

MS-CS Course Note (Non-Credit Course 3)

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This is my course note on “Trees And Graphs: The Basics” provided by Colorado University of Boulder. This is a non-credit prep course for an MS-CS degree.

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1 Binary Search Trees

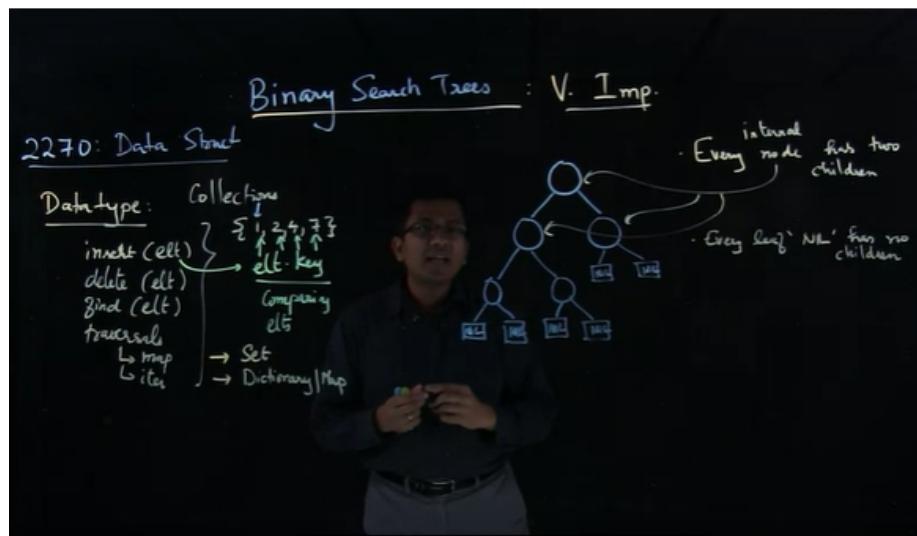
1.1 Basic Concepts

Binary search tree is a binary tree is a kind of data type with set of data elements without repetition.

We can insert, delete, search, and traverse the data elements in a binary search tree.

For each element in it, there will be a key of the element, which will always be a number.

With this setting in place, we can always compare different elements by comparing their keys, even if the elements are not numbers.



In the figure, we have a binary search tree with some nodes and leaves. Every node has two children nodes and those leaves, which

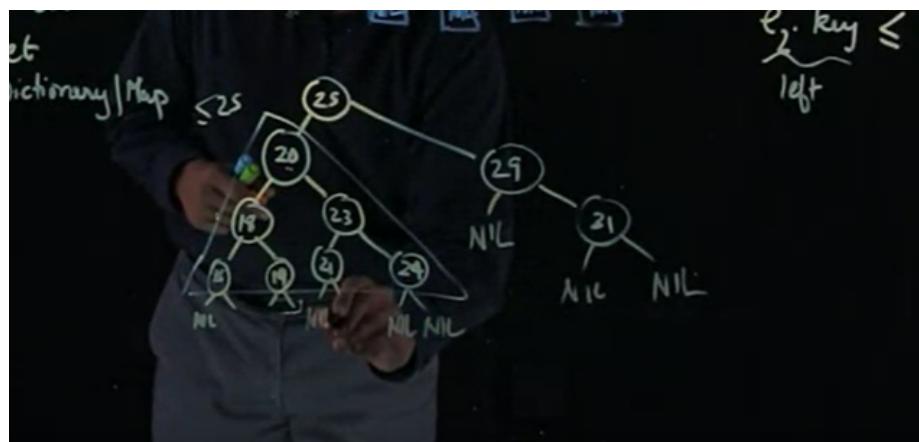
have no children nodes, are called nil nodes.

Every node has an element with a key, and the key of the left child node is always $<$ the key of the parent node, and the key of the right child node is always $>$ the key of the parent node.

The left and right child nodes are also binary search trees.

That is to say, the keys are always in a sorted order regardless of the structure of the tree. When we move the elements around, the keys will be different for each elements, in order to remain in the sorted order.

The leaves have no elements.



When there is a node with the key 25, every node in the left subtree will have a key $<$ 25, and every node in the right subtree will have a key $>$ 25.

The rule will also apply to all those subtrees.

Example:

```
25
 / \
15  50
/ \  / \
10 22 35 70
```

Question:

Binary Search Trees may look similar to Heaps, but it is important to consider their differences.

In a Min-Heap, the smallest element must be the root node of the tree.

In a Binary Search Tree, on the other hand, how would we find the smallest element?

A: We would traverse the left subtree of the root node until we reach a leaf node, which means a node with a NIL as its left child.

1.2 The Height of a Binary Search Tree

The height of a binary search tree is the number of edges on the longest path from the root node to a leaf node.

We will define the height of a leaf node as 0. Then the height of number 25, a.k.a. the root node of the below binary search tree is 2.

