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Variations on the theme of toric mirror symmetry

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Abstract

The theme of this thesis is an algebraic study of sheaves on real vector spaces. In the first part, we revisit the coherent-constructible correspondence for toric varieties, which describes sheaves on $\mathbb{R}^n$ with prescribed singular support in terms of torus-equivariant quasi-coherent sheaves on a toric variety. Among other things, we show that such a correspondence can be lifted to the universal base ring , the sphere spectrum. In the second part, we explain that this perspective can be used to describe all sheaves on $\mathbb{R}^n$ in the form of almost mathematics objects, following the work of Dmitry Vaintrob. This provides a natural interpretation of Efimov's calculation of continuous K-theory for sheaves on $\mathbb{R}^n$. We take this approach further to calculate the Picard group of sheaves on $\mathbb{R}^n$ under the convolution product.

The first part is a joint work with Yuxuan Hu, and the second part is a joint work with Robert Burklund.

Resumé

Temaet for denne afhandling er en algebraisk undersøgelse af sheaves på reelle vektorrum. I første del genbesøger vi den 'kohærente-konstruerbare korrespondance' for toriske varieteter, som beskriver sheaves på $\mathbb{R}^n$ med en given singulær støtte i form af torus-ækvivalente kvasi-kohærente sheaves på en torisk varietet. Blandt andet viser vi, at en sådan korrespondance kan løftes til den universelle basering . I anden del forklarer vi, at dette perspektiv kan bruges til at beskrive alle sheaves på $\mathbb{R}^n$ i form af 'almost mathematics' objekter, i tråd med Dmitry Vaintrobs arbejde. Dette giver en naturlig fortolkning af Efimov beregning af kontinuerlig K-teori for sheaves på $\mathbb{R}^n$. Vi videreføre denne tilgang til at beregne Picard-gruppen af sheaves på $\mathbb{R}^n$ under foldningsproduktet.

Første del er i samarbejde med Yuxuan Hu, og den anden del er i samarbejde med Robert Burklund.

Thesis Statement

This thesis consists of an introduction and two papers. The introduction is solely written by Qingyuan Bai.

Paper A (Title: Toric mirror symmetry for homotopy theorists) is a joint work of Qingyuan Bai and Yuxuan Hu. A draft of this manuscript has been presented on the preprint server arxiv:2501.06649 with DOI https://doi.org/10.48550/arXiv.2501.06649.

Paper B (Title: A study of sheaves on real vector spaces) is a joint work of Qingyuan Bai and Robert Burklund. This manuscript has never appeared publicly. An older version of this document was hosted on my GitHub repository under the name 'Some stuff about sheaves on real vector spaces'.

None of the manuscripts above have been submitted to journal for peer review at the moment of submission of this thesis.

All the authors involved have signed co-author statements. These statements are submitted along with the thesis.

Having recalled intrigues of former years
Having recalled a former love
Alexander Pushkin

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First of all, lots and lots of thank you to Dustin, for being my supervisor - you have been exceptionally kind and supportive to me! Reading the mathematics written by Dustin is a constant inspiration for me in the past several years. It is thus a huge privilege to have the freedom of working on the mathematics I am curious about while getting advice from *the* mathematician I very much look up to. As such, I feel even ashamed about how little I have done about the PhD project that Dustin gave me - how I wish I could tell you something interesting about it someday.

I have also been lucky enough to be co-supervised by Lars Hesselholt. The influence of Lars on how I approach the mathematics contained in this thesis is obvious. Lars has managed to pass his enthusiasm about Grothendieck's six operations to me. I should have also listened to Lars' kind advice on learning more about p-adic geometry. Well, maybe in a parallel universe...

Robert has been the main source of mathematical strength for me in the past three years. Most of the content in this thesis was clarified in our conversation, and I look forward to writing about the motivic little disc operad with you! Maths aside, I want to thank you for having me at Oberwolfach and offering me shots on Wednesday party night - it was a wonderful occasion.

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Introduction

We first provide some ideas about the general theme of the content of this thesis. Then we summarize the main content of the two papers, highlighting the mathematical ideas and technical strategies.

Unless stated otherwise, 'category' in this introduction means '$(\infty, 1)$-category'.

General theme

The objects of study in this thesis are categories arising from different contexts of geometry. The most important examples are the following two kinds:[1]

- The A-model categories: (Betti) topological sheaves. Let X be a locally compact Hausdorff topological space and k be any commutative ring. One can attach the category

$$\mathcal{S}\mathrm{hv}(X; \mathrm{Mod}_k)$$

  which is the category of sheaves of complexes of k-modules on X.

- The B-model categories: (quasi-)coherent sheaves. Let Y be an algebraic stack over the base commutative ring k. One can attach the category

$$\mathrm{QCoh}(Y)$$

  which is the category of complexes of quasi-coherent sheaves on Y.

In practice, one works with constructible sheaves on stratified topological spaces on the A-side so that its size matches with the B-side, but we will carelessly ignore this issue for now.

The very general form of homological mirror symmetry poses the following lists of questions:

1. Given X and Y as above, produce interesting functors between $\mathcal{S}\mathrm{hv}(X; \mathrm{Mod}_k)$ and $\mathrm{QCoh}(Y)$, ideally equivalences.

2. In the situation of 1, one can ask further for functors that respect auxiliary structures - tensor products, t-structures, semi-orthogonal decompositions etc.

[1] The name 'A/B-model' is borrowed from topological string theory.

3. More ambitiously, both of the assignments $\mathcal{S}\mathrm{hv}(-;\mathrm{Mod}_k)$ and $\mathrm{QCoh}(-)$ come with six operations, and one can ask in the situation of 1 and 2 if the functors are compatible with push-pull along morphisms.

The reader could reasonably ask what these questions are good for if, after all, one only cares about the geometry of X and Y. I find the following two perspectives particularly fascinating:

- **Computing invariants.** An equivalence between dualizable categories induces an isomorphism between their non-commutative invariants, say K-theory and Hochschild homology. Given an equivalence as in point 1, one can expect to pass across the equivalence and make easier computation of these invariants. On the other hand, these invariants are often related to the cohomology theory of the underlying spaces and hence reflect the geometry.
- **Lifting varieties.** The A-side categories $\mathcal{S}\mathrm{hv}(X;\mathrm{Mod}_k)$ naturally admits integral lifts in the sense that there exists equivalence of symmetric monoidal categories
$$\mathcal{S}\mathrm{hv}(X;\mathrm{Mod}_{\mathbb{Z}}) \otimes_{\mathrm{Mod}_{\mathbb{Z}}} \mathrm{Mod}_k \cong \mathcal{S}\mathrm{hv}(X;\mathrm{Mod}_k).$$
In fact one can lift further to a deeper, homotopical base ring . If one can produce equivalence of symmetric monoidal categories
$$\mathcal{S}\mathrm{hv}(X;\mathrm{Mod}_k) \cong \mathrm{QCoh}(Y)$$
for a variety Y defined over k, then the Tannakian reconstruction theorem will produce an integral lift of Y.

We further comment on the usefulness of point 2 and 3 above.

- **Functoriality.** Classically, to produce functions on a space X, one first produces functions locally and then check they glue together. On the level of categories, there is a very flexible notion of 'locality' - either via descent, or via semi-orthogonal decomposition. This gives many more ways to produce functors between categories - once we achieve a functorial mirror symmetry result, we can pass via limit/colimit argument to reach other examples. This approach turns out to be extremely useful in practice.

We end the general discussion with the following example.

Example (Fourier theory for torus)**.** Let $X = S^1$ and $Y = \mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}])$. We produce a functor
$$\mathrm{QCoh}(\mathbb{G}_m) \longrightarrow \mathcal{S}\mathrm{hv}(S^1;\mathrm{Mod}_k)$$
as follows: a quasi-coherent sheaf on $\mathbb{G}_m$ is the same as a pair (V, ϕ) where V is a k-module and
$$\phi : V \xrightarrow{\cong} V$$
is a k-linear automorphism of V. This pair defines for us a locally constant sheaf $\mathcal{L}_{(V,\phi)} \in \mathcal{S}\mathrm{hv}(S^1;\mathrm{Mod}_k)$ whose stalk at 1 is V and the monodromy automorphism is given by $\phi : V \longrightarrow V$. In fact this defines an equivalence
$$\mathrm{QCoh}(\mathbb{G}_m) \xrightarrow{\cong} \mathcal{S}\mathrm{hv}^{\mathrm{loc.const.}}(S^1;\mathrm{Mod}_k).$$
The equivalence also comes with a symmetric monoidal structure: it intertwines pointwise tensor product on the left and convolution product on the right.

In this thesis, we provide a case study of toric mirror symmetry, which is itself a generalization of the Fourier theory for torus above. We now outline the content of the two papers in the thesis.

Paper A: Toric mirror symmetry for homotopy theorists

The paper 'Toric mirror symmetry for homotopy theorists' is joint work with Yuxuan Hu. We revisit the following theorem of [2]:

Theorem. Let Y be a smooth projective toric variety over a field k of dimension n with torus $\mathbb{T}$. There is a fully faithful embedding

$$\kappa : \mathrm{Perf}(Y/\mathbb{T}) \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k).$$

Its image are precisely sheaves with perfect stalks and prescribed singular support Λ_Y depending on Y. The functor intertwines the pointwise tensor product on the left and the convolution product on the right.

Our main result is an enhancement of the above theorem:

Theorem. Let Y be a smooth projective toric spectral scheme of dimension n with torus $\mathbb{T}$. There is a fully faithful, colimit preserving and symmetric monoidal functor

$$\kappa : \mathrm{QCoh}(Y/\mathbb{T}) \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp}),$$

where the source has pointwise tensor product and the target has convolution product. Its image are precisely sheaves with prescribed singular support Λ_Y depending on Y.

Here are the major improvements we have made:

- We promote the statement to the level of large categories - instead of working with the category of perfect complexes, we show that in fact the functor κ defined on $\mathrm{QCoh}(Y/\mathbb{T})$ is also fully faithful.
- We lift the coefficient ring k to the universal base ring , the sphere spectrum.
- We carefully construct the symmetric monoidal structure on κ.
- We produce the compatibility of the functor κ with Fourier theory

  $$\mathrm{QCoh}(B\mathbb{T}) \cong \mathcal{S}\mathrm{hv}(\mathbb{Z}; \mathrm{Sp}).$$

  With this, we can apply the technique of de-equivariantization and deduce the version of above theorem without $\mathbb{T}$-equivariance [5] as a formal consequence. More precisely, we obtain a fully faithful, colimit preserving and symmetric monoidal functor

  $$\kappa : \mathrm{QCoh}(Y) \longrightarrow \mathcal{S}\mathrm{hv}((S^1)^{\times n}; \mathrm{Sp})$$

  and characterize its image in terms of singular support.

We refer the reader to Paper A for precise statements. Now we highlight some technical aspects of the work, and illustrate some examples.

1. In terms of general strategy, the majority of the work concerns construction of the functor κ and its symmetric monoidal structure. The functor κ is defined via gluing: we first construct functors
$$\mathrm{QCoh}(U) \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp})$$
for toric affine opens U inside of Y and then glue them together. To do so, we made the following observation: for each conic convex closed cone $Z \subseteq \mathbb{R}^n$, the object $\omega_Z \in \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp})$ has the structure of an idempotent algebra under convolution product. Localizing at ω_Z in $\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp})$ mirrors the localization at Zariski open in $\mathrm{QCoh}(Y/\mathbb{T})$. As we will see, this plays a central role in the later investigation.

2. Regarding category theory, we perform several constructions in higher algebra with multiplicative functoriality. On the QCoh side, we construct highly coherent symmetric monoidal functors taking (multi-)filtered spectra to quasi-coherent sheaves on spectral toric stacks. This is a mild generalization of the work in [6], but our method is different and more direct. On the $\mathcal{S}$hv side, we explain how the assignment
$$\mathrm{ClosedSubset}(\mathbb{R}^n) \to \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp}) : Z \mapsto \omega_Z$$
is multiplicative for the Minkowski sum on the source and convolution product on the target. This turns out to be a very general construction with abstract six operations and I am looking into its further application.

3. Regarding micro-local geometry, we carefully define singular support in the generality we need. It is unfortunate that the current literature doesn't support the full strength of micro-local analysis of sheaves in the setting of (unbounded) large category, and we dance around this problem making use of Fourier-Sato transform to define singular support for conic sheaves at the origin. We note that the main theorem in particular implies that sheaves on $\mathbb{R}^n$ with prescribed singular support Λ_Y are closed under convolution product. This is not at all a formal consequence - the interaction of convolution product (!-pushforward) and constructibility (singular support) is subtle - and we play a very intricate trick to prove it. This closure property has been used in [9], though not justified.

We end this part with the following examples.

Example. Let $\Lambda \subset T^*\mathbb{R}^1$ be the following subset:
$$\Lambda = \mathbb{R}^1 \times \{0\} \cup \bigcup_{n \in \mathbb{Z}} \{n\} \times \mathbb{R}_{\geqslant 0} \subseteq \mathbb{R}^2 \cong T^*\mathbb{R}^1$$
shaped as an 'infinite fork' pointing upwards. In the paper we have produced the following equivalence of symmetric monoidal categories:
$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \cong \mathrm{Fun}(\mathbb{Z}_{\leqslant}, \mathrm{Sp}) \cong \mathcal{S}\mathrm{hv}_\Lambda(\mathbb{R}^1; \mathrm{Sp}).$$
The left-hand side equivalence (as in [6]) provides the algebro-geometric way to describe filtered objects. The right-hand side equivalence provides a micro-local geometric way to think about filtered objects.

Example. Let $Y = \mathbb{P}^1$ be the flat projective line defined over the sphere spectrum. The main theorem implies

$$\mathrm{QCoh}(\mathbb{P}^1) \cong \mathrm{Cons}_{\mathrm{P}}(S^1; \mathrm{Sp})$$

where right-hand side is the category of sheaves on S^1 locally constant away from the origin. The reader should compare this to the Fourier theory for torus: here we allow for more sheaves than the locally constant ones. Passing via exodromy [4], we obtain a spectral lift of Beilinson's theorem:

$$\mathrm{QCoh}(\mathbb{P}^1) \cong \mathrm{Fun}(\bullet \rightrightarrows \bullet, \mathrm{Sp}).$$

Paper B: A study of sheaves on real vector spaces

This paper 'A study of sheaves on real vector spaces' is joint work with Robert Burklund. We answer the following elementary question:

Question. Let k be a field and consider the symmetric monoidal category

$$(\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k), *)$$

where the convolution product of two sheaves $\mathcal{F}$ and $\mathcal{G}$ is defined as

$$\mathcal{F} * \mathcal{G} = +_!(\mathcal{F} \boxtimes \mathcal{G}).$$

What are the convolution invertible objects in this category?

This is a special case of the more general question asked by Oscar Harr and Branko Juran :

Question. Let G be a Lie group. What are the convolution invertible objects in $\mathcal{S}\mathrm{hv}(G; \mathrm{Mod}_k)$?

It is easy to produce examples of convolution invertible objects in such a category.

Example. Let $g \in G$ be an element. Then the skyscraper sheaf $\underline{k}_g$ is convolution invertible. Its inverse is $\underline{k}_{g^{-1}}$.

I find it very surprising that one can produce many more examples than the skyscraper sheaves:

Example. In Paper A we have seen the symmetric monoidal functor

$$\kappa : \mathrm{QCoh}(\mathbb{P}^1_k) \longrightarrow \mathcal{S}\mathrm{hv}(S^1; \mathrm{Mod}_k).$$

The image of $\mathcal{O}(1) \in \mathrm{QCoh}(\mathbb{P}^1_k)$ under this functor is $\omega_{(0,1)}$ - the constructible sheaf which has stalk $k[1]$ away from the origin and stalk 0 at the origin. Since $\mathcal{O}(1)$ is tensor invertible, we thus know that $\omega_{(0,1)}$ is also convolution invertible. It is a fun exercise for the reader to find out its inverse.

Our main theorem is the following:

Theorem. The group of convolution invertible objects can be computed as

$$\pi_0\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k), *) = \mathrm{Func}^{\mathrm{cts}}(S^{n-1}, \mathbb{R}) \times \pi_0\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k), \otimes),$$

where $\mathrm{Func}^{\mathrm{cts}}$ is the additive group of continuous functions.

Example. When $n = 1$, we have $\mathrm{Func}^{\mathrm{cts}}(S^0, \mathbb{R}) \cong \mathbb{R} \times \mathbb{R}$, parametrizing the start point and the end point of an interval. The reader might compare this to the example with S^1 above.

Now we comment on the proof idea for $n = 1$. The following steps were implicitly taken when Efimov did computation with continuous K-theory of the same category in [1], which inspired our approach.

1. First off, one can do the computation by localizing at idempotent algebras constructed in Paper A and then gluing back. The idempotent algebra of interest is $\omega_{(0,\infty)}$ and we are led to study the category
$$\mathrm{Mod}_{\omega_{(0,\infty)}} \mathcal{S}\mathrm{hv}(\mathbb{R}^1; \mathrm{Mod}_k).$$

2. After localization, the category $\mathrm{Mod}_{\omega_{(0,\infty)}} \mathcal{S}\mathrm{hv}(\mathbb{R}^1; \mathrm{Mod}_k)$ has an explicit, algebraic description: it is the category of complete $\mathbb{R}$-filtered modules. In other words, it can be identified with the category
$$\{X(-) : \mathbb{R}_{\leqslant} \longrightarrow \mathrm{Mod}_k : \forall r \in \mathbb{R}, X(r) \cong \varprojlim_{r<s} X(s)\}.$$

3. Finally, one can do homological algebra with this category to compute its Picard group.

The main difficulty in the final step (and in general) is that the categories involved are not compactly generated and the unit of the monoidal structure is not compact. In the end there are very few tools from category theory to use to control the look of a tensor invertible object, and we have to do something very concrete.

It is also worth mentioning that the category $\mathrm{Mod}_{\omega_{(0,\infty)}} \mathcal{S}\mathrm{hv}(\mathbb{R}^1; \mathrm{Mod}_k)$ admits a presentation as an almost module category. This was already observed by Dmitry Vaintrob in [8] (who pushed this idea even further and produced for each n an almost mathematics object along with with an equivalence between its category of almost modules and $\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k)$). Our computation is very motivated by Vaintrob's construction. Ofer Gabber has taught us that the computation in the last step above and be done via homological algebra in almost mathematics.

Now we comment on the proof of the general case. We have suggested two ways to do the computations for $\mathbb{R}^n$.

1. The first approach is a direct generalization of the method above, with Radon transform as in [3]. However, the technology required to produce symmetric monoidal structure on the relevant functor is not there in the literature, so what we have is no more than a sketch.

2. The second approach, suggested to us by Peter Scholze, is done via the new technology of wild Betti sheaves [7]. This is very concrete and we have managed to prove along the way other interesting things about wild Betti sheaves.

Unfortunately, we have no idea how to answer the question of Harr and Juran in general. We don't have any clue for $G = SL_2(\mathbb{R})$.

Example. Finally we turn back to compare with the Fourier theory of torus. As noted before, it is very uncommon for an algebraic object to have the property that its $\mathrm{QCoh}(-)$ is equivalent to $\mathcal{S}\mathrm{hv}(-)$ of some topological space. However, in the case of S^1, the idea of Dmitry Vaintrob mentioned above actually produces an almost mathematic object $\mathcal{Y}$ such that

$$\mathcal{S}\mathrm{hv}(S^1; \mathrm{Sp}) \cong \mathrm{QCoh}(\mathcal{Y})$$

where we abusively denoted the category of almost modules by $\mathrm{QCoh}(\mathcal{Y})$. The object $\mathcal{Y}$ appears as an universal compactification of the torus $\mathbb{G}_m$, in particular it comes with an embedding

$$\mathrm{QCoh}(\mathbb{G}_m) \longrightarrow \mathrm{QCoh}(\mathcal{Y}),$$

which, in terms of sheaves, is the embedding of locally constant sheaves into all sheaves. It would be very interesting to find other uses of this object $\mathcal{Y}$. Moreover, if we think of $\mathcal{Y}$ as attached to the character variety of reductive group $\mathbb{G}_m$, it would be very interesting to find similar constructions for character varieties of other reductive group G.

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Paper A: Toric mirror symmetry for homotopy theorists

This chapter contains the following paper:

Qingyuan Bai and Yuxuan Hu. *Toric mirror symmetry for homotopy theorists.* 2025.

A preprint version of this manuscript can be found on the server arxiv:2501.06649

Toric Mirror Symmetry for Homotopy Theorists

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Abstract

We construct functors sending torus-equivariant quasi-coherent sheaves on toric schemes over the sphere spectrum to constructible sheaves of spectra on real vector spaces. This provides a spectral lift of the toric homological mirror symmetry theorem of Fang-Liu-Treumann-Zaslow. Along the way, we obtain symmetric monoidal structures and functoriality results concerning those functors, which are new even over a field. We also apply the de-equivariantization technique to prove a 'non-equivariant' version of the theorem, giving both a new proof and generalization of Kuwagaki's result over $\mathbb{C}$. As a concrete application, we obtain an alternative proof of Beilinson's linear algebraic description of quasi-coherent sheaves on projective spaces with spectral coefficients.

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1 Introduction

This paper proves a 'spectral mirror symmetry' theorem about toric schemes over the sphere spectrum $\mathbb{S}$. To state our result, we need the notion of toric geometry in spectral algebraic geometry.

Recollection. Given a toric fan Σ in $\mathbb{R}^n$, Lurie constructed in [26] a spectral algebraic scheme X_Σ defined over the base ring $\mathbb{S} \in \mathrm{CAlg}(\mathrm{Sp})$, equipped with an action by the torus $\mathbb{T} := \mathrm{Spét}(\mathbb{S}[\mathbb{Z}^n])$. The construction is almost the same as the classical construction of toric variety, but uses the spectral monoid algebra $\mathbb{S}[M]$ for a monoid M. The precise definition will be recalled later. We call X_Σ the toric scheme over $\mathbb{S}$ associated to Σ. It is a lift to the sphere spectrum of the classical toric variety $X_{\Sigma,\mathbb{C}}$ defined over the field $\mathbb{C}$: there is a canonical isomorphism $X_\Sigma \otimes_{\mathbb{S}} \mathbb{C} \cong X_{\Sigma,\mathbb{C}}$.

Our main result is the following construction:

Main Theorem. Let Σ be a smooth projective fan in $\mathbb{R}^n$. There is a colimit-preserving, fully faithful and symmetric monoidal functor

$$\kappa : \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp}).$$

The image of κ are precisely those sheaves on $\mathbb{R}^n$ which are constructible for a periodic hyperplane arrangement $\mathcal{S}_\Sigma$, and have singular supports contained in a conic Lagrangian $\Lambda_\Sigma \subseteq T^*\mathbb{R}^n$.

On the left-hand side is the ∞-category of quasi-coherent sheaves on the quotient stack of the toric scheme X_Σ by the torus $\mathbb{T}$ action, equipped with the standard pointwise tensor product. On the right-hand side is the ∞-category of sheaves of spectra on the real vector space $\mathbb{R}^n$, equipped with the convolution tensor product.

The notions of constructible sheaves and singular supports are rather technical. Here we emphasize that the stratification $\mathcal{S}_\Sigma$ and the conic Lagrangian Λ_Σ are defined via explicit formulas. This ensures that an explicit calculation of the image of κ is manageable.

Upon base change to a classical commutative ring, our construction recovers the main theorem of [10], which motivates the present work. We highlight the following major improvements.

- The authors of [10] work over a classical commutative ring R, while our work is carried out over the sphere spectrum $\mathbb{S}$. By base change, the result holds for any connective $\mathbb{E}_\infty$-ring spectrum.
- The construction in [10] uses dg-categories and establishes a monoidal functor, identifying the $\mathbb{E}_1$-monoidal structures on both sides. Working in the setting of ∞-categories, we construct κ as a symmetric monoidal functor, providing an $\mathbb{E}_\infty$-lift of the original statement.
- We show that various expected structural results about κ arise from a fundamental reinterpretation of the category $\Theta(\sigma)$ from [10]. This category parametrizes, naturally in $\sigma \in \Sigma^{\mathrm{op}}$, both (a) a family of abstract (discrete) monoids, and (b) a family of monoids in topological spaces. Case (a) leads us to quasi-coherent sheaves on the toric stack $[X_\sigma/\mathbb{T}]$, while (b) gives us constructible sheaves via the six-functor formalism.
- We demonstrate that these structural results enable us to formally deduce the *non-equivariant* version of the theorem, as envisioned in [10]. This yields a new, conceptually transparent proof of the main result[1] in [23].

[1] In the case where X_Σ is smooth projective.

We begin with the historical motivation involving toric vector bundles and polytopes. We then provide a detailed overview of our main results, including our new contributions and their applications, in Section 1.2. In Sections 1.3 and 1.4, we discuss related work, our inspirations, and the key ideas of our proof strategy.

1.1 Motivation: toric vector bundles and polytopes

In the classical study of smooth projective toric varieties over $\mathbb{C}$, there is a dictionary between ample line bundles and their moment polytopes as explained in [11, Section 3.4]. Morelli observed that vector bundles also fit into this dictionary. He proved in [28, Theorem 7] that there is an injective map from the torus-equivariant Grothendieck K-group of an n-dimensional smooth projective toric variety X to the set of $\mathbb{Z}$-valued constructible functions on the real vector space $\mathbb{R}^n$:

$$K_0^T(X) \longrightarrow \mathrm{Fun}^{\mathrm{cons}}(\mathbb{R}^n; \mathbb{Z}).$$

It becomes a map of commutative rings if one equips the set of constructible functions with pointwise addition and convolution product. This map generalizes the original dictionary: it takes the class of an ample line bundle to the characteristic function on the moment polytope.

Morelli's theorem admits a natural categorification. On the left-hand side, one replaces $K_0^T(X)$ by $D^b_T(X)$, the bounded derived category of torus-equivariant coherent sheaves on X. On the right-hand side, one replaces the ring of constructible functions on $\mathbb{R}^n$ by $D^b_{cc}(\mathbb{R}^n; \mathcal{S}_\Sigma)$, the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on $\mathbb{R}^n$ which are compactly supported and constructible [2] for a stratification $\mathcal{S}_\Sigma$. This stratification $\mathcal{S}_\Sigma$ is a periodic hyperplane arrangement determined by the toric fan Σ for X. The work of Fang-Liu-Treumann-Zaslow [10, Theorem 1.1] constructed a fully faithful functor between dg-categories (named as coherent-constructible correspondence)

$$\kappa_{\mathbb{C}} : D^b_T(X) \longrightarrow D^b_{cc}(\mathbb{R}^n; \mathcal{S}_\Sigma)$$

which recovers Morelli's theorem upon taking K_0. Furthermore, they have provided a description of the image of $\kappa_{\mathbb{C}}$ in terms of singular support.

From another perspective, one might anticipate eliminating the extra data of torus-equivariance. In [4], Bondal independently suggested that, given certain mild assumptions, there exists a fully faithful functor from the bounded derived category of coherent sheaves on X to the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on the topological torus

$$\overline{\kappa}_{\mathbb{C}} : D^b(X) \longrightarrow D^b(\mathbb{R}^n/\mathbb{Z}^n),$$

whose image is constructible for a specific stratification. As it turns out (see [37]) one can define a functor $\overline{\kappa}_{\mathbb{C}}$ in a way similar to $\kappa_{\mathbb{C}}$, and describe the image of $\overline{\kappa}_{\mathbb{C}}$ in terms of singular support. This line of work has been further pursued in [34, 42, 23].

Our work can be seen as a continuation of this narrative within spectral algebraic geometry.

[2] In the strong sense: the stalks have to be perfect.

1.2 What is done in this paper?

From now on we write ∞-category simply as 'category'.

This project begins with the realization that, on the 'constructible' aspect of the discussion, there is a clear extension to the sphere spectrum. Instead of working with the bounded derived category of sheaves of $\mathbb{C}$-vector spaces, one can consider the large category of sheaves of spectra on a real vector space,

$$\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp})$$

where the convolution product is naturally defined. This is made possible by recent advances [41] in the yoga of six-operations. On the 'coherent' side, it is generally difficult to lift varieties to the sphere spectrum. However, it is straightforward to construct lifts of toric varieties, as they are Zariski locally monoid schemes glued together along maps induced by monoid homomorphisms.

As recalled above, given a toric fan Σ in $\mathbb{R}^n$, Lurie has defined in [26] the toric scheme X_Σ over the sphere spectrum, equipped with an action of the torus $\mathbb{T} \cong \mathrm{Sp\acute{e}t}(\mathbb{S}[\mathbb{Z}^n])$. We write $[X_\Sigma/\mathbb{T}]$ for the quotient stack of the action. The main part of this paper concerns the following construction.

Theorem A. Let $\Sigma \subseteq \mathbb{R}^n$ be a smooth projective[3] fan. There exists a colimit-preserving, fully-faithful and symmetric monoidal functor

$$\kappa : \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp}).$$

We explicitly characterize the image of this functor:

$$\mathrm{Im}(\kappa) = \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(\mathbb{R}^n; \mathrm{Sp}) \subseteq \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp}).$$

On the right-hand side is the subcategory of sheaves such that[4]:

1. It is constructible[5] for the stratification $\mathcal{S}_\Sigma$ given by the periodic hyperplane arrangement H_Σ, consisting of the integral translations of the hyperplanes perpendicular to 1-cones in Σ

$$H_\Sigma := \{m + \sigma^\perp : m \in \mathbb{Z}^n, \sigma \in \Sigma(1)\}.$$

2. It has singular support contained in the conic Lagrangian Λ_Σ:

$$\Lambda_\Sigma := \bigsqcup_{m \in \mathbb{Z}^n; \sigma \in \Sigma} m + \sigma^\vee \times -\sigma \subseteq \mathbb{R}^n \times \mathbb{R}^n = T^*\mathbb{R}^n,$$

where the dual cone $\sigma^\vee \subseteq \mathbb{R}^n$ is the collection of vectors x such that the inner product $(x, -)$ is nonnegative on σ.

Remark 1.2.1. 1. For simplicity, we have identified the vector space $\mathbb{R}^n$ with its dual by the standard inner product. In the main text we use the cocharacter lattice N and the character lattice M to carefully distinguish between the two.

[3] See Notation 3.1.1

[4] It is possible to remove the assumption on constructibility once one has a good understanding of singular supports in greater generality, see Warning 5.1.10.

[5] Unless specified, we always mean constructible in the weak sense: there will be no constraints on the size of the stalk.

2. The category $\mathrm{QCoh}([X_\Sigma/\mathbb{T}])$ has the standard pointwise tensor product, while the category $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})$ has the convolution product defined in Construction 4.1.2.

3. By base change, the similar statement is valid for any other connective $\mathbb{E}_\infty$-ring spectrum E in place of $\mathbb{S}$.

We also provide compatibility of the functor κ with the action of $\mathrm{QCoh}(B\mathbb{T})$ on both sides.

Theorem B. The functor κ in Theorem A fits into the following diagram in $\mathrm{CAlg}(\mathrm{Pr}^L)$

$$\begin{tikzcd} \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \arrow[r, "\kappa"'] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(\mathbb{R}^n;\mathrm{Sp}) \\ \mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\pi^*"] \arrow[r, "\simeq"'] & \mathcal{S}\mathrm{hv}(\mathbb{Z}^n;\mathrm{Sp}). \arrow[u, "i_!"'] \end{tikzcd}$$

Remark 1.2.2. 1. Both categories on the left have standard tensor product and both categories on the right have convolution product.

2. The functor π^* is $*$-pullback along the projection $\pi : [X_\Sigma/\mathbb{T}] \to B\mathbb{T}$. The functor $i_!$ is !-pushforward along the inclusion of topological groups $i : \mathbb{Z}^n \to \mathbb{R}^n$.

3. We have used the identification of symmetric monoidal categories

$$\mathrm{QCoh}(B\mathbb{T}) \simeq \mathrm{Fun}(\mathbb{Z}^n, \mathrm{Sp}) \simeq \mathcal{S}\mathrm{hv}(\mathbb{Z}^n;\mathrm{Sp})$$

explained in Theorem 3.3.10 and the proof of Theorem 6.1.5.

The above Theorem A and B could be found in the main text as Definition 4.3.8 and Corollary 5.3.4.

From this we deduce several consequences. First of all, we apply the technique of de-equivariantization and obtain the following theorem, which is a 'non-equivariant' version of Theorem A.

Theorem C (Corollary 6.1.9)**.** Let $\Sigma \subseteq \mathbb{R}^n$ be a smooth projective fan. There exists a colimit-preserving, fully-faithful and symmetric monoidal functor

$$\overline{\kappa} : \mathrm{QCoh}(X_\Sigma) \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n/\mathbb{Z}^n;\mathrm{Sp}).$$

The image of $\overline{\kappa}$ can be described by constructibility and singular support in a similar way:

$$\mathrm{Im}(\overline{\kappa}) = \mathcal{S}\mathrm{hv}_{\overline{\Lambda}_\Sigma}(\mathbb{R}^n/\mathbb{Z}^n;\mathrm{Sp}) \subseteq \mathcal{S}\mathrm{hv}(\mathbb{R}^n/\mathbb{Z}^n;\mathrm{Sp}).$$

On the right-hand side is the subcategory of sheaves on the topological torus $\mathbb{R}^n/\mathbb{Z}^n$ which are constructible for $\overline{\mathcal{S}}_\Sigma$ and have singular support contained in $\overline{\Lambda}_\Sigma$.

Remark 1.2.3. 1. The category of quasi-coherent sheaves has the standard pointwise tensor product, while the category of sheaves on the topological torus has convolution product.

2. The stratification $\overline{\mathcal{S}}_\Sigma$ (conic Lagrangian $\overline{\Lambda}_\Sigma$) is the image of $\mathcal{S}_\Sigma$ (Λ_Σ) under projection map $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ ($d\pi : T^*\mathbb{R}^n \to T^*\mathbb{R}^n/\mathbb{Z}^n$).

Remark 1.2.4. In [10], the authors remarked that 'it is almost but not quite possible to deduce the non-equivariant results from equivariant results by some formal argument'. We show that such deduction is possible in our setting. The reason we can make such an argument is that we work systematically with large (presentably stable) categories, plus the full strength of the higher categorical technology of Tannakian formalism from [SAG, Chapter 9].

As an application, we reprove Beilinson's theorem about the projective line (and also projective spaces) by combining Theorem C and the exodromy equivalence from [14].

Theorem D (Example 6.2.1)**.** There is an equivalence of categories:

$$\mathrm{QCoh}(\mathbb{P}^1_{\mathbb{S}}) \simeq \mathrm{Fun}(\bullet \rightrightarrows \bullet, \mathrm{Sp}).$$

The following two remarks concern the conceptual consequences of this paper that are related to previous works.

Remark 1.2.5. The de-equivariantization in Theorem C can be thought of as performing base change along the symmetric monoidal functor

$$\mathrm{colim} : \mathrm{Fun}(\mathbb{Z}^n, \mathrm{Sp}) \longrightarrow \mathrm{Sp}.$$

More generally, one can obtain a relative (to a presentably symmetric monoidal category $\mathcal{C}$) version of toric constructions as in Definition 6.3.1 by base changing along other colimit-preserving symmetric monoidal functors out of $\mathrm{Fun}(\mathbb{Z}^n, \mathrm{Sp})$: in particular this recovers the result of the second named author and Pyongwon Suh [19] relating quasi-coherent sheaves on a toric fibration to a category of sheaves on the topological torus with twisted coefficient category, see Example 6.3.3.

