

# Proximal Point Method for Online Saddle Point Problem

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**Abstract.** This paper focuses on the online saddle point problem, which involves a sequence of two-player time-varying convex-concave games. Considering the nonstationarity of the environment, we adopt the duality gap and the dynamic Nash equilibrium regret as performance metrics for algorithm design. We present three variants of the proximal point method: the Online Proximal Point Method (OPPM), the Optimistic OPPM (OptOPPM), and the OptOPPM with multiple predictors. Each algorithm guarantees upper bounds for both the duality gap and dynamic Nash equilibrium regret, achieving near-optimality when measured against the duality gap. Specifically, in certain benign environments, such as sequences of stationary payoff functions, these algorithms maintain a nearly constant metric bound. Experimental results further validate the effectiveness of these algorithms. Lastly, this paper discusses potential reliability concerns associated with using dynamic Nash equilibrium regret as a performance metric. The technical appendix can be found at <https://arxiv.org/abs/2407.04591>.

**Keywords:** Online Saddle Point Problem · Proximal Point Method · Duality Gap · Multiple Predictors.

## 1 Introduction

The Online Saddle Point (OSP) problem, initially introduced by [8], involves a sequence of two-player time-varying convex-concave games. In round  $t$ , Players 1 and 2 *jointly* select a strategy pair  $(x_t, y_t) \in X \times Y$ . Here, Player 1 minimizes his payoff, while Player 2 maximizes his payoff. Both players make decisions without prior knowledge of the current or future payoff functions. Upon finalizing the strategy pair, the environment reveals a continuous payoff function  $f_t : X \times Y \rightarrow \mathbb{R}$ , which satisfies the following conditions:  $\forall y \in Y$ ,  $f_t(\cdot, y)$  is convex on  $X$ , and  $\forall x \in X$ ,  $f_t(x, \cdot)$  is concave on  $Y$ . No additional assumptions are imposed on the environment, thereby allowing potential regularity or even adversarial behavior. The goal is to provide players with decision-making algorithms that approximate Nash equilibria, ensuring that the players' decisions in most rounds are close to saddle points.

Observe that the OSP problem extends the application of Online Convex Optimization (OCO, [33]) to include interactions among two players and the environment. Consequently, it becomes straightforward to identify online scenarios that are well-suited

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for OSP. Such scenarios include dynamic routing [4,15] and online advertising auctions [21], which fall under the broader category of budget-reward trade-offs.

Given the nonstationarity of the environment, there are two optional metrics available for evaluating the performance of an OSP algorithm:

- 1) The duality gap (D-Gap, [32]), which is given by

$$\text{D-Gap}_T := \sum_{t=1}^T \max_{y \in Y} f_t(x_t, y) - \sum_{t=1}^T \min_{x \in X} f_t(x, y_t). \quad (1)$$

- 2) The dynamic Nash equilibrium regret (NE-Reg, [32]), which is defined as

$$\text{NE-Reg}_T := \left| \sum_{t=1}^T f_t(x_t, y_t) - \sum_{t=1}^T \max_{y \in Y} \min_{x \in X} f_t(x, y) \right|. \quad (2)$$

[32] first presented an algorithm that, under bilinear payoff functions, simultaneously guarantees upper bounds for three performance metrics: the duality gap, dynamic Nash equilibrium regret, and static individual regret. In their commendable work, the authors have directed their focus towards the algorithm's adaptability to a spectrum of metrics, attenuating the pursuit of algorithmic optimality. Notably, their method yields a duality gap upper bound of  $\tilde{O}(\sqrt{T})$  for sequences of stationary payoff functions, whereas we advocate for a sharper bound of  $O(1)$  that we believe is optimal in such scenarios. Our assertion is inspired by [5], who demonstrated that proximal methods can incur  $O(1)$ -level regret in online convex optimization when dealing with stationary loss function sequences.

In this paper, we propose three variants of the proximal point method for solving the OSP problem: the Online Proximal Point Method (OPPM), the Optimistic OPPM (OptOPPM), which allows for an arbitrary predictor, and OptOPPM with multiple predictors, enhancing the algorithm to handle multiple predictors. Our analysis yields the following results:

- 1) OPPM achieves that

$$\text{D-Gap}_T, \text{NE-Reg}_T \leq \tilde{O}\left(\min\left\{V_T, \sqrt{(1+C_T)T}\right\}\right),$$

- 2) OptOPPM enjoys that

$$\text{D-Gap}_T, \text{NE-Reg}_T \leq \tilde{O}\left(\min\left\{V'_T, \sqrt{(1+C_T)T}\right\}\right),$$

- 3) and OptOPPM with multiple predictors guarantees that

$$\text{D-Gap}_T, \text{NE-Reg}_T \leq \tilde{O}\left(\min\left\{V_T^1, \dots, V_T^d, \sqrt{(1+C_T)T}\right\}\right),$$

where  $V_T$  represents the cumulative difference of the convex-concave payoff function series,  $V'_T$  denotes the cumulative error of one single predictor, and  $V_T^k$  indicates the cumulative error of the  $k$ -th predictor. All algorithms are near-optimal, as they achieve a worst-case duality gap upper bound of  $\tilde{O}\left(\sqrt{(1+C_T)T}\right)$ . In particular, under favorable

scenarios such as stationary payoff function sequences, these algorithms attain a sharp bound of  $\tilde{O}(1)$ . Notably, the OptOPPM with multiple predictors can autonomously select the most effective predictor from a set of  $d$  options. Even when all predictors underperform, it preserves the worst-case bound, significantly enhancing the algorithm's practical utility.

It is imperative to highlight that in time-varying games, relying on dynamic Nash equilibrium regret for performance evaluation of algorithms may lead to concerns over its reliability. A sublinear duality gap suggests an approximation to a coarse correlated equilibrium, while a sublinear dynamic Nash equilibrium regret does not necessarily imply such a correlation. Consider a scenario where, in half of the iterations, the actual payoffs significantly exceed those at the Nash equilibrium, while in the other half, they are substantially lower. This results in a small dynamic Nash equilibrium regret, but it does not guarantee that the strategies in most rounds are close to the Nash equilibrium. See Example 1 for a more precise discussion.

## 2 Related Work

The OSP problem is a time-varying version of the minimax problem. The first minimax theorem was proven by [30]. Subsequent to the seminal work of [14], which unveiled the connections between the minimax problem and online learning, there has been a burgeoning interest in the development of no-regret algorithms tailored for static environments [2, 11, 12, 17, 24, 29]. In recent years, the research focus has expanded to encompass the OSP problem and its various variants [3, 8, 7, 13, 26, 32].

[8] were the pioneers in explicitly addressing the OSP problem, introducing the concept of saddle-point regret as follows:

$$\left| \sum_{t=1}^T f_t(x_t, y_t) - \min_{x \in X} \max_{y \in Y} \sum_{t=1}^T f_t(x, y) \right|.$$

In a subsequent work, [7] redefined saddle-point regret as Nash equilibrium regret and developed an FTRL-like algorithm capable of achieving a Nash equilibrium regret bound of  $\tilde{O}(\sqrt{T})$ . Moreover, they proved that it is impossible for any algorithm to simultaneously attain sublinear Nash equilibrium regret and sublinear individual regret for both players. Building on these findings, [32] refined the notion of dynamic Nash equilibrium regret by reassessing the Nash equilibrium regret, moving the minimax operation inside the summation, and delineating it as Equation (2). They also proposed an algorithm predicated on the meta-expert framework, which ensures upper bounds for three performance metrics: the duality gap, dynamic Nash equilibrium regret, and static individual regret, effectively covering beliefs from stationary to highly nonstationary.

In contrast to the approach by [32], which targets a broad spectrum of nonstationarity levels, this paper takes beliefs that the environment exhibits nonstationarity and accordingly designs algorithms to achieve near-optimal performance. Moreover, this study underscores the potential issues with the reliability of dynamic Nash equilibrium regret as a metric for evaluating algorithmic performance in time-varying games.

### 3 Preliminaries

Throughout this paper, we use the abbreviated notation  $1:n$  to represent  $1, 2, \dots, n$ , and define  $(\cdot)_+ := \max(\cdot, 0)$ . For asymptotic upper bounds, we employ big  $O$  notation, while  $\tilde{O}$  is utilized to hide poly-logarithmic factors. Denote by  $\langle \cdot, \cdot \rangle: \mathcal{X}^* \times \mathcal{X} \rightarrow \mathbb{R}$  the canonical dual pairing, where  $\mathcal{X}^*$  represents the dual space of  $\mathcal{X}$ .

The Fenchel coupling [22,23] induced by a proper function  $\varphi$  is defined as

$$B_\varphi(x, z) := \varphi(x) + \varphi^*(z) - \langle z, x \rangle, \quad \forall (x, z) \in \mathcal{X} \times \mathcal{X}^*,$$

where  $\varphi^*$  represents the convex conjugate of  $\varphi$  given by  $\varphi^*(z) := \sup_{x \in \mathcal{X}} \langle z, x \rangle - \varphi(x)$ . Fenchel coupling is the general form of Bregman divergence. By Fenchel-Young inequality,  $B_\varphi(x, z) \geq 0$ , and equality holds if and only if  $z$  is the subgradient of  $\varphi$  at  $x$ . We use the superscripted notation  $x^\varphi$  to abbreviate one subgradient of  $\varphi$  at  $x$ . Directly applying the definition of Fenchel coupling yields  $B_\varphi(x, y^\varphi) + B_\varphi(y, z) - B_\varphi(x, z) = \langle z - y^\varphi, x - y \rangle$ . A similar version can be found in [10].

$\varphi$  is  $\mu$ -strongly convex if  $\forall x \in \mathcal{X}, \forall y^\varphi \in \partial\varphi(y): B_\varphi(x, y^\varphi) \geq \frac{\mu}{2} \|x - y\|^2$ .

$B_\varphi$  is  $L$ -Lipschitz with respect to its first variable, if  $\exists L$ , such that  $\forall y \in \mathcal{X}, \forall y^\varphi \in \partial\varphi(y)$ , and  $\forall x_1, x_2 \in \mathcal{X}: |B_\varphi(x_1, y^\varphi) - B_\varphi(x_2, y^\varphi)| \leq L \|x_1 - x_2\|$ .

## 4 Main Results

This section begins by outlining the assumptions related to the OSP problem. It then evaluates performance metrics and establishes the lower bound for the Duality Gap, a crucial step in analyzing algorithmic optimality. Finally, this section introduces three variants of the proximal point method: the Online Proximal Point Method (OPPM), the Optimistic OPPM (OptOPPM) that incorporates an arbitrary predictor, and a variant of OptOPPM designed to accommodate multiple predictors, thereby enhancing the algorithm's practical utility.