Example:

```
25      -> height = 2
/
15  50    -> height = 1
/ \  / \
10 22 35 70  -> height = 0, since they are leaf nodes
```

Let's assume we have a balanced binary search tree with n internal nodes.

One each layer from the root node, there'll be 2^0 nodes, 2^1 nodes, 2^2 nodes, \dots , 2^h nodes.

The total number of nodes in the tree will be $2^0 + 2^1 + 2^2 + \dots + 2^h = 2^{h+1} - 1$.

So, the height of the tree will be $h = \log_2(n + 1) - 1$.

In the sense of big O notation, the height of a binary search tree is $O(\log_2(n))$, in a balanced binary search tree scenario.

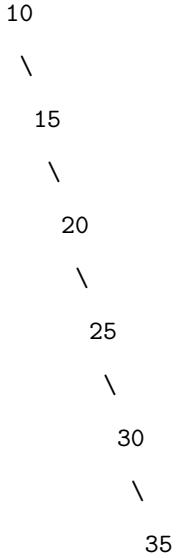
For example, $\log_2^8 = 3$, so the height of a binary search tree with 8 nodes is 3.

$\log_2^{15} = 4$, so the height of a binary search tree with 15 nodes is 4.

In the worst case scenario, where the binary search tree is not balanced, the height of the tree will be $O(n)$.

That is to say, the tree will be a linked list looks like this;

Linked List Example:



In this case, the height of the tree is 6, which is equal to the number of nodes in the tree.

The height of the tree is $O(n)$, which is the worst case scenario.

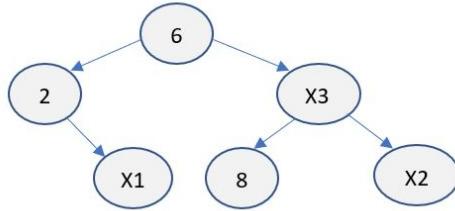
Normally, we will have something in between the best case scenario and the worst case scenario, $O(\log(n)) < \text{height} < O(n)$.

1.3 Basics of Binary Search Tree Quiz

1. Consider the following binary search tree below with missing values X_1 , X_2 and X_3 .

1 / 1 point

Note that the leaves labeled NIL are not shown, but please assume that they exist.



Select all true statements about the tree.

- X_1 can be any value less than or equal to 6.
 X_1 can be set to the number 5 while remaining a valid binary search tree.

Correct
X1 must also be ≥ 2 since it is the right child of 2, and $X_1 \leq 6$ since it is in the left subtree of the root 6. Therefore, 5 is a possible value.

- X_3 can be any number ≥ 6 .
 X_3 can be any number ≥ 8 and $\leq X_2$.

Correct
Correct

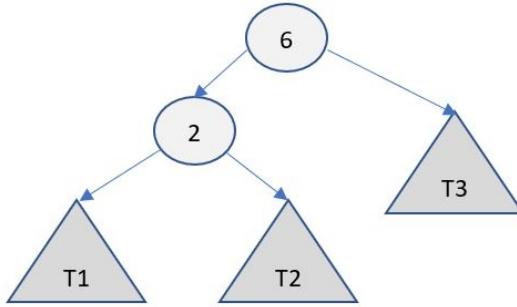
- X_2 must have a value ≥ 8 and $\geq X_3$.

Correct
Correct

- The height of the root node is 3.

Correct
Correct. Note that the root has a longest path of length 3 to a leaf.

2. Consider the following binary search tree with subtrees shown below. Select all true statements about it.



Every node in T_1 must have value ≤ 2 .

Correct

Correct since T_1 is the left subtree of node 2.

Every node in T_2 must have key ≥ 2 and ≤ 6 .

Correct

Correct since T_1 is in the right subtree of node 2 and left subtree of the node 6.

If the node with key 25 is found in the tree, we will find it in subtree T_2 .

If the node with key -10 is to be found in the tree, it can be found in subtree T_2 .

If the node with key 7 is to be found in the tree, it will be found in T_3 .

Correct

Correct since $7 > 6$ it will be found in the right subtree of the root node 6.

If the height of subtree T_1 is 4 and that of subtree T_2 is 2 then the height of node labeled 2 is 5.

Correct

Correct since $\max(4, 2) + 1 = 5$

3. Select all correct statements from the list below about binary search trees.

0.666666666666666
/1 point

In a fully balanced binary search tree with n total nodes (internal and leaf nodes), where $n = 2^k - 1$ for some k , we will have $(n + 1)/2$ leaves.

In the worst case, a binary search tree with n internal nodes can have height n .

Correct

Correct. Every node in the tree has a single child in the worst case

Assuming that all keys are distinct, the key at the root is the median among all keys of the binary search tree.

可能是这个

You didn't select all the correct answers

1.4 Insertion and Deletion in a Binary Search Tree

1.4.1 Insertion

We can insert a new element into a binary search tree by comparing the key of the new element with the key of the root node.

If the key of the new element is less than the key of the root node, we will insert the new element into the left subtree.