Remark 1.2.6. We also offer a conceptual framework for the 'log-perfectoid mirror symmetry' as introduced by Dmitry Vaintrob, ensuring that (a large category version of) [39, Theorem 2] holds symmetric monoidally over $\mathbb{S}$. See Remark 4.5.8 for the connection to Dmitry Vaintrob's work on log quasi-coherent sheaves. This may serve as a motivation for Efimov's computation with continuous K-theory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1; \mathrm{Sp})$ in [9]. See [2] for an expository account of these materials. In that note, this perspective is used to compute the Picard group of the symmetric monoidal category $\mathcal{S}\mathrm{hv}(\mathbb{R}^1; \mathrm{Sp})$ equipped with convolution product.

1.3 Related work and inspirations

Needless to say, there have been numerous papers on toric mirror symmetry and we can only mention an incomplete list of references in this introduction. We will now list some of them that inspired our project. We then provide some justifications for our (unfortunately, long) writing.

Remark 1.3.1 (Proof ideas from the literature)**.** Most of the ideas in this paper have appeared in one way or another in the literature. The main proof method is to adapt the constructions of [10] in the context of large categories, $\mathbb{S}$ coefficient, and with symmetric monoidal structures. The method of localization along idempotent algebras has been used in [23] disguised as the Tamarkin projector. The proof we present for the characterization of the image in terms of singular supports is taken from [42]. Finally, the idea of applying de-equivariantization in this story was spelled out in [35].

Remark 1.3.2 (Dropping assumptions on smoothness and projectivity)**.** The restriction on the smoothness and projectivity of the fan is removed in [23]. But we do not pursue the generality as in there.

Remark 1.3.3 (Necessity of higher algebra)**.** It is clear that in this story of coherent-constructible correspondence, higher categorical techniques are needed in constructing the functors and characterizing images. Here we give a presentation without directly using model categories or dg-categories. For comparison, it would be difficult to articulate the convolution product on the category of sheaves on a real vector space as a symmetric monoidal structure in terms of derived category of sheaves. These kinds of difficulties would only add up when one works with spectral coefficients It appears to us that applying the language of higher algebra is the most convenient way to spell out the details.

Remark 1.3.4 (Large categories)**.** In this paper, we systematically work with large (presentable stable) categories. This approach simplifies certain constructions involving 'generators', which tend to be more intricate with small categories. Another significant reason for maintaining this level of generality stems from our interest in $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})$: since it is not compactly generated, there is no obvious reason to hope for an algebro-geometric mirror object Y such that

$$\mathrm{QCoh}(Y) \xrightarrow{\simeq} \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp}).$$

The sheaf category is however dualizable in the sense of [SAG, Appendix D] and [9], with a presentably symmetric monoidal structure of convolution. Inspired by utility of such (dualizable but not compactly generated) categories in analytic geometry, one would hope to get a better understanding of them. For example, Dmitry Vaintrob [39] constructs an almost mathematics object Y as a mirror for $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})$. In other words, his 'log-perfectoid' construction provides such Y with $\mathrm{QCoh}(Y) \simeq \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})$. This should be thought of as algebraization of the sheaf category.

Remark 1.3.5 (Mirror symmetry over the sphere spectrum)**.** It is widely expected that one can define a version of Fukaya categories over the sphere spectrum for a (nice) symplectic manifold equipped with certain orientation data (see, for example, [1, 27, 31, 21] for various perspectives of works towards a definition). The $\mathbb{Z}$-linear equivalence between Fukaya categories and (microlocal) sheaf categories supplied by [12] should carry over to this new setting. In particular, modeling the $\mathbb{S}$-linear Fukaya category of $T^*\mathbb{R}^n$ (with stop given by Λ_Σ) by $\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(\mathbb{R}^n;\mathrm{Sp})$, our result may be interpreted as an instance of $\mathbb{S}$-linear mirror symmetry.

Remark 1.3.6 (Higher structures from mirror symmetry)**.** Another reason for us to implement mirror symmetry over $\mathbb{S}$ is the hope that it would motivate constructions in category theory and homotopy theory. A wonderful example of such an advance is provided in [25] where Lurie made the observation that Waldhausen's S-construction is corepresented by cosimplicial objects $\mathrm{Quiv}^\bullet$ and this family of objects has certain coparacyclic structure. In fact, symplectic geometry provides a motivation for such an observation (see [36, Section 1.2] for more on this): each Quiv^n can be seen (after 2-periodization) as an $\mathbb{S}$-linear topological Fukaya category on the 2-dimensional disc with $n+1$ stoppings on the boundary, and the (para)cyclic symmetry comes from rotations of the disc. The actual construction of $\mathrm{Quiv}^\bullet$ however, runs on the 'mirror' side, i.e., as the category of matrix factorizations in spectral algebraic geometry. It is possible to relate the content of this paper to the above story in the following way: it was explained in [30, 12] that topological Fukaya categories could be modeled locally, by (the microlocalization of) the category of sheaves with prescribed singular supports. We hope that the description of such categories in terms of algebraic geometry might help with constructions of higher structures, such as those suggested in [36, 8].

1.4 Proof strategy

Now we explain the proof strategy along with some technicalities in the paper.

Remark 1.4.1. The construction of the functor κ comes in two parts.

1. First, we construct κ for affine toric varieties indexed by $\sigma \in \Sigma$. This is implemented by the following correspondence:
$$\mathrm{QCoh}([X_\sigma/\mathbb{T}]) \xleftarrow{\simeq} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp}),$$
where the middle category is the category of presheaves on a poset. See the later remark for a quick idea of what this poset $\Theta(\sigma)$ looks like and how to construct these functors. Once the functors are constructed, one can follow the arguments from [29] to show that the functor on the left is an equivalence.

2. The second step involves gluing: for the inclusion of cones $\sigma \subseteq \tau$, there is a symmetric monoidal restriction functor
$$\mathrm{QCoh}([X_\tau/\mathbb{T}]) \longrightarrow \mathrm{QCoh}([X_\sigma/\mathbb{T}]).$$
One can think of this as a diagram indexed by $\sigma \in \Sigma^{\mathrm{op}}$ and Zariski descent implies that the limit of this diagram is the category $\mathrm{QCoh}([X_\Sigma/\mathbb{T}])$. The construction in the first step is compatible with the restriction functor, allowing one to take limits on the sheaf category side to obtain the functor κ.

Remark 1.4.2. The functors in the previous remark are constructed as follows.

1. The 1-category $\Theta(\sigma)$ is an explicit poset:
$$\Theta(\sigma) = \{m + \sigma^\vee : m \in \mathbb{Z}^n\},$$
the set of integral translations of the dual cone $\sigma^\vee$ under inclusion relation. It acquires a symmetric monoidal structure by the Minkowski sum. For example, with $\sigma = [0, \infty) \subseteq \mathbb{R}^1$, the corresponding poset is just $\mathbb{Z}_{\leqslant}$, the poset of integers equipped with addition.

2. To construct the correspondence, note that the presheaf category has the Day convolution monoidal structure, so it suffices to construct (lax) symmetric monoidal functors out of $\Theta(\sigma)$. It turns out this is not an easy task, and we explain how we overcome the difficulties below.

Remark 1.4.3 (Constructing functors into QCoh)**.** A prototypical example of the functor we will construct that maps into $\mathrm{QCoh}([X_\sigma/\mathbb{T}])$ is the symmetric monoidal functor
$$\mathrm{Fun}(\mathbb{Z}_{\leqslant}, \mathrm{Sp}) \longrightarrow \mathrm{QCoh}([\mathbb{A}^1/\mathbb{G}_m])$$
which classifies the universal line bundle $\mathcal{O}(1)$ and the universal section $\cdot x : \mathcal{O} \to \mathcal{O}(1)$ (see [29]). Note this says in particular that the line bundle $\mathcal{O}(1)$ is a strict Picard element. See [SAG, Warning 5.4.3.3] for more on this notion of strictness and why it is not completely trivial to construct such a symmetric monoidal functor - it asks in particular to trivialize all the cohomological operations on the object $\mathcal{O}(1)$ coming from the $\mathbb{E}_\infty$-operad. Our method of construction passes through an unstable (set-valued, actually) model of such data, which provides an alternative construction of the functor in the proof of [29, Theorem 4.1]. We will also construct a slightly generalized version of this with target being $\mathrm{QCoh}([\mathbb{A}^n/\mathbb{G}_m^n])$.

Remark 1.4.4 (Constructing functors into $\mathcal{S}hv$)**.** A prototypical example of the functor we will construct that maps into $\mathcal{S}hv(\mathbb{R}^n; Sp)$ (equipped with convolution) is a lax symmetric monoidal functor

$$\mathrm{Fun}(\mathbb{Z}_{\leqslant}, Sp) \longrightarrow \mathcal{S}hv(\mathbb{R}^1; Sp)$$

which sends $n \in \mathbb{Z}$ to the dualizing sheaf on the open half line $\omega_{(-\infty,n)}$. This is achieved by making a more general construction: given a commutative monoid M in the category of locally compact Hausdorff spaces, we construct a lax symmetric monoidal structure on the relative homology functor taking a pair $(X, f : X \to M)$ to $f_! f^! \omega_M \in \mathcal{S}hv(M; Sp)$. With this functor at hand, the problem is reduced to 1-categorical manipulations. This general construction is very much inspired by [13, Chapter 3], and we believe that it has other interesting uses.

Remark 1.4.5 (Gluing in $\mathcal{S}hv$)**.** To make the gluing procedure precise, we will prove a sheaf-theoretic counterpart of Zariski descent in $\mathcal{S}hv(\mathbb{R}^n; Sp)$. This is implemented with idempotent algebras as in [HA, Definition 4.8.2.8]. For example, the dualizing sheaf $\omega_{(-\infty,0)}$ is an idempotent algebra in $\mathcal{S}hv(\mathbb{R}^1; Sp)$ for the convolution product, and this phenomenon generalizes to other cones. Given a smooth projective fan Σ, we will produce a collection of idempotent algebras in $\mathcal{S}hv(\mathbb{R}^n; Sp)$ and show that their meet is the unit $\mathbb{1}_{\mathcal{S}hv(\mathbb{R}^n;Sp)}$ as an idempotent algebra.

Remark 1.4.6 (Singular supports for polyhedral sheaves)**.** To characterize the image of κ, we make use of the recent advances [7, 14] of the exodromy equivalence with large category of constructible sheaves. Following [10], we supply a definition of singular support for sheaves constructible for affine hyperplane arrangements - via the Fourier-Sato transform. Compare the general definition laid out in [21]. This definition of singular support provides a convenient setup for us to apply the non-characteristic deformation lemma [33] and obtain the characterization of image of κ. The proof presented here supplements some details missing in [42].

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1.6 Conventions

Notation 1.6.1 (Category theory)**.** We don't touch on set-theoretic issues in this paper. We write Cat for the $(\infty, 1)$-category of quasi-categories[6], functors, natural isomorphisms and so on. We refer to objects in Cat as 'categories' to avoid putting ∞ in front of everything. This however makes

[6] We follow [HTT] and use quasi-categories, but our argument is in fact independent of the model.

us write 'stable category' instead of more established name 'stable ∞-category'. We identify a 1-category with its nerve in Cat and stress that it is 1-category when we have one. We write Spc for the category of spaces (or homotopy types, or anima) and Sp for the stable category of spectra. We write Map for mapping spaces in a category and map for mapping spectra in a stable category.

Notation 1.6.2 (Simplicial objects)**.** By Δ we mean the (1-)category of non-empty ordered finite sets and order preserving maps between them. A (co)simplicial diagram in $\mathcal{C}$ is a functor from $(\Delta)\Delta^{op}$ to $\mathcal{C}$. We only draw face maps when visualizing a (co)simplicial diagram. We write d^i for the structure (face) maps in a cosimplicial diagram.

Notation 1.6.3 (Symmetric monoidal categories)**.** We write $(\mathcal{C}, \otimes)$ for a symmetric monoidal category and often refer to $\mathcal{C}$ as a symmetric monoidal category, omitting the monoidal structure. We write $\mathcal{C}^{\otimes}$ for the underlying operad of $(\mathcal{C}, \otimes)$. We write $\mathrm{CAlg}(\mathcal{C}, \otimes) := \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^{\otimes})$ for the category of $\mathbb{E}_\infty$-algebras in $\mathcal{C}$. And when there is no danger of confusion, we will omit the monoidal structure and write $\mathrm{CAlg}(C)$. For example, CAlg(Sp) would refer to the category of $\mathbb{E}_\infty$-ring spectra. In the special case of Set or Spc equipped with Cartesian symmetric monoidal structure, we also write CMon for the category of commutative monoids and CGrp for the category of commutative groups. All the presheaf categories are assumed to carry Day convolution structure when considered as a symmetric monoidal category.

Notation 1.6.4 ((Lax) symmetric monoidal functors)**.** For two symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ for the category of symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. We write $\mathrm{Fun}^{\mathrm{lax}\otimes}(\mathcal{C}, \mathcal{D})$ for the category of symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. We write SMCat for the category of symmetric monoidal categories and (strongly) symmetric monoidal functors between them.

Notation 1.6.5 (Algebraic geometry)**.** We approach spectral algebraic geometry through the functor of points. We write Stk for the full subcategory of fpqc sheaves inside $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathrm{Spc})$ (what's better, the objects we are dealing with in this paper are all geometric stacks in the sense of [SAG, Definition 9.3.0.1]), and we write $\mathrm{Sp\acute{e}t}(-)$ for the Yoneda functor $\mathrm{CAlg}^{\mathrm{cn,op}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathrm{Spc})$ which factors through Stk (In SAG, Spét was used for another construction, but Lurie has provided a comparison with this Yoneda point of view in [SAG, Proposition 1.6.4.2]).

Notation 1.6.6 (Topological spaces)**.** We write LCH for the (1-)category of locally compact Hausdorff spaces and continuous maps between them. However, all topological spaces of interest in this paper are finite dimensional manifolds. We often write $j_U : U \to X$ for the inclusion of an open subset and $i_Z : Z \to X$ for the inclusion of a closed subset. We say a map $f : X \to Y$ is étale if it is a local homeomorphism.

Notation 1.6.7 (Sheaf theory)**.** It will be very convenient for us to extract a 'six-functor formalism' out of [41] on the category of locally compact Hausdorff topological spaces. We write $\mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$ for the category of sheaves of spectra on a locally compact Hausdorff topological space X, and we write $f^* \dashv f_*$, $f_! \dashv f^!$ and $\otimes \dashv \mathrm{Hom}$ for the six functors that come with it. For an open $U \subseteq X$, we write $\underline{\mathbb{S}}_U \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$ for the sheafification of the $\mathbb{S}$-linearized presheaf represented by U. In other words, if we write $j_U : U \to X$ for the inclusion map and $\underline{\mathbb{S}} \in \mathcal{S}\mathrm{hv}(U; \mathrm{Sp})$ for the constant sheaf valued at $\mathbb{S}$, $\underline{\mathbb{S}}_U$ is equivalently

$$\underline{\mathbb{S}}_U := j_{U!}\underline{\mathbb{S}} \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$$

and we abusively call it the representable sheaf on U. Note that $\underline{\mathbb{S}}_X$ is just constant sheaf valued at $\mathbb{S}$ on X. Similarly for a closed subset $Z \subseteq X$ we write

$$\underline{\mathbb{S}}_Z := i_{Z*}\underline{\mathbb{S}} \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp}).$$

We reserve the symbol ω for dualizing sheaves. Let $p : X \to *$ be the canonical map to the final object. The dualizing sheaf of X is defined to be

$$\omega_X := p^!\underline{\mathbb{S}} \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp}).$$

Similarly, when we work with an open subset U or closed subset Z in X, we write

$$\omega_U := j_{U!}j_U^!\omega_X \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$$

and

$$\omega_Z := i_{Z!}i_Z^!\omega_X \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp}).$$

2 Combinatorial model

In [10, Section 3] the authors defined a poset $\Gamma(\Sigma, M)$ that interpolates between the category of quasi-coherent sheaves and the category of constructible sheaves. In this section, we recall their definition and present functoriality of the definition. Refer to Notation 3.1.1 for explanations of lattices, cones, fans, and associated topics if you are unfamiliar with them.

Definition 2.0.1 (Poset of cones)**.** Given a pair of lattice and fan (N, Σ), we consider Σ as a poset as follows: the objects of Σ are cones $\sigma \in \Sigma$ and morphisms between two cones are inclusions.

Definition 2.0.2. Let $\mathrm{Poly}(M_{\mathbb{R}})$ be the poset of closed polyhedral subsets in $M_{\mathbb{R}}$ (the subsets that can be written as a Minkowski sum of a polytope and a polyhedral cone) with the morphisms being inclusions. This is a symmetric monoidal 1-category if we take Minkowski sum $+$.

Definition 2.0.3 (The Θ category)**.** Fix a cone $\sigma \subset N_{\mathbb{R}}$, the (1-)category $\Theta(\sigma)$ is defined as the full subcategory of posets of closed subsets in $M_{\mathbb{R}}$:

$$\Theta(\sigma) \subseteq \mathrm{Poly}(M_{\mathbb{R}}).$$

It is spanned by objects of the form $m + \sigma^{\vee}$ for $m \in M$.

Observe that the association $\sigma \mapsto \Theta(\sigma)$ is functorial in σ that it assembles into a functor

$$\Theta(-) : \Sigma^{\mathrm{op}} \to \mathrm{Cat}.$$

Given an inclusion $i : \sigma \to \tau \in \sigma$ of cones, the induced functor is

$$\begin{aligned} \Theta(i) \colon \Theta(\tau) &\to \Theta(\sigma) \\ m + \tau^{\vee} &\mapsto m + \sigma^{\vee}. \end{aligned}$$

Remark 2.0.4 (Symmetric monoidal structures on $\Theta(-)$)**.** We make the following observations:

1. Each $\Theta(\sigma)$ has the structure of a symmetric monoidal (1-)category. This could be obtained by observing that as a full subcategory, $\Theta(\sigma)$ inherits a (non-unital) symmetric monoidal structure from the symmetric monoidal category $(\mathrm{Poly}(M_{\mathbb{R}}), +)$. To make it unital, it suffices to note that $\sigma^{\vee} \in \Theta(\sigma)$ acts as a tensor unit.

2. We might also observe that $\sigma^{\vee}$ is an idempotent algebra in $(\mathrm{Poly}(M_{\mathbb{R}}), +)$ and define $\Theta(\sigma)$ to be a full subcategory of $\mathrm{Mod}_{\sigma^{\vee}}(\mathrm{Poly}(M_{\mathbb{R}}))$, and it directly follows that $\Theta(\sigma)$ inherits a symmetric monoidal structure.

3. For each inclusion $i : \sigma \to \tau$, $\Theta(i)$ has the structure of a symmetric monoidal functor which can be observed directly since we are working with posets: there is no coherence issue. *In conlusion*, $\Theta(-)$ lifts to a functor $\Sigma^{\mathrm{op}} \to \mathrm{SMCat}$.

4. For later use, consider the discrete category of M with symmetric monoidal structure given by addition. There are symmetric monoidal functors

$$p_{\sigma} : M \longrightarrow \Theta(\sigma) : m \mapsto m + \sigma^{\vee}$$

and they assemble into a natural transformation between diagrams in SMCat indexed by $\sigma \in \Sigma^{\mathrm{op}}$ where the source is thought of as a constant diagram.

Remark 2.0.5 (Comparison with other models)**.** Our definition of $\Theta(-)$ works cone by cone, while in [40, Section 5][10, Section 3] global categories were proposed. Later we will see that one wants to compute

$$\lim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(-)^{\mathrm{op}}, \mathrm{Sp}).$$

An explicit presentation of the limit of such a diagram of presheaf categories (with arrows given by left Kan extensions of functors) remains unclear to us. However, '(co)sheaves for Morelli topology' as in [40, Section 6] seems like a combinatorial presentation of the limit.

3 Toric geometry

Classically, toric geometry builds on the linearization functor $\mathbb{Z}[-] : \mathrm{CMon}(\mathrm{Set}) \to \mathrm{CAlg}(\mathrm{Ab})$. For example, $\mathbb{Z}[\mathbb{N}] = \mathbb{Z}[X]$ is the one-variable polynomial ring. Toric schemes are constructed from the $\mathrm{Spec}(-)$ of these monoid algebras by gluing along maps coming from $\mathrm{CMon}(\mathrm{Set})$. In this section we present some basic materials on flat toric geometry over $\mathbb{S}$ [7]. Although we will not prove it, the toric schemes we define here will be flat over the base ring $\mathbb{S}$. The term 'flat' is also reminiscent of the fact that after base changing to $\mathbb{Z}$, it recovers the classical construction of toric schemes. The idea of looking at flat toric scheme over $\mathbb{S}$ is certainly well-known, going back to [26] and [SAG, Remark 5.4.1.9]. Most of the discussions here would be rather formal: we are mainly interested in the category of quasi-coherent sheaves and related categorical nonsense.

In the first part, we fix notations for the toric construction and explain how the action diagram presents the quotient stack by the torus action. In the second part, we recall the functoriality of quasi-coherent sheaves and provide an unstable model for quasi-coherent sheaves on the quotient stack. This is used in the third part, where we construct combinatorial-coherent comparison functor. Finally we follow the approach of [29] to show that this functor is an equivalence.

3.1 Recollections on toric geometry

Notation 3.1.1. We recall the following notations useful in the combinatorics of toric varieties.

- A lattice is a finitely generated free abelian group $N \in \mathrm{Ab} = \mathrm{CGrp}(\mathrm{Set})$.
- The dual lattice M of N is $M := \mathrm{Hom}_{\mathrm{Ab}}(N, \mathbb{Z}) \in \mathrm{Ab}$.
- A cone $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ for us is a rational polyhedral cone in $N_{\mathbb{R}}$.
- The dual cone of $\sigma \subset N_{\mathbb{R}}$ is $\sigma^{\vee} := \{m \in M_{\mathbb{R}} : \langle m, n \rangle \geqslant 0, \forall n \in \sigma\} \subseteq M_{\mathbb{R}}$.
- A fan Σ in N is a collection of strongly convex cones in N closed under taking faces, such that every pair of cones either are disjoint or meet along a common face.
- A fan Σ is smooth [11, Section 2.1] if each of the cone σ is spanned by part of a basis of N.
- A fan Σ is projective [11, Section 3.4] if it admits an integral moment polytope $P \subset M_{\mathbb{R}}$: a polytope such that its faces are in bijection with cones in Σ and the cone σ corresponding to a face F is precisely the dual cone of the angle spanned by P along F.

Construction 3.1.2 (Flat toric scheme)**.** Given a pair (N, Σ) of lattice and fan. The assignment

$$\sigma \mapsto S_{\sigma} := \sigma^{\vee} \cap M \in \mathrm{CMon}(\mathrm{Set})$$

gives rise to a functor $\Sigma^{\mathrm{op}} \to \mathrm{CMon}(\mathrm{Set}) = \mathrm{CAlg}(\mathrm{Set})$. On the other hand, the symmetric monoidal functors

$$\mathrm{Set} \hookrightarrow \mathrm{Spc} \xrightarrow{\Sigma^{\infty}_{+}} \mathrm{Sp}$$

[7] While it's possible to make sense of, say, a non-flat $\mathbb{P}^1$ as in [SAG, Construction 19.2.6.1], we don't know of a general construction of non-flat toric schemes.

induce a functor $\mathbb{S}[-] : \mathrm{CAlg(Set)} \to \mathrm{CAlg(Sp)}$. Consider the image of σ under this composite functor

$$\mathcal{O}_\sigma := \mathbb{S}[\sigma^\vee \cap M] \in \mathrm{CAlg}(\mathrm{Sp})$$

which should be thought of as the ring of functions on the affine toric scheme X_σ associated to the cone σ. Postcomposing with Spét, we get a functor $\Sigma \longrightarrow \mathrm{Stk}$:

$$\sigma \mapsto \mathrm{Sp\acute{e}t}(\mathcal{O}_\sigma).$$

The flat toric scheme X_Σ associated to (N, Σ) is defined to be the colimit of this diagram

$$X_\Sigma := \operatorname*{colim}_\Sigma \mathrm{Sp\acute{e}t}\, \mathcal{O}_\sigma \in \mathrm{Stk}.$$

computed in the category of stacks. For simplicity, we will often omit 'flat' and call X_Σ the toric scheme associated to (N, Σ).

Remark 3.1.3 (An alternative version of 'toric geometry'). Motivated by the fact that $\mathbb{N}^{\times k}$ is the free object on k points in CMon(Set) (and similarly $\mathbb{Z}^{\times k}$ is the free object on k points in CGrp(Set)), one might want to reimagine a toric geometry over the sphere spectrum building upon monoid algebra of free objects in CMon(Spc) (or CGrp(Spc)). We don't carry out the construction here, but only point out the following subtleties:

1. The natural numbers $\mathbb{N}$ (resp. $\mathbb{Z}$) is the free $\mathbb{E}_1$-monoid (resp. $\mathbb{E}_1$-group) on a point. However, when viewed as an $\mathbb{E}_\infty$-monoid, $\mathbb{N}$ is far from being a free object: a map in CMon(Spc) from $\mathbb{N}$ instead picks out a 'strictly commutative element' in the target.
2. The flat affine line $\mathrm{Sp\acute{e}t}(\mathbb{S}[\mathbb{N}])$ doesn't support the addition map, see [24, Section 3.5].

Example 3.1.4 (Flat tori over $\mathbb{S}$). If one picks the fan to consist only of the origin, the associated flat toric scheme (named $\mathbb{T}$) is the torus associated to M:

$$\mathbb{T} := \mathrm{Sp\acute{e}t}(\mathbb{S}[M]).$$

Note that $\mathbb{T}$ has the structure of a group object (and we will call it a group scheme) given that M is a cogroup object in $\mathrm{CMon(Spc)}$.

Recall that a toric variety over a field k contains a torus as an open-dense subset and the torus action extends continuously to the whole variety. Now we explain the torus action in the setting of flat toric geometry.

Construction 3.1.5. Recall that given a category $\mathcal{C}$ with all limits, and considering $\mathcal{C}$ as a Cartesian symmetric monoidal category, every object $X \in \mathcal{C}$ acquires a canonical commutative coalgebra structure, informally specified by regarding the diagonal as the comultiplication map

$$\Delta : X \to X \times X.$$

In particular, every map $f : Y \to X$ exhibits Y as a comodule over X, with the coaction map informally specified by

$$\mu : Y \xrightarrow{\Delta} Y \times Y \xrightarrow{(f, \mathrm{id})} X \times Y.$$

In fact this map is induced by the lift of $f : Y \to X$ to a map of coalgebras. Specializing to the situation $\mathcal{C} = \mathrm{CMon}(\mathrm{Set})$ [8], we see that every submonoid S_σ of M is canonically coacted on by M. Moreover, these coactions are compatible with inclusions among S_σ. Therefore, $\mathcal{O}_\sigma = \mathbb{S}[S_\sigma]$ acquires a canonical $\mathbb{S}[M]$-comodule structure. Further passing to Spét, this gives a compatible family of actions of the group scheme $\mathbb{T} = \mathrm{Sp\acute{e}t}\,\mathbb{S}[M]$ on $\mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma$, each encoded by a simplicial diagram

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma \times \mathbb{T} \times \mathbb{T} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma \times \mathbb{T} \rightrightarrows \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma.$$

Taking colimits along σ, we obtain the diagram

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} (\mathrm{colim}_\sigma \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma) \times \mathbb{T} \times \mathbb{T} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} (\mathrm{colim}_\sigma \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma) \times \mathbb{T} \rightrightarrows \mathrm{colim}_\sigma \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma,$$

because colimits are universal in Stk. [9] We therefore obtain an action of $\mathbb{T}$ on

$$X_\Sigma = \operatorname*{colim}_{\sigma \in \Sigma} \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma,$$

to which we refer as the torus action of $\mathbb{T}$ on X_Σ, and the corresponding simplicial diagram $(X_\Sigma // \mathbb{T})_\bullet$ the action diagram of $\mathbb{T}$ on X_Σ. Formally, one might think of each $\mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma$ as an object in $\mathrm{Mod}_{\mathrm{Sp\acute{e}t}(\mathbb{S}[M])}\mathrm{Stk}$ and take colimits along Σ in the module category. Given that the forgetful functor

$$\mathrm{Mod}_{\mathrm{Sp\acute{e}t}(\mathbb{S}[M])}(\mathrm{Stk}) \longrightarrow \mathrm{Stk}$$

commutes with colimits, one sees that X_Σ acquires an action of $\mathbb{T}$, and its action diagram can be identified with the above action diagram.

Definition 3.1.6. The quotient stack $[X_\Sigma/\mathbb{T}]$ is the geometric realization of the action diagram of $\mathbb{T}$ on X_Σ:

$$[X_\Sigma/\mathbb{T}] := \operatorname*{colim}_{\Delta^{\mathrm{op}}} \left(\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X_\Sigma \times \mathbb{T} \times \mathbb{T} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X_\Sigma \times \mathbb{T} \rightrightarrows X_\Sigma \right) \in \mathrm{Stk}.$$

Remark 3.1.7. The Čech nerve of the projection $X_\Sigma \to [X_\Sigma/\mathbb{T}]$ is canonically identified with the action diagram of $\mathbb{T}$ on X_Σ. This is a direct consequence of Lemma A.1.1 and the fact that every groupoid object in an ∞-topos is effective [HTT, Theorem 6.1.0.6].

Remark 3.1.8. Alternatively, one might take the quotient affine locally on each X_σ by defining

$$[X_\sigma/\mathbb{T}] := \operatorname*{colim}_{\Delta^{\mathrm{op}}} \left(\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X_\sigma \times \mathbb{T} \times \mathbb{T} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X_\sigma \times \mathbb{T} \rightrightarrows X_\sigma \right) \in \mathrm{Stk}$$

via the action diagram. Then one can perform gluing

$$[X_\Sigma/\mathbb{T}] = \operatorname*{colim}_{\sigma \in \Sigma} [X_\sigma/\mathbb{T}]$$

and obtain the same stack, since colimit commutes with colimit.

[8] Note that $\mathrm{CMon}(\mathrm{Set})$ is preadditive.

[9] In particular, taking colimits commutes with taking finite products.

3.2 Quasi-coherent sheaves

The functor

$$\mathrm{QCoh} : \mathrm{Stk}^{\mathrm{op}} \to \mathrm{Cat}$$

given in [SAG, Definition 6.2.2.1] is lax symmetric monoidal in view of [SAG, §6.2.6] and [HA, Theorem 2.4.3.18]. To each stack it assigns a symmetric monoidal category:

$$X \mapsto \mathrm{QCoh}(X) \in \mathrm{SMCat}$$

such that to affines $\mathrm{Sp\acute{e}t}(R)$ it assigns $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}(R)) \simeq \mathrm{Mod}_R(\mathrm{Sp})$. In fact this functor preserves limits as in [SAG, Proposition 6.2.3.1], hence one gets a presentation of quasi-coherent sheaves on quotient stack as

$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \simeq \lim_{\Sigma^{\mathrm{op}}} \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$

while in turn each piece is presented by

$$\mathrm{QCoh}([X_\sigma/\mathbb{T}]) \simeq \lim_{\Delta} \left(\cdots \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}} \mathrm{QCoh}(X_\sigma \times \mathbb{T} \times \mathbb{T}) \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}} \mathrm{QCoh}(X_\sigma \times \mathbb{T}) \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow}} \mathrm{QCoh}(X_\sigma) \right).$$

Note that this is actually a limit of symmetric monoidal categories [SAG, §6.2.6]. At first glance, it might seem difficult to write down objects explicitly in this category. Motivated by [SAG, Construction 5.4.2.1], we proceed by making the following unstable construction.

Construction 3.2.1 (Unstable analogue)**.** Fix a cone σ in a lattice N, recall that Construction 3.1.5 provides an coaction of M on $S_\sigma = \sigma^\vee \cap M$. The coaction is presented by the following cosimplicial diagram in $\mathrm{CMon}(\mathrm{Spc})$:

$$\cdots \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}} S_\sigma \times M \times M \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}} S_\sigma \times M \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow}} S_\sigma.$$

Passing to module categories (with the functors being extension-of-scalar), one obtains

$$\cdots \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}} \mathrm{Mod}_{S_\sigma \times M \times M}(\mathrm{Spc}) \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}} \mathrm{Mod}_{S_\sigma \times M}(\mathrm{Spc}) \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow}} \mathrm{Mod}_{S_\sigma}(\mathrm{Spc}).$$

This is a cosimplicial diagram of symmetric monoidal categories, and we write $\mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M$ for the limit.

Remark 3.2.2. Replacing Spc by Set in the above, we get a new category which we name as $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$. Recall from [18, Theorem 2.2] that in classical algebraic geometry, the 1-category of comodules over a Hopf algebroid (A, Γ) is the same as the 1-category of quasi-coherent sheaves on the stack $X_{(A,\Gamma)}$ it presents. In a similar way, one can think of the pair $(S_\sigma, S_\sigma \times M)$ as a Hopf algebroid in the symmetric monoidal category $\mathrm{CMon}(\mathrm{Set})$.[10] The limit definition of $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$ mimics the definition of the category quasi-coherent sheaves on a stack. Indeed, we can think of $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$ as the category of comodules over the Hopf algebroid $(S_\sigma, S_\sigma \times M)$. The object in such comodule category can be constructed with a finite amount of data while the category $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$ relates to the category of quasi-coherent sheaves, as we explain below.

[10]Recall that, traditionally, a Hopf algebroid is defined in the symmetric monoidal category Mod_R.

Remark 3.2.3 (1-categorical analogue and degeneracy)**.** Consider the limit presentation of $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$. As the categories involved are all 1-categories, the limit is canonically identified with the limit of the diagram restricted to $\Delta_{\leqslant 2}$ (see [16, Proposition A.1]). Note also that one can produce objects and morphisms in the limit with a finite amount of data (actually very little is needed). More precisely, consider a cosimplicial diagram of 1-categories $\mathcal{C}_\bullet$, the limit is still a 1-category whose objects are pairs (x, f) where x is an object in $\mathcal{C}_0$, $f : d^1x \to d^0x$ is an isomorphism in $\mathcal{C}_1$ such that $d^0 f \circ d^2 f = d^1 f$ in $\mathcal{C}_2$. A map from (x, f) to (y, g) is a map $\varphi : x \to y$ in $\mathcal{C}_0$ that commutes with the structure maps f and g.

Example 3.2.4 (Explicit construction of the tautological object in $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$)**.** Here we provide a concrete example of how to down objects in the category $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$. We supply a particular lift[11] of M to an object in this category, where M is an S_σ-module via the canonical inclusion. To provide the lift is to provide the structure map (note that the relative tensor products are induced by different maps $S_\sigma \to S_\sigma \times M$ where the left one is (id, inclusion) and the right one is (id, 0))

$$f : M \times_{S_\sigma} S_\sigma \times M \longrightarrow M \times_{S_\sigma} S_\sigma \times M \in \mathrm{Mod}_{S_\sigma \times M}(\mathrm{Set})$$

which is an isomorphism and we define f such that

$$f(m, s, n) := (m, s, n + m) \in M \times_{S_\sigma} S_\sigma \times M.$$

It is a tedious exercise to check that f satisfies the cocycle conditions as above and we leave it to the reader. Upon linearization, this object corresponds to the $*$-pushforward of $\mathcal{O}_{[\mathbb{T}/\mathbb{T}]}$ along the open immersion $[\mathbb{T}/\mathbb{T}] \hookrightarrow [X_\Sigma/\mathbb{T}]$.