### 4.1 Assumptions

Let  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$  and  $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$  be normed vector spaces. We can omit norm subscripts without ambiguity when the norm is determined by the space to which the element belongs. Let  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ . Now we introduce the following two assumptions:

- A1 The feasible sets  $X$  and  $Y$  are both compact and convex, with the diameter of  $X$  denoted as  $D_X$ , and the diameter of  $Y$  as  $D_Y$ .
- A2 All payoff functions satisfy the boundedness of subdifferentiation, i.e.,  $\exists G_X < +\infty, \exists G_Y < +\infty, \forall x \in X, \forall y \in Y, \forall t, \|\partial_x f_t(x, y)\| \leq G_X, \|\partial_y(-f_t)(x, y)\| \leq G_Y$ .

In accordance with Theorem 3 by [20], A1 ensures the existence of a saddle point. Specifically, there exists a pair  $(x_t^*, y_t^*) \in X \times Y$  such that for all  $x \in X$  and  $y \in Y$ :  $f_t(x_t^*, y) \leq f_t(x_t^*, y_t^*) \leq f_t(x, y_t^*)$ . One may write  $(x_t^*, y_t^*) = \arg \min_{x \in X} \max_{y \in Y} f_t(x, y)$ .

## 4.2 Discussion on Performance Metrics

This subsection first confirms that the individual regret (Reg) can guarantee both the duality gap and dynamic Nash equilibrium regret, and then proceeds to demonstrate the intrinsic issues associated with dynamic Nash equilibrium regret. Prior to this, we introduce the following notations:

$$x'_t = \arg \min_{x \in X} f_t(x, y_t), \quad y'_t = \max_{y \in Y} f_t(x_t, y).$$

Individual regrets of Players 1 and 2 are defined as follows:

$$\text{Reg}_T^1 := \sum_{t=1}^T f_t(x_t, y_t) - \sum_{t=1}^T f_t(x'_t, y_t), \quad \text{Reg}_T^2 := \sum_{t=1}^T f_t(x_t, y'_t) - \sum_{t=1}^T f_t(x_t, y_t).$$

The duality gap can be interpreted as the summation of the two players' individual regrets, namely,  $\text{D-Gap}_T = \text{Reg}_T^1 + \text{Reg}_T^2$ , which implies the following lemma.

**Lemma 1.** *If an online algorithm guarantees individual regrets for both players, it also ensures a duality gap guarantee.*

Considering that two individual regrets provide the same evaluative utility as the duality gap, we choose to use only the duality gap as the metric to assess an online algorithm's performance.

The dynamic Nash equilibrium regret can be bounded by the duality gap, namely,  $\text{NE-Reg}_T \leq \text{D-Gap}_T$  (refer to Proposition 11 of [32]), which deduces the following lemma.

**Lemma 2.** *If an online algorithm offers individual regret guarantees for both players, it also ensures a guarantee of dynamic Nash equilibrium regret.*

It is noteworthy that using dynamic Nash equilibrium regret as a performance metric could raise questions about its reliability. The following extreme example illustrates that the environment may intentionally mislead players, resulting in an internal cancellation of the absolute value of dynamic Nash equilibrium regret.

*Example 1.* Consider the OSP problem defined on the feasible domain  $[-1, 1]^2$ . In round  $t$ , the environment feeds back the payoff function of  $f_t(x, y) = (x - x_t^*)^2 - (y - y_t^*)^2$ , where

$$x_t^* = x_t + \frac{(-1)^t + 1}{2} (2\chi_{x_t < 0} - 1), \quad y_t^* = y_t - \frac{(-1)^t - 1}{2} (2\chi_{y_t < 0} - 1),$$

$\chi_A$  is the 0/1 indicator function with  $\chi_A = 1$  if and only if  $A$  is true. Regardless of the algorithm used, the players' strategy pair will never approach the saddle point, but  $\text{NE-Reg}_T = \left| \sum_{t=1}^T (-1)^t \right| \leq 1$ .

Although Remark 2 in [32] pointed out that performance metrics based on function values inherently possess certain limitations, the weakness of dynamic Nash equilibrium regret is easier to trigger.

### 4.3 Duality Gap Lower Bound

Given the reliability concerns of dynamic Nash equilibrium regret, this subsection only demonstrates the lower bound for the duality gap.

The following theorem states that the duality gap has a lower bound.

**Theorem 1 (Duality Gap Lower Bound).** *Regardless of the strategy pairs adopted by the two players, there always exist a sequence of convex-concave payoff functions  $f_{1:T}$  that adhere to Assumptions A1 and A2, ensuring a duality gap of at least  $\Omega(\sqrt{(1 + C_T)T})$ , where*

$$C_T = \sum_{t=1}^T (\|x'_t - x'_{t-1}\| + \|y'_t - y'_{t-1}\|).$$

An online algorithm that attains a duality gap upper bound of  $\tilde{O}(\sqrt{(1 + C_T)T})$  is considered near-optimal. However, achieving this upper bound is trivial. Specifically, if Players 1 and 2 independently select OCO algorithms, such as Ader [31] or Online Mirror Descent with doubling trick, they can ensure that

$$\text{Reg}_T^1 \leq \tilde{O}\left(\sqrt{\left(1 + \sum_{t=1}^T \|x'_t - x'_{t-1}\|\right)T}\right), \quad \text{Reg}_T^2 \leq \tilde{O}\left(\sqrt{\left(1 + \sum_{t=1}^T \|y'_t - y'_{t-1}\|\right)T}\right),$$

thereby securing  $\text{D-Gap}_T \leq \tilde{O}(\sqrt{(1 + C_T)T})$ .

In the following three subsections, we will design non-trivial algorithms that not only ensure an  $\tilde{O}(\sqrt{(1 + C_T)T})$  duality gap upper bound but also further reduce the duality gap in certain benign environments.

### 4.4 Online Proximal Point Method

The proximal point method, initially introduced in the seminal work by [25], has established itself as a classic first-order method for solving minimax problems. To tailor it for the solution of the OSP problem, this subsection introduces and analyzes the Online Proximal Point Method (OPPM). OPPM can be formalized as follows:

$$(x_{t+1}, y_{t+1}) = \arg \min_{x \in X} \max_{y \in Y} f_t(x, y) + \frac{1}{\eta_t} B_\phi(x, x_t^\phi) - \frac{1}{\gamma_t} B_\psi(y, y_t^\psi),$$

where  $\eta_t > 0$  and  $\gamma_t > 0$  represent learning rates of Players 1 and 2, respectively,  $x_t^\phi \in \partial \phi(x_t)$ ,  $y_t^\psi \in \partial \psi(y_t)$ . To facilitate our analysis, we assume that the regularization terms satisfy the following property.

*Property 1.* The functions  $\phi$  and  $\psi$  are both 1-strongly convex, and their Fenchel couplings satisfy Lipschitz continuity for the first variable. Specifically,  $\exists L_\phi < +\infty, \exists L_\psi < +\infty, \forall \alpha, x, x' \in X, \forall \beta, y, y' \in Y$ :

$$|B_\phi(x, \alpha^\phi) - B_\phi(x', \alpha^\phi)| \leq L_\phi \|x - x'\|, \quad |B_\psi(y, \beta^\psi) - B_\psi(y', \beta^\psi)| \leq L_\psi \|y - y'\|.$$

The following theorem provides the individual regret guarantee for OPPM.

**Theorem 2 (Individual Regret for OPPM).** *Under Assumptions A1 and A2, let regularizers satisfy Property 1, and let  $C$  be a preset upper bound of  $C_T$ . If the learning rates of two players follow from*

$$\eta_t = \frac{L(2D + C)}{\epsilon + \sum_{\tau=1}^{t-2} \Delta_\tau} = \gamma_t,$$

where the constant  $\epsilon > 0$  prevents initial learning rates from being infinite,  $\Delta_t = (\Sigma_t - \max_{\tau \in 1:t-1} \Sigma_\tau)_+$ ,  $\Sigma_t = \max\{\Sigma_t^1, \Sigma_t^2\}$ , and

$$\begin{aligned} \Sigma_T^1 &= \left( \sum_{t=1}^T (f_t(x_t, y_t) - f_t(x_{t+1}, y_{t+1}) + f_t(x'_{t+1}, y_{t+1}) - f_t(x'_t, y_t)) \right)_+, \\ \Sigma_T^2 &= \left( \sum_{t=1}^T (f_t(x_t, y'_t) - f_t(x_{t+1}, y'_{t+1}) + f_t(x_{t+1}, y_{t+1}) - f_t(x_t, y_t)) \right)_+. \end{aligned}$$

Then OPPM enjoys the following individual regret:

$$\text{Reg}_T^1, \text{Reg}_T^2 \leq O \left( \min \left\{ \sum_{t=1}^T \rho(f_t, f_{t-1}), \sqrt{(1+C)T} \right\} \right),$$

where  $\rho(f_t, f_{t-1}) = \max_{x \in X, y \in Y} |f_t(x, y) - f_{t-1}(x, y)|$  measures the distance between  $f_t$  and  $f_{t-1}$ .

It should be noted that Theorem 2 corresponds to Theorem 5.1 of [6]. However, Theorem 2 involves more complex parameter settings and a complicated proof process (see Appendix B) due to the mutual influence between players resulting from the joint update of OPPM. Furthermore, this joint update is essential. Merely having each player independently execute the implicit online mirror descent, as described by [6], would only guarantee the worst-case bound, without addressing the temporal variability term of  $\sum_{t=1}^T \rho(f_t, f_{t-1})$ .

In Theorem 2, learning rates are contingent upon the preset value  $C$ . To obviate this dependency, one might resort to the application of the doubling trick [27]. The pseudocode for OPPM is encapsulated in Algorithm 1.

Applying Lemmas 1 and 2, we deduce that Algorithm 1 offers assurances on both the duality gap and dynamic Nash equilibrium regret.