If the key of the new element is greater than the key of the root node, we will insert the new element into the right subtree.

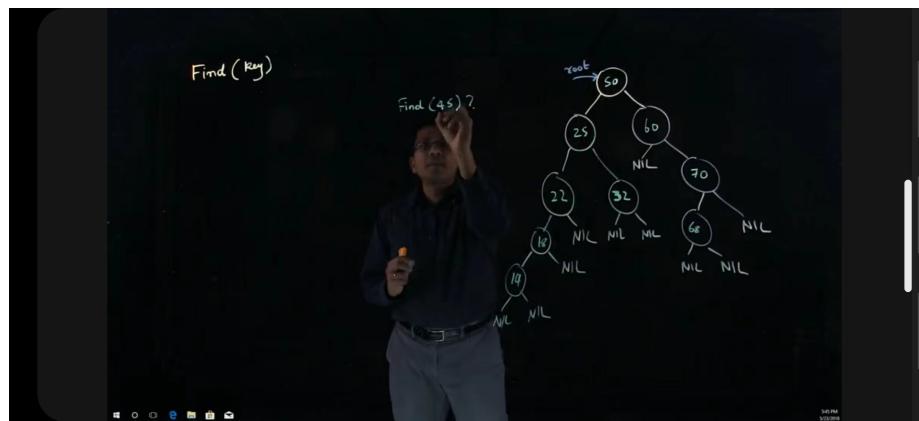
We will repeat the process until we reach a leaf node.

For example, we have a binary search tree with the following nodes;

Example:

```
25
 / \
15  50
 / \  / \
10 22 35 70
```

If we want to insert a new element with the key 40, we will compare 40 with 25, and then 40 with 50, and then 40 with 35. Since 40 is greater than 35, we will insert 40 as the right child node of 35.



The above example actually consists of two steps.

First, we will search for the element to be inserted, which is called the *find()* operation.

Then, after successfully locates the element, we will insert it into the binary search tree.

We will talk about find operation now.

Assuming we have an imperfect binary search tree and we want to locate the key 45, how should we do that?

We will start from the root node, and then compare the key of the root node with the key of the element to be located.

If the key of the root node is equal to the key of the element to be located, we will return the root node. If the key of the root node is greater than the key of the element to be located, we will search the left subtree. Otherwise, we will search the right subtree.

The overall process will be repeated until we reach a leaf node.

If we reach a leaf node and the key of the leaf node is not equal to the key of the element to be located, we will return NIL.

Here is the pseudo code for the find operation:

```
find(root, key)
    if root == NIL or root.key == key
        return root
    if root.key > key
        return find(root.left, key) # This is the recursive call.
    return find(root.right, key) # This is the recursive call.
```

As for the time complexity, the find operation will take $O(h)$ time, where h is the height of the binary search tree.

Now we will go ahead with the second step of the insertion operation.

We will insert the new element into the binary search tree by comparing the key of the new element with the key of the root node.

Here is the pseudo code for the insertion operation:

```
insert(root, key)
    if root == NIL
        return new Node(key)
    if key < root.key
```

```

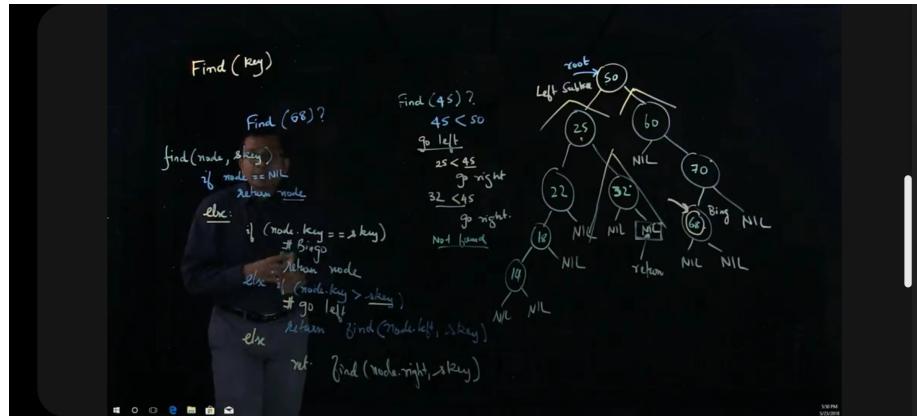
root.left = insert(root.left, key) # This is the recursive call.

else

    root.right = insert(root.right, key) # This is the recursive call.

return root

```



1.4.2 Deletion

Now we will talk about the deletion operation.

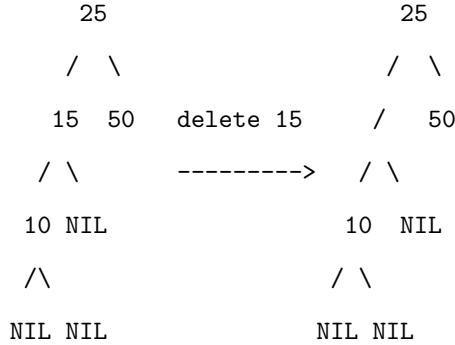
There are three cases to consider when deleting a node from a binary search tree.