Warning 3.2.5. Given a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ and $A \in \mathrm{CAlg}(\mathcal{C})$, it induces a functor $F_A : \mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_{F(A)}(\mathcal{D})$. If $\mathcal{C}$ and $\mathcal{D}$ have geometric realizations and tensor products in $\mathcal{C}$ and $\mathcal{D}$ commutes with geometric realizations, then both $\mathrm{Mod}_A(\mathcal{C})$ and $\mathrm{Mod}_{F(A)}(\mathcal{D})$ have symmetric monoidal structures given by relative tensor products. However, the functor F_A lifts to a symmetric monoidal functor only when F commutes with geometric realizations. The lift is functorial in the sense of [HA, Theorem 4.8.5.16] (see below). The example to keep in mind is the following:

$$\mathrm{Set} \to \mathrm{Spc} \to \mathrm{Sp}$$

is a sequence of symmetric monoidal functors; The latter preserves geometric realization while the former doesn't. For instance, the relative tensor product $X \times_{\mathbb{Z}} Y$ is in general not the same computed in Spc as done in Set. When X and Y are both a point, in Set the outcome is still a point, while in Spc one gets $B\mathbb{Z}$.

Remark 3.2.6 (An antidote to the warning)**.** As explained by above warning, for a given monoid $S \in \mathrm{CMon}(\mathrm{Set})$, we don't have a symmetric monoidal structure on the inclusion functor $\mathrm{Mod}_S(\mathrm{Set}) \to \mathrm{Mod}_S(\mathrm{Spc})$. One can, however, define a symmetric monoidal category sitting in both of them: take $\mathrm{Mod}_S(\mathrm{Spc})^{\mathrm{free}} \subset \mathrm{Mod}_S(\mathrm{Spc})$ to be the full subcategory spanned by coproducts of S. This category inherits a symmetric monoidal structure and can be identified, symmetric monoidally, with the full subcategory spanned by coproducts of S in $\mathrm{Mod}_S(\mathrm{Set})$. To be very rigorous with the construction that follows, one should construct symmetric monoidal functor directly into $\mathrm{Mod}_S(\mathrm{Spc})$, but we will construct functors into $\mathrm{Mod}_S(\mathrm{Set})$ and observe that they lift to $\mathrm{Mod}_S(\mathrm{Spc})$. We will also use the following fact obtained from [HA, Theorem 4.8.5.16].

[11] There are obviously others.

Proposition 3.2.7. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories admitting all geometric realizations. Let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. Assume that:

1. Tensor products in $\mathcal{C}$ and $\mathcal{D}$ commute with geometric realizations.
2. The functor F commutes with geometric realizations.

Then there is a diagram

$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, bend left, "\mathrm{Mod}_{(-)}(\mathcal{C})"] \arrow[r, bend right, "\mathrm{Mod}_{F(-)}(\mathcal{D})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{SMCat} \end{tikzcd} .$$

When evaluated at $A \to B \in \mathrm{CAlg}(\mathcal{C})$, the diagram reads

$$\begin{tikzcd} \mathrm{Mod}_A(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Mod}_B(\mathcal{C}) \arrow[d] \\ \mathrm{Mod}_{F(A)}(\mathcal{D}) \arrow[r] & \mathrm{Mod}_{F(B)}(\mathcal{D}) \end{tikzcd} .$$

Proof. See Proposition A.2.1. □

The linearization functor $\mathbb{S}[-] : \mathrm{Spc} \to \mathrm{Sp}$ is symmetric monoidal and preserves geometric realizations. So it induces, functorially, symmetric monoidal functors on module categories. This implies that there is a natural transformation from the cosimplicial diagram that presents $\mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M$ to the cosimplicial diagram that presents $\mathrm{QCoh}([X_\sigma/\mathbb{T}])$. We write

$$\mathcal{O}[-] : \mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M \to \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$

for the symmetric monoidal functor one obtains after taking limit along Δ. Note that both sides of above are indexed over $\sigma \in \Sigma^{\mathrm{op}}$, and for the same reason, $\mathcal{O}[-]$ assembles into a natural transformation of diagrams. In the next subsection we will use this natural transformation to produce a comparison functor from combinatorial models.

3.3 Combinatorial versus quasi-coherent

The goal of this subsection is to provide the following construction.

Proposition 3.3.1. There exists a symmetric monoidal equivalence of categories

$$\Phi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$

where the left-hand side has the Day convolution tensor product and the right-hand side has the standard tensor product of quasi-coherent sheaves. Moreover, these equivalences are functorial in $\sigma \in \Sigma^{\mathrm{op}}$ in that they assemble into a natural transformation of diagrams in SMCat indexed by Σ^{op}. Hence taking limit produces

$$\lim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\simeq} \lim_{\Sigma^{\mathrm{op}}} \mathrm{QCoh}([X_\sigma/\mathbb{T}]) \simeq \mathrm{QCoh}([X_\Sigma/\mathbb{T}]).$$

Remark 3.3.2 (Compatibility with the torus)**.** We will establish along the way an equivalence

$$\Phi_M : \mathrm{Fun}(M, \mathrm{Sp}) \simeq \mathrm{QCoh}(B\mathbb{T})$$

and will also provide compatibility of Φ_M with above equivalence Φ_σ (see Remark 3.3.9), i.e., the following diagram commutes

$$\begin{tikzcd}
\lim_{\Sigma^{op}} \mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \arrow[r, "\lim_{\Sigma^{op}} \Phi_\sigma"] & \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \\
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[u, "\lim_{\Sigma^{op}} (p_\sigma)_!"] \arrow[r, "\Phi_M"] & \mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\lim_{\Sigma^{op}} \pi_\sigma^*"']
\end{tikzcd} .$$

Remark 3.3.3 (The geometry of filtrations)**.** Take the pair $N = \mathbb{Z}$ and $\Sigma = \{0, \mathbb{R}_{\geqslant 0}\}$. The theorem above reads

$$\mathrm{Fun}(\mathbb{Z}_{\leqslant}, \mathrm{Sp}) \simeq \mathrm{QCoh}([\mathbb{A}^1/\mathbb{G}_m]),$$

which is [29, Theorem 1.1]. The proof presented in this subsection actually follows closely the approach in [29].

We begin by constructing the functor Φ_σ, then explain its naturality along $\sigma \in \Sigma^{op}$.

Construction 3.3.4. (Construction of the functor in the unstable case) Fix a cone σ in a lattice N, we define a functor

$$\phi_\sigma : \Theta(\sigma) \to \mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$$

as follows: for $V \in \Theta(\sigma)$, recall that V is an integral translation of $\sigma^\vee$. We define $\phi_\sigma(V)$ to be

$$(V \cap M, f|_{V \cap M \times_{S_\sigma} S_\sigma \times M}) \in \mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$$

where $V \cap M$ is a subobject of $M \in \mathrm{Mod}_{S_\sigma}(\mathrm{Set})$ and it inherits the structure map f from Example 3.2.4 since f preserves $V \cap M \times_{S_\sigma} S_\sigma \times M$.

Then we move on to morphisms. Given an inclusion $i : V \subset W \in \Theta(\sigma)$, we define $\phi_\sigma(i)$ to be the inclusion map

$$\phi(i) : V \cap M \to W \cap M \in \mathrm{Mod}_{S_\sigma}(\mathrm{Set}).$$

It remains to check that $\phi(i)$ is compatible with the structure maps of $\phi(V)$ and $\phi(W)$. This follows directly from that both structure maps are inherited from (M, f). So we know that $\phi(i)$ lifts to a map in $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$. The symmetric monoidal structure on the functor can be supplied and checked directly as it is a functor between 1-categories. The construction lands in $\mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^{\mathrm{free}}$ in each degree and hence lifts to a symmetric monoidal functor to $\mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M$. To conclude, we have obtained a symmetric monoidal functor

$$\phi_\sigma : \Theta(\sigma) \longrightarrow \mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M.$$

Remark 3.3.5. (Naturality along $\sigma \in \Sigma^{op}$) The functors ϕ_σ as above assemble into a natural transformation between diagrams in SMCat indexed by Σ^{op}:

$$\Theta(-) \to \mathrm{Mod}_{S_-}(\mathrm{Spc})^M.$$

Since we are working within 1-categories, the naturality could be checked directly.

Definition 3.3.6. We define Φ_σ to be the left Kan extension of $\mathcal{O}[\phi_\sigma]$ along the stable Yoneda embedding:

$$\Phi_\sigma := \mathrm{Lan}_h(\mathcal{O}[\phi_\sigma]) : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{QCoh}([X_\sigma/\mathbb{T}]),$$

where we have used the linearization functor

$$\mathcal{O}[-] : \mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M \to \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$

from the last paragraph of Section 3.2. Note that it is symmetric monoidal with the Day convolution product on the domain. From the discussion in Remark 3.3.5 and functoriality of the Day convolution (see Section A.3) we learn that Φ_σ is a natural transformation

$$\begin{tikzcd} \Sigma^{\mathrm{op}} \arrow[r, bend left, "\mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp})"{name=U}] \arrow[r, bend right, "\mathrm{QCoh}([X_\sigma/\mathbb{T}])"'{name=D}] & \mathrm{SMCat} \arrow[from=U, to=D, phantom, "\Phi_\sigma \Downarrow"] \end{tikzcd}$$

between diagrams in SMCat indexed by Σ.

Example 3.3.7 (Equivariant line bundles on the affine line)**.** Take the pair $N = \mathbb{Z}$ and $\Sigma = \{0, \mathbb{R}_{\geqslant 0}\}$. The construction above produces a family of line bundles from the symmetric monoidal functor

$$\Phi_{\mathbb{R}_{\geqslant 0}} : \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}_{\leqslant}; \mathrm{Sp}) \longrightarrow \mathrm{QCoh}([\mathbb{A}^1/\mathbb{G}_m]).$$

After base changing to $\mathbb{Z}$, it recovers the universal line bundles $\phi(n) = \mathcal{O}(n)$, universal sections $\cdot x : \mathcal{O}(n) \to \mathcal{O}(n+1)$, and isomorphisms $\mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(mn)$. In general, it is possible to globalize this construction and construct torus-equivariant line bundles on toric schemes.

Now we move on to proving the main theorem of this section: that each Φ_σ is an equivalence of categories. First we do some preparations.

Variant 3.3.8 (Compare with Construction 3.3.4)**.** We can define a symmetric monoidal functor

$$\phi_M : M \to \mathrm{Mod}_*(\mathrm{Set})^M$$

(where the point $*$ is the initial monoid) as follows. On objects, $m \in M$ is taken to the pair $(*, f_m)$. Here $* \in \mathrm{Set}$ is the underlying object and $f_m : * \times M \to * \times M$ is the isomorphism given by translation by m. Again one checks this satisfies the cocycle condition as in Example 3.2.4 so $(*, f_m)$ defines an object in $\mathrm{Mod}_*(\mathrm{Set})^M$. This assignment lifts to a symmetric monoidal functor by direct inspection. Hence we get a symmetric monoidal functor

$$\Phi_M := \mathrm{Lan}_h\, \mathcal{O}[\phi_M] : \mathrm{Fun}(M, \mathrm{Sp}) \to \mathrm{QCoh}(B\mathbb{T}).$$

Remark 3.3.9. By the very explicit construction, the equivalence Φ_M above is compatible with Definition 3.3.6: there is a symmetric monoidal functor $p_\sigma : M \to \Theta(\sigma)$ that sends m to $m + \sigma^\vee$ (see Remark 2.0.4) making the diagram

$$\begin{tikzcd} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Phi_\sigma"] & \mathrm{QCoh}([X_\sigma/\mathbb{T}]) \\ \mathrm{Fun}(M, \mathrm{Sp}) \arrow[u, "(p_\sigma)_!"] \arrow[r, "\Phi_M"] & \mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\pi_\sigma^*"'] \end{tikzcd}$$

commute, where $(p_\sigma)_!$ stands for left Kan extension of presheaves along p_σ and π_σ^* stands for $*$-pullback of quasi-coherent sheaves along $\pi_\sigma : [X_\sigma/\mathbb{T}] \to B\mathbb{T}$. The commutativity of the square comes from 1-categorical inspection before linearization. Moreover, the maps above are natural in $\sigma \in \Sigma^{op}$ that one can interpret it as a square of natural transformations of diagrams in SMCat indexed by $\sigma \in \Sigma^{op}$.

We will follow the approach taken in [29, Theorem 4.1] to prove the following:

Theorem 3.3.10. There is an equivalence of symmetric monoidal categories

$$\Phi_M : \mathrm{Fun}(M, \mathrm{Sp}) \simeq \mathrm{QCoh}(B\mathbb{T}),$$

where the left-hand side comes with the Day convolution tensor product and the right-hand side comes with the standard tensor product of quasi-coherent sheaves.

Proof. We interpret Φ_M as an augmentation of the cosimplicial diagram presenting $\mathrm{QCoh}(B\mathbb{T})$:

$$\cdots \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; \mathrm{QCoh}(* \times \mathbb{T} \times \mathbb{T}) \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; \mathrm{QCoh}(* \times \mathbb{T}) \;\substack{\longleftarrow\\[-2pt]\longleftarrow}\; \mathrm{QCoh}(*) \longleftarrow \mathrm{Fun}(M, \mathrm{Sp}) \ .$$

Then the theorem follows from a direct application of [HA, Corollary 4.7.5.3] in its comonadic form (as used in the proof of [SAG, Theorem 5.6.6.1]). So we want to check the following:

1. The functor $d^0 : \mathrm{Fun}(M, \mathrm{Sp}) \to \mathrm{QCoh}(*) = \mathrm{Sp}$ is comonadic.
2. The Beck-Chevalley condition holds: for each $\alpha : [m] \to [n]$ in Δ_+, the diagram

$$\begin{tikzcd} \mathcal{C}^m \arrow[r, "d^0"] \arrow[d, "\alpha"'] & \mathcal{C}^{m+1} \arrow[d, "\alpha+1"] \\ \mathcal{C}^n \arrow[r, "d^0"] & \mathcal{C}^{n+1} \end{tikzcd}$$

   is right adjointable (for horizontal maps).

We first show $d^0 : \mathrm{Fun}(M, \mathrm{Sp}) \to \mathrm{Sp}$ is comonadic. By construction, d^0 takes an M-family of spectra $\{X_m\}$ to the coproduct $\oplus X_m$. The crucial observation is that each X_m is a retract of $\oplus X_m$. If $\oplus X_m \simeq 0$, then each X_m is a retract of 0, so the family $\{X_m\}$ is 0. Therefore, d^0 is conservative. It remains to show that d^0 preserves limits of cosimplicial diagrams in $\mathrm{Fun}(M, \mathrm{Sp})$ with split images in Sp.

A cosimplicial diagram $X^\bullet$ in $\mathrm{Fun}(M, \mathrm{Sp})$ is just an M-family of cosimplicial diagrams $\{X_m^\bullet\}$ in Sp. Under this identification, $d^0(X^\bullet) = \oplus X_m^\bullet$. Denote by $X^{-\infty} = \{X_m^{-\infty}\}$ a limit of $X^\bullet$. Note that each $X_m^\bullet$ is a retract of the split cosimplicial object $\oplus X_m^\bullet$, which itself must be split by [HA, Corollary 4.7.2.13]. Therefore each of the augmented cosimplicial object $X_m^{-\infty} \to X_m^\bullet$ is split. It follows that the coproduct

$$d^0(X^{-\infty}) \simeq \oplus_m X_m^{-\infty} \to \oplus_m X_m^\bullet \simeq d^0(X_m^\bullet)$$

is also split, thus a limit diagram, as desired.

Now we move on to checking the adjointability. For $\alpha : [m] \to [n]$, if $m \neq -1$, one can look at the corresponding groupoid object in Stk:

$$\begin{tikzcd}
* \times \mathbb{T}^{\times n+1} \arrow[r, "\alpha+1"] \arrow[d, "d^0"] & * \times \mathbb{T}^{\times m+1} \arrow[r, "\{0,1\}"] \arrow[d, "d^0"] & * \times \mathbb{T} \arrow[d, "d^0"] \\
* \times \mathbb{T}^{\times n} \arrow[r, "\alpha"] & * \times \mathbb{T}^{\times m} \arrow[r, "\{0\}"] & *
\end{tikzcd}.$$

By the Segal condition [HTT, Proposition 6.1.2.6], both the right square and the outer rectangle are pullback squares, so the left square is also a pullback in Stk. So we conclude that after applying $\mathrm{QCoh}(-)$ the left square is right adjointable by [SAG, Lemma D.3.5.6]. For $\alpha : [-1] \to [n]$, we first check that

$$\begin{tikzcd}
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "d^0"] \arrow[d, "\alpha=d^0"'] & \mathrm{Sp} \arrow[d, "\alpha+1=d^1"'] \\
\mathrm{Sp} \arrow[r, "d^0"] & \mathrm{QCoh}(\mathbb{T})
\end{tikzcd}$$

is right adjointable. We will use the following notational convention. Put $p : M \to *$ to be the projection of set M to a point, and we write $p_! \dashv p^*$ for the adjunction between left Kan extension and restriction of presheaves. Put $\pi : \mathbb{T} \to *$ to be the projection of stack $\mathbb{T}$ to a point, and we write $\pi^* \dashv \pi_*$ for the adjunction between $*$-pullback and $*$-pushforward of quasi-coherent sheaves. Under this notation, the diagram above reads:

$$\begin{tikzcd}
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "p_!"] \arrow[d, "p_!"'] & \mathrm{Sp} \arrow[d, "\pi^*"'] \\
\mathrm{Sp} \arrow[r, "\pi^*"] & \mathrm{QCoh}(\mathbb{T})
\end{tikzcd},$$

and the 2-cell filling the square comes from the construction in Variant 3.3.8. Be careful that the 2-cell is not the trivial one (and the trivial one won't be right adjointable). We need to show that

$$p_! p^* \to \pi_* \pi^* p_! p^* \to \pi_* \pi^* p_! p^* \to \pi_* \pi^*$$

is an equivalence of functors, where the maps involed are given, in turn, by the unit for $\pi^* \dashv \pi_*$, the homotopy of the 2-cell filling the diagram $\pi^* p_! \simeq \pi^* p_!$ and the counit for $p_! \dashv p^*$. Note that since both $p_! p^*$ and $\pi_* \pi^*$ are colimit-preserving, it suffices to check on $\mathbb{S} \in \mathrm{Sp}$. Unwinding the definition, the map reads

$$\begin{tikzcd}
p_! p^* \mathbb{S} \arrow[r] \arrow[d, equal] & \pi_* \pi^* p_! p^* \mathbb{S} \arrow[r] \arrow[d, equal] & \pi_* \pi^* p_! p^* \mathbb{S} \arrow[r] \arrow[d, equal] & \pi_* \pi^* \mathbb{S} \arrow[d, equal] \\
\bigoplus_M \mathbb{S} \arrow[r, "\oplus \epsilon"] & \bigoplus_M \mathbb{S}[M] \arrow[r, "\oplus m \cdot"] & \bigoplus_M \mathbb{S}[M] \arrow[r, "\mathrm{sum}"] & \mathbb{S}[M]
\end{tikzcd}.$$

The first map is the coproduct of unit maps $\mathbb{S} \to \mathbb{S}[M]$ for the algebra $\mathbb{S}[M]$. The second map is the coproduct of the maps $\cdot m : \mathbb{S}[M] \to \mathbb{S}[M]$ on each direct summand $m \in M$. The third map is induced by identity map $\mathrm{id} : \mathbb{S}[M] \to \mathbb{S}[M]$ on each summand, i.e., forming summation. The composition, which is $\cdot m : \mathbb{S} \to \mathbb{S}[M]$ on each summand, is readily an equivalence of spectra.

Recall that we are showing diagrams involving $[-1]$ are right adjointable, and we have only checked one of them. Now for a general map $\alpha : [-1] \to [n]$, observe that there is a commutative diagram (note the different orientation of the diagram)

$$\begin{tikzcd}
{[-1]} \arrow[r, "\alpha'"'] \arrow[d, "d^0"] & {[0]} \arrow[r, "\beta"'] \arrow[d, "d^0"] & {[n]} \arrow[d, "d^0"'] \\
{[0]} \arrow[r, "\alpha'+1=d^1"] & {[1]} \arrow[r, "\beta+1"] & {[n+1]}
\end{tikzcd}$$

in Δ_+ where $\beta \circ \alpha' = \alpha$. This is taken to a diagram of categories where both of the squares are right adjointable (now along the vertical edges). We hence conclude that the outer rectangle is also right adjointable, as desired. $\square$

We now state a technical claim about adjointability of diagrams that we will use in proving Proposition 3.3.1. The proof will be offered later.

Lemma 3.3.11. The diagram in Remark 3.3.9 is right adjointable for taking right adjoints of $(p_\sigma)_!$ and π_σ^*. In other words, the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \arrow[r, "\Phi_\sigma"] \arrow[d, "(p_\sigma)^*"'] & \mathrm{QCoh}([X_\sigma/\mathbb{T}]) \arrow[d, "\pi_{\sigma *}"] \\
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "\Phi_M"] & \mathrm{QCoh}(B\mathbb{T})
\end{tikzcd}$$

commutes, with the homotopy specified by

$$\Phi_M p_\sigma^* \to \pi_{\sigma *}\pi_\sigma^* \Phi_M p_\sigma^* \to \pi_{\sigma *}\Phi_\sigma p_{\sigma !} p_\sigma^* \to \pi_{\sigma *}\Phi_\sigma$$

where the maps involved are given in turn by the unit for $\pi_\sigma^* \dashv \pi_{\sigma *}$, the homotopy of the 2-cell in the diagram $\pi_\sigma^*\Phi_M \simeq \Phi_\sigma p_{\sigma !}$ and the counit for $p_{\sigma !} \dashv p_\sigma^*$.

We are now ready to prove the main theorem of the section.

Proof of Proposition 3.3.1. Naturality of the functors has been explained in Remark 3.3.9. What's left to check is that for each σ, Φ_σ is an equivalence of categories. Given Lemma 3.3.11 we are in the situation of comparing monadic adjunctions [HA, Proposition 4.7.3.16]: each of the category is monadic over another category. We claim that the assumptions in [HA, Proposition 4.7.3.16] are readily true in our case: (1) is true as our diagrams are obtained by taking right adjoints of right adjointable diagrams; (2) and (3) follow from that both p_σ^* and $\pi_{\sigma *}$ are colimit-preserving functors; (4) is true because π is affine so $*$-pushforward along π is conservative. Note that the functor p^* (which is restriction of presheaves) is also conservative since p is an essentially surjective functor; and (5) requires essentially to check if the diagram is left adjointable, which again follows from the fact that the diagram itself comes from taking right adjoints of a right adjointable diagram, see [HTT, Remark 7.3.1.3]. $\square$

Proof of Lemma 3.3.11. We look at the map between augmented action diagrams which presents the map $[X_\sigma/\mathbb{T}] \to B\mathbb{T}$

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_\sigma \times \mathbb{T} \times \mathbb{T} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & X_\sigma \times \mathbb{T} \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & X_\sigma \arrow[r] \arrow[d] & {[X_\sigma/\mathbb{T}]} \arrow[d] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbb{T} \times \mathbb{T} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{T} \arrow[r, shift left=1] \arrow[r, shift right=1] & * \arrow[r] & B\mathbb{T}
\end{tikzcd}.$$

For each $\alpha : [m] \to [n] \in \Delta$, we have the diagram

$$\begin{tikzcd}
X_\sigma \times \mathbb{T}^{\times n} \arrow[r, "\alpha"] \arrow[d] & X_\sigma \times \mathbb{T}^{\times m} \arrow[r] \arrow[d] & {[X_\sigma/\mathbb{T}]} \arrow[d] \\
\mathbb{T}^{\times n} \arrow[r, "\alpha"] & \mathbb{T}^{\times m} \arrow[r] & B\mathbb{T}
\end{tikzcd},$$

where both the outer rectangle and the inner right square are pullbacks, so the left square is also a pullback (see Lemma A.1.1). Hence from [SAG, Lemma D.3.5.6], we learn that after taking QCoh, the left square becomes

$$\begin{tikzcd}
\mathrm{QCoh}(X_\sigma \times \mathbb{T}^{\times n}) & \mathrm{QCoh}(X_\sigma \times \mathbb{T}^{\times m}) \arrow[l, "\alpha"'] \\
\mathrm{QCoh}(\mathbb{T}^{\times n}) \arrow[u] & \mathrm{QCoh}(\mathbb{T}^{\times m}) \arrow[l, "\alpha"'] \arrow[u]
\end{tikzcd}$$

which is right adjointable (for vertical maps). By [HA, Corollary 4.7.4.18] this implies that $\mathrm{QCoh}(-)$ of the action diagram, viewed as $[n] \mapsto [\mathrm{QCoh}(\mathbb{T}^{\times n}) \to \mathrm{QCoh}(X_\sigma \times \mathbb{T}^{\times n})]$, lifts to a simplicial object in $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat})$, and $\mathrm{QCoh}(-)$ of the augmented action diagram is a limit diagram in $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat})$. Now one can similarly view the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Phi_\sigma"] & \mathrm{QCoh}([X_\sigma/\mathbb{T}]) \\
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "\Phi_M"] \arrow[u, "(p_\sigma)_!"] & \mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\pi_\sigma^*"']
\end{tikzcd}$$

as an augmentation to the simplicial object $[n] \mapsto [\mathrm{QCoh}(\mathbb{T}^{\times n}) \to \mathrm{QCoh}(X_\sigma \times \mathbb{T}^{\times n})]$ in $\mathrm{Fun}(\Delta^1, \mathrm{Cat})$. Determining its right adjointability reduces to asking if this augmentation lifts to $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat})$. The only thing left to check is right adjointability of the diagram (for taking right adjoints of the vertical arrows)

$$\begin{tikzcd}
\mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Phi_\sigma"] & \mathrm{QCoh}(X_\sigma) \\
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "\Phi_M"] \arrow[u, "p_{\sigma!}"] & \mathrm{QCoh}(*) \arrow[u, "\pi^*"']
\end{tikzcd}.$$

This is readily true once one unwinds the definition as in the proof of Theorem 3.3.10.

$\square$

4 Constructible sheaves

In the seminal book of [HA], Lurie sets up a general theory of constructible sheaves of spaces on a stratified topological space. In particular, this theory has been worked out with no finiteness assumptions on the sheaves involved. For a compactly generated presentable coefficient category $\mathcal{C}$, the theory of constructible sheaves on a stratified topological space valued in $\mathcal{C}$ could be worked out in a similar way (for example, as in [27]). We will follow this convention and setup various functors involved in the coherent-constructible correspondence. This approach makes several constructions easier. First of all, the yoga of six-functor provides a neat way to write down convolution products defined on the category of sheaves of spectra on a (locally compact Hausdorff) topological group and related functors. Secondly recent advances in exodromy [14, 7] make it gracefully simple to work with large categories of constructible sheaves.

The main goal of this section is to write down a symmetric monoidal functor from the combinatorial model to the category of sheaves on a real vector space. To do so, we first recall some generalities on convolution products for sheaves on real vector spaces. Then we move onto a digression on the lax symmetric monoidal structure on the relative homology functor. This is used in the next part to provide a combinatorial-constructible comparison functor along with its lax symmetric monoidal structure. After that we take a turn to recall some generalities on constructible sheaves and pin down a stratification following [10]. As a consequence, we show that the comparison functor is fully faithful for a smooth fan and its image consists of sheaves constructible for the stratification we introduced. Finally we take a detour to prove a technical fact about descent along idempotent algebras in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$. Putting everything together, we conclude that for a smooth projective fan, the combinatorial-constructible comparison functor we constructed is fully faithful and symmetric monoidal. We leave the characterization of the image to the next section.

4.1 Convolution product for sheaves on real vector spaces

Remark 4.1.1 (Hypercompleteness)**.** One needs not to worry about hypercompleteness in our situation, as we will only deal with sheaves on finite dimensional manifolds. In particular, equivalence of sheaves can be detected stalk-wise.

Take a finite dimensional real vector space $V \simeq \mathbb{R}^{\oplus n}$. It acquires the structure of a commutative algebra in $(\mathrm{LCH}, \times)$ via the addition of vectors

$$+ : V \times V \to V.$$

This equips $\mathcal{S}\mathrm{hv}(V;\mathrm{Sp})$ with a binary operation

$$* : \mathcal{S}\mathrm{hv}(V;\mathrm{Sp}) \times \mathcal{S}\mathrm{hv}(V;\mathrm{Sp}) \to \mathcal{S}\mathrm{hv}(V;\mathrm{Sp})$$

defined by

$$\mathcal{F} * \mathcal{G} := +_!(\mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{pr}_2^*\mathcal{G}).$$

This operation could be made coherently into a symmetric monoidal structure as follows.

Construction 4.1.2 (Convolution product)**.** Recall that the 'six-functor formalism' on LCH is a lax symmetric monoidal functor

$$\mathcal{D} : \mathrm{Corr}(\mathrm{LCH}, \mathrm{all}) \longrightarrow \mathrm{Cat}$$

and we have another symmetric monoidal functor ('Reg' for right leg)

$$\mathrm{Reg} : \mathrm{LCH} \to \mathrm{Corr}(\mathrm{LCH}, \mathrm{all})$$

which on objects acts as $X \mapsto X$ and on morphisms acts as

$$[X \xrightarrow{f} Y] \mapsto \left[\begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \right].$$

We define the composition

$$D_!(-) := \mathcal{D} \circ \mathrm{Reg} : \mathrm{LCH} \to \mathrm{Cat}$$

which is again a lax symmetric monoidal functor. This implies that for every commutative algebra $A \in \mathrm{CAlg}(\mathrm{LCH})$, the category $D_!(A) = \mathcal{S}\mathrm{hv}(A; \mathrm{Sp})$ acquires a symmetric monoidal structure through the functoriality of $D_!$. We name the resulting monoidal product convolution and write it as $*$.

Proposition 4.1.3. Let V be a real vector space and $*$ be the convolution product operation on sheaves.

1. The convolution product $*$ is cocontinuous in each variable.
2. Let $X, Y \subseteq V$ be polyhedral open subsets of a real vector space. We can compute very explicitly
$$\underline{\mathbb{S}}_X * \underline{\mathbb{S}}_Y \simeq \underline{\mathbb{S}}_{X+Y}[-\dim(V)]$$
where
$$X + Y := \{x + y : x \in X, y \in Y\}$$
is the Minkowski sum of the subsets.

Proof. Point 1 follows from the fact that $*$-pullback, $\otimes$ of sheaves and !-pushforward all preserve colimits. For point 2, we apply proper base change and learn that

$$\underline{\mathbb{S}}_X * \underline{\mathbb{S}}_Y \simeq +_{|X \times Y!} \underline{\mathbb{S}}_{X \times Y}$$

where $+$ is restricted to a map $X \times Y \to \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. By the fact that X and Y are polyhedral open subsets, one can prove this map $+$ is a smooth $\mathbb{R}^n$ bundle over its image $X + Y \subseteq \mathbb{R}^n$. It follows that the !-pushforward of $\underline{\mathbb{S}}_{X \times Y}$ along the addition map is locally constant. The computation reduces to the fact that for the projection $p : Z \times \mathbb{R}^n \to Z$, one has

$$p_! \underline{\mathbb{S}} = \underline{\mathbb{S}}[-n].$$

Keep in mind that $X + Y$ is contractible. $\square$

Remark 4.1.4. As a side remark, polyhedral opens form a basis for the topology. In principle, one can compute the convolution of any two sheaves using the above facts.

4.2 Digression: multiplicative structures on Betti homology

As we have seen earlier, the addition operation on the finite dimensional real vector space $M_{\mathbb{R}}$ makes it into a commutative monoid in the 1-category LCH. Thus the slice category $\mathrm{LCH}_{/M_{\mathbb{R}}}$ acquires a symmetric monoidal structure which can be informally defined as follows:

$$(X, f) \otimes (Y, g) := (X \times Y, f + g)$$

(see [HA, Theorem 2.2.2.4] for the general construction). We denote by $(\mathrm{LCH}_{/M_{\mathbb{R}}}, \otimes)$ this symmetric monoidal category. The structure of a commutative monoid on $M_{\mathbb{R}}$ also has been used to provide a convolution product on the category of sheaves on $M_{\mathbb{R}}$, and these two categories are indeed related. The goal of this digression is to explain the following construction.

Proposition 4.2.1 (Taking homology is symmetric monoidal)**.** There is a lax symmetric monoidal functor

$$\Gamma_{M_{\mathbb{R}}} : (\mathrm{LCH}_{/M_{\mathbb{R}}}, \otimes) \longrightarrow (\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}), *)$$

which on objects acts by

$$(X, f) \longmapsto f_! f^! \omega_{M_{\mathbb{R}}},$$

where $\omega_{M_{\mathbb{R}}}$ is the dualizing sheaf on $M_{\mathbb{R}}$.

Remark 4.2.2 (A similar construction in the literature)**.** Let us immediately point out that, a very similar construction has been carried out (in the ℓ-adic context) by Gaitsgory-Lurie in [13, Chapter 3]. An elaboration (in the Betti context) of the ideas in that paper would produce a more general construction that easily provides the functor as above (for example, one could allow the base groups G to vary). We have, however, decided to give an ad-hoc construction of the functor that we need in this note to simplify our exposition (also because the situation we are dealing with here is extremely simple). We will return to this construction elsewhere.

The construction is technical in contrast to the simple application we have in mind. The reader is advised to skip the rest of this section and come back later. Before we go into the construction, here is a rough plan.

Remark 4.2.3 (Preview of strategy)**.** We will define a symmetric monoidal category $\mathcal{S}\mathrm{hv}_!$ which comes with a symmetric monoidal functor

$$p : \mathcal{S}\mathrm{hv}_! \to \mathrm{LCH}_{/M_{\mathbb{R}}}.$$

We will then produce a lax symmetric monoidal functor as a section of p:

$$s : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}\mathrm{hv}_!,$$

and another symmetric monoidal functor

$$t : \mathcal{S}\mathrm{hv}_! \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}),$$

so that the composition

$$t \circ s : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

is what we want.

Remark 4.2.4 (A rough description of the players)**.** We give an informal description of the categories and functors appearing in the previous remark. One can describe the category $\mathcal{S}\mathrm{hv}_!$ as follows. An object in $\mathcal{S}\mathrm{hv}_!$ is a pair $(X, f, \mathcal{F})$ where (X, f) is an object of $\mathrm{LCH}_{/M_{\mathbb{R}}}$ and $\mathcal{F} \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$. A map (h, ϕ) from $(X, f, \mathcal{F})$ to $(Y, g, \mathcal{G})$ consists of a map $h : (X, f) \to (Y, g)$ in $\mathrm{LCH}_{/M_{\mathbb{R}}}$ and a map $\phi : h_! \mathcal{F} \to \mathcal{G}$ in $\mathcal{S}\mathrm{hv}(Y; \mathrm{Sp})$. The symmetric monoidal structure is a mixture of tensor product in $\mathrm{LCH}_{/M_{\mathbb{R}}}$ and exterior product of sheaves: $(X, f, \mathcal{F}) \otimes (Y, g, \mathcal{G}) = (X \times Y, f + g, \mathcal{F} \boxtimes \mathcal{G})$. With these we can also roughly describe the functors. The functor

$$p : \mathcal{S}\mathrm{hv}_! \to \mathrm{LCH}_{/M_{\mathbb{R}}}$$

is the forgetful functor taking $(X, f, \mathcal{F})$ to (X, f). The functor

$$s : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}\mathrm{hv}_!$$

takes (X, f) to $(X, f, f^! \omega_{M_{\mathbb{R}}}) \in \mathcal{S}\mathrm{hv}_!$. The functor

$$t : \mathcal{S}\mathrm{hv}_! \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

takes $(X, f, \mathcal{F})$ to $f_! \mathcal{F} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$. This casual description suggests that $t \circ s$ supplies the construction we need. Note that we are not even mentioning what these functor does to maps or higher coherences, nor multiplicative structure. This is what makes the construction technical.