**Theorem 3 (Performance of Algorithm 1).** *If two players output strategy pairs according to Algorithm 1, then*

$$\text{D-Gap}_T, \text{NE-Reg}_T \leq O \left( \min \left\{ V_T, \sqrt{(1 + \log_2 C_T + C_T)T} \right\} \right),$$

where  $V_T = \sum_{t=1}^T \rho(f_t, f_{t-1})$ . Specially, in scenarios where the sequence of payoff functions is stationary, then

$$\text{D-Gap}_T, \text{NE-Reg}_T, \text{Reg}_T^1, \text{Reg}_T^2 = O(1).$$

**Algorithm 1** OPPM with Adaptive Learning Rates

**Require:** Feasible sets  $X$  and  $Y$  satisfy A1. The payoff function  $f_t$  satisfies A2. Regularizers satisfy Property 1. The constant  $\epsilon > 0$  prevents initial learning rates from being infinite

**Initialize:**  $C$

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1: for  $t = 1, 2, \dots$  do
2:   Output  $(x_t, y_t)$ , and then observe a continuous convex-concave payoff function  $f_t$ 
3:    $x'_t = \arg \min_{x \in X} f_t(x, y_t)$ ,  $y'_t = \max_{y \in Y} f_t(x_t, y)$ 
4:   if  $\sum_{\tau=1}^t (\|x'_\tau - x'_{\tau-1}\| + \|y'_\tau - y'_{\tau-1}\|) > C$  then
5:      $C \leftarrow 2C$  ▷ Doubling trick
6:   end if
7:   Update  $\eta_t$  and  $\gamma_t$  according to the setting of Theorem 2
8:   Update  $(x_{t+1}, y_{t+1}) = \arg \min_{x \in X} \max_{y \in Y} f_t(x, y) + B_\phi(x, \tilde{x}_t^\phi) / \eta_t - B_\psi(y, \tilde{y}_t^\psi) / \gamma_t$ 
9: end for

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The worst-case bound delineated in the first statement of Theorem 3 suggests that the OPPM approaches optimality. For sequences of stationary payoff functions, we observe that the duality gap, dynamic Nash equilibrium regret, and two individual regrets all enjoy a sharp bound of  $O(1)$ , which surpasses the  $\tilde{O}(\sqrt{T})$  duality gap and slightly refines the  $\tilde{O}(1)$  dynamic Nash equilibrium regret, both reported by [32], and matches the  $O(1)$  static individual regret in [18].

Typically, OPPM lacks a general solution and necessitates a case-by-case computation tailored to the specific nature of the payoff functions. In essence, each iteration is tasked with solving a strongly-convex-strongly-concave saddle point problem. In certain instances, OPPM can be expressed in a closed form, such as when the regularizers  $\phi$  and  $\psi$  are both square norms, and  $f_t$  is bilinear or quadratic. When a closed form solution is unavailable, numerical techniques can be employed for efficient approximation [1, 9, 19].

#### 4.5 Optimistic OPPM

The performance of the OPPM algorithm is partially contingent upon the stationarity of the sequence of payoff functions. However, taking a highly nonstationary periodic environment as an example, proactive prediction of the upcoming payoff function can significantly improve the performance of online algorithms. This approach, known as ‘optimism’, replaces the stationarity of the payoff function sequence with prediction accuracy. In this subsection, we explore the optimistic variant of OPPM and introduce the following update:

$$\begin{aligned}
 (x_t, y_t) &= \arg \min_{x \in X} \max_{y \in Y} h_t(x, y) + \frac{1}{\eta_t} B_\phi(x, \tilde{x}_t^\phi) - \frac{1}{\gamma_t} B_\psi(y, \tilde{y}_t^\psi), \\
 \tilde{x}_{t+1} &= \arg \min_{x \in X} \eta_t f_t(x, y_t) + B_\phi(x, \tilde{x}_t^\phi), \quad \tilde{y}_{t+1} = \arg \max_{y \in Y} \gamma_t f_t(x_t, y) - B_\psi(y, \tilde{y}_t^\psi),
 \end{aligned}$$

where  $h_t$  is an arbitrary convex-concave predictor. We refer to the above update as the Optimistic OPPM (OptOPPM).

The following theorem provides the individual regret guarantee for OptOPPM.



**Theorem 4 (Individual Regret for OptOPPM).** *Under Assumptions A1 and A2, let regularizers satisfy Property 1, let the predictor  $h_t$  satisfy Assumption A2, and let  $C^1$  and  $C^2$  be the preset upper bounds for  $C_T^1$  and  $C_T^2$ , respectively. If the learning rates of two players follow from*

$$\eta_t = \frac{L_\phi(D_X + C^1)}{\epsilon + \sum_{\tau=1}^{t-1} \delta_\tau^1}, \quad \gamma_t = \frac{L_\psi(D_Y + C^2)}{\epsilon + \sum_{\tau=1}^{t-1} \delta_\tau^2},$$

where the constant  $\epsilon > 0$  prevents initial learning rates from being infinite, and

$$\begin{aligned} 0 \leq \delta_t^1 &= f_t(x_t, y_t) - h_t(x_t, y_t) + h_t(\tilde{x}_{t+1}, y_t) - f_t(\tilde{x}_{t+1}, y_t) \\ &\quad - B_\phi(\tilde{x}_{t+1}, x_t^\phi) / \eta_t, \\ 0 \leq \delta_t^2 &= f_t(x_t, \tilde{y}_{t+1}) - h_t(x_t, \tilde{y}_{t+1}) + h_t(x_t, y_t) - f_t(x_t, y_t) \\ &\quad - B_\psi(\tilde{y}_{t+1}, y_t^\psi) / \gamma_t, \end{aligned}$$

Then the individual regrets of OptOPPM satisfy

$$\text{Reg}_T^i \leq O \left( \min \left\{ \sum_{t=1}^T \rho(f_t, h_t), \sqrt{(1 + C^i)T} \right\} \right), \quad \forall i \in \{1, 2\}.$$

Theorem 4 can be regarded as an optimistic and concave-convex variant of Theorem 5.1 of [6]. Interestingly, the proof of Theorem 4 (refer to Appendix D) reveals that OptOPPM necessitates a joint update for the strategy pair  $(x_t, y_t)$ , whereas the production of auxiliary strategies  $\tilde{x}_{t+1}$  and  $\tilde{y}_{t+1}$  demands separate updates.

Similar to Algorithm 1, the dependence on  $C^1$  and  $C^2$  for the learning rates in OptOPPM can be obviated by employing the doubling trick. Details on the adjustment of learning rates are delineated in Algorithm 2.

Utilizing Lemmas 1 and 2, it can be inferred that Algorithm 2 provides guarantees regarding both the duality gap and the dynamic Nash equilibrium regret.

**Theorem 5 (Performance of Algorithm 2).** *If two players output strategy pairs according to Algorithm 2, then*

$$\text{D-Gap}_T, \text{NE-Reg}_T \leq O \left( \min \left\{ V'_T, \sqrt{(1 + \log_2 C_T + C_T)T} \right\} \right),$$

where  $V'_T = \sum_{t=1}^T \rho(f_t, h_t)$ . Furthermore, If  $f_t$  and  $h_t$  are both bilinear functions, then

$$\text{D-Gap}_T, \text{NE-Reg}_T \leq O \left( \min \left\{ V'_T, \sqrt{(1 + \log_2 C_T + C_T)V''_T} \right\} \right),$$

where  $V''_T = \sum_{t=1}^T \rho^2(f_t, h_t)$ .

Algorithm 2 is designed to surpass Algorithm 1. By selecting  $h_t = f_{t-n}$ , where  $n$  is a positive integer, OptOPPM ensures a sharp metric bound of  $O(1)$  in environments with a periodicity of  $k$ , provided that  $k$  is a factor of  $n$ . Since a sequence of stationary payoff functions is equivalent to an environment with a periodicity of 1, OptOPPM thus

**Algorithm 2** OptOPPM with Adaptive Learning Rates

**Require:** Feasible sets  $X$  and  $Y$  satisfy A1. Both the payoff function  $f_t$  and the predictor  $h_t$  satisfy A2. Regularizers satisfy Property 1. The constant  $\epsilon > 0$  prevents initial learning rates from being infinite