The node to be deleted can be a leaf node, a node with only one child, or one with two child nodes.

If both child nodes are NIL, we can simply delete the node.

If one of the child nodes is NIL, we can delete the node, then reconnect between the past-parent node and past-child node, like this;

Only One Child Node Example:



What if we want to delete a node that has two non-NIL children?

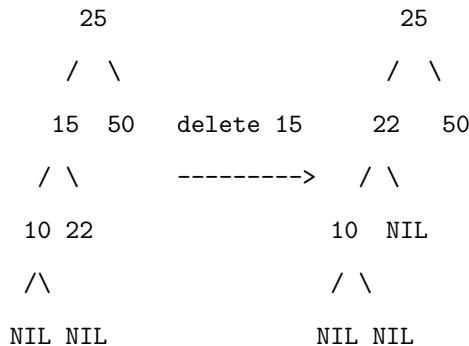
We will find the smallest node in the right subtree of the node to be deleted, and then replace the node to be deleted with the smallest node.

To perform this operation, we will ‘walk’ right from the root node by one step and then ‘walk’ left until we reach a leaf node.

During each step, we will compare the key of the current node with the key of the node to be deleted.

The leaf node we reached will be the successor of the node we deleted.

Two Child Nodes Example:



Here is the pseudo code for the deletion operation:

```

delete(root, key)
    if root == NIL
        return root
    if key < root.key
        root.left = delete(root.left, key) # This is the recursive call.
    else if key > root.key
        root.right = delete(root.right, key) # This is the recursive call.
    else
        if root.left == NIL
            return root.right
        else if root.right == NIL
            return root.left
        root.key = minValue(root.right)
        root.right = delete(root.right, root.key) # This is the recursive call.
    return root

```

1.4.3 Tree Traversal

There are three ways to traverse a binary search tree.

In-order traversal, pre-order traversal, and post-order traversal.

Inorder traversal visits nodes in a binary tree in the following order:

Visit the left subtree.

Visit the root node.

Visit the right subtree.

This traversal method is particularly useful for binary search trees (BSTs) because it visits the nodes in ascending order.

Example:

Consider the following binary search tree (BST):

```
        4
       / \
      2   6
     / \ / \
    1  3 5  7
```

Inorder traversal of this tree would be: 1, 2, 3, 4, 5, 6, 7.

Python Code:

```
class Node:

    def __init__(self, key):
        self.left = None
        self.right = None
        self.val = key

    def inorder_traversal(root):

        if root:
            # Traverse the left subtree
            inorder_traversal(root.left)

            # Visit the root node
            print(root.val, end=' ')

            # Traverse the right subtree
            inorder_traversal(root.right)
```

Preorder traversal visits nodes in the following order:

Visit the root node.

Visit the left subtree.

Visit the right subtree.

Preorder traversal is useful for creating a copy of the tree or getting a prefix expression of an expression tree.

Example:

Consider the same BST:

```
4
/
2   6
/ \ / \
1 3 5 7
```

Preorder traversal of this tree would be: 4, 2, 1, 3, 6, 5, 7.

Python Code:

```
def preorder_traversal(root):
    if root:
        # Visit the root node
        print(root.val, end=' ')
        # Traverse the left subtree
        preorder_traversal(root.left)
        # Traverse the right subtree
        preorder_traversal(root.right)
```

Postorder traversal visits nodes in the following order:

Visit the left subtree.

Visit the right subtree.

Visit the root node.

Postorder traversal is useful for deleting a tree or evaluating postfix

expressions of an expression tree.

Example:

Consider the same BST:

```
        4
       / \
      2   6
     / \ / \
    1  3 5  7
```

Postorder traversal of this tree would be: 1, 3, 2, 5, 7, 6, 4.

Python Code:

```
def postorder_traversal(root):
    if root:
        # Traverse the left subtree
        postorder_traversal(root.left)
        # Traverse the right subtree
        postorder_traversal(root.right)
        # Visit the root node
        print(root.val, end=' ')
```

1.5 BST Quiz

The figure consists of two side-by-side screenshots of a mobile application interface, likely from an iPhone, displaying a quiz question about Binary Search Trees (BST).

Left Screenshot (Question 1/4):

- Header:** 09:46, Battery 71%.
- Title:** Binary Search Tree: Insert and Delete - 1 of 4
- Text:** Question 1/4, 0.75 / 1 point.
- Description:** Suppose we wish to insert nodes with keys -5, 11, and 10, in that order, into the following Binary Search Tree (BST).
A diagram of a BST is shown:

```
graph TD; 6((6)) --> 2((2)); 6 --> 17((17)); 2 --> 5((5)); 2 --> 8((8)); 17 --> 22((22));
```
- Instruction:** Select all true statements from the list below.
- Options:**
 - When the node with key 11 is inserted, it becomes the left child of the node with key 8.
 - When the node with key 10 is inserted, it becomes the left child of the node with key 11.
- Buttons:** Next → at the bottom.