We start by constructing $\mathcal{S}\mathrm{hv}_!$.

Notation 4.2.5. The forgetful functor forgetful : $\mathrm{LCH}_{/M_{\mathbb{R}}} \longrightarrow \mathrm{LCH}$ is symmetric monoidal and we have a composition of functors

$$\mathrm{LCH}_{/M_{\mathbb{R}}} \xrightarrow{\text{forgetful}} \mathrm{LCH} \xrightarrow{D_!} \mathrm{Cat}$$

where the latter functor comes from Construction 4.1.2. We abuse notation and again write the composition as

$$D_! : \mathrm{LCH}_{/M_{\mathbb{R}}} \longrightarrow \mathrm{Cat}$$

when there is no danger of confusion. Note that this composition is also a lax symmetric monoidal functor.

The category $\mathcal{S}\mathrm{hv}_!$ is just the unstraightening (i.e. Grothendieck construction) of the functor $D_! : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathrm{Cat}$, and the symmetric monoidal structure actually comes with unstraightening - using a symmetric monoidal version of the Grothendieck construction that we recall as follows.

Theorem 4.2.6 (Symmetric monoidal Grothendieck construction)**.** (See [17, A.2.1][13, Proposition 3.3.4.11] [32, Theorem 2.1] for a history of the theorem.) Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. There is an equivalence of categories

$$\mathrm{coCart}_{\mathcal{C}}^{\mathbb{E}_\infty} \simeq \mathrm{Fun}^{\mathrm{lax}\otimes}(\mathcal{C}, \mathrm{Cat})$$

which is compatible with the straightening-unstraightening equivalence

$$\mathrm{coCart}_{\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}).$$

Let's immediately recall the definition of the objects appearing in the theorem.

1. For a category $\mathcal{C}$, the category $\mathrm{coCart}_{\mathcal{C}}$ is the category of coCartesian fibrations over $\mathcal{C}$ with coCartesian edge preserving functors over $\mathcal{C}$ as morphisms.
2. If $(\mathcal{C}, \otimes)$ is a symmetric monoidal category with $\mathcal{C}^{\otimes} \to \mathbb{E}_{\infty}^{\otimes}$[12] being the underlying operad, the category $\mathrm{coCart}_{\mathcal{C}}^{\mathbb{E}_{\infty}^{\otimes}}$ is the category of $\mathbb{E}_{\infty}^{\otimes}$-monoidal coCartesian fibrations over $\mathcal{C}$ of [32, Definition 1.11]. It is defined to be the full subcategory of $\mathrm{coCart}_{\mathcal{C}^{\otimes}}$ spanned by those coCartesian fibrations $\mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ such that the underlying $\mathcal{D} \to \mathcal{C}$ is a coCartesian fibration and that the $\mathbb{E}_{\infty}^{\otimes}$-monoidal operations preserve coCartesian edge.

Definition 4.2.7. Applying the symmetric monoidal Grothendieck construction to the lax symmetric monoidal functor $D_! : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathrm{Cat}$ produces an $\mathbb{E}_{\infty}^{\otimes}$-monoidal coCartesian fibration

$$p^{\otimes} : \mathcal{S}\mathrm{hv}_!^{\otimes} \longrightarrow \mathrm{LCH}^{\otimes}_{/M_{\mathbb{R}}}.$$

We write

$$p : \mathcal{S}\mathrm{hv}_! \longrightarrow \mathrm{LCH}_{/M_{\mathbb{R}}}$$

for the underlying map making $\mathcal{S}\mathrm{hv}_!$ a coCartesian fibration over $\mathrm{LCH}_{/M_{\mathbb{R}}}$.

In view of [HA, Remark 2.1.2.14] and Lemma 4.2.13, the structure map $p^{\otimes}$ is a map of $\mathbb{E}_{\infty}^{\otimes}$-monoidal category. In other words, it presents p as a symmetric monoidal functor. This functor p won't appear in the final construction, but we will introduce other players that revolve around $\mathcal{S}\mathrm{hv}_!$ and p. We start with introducing the following diagram

$$\begin{tikzcd} \mathrm{LCH}_{/M_{\mathbb{R}}} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\underline{M_{\mathbb{R}}}"'] \arrow[r, phantom, "\Downarrow h"] & \mathrm{LCH}_{/M_{\mathbb{R}}} \arrow[r, "D_!"] & \mathrm{Cat} \end{tikzcd},$$

where $\underline{M_{\mathbb{R}}}$ is the constant functor at $(M_{\mathbb{R}}, \mathrm{id}) \in \mathrm{LCH}_{/M_{\mathbb{R}}}$ and h is the natural transformation to the constant functor on the terminal object. Note that h is actually a natural transformation between lax symmetric monoidal functors. Now we apply the Grothendieck construction to $D_!(h) : D_! \circ \mathrm{id} \to D_! \circ \underline{M_{\mathbb{R}}}$ and get the following diagram

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}_! \arrow[rr, "\mathrm{Un}(D_!(h))"] \arrow[dr, "p"'] & & \mathrm{LCH}_{/M_{\mathbb{R}}} \times \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \arrow[dl, "q"] \\ & \mathrm{LCH}_{/M_{\mathbb{R}}} & \end{tikzcd}$$

underlying the diagram of operads supplied by the symmetric monoidal Grothendieck construc-

[12] Note that $\mathbb{E}_{\infty}^{\otimes}$ is just a fancy name for Fin_*.

tion

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}_!^{\otimes} \arrow[rr, "\mathrm{Un}(D_!(h))^{\otimes}"] \arrow[dr, "p^{\otimes}"] \arrow[ddr, bend right, "\pi_2^{\otimes}"'] & & (\mathrm{LCH}_{/M_{\mathbb{R}}} \times \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}))^{\otimes} \arrow[dl, "q^{\otimes}"'] \arrow[ddl, bend left, "\pi_3^{\otimes}"] \\
& \mathrm{LCH}^{\otimes}_{/M_{\mathbb{R}}} \arrow[d, "\pi_1^{\otimes}"] & \\
& \mathbb{E}_{\infty}^{\otimes} &
\end{tikzcd}.$$

In the diagram, $\pi_i^{\otimes}$ are the structure maps of the operads. Our first goal is to produce the right adjoint r of $\mathrm{Un}(D_!(h))$ along with the lax symmetric monoidal structure on it.

Proposition 4.2.8. The functor $\mathrm{Un}(D_!(h)) : \mathcal{S}\mathrm{hv}_! \to \mathrm{LCH}_{/M_{\mathbb{R}}} \times \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$ admits a right adjoint r. Moreover, r admits a lax symmetric monoidal structure.

Proof. To begin with, we want to show that $\mathrm{Un}(D_!(h))$ has a right adjoint functor r. We know the following facts about $\mathrm{Un}(D_!(h))$: that the restriction of $\mathrm{Un}(D_!(h))$ to each fiber over $\mathrm{LCH}_{/M_{\mathbb{R}}}$ has a right adjoint and that $\mathrm{Un}(D_!(h))$ preserves coCartesian edges since it is unstraightened from a natural transformation. With these one can apply [HA, Proposition 7.3.2.6] and learn that it has a right adjoint (even relative to $\mathrm{LCH}_{/M_{\mathbb{R}}}$). By construction, r restricts to fiberwise right adjoints.

Now we explain the lax symmetric monoidal structure on r. From Lemma 4.2.13 we learn that $\mathrm{Un}(D_!(h)^{\otimes})$ is a map of $\mathbb{E}_{\infty}^{\otimes}$-monoidal categories, i.e. $\mathrm{Un}(D_!(h))$ is a symmetric monoidal functor. Now one can invoke [HA, Corollary 7.3.2.7] and learn that r has a structure of lax symmetirc monoidal functor. □

We have achieved our first goal. Our next player is the functor

$$\mathrm{id} \times \underline{\omega_{M_{\mathbb{R}}}} : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathrm{LCH}_{/M_{\mathbb{R}}} \times \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}).$$

As the name suggests, it is induced by $\mathrm{id} : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathrm{LCH}_{/M_{\mathbb{R}}}$ and the constant functor $\underline{\omega_{M_{\mathbb{R}}}}$: $\mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$. Recall that we have the dualizing sheaf $\omega_{M_{\mathbb{R}}}$ defined by

$$\omega_{M_{\mathbb{R}}} := \pi^! \mathbb{1}_{\mathcal{S}\mathrm{hv}(*;\mathrm{Sp})} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}),$$

where $\pi : M_{\mathbb{R}} \to *$ is the map from $M_{\mathbb{R}}$ to the terminal object $*$. Let's make an observation on $\omega_{M_{\mathbb{R}}}$.

Proposition 4.2.9. The dualizing sheaf $\omega_{M_{\mathbb{R}}}$ acquires the structure of a commutative algebra for the convolution product.

Proof. This follows from the fact that $\pi^! : \mathcal{S}\mathrm{hv}(*;\mathrm{Sp}) \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$ has the structure of a lax symmetric monoidal functor where both side has the convolution symmetric monoidal structure. In addition, the convolution product on $\mathcal{S}\mathrm{hv}(*;\mathrm{Sp})$ is the same as the point-wise tensor product on $\mathcal{S}\mathrm{hv}(*;\mathrm{Sp}) \simeq \mathrm{Sp}$ that is usually used. The lax symmetric monoidal structure on $\pi^!$ is given by the (strong) symmetric monoidal structure on its left adjoint $\pi_!$. To be more precise: the map π is actually a map of commutative monoids in LCH. Hence by construction of the convolution tensor product, π induces a symmetric monoidal functor

$$\pi_! : \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(*;\mathrm{Sp}).$$

We again take advantage of [HA, Corollary 7.3.2.7] and get a lax symmetric monoidal structure on its right adjoint

$$\pi^! : \mathcal{S}\mathrm{hv}(*;\mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}).$$

In particular it takes $\mathbb{1}_{\mathcal{S}\mathrm{hv}(*;\mathrm{Sp})}$ to a commutative algebra, as desired. □

The commutative algebra structure on $\omega_{M_\mathbb{R}}$ furnishes the constant functor

$$\underline{\omega_{M_\mathbb{R}}} : \mathrm{LCH}_{M_\mathbb{R}} \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp})$$

with a lax symmetric monoidal structure. From this discussion, one learns that

Proposition 4.2.10. The functor $\mathrm{id} \times \underline{\omega_\mathbb{R}}$ admits a lax symmetric monoidal structure.

Proof. By previous discussion, it is a product of two lax symmetric monoidal functors, hence has a lax symmetric monoidal structure. □

We arrive at the following diagram

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}_! \arrow[rr, "\mathrm{Un}(D_!(h))"'] \arrow[dr, "p"'] & & \mathrm{LCH}_{/M_\mathbb{R}} \times \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \arrow[ll, bend right, "r"'] \arrow[dl, "q"] \arrow[r, "p_2"] & \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \\
& \mathrm{LCH}_{/M_\mathbb{R}} \arrow[ur, bend right, "\mathrm{id}\times\underline{\omega_{M_\mathbb{R}}}"'] & &
\end{tikzcd}$$

where we are going to make use of the red-colored functors, which are lax symmetric monoidal. We conclude the construction by a composition of these four functors: according to the plan, we have constructed the following lax symmetric monoidal functors

$$s = r \circ (\mathrm{id} \times \underline{\omega_{M_\mathbb{R}}}) : \mathrm{LCH}_{/M_\mathbb{R}} \to \mathcal{S}\mathrm{hv}_!$$

and

$$t = p_2 \circ \mathrm{Un}(D_!(h)) : \mathcal{S}\mathrm{hv}_! \to \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp})$$

so that the composition

$$t \circ s : \mathrm{LCH}_{/M_\mathbb{R}} \to \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp})$$

is what we aimed for.

Definition 4.2.11 (Sheaf of relative homology)**.** We define the lax symmetric monoidal functor

$$\Gamma_{M_\mathbb{R}} = t \circ s : \mathrm{LCH}_{/M_\mathbb{R}} \to \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp})$$

as the output of the construction. And we call $\Gamma_{M_\mathbb{R}}(X, f)$ the sheaf of homology of X relative to $M_\mathbb{R}$. The naming follows from the fact that after further !-pushfoward to a point, the sheaf $\Gamma_{M_\mathbb{R}}(X, f)$ is taken to the homology of X:

$$\pi_! \Gamma_{M_\mathbb{R}}(X, f) \simeq C_c^*(X, \omega_X).$$

Note, however, the name might be misleading since the stalk of the sheaf $\Gamma_{M_\mathbb{R}}(X, f)$ needs not to be the homology of the fiber.

Variant 4.2.12. For later purposes, we also by abuse of notation write the restriction of the functor $\Gamma_{M_{\mathbb{R}}}$ to the full subcategory of polyhedral subsets as

$$\Gamma_{M_{\mathbb{R}}} : \mathrm{Poly}(M_{\mathbb{R}}) \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

Moreover, the category $\mathrm{Poly}(M_{\mathbb{R}})$ carries a symmetric monoidal structure given by Minkowski sum that makes the inclusion functor

$$\mathrm{Poly}(M_{\mathbb{R}}) \longrightarrow \mathrm{LCH}_{/M_{\mathbb{R}}}$$

lax symmetric monoidal. We hence conclude that the functor

$$\Gamma_{M_{\mathbb{R}}} : \mathrm{Poly}(M_{\mathbb{R}}) \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

is also lax symmetric monoidal.

We end the section by recording the following elaboration of the argument in [HA, Proposition 2.1.2.12]. See also [Kerodon, 01UL].

Lemma 4.2.13. We have the following facts concerning coCartesian fibrations:

1. Consider the following commuting diagram of categories:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "p"] \arrow[dr, "q \circ p"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{E} & \end{tikzcd}.$$

   If both q and p are coCartesian fibraitons, then so is $q \circ p$. Moreover, given an edge $f \in \mathcal{E}$ and a $q \circ p$-coCartesian lift $f' \in \mathcal{C}$ of f, there exists an edge $f'' \in \mathcal{D}$ which is a q-coCartesian lift of f and $p(f')$ is equivalent to f''. Consequently, p preserves coCartesian lifts from $\mathcal{E}$.

2. Consider the following commuting diagram of categories:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "p"] \arrow[dr, "q \circ p"'] \arrow[ddr, "\pi_2"'] & & \mathcal{D} \arrow[dl, "q"] \arrow[ddl, "\pi_3"] \\ & \mathcal{E} \arrow[d, "\pi_1"] & \\ & \mathcal{O} & \end{tikzcd}.$$

   Assume that q, $q \circ p$ and π_1 are coCartesian fibrations. Assume further that p preserves coCartesian lifts from $\mathcal{E}$. Then p preserves coCartesian lifts from $\mathcal{O}$.

Proof. 1. That a composition of coCartesian fibration is coCartesian fibration is proved in [HTT, Proposition 2.4.2.3]. For the second part, given $f \in \mathcal{E}$ and a $q \circ p$ coCartesian lift $f' \in \mathcal{C}$ of f, one can choose $f'' \in \mathcal{D}$ to be a q-coCartesian lift of f. Let $\bar{f}' \in \mathcal{C}$ be a p-coCartesian lift of f'', then $\bar{f}'$ would also be a $q \circ p$-coCartesian lift of f using [HTT, Proposition 2.4.1.3]. We conclude that $\bar{f}'$ is equivalent to f' and hence $p(f')$ is equivalent to $p(\bar{f}') = f''$. The last claim about p preserving coCartesian lifts from $\mathcal{E}$ then follows.

2. Let $f' \in \mathcal{C}$ be a π_2-coCartesian lift of $f \in \mathcal{O}$. By the previous item, we might assume f' is a $q \circ p$-coCartesian lift of $q \circ p(f')$. Then by assumption on p, the image $p(f') \in \mathcal{D}$ is a q-coCartesian lift of $q \circ p(f')$, hence is a π_3-coCartesian lift of $f \in \mathcal{O}$ as desired.

□

4.3 Combinatorial versus constructible

Now we take advantage of the functor $\Gamma_{M_\mathbb{R}}$ from the previous section to write down the combinatorial-constructible comparison functor. First, we give a quick idea of the construction.

Fix toric data (N, Σ) and pick a cone $\sigma \in \Sigma$. Recall that we have defined the combinatorial category $\Theta(\sigma)$ to be a full subcategory of $\mathrm{Poly}(M_\mathbb{R})$. The category $\mathrm{Poly}(M_\mathbb{R})$ has a symmetric monoidal structure given by Minkowski sum and one can think of the symmetric monoidal structure on $\Theta(\sigma)$ as inherited from the inclusion (to be very precise, $\Theta(\sigma)$ includes into the full subcategory $\mathrm{Mod}_{\sigma^\vee}\mathrm{Poly}(M_\mathbb{R})$ over the idempotent algebra $\sigma^\vee \in \mathrm{Poly}(M_\mathbb{R})$ and this inclusion is symmetric monoidal). Post-composing this inclusion with $\Gamma_{M_\mathbb{R}}$ that we have defined earlier, we get a combinatorial-to-constructible comparison functor. The goal of this section is to construct this functor and present its functoriality along Σ.

We start with constructing a family of idempotent algebras in $\mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$. Here is a technical observation of the interaction of $\Gamma_{M_\mathbb{R}}$ with polytopes which is conceptually helpful, albeit not necessarily needed.

Lemma 4.3.1. For a closed polyhedral subset (of top dimension) $\overline{U} \subseteq M_\mathbb{R}$ and its interior U, the map of sheaves

$$\Gamma_{M_\mathbb{R}}(U) \to \Gamma_{M_\mathbb{R}}(\overline{U})$$

induced from $U \to \overline{U}$ is an equivalence. Note that left hand side is a more familiar object: the extension-by-zero of a shift of constant sheaf on an open subset.

Proof. This can be proved by comparing the recollement sequences for U and $\overline{U}$. Here we give a direct proof. In this case, one can check equivalence on stalks. By proper base change, it is easy to check for $x \notin \partial\overline{U}$ the map is an equivalence on stalk at x. It remains to check that at $x \in \partial\overline{U}$ the stalk of the right-hand side vanishes (again by proper base change it vanishes on the left-hand side). To compute the stalk, one can pick a family of open balls D_i of shrinking radii centered at x and compute

$$\Gamma_{M_\mathbb{R}}(\overline{U})_x \simeq \operatorname{colim} \Gamma_{M_\mathbb{R}}(\overline{U})(D_i).$$

To compute the right-hand side, use the identification $\omega_{M_\mathbb{R}} \simeq \underline{\mathbb{S}}[n]$ and apply proper base change to get

$$\Gamma_{M_\mathbb{R}}(\overline{U})(D_i) \simeq (i_{\overline{U}!} i^!_{\overline{U}} \underline{\mathbb{S}}[n])(D_i) \simeq \mathrm{fib}[(\underline{\mathbb{S}}(D_i) \to \underline{\mathbb{S}}(D_i \setminus \overline{U})][n].$$

Since $\overline{U}$ is polyhedral, for a sufficiently small ball $D_i \to D_i \setminus \overline{U}$ is a homotopy equivalence and hence the stalk vanishes, as desired. □

Proposition 4.3.2 (Dualizing sheaf of a cone is an idempotent algebra)**.** For each $\sigma \in \Sigma$, the object $\sigma^\vee \in \mathrm{Poly}(M_\mathbb{R})$ has the structure of an idempotent algebra. Thus, we might think of $\sigma^\vee$ as a diagram of idempotent algebras in $\mathrm{Poly}(M_\mathbb{R})$ indexed by $\sigma \in \Sigma^{\mathrm{op}}$. Moreover, the image of each $\sigma^\vee$ under $\Gamma_{M_\mathbb{R}}$ is also an idempotent algebra. Therefore, we get $\Gamma_{M_\mathbb{R}}(\sigma^\vee) = \omega_{\sigma^\vee}$ as a diagram of idempotent algebras in $\mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$ indexed by Σ^{op}.

Proof. The first observation is direct, using that $\sigma^\vee + \sigma^\vee = \sigma^\vee$ for any cone σ. For the second assertion, one needs to compute that the multiplication map of the algebra $\Gamma_{M_\mathbb{R}}(\sigma^\vee)$ is an isomorphism

$$\Gamma_{M_\mathbb{R}}(\sigma^\vee) * \Gamma_{M_\mathbb{R}}(\sigma^\vee) \xrightarrow{\simeq} \Gamma_{M_\mathbb{R}}(\sigma^\vee).$$

By the previous lemma, it is equivalent to showing that $\Gamma_{M_\mathbb{R}}(\sigma^{\vee,\circ})$ is an idempotent algebra. Now that we are working with a polyhedral open subset, we can unpack the definition of multiplication maps and this reduces to the same computation as in Proposition 4.1.3 up to a shift. $\square$

Corollary 4.3.3. There is a diagram in SMCat indexed by Σ^{op} given by

$$\sigma \mapsto \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \in \mathrm{SMCat}.$$

Furthermore, there is a symmetric monoidal left adjoint functor

$$L : \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \longrightarrow \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

given by tensoring with $\omega_{\sigma^\vee}$ in each component. In particular, this functor has a lax symmetric monoidal right adjoint

$$R : \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}).$$

Note that since each $\omega_{\sigma^\vee}$ is an idempotent algebra, the forgetful functor

$$\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

is a fully faithful functor. Hence R is also fully faithful. One can describe the functor R explicitly as follows: given an object in the limit, one applies forgetful functor to $\mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$ pointwise to get a diagram in $\mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$ and then take the limit. See Proposition 4.5.4 for more on this functor R and that it is always an equivalence for a smooth projective fan, hence in particular symmetric monoidal.

We move on to the main construction. The following proposition sketches our goal and the construction will be provided right after.

Proposition 4.3.4. There is a symmetric monoidal functor

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

where the left-hand side has the Day convolution tensor product and right-hand side has the convolution product of sheaves. Moreover, these functors are natural in $\sigma \in \Sigma^{\mathrm{op}}$ that they assemble into a natural transformation of diagrams in SMCat indexed by $\sigma \in \Sigma^{\mathrm{op}}$. Hence taking limit produces

$$\lim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\lim \Psi_\sigma} \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \xrightarrow{R} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}).$$

The first functor is symmetric monoidal. It is fully faithful when the fan is smooth, as shown in Corollary 4.4.19. The latter functor is the right adjoint functor R in Corollary 4.3.3 which is lax symmetric monoidal and fully faithful. It is symmetric monoidal when the fan is smooth and projective, as shown in Proposition 4.5.4. In conclusion, when the fan Σ is smooth and projective we have a symmetric monoidal fully faithful functor

$$\Psi_\Sigma : \lim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\lim \Psi_\sigma} \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}).$$

We first construct Ψ_σ pointwise.

Construction 4.3.5. Fix $\sigma \in \Sigma$, consider the composition of lax symmetric monoidal functors

$$\Theta(\sigma) \longrightarrow \mathrm{Poly}(M_{\mathbb{R}}) \xrightarrow{\Gamma_{M_{\mathbb{R}}}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}),$$

where the first functor is the canonical inclusion (recall that $\Theta(\sigma)$ is by definition a full subcategory of $\mathrm{Poly}(M_{\mathbb{R}})$) and the second functor is $\Gamma_{M_{\mathbb{R}}}$ constructed in Definition 4.2.11. Now we observe that for each σ, the image of $\Theta(\sigma)$ lies in the full subcategory $\mathrm{Mod}_{\omega_{\sigma^\vee}}\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$ of $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$. It follows that we have a lax symmetric monoidal functor

$$\psi_\sigma : \Theta(\sigma) \to \mathrm{Mod}_{\omega_{\sigma^\vee}}\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$$

which is readily symmetric monoidal by Proposition 4.1.3. Now one can left Kan extend this to a symmetric monoidal functor

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}},\mathrm{Sp}) \to \mathrm{Mod}_{\omega_{\sigma^\vee}}\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$$

which is what we want.

Now we construct Ψ_σ with functoriality along σ.

Remark 4.3.6 (Functoriality of Ψ_σ along σ)**.** To provide the functoriality of symmetric monoidal functors

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}},\mathrm{Sp}) \to \mathrm{Mod}_{\omega_{\sigma^\vee}}\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$$

along $\sigma \in \Sigma^{\mathrm{op}}$, we make the following constructions. Consider the subcategory $\mathrm{Poly}^*(M_{\mathbb{R}}) \subseteq \mathrm{Poly}(M_{\mathbb{R}})$ spanned by the origin and top dimensional polyhedral subsets. This category inherits a symmetric monoidal structure and Proposition 4.1.3 implies that the restriction

$$\Gamma_{M_{\mathbb{R}}} : \mathrm{Poly}^*(M_{\mathbb{R}}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$$

is symmetric monoidal[13]. Now we consider the presheaf category [14]

$$\mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}},\mathrm{Spc})$$

equipped with the Day convolution product. We have a family of idempotent algebras in $\mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}},\mathrm{Spc})$ given by

$$\sigma^\vee \in \mathrm{CAlg}(\mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}},\mathrm{Spc}))$$

coming from Corollary 4.3.3. We have by abuse of notation, identified $\sigma^\vee$ with its Yoneda image. Now $\Gamma_{M_{\mathbb{R}}}$ induces a symmetric monoidal colimit-preserving functor

$$\mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}},\mathrm{Spc}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$$

so we can apply Proposition A.2.1 and obtain a diagram in SMCat indexed by $\sigma \in \Sigma^{\mathrm{op}}$:

$$\mathrm{Mod}_{\sigma^\vee}\,\mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}},\mathrm{Spc}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}}\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}).$$

[13] It is true that $\Gamma_{M_{\mathbb{R}}}$ is symmetric monoidal on $\mathrm{Poly}(M_{\mathbb{R}})$, but it takes more effort to show. We restrict to $\mathrm{Poly}^*(M_{\mathbb{R}})$ as it suffices for our purpose here.

[14] Note that we have played the same trick of passing to presheaf category in Definition 3.3.6.

It remains to write down a natural transformation in SMCat of diagrams indexed by Σ^{op}

$$\psi_\sigma : \Theta(\sigma) \longrightarrow \mathrm{Mod}_{\sigma^\vee} \mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc}).$$

This reduces to 1-categorical considerations: for example, one way to do this is to identify (as a symmetric monoidal category) $\Theta(\sigma)$ with the full subcategory of $\mathrm{Mod}_{\sigma^\vee} \mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc})$ spanned by Yoneda image of integral translations of $\sigma^\vee$ as in Remark 2.0.4. Given $\tau \subseteq \sigma$ in Σ, the symmetric monoidal functors given by base change

$$\mathrm{Mod}_{\sigma^\vee} \mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc}) \longrightarrow \mathrm{Mod}_{\tau^\vee} \mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc})$$

restricts to structure maps

$$\Theta(\sigma) \longrightarrow \Theta(\tau).$$

Consequently we have a natural transformation between diagrams in SMCat indexed by Σ:

$$\Theta(\sigma) \longrightarrow \mathrm{Mod}_{\sigma^\vee} \mathrm{Fun}(\mathrm{Poly}(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

Now one can left Kan extend as in Section A.3 and obtain the symmetric monoidal functors naturally along Σ^{op}

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

For future use, we record here the interaction of Ψ_σ with translation action by lattice M.

Construction 4.3.7 (Compatibility with the lattice)**.** For convenience, we assume that the fan is smooth and projective. For each σ, recall that we have a symmetric monoidal inclusion $p_\sigma : M \to \Theta(\sigma) : m \mapsto m + \sigma^\vee$ and one obtains its left Kan extension as a symmetric monoidal functor:

$$p_{\sigma!} : \mathrm{Fun}(M, \mathrm{Sp}) \to \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}).$$

This functor is natural in σ when we view $\mathrm{Fun}(M, \mathrm{Sp})$ as a constant diagram indexed by $\sigma \in \Sigma^{\mathrm{op}}$. It follows that we have the following diagram in SMCat after taking limit:

$$\begin{tikzcd} \lim_\sigma \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Psi_\Sigma"] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \\ \mathrm{Fun}(M, \mathrm{Sp}) \arrow[u, "\lim_\sigma p_{\sigma!}"] \arrow[ur, "\Psi_\Sigma \circ \lim_\sigma p_{\sigma!}"'] & \end{tikzcd}.$$

It turns out that one can identify the diagonal functor with a more familiar one when working with a smooth projective fan Σ: (on the bottom we view M as a discrete topological group, and $i_!$ is the !-pushforward along the inclusion of topological groups which is symmetric monoidal for sheaf categories with convolution products)

$$\begin{tikzcd} \lim_\sigma \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Psi_\Sigma"] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \\ \mathrm{Fun}(M, \mathrm{Sp}) \arrow[u, "\lim_\sigma p_{\sigma!}"] \arrow[r, "\simeq"'] & \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \arrow[u, "i_!"'] \end{tikzcd}.$$

As we don't need it for now, we defer its proof to Theorem 6.1.5.

We summarize the constructions so far by making the following definition.

Definition 4.3.8. Let Σ be a smooth projective fan. Combining Proposition 3.3.1 and Proposition 4.3.4, we have the following functors

$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}] \xleftarrow{\simeq} \lim_{\sigma\in\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}};\mathrm{Sp}) \longrightarrow \lim_{\sigma\in\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}),$$

where the first two functors are symmetric monoidal functors supplied by $\lim \Phi_\sigma$ and $\lim \Psi_\sigma$. We take the inverse of the first functor and obtain the coherent-constructible correspondence functor

$$\kappa : \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}),$$

which is symmetric monoidal and fully faithful when Σ is smooth projective, in view of Proposition 4.3.4. Given Remark 3.3.2 and Construction 4.3.7, we also have the following diagram in SMCat:

$$\begin{tikzcd} \mathrm{QCoh}(X_\Sigma/\mathbb{T}) \arrow[r, "\kappa"'] & \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \\ \mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\pi^*"] \arrow[r, "\simeq"'] & \mathcal{S}\mathrm{hv}(M;\mathrm{Sp}) \arrow[u, "i_!"'] \end{tikzcd}.$$

4.4 Polyhedral stratification

The goal of this subsection is twofold: on the one hand, we show that the functors Ψ_σ previously constructed are fully-faithful; on the other hand, we pin down a first-order approximation of the characterization of the image of κ. That is to say, we will not actually work with the whole (gigantic) category of sheaves, but only a subcategory: those constructible for some fixed stratification. Moreover, the stratification has an elementary description in terms of the fan data. We start with a quick review on constructible sheaves following [7].

Definition 4.4.1. A poset P satisfies the ascending chain condition if every strictly increasing chain in P stops after finitely many steps. A poset P is locally finite if each $P_{\geqslant q} := \{p : p \geqslant q\}$ is finite.

Remark 4.4.2. Note that locally finite implies the ascending chain condition but not vice versa.

Definition 4.4.3. A stratified topological space is a continuous map $\pi : X \to P$ where X is a topological space and P is a poset equipped with the Alexandroff topology [15]. We often write (X, P) for a stratified topological space and omit the map π. For each $p \in P$, the preimage $\pi^{-1}(p) \subseteq X$ is called its p-stratum X_p. The stratum X_p is a closed subspace of $\mathrm{Star}_p := \pi^{-1}\{q : p \leqslant q\} \subseteq X$, the open star around p.

Definition 4.4.4. Given two maps $f, g : (X, P) \to (Y, Q)$ between stratified topological spaces. A stratified homotopy between f and g is a map $H : X \times [0, 1] \to Y$ between stratified topological spaces that restricts to f on $X \times 0$ and g on $X \times 1$. Here $X \times [0, 1]$ inherits the stratification from X.

[15] Recall that a subset $U \subseteq P$ is open in the Alexandroff topology if and only if for $p \in U$, $p \leqslant q$ implies $q \in U$. In other words, U is a 'cosieve': a subset that is upward closed for the partial order of P.

Definition 4.4.5. A map of stratified topological space $f : (X, P) \to (Y, Q)$ is a stratified homotopy equivalence if there is a map $g : (Y, Q) \to (X, P)$ going in the other direction, such that both of $f \circ g$ and $g \circ f$ are stratified homotopic to identity.

Definition 4.4.6. Fix a compactly generated category $\mathcal{C}$ (we will only care about Spc or Sp) as coefficient and a stratified topological space $\pi : X \to P$. A sheaf on X valued in $\mathcal{C}$ is P-constructible [16] if its restriction to each stratum X_p is locally constant. We write $\mathrm{Cons}_P(X; \mathcal{C})$ for the full subcategory of P-construcible sheaves.

We want to take advantage of the exodromy equivalence to identify a family of compact generators for the category of constructible sheaves. We start by importing the following theorem which realizes the exodromy equivalence for a class of particularly simple stratified topological spaces.

Theorem 4.4.7. [7, Theorem 3.4] Let $\pi : X \to P$ be a stratified topological space with π surjective and P satisfying the ascending chain condition. Suppose there is a collection $\mathcal{B}$ of open subsets of X such that

1. the representable sheaves h_U for $U \in \mathcal{B}$ generate the topos $\mathcal{S}\mathrm{hv}(X; \mathrm{Spc})$.
2. for all $U \in \mathcal{B}$, there is a $p \in P$ such that U includes into Star_p by a stratified homotopy equivalence.

Then the pullback map

$$\pi^* : \mathrm{Fun}(P, \mathrm{Spc}) \to \mathcal{S}\mathrm{hv}(X; \mathrm{Spc})$$

preserves all limits and colimits and is fully faithful with essential image $\mathrm{Cons}_P(X; \mathrm{Spc})$.

Remark 4.4.8. The theorem in [7] was stated and proved for sheaves valued in Spc. The proof works verbatim for Sp coefficient. It is also true for other compactly generated coefficient categories, which isn't needed for our exposition.

This gives, for a locally finite poset P and stratification $X \to P$ as above, an explicit realization of the exodromy equivalence

$$\pi^* : \mathrm{Fun}(P, \mathrm{Sp}) \to \mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$$

which is the left adjoint of $\mathcal{S}\mathrm{hv}(X; \mathrm{Sp}) \to \mathrm{Fun}(P, \mathrm{Sp})$ sending $\mathcal{F}$ to $[q \mapsto \mathcal{F}(\mathrm{Star}_q)]$. Tracing through the equivalences, one sees that for $q \in P$, the image of q under stable Yoneda embedding (i.e. $\mathbb{S}[\mathrm{Map}_P(q, -)]$) is taken to $i_{\mathrm{Star}_q!}(\underline{\mathbb{S}})$ where i_{Star_q} is the inclusion of Star_q into X.

Corollary 4.4.9. Let $\pi : X \to P$ be as in Theorem 4.4.7. The category $\mathrm{Cons}_P(X; \mathrm{Sp})$ is generated by the compact objects $\{i_{\mathrm{Star}_q!}(\underline{\mathbb{S}})\}_{q \in P}$ in the following sense: the smallest cocomplete stable subcategory of $\mathrm{Cons}_P(X; \mathrm{Sp})$ that contains these objects is itself.