**Initialize:**  $C^1, C^2$

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1: for  $t = 1, 2, \dots$  do
2:   Receive  $h_t$ , update  $\eta_t$  and  $\gamma_t$  according to the setting of Theorem 4
3:   Update  $(x_t, y_t) = \arg \min_{x \in X} \max_{y \in Y} h_t(x, y) + B_\phi(x, \tilde{x}_t^\phi)/\eta_t - B_\psi(y, \tilde{y}_t^\psi)/\gamma_t$ 
4:   Output  $(x_t, y_t)$ , and then observe a continuous convex-concave payoff function  $f_t$ 
5:    $x'_t = \arg \min_{x \in X} f_t(x, y_t)$ ,  $y'_t = \arg \max_{y \in Y} f_t(x_t, y)$ 
6:   if  $\sum_{\tau=1}^t \|x'_\tau - x'_{\tau-1}\| > C^1$  then
7:      $C^1 \leftarrow 2C^1$  ▷ Player 1 doubles his preset
8:   end if
9:   if  $\sum_{\tau=1}^t \|y'_\tau - y'_{\tau-1}\| > C^2$  then
10:     $C^2 \leftarrow 2C^2$  ▷ Player 2 doubles his preset
11:  end if
12:   $\tilde{x}_{t+1} = \arg \min_{x \in X} f_t(x, y_t) + B_\phi(x, \tilde{x}_t^\phi)/\eta_t$ ,  $\tilde{y}_{t+1} = \arg \max_{y \in Y} f_t(x_t, y) - B_\psi(y, \tilde{y}_t^\psi)/\gamma_t$ 
13: end for

```

exhibits superior adaptability compared to OPPM. Additionally, the arbitrariness of the predictor  $h_t$  allows for the development of predictive models across a broader spectrum of possible environments, with precise models yielding tighter metric bounds.

In Algorithm 2, the update rules for  $\tilde{x}_{t+1}$  and  $\tilde{y}_{t+1}$  can be determined numerically as detailed by [28], or transformed into a closed-form expression where feasible.

#### 4.6 Optimistic OPPM with Multiple Predictors

The OptOPPM algorithm is not only near-optimal, but also capable of further reducing the duality gap when the predictor sequence is sufficiently accurate. However, OptOPPM currently supports only a single predictor  $h_t$ . Given the continuously changing and uncertain nature of the environment, multiple prediction models are often under consideration. Should an online algorithm be able to automatically track the best among numerous predictors, and maintain its optimality even when all predictors are underperforming, it would greatly enhance the algorithm's versatility. This section extends OptOPPM to support multiple predictors.

Let's consider that there are  $d$  models, denoted by  $M^k$ ,  $k = 1:d$ . In round  $t$ , each model  $M^k$  provides its predictor  $h_t^k$ . Given that OptOPPM is limited to a single predictor, a logical approach is to compute the weighted average of these  $d$  predictors, formulated as  $h_t = \sum_{k=1}^d \omega_t^k h_t^k$ , with the weights  $\omega_t$  being derived from a specific implementation of "learning from expert advice".

Consider employing the "clipped" variant of the Hedge algorithm to generate the weight coefficients. The clipped Hedge can be formalized as follows:

$$\omega_{t+1} = \arg \min_{\omega \in \Delta_d^\alpha} \theta_t \langle L_t, \omega \rangle + \text{KL}(\omega, \omega_t),$$

or equivalently,

$$\omega_{t+1} = \arg \min_{\omega \in \Delta_d^\alpha} \left\langle \ln \frac{\omega}{\omega_t \cdot \exp(-\theta_t L_t)}, \omega \right\rangle, \quad (3)$$

where  $\Delta_d^\alpha = \{w \in \mathbb{R}_+^d \mid \|w\|_1 = 1, w^i \geq \alpha/d, \forall i \in 1:d\}$ ,  $L_t$  is a linearized loss,  $\theta_t > 0$  represents the learning rate, and KL represents the Kullback-Leibler divergence.

Prior to delving into the reasons for the effectiveness of clipped Hedge, it is pertinent to first consider the static regret guarantee it provides.

**Lemma 3 (Static Regret for Clipped Hedge).** *Assume that  $L_t \geq 0$ ,  $\max_{t \in 1:T} \|L_t\|_\infty = L_\infty$  and  $\alpha = d/T < 1$ . If the learning rate follow from  $\theta_t = (\ln T)/(\epsilon + \sum_{\tau=1}^{t-1} \sigma_\tau)$ , where the constant  $\epsilon > 0$  prevent  $\theta_1$  from being infinite, and  $\sigma_t = \langle L_t, \omega_t - \omega_{t+1} \rangle - \text{KL}(\omega_{t+1}, \omega_t)/\theta_t$ , Then the clipped Hedge enjoys the following static regret:*

$$\text{Reg}_T u \leq 2 \sqrt{(1 + \ln T) L_\infty \sum_{t=1}^T \langle L_t, u \rangle} + O(\ln T), \quad \forall u \in \Delta_d^0.$$

*Remark 1.* Lemma 3 is the static version of Corollary B.0.1 of [6]. One may obtain the desired static regret bound by setting the path length to null. The clipping operation is designed to meet the requirements of adaptive learning rates.

Next, we delve into the analysis that the clipped Hedge indeed provides multi-predictor support for OptOPPM. Note that in the proof of Theorem 4, the relaxed form inequalities  $\delta_t^1, \delta_t^2 \leq 2\rho(f_t, h_t)$  are employed. In actuality, by selecting  $L_t$  according to Line 5 of Algorithm 3, we can deduce that  $\delta_t^1, \delta_t^2 \leq 2 \langle L_t, \omega_t \rangle$ . Consequently, the benign bound  $V_T'$  mentioned in Theorem 5 can be substituted with  $\sum_{t=1}^T \langle L_t, \omega_t \rangle$ . Drawing from Lemma 3, we have that

$$\begin{aligned} \sum_{t=1}^T \langle L_t, \omega_t \rangle &\leq \sum_{t=1}^T \langle L_t, 1_k \rangle + O(\sqrt{\ln T}) \sqrt{\sum_{t=1}^T \langle L_t, 1_k \rangle} + O(\ln T) \\ &\leq 2 \sum_{t=1}^T \langle L_t, 1_k \rangle + O(\ln T) \leq 2 \sum_{t=1}^T \rho(f_t, h_t^k) + O(\ln T), \quad \forall k \in 1:d, \end{aligned}$$

where  $1_k$  denotes the  $d$ -dimensional one-hot vector with the  $k$ -th element being 1. Given the arbitrariness of  $k$ , we obtain

$$\sum_{t=1}^T \langle L_t, \omega_t \rangle \leq 2 \min_{k \in 1:d} \sum_{t=1}^T \rho(f_t, h_t^k) + O(\ln T),$$

which implies that the benign bound is almost determined by the most accurate predictor.

The analysis presented above leads to the development of Algorithm 3, which is supported by the performance guarantees outlined in Theorem 6.

*Property 2.* All prediction errors are bounded, that is,  $\exists L_\infty < +\infty, \forall t, \forall k \in 1:d$ , we have that  $|\rho(f_t, h_t^k)| \leq L_\infty$ .

**Algorithm 3** Multi-Predictor Support Subroutine for OptOPPM**Require:** All predictors satisfy Assumption A2 and Property 2**Initialize:**  $T$ 


---