Right Screenshot (Question 1/4):

- Header:** 09:46, Battery 71%.
- Title:** Binary Search Tree: Insert and Delete - 1 of 4
- Text:** 8.
- List:**
 - When the node with key 10 is inserted, it becomes the left child of the node with key 11.
Correct
 - The node with key -5 will become a left child of the node with key 2.
Correct
 - When the node with key 11 is inserted, it displaces the node with key 8, which becomes its left child.
Incorrect
Incorrect – this is not how we do insertion.
Inserted nodes always take leaf positions in the existing tree.
- Buttons:** Next → at the bottom.

Figure 1: Question 1

The figure consists of two side-by-side screenshots of a mobile application interface. Both screens show the same question at different stages of completion.

Top Bar: Shows the time as 09:47, signal strength, battery level at 71%, and a back arrow.

Section Header: "Binary Search Tree: Insert and Delete - 2 of 4"

Text: "Question 2/4 0.75 / 1 point".

Text: "Starting from an empty tree, we insert the nodes with keys $[1, \dots, n]$ in some order. Select all the true statements from the list below."

Statement 1: For $n = 7$, inserting the nodes in the order $[4, 2, 1, 3, 6, 5, 7]$ yields a fully balanced binary tree of depth 3. **Feedback:** Correct

Statement 2: The tree can have depth between $\log n$ and n , depending on the actual order which the keys are inserted. **Feedback:** Correct

Statement 3: If the nodes are inserted in descending order, then the resulting tree has height n . **Feedback:** Correct

Statement 4: For $n = 7$, the only two insertion sequences that yield a tree of depth n are when the keys are inserted in ascending or in descending order. **Feedback:** Incorrect
This is incorrect. For instance, inserting nodes in the order, $7, 1, 2, 6, 3, 5, 4$ also yields a tree of depth n .

Buttons: "Previous" and "Next" at the bottom of each screen.

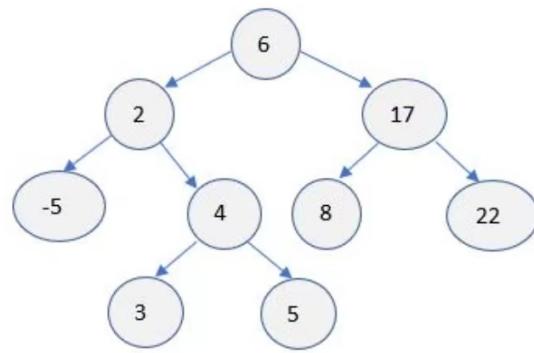
Figure 2: Question 2

← Binary Search Tree: Insert and Delete - 3 of 4

Question 3/4

1.0 / 1 point

Consider the following Binary Search Tree.



Select the single true statement from the list below.



If we delete the root (6), it will be replaced by one of its children.



If we wish to delete the node 2, we can replace it with its successor node 3. In this case, the node 4 will be left with just one child.



Correct

09:48

Binary Search Tree: Insert and Delete - 4 of 4

✗ **Incorrect**
You didn't select all the correct answers

Question 4/4
0.75 / 1 point

Consider the Binary Search Tree below.

Select all the true statements from the list below.

Post-order traversal of a BST produces the reversal of the list obtained from its pre-order traversal.

Pre-order traversal of the BST above yields the list [6, 2, -5, 4, 3, 5, 17, 8, 22].

✓ **Correct**

[← Previous](#) [Done](#)

09:48

Binary Search Tree: Insert and Delete - 4 of 4

Select all the true statements from the list below.

Post-order traversal of a BST produces the reversal of the list obtained from its pre-order traversal.

Pre-order traversal of the BST above yields the list [6, 2, -5, 4, 3, 5, 17, 8, 22].

✓ **Correct**

In-order traversal of a Binary Search Tree always leads to a sorted list of keys.

All traversals require as much time as the number of nodes in the tree.

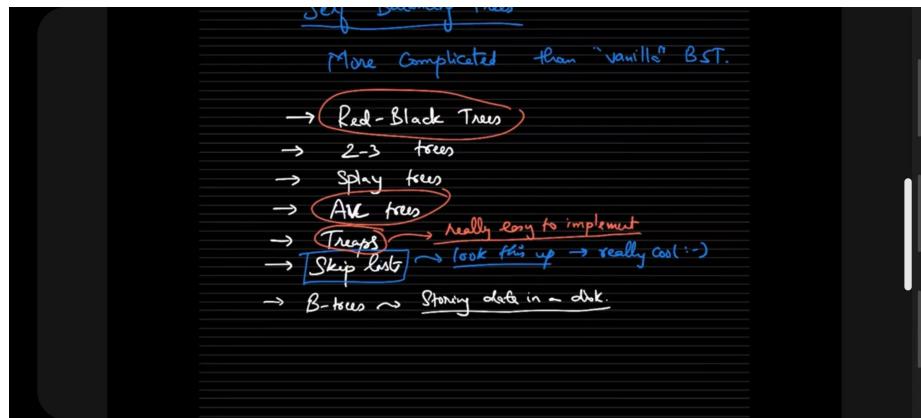
✓ **Correct**

[← Previous](#) [Done](#)

Figure 4: Question 4

2 Algorithms on Trees

2.1 Red-Black Trees Basics



Normally the performance of a BST depends on its height or depth.