For future use we record a description of compact objects in functor categories here:

Lemma 4.4.10. Let P be a locally finite poset. A functor $X \in \mathrm{Fun}(P, \mathrm{Sp})$ is compact if and only if its value is none-zero on finitely many objects and each of its value is a finite spectrum.

[16] These are sometimes called quasi-constructible in the literature, where the word constructible is reserved for objects also satisfying a finiteness condition which we don't impose here.

Proof. This is the same as [25, Proposition 2.2.6] which proves the case where P is finite. If a functor F is compact, then as in [25, Proposition 2.2.5], it is a retract of an object in the smallest stable subcategory of $\mathrm{Fun}(\mathsf{P}, \mathrm{Sp})$ containing the image of stable Yoneda embedding. By the assumption that P is locally finite, each of the Yoneda functor $\mathbb{S}[\mathrm{Map}(x, -)]$ is non-zero on finitely many objects and each of its value is a finite spectrum. Such condition cuts out a stable subcategory $\mathcal{D}$ of $\mathrm{Fun}(\mathsf{P}, \mathrm{Sp})$ which is closed under retract, so we know that F is in $\mathcal{D}$. On the other hand, if F is non-zero on finitely many objects and each of its value is a finite spectrum, we may take the subposet $\mathrm{supp}^{\star}(\mathsf{F}) := \{y \in \mathsf{P} : \exists x \leqslant y, \mathsf{F}(x) \neq 0\}$. By assumption, this is a finite poset and the restriction of F to $\mathrm{supp}^{\star}(\mathsf{F})$ is thus compact. Now note that F is the left Kan extension of its restriction to $\mathrm{supp}^{\star}(\mathsf{F})$ and that left Kan extension preserves compact objects. □

Now we specialize to the case of our interest:

Definition 4.4.11 (FLTZ stratification)**.** (See also [37, Definition 4.3]) Fix a pair (N, Σ) of lattice and fan, and assume further that Σ spans $N_{\mathbb{R}}$ as an $\mathbb{R}$-vector space. We define a stratification $\mathcal{S}_{\Sigma}$ on $M_{\mathbb{R}}$ as follows. To start with, one has an affine hyperplane arrangement in $M_{\mathbb{R}}$ given by

$$H_{\Sigma} := \{m + \sigma^{\perp} : m \in M, \sigma \in \Sigma(1)\},$$

where $\sigma^{\perp} := \{m \in M : (m, n) = 0 \,\forall\, n \in \sigma\}$. One has the following induction procedure to specify strata of a stratification: first look at the complement

$$V := M_{\mathbb{R}} \setminus \bigcup_{h \in H_{\Sigma}} h,$$

where each of the connected component of V should be considered as a single stratum. For each $h \in H_{\Sigma}$, intersecting $h' \in H_{\Sigma}$ with h produces an affine hyperplane arrangement on h. Thus one can work inductively and define a poset of strata $\mathcal{S}_{\Sigma}$ of $M_{\mathbb{R}}$ (note they are locally closed). The closure of each stratum is a union of strata and one specify a poset structure by closure-inclusion. The map sending each point in $M_{\mathbb{R}}$ to the stratum it belongs to in $\mathcal{S}_{\Sigma}$ will be a continuous map and this gives a stratification on $M_{\mathbb{R}}$. We refer to this stratification $\mathcal{S}_{\Sigma}$ as the FLTZ stratification for Σ and we will often omit mentioning Σ when it is clear from the context.

Remark 4.4.12. Note that the FLTZ stratification only depends on the collection of 1-cones in Σ.

Remark 4.4.13. The stratification $\mathcal{S}_{\Sigma}$ on $M_{\mathbb{R}}$ is apparently sub-analytic in the sense of [14, Definition 5.3.7] so the exodromy equivalence applies directly. We choose to apply exodromy equivalence from [7] since it's more elementary.

We wish to use exodromy equivalence Theorem 4.4.7 to understand $\mathcal{S}_{\Sigma}$-constructible sheaves. For that we need:

Proposition 4.4.14. The stratification $\mathcal{S}_{\Sigma}$ on $M_{\mathbb{R}}$ meets the assumptions of Theorem 4.4.7 above.

Proof. We need to provide a basis of opens for $M_{\mathbb{R}}$ with desired properties. Consider the standard basis

$$\mathcal{B} := \{D(x, r) : \text{open ball of radius } r \text{ centered at } x \in M_{\mathbb{R}}\}$$

and a subset of it.

$$\mathcal{B}(\mathcal{S}_{\Sigma}) := \{D(x, r) \in \mathcal{B} : D(x, r) \text{ is stratified homotopy equivalent to the open star at } x\}$$

By definition each $D(x,r) \in \mathcal{B}(\mathcal{S}_\Sigma)$ would go through point 2. It suffices to check point 1, that it is a basis (or at the very least, nonempty). We claim that: for each $x \in M_\mathbb{R}$ there exists $r_x > 0$ such that $r < r_x$ implies $D(x,r) \in \mathcal{B}(\mathcal{S}_\Sigma)$. This directly implies that $\mathcal{B}(\mathcal{S}_\Sigma)$ is a basis of opens for $M_\mathbb{R}$. To prove the claim, a first observation is that for sufficiently small r, $D(x,r)$ with restricted stratification of $\mathcal{S}_\Sigma$ is (stratified) isomorphic to a real vector space with stratification given by a family of hyperplane arrangements. There is no other stratum coming into the picture than those passing through x. Fix such small r_x, then for all $r \leqslant r_x$, each $D(x,r)$ includes into each other as a stratified homotopy equivalence. It remains to prove that $D(x,r_x)$ is stratified homotopy equivalent to the open star at x. For this a straight-line linear homotopy should do the work. Note this works because the open star is convex and the linear scaling towards x respects the stratification. □

Remark 4.4.15. Note also that for a smooth projective fan Σ the poset P underlying the stratification $\mathcal{S}_\Sigma$ is locally finite, as each stratum is only in the closure of finitely many other strata, and each exit-path will enter a higher dimensional stratum, so must stop after finitely many steps.

The reason to introduce $\mathcal{S}_\Sigma$ is the following:

Proposition 4.4.16. Fix a pair (N,Σ) of lattice and fan. One might post-compose the functor

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

in Proposition 4.3.4 with forgetful into $\mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$, then its image all lands into the subcategory $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ of sheaves constructible for the FLTZ stratification. As a consequence, the functor

$$\varprojlim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

of Proposition 4.3.4 also lands into $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$.

Proof. It suffices to note that each $U \in \Theta(\sigma)$ is given by a cone bound by the hyperplane arrangement H_Σ. Any stratum of the stratification would be either contained in it or be disjoint from it. Using Lemma 4.3.1 and proper base change, it follows that $\Gamma_{M_\mathbb{R}}(U)$ is constructible for the FLTZ stratification. Now the image of $\Theta(\sigma)$ is colimit generated by these objects as a stable category, and constructible sheaf category is also closed under colimits, so we are done. □

We give a standard example to illustrate the ideas of the definitions so far.

Example 4.4.17. Take the fan spanned by $\{e_1, \ldots, e_n\} \subset \mathbb{Z}^n = N$. To be more precise, $\Sigma = \{\mathrm{span}(S) : S \subseteq \{e_1, \ldots, e_n\}\}$. This is the fan corresponding to $\mathbb{A}^n$ in toric geometry. It specifies the standard grid in $M_\mathbb{R} \simeq \mathbb{R}^n$ as the FLTZ stratification. The strata of $M_\mathbb{R} \to \mathcal{S}_\Sigma$ are faces of the unit hypercubes whose vertices have integer coordinates. More precisely, each stratum is cut out by equalities $\{x_i = n_i : i \in I\}$ and inequalities $\{x_j \in (n_j, n_j + 1) : j \in J\}$ where n_i and n_j are integers and the pair (I, J) is a decomposition of $\{1, \ldots, n\}$. The open stars in this case are also very explicit: they are certain hyper-rectangles whose vertices have integer coordinates. Using Proposition 4.1.3 one can compute the convolution product of representable sheaves on these open stars and it turns out to be again $\mathcal{S}_\Sigma$-constructible. It follows that in this case $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ is closed under convolution product.

Warning 4.4.18. The convolution product usually doesn't interact well with the FLTZ stratification $\mathcal{S}_\Sigma$. More precisely, for a fixed Σ, the convolution product of two $\mathcal{S}_\Sigma$-constructible sheaves needs not to stay $\mathcal{S}_\Sigma$-constructible. We will see later how to fix this.

Corollary 4.4.19. For a pair (N, Σ) with the fan Σ being smooth, the functor Ψ_Σ constructed in Proposition 4.3.4 is fully faithful. Moreover, for each $\sigma \in \Sigma$, the functor Ψ_σ is fully faithful.

Proof. Fix such σ, by the assumption on the smoothness, one can perform a linear transform in $SL(n, \mathbb{Z})$ which takes σ to the cone $\{e_1, \dots, e_k\}$ in the standard fan $\{e_1, \dots, e_n\} \subset N = \mathbb{Z}^n$ as in the previous example. So without loss of generality, we will prove for this standard case the functor Ψ_σ is fully faithful. Recall that Ψ_σ is of the form

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

and we note that it first of all factors through the full subcategory $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) \cap \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$ of FLTZ constructible sheaves inside $\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$ (for the standard fan Σ spanned by $\{e_1, \dots, e_n\}$ as above). The domain category is a compactly generated presentable stable category, with a set of compact generators supplied by the stable Yoneda image of representables. By construction of the functor Ψ_σ, it is fully faithful on this set of compact generators. Let's try to give an explicit description of the intersection $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) \cap \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$. We make the following observations:

1. The image of $\Psi_\sigma(\sigma^\vee)$ is an idempotent algebra for the constructible sheaf category $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ equipped with convolution monoidal structure. As before, we denote $\omega_{\sigma^\vee}$ for this algebra and consider the category $\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$. This is a category compactly generated by the convolution of representable sheaves on open stars with $\omega_{\sigma^\vee}$. From the previous example we know explicitly these open stars are integral hyper-rectangles, and the convolution products are (shifts of) representable sheaves on $\sigma^{\vee,\circ} + m$ for $m \in M$. Note that these are precisely the image of $\sigma^{\vee,\circ} + m$ under Ψ_σ.

2. It follows that wehave

$$\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \cap \mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) = \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$$

   and the functor Ψ_σ lands in this full subcategory. Moreover Ψ_σ takes a set of compact generators (representable presheaves on $\sigma^{\vee,\circ} + m$) to compact objects in the target $\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$, and is fully faithful on these compact generators.

We apply the following Lemma 4.4.20 and learn that Ψ_σ is fully faithful. It follows that $\lim_{\Sigma^{op}} \Psi_\sigma$ is also fully faithful. Now Ψ_Σ is a composition of two fully faithful functors, and hence is itself fully faithful. $\square$

Lemma 4.4.20. Let $\mathcal{C}$ be a compactly generated presentable stable category, with a chosen set of compact generators S (in other words, the smallest stable full subcategory of $\mathcal{C}$, which is closed under colimit in $\mathcal{C}$ and contains S, is $\mathcal{C}$ itself). Given a cocontinuous functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ with $\mathcal{D}$ a presentable stable category. Assume that F is fully faithful on S, and it takes S to compact objects in $\mathcal{D}$. Then F is fully faithful on all of $\mathcal{C}$.

4.5 Digression: gluing of idempotents in the sheaf category

This subsection is meant to answer the following question: can one give a description of $\mathcal{S}hv(M_{\mathbb{R}};\mathrm{Sp})$ similar to the limit diagram provided by Zariski descent for $\mathrm{QCoh}(-)$? In order to answer this question, we first recall how descent works in a presentably symmetric monoidal category with idempotent algebras. The following material is taken from [6, Lecture 8].

Definition 4.5.1. [HA, Definition 4.8.2.1] Let $\mathcal{C}$ be a presentably symmetric monoidal category. The category of idempotent objects $\mathcal{C}^{\mathrm{idem}} \subset \mathrm{Fun}([1],\mathcal{C})$ is the full subcategory of pairs $(A, f : 1_{\mathcal{C}} \to A)$ such that $f \otimes A : A \to A \otimes A$ is an equivalence.

We also recall the following facts:

1. [HA, Proposition 4.8.2.9] Take $\mathrm{CAlg}(\mathcal{C})^{\mathrm{idem}}$ to be the full subcategory of $\mathrm{CAlg}(\mathcal{C})$ spanned by $A \in \mathrm{CAlg}(\mathcal{C})$ such that the unit map makes A into an idempotent object of $\mathcal{C}$. The forgetful functor $\mathrm{CAlg}(\mathcal{C})^{\mathrm{idem}} \to \mathcal{C}^{\mathrm{idem}}$ is an equivalence. In particular every idempotent object acquires uniquely a commutative algebra structure.

2. [HA, Proposition 4.8.2.4] Take $A \in \mathcal{C}^{\mathrm{idem}}$. The functor $\mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ is a localization. In particular the forgetful $\mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ is fully faithful, with image those $X \in \mathcal{C}$ such that $X \to X \otimes A$ is an equivalence.

3. [6, Lemma 5 of Lecture 8] The category $\mathcal{C}^{\mathrm{idem}}$ is a poset.

4. [6, Lemma 5 of Lecture 8] As a poset $\mathcal{C}^{\mathrm{idem}}$ has all joins (unions) and finite meets (intersections). The join of A and B is computed as $A \otimes B$, and the join of an infinite family $\{A_i : i \in I\}$ is computed as the filtered colimit over the join of finite subsets (in the underlying category).

$$A \vee B = A \otimes B$$

$$\vee_{i\in I} A_i = \operatorname*{colim}_{J \subset I,\, \text{finite}} \bigotimes_{j\in J} A_j$$

The meet of A and B is computed as fiber of $A \times B \to A \otimes B$ and the meet of a *finite* family $\{A_i : i \in I\}$ is computed as the limit over the poset of nonempty subsets $J \subset I$ of the functor $J \mapsto \otimes_{j\in J} A_j$ (in the underlying category). Note that the limit diagram would be a cubical diagram as [HA, Proposition 1.2.4.13].

$$A \wedge B = A \underset{A\otimes B}{\times} B$$

$$\wedge_{i\in I} A_i = \lim_{J \subset I,\, \text{nonempty}} \bigotimes_{j\in J} A_j$$

5. [6, Theorem 4 of Lecture 8] One can define a Grothendieck topology on $\mathcal{C}^{\mathrm{idem,op}}$ as follows: a family of maps $\{f_i : A \to A_i \in \mathcal{C}^{\mathrm{idem}}\}$ is a cover if it contains a finite sub-family of maps $\{f_i : A \to A_i \in \mathcal{C}^{\mathrm{idem}}\}$ presenting A as the meet for $\{A_i\}$ in $\mathcal{C}^{\mathrm{idem}}$.

Theorem 4.5.2. The presheaf $\mathrm{Mod}_{(-)}(\mathcal{C}) : \mathcal{C}^{\mathrm{idem}} \to \mathrm{SMCat}$ which takes A to $\mathrm{Mod}_A(\mathcal{C})$ is a sheaf for above topology.

Proof. This is the same as Theorem 4 in [6, Lecture 8]. $\square$

Now we run this machine in practice. The most important example for us is the following:

Example 4.5.3 (Zariski descent in algebraic geometry)**.** For a scheme X and an open $U \subset X$, $*$-pushforward of the structure sheaf $i_*\mathcal{O}_U$ is an idempotent algebra in $\mathrm{QCoh}(X)$ (equipped with standard tensor product of quasi-coherent sheaves). The category of modules can be identified as

$$\mathrm{Mod}_{i_*\mathcal{O}_U}\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(U).$$

If a finite family $\{U_i\}$ form a Zariski cover of X, one can show that the family $\{\mathbb{1}_{\mathrm{QCoh}(X)} \to i_*\mathcal{O}_U\}$ is a cover. Evaluating $\mathrm{Mod}_{(-)}(\mathrm{QCoh}(X))$ on this cover recovers Zariski descent for $\mathrm{QCoh}(-)$.

Our goal is to formulate a convolution-of-sheaf version of such phenomenon. We fix a smooth projective fan Σ on N until the end of the subsection. Let's consider the family of idempotent algebras for the convolution product

$$\omega_{\sigma^\vee} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

introduced in Proposition 4.3.2.

Proposition 4.5.4. Let Σ be a smooth projective fan. Take the subset $\Sigma(n) \subset \Sigma$ of the top dimensional cones, then $\{\mathbb{1}_{\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})} \to \omega_{\sigma^\vee} : \sigma \in \Sigma(n)\}$ is a cover of $\mathbb{1}_{\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})}$. More explicitly, one has the following equivalences

$$\mathbb{1}_{\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})} \xrightarrow{\simeq} \lim_{\sigma \in \Sigma^{\mathrm{op}}} \omega_{\sigma^\vee} \xrightarrow{\simeq} \lim_{S \in \mathcal{P}_{\neq \emptyset}(\Sigma(n))} \bigotimes_{\tau \in S} \omega_{\tau^\vee},$$

where the second map is an equivalence by the following observations.

Remark 4.5.5. We make the following observations about the diagrams.

- Fix a smooth projective fan Σ. There exits an adjunction

  $$l : \mathcal{P}_{\neq \emptyset}(\Sigma(n)) \rightleftharpoons \Sigma^{\mathrm{op}} : r$$

  between the poset $\Sigma(n)$ of nonempty subsets of top dimensional cones and the opposite poset of all cones in the fan Σ. The map l sends a subset $S \subseteq \Sigma(n)$ to the intersection

  $$l(S) := \bigcap_{\sigma \in S} \sigma \in \Sigma^{\mathrm{op}}.$$

  The map r sends a cone $\tau \in \Sigma^{\mathrm{op}}$ to all the top dimensional cones containing it

  $$r(\tau) := \{\sigma \in \Sigma(n) : \tau \subseteq \sigma\}.$$

  The adjunction reduces to the following observation: the intersection of cones in a subset S contains τ if and only if S is contained in $r(\tau)$ which is the subset of all the cones containing τ. In particular, the map l is a final functor, or a limit-equivalence. Moreover, note that the composition $l \circ r$ is the identity map on Σ^{op}.

- The composite of functors

$$\mathcal{P}_{\neq\emptyset}(\Sigma(n)) \xrightarrow{l} \Sigma^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}))^{\mathrm{idem}}$$

$$S \mapsto l(S) \mapsto \omega_{l(S)^\vee}$$

can be identified with the Čech diagram

$$\mathcal{P}_{\neq\emptyset}(\Sigma(n)) \longrightarrow \mathrm{CAlg}(\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}))^{\mathrm{idem}}$$

$$S \mapsto \bigotimes_{\tau\in S} \omega_{\tau^\vee}.$$

This follows from that $\omega_{\sigma^\vee} * \omega_{\tau^\vee} \simeq \omega_{(\sigma\cap\tau)^\vee}$, which is a consequence of the combinatorial fact $\sigma^\vee + \tau^\vee = (\sigma\cap\tau)^\vee$ and the computation of convolution (as in Proposition 4.1.3). The combinatorial fact $\sigma^\vee + \tau^\vee = (\sigma\cap\tau)^\vee$ is a direct consequence of the separation lemma in [11, (11) and (12) of Section 1.2].

Proof of Proposition 4.5.4. By finality, we can switch to the diagram indexed by Σ^{op}. We need to show that

$$\mathbb{1}_{\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})} \to \lim_{\sigma\in\Sigma^{\mathrm{op}}} \omega_{\sigma^\vee}$$

is an equivalence. Let's compute the stalk of the limit. At the origin, the stalk is

$$\lim_{\sigma\in\Sigma^{\mathrm{op}}} (\omega_{\sigma^\vee})_0 \simeq \lim_{\sigma\in\Sigma^{\mathrm{op}}} \mathbb{S}_{\{0\}}[n] \in \mathrm{Sp},$$

where $\mathbb{S}_{\{0\}} : \Sigma^{\mathrm{op}} \to \mathrm{Sp}$ is the presheaf on Σ that takes value $\mathbb{S}$ at the origin and zero otherwise. To evaluate the limit, note that there is a fiber sequence in $\mathrm{Fun}(\Sigma^{\mathrm{op}}, \mathrm{Sp})$

$$\mathbb{S}_{\{0\}} \to \underline{\mathbb{S}} \to \mathbb{S}_{\Sigma^{\mathrm{op}}\setminus\{0\}},$$

where $\underline{\mathbb{S}}$ is the constant presheaf and $\mathbb{S}_{\Sigma^{\mathrm{op}}\setminus\{0\}}$ is the right Kan extension of the constant presheaf on $\Sigma \setminus \{0\}$. Taking global sections, we get the fiber sequence

$$\lim_{\sigma\in\Sigma^{\mathrm{op}}} \mathbb{S}_{\{0\}} \to \mathbb{S} \to \lim_{\sigma\in\Sigma^{\mathrm{op}}} \mathbb{S}_{\Sigma^{\mathrm{op}}\setminus\{0\}},$$

where the last term can be further computed by

$$\begin{aligned}\lim_{\sigma\in\Sigma^{\mathrm{op}}} \mathbb{S}_{\Sigma^{\mathrm{op}}\setminus\{0\}} &\simeq \lim_{\sigma\in\Sigma^{\mathrm{op}}\setminus\{0\}} \underline{\mathbb{S}} \\ &\simeq \mathbb{S}\oplus\mathbb{S}[-n+1].\end{aligned}$$

Indeed, $\Sigma^{\mathrm{op}} \setminus \{0\}$ can be identified with the opposite of the exit path category of S^{n-1} with the stratification induced by the fan, and taking global sections of the constant presheaf $\underline{\mathbb{S}}$ thus computes the cotensor

$$\mathbb{S}^{S^{n-1}} \simeq \mathbb{S}\oplus\mathbb{S}[-n+1].$$

Under this identification, $\mathbb{S} \to \mathbb{S}\oplus\mathbb{S}[-n+1]$ is the inclusion of the first factor. Consequently,

$$\lim_{\sigma\in\Sigma^{\mathrm{op}}} \mathbb{S}_{\{0\}} \simeq \mathbb{S}[-n]$$

and thus

$$\lim_{\sigma\in\Sigma^{\mathrm{op}}}(\omega_{\sigma^\vee})_0 \simeq \lim_{\sigma\in\Sigma^{\mathrm{op}}} \mathbb{S}_{\{0\}}[n] \simeq \mathbb{S}$$

as desired.

Next we compute the stalk of the limit at $m \in M_{\mathbb{R}}$ (which is away from the origin). Similarly, we look at the limit

$$\lim_{\sigma\in\Sigma^{\mathrm{op}}} \mathbb{S}_{m,+}[n],$$

where $\mathbb{S}_{m,+}$ is the functor which evaluates on σ to be $\mathbb{S}$ if $m \in \sigma^{\vee,\circ}$ and 0 otherwise. To be precise, it fits into a fiber sequence

$$\mathbb{S}_{m,+} \to \underline{\mathbb{S}} \to \mathbb{S}_{m,-}$$

in $\mathrm{Fun}(\Sigma^{\mathrm{op}}, \mathrm{Sp})$. Here $\underline{\mathbb{S}}$ is the constant functor and $\mathbb{S}_{m,-}$ is right Kan extended from the constant presheaf on the sub-poset

$$\Sigma_{m,-} := \{\sigma : m \notin \sigma^{\vee,\circ}\} \subseteq \Sigma$$

(one can check that the right Kan extension takes everything outside of $\Sigma^{\mathrm{op}}_{m,-}$ to 0). We claim that the limit along Σ^{op} of the map

$$\underline{\mathbb{S}} \to \mathbb{S}_{m,-}$$

is an isomorphism $\mathbb{S} \to \mathbb{S}$. It suffices to show that the poset $\Sigma^{\mathrm{op}}_{m,-}$ is contractible. For that, we make the following combinatorial argument.

We will adapt the proof of [10, Proposition 3.7] to our situation. We fix a moment polytope P for the fan Σ. Consider the poset $F(P)$ of faces of P under inclusion, then there is an (inclusion) order reversing bijection between $F(P)$ and Σ. For example, the codimension 0 face of P (which is P itself) corresponds to the 0 dimensional cone of the origin. Now we consider the following subposet: (informally, the subset of $F(P)$ that's visible from ∞ through the direction m)

$$F(P)_{m,-} := \{C \in F(P) : \forall c \in C, \text{the ray } c + \mathbb{R}_{>0}\cdot m \text{ doesn't meet } P^\circ\}.$$

We claim that the canonical bijection between $F(P)$ and Σ^{op} induces a bijection between $F(P)_{m,-}$ and $\Sigma^{\mathrm{op}}_{m,-}$. This follows readily from the definition: if $m \notin \sigma^{\vee,\circ}$, then m is also not in $\tau^{\vee,\circ}$ for $\sigma \subseteq \tau$. Consider the corresponding face C_σ in P, at each point $c \in C$, the angle spanned by P is $\tau^\vee$ for some $\sigma \subseteq \tau$, which means that the ray $c + t\cdot m$ will not pass through P°. Conversely, let $m \in \sigma^{\vee,\circ}$ for some σ (so $\sigma \notin \Sigma^{\mathrm{op}}_{m,-}$) and C_σ be the corresponding face in P, it follows that at a relative interior point c of C_σ, the angle spanned by P is precisely $\sigma^\vee$, and that $m \in \sigma^{\vee,\circ}$ means that the ray $c + t\cdot m$ will pass through P°. Now the topological space $P_{m,-}$ of union of faces in $F(P)_{m,-}$ is contractible because if one fixes a hyperplane H perpendicular to m and consider projection to H along m, the image of $P_{m,-}$ is the same as P, which is convex. However, the map from $P_{m,-}$ to its image is a homotopy equivalence. Hence we conclude that $P_{m,-}$ is contractible. Now $F(P)_{m,-}$ is the exit-path category for the stratification on $P_{m,-}$ by the faces, hence also contractible. We conclude that $\Sigma^{\mathrm{op}}_{m,-}$ is also contractible as desired. □

Remark 4.5.6. For the fan corresponding to $\mathbb{P}^n$, one can give a slick proof by noting that the limit diagram for Σ^{op} is the same as the Čech diagram for the open cover of $M_{\mathbb{R}}$ as a topological space by $\{\sigma^{\vee,\circ} \to M_{\mathbb{R}} : \sigma \in \Sigma(1)\}$ and use [HA, Proposition 1.2.4.13]. However, it is not true in general that the diagram as above is the Čech diagram for an open cover of $M_{\mathbb{R}}$. Therefore, we opt for a different proof as above instead.

Corollary 4.5.7. For a smooth projective fan Σ, the functor Ψ_Σ assembled in Proposition 4.3.4 is symmetric monoidal.

Proof. Recall from Proposition 4.3.4 that Ψ_Σ is a composition:

$$\Psi_\Sigma : \lim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\lim \Psi_\sigma} \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

The first functor is always symmetric monoidal, and we are concerned with the second functor. Note that it is defined as a right adjoint to the functor

$$\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}),$$

which is an equivalence when the fan is smooth and projective, given Proposition 4.5.4. So we conclude that the second functor is also symmetric monoidal, and so is Ψ_Σ. $\square$

Remark 4.5.8. More generally, the result of Dmitry Vaintrob in [39] could be interpreted to suggest that the limit of the family of idempotent algebras in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$ as in Proposition 4.5.4 should only depend on the support, but not a particular fan. This is closely related to his construction [38] of log quasi-coherent category of toroidal compactifications. A direct adaptation of the construction of our comparison functor to Dmitry Vaintrob's setting will produce a symmetric monoidal equivalence between the category of 'almost' quasi-coherent sheaves on smooth projective toric schemes and the categories of sheaves of spectra on real vector spaces without constructibility constraints.

5 Singular support

The aim of this section is to characterize $\mathrm{Im}(\kappa)$ for a smooth projective fan Σ in terms of singular support. The idea of using singular support to describe the image of κ was originally due to [10]. Let Λ_Σ be the conic Lagrangian subset of the cotangent bundle $T^*M_{\mathbb{R}}$ given in Definition 5.1.17. We will define a full subcategory of $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$ spanned by sheaves with singular support contained in Λ_Σ. It follows directly from the definition that the functor κ factors through $\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp})$, so

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \supseteq \mathrm{Im}(\kappa).$$

Then we follow the idea of [42] to show that the inclusion is an equality. The benefit of our approach is that along the way we will construct an explicit family of compact generators of $\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp})$.

We will first take a quick tour of the theory of singular support for polyhedral sheaves. This is particularly simple, since locally we are working with conic sheaves on a real vector space. Then we revisit the interplay between twisted polytopes and sheaves. Finally we invoke the non-characteristic deformation lemma from [33] to prove the result.

The reason why our proof is less straightforward as opposed to what appears in [10, 42] is the following. We find that there is a lack of a general theory of singular supports for sheaves of spectra, so that the arguments one can make in its classical counterpart [22] would carry over without much modification (see, however, [21] for an exposition in this direction.) We hope that this section serves as an invitation to homotopy theorists to revisit the notion of singular supports in greater generality and to investigate questions like Remark 5.3.1.

5.1 Singular support for polyhedral sheaves

Following [10, Section 4], we define the notion of singular supports for polyhedrally constructible sheaves on real vector spaces (and also tori). 'Polyhedral' means that we fix a stratification P on a real vector space V, specified (as in Definition 4.4.11) by an affine hyperplane arrangement. We will consider sheaves which are constructible for such 'polyhedral' stratification. Locally, these sheaves are modeled on conic sheaves F on a real vector space V, which we will first study. All vector spaces appearing here will be finite dimensional.

Remark 5.1.1. We will make use of results in [22], but the reader should be warned that the book was written in the classical language of bounded derived category of sheaves. So it is not directly applicable in our situation. However, the results we make use of could be verified with the same proof from there. We will revisit these facts in future work.

Definition 5.1.2. Recall that the topological group $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$ acts continuously on a real vector space V via multiplication. We define the category of conic sheaves on V to be the full subcategory of sheaves that are constant when restricted to each orbit, and write it as

$$\mathcal{S}\mathrm{hv}^{\mathrm{conic}}(V;\mathrm{Sp}) \subseteq \mathcal{S}\mathrm{hv}(V;\mathrm{Sp}).$$

Definition 5.1.3 (Fourier-Sato transform)**.** Let V be a real vector space with dual V^*. The Fourier-Sato transform is defined to be

$$\mathcal{FS} : \mathcal{S}\mathrm{hv}^{\mathrm{conic}}(V;\mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}^{\mathrm{conic}}(V^*;\mathrm{Sp})$$

$$F \mapsto p_! q^* F$$

where $p : K \to V^*$ and $q : K \to V$ are projections from the kernel

$$K := \{(x, y) \in V \times V^* : \langle x, y \rangle \leqslant 0\} \subset V \times V^*.$$

We define the singular support at the origin of a conic sheaf F to be the support (i.e. closure of the points where the stalk doesn't vanish) of $\mathcal{FS}(F) \subsetneq V^*$, where we identify V^* with the cotangent space of V at the origin

$$\mu\mathrm{supp}_0(F) := \mathrm{supp}(\mathcal{FS}(F)) \subset V^* \simeq T_0^*(V).$$

Proposition 5.1.4 ([22, Theorem 3.7.9])**.** The Fourier-Sato transform is an equivalence of categories between conic sheaves on V and conic sheaves on V^* :

$$\mathcal{FS} : \mathcal{S}hv^{\mathrm{conic}}(V; \mathrm{Sp}) \xrightarrow{\simeq} \mathcal{S}hv^{\mathrm{conic}}(V^*; \mathrm{Sp}).$$

Remark 5.1.5 (An alternative definition)**.** One can also define a notion of singular supports using a Morse-type construction as in [33, Definition 4.5]. It coincides with this definition, but we will not use it here.

One particular feature of such definition we will use is that it interacts nicely with cones.

Lemma 5.1.6 ([22, Lemma 3.7.10])**.** Let V be a real vector space with V^* its dual. Let $\tau \subsetneq V$ be an open convex cone and $-\tau^\vee \subsetneq V^*$ be the negative of its dual cone. Then

$$\mathcal{FS}(\omega_\tau) = \underline{\mathbb{S}}_{-\tau^\vee}.$$

In particular, the singular support at the origin of ω_τ is

$$\mu\mathrm{supp}(\omega_\tau)_0 = -\tau^\vee.$$

Now we globalize the previous definition:

Definition 5.1.7 (Singular support)**.** Let V be a vector space equipped with the stratification P specified by an affine hyperplane arrangement as in Definition 4.4.11. For a constructible sheaf $F \in \mathrm{Cons}_P(V; \mathrm{Sp})$, one can specify a subset of the cotangent bundle of V,

$$\mu\mathrm{supp}(F) \subsetneq T^*V \simeq V \times V^*$$

to be the (global) singular support of F. Its fiber at a point $\nu \in V$, denoted by $\mu\mathrm{supp}_\nu(F)$ is determined as follows: pick an open ball U centered at ν that only meets the hyperplanes passing through ν. Pick an exponential map from the tangent space:

$$\exp : V \xrightarrow{\simeq} U$$

and it pulls F back to a conic sheaf $\exp^* F \in \mathcal{S}hv^{\mathrm{conic}}(V; \mathrm{Sp})$. We define $\mu\mathrm{supp}(F)_\nu := \mu\mathrm{supp}_0(\exp^* F) \subsetneq V^*$ and we identify canonically V^* with $T_\nu^* V$.

Remark 5.1.8 (Singular support is well-defined)**.** We remark that at each point ν the subset $\mu\mathrm{supp}_\nu(F)$ doesn't depend on the choice of the open ball U nor the exponential map exp. To compare different choices we end up with a transition map

$$V \to V$$

which is given by multiplication of a continuous function valued in $\mathbb{R}_+$ on V. Since all the orbits are contractible and the sheaf involved is conic, one can produce an equivalence between sheaves $\exp^*(F)$ under different choices. We don't spell out the details here.

Definition 5.1.9 (Sheaves with prescribed singular support)**.** Following the notation as Definition 5.1.7. Let $\Lambda \subset T^*V \simeq V \times V^*$ be a subset. We define a full subcategory $\mathcal{S}\mathrm{hv}_\Lambda(V;\mathrm{Sp})$ of $\mathrm{Cons}_P(V;\mathrm{Sp})$ to be

$$\mathcal{S}\mathrm{hv}_\Lambda(V;\mathrm{Sp}) := \{F : \mu\mathrm{supp}(F) \subseteq \Lambda\}.$$

This is the subcategory of P-constructible sheaves with singular support contained in Λ.

Warning 5.1.10. Note that the notation didn't make explicit the dependence on P, but we always fix such a stratification and work inside the full subcategory of P-constructible sheaves. This should not cause confusion as we will work with a single fixed stratification at a time. It is true that $\mu\mathrm{supp}(F)$ doesn't depend on the ambient stratification - and in fact one can define singular support of a sheaf without the help of constructibility and arrive at the same notion. But beware that, given Λ, the category of P-constructible sheaves with singular support contained in Λ can vary as P changes. It is also true that they will be the same as long as conormal variety of P contains Λ. We will not prove these facts nor use them.