```

1: for  $t = 1, 2, \dots$  do
2:   Receive a group of predictors  $h_t^1, h_t^2, \dots, h_t^d$ 
3:   Provide predictor  $h_t = \sum_{k=1}^d \omega_t^k h_t^k$  to OptOPPM
4:   Obtain  $f_t, x_t, \tilde{x}_{t+1}, y_t$ , and  $\tilde{y}_{t+1}$  from Algorithm 2
5:    $L_t = \left[ \max_{k \in 1:d} \begin{Bmatrix} |f_t(x_t, y_t) - h_t^k(x_t, y_t)|, \\ |f_t(\tilde{x}_{t+1}, y_t) - h_t^k(\tilde{x}_{t+1}, y_t)|, \\ |f_t(x_t, \tilde{y}_{t+1}) - h_t^k(x_t, \tilde{y}_{t+1})| \end{Bmatrix} \right]_{k \in 1:d}$ 
6:   if  $t > T$  then
7:      $T \leftarrow 2T$  ▷ Doubling trick
8:   end if
9:    $\theta_t = (\ln T) / (\epsilon + \sum_{\tau=1}^{t-1} \sigma_\tau)$ 
10:   $\omega_{t+1} = \arg \min_{\omega \in \Delta_d^{d/T}} \theta_t \langle L_t, \omega \rangle + \text{KL}(\omega, \omega_t)$ 
11:   $\sigma_t = \langle L_t, \omega_t - \omega_{t+1} \rangle - \text{KL}(\omega_{t+1}, \omega_t) / \theta_t$ 
12: end for

```

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**Theorem 6 (Performance of Algorithm 3).** *Let there be  $d$  predictors satisfying Property 2. If two players output strategy pairs according to Algorithm 2 and integrate the  $d$  predictors following Algorithm 3, then*

$$\text{D-Gap}_T, \text{NE-Reg}_T \leq \tilde{O} \left( \min \left\{ V_T^1, \dots, V_T^d, \sqrt{(1 + C_T)T} \right\} \right),$$

where  $V_T^k = \sum_{t=1}^T \rho(f_t, h_t^k)$  represents the cumulative error of the  $k$ -th predictor.

Algorithm 3 is designed to extend multi-predictor support for OptOPPM. We refer to the integration of Algorithms 2 and 3 as *OptOPPM with multiple predictors*. This algorithm ensures that, in instances where any predictor yields an accurate forecast, there will be an accelerated convergence of the players' strategy pairs toward saddle points. Conversely, in instances where all predictors fail, the algorithm reverts to the performance applicable in the worst-case scenario.

For the clipped Hedge iteration (as shown in Equation (3)), an efficient solution is attainable by minor adjustments to the algorithm depicted in Figure 3 of [16]. The modified algorithm is elaborated in Algorithm 4, wherein the primary alteration is the removal of the constraint  $\|W\|_1 = 1$ .

## 5 Experiments

This section validates the effectiveness of our algorithms through experimentation and demonstrates the cancellation phenomenon that occurs within the absolute value of dynamic Nash equilibrium regret. Section 5.1 describes the experimental setup, and Section 5.2 presents the experimental results.

### 5.1 Setup

Consider the following synthesis problem: in round  $t$ , Players 1 and 2 jointly select a strategy pair  $(x_t, y_t) \in X \times Y$ , where  $X = [-4, 4]$  and  $Y = [-4, 4]$ , and then the environment feeds back a convex-concave payoff function  $f_t$ :

$$f_t(x, y) = \frac{1}{2} (x - x_t^*)^2 - \frac{1}{2} (y - y_t^*)^2 + (x - x_t^*) (y - y_t^*),$$

where  $(x_t^*, y_t^*) \in X \times Y$  is the saddle point of  $f_t$ . It is evident that Assumptions A1 and A2 are satisfied. To determine the payoff function  $f_t$ , it suffices to fix the saddle point  $(x_t^*, y_t^*)$ . We set up four cases, which are listed in Table 1. Case I is characterized by an asymptotic stability, where the saddle points of the payoff functions exhibit a decelerating trend over time. Cases II and III display traits of periodic oscillation; the saddle points under Case II vacillate between two branches, whereas those under Case III alternate among three branches. It is noteworthy that with the progression of time  $t$ , the incremental values of  $z_1$  and  $z_2$  tend to decelerate. Thus, Cases I, II, and III each correspond to benign environments where  $\rho(f_t, f_{t-1}) \rightarrow 0$ ,  $\rho(f_t, f_{t-2}) \rightarrow 0$ , and  $\rho(f_t, f_{t-3}) \rightarrow 0$ , respectively. Case IV is indicative of an adversarial setting. As shown in its figure, upon the players' selection of the strategy pair  $(x_t, y_t)$ , the environment engenders the saddle point  $(x_t^*, y_t^*)$  by initially rotating the strategy pair by  $8\pi/9$ , followed by its projection onto a circle with a radius of  $\sqrt{2}$ . This setting results in the absence of any algorithm that approximates saddle points.

Next, we proceed to instantiate three algorithms: OPPM, OptOPPM, and OptOPPM with multiple predictors. Let  $\phi(x) = x^2/2$  and  $\psi(y) = y^2/2$ . Consequently, both  $B_\phi$  and  $B_\psi$  are bounded and exhibit Lipschitz continuity with respect to their first variables. All algorithms are initialized with random initial values, as previous analysis has shown that all metric bounds are invariant to initial conditions. To circumvent an infinitely large initial learning rate, we set  $\epsilon = 0.1$ . For the OptOPPM algorithm, we employ the predictor  $h_t = f_{t-4}$ , enabling it to attain a sharp metric bound of  $O(1)$  in environments that are either stationary or exhibit periodicity with cycles of 2 or 4. For the OptOPPM with multiple predictors, we configure three predictors:  $h_t^1 = f_{t-4}$ ,  $h_t^2 = f_{t-5}$ , and  $h_t^3 = f_{t-6}$ , allowing it to enjoy a sharp metric bound of  $\tilde{O}(1)$  in environments that are stationary or have periodicity with cycles of 2, 3, 4, 5, or 6.

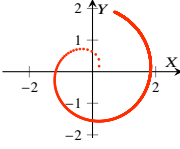
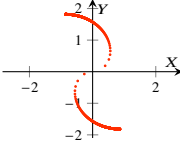
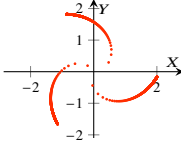
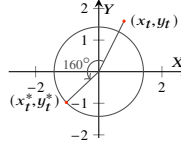
Finally, we adopt the algorithm delineated by [32] as the foundational benchmark for comparison.

### 5.2 Results

We run the repeated game for  $10^5$  rounds and record the time averaged duality gap  $\frac{1}{T}\text{D-Gap}_T$  and the time averaged dynamic Nash equilibrium regret  $\frac{1}{T}\text{NE-Reg}_T$ . The trajectory of the average duality gap approaches zero, indicating that the players' decisions in most rounds are close to saddle points.

All experimental outcomes align with the theoretical expectations. In the benign environment delineated by Case I, the average duality gap trajectories of all algorithms demonstrate a trend towards convergence (refer to Figure 1a), and the average dynamic

Table 1: Four Settings for Synthesis Experiments. In this table,  $z_1(t) = \ln(1+t)$  is a logarithmic growth function of  $t$ , and  $z_2(t) = \ln \ln(e+t)$  is a log-logarithmic growth function of  $t$ . We represent the saddle point  $(x_t^*, y_t^*)$  in the complex form  $x_t^* + iy_t^*$ , where  $i$  is the imaginary unit and satisfying the equation  $i^2 = -1$ .

Cases	I	II	III	IV
$x_t^* + iy_t^*$	$z_2(t)e^{iz_1(t)}$	$z_2(t)e^{i\pi t + iz_2(t)}$	$z_2(t)e^{i\frac{2\pi}{3}t + iz_2(t)}$	$\sqrt{2}e^{i(\frac{8\pi}{9} + \arg(x_t + iy_t))}$
Trajectories				
Remark	$\rho(f_t, f_{t-1}) \rightarrow 0$	$\rho(f_t, f_{t-2}) \rightarrow 0$	$\rho(f_t, f_{t-3}) \rightarrow 0$	Adversarial

Nash equilibrium regret trajectories follow a similar convergence pattern (refer to Figure 1b). In the periodically oscillatory environment specified by Case II, the average duality gap trajectories of OptOPPM and OptOPPM with multiple predictors exhibit a gradual convergence (refer to Figure 1c), with the average dynamic Nash equilibrium regret trajectories also converging (refer to Figure 1d). In the oscillatory scenario established by Case III, the average duality gap trajectory of OptOPPM with multiple predictors reaches convergence (refer to Figure 1e), and the average dynamic Nash equilibrium regret trajectory likewise converges (refer to Figure 1f). Notably, the average dynamic Nash equilibrium regret trajectory of OPPM converges amidst pronounced fluctuations (refer to Figure 1f), signifying an internal cancellation phenomenon within the absolute value of NE-Reg. In the adversarial setting defined by Case IV, no algorithm can approximate the saddle points, resulting in the average duality gap trajectories of all algorithms failing to converge (refer to Figure 1g). In contrast, the average dynamic Nash equilibrium regret trajectories exhibit intense fluctuations in convergence (refer to Figure 1h), with the severe oscillations indicating an internal cancellation phenomenon within the absolute value of NE-Reg.