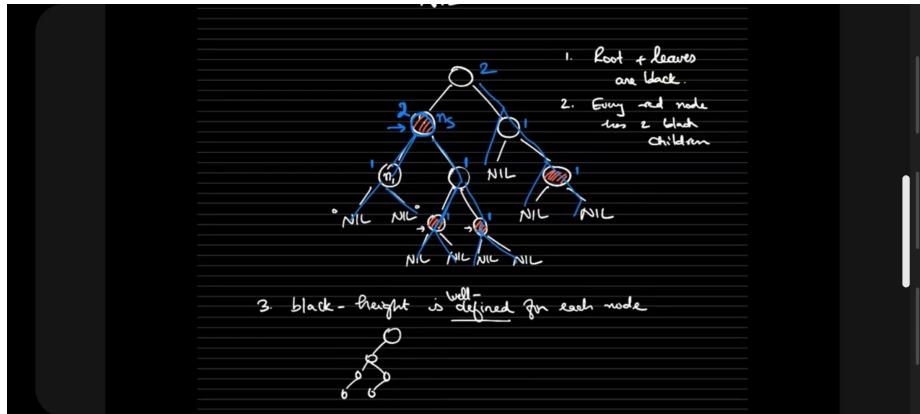
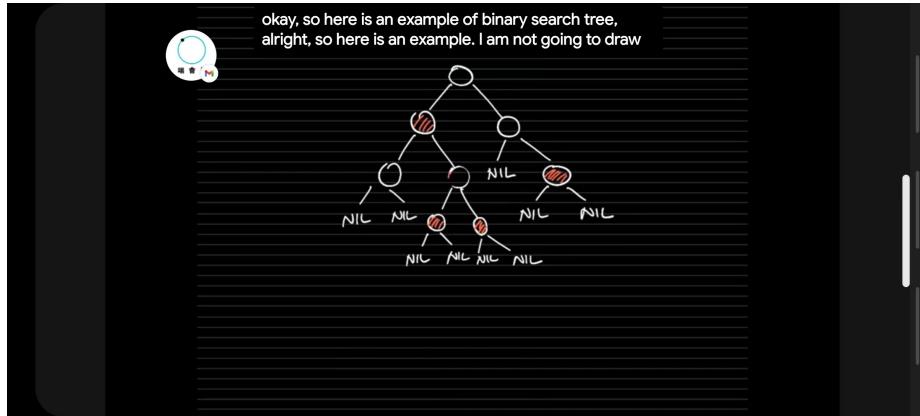
If we have a bad (not well-balanced) binary search tree, the height of the tree will be $O(n)$.

In this case, the performance of the tree will be the same as that of a linked list.

If it's balanced, the running time will be as good as $O(\log(n))$. This is not good enough.

What we need is self-balancing BSTs.

Here are some different types of self-balancing BSTs.



Red-black trees are a type of self-balancing binary search tree.

They are named after their property of having red and black nodes.

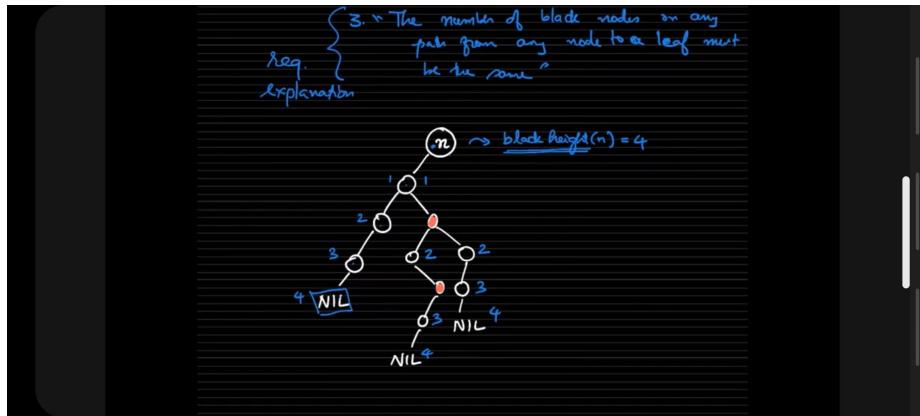
The red-black tree has the following properties:

1. Every node is either red or black.
2. The root node is always black.
3. Every leaf node is black.
4. If a red node has children, the children must be black.