Variant 5.1.11. The definition makes sense also for a quotient of a vector space by a lattice V/Γ, in particular for tori $\mathbb{R}^n/\mathbb{Z}^n$: fix a polyhedral stratification P on V/Γ and a constructible sheaf F for $(V/\Gamma, P)$, one can define a subset $\mu\mathrm{supp}(F) \subseteq T^*V/\Gamma$, and thus talk about the subcategory of P-constructible sheaves with prescribed singular support. We will make use of this notion in the final section.

Then we make several quick observations with the definition.

Remark 5.1.12 (Locality)**.** The definition is local in nature. This in particular implies that one can check if a constructible sheaf F on V/Γ has the prescribed singular support by pulling back and checking on V, since the projection map is a local homeomorphism preserving the linear structure.

Remark 5.1.13 (Closed under colimits)**.** Given a polyhedral stratification P on V and a subset Λ in T^*V. The subcategory $\mathcal{S}\mathrm{hv}_\Lambda(V;\mathrm{Sp})$ is a stable subcategory closed under colimits in $\mathrm{Cons}_P(V;\mathrm{Sp})$ and hence also in $\mathcal{S}\mathrm{hv}(V;\mathrm{Sp})$. This follows from that the $*$-pullback functor and the Fourier-Sato transform preserve colimits, and that support condition is closed under colimits.

The most important example of the computation with global singular support is the following:

Lemma 5.1.14. [10, Proposition 5.1] Let Σ be a smooth projective fan and consider the category of $\mathcal{S}_\Sigma$-constructible sheaves. We can bound the singular support of the sheaf $\omega_{m+\sigma^\vee} \in \mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R};\mathrm{Sp})$ for $\sigma \in \Sigma$:

$$\mu\mathrm{supp}(\omega_{m+\sigma^\vee}) \subseteq \bigsqcup_{\tau \subset \sigma} m + \tau^\perp \times -\tau \subset M_\mathbb{R} \times N_\mathbb{R} \simeq T^*M_\mathbb{R}.$$

We refer to the original treatment for the proof: it is a direct application of Lemma 5.1.6.

One feature of the notion of singular support is that it supports Morse theory. In our context, the foundational non-characteristic deformation lemma is supplied by [33, Theorem 4.1]:

Proposition 5.1.15. Let $M \in \mathrm{LCH}$ and $F \in \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(M;\mathrm{Sp})$ be hypercomplete. Let $\{U_s\}_{s\in\mathbb{R}}$ be a family of open subsets of M. Assume:

1. For all $t \in \mathbb{R}$, $U_t = \cup_{s<t} U_s$.

2. For all pairs $s \leqslant t$, the set $\overline{U_t \setminus U_s} \cap \mathrm{supp}(F)$ is compact.

3. Setting $Z_s := \cap_{t>s} \overline{U_t \setminus U_s}$, we have for all pairs $s \leqslant t$ and all $x \in Z_s$:

$$i^!(F)_x = 0$$

where $i : X \setminus U_t \to X$ is the inclusion. Note that by the recollement sequence where $j : U_t \to X$ is the inclusion

$$i_! i^!(F) \longrightarrow F \longrightarrow j_* j^*(F)$$

this is the same as asking $F_x \to j_* j^*(F)_x$ be an isomorphism for each $x \in Z_s$.

Then we have for all $t \in \mathbb{R}$:

$$F(\bigcup_s U_s) \xrightarrow{\simeq} F(U_t).$$

Remark 5.1.16. As we will be working with a finite dimensional real vector space, every sheaf is automatically hypercomplete. Beware that it is crucial that the coefficient category Sp is compactly generated presentable - otherwise one needs to change the definition of singular support. See [9, Remark 4.24].

Having prepared ourselves with enough abstract nonsense, here we present the crucial part of this subsection: we will provide a refinement of the category of $\mathcal{S}_\Sigma$-constructible sheaves such that the image of κ lies in it:

Definition 5.1.17 (FLTZ skeleton)**.** [17] Let Σ be a smooth projective fan. We define a conic Lagrangian subset of T^*M as follows:

$$\Lambda_\Sigma := \bigsqcup_{m \in M, \sigma \in \Sigma} m + \sigma^\vee \times -\sigma \subseteq M_\mathbb{R} \times N_\mathbb{R} \simeq T^*M_\mathbb{R}.$$

From now on we will focus on the category $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ of $\mathcal{S}_\Sigma$-constructible sheaves with singular support in Λ_Σ.

Lemma 5.1.18. The image of κ lies in $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$.

Proof. The category $\mathrm{Im}(\kappa)$ is generated as a stable category under colimit by the objects of the form $\omega_{m+\sigma^\vee}$, and each of them has singular support contained in Λ_Σ by Lemma 5.1.14. By Remark 5.1.13, the category of sheaves with prescribed singular support is closed under colimits, so the proof is done. □

5.2 Combinatorics of a smooth projective fan

One distinguishing feature of a smooth projective fan Σ in $N_\mathbb{R}$ is that it can be presented as the dual fan of an integral polytope P. See [11, Section 1.5] for the construction. Such polytope P has the following properties:

[17] The name 'FLTZ skeleton' is borrowed from symplectic geometry.

1. The Minkowski sum of P with any dual cone of $\sigma \in \Sigma$ is an integral translation of the dual cone of σ.

2. Each of the dual cone $\sigma^\vee$ can be written as an increasing union of translations of polytopes of the form $n \cdot P$, where each $n \cdot P$ is an integral multiple of the polytope P.

We will see that these properties imply that after fixing one such P, the objects $\{\omega_{m+n\cdot P}\}$ for varying n and translation along $m \in M$ form an explicit collection of compact generators for $\mathrm{Im}(\kappa)$. On the mirror side, this is reminiscent of the familiar fact from algebraic geometry: tensor powers of ample line bundles generate the category of quasi-coherent sheaves under colimits.

We will explain how the association $P \mapsto \omega_P$ generalizes to a bigger collection of combinatorial objects, namely, twisted polytopes. To start with, we will make use of the following description of $\mathrm{Im}(\kappa)$.

Proposition 5.2.1. The category $\mathrm{Im}(\kappa)$ enjoys the following properties and characterizations.

1. The category $\mathrm{Im}(\kappa)$ is closed under colimits and shifts in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$.

2. The category $\mathrm{Im}(\kappa)$ can be characterized explicitly as

$$\{\mathcal{F} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) : \mathcal{F} * \omega_{\sigma^\vee} \in \langle \omega_{m+\sigma^\vee} : m \in M \rangle\}.$$

3. The category $\mathrm{Im}(\kappa)$ is generated under colimits and shifts of the following collection of objects:

$$\{\omega_{m+\sigma^\vee} : \sigma \in \Sigma, m \in M\}.$$

4. The category $\mathrm{Im}(\kappa)$ is closed under convolution products in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$.

Proof. The first point comes from the fact that κ is a fully faithful, colimit-preserving functor from a presentable stable category, as κ is constructed from taking limit of a diagram in Pr^{L}. The second point follows directly from the limit description of κ. For the third point, using descent along idempotent algebras, every object $X \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$ is a finite limit of a diagram, whose terms are of the form $X * \omega_{\sigma^\vee}$. Each of them lies in the category spanned by $\omega_{m+\sigma^\vee}$ by point two, so X also lies in the category spanned by $\omega_{m+\sigma^\vee}$ as desired. Finally, since κ is symmetric monoidal, its image is closed under tensor products. $\square$

With this knowledge at hand, let's try to write down some objects in the category $\mathrm{Im}(\kappa)$.

Proposition 5.2.2. Let Σ be a smooth projective fan, there exist (in fact, many) polytopes P in $M_{\mathbb{R}}$ with integral vertices such that Σ can be realized as the dual fan of P. Coversely P might be called a moment polytope of Σ (actually, associated to some line bundle). More precisely, P has the following properties:

- The Minkowski sum of P with any dual cone $\sigma^\vee$ of $\sigma \in \Sigma$ is an integral translation of the dual cone of σ. Concretely, this says that for each $\sigma \in \Sigma$, there exists some $m \in M$ such that

$$P + \sigma^\vee = m + \sigma^\vee.$$

- Each of the dual cone $\sigma^\vee$ can be written as an increasing union of integral translations of polytopes of the form $n \cdot P$, where each $n \cdot P$ is an integral multiple of the polytope P. Concretely this says that for each $\sigma \in \Sigma$, one can pick a collection of $m_i \in M$ and form a increasing union
$$\bigcup_{i \geqslant 0} m_i + i \cdot P = \sigma^\vee.$$

Proof. The existence of such a polytope is assumed by the definition of a projective fan (see also [11, Section 1.5]). Both claims about the polytopes are direct combinatorics and we omit the proof. □

We will consider the object $\omega_P \in \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$.

Proposition 5.2.3. For such a moment polytope P as above:

1. The object ω_P lies in $\mathrm{Im}(\kappa)$.
2. The object ω_P is a compact object in $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ and hence also compact in $\mathrm{Im}(\kappa)$.
3. The same is true for $\omega_{m+n \cdot P}$ for each $m \in M$ and $n \in \mathbb{Z}_{>0}$. Moreover, these objects compactly generate the category $\mathrm{Im}(\kappa)$.

Proof. The first point comes from the characterization of $\mathrm{Im}(\kappa)$ in point (2) of Proposition 5.2.1. We can compute
$$\omega_P * \omega_{\sigma^\vee} \simeq \omega_{P+\sigma^\vee} \simeq \omega_{m+\omega_{\sigma^\vee}}$$
using that P is a moment polytope. The second point comes from an application of the exodromy equivalence and the description of compact objects in the functor categories by Lemma 4.4.10, using that such polytope P is assumed to be bounded. For the final point, one can write each $\sigma^\vee$ as an increasing union of polytopes of the form $m + n \cdot P$. It follows that there is a filtered colimit presentation
$$\operatorname*{colim}_{m+n \cdot P \subseteq \sigma^\vee} \omega_{m+n \cdot P} \simeq \omega_{\sigma^\vee}.$$
Up to translation, this shows that every $\omega_{m+\sigma^\vee}$ can be written as a colimit of $\omega_{m+n \cdot P}$. Hence $\mathrm{Im}(\kappa)$ is generated by $\omega_{m+n \cdot P}$ for varying $m \in M$ and $n > 0$. □

Remark 5.2.4 (Divisors and piecewise linear functions)**.** Here we give two more combinatorial ways to present the data of such polytope P. Firstly as 'divisors': the polytope P is the intersection of several half-spaces in $M_\mathbb{R}$, indexed by the 1-cones $\eta \in \Sigma(1)$. Let us fix the primitive integral vectors $v_\eta \in N$ for each $\eta \in \Sigma(1)$, then we can write
$$P = \bigcap_{\eta \in \Sigma(1)} \{m \in M_\mathbb{R} : \langle m, v_\eta \rangle \geqslant -n_\eta \in \mathbb{Z}\}.$$

Hence we can recover the polytope P from the collection of integers $\{n_\eta : \eta \in \Sigma(1)\}$. More generally by a divisor we would mean such a sequence of integers $\{n_\eta : \eta \in \Sigma(1)\}$ and we write D for a divisor. In case of a moment polytope P we write D_P for the associated divisor as above. Note that one can make sense of the addition of divisors as componentwise addition.

Secondly as piecewise linear functions: given a divisor $D_P = \{n_\eta : \eta \in \Sigma(1)\}$ coming from a moment polytope P, one may extend the assignment $v_\eta \mapsto -n_\eta$ $\mathbb{R}$-linearly on each cone to obtain an $\mathbb{R}$-valued function f_P on $N_{\mathbb{R}}$ (here we use that the fan is smooth and projective). For each top dimensional cone σ, there is a unique $m_\sigma \in M$ such that when restricted to σ

$$\langle m_\sigma, - \rangle = f_P(-)_{|\sigma}.$$

Such $\{m_\sigma\}$ is precisely the collection of vertices of P, see [11, Section 3.4]. So one might recover the polytope P from the data of f_P. This is part of the beautiful connection between line bundles, divisors and piecewise linear functions, as explained in Fulton's book.

Variant 5.2.5 (Twisted polytopes)**.** It is not true that every divisor $D = \{n_\eta\}$ or every integral piecewise linear function f corresponds to a polytope. However, one can still write down an object in $\mathrm{Im}(\kappa)$ starting from such data. Let us explain the idea here: fix a collection of integers $\{n_\eta\}$ as a divisor D. We may construct an piecewise linear integral function f on $M_{\mathbb{R}}$ in the same way as above. This piecewise linear integral function f determines and is determined by a collection of elements $\{m_\sigma \in M : \sigma \in \Sigma(n)\}$. We might consider the collection of closed subsets

$$\{m_\sigma + \sigma^\vee \subseteq M_{\mathbb{R}} : \sigma \in \Sigma(n)\}.$$

The fact that m_σ and m_τ agrees as functions on $\sigma \cap \tau$ (as they both agree with f) implies that

$$m_\sigma + (\sigma \cap \tau)^\vee = m_\tau + (\sigma \cap \tau)^\vee.$$

In fact, the function f (or the divisor D) determines an integral element

$$m_\sigma \in M/\sigma^\perp$$

for each $\sigma \in \Sigma$. Thus the subset

$$m_\sigma + \sigma^\vee \subseteq M_{\mathbb{R}}$$

is well defined. Note that, by definition, we have

$$(m_\sigma + \sigma^\vee) + \tau^\vee = m_\tau + \tau^\vee$$

for $\tau \subseteq \sigma$. We claim that the collection of objects

$$\{\omega_{m_\sigma + \sigma^\vee} \in \mathrm{Mod}_{\omega_{\sigma^\vee}} : \sigma \in \Sigma\}$$

determines an object in $\mathrm{Im}(\kappa)$ using descent along idempotent algebras, as described in Section 4.5. In other words, we claim that there exists an object $\omega(D) \in \mathrm{Im}(\kappa)$ such that, functorially in σ,

$$\omega(D) * \omega_{\sigma^\vee} \simeq \omega_{m_\sigma + \sigma^\vee}.$$

To do so, it requires to provide isomorphisms

$$\omega_{m_\sigma + \sigma^\vee} * \omega_{\tau^\vee} \xrightarrow{\simeq} \omega_{m_\tau + \tau^\vee}$$

for $\tau \subset \sigma$, and the homotopies between compositions and so on. The existence of such isomorphisms should follow from the equality

$$(m_\sigma + \sigma^\vee) + \tau^\vee = m_\tau + \tau^\vee$$

above. Formally, to construct $\omega(D)$, one can apply the functor $\Gamma_{M_{\mathbb{R}}}$ to the collection of subsets $\{m_\sigma + \sigma^\vee\}$ with inclusions between them. We leave the details of the proof to the readers.

If the divisor D corresponds to an actual polytope P, this construction recovers ω_P. We call a divisor twisted polytope as it needs not to come from a polytope and the assignment $D \mapsto \omega(D)$ generalizes $P \mapsto \omega_P$.

Remark 5.2.6. The passage from moment polytopes P to divisors D_P is additive in the sense that it takes Minkowski sum of moment polytopes to componentwise addition of divisors. In a similar fashion, the passage from divisors D to sheaves ω_D is additive: it takes componentwise addition of divisors to convolution product of sheaves

$$\omega(D_1 + D_2) \simeq \omega(D_1) * \omega(D_2).$$

This can be checked after convolution with each $\omega_{\sigma^\vee}$: one has

$$\omega_{m_1+\sigma^\vee} * \omega_{m_2+\sigma^\vee} \simeq \omega_{m_1+m_2+\sigma^\vee}.$$

One can carefully prove that the assignment $D \mapsto \omega(D)$ is a symmetric monoidal functor, but we will not do so.

Remark 5.2.7 (Every divisor is dominated by an ample divisor)**.** Even though not every divisor D comes from a polytope, it is true that after adding a large multiple of a divisor D_p coming from a polytope, the divisor $D + n \cdot D_P$ corresponds to a polytope. To see this, use the characterization of the divisors corresponding to a polytope as strictly convex functions, proved in [11, Section 3.4]. For algebraic geometers, this is similar to the fact that a line bundle will become ample after tensoring with a sufficiently ample (positive) line bundle.

Variant 5.2.8 (Sheaves and polytopes from $\mathbb{R}$-coefficient divisors)**.** The assumption that a divisor $D = \{n_\eta\}$ is a collection of integers or a piecewise linear function f_P is integral on each cone is not essential in the above discussion: one can write down objects in $\mathcal{S}hv(M_\mathbb{R}; \mathrm{Sp})$ from the data of an $\mathbb{R}$-coefficient 'divisor' $\{r_\eta\}$, or equivalently, a piecewise linear function f on $N_\mathbb{R}$. We leave the details to the reader as we will not use them in our exposition.

5.3 Microlocal characterization of the image of κ

In this section we prove the promised characterization of $\mathrm{Im}(\kappa)$. Before presenting the details of the proof, we briefly explain the proof idea here. We are going to show that the right orthogonal of the image $\mathrm{Im}(\kappa)$ in $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ is zero. This follows from the following explicit construction: for each $x \in M_\mathbb{R}$, we construct an object $\omega(D_x) \in \mathrm{Im}(\kappa)$ in the image of κ, such that $\omega(D_X)$ corepresents the functor of taking stalk at x in $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$. Formally,

$$\mathrm{map}(\omega(D_x), \mathcal{F})[n] \simeq \mathcal{F}_x$$

holds for all $\mathcal{F} \in \mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$. It follows that an object in the right orthogonal of $\mathrm{Im}(\kappa)$ will have vanishing stalks everywhere, so such object has to be zero. With the help of adjoint functor theorem, this proves that

$$\mathrm{Im}(\kappa) = \mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp}).$$

To prove such a statement about $\omega(D_x)$, we will apply the non-characteristic deformation lemma, after convoluting with a large enough multiple of ω_P, where P is a moment polytope of Σ.

The use of convolution product in the proof introduces some complication - as we don't know a priori if the category $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ is closed under convolution product (we will prove it nonetheless is, a fortiori). This is where our narrative diverges from [42]: we get around this issue by introducing an intermediate category as in Warning 5.3.13. Note that in [42] this complication was not explicitly addressed.

Remark 5.3.1. As a consequence of the proof, we will show that the category $\mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}};Sp)$ is compactly generated, and specify an explicit collection of generators. Now, in general, for each conic Lagrangian L in the cotangent bundle of a manifold X, one can define (as in [21]) a category of sheaves (of spectra) with singular support lying inside L. We are curious if said category is always compactly generated and if there is a natural procedure to pick out compact generators in that category. Specifically, as the microlocal stalk functor is one profitable perspective offered by the microlocal analysis of sheaves, we are curious if there is any natural way to write down corepresenting objects for microlocal stalk functor and compute the mapping spectra between them.

We begin by defining the object $\omega(D_x)$ mentioned above.

Definition 5.3.2. [42, Definition 4.1] For a point $x \in M_{\mathbb{R}}$, we define the probing sheaf at x

$$\omega(D_x) \in \mathcal{S}hv(M_{\mathbb{R}};Sp)$$

to be the object associated to the divisor

$$D_x = \{n_\eta(D_x) = \lfloor -\langle x, v_\eta\rangle \rfloor + 1 : \eta \in \Sigma(1)\}$$

via the construction of Variant 5.2.5. The integer $\lfloor -\langle x, v_\eta\rangle \rfloor + 1$ is the smallest integer strictly larger than $-\langle x, v_\eta\rangle$. Note that by Proposition 5.2.1 we have $\omega(D_x) \in \mathrm{Im}(\kappa)$.

The naming comes from the following theorem, whose proof takes up the rest of the section:

Theorem 5.3.3. For an arbitrary sheaf $\mathcal{F} \in \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}};Sp)$, there exists an isomorphism (which we will construct explicitly in the proof)

$$\mathrm{map}(\omega(D_x), \mathcal{F})[n] \xrightarrow{\simeq} \mathcal{F}_x \in Sp.$$

Given this, one can look at the inclusion $\mathrm{Im}(\kappa) \to \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}};Sp)$: the right orthogonal of $\mathrm{Im}(\kappa)$ vanishes because any object in there would have vanishing stalks everywhere. Applying the adjoint functor theorem to the inclusion functor, one obtains a right adjoint $\mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}};Sp) \to \mathrm{Im}(\kappa)$ such that the composition with inclusion is identity on $\mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}};Sp)$ - which proves that the inclusion is essentially surjective. We have obtained the following corollary.

Corollary 5.3.4. There is an identification of full subcategories in $\mathcal{S}hv(M_{\mathbb{R}};Sp)$:

$$\mathrm{Im}(\kappa) = \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}};Sp).$$

Notation 5.3.5. From now on we fix a moment polytope P for Σ, and we assume that the origin is contained in the interior of P. The polytope P is given by the combinatorial data of a divisor (Remark 5.2.4) as a collection of integers $\{n_\eta(P) : \eta \in \Sigma(1)\}$. In particular, this gives a presentation of P as the intersection of half spaces

$$P = \bigcap_{\eta \in \Sigma(1)} \{m : \langle m, v_\eta\rangle \geqslant -n_\eta(P)\}$$

(recall that v is a fixed primitive element of η). Because the origin is in the interior of P, we also know that

$$n_\eta(P) > 0$$

for each η. Moreover, we fix a fundamental domain $W \subset M_{\mathbb{R}}$ of $M_{\mathbb{R}}/M$ as follows. Pick a basis $\{m_i\}$ for the lattice M and take the half-closed hypercube

$$W := \{\Sigma_i r_i m_i : m_i \in M; r_i \in [0,1)\}.$$

By replacing P with some large multiple $n \cdot P$, we assume for each $x \in W$, the divisor

$$D_x + D_P$$

also corresponds to a moment polytope, which we call P_x. One can achieve this by observing that there are only finitely many different divisors D_x for $x \in W$. For each fixed D_x we can dominate it with a large multiple of P by Remark 5.2.7.

Remark 5.3.6. We will prove that $\omega(D_x)$ corepresents taking stalks at $x \in W$. The same statement for every point $x \in M_{\mathbb{R}}$ will then follow. Indeed, observe that for $m \in M$,

$$\omega(D_{x+m}) \simeq \omega_m * \omega(D_x)$$

while convolution with ω_m is just !-pushforward along translation by m. So we can translate other points into the fundamental domain and obtain the above statement for other points. Another way to see this is that, such P as above actually dominates D_x for all points x so the proof carries through.

Now we consider a family of polytopes deforming P_x.

Definition 5.3.7 (Non-characteristic deformation of the probing sheaf)**.** Fix $x \in W$ and a small positive real number $0 < \epsilon \ll 1$ so that

$$-\langle x, v_\eta \rangle + \epsilon \cdot n_\eta < \lfloor -\langle x, v_\eta \rangle \rfloor + 1$$

for all $\eta \in \Sigma(1)$. Consider the following increasing family of polytopes indexed by $s \in [0,1]$:

$$P_{x,s} := s \cdot P_x + (1-s) \cdot (x + (1+\epsilon) \cdot P).$$

It grows from (when $s = 0$) $x + (1+\epsilon) \cdot P$ to (when $s = 1$) P_x. Note the definition of $P_{x,s}$ depends on ϵ implicitly.

We will apply the non-characteristic deformation lemma to this family. To do so, we start with an observation about its interaction with Λ_Σ.

Lemma 5.3.8. Write $P_{x,s}$ as the intersection of half-planes

$$P_{x,s} = \bigcap_{\eta \in \Sigma(1)} \{m \in M_{\mathbb{R}} : \langle m, v_\eta \rangle \geqslant -n_{\eta,x,s} \in \mathbb{R}\}.$$

Then for $s \in [0,1)$, none of the real numbers $-n_{\eta,x,s}$ will be an integer. (In terms of Variant 5.2.8, these $\{n_{\eta,x,s}\}$ give the real coeffient divisors for $P_{x,s}$.)

Proof. Since the passage from polytopes to divisors is linear, we might look at the two ends of the interpolation, and inspect the coefficients of the corresponding divisors

$$n_{\eta,x,1} = n_\eta(P_x) = n_\eta(P) + \lfloor -\langle x, v_\eta \rangle \rfloor + 1,$$

$$n_{\eta,x,0} = n_\eta(P) - \langle x, v_\eta \rangle + \epsilon \cdot n_\eta(P).$$

As long as

$$-\langle x, v_\eta \rangle + \epsilon \cdot n_\eta < \lfloor -\langle x, v_\eta \rangle \rfloor + 1$$

for each $\eta \in \Sigma(1)$, there will be no integer between $n_{\eta,x,0}$ and $n_{\eta,x,1}$. The claim follows. □

Lemma 5.3.9. [42, Lemma 3.13] Let $s \in [0,1)$. Let $y \in \partial P_{x,s}$ be on the boundary of the polytope $P_{x,s}$, then we have the following estimate on the fiber of Λ_Σ at y:

$$\Lambda_{\Sigma,y} \cap -\sigma(y) = 0 \subseteq N_\mathbb{R} \simeq T^*_y(M_\mathbb{R}).$$

Here $\sigma(y) \in \Sigma$ is the cone dual to the angle spanned by $P_{x,s}$ at y, formally determined as follows. The vectors $\{p - y : p \in P_{x,s}\} \subseteq M_\mathbb{R}$ span a cone $\sigma(y)^\vee$ in $M_\mathbb{R}$. The cone $\sigma(y)$ is defined to be the dual cone of $\sigma(y)^\vee$.

Proof. For each $s \in [0,1)$, $P_{x,s}$ is also a moment polytope (but with non-integral vertices, as it is a convex linear combination of moment polytopes), so $\sigma(y) \in \Sigma$. Suppose, for sake of contradiction, that there exists a non-zero cotangent vector

$$0 \neq u \in -\sigma(y) \cap \Lambda_{\Sigma,y},$$

one can find some $\tau \in \Sigma$ and $m \in M$ such that

$$(y, u) \in m + \tau^\perp \times -\tau$$

and thus $u \in -\sigma(y) \cap -\tau$. This implies $\sigma(y) \cap \tau \neq \{0\}$. Note that τ cannot be the origin, so it must contain some 1-cone ρ. It follows that $\langle y, v_\rho \rangle = \langle m, v_\rho \rangle$ is an integer. On the other hand, $\rho \subseteq \sigma(y)$ is equivalent to

$$\langle v_\rho, p - y \rangle \geqslant 0$$

for any $p \in P_{x,s}$. This implies that v_ρ attains its minimum at y on $P_{x,s}$, which means $-n_{\rho,x,s} = \langle y, v_\rho \rangle$ is an integer. This contradicts Lemma 5.3.8. □

With this we study the family of open polytopes given by the interiors $P^\circ_{x,s}$ for $s \in [0,1)$.

Lemma 5.3.10. Consider a sheaf $F \in \mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ and the family of open polytopes given by the interior $P^\circ_{x,s}$ for $s \in (-1,1) \simeq \mathbb{R}$, where we extend the original family over $[0,1)$ by constant to the left: $P_{x,s} := P_{x,0}$ for $s < 0$. Then the assumptions of the non-characteristic deformation lemma Proposition 5.1.15 are satisfied for the sheaf F and the family of open subsets $U_s = P^\circ_{x,s}$.

Proof. The point 1 and 2 in the assumptions of Proposition 5.1.15 follow directly from the definition of $P^\circ_{x,s}$. Unwinding the final point, we see that Z_s is empty for $s \in (-1,0)$ and $Z_s = \partial P_{x,s}$ for $s \in [0,1)$. Following the notations from Proposition 5.1.15, we write

$$i : M_\mathbb{R} \setminus U_s \longrightarrow M_\mathbb{R}$$

for the inclusion maps. The goal is to show $(i^! F)_y = 0$ for every $y \in Z_s$. Applying the recollement sequence for the open-closed decomposition $M_\mathbb{R} = U_s \cup M_\mathbb{R} \setminus U_s$, this is equivalent to showing that the canonical map

$$F_y \to j_* j^*(F)_y$$

is an isomorphism for $y \in \partial P_{x,s}$ and $s \in [0,1)$, where $j : U_s = P^\circ_{x,s} \to M_{\mathbb{R}}$ is the inclusion map. Since the stalk at y only depends on the sheaf locally, we might choose a small enough open ball U centered at y and pullback F along an exponential map

$$\exp : M_{\mathbb{R}} \xrightarrow{\simeq} U$$

as in Definition 5.1.7. This reduces the question to the situation of a sheaf $\mathcal{F}$ on a vector space $M_{\mathbb{R}}$ constructible for a stratification by linear subspace (hence in particular, conic). The sheaf $\mathcal{F}$ has the same singular support at origin as F at y. In this case we are asking if the comparison map of the stalks at origin is an isomorphism:

$$\mathcal{F}_0 \to j_* j^* (\mathcal{F})_0$$

where $j : \sigma^{\vee,\circ}(y) \to M_{\mathbb{R}}$ is the inclusion of an open cone $\sigma^{\vee,\circ}(y)$ determined as in Lemma 5.3.9 (whose dual is named $\sigma(y) \subseteq N_{\mathbb{R}}$). By stratified homotopy invariance [7, Corollary 3.3] (or [22, Corollary 3.7.3]), we may identify this map with the restriction map

$$\mathcal{F}(M_{\mathbb{R}}) \to \mathcal{F}(\sigma^{\vee,\circ}(y)).$$

Now we can apply the Fourier-Sato transform: the map becomes

$$\mathrm{map}(\mathcal{FS}(\underline{\mathbb{S}}_{M_{\mathbb{R}}}), \mathcal{FS}(\mathcal{F})) \longrightarrow \mathrm{map}(\mathcal{FS}(\underline{\mathbb{S}}_{\sigma^{\vee,\circ}(y)}), \mathcal{FS}(\mathcal{F})).$$

To show that it is an isomorphism, it suffices to show

$$\mathrm{map}(\mathrm{cofib}(\mathcal{FS}(\underline{\mathbb{S}}_{\sigma^{\vee,\circ}(y)}) \to \mathcal{FS}(\underline{\mathbb{S}}_{M_{\mathbb{R}}})), \mathcal{FS}(\mathcal{F})) = 0.$$

By Lemma 5.1.6, we know that

$$\mathcal{FS}(\underline{\mathbb{S}}_{M_{\mathbb{R}}}) \simeq \underline{\mathbb{S}}_{\{0\}}[-n] \in \mathcal{S}\mathrm{hv}(N_{\mathbb{R}}; \mathrm{Sp})$$

$$\mathcal{FS}(\underline{\mathbb{S}}_{\sigma^{\vee,\circ}(y)}) \simeq \underline{\mathbb{S}}_{-\sigma(y)}[-n] \in \mathcal{S}\mathrm{hv}(N_{\mathbb{R}}; \mathrm{Sp}),$$

and the map between them is induced by inclusion. It follows that we can identify the cofiber as

$$\mathrm{cofib}(\mathcal{FS}(\underline{\mathbb{S}}_{\sigma^{\vee,\circ}(y)}) \to \mathcal{FS}(\underline{\mathbb{S}}_{M_{\mathbb{R}}})) \simeq \mathrm{cofib}(\underline{\mathbb{S}}_{-\sigma(y)} \longrightarrow \underline{\mathbb{S}}_{\{0\}})[-n] \simeq h_! \underline{\mathbb{S}}[1-n].$$

The map $h : -\sigma(y) \setminus \{0\} \to N_{\mathbb{R}}$ is the inclusion. Now the assumption on singular support $\mu\mathrm{supp}(\mathcal{F})_0 = \mu\mathrm{supp}(F)_y \subseteq \Lambda_{\Sigma,y}$ implies

$$\mathrm{supp}(\mathcal{FS}(\mathcal{F})) \subseteq \Lambda_{\Sigma,y} \subseteq N_{\mathbb{R}}.$$

Moreover, from Lemma 5.3.9 we learn that $\mathrm{supp}(\mathcal{FS}(\mathcal{F})) \cap -\sigma(y) \subseteq \{0\}$. This implies that the map h above factors through the open complement of support of $\mathcal{FS}(\mathcal{F})$, thus we must have

$$\mathrm{map}(\mathrm{cofib}(\mathcal{FS}(\underline{\mathbb{S}}_{\sigma^{\vee,\circ}(y)}) \to \mathcal{FS}(\underline{\mathbb{S}}_{M_{\mathbb{R}}})), \mathcal{FS}(\mathcal{F})) = \mathrm{map}(h_! \underline{\mathbb{S}}[1-n], \mathcal{FS}(\mathcal{F})) = 0.$$

This concludes the proof. $\square$

Corollary 5.3.11. For $F \in \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp})$ and ϵ sufficiently small, the restriction map

$$F(P^\circ_x) \longrightarrow F(x + (1+\epsilon) \cdot P^\circ)$$

is an isomorphism.

We will use this to prove that $\omega(D_x)$ corepresents taking stalk at x. To do so, we first prove a statement slightly different from Theorem 5.3.3.

Proposition 5.3.12. Let $\mathcal{G} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$. If

$$\mathcal{G} * \omega_P \in \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}),$$

then for sufficiently small ϵ and $x \in W$, we have

$$\mathcal{G}(x + \epsilon \cdot P^\circ) \xrightarrow{\simeq} \mathrm{map}(\omega(D_x), \mathcal{G})[n].$$

Taking colimit along $\epsilon \to 0$ provides an isomorphism

$$\mathcal{G}_x \xrightarrow{\simeq} \mathrm{map}(\omega(D_x), \mathcal{G})[n]$$

for $x \in W$. The same is true for all $x \in M_{\mathbb{R}}$.

Proof. Given that ω_P is a convolution invertible object, we have

$$\mathcal{G}(x + \epsilon \cdot P^\circ) \simeq \mathrm{map}(\underline{\mathbb{S}}_{x+\epsilon\cdot P^\circ}, \mathcal{G}) \xrightarrow{\simeq} \mathrm{map}(\underline{\mathbb{S}}_{x+\epsilon\cdot P^\circ} * \omega_P, \mathcal{G} * \omega_P) \simeq (\mathcal{G} * \omega_P)(x + (1+\epsilon)\cdot P^\circ).$$

Now by the assumption that $\mathcal{G} * \omega_P$ lies in $\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp})$, we can apply Corollary 5.3.11 and learn that the restriction map

$$(\mathcal{G} * \omega_P)(x + (1+\epsilon)\cdot P^\circ) \xleftarrow{\simeq} (\mathcal{G} * \omega_P)(P_x^\circ)$$

is an isomorphism. Finally again using ω_P is convolution invertible, we have (recall that P_x is associated to the divisor $D_x + D_P$)

$$\mathrm{map}(\omega(D_x), \mathcal{G}) \xrightarrow{\simeq} \mathrm{map}(\omega(D_x) * \omega_P, \mathcal{G} * \omega_P) \simeq \mathrm{map}(\omega_{P_x}, \mathcal{G} * \omega_P) \simeq (\mathcal{G} * \omega_P(P_X^\circ))[-n].$$

Putting the above equivalences together we arrive at

$$\mathcal{G}(x + \epsilon \cdot P^\circ) \simeq \mathrm{map}(\omega(D_x), \mathcal{G})[n].$$

This isomorphism is compatible with the restriction maps along shrinking ϵ, and hence we get

$$\mathcal{G}_x \simeq \mathrm{map}(\omega(D_x), \mathcal{G})[n]$$

as promised, for $x \in W$. As explained in Remark 5.3.6, the same result holds for any $x \in M_{\mathbb{R}}$. $\square$

Warning 5.3.13. This does not conclude the proof of Theorem 5.3.3: the missing point is that we don't know if $(-) * \omega_P$ preserves the subcategory

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \subseteq \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

To circumvent this problem, we consider the following subcategory of $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$:

$$\mathcal{C} := \{\mathcal{G} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) : \mathcal{G} * \omega_P \in \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp})\}.$$

A quick observation is that, since $\mathrm{Im}(\kappa)$ is contained in $\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp})$ and closed under convolutions, we have $\mathrm{Im}(\kappa) \subseteq \mathcal{C}$. The above argument effectively shows the following.