The experimental results demonstrate that OPPM and its variants surpass the algorithm delineated by [32] in terms of duality gap performance. Notably, OptOPPM with multiple predictors consistently and effectively approximates saddle points in Cases I, II, and III.

Given the insights garnered from both theoretical analysis and empirical results, it seems prudent to reconsider the reliance on dynamic Nash equilibrium regret as a performance metric.

## 6 Conclusion

This study addresses the online saddle point problem. In response to the nonstationary nature of online games, we introduce three proximal point method adaptations: the Online Proximal Point Method (OPPM), the Optimistic OPPM (OptOPPM), and the

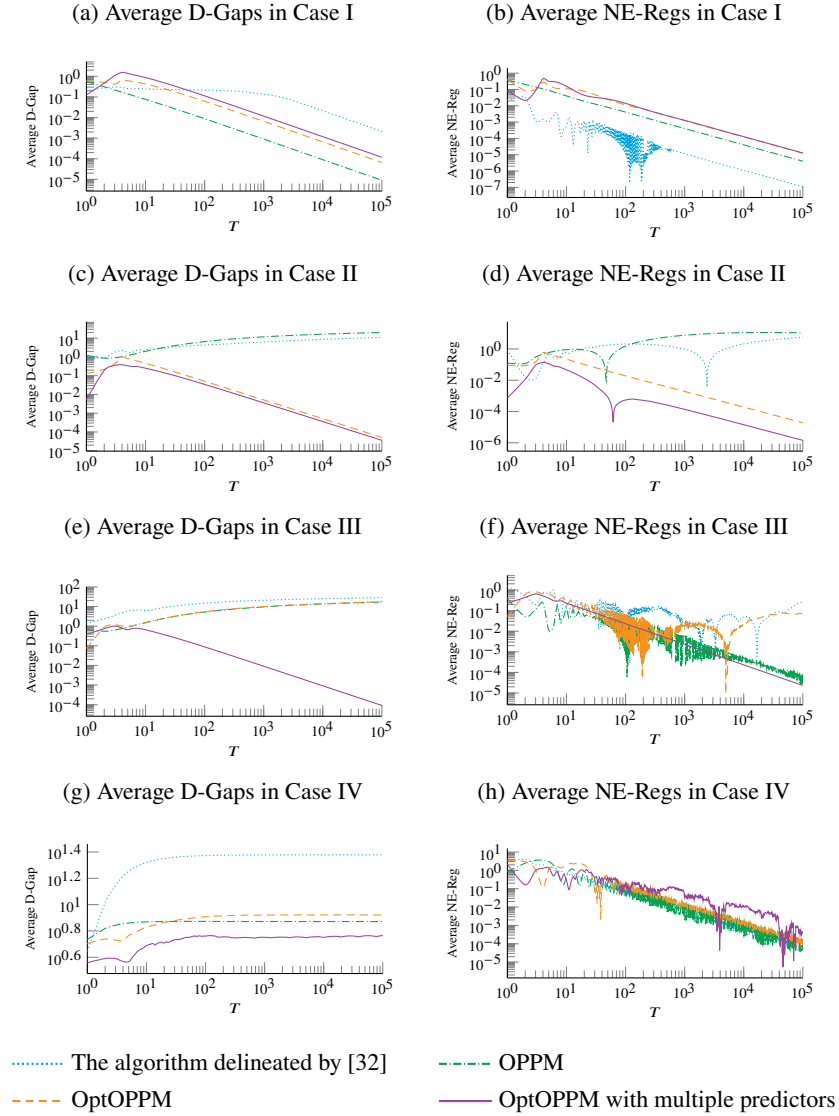


Fig. 1: Average D-Gaps and Average NE-Regs of Algorithms

OptOPPM with multiple predictors. These variations are crafted to secure upper bounds on the duality gap and dynamic Nash equilibrium regret, ensuring near-optimal performance in relation to the duality gap. Particularly, under favorable scenarios such as stationary payoff function sequences, these algorithms consistently achieve near-constant metric bounds. Furthermore, this study highlights potential reliability concerns when employing dynamic Nash equilibrium regret as a performance metric. We also conduct experiments with synthetic data to substantiate the effectiveness of our algorithms.

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## Technical Appendix

### A Proof of Theorem 1

*Proof (Proof of Theorem 1).* Let  $\mathcal{F}$  be the set of all convex-concave functions satisfying Assumption A2, and let  $\mathcal{L}_X^{G_X} = \{\ell \mid \ell(x) + 0y \in \mathcal{F}\}$ , let  $\mathcal{L}_Y^{G_Y} = \{\ell \mid 0x - \ell(y) \in \mathcal{F}\}$ . Formally, we need to prove that

$$\exists f_{1:T} \in \mathcal{F}, \quad \text{such that} \quad \sum_{t=1}^T \max_{y \in Y} f_t(x_t, y) - \sum_{t=1}^T \min_{x \in X} f_t(x, y_t) \geq \Omega(\sqrt{(1 + C_T)T}).$$

Indeed, we may choose  $f_t(x, y) = \alpha_t(x) - \beta_t(y) \in \mathcal{F}$ , where  $\alpha_t$  and  $\beta_t$  are both convex functions.  $\forall C \in [0, T(D_X + D_Y)]$  and  $\forall P \in [0, C]$ ,  $\exists \alpha_{1:T}$  and  $\exists \beta_{1:T}$ , such that

$$\begin{aligned} & \sum_{t=1}^T \max_{y \in Y} f_t(x_t, y) - \sum_{t=1}^T \min_{x \in X} f_t(x, y_t) \\ &= \max_{\forall u_{1:T} \in X, \forall v_{1:T} \in Y} \sum_{t=1}^T (f_t(x_t, v_t) - f_t(u_t, y_t)) \\ &\geq \max_{\sum_{t=1}^T \|u_t - u_{t-1}\| \leq P, \sum_{t=1}^T \|v_t - v_{t-1}\| \leq C - P} \sum_{t=1}^T (f_t(x_t, v_t) - f_t(u_t, y_t)) \\ &= \max_{\sum_{t=1}^T \|u_t - u_{t-1}\| \leq P} \sum_{t=1}^T (\alpha_t(x_t) - \alpha_t(u_t)) + \max_{\sum_{t=1}^T \|v_t - v_{t-1}\| \leq C - P} \sum_{t=1}^T (\beta_t(y_t) - \beta_t(v_t)) \\ &\geq \Omega(\sqrt{(1 + P)T}) + \Omega(\sqrt{(1 + C - P)T}) \\ &= \Omega(\sqrt{(1 + C)T}), \end{aligned} \tag{4}$$

where the last “ $\leq$ ” follows from Lemma 4. Note that Equation (4) holds for arbitrary  $C \in [0, T(D_X + D_Y)]$ . Choosing  $C = C_T$  yields the desired result.

**Lemma 4 (Theorem 2 of [31]).** *In the context of Online Convex Optimization, let the feasible set  $X$  be compact and convex, and let  $\mathcal{L}_X^G$  be the set of all convex loss functions defined on  $X$  with subgradients bounded by  $G$ . Regardless of the strategy adopted by the player, there always exists a comparator sequence  $u_{1:T}$  satisfying  $\sum_{t=1}^T \|u_t - u_{t-1}\| \leq P$ , and a sequence of loss functions  $\ell_{1:T} \in \mathcal{L}_X^G$ , ensuring that the regret is not less than  $\Omega(\sqrt{(1 + P)T})$ .*

### B Proof of Theorem 2

*Proof (Proof of Theorem 2).* The first-order optimality condition of OPPM implies that

$$\begin{aligned} & \exists \nabla_x f_t(x_{t+1}, y_{t+1}), \quad \forall x' \in X, \quad \langle \eta_t \nabla_x f_t(x_{t+1}, y_{t+1}) + x_{t+1}^\phi - x_t^\phi, x_{t+1} - x' \rangle \leq 0, \\ & \exists \nabla_y(-f_t)(x_{t+1}, y_{t+1}), \quad \forall y' \in Y, \quad \langle \gamma_t \nabla_y(-f_t)(x_{t+1}, y_{t+1}) + y_{t+1}^\psi - y_t^\psi, y_{t+1} - y' \rangle \leq 0. \end{aligned}$$

Let's take Player 1 as an example. The identity transformation on the instantaneous individual regret is as follows:

$$\begin{aligned} f_t(x_t, y_t) - f_t(x'_t, y_t) &= f_t(x_t, y_t) - f_t(x_{t+1}, y_{t+1}) + f_t(x'_{t+1}, y_{t+1}) - f_t(x'_t, y_t) \\ &\quad + \underbrace{f_t(x_{t+1}, y_{t+1}) - f_t(x'_{t+1}, y_{t+1})}_{(5a)}. \end{aligned} \quad (5)$$

Applying convexity and first-order optimality condition, we get

$$\begin{aligned} \text{Equation (5a)} &\leq \langle \nabla_x f_t(x_{t+1}, y_{t+1}), x_{t+1} - x'_{t+1} \rangle \leq \frac{1}{\eta_t} \langle x_t^\phi - x_{t+1}^\phi, x_{t+1} - x'_{t+1} \rangle \\ &= \frac{1}{\eta_t} \left[ B_\phi(x'_{t+1}, x_t^\phi) - B_\phi(x'_{t+1}, x_{t+1}^\phi) - B_\phi(x_{t+1}, x_t^\phi) \right]. \end{aligned}$$

Summing Equation (5) over time yields

$$\begin{aligned} \text{Reg}_T^1 &\leq \underbrace{\sum_{t=1}^T \frac{1}{\eta_t} \left( B_\phi(x'_{t+1}, x_t^\phi) - B_\phi(x'_{t+1}, x_{t+1}^\phi) \right)}_{(6a)} \\ &\quad + \underbrace{\sum_{t=1}^T \left( f_t(x_t, y_t) - f_t(x_{t+1}, y_{t+1}) + f_t(x'_{t+1}, y_{t+1}) - f_t(x'_t, y_t) \right)}_{(6b)}. \end{aligned} \quad (6)$$

Since the learning rate  $\eta_t$  does not increase,  $B_\phi$  is upper bounded by  $L_\phi D_X$  and is  $L_\phi$ -Lipschitz for the first variable, we have that

$$\begin{aligned} &\text{Equation (6a)} \\ &\leq \sum_{t=1}^T \frac{1}{\eta_t} \left( B_\phi(x'_{t+1}, x_t^\phi) - B_\phi(x'_t, x_t^\phi) \right) + \frac{B_\phi(x'_1, x_1^\phi)}{\eta_0} + \sum_{t=1}^T \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) B_\phi(x'_t, x_t^\phi) \\ &\leq \frac{L_\phi D_X}{\eta_T} + \sum_{t=1}^T \frac{L_\phi}{\eta_t} \|x'_{t+1} - x'_t\| \leq \frac{2L_\phi D_X}{\eta_T} + \sum_{t=1}^T \frac{L_\phi}{\eta_t} \|x'_t - x'_{t-1}\|. \end{aligned}$$

Let  $\Sigma_T^1 = (\text{Equation (6b)})_+$ , then Equation (6) can be rearranged as follows:

$$\text{Reg}_T^1 \leq L_\phi \left( 2 \frac{D_X}{\eta_T} + \sum_{t=1}^T \frac{1}{\eta_t} \|x'_t - x'_{t-1}\| \right) + \Sigma_T^1.$$

Likewise,

$$\text{Reg}_T^2 \leq L_\psi \left( 2 \frac{D_Y}{\gamma_T} + \sum_{t=1}^T \frac{1}{\gamma_t} \|y'_t - y'_{t-1}\| \right) + \Sigma_T^2,$$

where

$$\Sigma_T^2 = \left( \sum_{t=1}^T \left( f_t(x_t, y'_t) - f_t(x_{t+1}, y'_{t+1}) + f_t(x_{t+1}, y_{t+1}) - f_t(x_t, y_t) \right) \right)_+.$$

Let  $L = \max\{L_\phi, L_\psi\}$ ,  $D = \max\{D_X, D_Y\}$ ,  $C_T = \sum_{t=1}^T (\|x'_t - x'_{t-1}\| + \|y'_t - y'_{t-1}\|)$ , and let learning rates satisfy  $\eta_t = \gamma_t$ . Consequently, the individual regrets are subject to the following public upper bound:

$$\text{Reg}_T^1, \text{Reg}_T^2 \leq L \frac{2D + C_T}{\eta_T} + \max\{\Sigma_T^1, \Sigma_T^2\}.$$

Let  $\Sigma_T = \max\{\Sigma_T^1, \Sigma_T^2\}$ ,  $\Delta_t = (\Sigma_t - \max_{\tau \in 1:t-1} \Sigma_\tau)_+$ . We claim that  $\sum_{\tau=1}^t \Delta_\tau = \max_{\tau \in 1:t} \Sigma_\tau$ . This claim can be proved by induction. It is obvious that  $\Delta_1 = \Sigma_1$ . Now we assume the claim holds for  $t-1$  and prove it for  $t$ :

$$\begin{aligned} \sum_{\tau=1}^t \Delta_\tau &= \Delta_t + \sum_{\tau=1}^{t-1} \Delta_\tau = \left( \Sigma_t - \max_{\tau \in 1:t-1} \Sigma_\tau \right)_+ + \max_{\tau \in 1:t-1} \Sigma_\tau \\ &= \begin{cases} \Sigma_t & \Sigma_t \geq \max_{\tau \in 1:t-1} \Sigma_\tau \\ \max_{\tau \in 1:t-1} \Sigma_\tau & \Sigma_t < \max_{\tau \in 1:t-1} \Sigma_\tau \end{cases} = \max_{\tau \in 1:t} \Sigma_\tau. \end{aligned}$$

