choose the best combination of the portfolio from the list of efficient portfolios according to their preferences about risk and return.

The purpose of dividing the portfolio algorithm into three main categories is to meet the needs of different investors in real life. When investors have certain concerns about the portfolio risk, they can pick the first algorithm, which obtains the maximum portfolio return with a fixed portfolio risk rate. When the investor has a specific requirement about the rate of return on the investment, they can choose the second algorithm, which minimizes portfolio risk with a preset rate of portfolio return. If the investor does not have a clear preference for the level of risk or amount of return, he can choose to maximize the utility function by finding the balance point between the portfolio risk and portfolio return. The last method can also be used to help investors get the best portfolio compositions for their expectations by understanding their risk preferences and setting the risk aversion.

In summary, since different investors have different needs, it is impossible to decide on the best model. However, satisfying the investors' needs will be our first priority, no matter which model is to be used.

7 References

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8 Appendix

1. Covariance matrix of the 22 companies

	SPX	AXP	BA	CAT	CSCO	CVX	DO	DIS	GE	GS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE
SPX	0.000709	0.000801	0.000691	0.000731	0.000849	0.000877	0.000872	0.000661	0.000803	0.001197	0.000783	0.000681	0.000540	0.000576	0.001314	0.000592	0.000500	0.000805	0.000376	0.000337	0.000483	0.000550
AXP	0.000801	0.002091	0.001554	0.000474	0.001115	0.000844	0.001128	0.000858	0.001059	0.001164	0.001046	0.000643	0.001107	0.000554	0.001675	0.001080	0.000442	0.001134	0.000179	0.000805	0.000652	0.000431
BA	0.000691	0.001554	0.003026	0.000504	0.001215	0.000567	0.001434	0.000749	0.000991	0.000893	0.001119	0.000125	0.000824	0.000698	0.001237	0.001085	0.000435	0.001157	0.000279	0.000882	0.000940	0.000273
CAT	0.000731	0.000474	0.000504	0.002948	0.001308	0.001185	0.000502	0.000248	0.000510	0.001675	0.000716	0.000625	0.000683	0.000235	0.001530	0.000188	0.000719	0.000743	0.000513	0.000438	-0.000006	0.000184
CSCO	0.000849	0.001115	0.001215	0.001308	0.004957	0.001148	0.001644	0.000362	0.001167	0.002253	0.001460	0.001209	0.001132	0.000360	0.002949	0.000237	0.000751	0.001322	-0.000158	0.000648	0.000177	0.000064
CVX	0.000877	0.000844	0.000567	0.001185	0.001148	0.002002	0.001040	0.000933	0.001288	0.001485	0.001010	0.001128	0.000935	0.000787	0.001608	0.000818	0.000690	0.001199	0.000667	0.000501	-0.000185	0.000466
DO	0.000872	0.001128	0.001434	0.000502	0.001644	0.001040	0.002655	0.001259	0.001267	0.001331	0.000924	0.001058	0.001098	0.000761	0.001719	0.000920	0.000781	0.001244	0.000516	0.000904	0.000966	0.000559
DIS	0.000661	0.000858	0.000749	0.000248	0.000362	0.000933	0.001259	0.001827	0.001143	0.000589	0.001006	0.000685	0.000864	0.000521	0.000633	0.000766	0.000489	0.000872	0.000431	0.000347	0.000587	0.000537
GE	0.000803	0.001059	0.000991	0.000510	0.001167	0.001288	0.001267	0.001143	0.002195	0.001319	0.001167	0.001111	0.000459	0.000868	0.001589	0.001119	0.000614	0.001392	0.000340	0.000014	-0.000289	0.000729
GS	0.001197	0.001164	0.000893	0.001675	0.002253	0.001485	0.001331	0.000589	0.001319	0.004056	0.001416	0.001548	0.000924	0.000734	0.003982	0.000607	0.000591	0.001329	0.000439	0.000497	0.000580	0.000697
HD	0.000783	0.001046	0.001119	0.000716	0.001460	0.001010	0.000924	0.001006	0.001167	0.001416	0.002169	0.000862	0.000391	0.000643	0.001679	0.000964	0.000397	0.001000	0.000695	0.000615	0.000149	0.000722
IBM	0.000681	0.000643	0.000125	0.000625	0.001209	0.001128	0.001058	0.000685	0.001111	0.001548	0.000862	0.001812	0.000674	0.000400	0.001691	0.000620	0.000439	0.000885	0.000514	0.000118	0.000044	0.000529
INTC	0.000540	0.001107	0.000824	0.000683	0.001132	0.000935	0.001098	0.000864	0.000459	0.000924	0.000391	0.000674	0.002841	0.000361	0.001229	0.000700	0.000336	0.000600	-0.000053	0.001296	0.000471	-0.000119
JNJ	0.000576	0.000554	0.000698	0.000235	0.000360	0.000787	0.000761	0.000521	0.000868	0.000734	0.000643	0.000400	0.000361	0.001371	0.001100	0.000883	0.000599	0.000694	0.000696	0.000236	-0.000118	0.000902
JPM	0.001314	0.001675	0.001237	0.001530	0.002949	0.001608	0.001719	0.000633	0.001589	0.003982	0.001679	0.001691	0.001229	0.001100	0.005405	0.001002	0.000773	0.001404	0.000963	0.001381	-0.000068	0.001024
KO	0.000592	0.001080	0.001085	0.000188	0.000237	0.000818	0.000920	0.000766	0.001119	0.000607	0.000964	0.000620	0.000700	0.000883	0.001002	0.001747	0.000386	0.000775	0.000575	0.000331	0.000084	0.000468
MCD	0.000500	0.000442	0.000435	0.000719	0.000751	0.000690	0.000781	0.000489	0.000614	0.000591	0.000397	0.000439	0.000336	0.000599	0.000773	0.000386	0.000974	0.000593	0.000385	0.000372	0.000810	0.000597
MMM	0.000805	0.001134	0.001157	0.000743	0.001322	0.001199	0.001244	0.000872	0.001392	0.001329	0.001000	0.000885	0.000600	0.000694	0.001404	0.000775	0.000593	0.001463	0.000318	0.000206	0.000406	0.000589
MRK	0.000376	0.000179	0.000279	0.000513	-0.000158	0.000667	0.000516	0.000431	0.000340	0.000439	0.000695	0.000514	-0.000053	0.000696	0.000963	0.000575	0.000385	0.000318	0.001783	0.000435	-0.000712	0.000738
MSFT	0.000337	0.000805	0.000882	0.000438	0.000648	0.000501	0.000904	0.000347	0.000014	0.000497	0.000615	0.000118	0.001296	0.000236	0.001381	0.000331	0.000372	0.000206	0.000435	0.002607	0.000227	0.000018
NKE	0.000483	0.000652	0.000940	-0.000006	0.000177	-0.000185	0.000966	0.000587	-0.000289	0.000580	0.000149	0.000044	0.000471	-0.000118	-0.000068	0.000084	0.000810	0.000406	-0.000712	0.000227	0.004017	0.000301
PFE	0.000550	0.000431	0.000273	0.000184	0.000064	0.000466	0.000559	0.000537	0.000729	0.000697	0.000722	0.000529	-0.000119	0.000902	0.001024	0.000468	0.000597	0.000589	0.000738	0.000018	0.000301	0.001375