Proposition 5.3.14. The functor of taking stalk at x is corepresented by $\omega(D_x)$ (up to a shift) in $\mathcal{C}$.

A second observation we will need is that the category $\mathcal{C}$ is closed under colimits and limits in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$, and in particular presentable (but we actually only need cocompleteness for our argument).

Proposition 5.3.15. The inclusion $\mathrm{Im}(\kappa) \subseteq \mathcal{C}$ is an equality.

Proof. Same as the argument immediately following Theorem 5.3.3. □

The final observation we will use is that, since ω_P is a convolution-invertible object in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$, we have a functor

$$(-) * \omega_P^{-1} : \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \to \mathcal{C}.$$

Applying the above proposition, we learn that for each $\mathcal{F} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$,

$$\mathcal{F} * \omega_P^{-1} \in \mathcal{C} = \mathrm{Im}(\kappa).$$

However, now that $\mathrm{Im}(\kappa)$ is closed under convolution, one learns that

$$\mathcal{F} = \mathcal{F} * \omega_P^{-1} * \omega_P \in \mathrm{Im}(\kappa).$$

Consequently, we have

$$\mathrm{Im}(\kappa) = \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}),$$

and Theorem 5.3.3 follows easily (beware of the flip of logic here).

6 Epilogue

In the final section, we exploit the results developed thus far to derive some tangible ramifications. Firstly, we apply the folklore method of de-equivariantization to obtain the 'non-equivariant' version of the equivalence. Next, as a concrete consequence, we provide a proof of Beilinson's equivalence for flat $\mathbb{P}^n$ over $\mathbb{S}$. More generally, we introduce a definition of the toric construction in an abstract setting and explain how the equivalence fits into this framework. As an example, we demonstrate how this method recovers a family version of the equivalence as in [19].

Throughout the section we always work with a smooth projective fan.

6.1 De-equivariantization

One of the most basic notions in the theory of stacks is that of quotient stacks. Given a group object G acting on $X \in \mathrm{Stk}$, one can form the quotient stack $[X/G] \in \mathrm{Stk}$. The fundamental insight is that it encodes all the G-equivariant information about X. In this regard, $\mathrm{QCoh}([X/G])$ is just the category of objects in $\mathrm{QCoh}(X)$ together with a G-action, i.e., the category of G-modules in $\mathrm{QCoh}(X)$. Therefore, $\mathrm{QCoh}([X/G])$ is completely determined by $\mathrm{QCoh}(X)$, along with the action of G on $\mathrm{QCoh}(X)$.

This process of determining $F([X/G])$ from $F(X)$, together with the information of a G-action on $F(X)$, is colloquially referred to as equivariantization, where F is a sheaf, with $F = \mathrm{QCoh}(-)$ in the previous example.

A less-exploited point of view, dubbed de-equivariantization, allows us to sometimes go in the other direction. Indeed, observe that there is a pullback diagram in Stk

$$\begin{tikzcd} X \arrow[r] \arrow[d] & {[X/G]} \arrow[d] \\ * \arrow[r] & BG \end{tikzcd} .$$

With both $[X/G]$ and BG being perfect, $F = \mathrm{QCoh}(-)$ takes this pullback square to a pushout square in $\mathrm{CAlg}(\mathrm{Pr}^L)$ and we have

$$\mathrm{QCoh}(X) \simeq \mathrm{QCoh}([X/G]) \otimes_{\mathrm{QCoh}(BG)} \mathrm{QCoh}(*),$$

where the relative tensor product is taken in Pr^L (see [3, Proposition 4.6][SAG, Corollary 9.4.2.3]). Now we apply this method to the case that is interesting to us. First, we need some preparations.

Lemma 6.1.1. For each $\sigma \in \Sigma$, the stack $[X_\sigma/\mathbb{T}]$ is a perfect stack in the sense of [SAG, Definition 9.4.4.1]. Similarly, the stack $B\mathbb{T}$ is also a perfect stack.

Proof. We only present the proof for $[X_\sigma/\mathbb{T}]$, the other case could be proved with similar arguments. We need to check three things:

- The stack $[X_\sigma/\mathbb{T}]$ is a quasi-geometric stack. It is in fact geometric. Given [SAG, Corollary 9.3.1.4], this follows (in the same way as [29, Remark 2.1]) from the fact that $[X_\sigma/\mathbb{T}]$ is a colimit of the action diagram of $\mathbb{T}$ acting on X_σ, where the degree 0 term X_σ is affine and the map $d_0 : X_\sigma \times \mathbb{T} \to X_\sigma$ is representable, affine and faithfully flat.

- The structure sheaf $\mathcal{O}$ is a compact object in $\mathrm{QCoh}([X_\sigma/\mathbb{T}])$. Via Proposition 3.3.1, the structure sheaf is sent to a representable presheaf, which is certainly a compact object.
- The category $\mathrm{QCoh}([X_\sigma/\mathbb{T}])$ is generated by compact objects. Via Proposition 3.3.1 this reduces to the fact that the spectral presheaf category is compactly generated.

□

Theorem 6.1.2 (De-equivariantization for QCoh)**.** De-equivariantization applies to the following stacks:

- For each $\sigma \in \Sigma$, we have a symmetric monoidal equivalence
$$\mathrm{QCoh}([X_\sigma/\mathbb{T}]) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}(*) \xrightarrow{\simeq} \mathrm{QCoh}(X_\sigma).$$
- We have a symmetric monoidal equivalence
$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}(*) \xrightarrow{\simeq} \mathrm{QCoh}(X_\Sigma).$$

Proof. The first point is a direct application of [SAG, Corollary 9.4.2.3], given that both $[X_\sigma/\mathbb{T}]$ and $B\mathbb{T}$ are perfect stacks. For the second point, note that by the colimit presentation of $[X_\Sigma/\mathbb{T}]$ one has
$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \simeq \lim_{\Sigma^{op}} \mathrm{QCoh}([X_\sigma/\mathbb{T}]).$$
Hence the relative tensor product gives
$$\begin{aligned}\mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}(*) &\simeq (\lim_{\Sigma^{op}} \mathrm{QCoh}([X_\sigma/\mathbb{T}])) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}(*) \\ &\simeq \lim_{\Sigma^{op}} (\mathrm{QCoh}([X_\sigma/\mathbb{T}]) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}(*)) \\ &\simeq \lim_{\Sigma^{op}} \mathrm{QCoh}(X_\sigma) \\ &\simeq \mathrm{QCoh}(X_\Sigma),\end{aligned}$$
where the relative tensor product commutes with limits since $\mathrm{QCoh}(*)$ is dualizable over $\mathrm{QCoh}(B\mathbb{T})$ by [SAG, Corollary 9.4.2.2]. □

Remark 6.1.3. It seems plausible to directly show that $[X_\Sigma/\mathbb{T}]$ is a perfect stack by adapting the proof of [3, Proposition 3.21] in the spectral setting. We opt for the above proof because it is straightforward from what we have done so far.

We move on to the mirror side. Note that for the category of sheaves on real vector spaces, the de-equivariantization is reflected as the equivariantization, as we explain now. We need the following fact whose proof will be provided later.

Lemma 6.1.4. The lax symmetric monoidal functor given in Construction 4.1.2
$$D_!(-) : \mathrm{LCH} \longrightarrow \mathrm{Cat}$$
lifts to a symmetric monoidal functor (by abuse of notation, we give it the same name)
$$D_!(-) : \mathrm{LCH} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}.$$

Theorem 6.1.5 (Equivariantization for $\mathcal{S}\mathrm{hv}$)**.** There is a commutative square of symmetric monoidal categories

$$\begin{tikzcd} \mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, hook] \arrow[d, "\simeq"] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \arrow[d, "="] \\ \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \arrow[r, hook, "i_!"'] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \end{tikzcd}.$$

The symmetric monoidal functor

$$\mathrm{Fun}(M, \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

is from Construction 4.3.7. The map $i : M \to M_{\mathbb{R}}$ is the inclusion of the topological groups, hence !-pushforward along i induces a fully faithful symmetric monoidal functor

$$i_! : \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

where both categories are equipped with the convolution monoidal structure. Moreover, $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M; \mathrm{Sp})$ can be identified with the relative tensor product:

$$\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M; \mathrm{Sp}) \simeq \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}).$$

Proof. Recall from Construction 4.3.7 that the functor

$$\mathrm{Fun}(M, \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

is defined as a composition

$$\mathrm{Fun}(M, \mathrm{Sp}) \longrightarrow \lim_{\sigma} \mathrm{Fun}(\Theta(\sigma), \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

By Proposition 4.5.4, we know that it takes $m \in M$ to the skyscraper $\underline{\mathbb{S}}_{\{m\}} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$. Note that it preserves colimits, so its image is contained in the image of the fully faithful functor $i_!$. Hence we get a symmetric monoidal factorization

$$\mathrm{Fun}(M, \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \xrightarrow{i_!} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}),$$

and the first functor is readily checked to be an equivalence. Next we study the relative tensor product

$$\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp}.$$

Note that we may replace $\mathrm{Fun}(M, \mathrm{Sp})$ by $\mathcal{S}\mathrm{hv}(M; \mathrm{Sp})$ and Sp by $\mathcal{S}\mathrm{hv}(*; \mathrm{Sp})$. Hence we might as well study the relative tensor product

$$\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathcal{S}\mathrm{hv}(M;\mathrm{Sp})} \mathcal{S}\mathrm{hv}(*; \mathrm{Sp}),$$

formed along the symmetric monoidal functors

$$i_! : \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

and

$$p_! : \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(*; \mathrm{Sp}).$$

From Lemma 6.1.4, we have a symmetric monoidal functor

$$D_!(-) : \mathrm{LCH} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$$

and one can left Kan extend it to a symmetric monoidal colimit-preserving functor on the category of presheaves [18] on LCH (ignoring size issues)

$$D_!(-) : \mathrm{Fun}(\mathrm{LCH}^{\mathrm{op}}, \mathrm{Spc}) \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}.$$

By Proposition A.2.1, we know that $D_!(-)$ is compatible with forming relative tensor products. In particular, we get an identification

$$D_!(M_{\mathbb{R}}) \otimes_{D_!(M)} D_!(*) \xrightarrow{\simeq} D_!(h_{M_{\mathbb{R}}} \times_{h_M} h_*) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$$

where the underlying object of the right-hand side is computed as the colimit of $D_!(-)$ applied to the Bar complex of the relative tensor product

$$h_{M_{\mathbb{R}}} \times_{h_M} h_* \in \mathrm{Fun}(\mathrm{LCH}^{\mathrm{op}}, \mathrm{Spc}).$$

It remains to identify this colimit with $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M; \mathrm{Sp})$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. We have a map

$$h_{M_{\mathbb{R}}} \times_{h_M} h_* \longrightarrow h_{M_{\mathbb{R}}/M} \in \mathrm{CAlg}(\mathrm{Fun}(\mathrm{LCH}^{\mathrm{op}}, \mathrm{Spc}))$$

and we claim it becomes an equivalence once we apply $D_!(-)$. This essentially follows from étale descent for the functor $\mathcal{S}\mathrm{hv}(-)$. We supply a detailed explanation as follows: given the map, it suffices to show that after applying $D_!(-)$ one gets an equivalence of categories. We will identify the Bar complex computing the relative tensor product

$$h_{M_{\mathbb{R}}} \times_{h_M} h_* \in \mathrm{Fun}(\mathrm{LCH}^{\mathrm{op}}, \mathrm{Spc})$$

with the Yoneda image of Čech nerve of the covering map

$$M_{\mathbb{R}} \longrightarrow M_{\mathbb{R}}/M \in \mathrm{LCH}.$$

Note that we are comparing two simplicial diagrams sitting inside in a sub-1-category in $\mathrm{Fun}(\mathrm{LCH}^{\mathrm{op}}, \mathrm{Spc})$. It is direct to check that these two diagrams agree. It follows that after applying $D_!(-)$ we have an identification of simplicial diagram of categories

$$[n \mapsto \mathcal{S}\mathrm{hv}(M; \mathrm{Sp})^{\otimes n} \otimes \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})] \simeq [n \mapsto \mathcal{S}\mathrm{hv}(M^{\times n} \times M_{\mathbb{R}})],$$

where the colimit of the left-hand side by definition computes $D_!(h_{M_{\mathbb{R}}} \times_{h_M} h_*)$. Now the question is reduced to checking that

$$\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{S}\mathrm{hv}(M \times M \times M_{\mathbb{R}}; \mathrm{Sp}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{S}\mathrm{hv}(M \times M_{\mathbb{R}}; \mathrm{Sp}) \substack{\longrightarrow\\ \longrightarrow} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M; \mathrm{Sp})$$

is a colimit diagram in Pr^{L}, where all the arrows are given by !-pushforward. Equivalently, we can take right adjoints everywhere and check that the outcome is a limit diagram in Cat. Note that all

[18] Alternatively, one can left Kan extend to the category of étale sheaves and simplify some of the arguments below.

the non-degenerate maps are étale so !-pullback is canonically identified with $*$-pullback. Using that taking $\mathcal{S}\mathrm{hv}(-)$ with $*$-pullback has étale descent, it follows that

$$\mathrm{D}_!(h_{M_\mathbb{R}} \times_{h_M} h_*) \xrightarrow{\simeq} \mathrm{D}_!(h_{M_\mathbb{R}/M}) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}).$$

Thus we have a symmetric monoidal equivalence

$$\mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \simeq \mathcal{S}\mathrm{hv}(M_\mathbb{R}/M;\mathrm{Sp}) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}). \qquad \square$$

Remark 6.1.6. More informally and concretely, we can interpret the above argument as follows. To compute the tensor product

$$\mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \otimes_{\mathcal{S}\mathrm{hv}(M;\mathrm{Sp})} \mathcal{S}\mathrm{hv}(*;\mathrm{Sp}),$$

one can look at the colimit of the simplicial diagram in Pr^{L}

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{S}\mathrm{hv}(M;\mathrm{Sp}) \otimes \mathcal{S}\mathrm{hv}(M;\mathrm{Sp}) \otimes \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{S}\mathrm{hv}(M;\mathrm{Sp}) \otimes \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \;\substack{\longrightarrow\\\longrightarrow}\; \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp})$$

given by the Bar complex calculating the relative tensor product. By the Künneth formula [41, Proposition 2.30], one might identify each term with

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{S}\mathrm{hv}(M \times M \times M_\mathbb{R};\mathrm{Sp}) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{S}\mathrm{hv}(M \times M_\mathbb{R};\mathrm{Sp}) \;\substack{\longrightarrow\\\longrightarrow}\; \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \;,$$

where all the functors are now given by !-pushforward. In other words, this diagram is the outcome of applying $\mathrm{D}(-)_!$ to the diagram of Čech nerve of the map

$$M_\mathbb{R} \longrightarrow M_\mathbb{R}/M \in \mathrm{LCH}.$$

Now one can take right adjoints and compute the limit of the following diagram in Cat

$$\cdots \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\; \mathcal{S}\mathrm{hv}(M \times M \times M_\mathbb{R};\mathrm{Sp}) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; \mathcal{S}\mathrm{hv}(M \times M_\mathbb{R};\mathrm{Sp}) \;\substack{\longleftarrow\\\longleftarrow}\; \mathcal{S}\mathrm{hv}(M_\mathbb{R};\mathrm{Sp}) \;,$$

where all the functors are now !-pullback. Since all the maps are étale, the !-pullback are canonically identified with $*$-pullback. Thus the diagram is identified with the outcome of taking $\mathcal{S}\mathrm{hv}(-)$ and $*$-pullback of the Čech nerve of the covering map $M_\mathbb{R} \to M_\mathbb{R}/M$. So we might conclude that the limit

$$\lim_{\Delta} \mathcal{S}\mathrm{hv}(M^{\times n} \times M_\mathbb{R};\mathrm{Sp}) \simeq \mathcal{S}\mathrm{hv}(M_\mathbb{R}/M;\mathrm{Sp}),$$

by étale descent of taking $\mathcal{S}\mathrm{hv}(-)$ and $*$-pullback.

Proof of Lemma 6.1.4. It follows from [HA, Remark 4.8.1.9] and

- On objects each X is taken to a stable presentable category $\mathcal{S}\mathrm{hv}(X;\mathrm{Sp})$.
- On morphisms each f is taken to a colimit-preserving functor $f_!$
- The box tensor product on $\mathcal{S}\mathrm{hv}(X;\mathrm{Sp})$ is colimit-preserving in each variable.

- The Künneth formula holds [41, Proposition 2.30].

□

Finally, we can apply equivariantization to the category of sheaves with prescribed singular support:

Recollection 6.1.7. The condition of being constructible and having prescribed singular support is preserved and can be checked after pullback along an étale cover map. This follows from the local nature of the definition Remark 5.1.12. See [20, Lemma 3.7] for a related result that one can check local constancy and constructibility étale locally.

Corollary 6.1.8 (Equivariantization for $\mathcal{S}\mathrm{hv}_{\Lambda}$)**.** There is a symmetric monoidal equivalence

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \simeq \mathcal{S}\mathrm{hv}_{\overline{\Lambda}_\Sigma}(M_{\mathbb{R}}/M;\mathrm{Sp})$$

where the right-hand side is the subcategory of sheaves of spectra on $M_{\mathbb{R}}/M$ characterized by the following two conditions:

- It is constructible for the stratification $\overline{\mathcal{S}}_\Sigma := \pi(\mathcal{S}_\Sigma)$ inherited from the projection map π.
- It has singular support lying in $\overline{\Lambda}_\Sigma := d\pi(\Lambda_\Sigma) \subset T^*M_{\mathbb{R}}/M$ inherited form the projection map π.

The convolution symmetric monoidal structure on $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M;\mathrm{Sp})$ restricts to a symmetric monoidal structure on $\mathcal{S}\mathrm{hv}_{\overline{\Lambda}_\Sigma}(M_{\mathbb{R}}/M;\mathrm{Sp})$.

Proof. Functoriality of the relative tensor product provides a functor

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \simeq \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M;\mathrm{Sp}) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

We will show that this is fully faithful and describe its image in terms of singular support. To do so, we consider the following map between the Bar complexes.

$$\begin{tikzcd} \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \otimes \mathrm{Fun}(M,\mathrm{Sp}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \arrow[d] \\ \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}) \otimes \mathrm{Fun}(M,\mathrm{Sp}) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}) \end{tikzcd}.$$

After taking colimit, it recovers the above functor. All the vertical functors are fully faithful (since tensoring with a dualizable category preserves fully faithful functors [9, Theorem 2.2]). We may identify each term in the top row with its image along the vertical functors:

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \otimes \mathrm{Fun}(M,\mathrm{Sp})^{\otimes n} \simeq \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}} \times M^{\times n};\mathrm{Sp}) \subseteq \mathcal{S}\mathrm{hv}(M_{\mathbb{R}} \times M^{\times n};\mathrm{Sp})$$

where we have implicitly used Künneth formula for the bottom row. The right-hand side is the category of sheaves $\mathcal{F}$ on $M_{\mathbb{R}} \times M^{\times n}$ such that on each component $M_{\mathbb{R}}$, $\mathcal{F}$ is constructible for $\mathcal{S}_\Sigma$ and has singular support contained in Λ_Σ. Now we observe the following: the right adjoint of each functor in the bottom row is the $*$-pullback along an étale map. In particular it preserves the condition of constructibility and singular support. Thus taking right adjoints of the bottom row restricts to taking right adjoints of the top row:

$$\begin{tikzcd} \cdots & \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}} \times M; \mathrm{Sp}) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] & \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \arrow[l] \arrow[d] \\ \cdots & \mathcal{S}hv(M_{\mathbb{R}} \times M; \mathrm{Sp}) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & \mathcal{S}hv(M_{\mathbb{R}}; \mathrm{Sp}) \arrow[l, shift left] \arrow[l, shift right] & \mathcal{S}hv(M_{\mathbb{R}}/M; \mathrm{Sp}) \arrow[l] \end{tikzcd}$$

Note that both rows are now limit diagrams in Cat. We thus learn that there is a fully faithful functor

$$\mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \hookrightarrow \mathcal{S}hv(M_{\mathbb{R}}/M; \mathrm{Sp}).$$

As a full subcategory, $\mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp}$ is spanned by the objects which are sent to the full subcategory

$$\mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \hookrightarrow \mathcal{S}hv(M_{\mathbb{R}}; \mathrm{Sp})$$

through $*$-pullback along the projection

$$M_{\mathbb{R}} \longrightarrow M_{\mathbb{R}}/M.$$

By the observation we have made in the very beginning that we can check constructibility and singular support locally, this is precisely the category of sheaves on $M_{\mathbb{R}}/M$ which are constructible for $\overline{\mathcal{S}}_\Sigma$ and has prescribed singular support contained in $\overline{\Lambda}_\Sigma$. The proof is now done. □

Corollary 6.1.9. There is a symmetric monoidal equivalence

$$\overline{\kappa} : \mathrm{QCoh}(X_\Sigma) \xrightarrow{\simeq} \mathcal{S}hv_{\overline{\Lambda}_\Sigma}(M_{\mathbb{R}}/M; \mathrm{Sp})$$

where the right-hand side is the category appearing in Corollary 6.1.8.

Proof. It follows from the commutative diagram of Theorem B that the relative tensor products are identified in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{Sp} \simeq \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp}$$

Now the result follows from Theorem 6.1.2 and Corollary 6.1.8. □

6.2 Beilinson's theorem on projective spaces

As a concrete example of the abstract nonsense we have developed, we now give an explanation of Beilinson's linear algebraic description of quasi-coherent sheaves of $\mathbb{P}^1_{\mathbb{S}}$, the flat projective line over $\mathbb{S}$. Recall that the toric data corresponding to the projective line is given by the lattice $N = \mathbb{Z}$ and the fan $\{\{0\}, \mathbb{R}_{\geqslant 0}, \mathbb{R}_{\leqslant 0}\}$ inside $\mathbb{R}^1$.

Example 6.2.1. There are equivalences of categories:

$$\mathrm{QCoh}(\mathbb{P}^1_{\mathbb{S}}) \simeq \mathrm{Cons}_{\overline{\mathcal{S}}_\Sigma}(S^1; \mathrm{Sp}) \simeq \mathrm{Fun}(\bullet \rightrightarrows \bullet; \mathrm{Sp})$$

where the stratification $\overline{\mathcal{S}}_\Sigma$ has two strata: the origin and its complement. The first equivalence is given by $\overline{\kappa}$ and the second is given by exodromy [14].

Proof. The first functor is $\overline{\kappa}$ supplied by Corollary 6.1.9. More precisely, it embeds $\mathrm{QCoh}(\mathbb{P}^1_{\mathbb{S}})$ as a full subcategory in $\mathrm{Cons}_{\overline{\mathcal{S}}_\Sigma}(S^1; \mathrm{Sp})$. However, one checks readily that the condition on singular support is vacuous. Away from the origin, every $\overline{\mathcal{S}}_\Sigma$ constructible sheaf becomes locally constant, hence the singular support is always contained in the zero section. At the origin, the singular support asks for the support of some sheaf on $\mathbb{R}^1$ to have support contained in $\mathbb{R}^1$, which is again no restriction. We thus conclude that the first functor is an equivalence. The second functor is an direct application of exodromy equivalence from [14]. Note that the exit path category of $(S^1, \overline{\mathcal{S}}_\Sigma)$ is precisely the quiver $\bullet \rightrightarrows \bullet$. $\square$

Remark 6.2.2. It is possible to obtain the similar result for $\mathbb{P}^n_{\mathbb{S}}$ which states that the category $\mathrm{QCoh}(\mathbb{P}^n_{\mathbb{S}})$ is compactly generated by a collection of objects $\mathcal{O}(1), \cdots, \mathcal{O}(n+1)$, and they form an exceptional collection. This, however is more involved since the condition on singular support puts an actual constraint so one needs further arguments beyond applying exodromy equivalence. We only present a sketch of the proof idea here. Pick some equivariant lifts $\tilde{\mathcal{O}}(i) \in \mathrm{QCoh}([\mathbb{P}^n_{\mathbb{S}}/\mathbb{T}])$. The image of these $\tilde{\mathcal{O}}(i)$ under κ are dualizing sheaves on some explicit moment polytopes in $\mathbb{R}^n$ as in Section 5.2. To show that they generate, one can run the argument in Section 5.3 to see that these images $\kappa(\tilde{\mathcal{O}}(i))$ corepresents taking stalks at each points in a fundamental domain of $\mathbb{R}^n/\mathbb{Z}^n$, so by adjunction $\overline{\kappa}(\mathcal{O}(i))$ also corepresents taking stalks at each point on $\mathbb{R}^n/\mathbb{Z}^n$. This proves that they generate, and the mapping spectra can be directly computed by looking at the intersections of these moment polytopes, which we omit. This computation also recovers the presentation of $\mathrm{QCoh}(\mathbb{P}^n_{\mathbb{S}})$ as the category of presheaves of spectra on an explicit quiver with relations defined by Beilinson.

Remark 6.2.3. This suggests we might dream of the exodromy for constructible sheaves with prescribed singular supports: can one read off Beilinson's quiver directly from the singular support $\overline{\Lambda}_{\Sigma_n}$ where Σ_n is the fan for $\mathbb{P}^n$? In general, we might ask how to describe the category of constructible sheaves with prescribed singular supports in terms of presheaf categories: this is not always possible, and the question of when it is possible remains largely open. However, see below for the example of $\mathbb{P}^2$ where it is indeed possible.

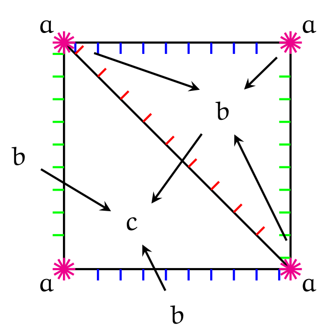

Figure 1: An illustration of a sheaf in $\mathcal{S}\mathrm{hv}_{\overline{\Lambda}_{\mathbb{P}^2}}(\mathbb{R}^2/\mathbb{Z}^2)$, drawn in a fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$. The short directional strokes—drawn along the edges and diagonal, fanning out at the corners—schematically represent $\overline{\Lambda}_{\mathbb{P}^2}$ in each cotangent fiber. Three distinguished stalks and ways that they are allowed to exit are drawn.

Remark 6.2.4. When the fan Σ is zonotopal and unimodular[19] (see [37, Definition 4.2]), the conic Lagrangian Λ_Σ is identified with the conormal variety of the stratification $\mathcal{S}_\Sigma$ (similarly for $\overline{\Lambda}_\Sigma$) [37, Theorem 4.4]. From this one can argue that the singular support condition is automatically satisfied for all $\mathcal{S}_\Sigma$- (resp. $\overline{\mathcal{S}}_\Sigma$-) constructible sheaves. Thus Corollary 6.1.9 identifies $\mathrm{QCoh}(X_\Sigma)$ with the category of constructible sheaves

$$\mathrm{QCoh}(X_\Sigma) \simeq \mathrm{Cons}_{\overline{\mathcal{S}}_\Sigma}(M_\mathbb{R}/M; \mathrm{Sp})$$

where exodromy [14] applies. However, it is not clear to us how to write down the exit path category explicitly from the combinatorics of the fan.

6.3 Relative toric bundle

The proof of Corollary 6.1.9 depends on the base change functor

$$(-) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) : \mathrm{CAlg}(\mathrm{Pr}^L)_{\mathrm{QCoh}(B\mathbb{T})/} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^L)$$

applied to the symmetric monoidal functor

$$\mathrm{QCoh}(B\mathbb{T}) \simeq \mathrm{Fun}(M, \mathrm{Sp}) \xrightarrow{\mathrm{colim}} \mathrm{Sp} \in \mathrm{CAlg}(\mathrm{Pr}^L).$$

There is no reason to stop at this case, and we can make the formal definition:

Definition 6.3.1 (Relative toric construction)**.** Fix a lattice N and fan Σ. Given a symmetric monoidal functor

$$f : M \longrightarrow \mathcal{C}$$

where $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$, it induces a map

$$F : \mathrm{QCoh}(B\mathbb{T}) \simeq \mathrm{Fun}(M; \mathrm{Sp}) \longrightarrow \mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$$

and we define

$$\mathrm{Mod}_{X_{\Sigma,f}}\mathcal{C} := \mathcal{C} \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \in \mathrm{CAlg}(\mathrm{Pr}^L)$$

to be the relative toric bundle over $\mathcal{C}$ associated with Σ and f.

From the definition it follows that both $\mathrm{QCoh}([X_\Sigma/\mathbb{T}])$ and $\mathrm{QCoh}(X_\Sigma)$ are examples of the relative toric construction.

Remark 6.3.2. Tautologically, we have

$$\mathrm{Mod}_{X_{\Sigma,f}}\mathcal{C} \simeq \mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathcal{C}.$$

We believe the right hand side admits a sheaf theoretic interpretation. In particular, it should by descent describe the category of (twisted) sheaves on the torus $M_\mathbb{R}/M$ valued in a local system of categories specified by the delooping

$$Bf : BM \simeq \Pi_\infty(M_\mathbb{R}/M) \to \mathrm{CAlg}(\mathrm{Pr}^L).$$

A general theory of twisted sheaves that allows us to make such descriptions will be pursued in future work.

[19] Unfortunately these assumptions are quite restrictive.

Example 6.3.3. In [19], the second named author with Pyongwon Suh considered the data of a classical scheme S and n line bundles $\{L_n \in \mathrm{Pic}(S)\}$ on S. Such collection of line bundles defines a symmetric monoidal functor

$$f : \mathbb{Z}^n \longrightarrow \mathrm{QCoh}(S)$$

and the relative toric bunlde over $\mathrm{QCoh}(S)$ associated with Σ and f could be identified with the category of quasi-coherent sheaves on an S-scheme $\mathcal{X}_{\Sigma,f}$:

$$\mathrm{Mod}_{X_{\Sigma,f}}\mathrm{QCoh}(S) \simeq \mathrm{QCoh}(\mathcal{X}_{\Sigma,f}).$$

The relative toric scheme (or so-called toric fibration) $\mathcal{X}_{\Sigma,f}$ is constructed affine locally on S, as a toric scheme with respect to the torus associated with $\oplus L_i$ over S. Equivalently, it can be identified with the base change of $[X_\Sigma/\mathbb{T}] \to B\mathbb{T}$ along the map $S \to B\mathbb{T}$ classifying these line bundles $\{L_i\}$. On the mirror side, the base change can be interpreted as sheaves on the torus $\mathbb{R}^n/\mathbb{Z}^n$ with twisted-coefficient (see Remark 6.3.2) - roughly the stalk of the coefficient category is $\mathrm{QCoh}(S)$ and the monodromy is given by tensoring with L_i.

Remark 6.3.4. Such $f : \mathbb{Z}^n \longrightarrow \mathcal{C}$ classifies n strict Picard elements in $\mathcal{C}$ that also strictly commute with each other. Beware that such datum is rare in the wild, see [5].

Appendix A Categorical generalities

A.1 Modules over grouplike monoid

We find the following lemma straightforward, but can not locate a proof in the literature.

Lemma A.1.1. Let T be an ∞-category admitting finite limits, $\mathsf{G} \in \mathrm{Mon}(\mathsf{T})$, and X a G-module. If G is grouplike, then $(\mathsf{X}//\mathsf{G})_\bullet$ is a groupoid object.

Proof. Unwinding the definitions, there is a canonical map

$$p : (X//G)_\bullet \to (*//G)_\bullet,$$

where the latter can be identified with the underlying simplicial object of G, hence a groupoid object [HA, Remark 5.2.6.5]. Therefore it suffices to show that this map is a Cartesian natural transformation (see [HTT, Definition 6.1.3.1].)

In other words, we want to show that for every $\alpha : [m] \to [n]$, the diagram

$$\begin{tikzcd}
X \times G^n \arrow[r, "\simeq"] & (X//G)_n \arrow[r, "\alpha"] \arrow[d] & (X//G)_m \arrow[d] & X \times G^m \arrow[l, "\simeq"'] \\
G^n \arrow[r, "\simeq"] & (*//G)_n \arrow[r, "\alpha"] & (*//G)_m & G^m \arrow[l, "\simeq"']
\end{tikzcd}$$

is a pullback, i.e., $p(\alpha) : p([n]) \to p([m]) \in \mathrm{Fun}([1], T)$ is a Cartesian morphism.

We proceed by induction and show that $p|_{\Delta^{op}_{\leqslant n}}$ is a Cartesian transformation for each n. For the base case $n = 0$, there is nothing to prove. For $n \geqslant 1$, note that every map in $\Delta_{\leqslant n}$ can be factored into a sequence of maps in which each is either in $\Delta_{\leqslant n-1}$ or one of the following: the injective maps $\delta_k : [n-1] \to [n]$ and the surjective maps $\sigma_k : [n] \to [n-1]$. Therefore it suffices to show that $p(\delta_k)$ and $p(\sigma_k)$ are Cartesian morphisms.

For $p(\delta_k)$, we claim that it suffices to prove $p(\delta_0)$ and $p(\delta_n)$ are Cartesian: indeed, for $0 < k < n$, consider the decomposition $[0, k] \cup [k, n] = [n]$ and the diagram

$$\begin{tikzcd}
p([n]) \arrow[r, "|" marking] \arrow[d, "-" marking] \arrow[dr, dashed, start anchor=south east, to path={-- ([yshift=-3em]\tikztotarget.north)}] & p([0,k]) \arrow[d, squiggly] \arrow[ddr, squiggly] & \\
p([k,n]) \arrow[r, squiggly] \arrow[ddr, squiggly] & p(\{k\}) & \\
 & p(\{\cdots < k-1 < k+1 < \cdots\}) \arrow[r, squiggly] \arrow[d, squiggly] & p([0,k-1]) \\
 & p([k+1,n]) &
\end{tikzcd} .$$

By induction hypothesis, all the squiggly arrows are Cartesian. By the 2-out-of-3 property of Cartesian morphisms, to show the dashed arrow is Cartesian (and hence every arrow is Cartesian), it suffices to show either of the barred arrows is Cartesian. However $[0, k] \hookrightarrow [n]$ factors as a map in $\Delta_{\leqslant n-1}$ followed by δ_n.

Using the identifications
$$(X//G)_n \simeq X \times G^n,$$
and
$$\prod_i ([i < i+1] \hookrightarrow [n])^* : (*//G)_n \simeq G^n,$$
$p(\delta_0)$ is equivalent to
$$\begin{tikzcd} X \times G^n \arrow[r] \arrow[d] & G^n \arrow[d] \\ X \times G^{n-1} \arrow[r] & G^{n-1} \end{tikzcd},$$
where all the maps are projection, hence Cartesian.
Similarly, $p(\delta_n)$ is equivalent to the product of
$$\begin{tikzcd} X \times G \arrow[r] \arrow[d, "a"'] & G \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$
with G^{n-1}. Therefore it suffices to show the map $X \times G \xrightarrow{(a,\mathrm{pr})} X \times G$ is an equivalence, which is indeed true as it admits a homotopy inverse given by shearing.

To see $p(\sigma_k)$ is Cartesian, simply note that both its source and target are (induced by) diagonal maps. $\square$

A.2 Functoriality of module categories

The following is a direct consequence of [HA, Theorem 4.8.5.16]. The reader might compare it to [HA, Remark 4.8.5.19].