Let  $C$  be a preset upper bound of  $C_T$ , then applying the prescribed learning rate yields

$$\text{Reg}_T^1, \text{Reg}_T^2 \leq \epsilon + 2 \sum_{t=1}^T \Delta_t = \epsilon + 2 \max_{t \in 1:T} \Sigma_t. \quad (7)$$

Next, we estimate the upper bound of the r.h.s. of the above inequality in two ways. The first one:

$$\max_{t \in 1:T} \Sigma_t \leq O(1) + 2 \sum_{t=1}^T \rho(f_t, f_{t-1}). \quad (8)$$

Indeed,

$$\begin{aligned} \Sigma_t^1 &= \left( \sum_{\tau=1}^t \left( f_\tau(x_\tau, y_\tau) - f_\tau(x_{\tau+1}, y_{\tau+1}) + f_\tau(x'_{\tau+1}, y_{\tau+1}) - f_\tau(x'_\tau, y_\tau) \right) \right)_+ \\ &\leq \left| f_0(x_1, y_1) - f_0(x'_1, y_1) - f_t(x_{t+1}, y_{t+1}) + f_t(x'_{t+1}, y_{t+1}) \right| \\ &\quad + \sum_{\tau=1}^t \left| f_\tau(x_\tau, y_\tau) - f_{\tau-1}(x_\tau, y_\tau) \right| + \sum_{\tau=1}^t \left| f_\tau(x'_\tau, y_\tau) - f_{\tau-1}(x'_\tau, y_\tau) \right| \\ &\leq 2(D_X G_X + D_Y G_Y) + 2 \sum_{\tau=1}^t \rho(f_\tau, f_{\tau-1}), \end{aligned}$$

where the last “ $\leq$ ” follows from Lemma 5. Likewise,

$$\Sigma_t^2 \leq 2(D_X G_X + D_Y G_Y) + 2 \sum_{\tau=1}^t \rho(f_\tau, f_{\tau-1}).$$

So Equation (8) holds. The other one:

$$\sum_{t=1}^T \Delta_t \leq O\left(\sqrt{(1+C)T}\right). \quad (9)$$

The derivation is as follows. According to Lemma 6,  $\|x_t - x_{t+1}\| \leq \min\{D_X, \eta_t G_X\}$ , and  $\|y_t - y_{t+1}\| \leq \min\{D_Y, \eta_t G_Y\}$ . Let  $G = \max\{G_X, G_Y\}$ . Thus we have that

$$\begin{aligned}
(\Sigma_t^1 - \Sigma_{t-1}^1)_+ &\leq (f_t(x_t, y_t) - f_t(x_{t+1}, y_{t+1}) + f_t(x'_{t+1}, y_{t+1}) - f_t(x'_t, y_t))_+ \\
&\leq |f_t(x_t, y_t) - f_t(x_t, y_{t+1})| + |f_t(x_t, y_{t+1}) - f_t(x_{t+1}, y_{t+1})| \\
&\quad + |f_t(x'_{t+1}, y_{t+1}) - f_t(x'_{t+1}, y_t)| + |f_t(x'_{t+1}, y_t) - f_t(x'_t, y_t)| \\
&\leq G_Y \|y_t - y_{t+1}\| + G_X \|x_t - x_{t+1}\| + G_Y \|y_t - y_{t+1}\| + G_X \|x'_t - x'_{t+1}\| \\
&\leq \min\{D_X G_X + 2D_Y G_Y, \eta_t (G_X^2 + 2G_Y^2)\} + G_X \|x'_t - x'_{t+1}\| \\
&\leq \min\{3DG, 3\eta_t G^2\} + G (\|x'_t - x'_{t+1}\| + \|y'_t - y'_{t+1}\|).
\end{aligned}$$

Likewise,

$$(\Sigma_t^2 - \Sigma_{t-1}^2)_+ \leq \min\{3DG, 3\eta_t G^2\} + G (\|x'_t - x'_{t+1}\| + \|y'_t - y'_{t+1}\|).$$

Note that  $\Sigma_t - \Sigma_{t-1} = \max\{\Sigma_t^1 - \Sigma_{t-1}^1, \Sigma_t^2 - \Sigma_{t-1}^2\} \leq \max\{\Sigma_t^1 - \Sigma_{t-1}^1, \Sigma_t^2 - \Sigma_{t-1}^2\}$ , so we get

$$\begin{aligned}
\Delta_t &= \left(\Sigma_t - \max_{\tau \in 1:t-1} \Sigma_\tau\right)_+ \leq (\Sigma_t - \Sigma_{t-1})_+ \\
&\leq \min\{3DG, 3\eta_t G^2\} + G (\|x'_t - x'_{t+1}\| + \|y'_t - y'_{t+1}\|).
\end{aligned}$$

Let  $\xi_t = (\Delta_t - G(\|x'_t - x'_{t+1}\| + \|y'_t - y'_{t+1}\|))_+$ , then  $\xi_t \leq \min\{3DG, 3\eta_t G^2\}$ , and thus,

$$\begin{aligned}
\left(\sum_{t=1}^{T-1} \xi_t\right)^2 &= \sum_{t=1}^{T-1} \xi_t^2 + 2 \sum_{t=1}^{T-1} \xi_t \sum_{\tau=1}^{t-2} \xi_\tau \leq \sum_{t=1}^{T-1} \xi_t^2 + 2 \sum_{t=1}^{T-1} \xi_t \sum_{\tau=1}^{t-2} \Delta_\tau \\
&= \sum_{t=1}^{T-1} \xi_t^2 + 2 \sum_{t=1}^{T-1} \xi_t \left(\frac{L(2D+C)}{\eta_t} - \epsilon\right) \leq \sum_{t=1}^{T-1} 9D^2 G^2 + \sum_{t=1}^{T-1} 6G^2 L(2D+C),
\end{aligned}$$

where the last “ $\leq$ ” uses the first and second terms in the minimum of the bound for  $\xi_t$  in turn. Now we get

$$\sum_{t=1}^T \Delta_t \leq O(C_T) + \sum_{t=1}^T \xi_t \leq O(C_T) + O\left(\sqrt{(1+C)T}\right) \leq O\left(\sqrt{(1+C)T}\right).$$

So Equation (9) holds. Substituting Equations (8) and (9) into Equation (7) gives the following conclusion:

$$\text{Reg}_T^1, \text{Reg}_T^2 \leq O\left(\min\left\{\sum_{t=1}^T \rho(f_t, f_{t-1}), \sqrt{(1+C)T}\right\}\right).$$

**Lemma 5.** *Assumptions A1 and A2 guarantee the finiteness of the instantaneous duality gap.*

*Proof (Proof of Lemma 5).*  $\forall x, x' \in X, \forall y, y' \in Y$ ,

$$\begin{aligned} \langle \partial_x f_t(x', y), x - x' \rangle &\leq f_t(x, y) - f_t(x', y) \leq \langle \partial_x f_t(x, y), x - x' \rangle, \\ \langle \partial_y(-f_t)(x, y'), y - y' \rangle &\leq f_t(x, y') - f_t(x, y) \leq \langle \partial_y(-f_t)(x, y), y - y' \rangle, \end{aligned}$$

which implies that  $|f_t(x, y) - f_t(x', y)| \leq D_X G_X$ ,  $|f_t(x, y') - f_t(x, y)| \leq D_Y G_Y$ , and  $|f_t(x, y') - f_t(x', y)| \leq |f_t(x, y') - f_t(x, y)| + |f_t(x, y) - f_t(x', y)| \leq D_X G_X + D_Y G_Y$ .

**Lemma 6.** *Under Assumption A2 and Property 1, OPKM guarantees that  $\|x_t - x_{t+1}\| \leq \eta_t G_X$  and  $\|y_t - y_{t+1}\| \leq \gamma_t G_Y$ .*

*Proof (Proof of Lemma 6).* Let  $F_t(x, y) = f_t(x, y) + B_\phi(x, x_t^\phi)/\eta_t - B_\psi(y, y_t^\psi)/\gamma_t$ . Let  $\Phi = F_t(\cdot, y_{t+1})$ . Note that  $\Phi$  is  $\eta_t^{-1}$ -strongly convex, so we have that

$$B_\phi(x_{t+1}, x_t^\phi) \geq \frac{1}{2\eta_t} \|x_t - x_{t+1}\|^2, \quad B_\phi(x_t, x_{t+1}^\phi) \geq \frac{1}{2\eta_t} \|x_t - x_{t+1}\|^2.$$

Choosing  $x_{t+1}^\phi = 0$ , and adding the above two inequalities yields

$$\frac{1}{\eta_t} \|x_t - x_{t+1}\|^2 \leq \langle x_t^\phi - x_{t+1}^\phi, x_t - x_{t+1} \rangle \leq \|x_t^\phi\| \|x_t - x_{t+1}\|.$$

Note that  $\|x_t^\phi\| = \|\nabla_x f_t(x_t, y_{t+1})\| \leq G_X$ . Thus we have that  $\|x_t - x_{t+1}\| \leq \eta_t G_X$ . Likewise,  $\|y_t - y_{t+1}\| \leq \gamma_t G_Y$ .

### C Proof of Theorem 3

*Proof (Proof of Theorem 3).* Consider the Algorithm 1 initially at stage 0. Each time the value  $C$  is doubled, the algorithm advances to the next stage. We use  $C_n$  to represent the value of  $C$  at stage  $n$ , and let  $S_n$  be the set of indices for all rounds in the  $n$ -th stage. Assume that the game pauses after the  $T$ -th round is completed, at which point the OPKM is at stage  $s$ . Applying Theorem 2 yields

$$\begin{aligned} \text{Reg}_T^i &= \sum_{n=1}^s \text{Reg}_{S_n}^i \leq \sum_{n=1}^s O \left( \min \left\{ \sum_{t \in S_n} \rho(f_t, f_{t-1}), \sqrt{(1 + 2^n C_0) |S_n|} \right\} \right) \\ &\leq O \left( \min \left\{ \sum_{t=1}^T \rho(f_t, f_{t-1}), \sqrt{s + \sum_{n=1}^s 2^n C_0 \sqrt{\sum_{n=1}^s |S_n|}} \right\} \right), \quad \forall i = 1, 2. \end{aligned}$$

Note that  $2^{s-1} C_0 = C_{s-1} < C_T \leq C_s = 2^s C_0$ , and  $T = \sum_{n=1}^s |S_n|$ . We have that

$$\text{Reg}_T^i \leq O \left( \min \left\{ \sum_{t=1}^T \rho(f_t, f_{t-1}), \sqrt{(1 + \log_2 C_T + 4C_T) T} \right\} \right), \quad \forall i = 1, 2.$$

Applying Lemmas 1 and 2 yields the conclusion to be proved.

## D Proof of Theorem 4

*Proof (Proof of Theorem 4).* The first-order optimality condition of OptOPPM implies that

$$\begin{aligned}
& \exists \nabla_x h_t(x_t, y_t), \quad \forall x' \in X, \quad \langle \eta_t \nabla_x h_t(x_t, y_t) + x_t^\phi - \tilde{x}_t^\phi, x_t - x' \rangle \leq 0, \\
& \exists \nabla_y(-h_t)(x_t, y_t), \quad \forall y' \in Y, \quad \langle \gamma_t \nabla_y(-h_t)(x_t, y_t) + y_t^\psi - \tilde{y}_t^\psi, y_t - y' \rangle \leq 0, \\
& \exists \nabla_x f_t(\tilde{x}_{t+1}, y_t), \quad \forall x' \in X, \quad \langle \eta_t \nabla_x f_t(\tilde{x}_{t+1}, y_t) + \tilde{x}_{t+1}^\phi - \tilde{x}_t^\phi, \tilde{x}_{t+1} - x' \rangle \leq 0, \\
& \exists \nabla_y(-f_t)(x_t, \tilde{y}_{t+1}), \quad \forall y' \in Y, \quad \langle \gamma_t \nabla_y(-f_t)(x_t, \tilde{y}_{t+1}) + \tilde{y}_{t+1}^\psi - \tilde{y}_t^\psi, \tilde{y}_{t+1} - y' \rangle \leq 0.
\end{aligned}$$

Let's take Player 1 as an example. We first perform identity transformation on the instantaneous individual regret:

$$\begin{aligned}
f_t(x_t, y_t) - f_t(x'_t, y_t) &= \underbrace{f_t(x_t, y_t) - h_t(x_t, y_t) + h_t(\tilde{x}_{t+1}, y_t) - f_t(\tilde{x}_{t+1}, y_t)}_{(10a)} \\
&\quad + \underbrace{h_t(x_t, y_t) - h_t(\tilde{x}_{t+1}, y_t) + f_t(\tilde{x}_{t+1}, y_t) - f_t(x'_t, y_t)}_{(10b)}. \tag{10}