5. Every path from a node to its descendant NIL nodes must have the same number of black nodes.

The height of a red-black tree is $O(\log(n))$.

The red-black tree is a balanced binary search tree.



When counting the number of black nodes, we do not count the red nodes in between.

True/False: black height of a parent node with a red child is the same as that of the red child whereas for a node with a black child, black height of parent is one more than that of the black child.

True

False

Skip **Submit**

Here is a question regarding red-black trees.

We can come up with two scenarios for it.

Scenario 1:

Parent node with RED child:

B (parent node: height of 1)

\
red
/ \

NIL NIL (height of 1)

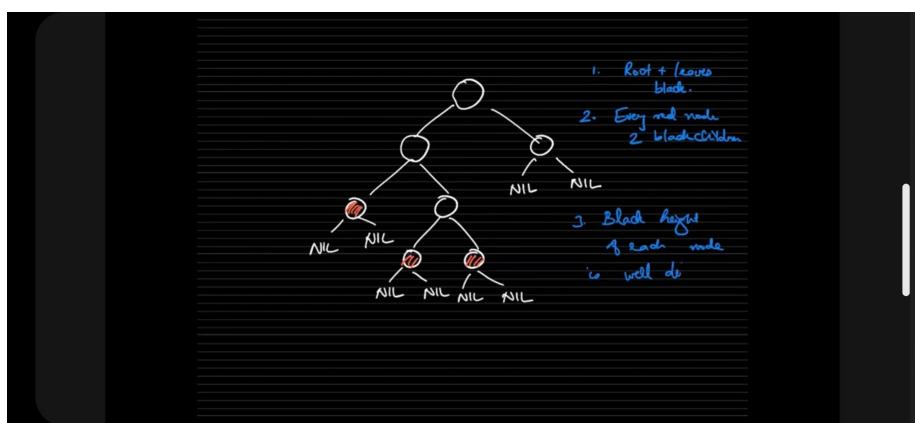
Scenario 2:

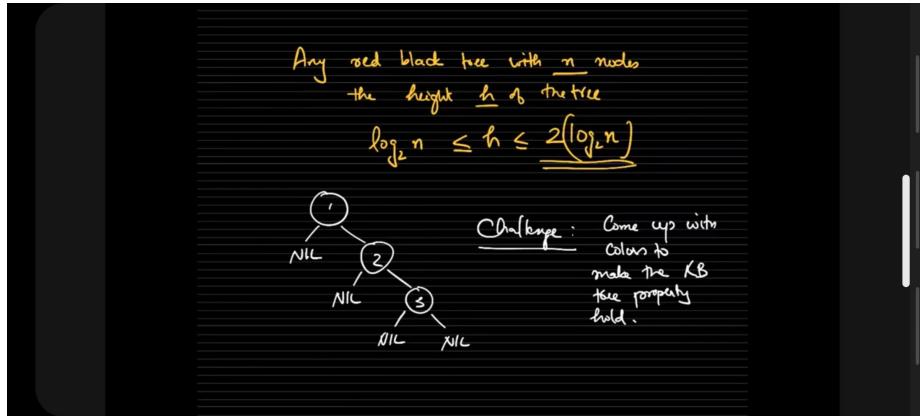
Parent node with BLACK child:

B (parent node: height of 2)

\
B (height of 1)
/ \
NIL red
/ \

NIL NIL (height of 2)





As we can see, the tree in the figure is not a red-black tree.

On the left corner, the height of the root node is 2, however, the height of the middle subtree is 3.

For any red-black tree with n nodes, the height h must obey the following rule:

$$\log(n) \leq h \leq 2 * \log(n).$$

Here is another example of a unbalanced tree that cannot satisfy the red-black tree properties.

09:40 X C •

HD 4G R 66%

✓ Correct

Why is it the case that at least half the nodes on a path are black?



Because a red node must have at least two black children and the root/leaf have to be black.

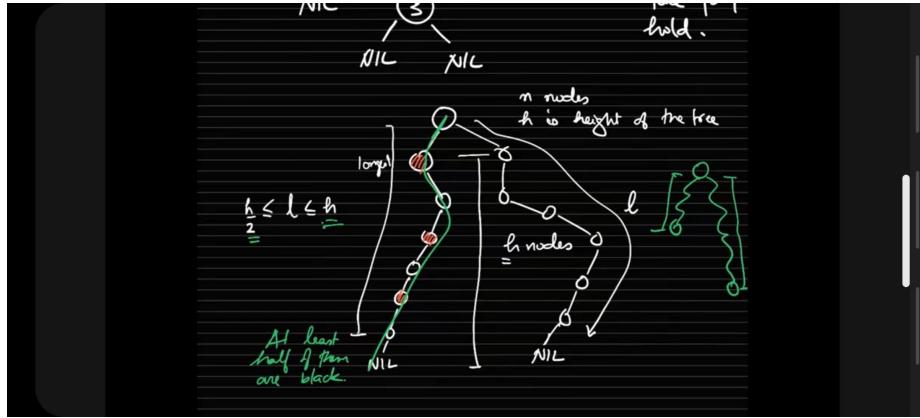
Correct

Correct



The statement is incorrect: it is possible that all nodes in the path are red.