Figure 3: Covariance Matrix

2. Codes for Part 3

```python
from mosek.fusion import *
import numpy as np
import sys
import pandas as pa

m = np.array([0.012810829, 0.018543904, 0.018973424,-0.001114397 ,0.01018373,   0.007442031,   0.01356803,    0.027093189,
0.012845066,   0.01696075,    0.024829794,   0.001207155,   0.011752241,   0.017902659,   0.017303545,   0.009811993,   0.001415824,   0.017604591,
0.016952363,   0.016685227,   0.01819924,    0.013235537])

# x = pa.read_excel("309.xlsx", sheet_name="Sheet5")
S = np.array(pa.read_excel("309.xlsx", sheet_name="Sheet5", header=None))
N = m.shape[0]  # Number of securities

gamma2 = 0.001 # Risk limit

# Cholesky factor of S to use in conic risk constraint
G = np.linalg.cholesky(S)
#n = 22

#er, x = BasicMarkowitz(n,m,GT,x0,w,gamma)

with Model("markowitz") as M:
    # Settings
    M.setLogHandler(sys.stdout)

    # Decision variable (fraction of holdings in each security)
    # The variable x is the fraction of holdings in each security.
    # x must be positive, this imposes the no short-selling constraint.
    x = M.variable("x", N)

    # Budget constraint
    M.constraint('budget', Expr.sum(x), Domain.equalsTo(1))

    # Objective
    M.objective('obj', ObjectiveSense.Maximize, Expr.dot(m, x))

    # Imposes a bound on the risk
    M.constraint('risk', Expr.vstack(gamma2, 0.5, Expr.mul(G.T, x)),
                 Domain.inRotatedQCone())

    # Solve optimization
    M.solve()

    returns = M.primalObjValue()
    portfolio = x.level()


print("----------------------------------------------------")
print(returns)
print("----------------------------------------------------")
print(portfolio)
```

Figure 4: Python Codes

```
Problem
  Name                   : markowitz
  Objective sense        : maximize
  Type                   : CONIC (conic optimization problem)
  Constraints            : 1
  Affine conic cons.     : 1
  Disjunctive cons.      : 0
  Cones                  : 0
  Scalar variables       : 23
  Matrix variables       : 0
  Integer variables      : 0

Optimizer started.
Presolve started.
Linear dependency checker started.
Linear dependency checker terminated.
Eliminator started.
Freed constraints in eliminator : 9
Eliminator terminated.
Eliminator - tries                  : 1                 time                   : 0.00
Lin. dep.  - tries                  : 1                 time                   : 0.00
Lin. dep.  - number                 : 0
Presolve terminated. Time: 0.00
Optimizer terminated. Time: 0.00


Interior-point solution summary
  Problem status  : PRIMAL_AND_DUAL_FEASIBLE
  Solution status : OPTIMAL
  Primal.  obj: 4.4553136795e-02    nrm: 2e+00    Viol.  con: 0e+00    var: 0e+00    acc: 0e+00
  Dual.    obj: 4.4553136795e-02    nrm: 2e+01    Viol.  con: 0e+00    var: 9e-18    acc: 0e+00
-----------------------------------------------------
0.044553136794925154
-----------------------------------------------------
[-0.728585    0.22051849 -0.33543364  0.28341666  0.49455885 -0.12700677
 -0.43037054  0.44167028  0.7102027   0.11541538 -0.37802879 -0.16733716
 -0.13111293  0.94520729 -0.2786594  -0.12594562 -1.68648826  0.12005139
  0.80939231  0.46730102  0.77777835  0.00345538]
```

Figure 5: Output of Python Codes