\end{aligned}$$

By using convexity and first-order optimality conditions, we get

$$\begin{aligned}
& \text{Equation (10b)} \leq \langle \nabla_x h_t(x_t, y_t), x_t - \tilde{x}_{t+1} \rangle + \langle \nabla_x f_t(\tilde{x}_{t+1}, y_t), \tilde{x}_{t+1} - x'_t \rangle \\
& \leq \langle \tilde{x}_t^\phi - x_t^\phi, x_t - \tilde{x}_{t+1} \rangle / \eta_t + \langle \tilde{x}_t^\phi - \tilde{x}_{t+1}^\phi, \tilde{x}_{t+1} - x'_t \rangle / \eta_t \\
& = [B_\phi(\tilde{x}_{t+1}, \tilde{x}_t^\phi) - B_\phi(\tilde{x}_{t+1}, x_t^\phi) - B_\phi(x_t, \tilde{x}_t^\phi)] / \eta_t \\
& \quad + \underbrace{[B_\phi(x'_t, \tilde{x}_t^\phi) - B_\phi(x'_t, \tilde{x}_{t+1}^\phi)] / \eta_t - B_\phi(\tilde{x}_{t+1}, \tilde{x}_t^\phi) / \eta_t}_{=: \Phi_t}. \tag{11}
\end{aligned}$$

Let  $\delta_t^1 = \text{Equation (10a)} - B_\phi(\tilde{x}_{t+1}, x_t^\phi) / \eta_t$ , so we have that  $f_t(x_t, y_t) - f_t(x'_t, y_t) \leq \Phi_t + \delta_t^1$ . Note that  $\eta_t$  is non-increasing over time,  $B_\phi$  is  $L_\phi$ -Lipschitz w.r.t. the first variable, and  $L_\phi D_X$  is the supremum of  $B_\phi$ . Thus,

$$\begin{aligned}
& \sum_{t=1}^T \Phi_t \\
& \leq \frac{B_\phi(x'_0, \tilde{x}_1^\phi)}{\eta_0} + \sum_{t=1}^T \frac{1}{\eta_t} (B_\phi(x'_t, \tilde{x}_t^\phi) - B_\phi(x'_{t-1}, \tilde{x}_t^\phi)) + \sum_{t=1}^T \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) B_\phi(x'_{t-1}, \tilde{x}_t^\phi) \\
& \leq \frac{L_\phi D_X}{\eta_T} + \sum_{t=1}^T \frac{L_\phi}{\eta_t} \|x'_t - x'_{t-1}\|,
\end{aligned}$$

Now we get

$$\text{Reg}_T^1 \leq \frac{L_\phi D_X}{\eta_T} + \sum_{t=1}^T \frac{L_\phi}{\eta_t} \|x'_t - x'_{t-1}\| + \sum_{t=1}^T \delta_t^1.$$



Likewise,

$$\text{Reg}_T^2 \leq \frac{L_\psi D_Y}{\gamma_T} + \sum_{t=1}^T \frac{L_\psi}{\gamma_t} \|y'_t - y'_{t-1}\| + \sum_{t=1}^T \delta_t^2,$$

where  $\delta_t^2 = f_t(x_t, \tilde{y}_{t+1}) - h_t(x_t, \tilde{y}_{t+1}) + h_t(x_t, y_t) - f_t(x_t, y_t) - B_\psi(\tilde{y}_{t+1}, y'_t)/\gamma_t$ . Next, we verify  $\delta_t^1, \delta_t^2 \geq 0$ . To verify  $\delta_t^1 \geq 0$ , it suffices to combine the following two inequalities:

$$\begin{aligned} f_t(x_t, y_t) + B_\phi(x_t, \tilde{x}_t^\phi)/\eta_t &\geq f_t(\tilde{x}_{t+1}, y_t) + B_\phi(\tilde{x}_{t+1}, \tilde{x}_t^\phi)/\eta_t, \\ -h_t(x_t, y_t) + h_t(\tilde{x}_{t+1}, y_t) &\geq -[B_\phi(\tilde{x}_{t+1}, \tilde{x}_t^\phi) - B_\phi(\tilde{x}_{t+1}, x_t^\phi) - B_\phi(x_t, \tilde{x}_t^\phi)]/\eta_t. \end{aligned}$$

The first inequality takes advantage of the optimality condition, and the second inequality is part of Equation (11). Likewise,  $\delta_t^2 \geq 0$ . So all learning rates are non-increasing. Let's go back to the focus on Player 1. The prescribed learning rate guarantees that

$$\text{Reg}_T^1 \leq \frac{L_\phi}{\eta_T} (D_X + C_T^1) + \sum_{t=1}^T \delta_t^1 \leq \epsilon + 2 \sum_{t=1}^T \delta_t^1.$$

On the one hand,  $\delta_t^1 \leq 2\rho(f_t, h_t)$  causes

$$\text{Reg}_T^1 \leq \epsilon + 4 \sum_{t=1}^T \rho(f_t, h_t). \quad (12)$$

On the other hand, notice that

$$\begin{aligned} \delta_t^1 &\leq \langle \nabla_x f_t(x_t, y_t) - \nabla_x h_t(\tilde{x}_{t+1}, y_t), x_t - \tilde{x}_{t+1} \rangle - B_\phi(\tilde{x}_{t+1}, x_t^\phi)/\eta_t \\ &\leq 2G_X \|x_t - \tilde{x}_{t+1}\| - B_\phi(\tilde{x}_{t+1}, x_t^\phi)/\eta_t \leq \min\{2D_X G_X, 2\eta_t G_X^2\}, \end{aligned} \quad (13)$$

which implies that

$$\begin{aligned} \left( \sum_{t=1}^T \delta_t^1 \right)^2 &= \sum_{t=1}^T (\delta_t^1)^2 + 2 \sum_{t=1}^T \delta_t^1 \sum_{\tau=1}^{t-1} \delta_\tau^1 = \sum_{t=1}^T (\delta_t^1)^2 + 2 \sum_{t=1}^T \delta_t^1 \left( \frac{L_\phi(D_X + C^1)}{\eta_t} - \epsilon \right) \\ &\leq \sum_{t=1}^T 4G_X^2 D_X^2 + \sum_{t=1}^T 4G_X^2 L_\phi (D_X + C^1). \end{aligned}$$

This results in the following regret bound:

$$\text{Reg}_T^1 \leq \epsilon + 4G_X \sqrt{(D_X^2 + L_\phi D_X + L_\phi C^1) T}. \quad (14)$$

Combining Equations (12) and (14) yields

$$\text{Reg}_T^1 \leq \epsilon + 4 \min \left\{ \sum_{t=1}^T \rho(f_t, h_t), G_X \sqrt{(D_X^2 + L_\phi D_X + L_\phi C^1) T} \right\}.$$

Likewise, the individual regret of Player 2 satisfies

$$\text{Reg}_T^2 \leq \epsilon + 4 \min \left\{ \sum_{t=1}^T \rho(f_t, h_t), G_Y \sqrt{(D_Y^2 + L_\psi D_Y + L_\psi C^2) T} \right\}.$$

## E Proof of Theorem 5

*Proof (Proof of Theorem 5).* The first statement of Theorem 5 can refer to the proof of Theorem 3. Let's focus on the second statement. If  $f_t$  and  $h_t$  are both bilinear functions, then Equation (13) can be replaced by  $\delta_t^1 \leq \|Y\| \rho(f_t, h_t) \|x_t - \tilde{x}_{t+1}\| - B_\phi(\tilde{x}_{t+1}, x_t^\phi)/\eta_t$ , where  $\|Y\| = \sup_{y \in Y} \|y\|$ . In short,  $G_X$  in the round  $t$  can be replaced by  $\rho(f_t, h_t)$ . Therefore, it is straightforward to arrive at the conclusion to be proved.

## F Proof of Theorem 6

*Proof (Proof of Theorem 6).* Note that in the proof of Theorem 4, the relaxed form inequalities  $\delta_t^1, \delta_t^2 \leq 2\rho(f_t, h_t)$  are employed. In fact,  $L_t$  induces a tighter upper bound.

$$\begin{aligned} \delta_t^1 &\leq f_t(x_t, y_t) - h_t(x_t, y_t) + h_t(\tilde{x}_{t+1}, y_t) - f_t(\tilde{x}_{t+1}, y_t) \\ &\leq \sum_{k=1}^K \omega_t^k \left( |f_t(x_t, y_t) - h_t^k(x_t, y_t)| + |f_t(\tilde{x}_{t+1}, y_t) - h_t^k(\tilde{x}_{t+1}, y_t)| \right) \leq 2 \langle L_t, \omega_t \rangle. \end{aligned}$$

Likewise,  $\delta_t^2 \leq 2 \langle L_t, \omega_t \rangle$ . Now the benign bound  $V_T'$  in Theorem 5 can be substituted with  $\sum_{t=1}^T \langle L_t, \omega_t \rangle$ . Drawing from Lemma 3, we have that

$$\begin{aligned} \sum_{t=1}^T \langle L_t, \omega_t \rangle &\leq \sum_{t=1}^T \langle L_t, 1_k \rangle + 2 \sqrt{(1 + \ln T) L_\infty \sum_{t=1}^T \langle L_t, 1_k \rangle} + O(\ln T) \\ &= \sum_{t=1}^T \langle L_t, 1_k \rangle + O(\sqrt{\ln T}) \sqrt{\sum_{t=1}^T \langle L_t, 1_k \rangle} + O(\ln T) \\ &\leq 2 \sum_{t=1}^T \langle L_t, 1_k \rangle + O(\ln T) \leq 2 \sum_{t=1}^T \rho(f_t, h_t^k) + O(\ln T), \quad \forall k \in 1:d, \end{aligned}$$

where  $1_k$  denotes the  $d$ -dimensional one-hot vector with the  $k$ -th element being 1. Given the arbitrariness of  $k$ , we obtain

$$\sum_{t=1}^T \langle L_t, \omega_t \rangle \leq 2 \min_{k \in 1:d} \sum_{t=1}^T \rho(f_t, h_t^k) + O(\ln T),$$

In conclusion, the benign bound  $V_T'$  in Theorem 5 can be substituted with

$$\min_{k \in 1:d} \sum_{t=1}^T \rho(f_t, h_t^k) + O(\ln T).$$

Ignoring the dependence on poly-logarithmic factors yields the conclusion to be proved.

## G Solver for Clipped Hedge

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**Algorithm 4** Program to Solve  $w^* = \arg \min_{w \in \Delta_d^\alpha} \langle \ln(w/W), w \rangle$

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**Input:**  $W, \alpha$

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1:  $d \leftarrow |W|$ ,  $I \leftarrow 1:d$ ,  $C_\# \leftarrow 0$ ,  $C_{\%_0} \leftarrow 0$ 
2: while  $I \neq \emptyset$  do
3:    $w \leftarrow$  the median of  $W_I$ 
4:    $L \leftarrow \{i \mid i \in I, W_i < w\}$ ,  $M \leftarrow \{i \mid i \in I, W_i = w\}$ ,  $H \leftarrow \{i \mid i \in I, W_i > w\}$ 
5:   if  $w \frac{d - (C_\# + |L|)\alpha}{\|W\|_1 - (C_{\%_0} + \|W_L\|_1)} < \alpha$  then
6:      $C_\# \leftarrow C_\# + |L| + |M|$ ,  $C_{\%_0} \leftarrow C_{\%_0} + \|W_L\|_1 + \|W_M\|_1$ 
7:     if  $H = \emptyset$  then
8:        $w \leftarrow \min\{W_i \mid i \in I, W_i > w\}$ 
9:     end if
10:     $I \leftarrow H$ 
11:  else
12:     $I \leftarrow L$ 
13:  end if
14: end while

```

**Output:**  $\forall i \in 1:d$ ,  $w_i^* \leftarrow \begin{cases} \frac{\alpha}{d} & W_i < w \\ \frac{W_i}{d} \frac{d - C_\# \alpha}{\|W\|_1 - C_{\%_0}} & W_i \geq w \end{cases}$

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