Continue



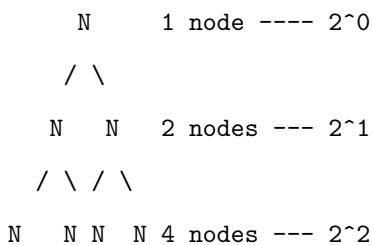
Now we can try to figure out how can we find a red-black tree.

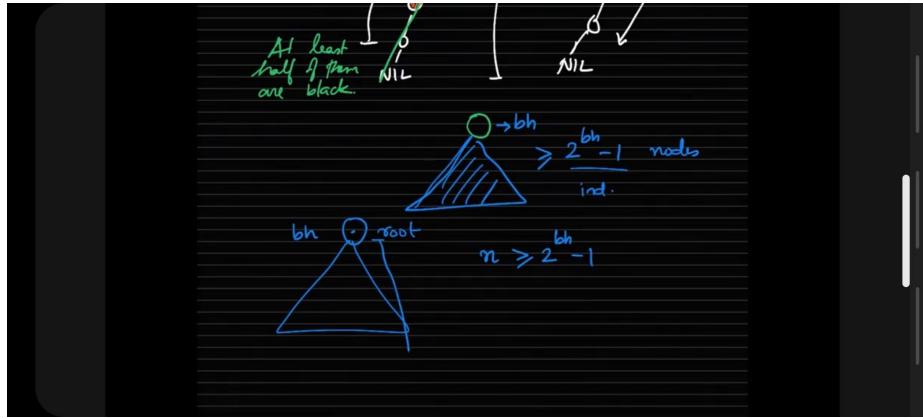
Suppose we have a n node, h height binary search tree.

If it is a red-black tree, any path l in the tree must be $h/2 < l < h$, since on any given path, at least half of the nodes must be black.

Let's assume the black height of a specific node is bh . The subtree under that node will have a total node count $nc \geq 2^{bh} - 1$.

This can be proved by induction, or we can also directly see the logic via such graph:





Since we have the property that any path $l \geq h/2$, we can say that

$$nc \geq 2^{bh} - 1 \geq 2^{h/2} - 1.$$

Then we can conclude that $h \leq 2 * \log(n + 1)$.

2.1.1 RBT Quiz

10:48 ⓘ E •

Red-Black Tree Basics - 1 of 3

Question 1/3
1.0 / 1 point

Consider the following tree with nodes colored red/black. Sentinels (NIL) are shown by black squares.

Answer the following questions below based on the black height of various nodes.

Each leaf has black height 0.

The node labelled 9 has black height 1.

Correct
Correct – black height does not include the node you are starting from.

Next →

10:48 ⓘ E •

Red-Black Tree Basics - 1 of 3

The node labelled 7 has black height 2.

Correct
Every path from the node to a leaf has one black node that includes the sentinel node itself.

The root node has black height 2.

Correct
Look at every path from the node 7 to a sentinel. It has two black nodes including the sentinel.

The tree is a valid red-black tree that satisfies all the conditions of a red-black tree.

Correct
This is correct.

Next →

Figure 5: RBT Question 1

10:49 E • HD . 4G R. 66%

← Red-Black Tree Basics - 2 of 3

Question 2/3
1.0 / 1 point

Consider the tree below with nodes labeled red/black.

Select the correct fact from the list below.

There is a red node which has a red child.

The black height at node n2 is not well defined.

The black height at node n1 is not well defined.

The black height at the root is 2.

Correct

This is a valid red/black tree.

Correct

[← Previous](#) [Next →](#)

10:49 E • HD . 4G R. 66%

← Red-Black Tree Basics - 2 of 3

Select the correct fact from the list below.

There is a red node which has a red child.

The black height at node n2 is not well defined.

The black height at node n1 is not well defined.

The black height at the root is 2.

Correct

This is a valid red/black tree.

Correct

[← Previous](#) [Next →](#)

Figure 6: RBT Question 2

← Red-Black Tree Basics - 3 of 3

Consider a red-black tree with $n \geq 128$ nodes.

Select all the true facts about the tree.



The tree can have height more than $n/2$.



Finding a key will take time $\Theta(\log n)$.



Correct



The difficulty in red-black trees consists of maintaining the red-black property when we insert/delete elements.



Correct



If the longest path from root to leaf is 12 then every path must have size at least 6.

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Correct

note that the black height must be the same. In the worst case every other node in the longest path is a red node. This means that the shortest path must

2.2 Red-Black Tree Rotation, Algorithms for Insertion/Deletion

TBC from notebook.