Proposition A.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories admitting all geometric realizations. Let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. Assume that:

1. Tensor products in $\mathcal{C}$ and $\mathcal{D}$ commute with geometric realizations.
2. The functor F commutes with geometric realizations.

Then there is a diagram
$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, bend left, "\mathrm{Mod}_{(-)}(\mathcal{C})"] \arrow[r, bend right, "\mathrm{Mod}_{F(-)}(\mathcal{D})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{CAlg}(\mathrm{Cat}) \end{tikzcd}.$$
When evaluated at $A \to B \in \mathrm{CAlg}(\mathcal{C})$, the diagram reads
$$\begin{tikzcd} \mathrm{Mod}_A(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Mod}_B(\mathcal{C}) \arrow[d] \\ \mathrm{Mod}_{F(A)}(\mathcal{D}) \arrow[r] & \mathrm{Mod}_{F(B)}(\mathcal{D}) \end{tikzcd}.$$

Proof. We pick up notations in [HA, Theorem 4.8.5.16] and fix $\mathcal{K}$ to be just $\{\Delta^{\mathrm{op}}\}$ (in particular the following items refer to items there). The symmetric monoidal coCartesian fibrations in (1) and the functor $\Theta^{\otimes}$ in (3) straighten (symmetric monoidally) to lax symmetric monoidal functors and natural transformations [20]

$$\begin{tikzcd} \mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat}) \arrow[r, bend left, "\mathrm{Alg}(-)"] \arrow[r, bend right, "\mathrm{Mod}_{(-)}(\mathrm{Cat}(\mathcal{K}))"'] \arrow[r, phantom, "\Downarrow\overline{\Theta}"] & \mathrm{Cat} \end{tikzcd} .$$

One applies further CAlg on both sides and obtain

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat})) \arrow[r, bend left, "\mathrm{Alg}(-)"] \arrow[r, bend right, "\mathrm{Mod}_{(-)}(\mathrm{Cat}(\mathcal{K}))"'] \arrow[r, phantom, "\Downarrow\overline{\Theta}"] & \mathrm{CAlg}(\mathrm{Cat}) \end{tikzcd} .$$

The assumption on $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ ensures that it lifts to a map in $\mathrm{CAlg}(\mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat}))$. We evaluate the above natural transformation $\overline{\Theta}$ on F and obtain a commuting diagram in $\mathrm{CAlg}(\mathrm{Cat})$

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{C}) \arrow[r, "\mathrm{Mod}_{(-)}(\mathcal{C})"] \arrow[d, "\mathsf{F}"] & \mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}(\mathcal{K})) \arrow[d, "(-)\otimes_{\mathcal{C}}\mathcal{D}"] \\ \mathrm{Alg}(\mathcal{D}) \arrow[r, "\mathrm{Mod}_{(-)}(\mathcal{D})"] & \mathrm{Mod}_{\mathcal{D}}(\mathrm{Cat}(\mathcal{K})) \end{tikzcd} .$$

Applying CAlg again to the diagram gives the commutative square in Cat

$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, "\mathrm{Mod}_{(-)}(\mathcal{C})"] \arrow[d, "\mathsf{F}"] & \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}(\mathcal{K}))) \arrow[d, "(-)\otimes_{\mathcal{C}}\mathcal{D}"] \\ \mathrm{CAlg}(\mathcal{D}) \arrow[r, "\mathrm{Mod}_{(-)}(\mathcal{D})"] & \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{D}}(\mathrm{Cat}(\mathcal{K}))) \end{tikzcd} .$$

Note that by [HA, Lemma 4.8.4.2], the functor $(-) \otimes_{\mathcal{C}} \mathcal{D}$ is a symmetric monoidal left adjoint

$$(-) \otimes_{\mathcal{C}} \mathcal{D} : \mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}(\mathcal{K})) \longrightarrow \mathrm{Mod}_{\mathcal{D}}(\mathrm{Cat}(\mathcal{K})).$$

It follows that there is an adjunction $(-) \otimes_{\mathcal{C}} \mathcal{D} \dashv \mathrm{fgt}$ between $\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}(\mathcal{K})))$ and $\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{D}}(\mathrm{Cat}(\mathcal{K})))$. Combining this adjunction with the commutative square, we obtain a natural transformation

$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, bend left, "\mathrm{Mod}_{(-)}(\mathcal{C})"] \arrow[r, bend right, "\mathrm{Mod}_{\mathsf{F}(-)}(\mathcal{D})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}(\mathcal{K}))) \end{tikzcd} .$$

Post-composing with the forgetful (see [HA, Corollary 4.8.1.4]) to $\mathrm{CAlg}(\mathrm{Cat})$ gives what we claimed. □

[20] Note that $\Theta^{\otimes}$ preserves coCartesian edges over $\mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat})^{\otimes}$. This follows from the following two facts: from (4) we know it is a symmetric monoidal functor hence preserves coCartesian lift from Fin_* and from [HA, Proposition 4.8.5.1] we know the underlying functor Θ preserves coCartesian edges over $\mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat})$.

A.3 Reminders on Day convolutions

Remark A.3.1. Given a small symmetric monoidal category $(\mathcal{C}, \otimes)$, there is a symmetric monoidal structure on the spectral presheaf category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ called 'Day convolution'. The stable Yoneda embedding h[21] has a structure of symmetric monoidal functor and has the following universal property.

For any presentably symmetric monoidal stable category $\mathcal{D}$ with a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$, there exists a unique symmetric monoidal, colimit-preserving lift to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\mathrm{h}"] \arrow[dr, "F"'] & & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \arrow[dl, dashed, "\exists!"] \\
& \mathcal{D} &
\end{tikzcd}.$$

To be precise, one learns from [HA, Proposition 4.8.1.10] that for each small symmetric monoidal category $(\mathcal{C}, \otimes)$, the presheaf category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc})$ has the structure of a presentably symmetric monoidal category, and the (unstable) Yoneda functor

$$\mathrm{h} : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc})$$

has a structure of symmetric monoidal functor. Moreover, the restriction map

$$\mathrm{Fun}^{\mathrm{lax}\otimes, \mathrm{L}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc}), \mathcal{D}) \xrightarrow{\mathrm{h}^*} \mathrm{Fun}^{\mathrm{lax}\otimes}(\mathcal{C}, \mathcal{D})$$

is an equivalence for any presentably symmetric monoidal category $\mathcal{D}$. The restriction of above functor to the full subcategory of symmetric monoidal functors

$$\mathrm{Fun}^{\otimes, \mathrm{L}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc}), \mathcal{D}) \xrightarrow{\mathrm{h}^*} \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$$

is also an equivalence. Using the symmetric monoidal adjunction

$$\begin{tikzcd}
\mathrm{Pr}^{\mathrm{L}} \arrow[r, bend left, "-\otimes \mathrm{Sp}"] & \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \arrow[l, bend left, "\text{forgetful}"]
\end{tikzcd}$$

one learns that the stable analogues

$$\mathrm{h}^* \colon \mathrm{Fun}^{\mathrm{lax}\otimes, \mathrm{L}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lax}\otimes}(\mathcal{C}, \mathcal{D})$$

$$\mathrm{h}^* \colon \mathrm{Fun}^{\otimes, \mathrm{L}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$$

also hold for any presentably symmetric monoidal stable category $\mathcal{D}$.

[21] We abuse notation by writing h for both the unstable and the stable Yoneda embedding, when there is no danger of confusion.

Remark A.3.2 (Day convolution as a partial adjunction)**.** The equivalence above could be understood as a partial adjunction between forgetful and forming category of presheaves:

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{CAT}) & \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}) \arrow[l, "\text{forgetful}"'] \\ \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{small}}) \arrow[u, "i"] \arrow[ur, "\mathrm{Fun}(-^{\mathrm{op}},\mathrm{Sp})"'] & \end{tikzcd}.$$

See, for example, [15, 1.32] on how to extract the adjoints functorially. In particular, the equivalences

$$\mathrm{Fun}^{\mathrm{lax}\otimes,\mathrm{L}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp}),\mathcal{D})^{\simeq} \xrightarrow{h^*} \mathrm{Fun}^{\mathrm{lax}\otimes}(\mathcal{C},\mathcal{D})^{\simeq}$$

$$\mathrm{Fun}^{\otimes,\mathrm{L}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp}),\mathcal{D})^{\simeq} \xrightarrow{h^*} \mathrm{Fun}^{\otimes}(\mathcal{C},\mathcal{D})^{\simeq}$$

are functorial in $\mathcal{C}$ and $\mathcal{D}$[22]. This implies that the construction

$$\begin{aligned} \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{small}}) &\to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}) \\ \mathcal{C} &\mapsto \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})^{\textbf{Day-}\otimes} \end{aligned}$$

exhibits $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})^{\textbf{Day-}\otimes}$ as the symmetric monoidal stable cocompletion of $\mathcal{C}$.

[22]This essentially boils down to the universal property of Day convolution as stated above. It is however convenient for us to phrase it in terms of partial adjunction.

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Paper B: A study of sheaves on real vector spaces

This chapter contains the following paper:

Qingyuan Bai and Robert Burklund. *A study of sheaves on real vector spaces.* 2025.

A study of sheaves on real vector spaces

Qingyuan Bai and Robert Burklund

In the first section we recall the computation of $K^{cont}(\mathcal{S}hv(\mathbb{R}^1; Sp))$ sketched by Sasha Efimov in his talk. No knowledge of singular support will be assumed. We also provide a motivation for this computation through the looking-glass of homological mirror symmetry. Note that Marc Hoyois already provided a computation of K^{cont} for any locally compact Hausdorff spaces in [3].

In the second section we compute the Picard groupoid of sheaves (of k-modules for a field k) on $\mathbb{R}^1$ under convolution product.

In the third section we propose some speculations for Picard groupoid of sheaves on $\mathbb{R}^n$ under convolution product.

Finally, in the fourth section we sketch another approach to the computation for Picard groupoid of sheaves on $\mathbb{R}^n$ (of k-modules for a field k) under convolution product, which builds on the notion of Fourier transform for wild Betti sheaves. The general strategy in this section was explained to us by Peter Scholze (but the inaccuracies are of course our own faults).

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1 K-theory of sheaves on $\mathbb{R}^1$

We will explain the following computation:

Theorem 1. The continuous K-theory of the category of sheaves of spectra on the real line can be computed as

$$K^{cont}(\mathcal{S}hv(\mathbb{R}^1; Sp)) \cong \Omega(K(\mathbb{S})).$$

The key ingredient is the following category.

Definition 2. Let $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})$ be the full subcategory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ spanned by

$$\{\mathcal{F} \in \mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}) : \mathcal{F}((-\infty,a)) \xrightarrow{\cong} \mathcal{F}((b,a)) \forall b < a\}.$$

And one defines $\mathcal{S}\mathrm{hv}^-(\mathbb{R}^1;\mathrm{Sp})$ similarly as the full subcategory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ spanned by

$$\{\mathcal{F} \in \mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}) : \mathcal{F}((b,+\infty)) \xrightarrow{\cong} \mathcal{F}((b,a)) \forall b < a\}.$$

We also write $\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp}) := \mathcal{S}\mathrm{hv}^{\mathrm{lc}}(\mathbb{R}^1;\mathrm{Sp})$ for the full subcategory of locally constant (hence constant) sheaves on $\mathbb{R}^1$.

The fact that we will prove about $\mathcal{S}\mathrm{hv}^+$ translates verbatimly to $\mathcal{S}\mathrm{hv}^-$. To start with, there are many more equivalent descriptions for $\mathcal{S}\mathrm{hv}^+$.

Proposition 3. The following subcategories of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ are the same.

1. The subcategory $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})$ as we described in Definition 2.

2. The subcategory spanned by the image of a fully faithful functor of pulling back sheaves

$$\pi^* : \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp}) \to \mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}).$$

   The topological space $\mathbb{R}^1_+$ has underlying set $\mathbb{R}^1$ and a weaker topology, with opens specified by

$$\{U_a := (-\infty,a) \subseteq \mathbb{R}^1 : a \in [-\infty,+\infty]\}.$$

   The continuous map $\pi : \mathbb{R}^1 \to \mathbb{R}^1_+$ is given by identity on the underlying set.

3. The cocomplete stable subcategory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ spanned by $h(U_a)^{\mathrm{sh}}$, the sheafification of the stable Yoneda image of $U_a = (-\infty,a) \in \mathrm{Open}(\mathbb{R}^1)$ (we will abusively call them *representable sheaves*).

4. The subcategory $\mathrm{Mod}_{\theta_+}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}))$. We equip the category $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ with a convolution symmetric monoidal structure, and there is an idempotent algebra $\theta_+ := h(U_0)^{\mathrm{sh}}[1]$.

5. (For fans of singular support, we won't use this) The subcategory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ spanned by sheaves with singular support contained in $\mathbb{R}^1 \times \mathbb{R}^1_{\geqslant 0} \subseteq T^*\mathbb{R}^1$.

Proof. We will focus on comparing the first 4 descriptions.

- 2 versus 3: it is not so obvious that π^* is fully faithful. To prove this, one has an adjunction between the locales:

$$\begin{tikzcd} \mathrm{Open}(\mathbb{R}^1) \arrow[rr, bend left, "\tau: U \mapsto U + U_0"] \arrow[rr, phantom, "\perp"] & & \mathrm{Open}(\mathbb{R}^1_+) \arrow[ll, bend left, "\pi^{-1}"] \end{tikzcd}$$

  where the addition means Minkowski sum. Hence we get τ^* left adjoint to π^* on sheaf category. Now it suffices to check that on representable sheaf x, the counit map $\tau^*\pi^*x \to x$ is an isomorphism: this is true from the definition. It follows that π^* is fully faithful. Now we need to see that the functor π^* lands into the subcategory given in 3 and is essentially surjective. This follows from the following two facts: π^* takes representable sheaves to representable sheaves and they generate under colimits in both categories.

- 3 versus 4: we first explain a bit more about 4. The convolution product is given by

$$\mathcal{F} * \mathcal{G} := +_!(\mathcal{F} \boxtimes \mathcal{G})$$

where $+ : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is the addition map. As a motivation, one can directly compute

$$\theta_+ * \theta_+ \cong \theta_+.$$

The recollement sequence for $(-\infty, 0] = (-\infty, 0) \cup \{0\}$ provides a map $\mathbb{1}_{\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})} \to \theta_+$ and makes the target into an idempotent algebra. Now the module category, as a localization, is colimit generated by the image of representable sheaves under convolution with θ_+: and one directly computes that the convolution of representable sheaves with θ_+ gives representable sheaves on U_a.

- 4 versus 1. The module category as a localization can be equally described by the collection of local objects:

$$\mathrm{Mod}_{\theta_+}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) = \{Y : \mathrm{Map}(X \otimes \theta_+, Y) = \mathrm{Map}(X, Y)\}.$$

(we write Map for mapping space and map for mapping spectrum, but note in above one might as well use mapping spectrum for the local condition) The local condition is closed under colimit in testing object X so one might as well check on representable sheaves X. By our discussion above this is exactly what 1 says.

- (*) 1 versus 5: the reader might consult the definition of singular support from [4] or [1]. It follows from the definition that condition in 1 implies condition in 5. Conversely, use non-characteristic deformation as in [5, Theorem 4.1].

□

Remark 4. The history of this category dates back to [4, Section 3.5] where it was studied from point 2 above. The localization kernel θ_+ was introduced by D.Tamarkin.

Proposition 5. The categories in Definition 2 fits in to a Cartesian diagram in Cat:

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}) \arrow[r] \arrow[d] & \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp}) \arrow[d] \\
\mathcal{S}\mathrm{hv}^-(\mathbb{R}^1;\mathrm{Sp}) \arrow[r] & \mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp})
\end{tikzcd}$$

More precisely, from the view point of 4 above, we have the following diagram

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{1}}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \arrow[r, "*\theta_+"] \arrow[d, "*\theta_-"'] & \mathrm{Mod}_{\theta_+}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \arrow[d, "*\theta_-"] \\
\mathrm{Mod}_{\theta_-}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \arrow[r, "*\theta_+"'] & \mathrm{Mod}_{\theta_+ * \theta_-}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}))
\end{tikzcd}$$

which after identification gives the above diagram.

Proof. We first need to supply an identification between the categories on the down right corner - $\mathcal{S}hv^0(\mathbb{R}^1;\mathrm{Sp}) := \mathcal{S}hv^{lc}(\mathbb{R}^1;\mathrm{Sp})$ and $\mathrm{Mod}_{\theta_+ * \theta_-}(\mathcal{S}hv(\mathbb{R}^1;\mathrm{Sp}))$. One can follow exactly the same argument as comparing 4 versus 1 in Proposition 3, after identifying $\theta_+ * \theta_- \cong \underline{\mathbb{S}}[1]$.

We also need to show that the diagram is Cartesian. This follows from the standard consequence of descent along idempotent algebra, and the Cartesian diagram of idempotent algebras

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d] & \theta_+ \arrow[d] \\ \theta_- \arrow[r] & \theta_+ * \theta_- \end{tikzcd}$$

which could be verified directly: it is the recollement sequence for $\mathbb{R}^1 = \{0\} \coprod \mathbb{R}^\times$.

$\square$

Note that the diagram lifts to $\mathrm{Cat}^{\mathrm{dual}}$ and all edges are moreover localizations. Applying excision (as in [3, Corollary 13], but for dualizable categories) we learn that after applying $\mathrm{K}^{\mathrm{cont}}$, the above diagram becomes Cartesian:

$$\begin{tikzcd} \mathrm{K}^{\mathrm{cont}}(\mathcal{S}hv(\mathbb{R}^1;\mathrm{Sp})) \arrow[r] \arrow[d] & \mathrm{K}^{\mathrm{cont}}(\mathcal{S}hv^+(\mathbb{R}^1;\mathrm{Sp})) \arrow[d] \\ \mathrm{K}^{\mathrm{cont}}(\mathcal{S}hv^-(\mathbb{R}^1;\mathrm{Sp})) \arrow[r] & \mathrm{K}^{\mathrm{cont}}(\mathcal{S}hv^0(\mathbb{R}^1;\mathrm{Sp})) \end{tikzcd}$$

where we immediately note that

$$\mathrm{K}^{\mathrm{cont}}(\mathcal{S}hv^0(\mathbb{R}^1;\mathrm{Sp})) = \mathrm{K}^{\mathrm{cont}}(\mathcal{S}hv^{lc}(\mathbb{R}^1;\mathrm{Sp})) = \mathrm{K}^{\mathrm{cont}}(\mathrm{Sp}) = \mathrm{K}(\mathbb{S}).$$

This reduces the computation to the following fact:

Proposition 6. The continuous K-theory of $\mathcal{S}hv^+(\mathbb{R}^1;\mathrm{Sp})$ vanishes (and similarly for $\mathcal{S}hv^-$).

To show this we provide a localization sequence for $\mathcal{S}hv^+(\mathbb{R}^1;\mathrm{Sp})$. For that we switchfoot and take the view of point 2 from Proposition 3.

Proposition 7. There is a Verdier sequence of the form

$$\begin{tikzcd} \prod_{\mathbb{R}} \mathrm{Sp} \arrow[r, "i" description] & \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) \arrow[l, bend right, "L_0"', "\perp"] \arrow[l, bend left, "R_0", "\perp"'] \arrow[r, "(-)^{\mathrm{sh}}" description] & \mathcal{S}hv(\mathbb{R}^1_+;\mathrm{Sp}) \arrow[l, bend right, "L_1"', "\perp"] \arrow[l, bend left, "R_1", "\perp"'] \end{tikzcd} \tag{1}$$

with

$$i : \prod_{\mathbb{R}} \mathrm{Sp} \to \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) : a \mapsto \operatorname*{colim}_{b<a} h(b)$$

and

$$(-)^{\mathrm{sh}} : \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) \to \mathcal{S}hv(\mathbb{R}_{\leqslant}, \mathrm{Sp}) = \mathcal{S}hv(\mathbb{R}^1_+;\mathrm{Sp})$$

being sheafification, using identification in Lemma 9 below. The other functors will be made explicit later, see Diagram 2.

Before we provide the construction, here is a definition and a technical fact (you should skip it and fast-forward to the proof of Proposition 7 on the first reading):

Definition 8. We define $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})^{+} \subseteq \mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})$ to be the subcategory of semi-continuous presheaves:

$$\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})^{+} := \{\mathcal{F} : \forall a \in \mathbb{R}, \mathcal{F}(a) \xrightarrow{\cong} \lim_{b<a} \mathcal{F}(b)\}.$$

This could be equivalently phrased as a sheaf condition for the induced topology on $\mathbb{R}_{\leqslant}$, if we include $\mathbb{R}_{\leqslant}$ into $[-\infty, +\infty] = \mathrm{Open}(\mathbb{R}^1_+)$. A subtlety is that sheaf condition seems to be a bit stronger than semi-continuity. They are actually equivalent because every covering of $a \in \mathbb{R}$ which doesn't contain a will be a cofinal subposet of $\{b \in \mathbb{R} : b < a\}$.

Lemma 9. *Following above identification, restriction of sheaves induces an equivalence*

$$\pi_* : \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+; \mathrm{Sp}) \xrightarrow{\cong} \mathcal{S}\mathrm{hv}(\mathbb{R}_{\leqslant}; \mathrm{Sp}) = \mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})^{+}.$$

Proof. This is almost a direct consequence of $\mathbb{R}_{\leqslant}$ being a basis of $\mathrm{Open}(\mathbb{R}^1_+)$. More precisely: a presheaf on $\mathbb{R}_{\leqslant}$ is a sheaf if and only if its right Kan extension to $\mathrm{Open}(\mathbb{R}^1_+)$ is a sheaf. From this we obtain a geometric embedding: (where Res is restriction and Ran is right Kan extension)

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(\mathbb{R}_{\leqslant}, \mathrm{Sp}) \arrow[rr, bend right, "\mathrm{Ran}"'] \arrow[rr, phantom, "\perp"] && \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+; \mathrm{Sp}) \arrow[ll, bend right, "\mathrm{Res}"'] \end{tikzcd} .$$

Now one can directly inspect the functor Res and show it's fully faithful and essentially surjective. We leave the details to the reader. □

Proof of Proposition 7. We will present another Verdier localization sequence then identify the terms with Diagram 1. The category $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})$ carries a symmetric monoidal structure $\otimes$ called Day convolution: this structure is inherited from the fact that $\mathbb{R}_{\leqslant}$ is a symmetric monoidal category, and it makes the stable Yoneda embedding:

$$h : \mathbb{R}_{\leqslant} \longrightarrow \mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})$$

carry a structure of symmetric monoidal functor. Now look at the following cofiber sequence in $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})$:

$$\operatorname*{colim}_{b<0} h(b) \longrightarrow h(0) \longrightarrow \mathbb{S}_{\{0\}}$$

where the later one is 'skyscraper' presheaf at $0 \in \mathbb{R}_{\leqslant}$. Note that $h(0) = \mathbb{1}_{\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}, \mathrm{Sp})}$ and we claim this map $h(0) \to \mathbb{S}_{\{0\}}$ presents $\mathbb{S}_{\{0\}}$ as an idempotent algebra. This equivalent to the following

$$\operatorname*{colim}_{b<0} h(b) \otimes \operatorname*{colim}_{b<0} h(b) \xrightarrow{\cong} \operatorname*{colim}_{b<0} h(b)$$

which is true by virtue that $\otimes$ commutes with colimit in each variable and one knows $h(a) \otimes h(b) = h(a+b)$, so one can directly evaluate both side and compare. (Note that the fiber $I =$

$\mathrm{colim}_{b<0}\, h(b)$ of $h(0) \to \mathbb{S}_{\{0\}}$ 'sits in homological degree 0', as an incarnation of almost mathematics situation.) One writes down the Verdier sequence for localization at this idempotent algebra:

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{S}_{\{0\}}}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})) \arrow[r, "\text{inclusion}" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) \arrow[l, bend right, "\mathbb{S}_{\{0\}}\otimes -"'] \arrow[l, bend left, "\underline{\mathrm{Map}}(\mathbb{S}_{\{0\}}, -)"] \arrow[r, "\mathrm{I}\otimes -" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}} \arrow[l, bend right, "\text{inclusion}"'] \arrow[l, bend left, "\underline{\mathrm{Map}}(\mathrm{I}, -)"]
\end{tikzcd} \quad . \tag{2}$$

To compare with Diagram 1, let's start by identifying the first term

$$\mathrm{Mod}_{\mathbb{S}_{\{0\}}}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})) \cong \prod_{\mathbb{R}} \mathrm{Sp}.$$

To do so, let's examine the condition on $\mathcal{F} \in \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})$ to be a module over $\mathbb{S}_{\{0\}}$. This is requiring:

$$\mathcal{F} \xrightarrow{\cong} \mathcal{F} \otimes h(0) \to \mathcal{F} \otimes \mathbb{S}_{\{0\}} \xrightarrow{\cong} \mathrm{cofib}[\operatorname*{colim}_{b<0} h(b) \otimes \mathcal{F} \to h(0) \otimes \mathcal{F}]$$

to be an equivalence, which is the same as

$$\operatorname*{colim}_{b<0} h(b) \otimes \mathcal{F} = 0.$$

Evaluating above on each $a \in \mathbb{R}$ we learn that

$$\left(\operatorname*{colim}_{b<0} h(b) \otimes \mathcal{F}\right)(a) = \operatorname*{colim}_{b<0} \mathcal{F}(a-b) = 0.$$

Note that for all $a < c$, the restriction map $\mathcal{F}(c) \to \mathcal{F}(a)$ factorizes through $\operatorname*{colim}_{b<0} \mathcal{F}(a-b) = 0$, so we conclude that for such $\mathcal{F}$ all restriction maps have to be zero map. Conversely, if $\mathcal{F}$ is a presheaf such that all the restriction maps are zero, one can directly supply an equivalence

$$\bigoplus_{a \in \mathbb{R}} \mathcal{F}(a)_{\{a\}} \xrightarrow{\cong} \mathcal{F}.$$

On the left hand side, $\mathcal{F}(a)_{\{a\}}$ is the 'skyscraper' presheaf at $a \in \mathbb{R}$ whose value is $\mathcal{F}(a)$ at a and 0 otherwise. Each $\mathcal{F}(a)_{\{a\}}$ is a module over $\mathbb{S}_{\{0\}}$ and it follows that $\mathcal{F}$ is a module over $\mathbb{S}_{\{0\}}$. To conclude, we have an identification:

$$\mathrm{Mod}_{\mathbb{S}_{\{0\}}}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})) = \{\mathcal{F} : \text{all restriction maps are zero}\} \xrightarrow{\cong} \mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp}) = \prod_{\mathbb{R}} \mathrm{Sp}.$$

For the middle equivalence, one can look at the functor of restriction of presheaves

$$\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) \to \mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp})$$

and check directly it is fully-faithful and essentially surjective on the subcategory.

Our next mission is to provide an identification of the categories

$$\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}} \cong \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+; \mathrm{Sp}).$$

First off Lemma 9 already provides an equivalence

$$\mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp}) \xrightarrow{\cong} \mathrm{Fun}(\mathbb{R}_{\leqslant},\mathrm{Sp})^+.$$

So it suffices to identify $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}}$ with $\mathrm{Fun}(\mathbb{R}_{\leqslant},\mathrm{Sp})^+$. Let's examine the condition on $\mathcal{F} \in \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})$ to be semi-continuous. For such a presheaf we must have for all $a \in \mathbb{R}$

$$\mathcal{F}(a) \xrightarrow{\cong} \lim_{b<a} \mathcal{F}(b).$$

On the other hand, with the lower localization sequence, we can identify $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}}$ with the kernel of the internal hom functor

$$\underline{\mathrm{Map}}(\mathbb{S}_{\{0\}},-) : \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp}) \to \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp}).$$

And we have

$$\begin{aligned}
\mathcal{F} \in \ker(\underline{\mathrm{Map}}(\mathbb{S}_{\{0\}},-)) &\Leftrightarrow \forall a \in \mathbb{R}, \mathrm{map}(h(a), \underline{\mathrm{Map}}(\mathbb{S}_{\{0\}},\mathcal{F}) = 0 \\
&\Leftrightarrow \forall a \in \mathbb{R}, \mathrm{map}(h(a) \otimes \mathbb{S}_{\{0\}},\mathcal{F}) = 0 \\
&\Leftrightarrow \forall a \in \mathbb{R}, \mathrm{map}(\mathbb{S}_{\{a\}},\mathcal{F}) = 0 \\
&\Leftrightarrow \forall a \in \mathbb{R}, \mathcal{F}(a) \xrightarrow{\cong} \lim_{b<a} \mathcal{F}(b)
\end{aligned}$$

which is exactly the condition on $\mathcal{F}$ to be semi-continuous. Hence we conclude that

$$\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}} \cong \ker(\underline{\mathrm{Map}}(\mathbb{S}_{\{0\}},-)) \cong \mathrm{Fun}(\mathbb{R}_{\leqslant},\mathrm{Sp})^+ \cong \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp}).$$

□

Finally we can prove the promised Proposition 6.

Proof of Proposition 6. Let's look at the upper localization sequence of Diagram 1. Under the identification with later Diagram 2 it looks like

$$\mathrm{Mod}_{\mathbb{S}_{\{0\}}}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})) \xleftarrow{\mathbb{S}_{\{0\}}\otimes -} \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp}) \xleftarrow{\text{inclusion}} \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}} .$$

So we learn that

$$\mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp}) \cong \ker\left[L_0 : \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp}) \to \mathrm{Fun}(\mathbb{R}^{\mathrm{disc}},\mathrm{Sp}) = \prod_{\mathbb{R}} \mathrm{Sp}\right]$$

where L_0 is determined by

$$L_0(h(a)) = h(a)$$

and is strongly cocontinuous (more concretely, $L_0(\mathcal{F})(b) = \mathrm{cofib}[\mathrm{colim}_{a>b}\,\mathcal{F}(a) \to \mathcal{F}(b)]$). Now both categories are compactly generated and $L_0 = \mathrm{Ind}(l_0)$ where

$$l_0 : \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\omega} \to (\prod_{\mathbb{R}} \mathrm{Sp})^{\omega}$$

is the functor between compact objects (note that l_0 takes representables to representables, but doesn't come from left Kan extension of a functor between the posets!). To show $K^{\mathrm{cont}}(\mathcal{S}hv(\mathbb{R}^1_+;\mathrm{Sp}))$ vanishes it suffices to prove that L_0 (equivalently, l_0) induces an isomorphism on $K(-)$. So we've reduced to studying the map induced on $K(-)$ by a map in $\mathrm{Cat}^{\mathrm{perf}}$. Moreover, each of the categories involved have an 'infinite full exceptional collection'(IFEC): they are given by representables in both categories, and the functor l_0 takes IFEC of one to another. So we can apply the following technical lemma and win. $\square$

Definition 10. Let's recall that for a category $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$, a full exceptional collection (FEC) is a finite set of objects $E = \{X_\alpha : 0 \leqslant \alpha \leqslant n\}$ such that

- $\mathrm{map}(X_\alpha, X_\beta) = 0$ for all $\alpha > \beta$.
- $\mathrm{map}(X_\alpha, X_\alpha) = \mathbb{S}$.
- The inclusion of the span on each object $\langle X_\alpha \rangle$ into $\mathcal{C}$ admits a right adjoint.
- The span $\langle X_\alpha : 0 \leqslant \alpha \leqslant n\rangle$ is all of $\mathcal{C}$.

where by span we mean the idempotent completion of the stable subcategory spanned by a collection of objects.[1] An infinite full exceptional collection (IFEC) on $\mathcal{C}$ is a set of objects $E = \{X_\alpha : \alpha \in \mathbb{R}\}$ indexed by real numbers such that the span $\langle X_\alpha : \alpha \in \mathbb{R}\rangle = \mathcal{C}$ and for each finite subset $S \subseteq \mathbb{R}$, $\{X_\alpha : \alpha \in S\}$ is a full exceptional collection of $\langle X_\alpha : \alpha \in S\rangle$.

Lemma 11. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor in $\mathrm{Cat}^{\mathrm{perf}}$. If $\mathcal{C}$ has an IFEC and F takes this IFEC to an IFEC in $\mathcal{D}$, then F induces isomorphism on $K(-)$.*

Proof. Using that $K(-)$ commutes with filterted colimit in $\mathrm{Cat}^{\mathrm{perf}}$, one reduces to the following claim: If $G : \mathcal{E} \to \mathcal{F}$ is in $\mathrm{Cat}^{\mathrm{perf}}$ and G takes a finite full exceptional collection of $\mathcal{E}$ to a finite full exceptional collection of $\mathcal{F}$. Then G induces isomorphism on $K(-)$. This is left to the reader as an exercise. $\square$

This finishes the proof. Now we take advantage of reader's attention and speculate a little bit:

Remark 12. Two of the ideas in the proof could be motivated through homological mirror symmetry of toric varieties. We remind the reader about homological mirror symmetry for projective line in point 0 below, then present these motivations.

0. The torus-equivariant homological mirror symmetry of projective line provides an equivalence of the following categories:

$$\mathrm{QCoh}(\mathbb{P}^1/\mathbb{G}_m) \xrightarrow{\cong} \mathrm{Cons}_P(\mathbb{R}^1;\mathrm{Sp})$$

   where left hand side is quasi-coherent modules over the flat toric scheme defined over $\mathbb{S}$ and right hand side is the full subcategory of $\mathcal{S}hv(\mathbb{R}^1;\mathrm{Sp})$ spanned by sheaves locally constant away from $\mathbb{Z} \subseteq \mathbb{R}^1$. This subcategory inherits the symmetric monoidal structure of convolution product and the equivalence upgrades to a symmetric monoidal one.

[1] Note that our definition is weaker than the usual non-commutative geometry terminology.

1. The idempotent algebra θ_+ is the image of idempotent algebra

$$j_*\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m} \in \mathrm{QCoh}(\mathbb{P}^1/\mathbb{G}_m)$$

under the equivalence above. The diagram in Proposition 5 is the mirror picture (albeit in the bigger category $\mathcal{S}\mathrm{hv}$ instead of Cons) of the Zariski descent diagram for QCoh, with $\mathbb{P}^1/\mathbb{G}_m$ being covered by two pieces of $\mathbb{A}^1/\mathbb{G}_m$.

2. One can ask what is the algebro-geometric mirror of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$, and Dmitry Vaintrob provided an answer in [7]: it is (an equivariant flavor of) the universal compactification of the torus $\mathbb{G}_m$, implemented by almost mathematics. To be more precise, given a toric fan, instead of producing a scheme glued from pieces of monoid algebra, one can define directly, for each cone, a category of almost modules then glue them. One declares the gluing outcome to be the category of quasicoherent sheaves of the universal (partial) compactification corresponding to the fan. If we apply this construction to the fan of $\mathbb{A}^1$ we arrive at an almost mathematics situation where the Verdier localisation sequence for the almost module category is the mirror picture of Proposition 7. If we do this to the fan of $\mathbb{P}^1$ we get an algebro-geometric mirror to $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$:

$$\mathrm{QCoh}^a(\mathbb{P}^{1,\mathrm{nov}}/\mathbb{G}_m^{\mathrm{nov}}) \cong \mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$$

which is even a symmetric monoidal equivalence where we use tensor product of quasi-coherent sheaves on the left and convolution product of sheaves on the right. For actual definition (with coefficient in ordinary rings), read around [7, Theorem 4]. The construction as in point 0 will lift this equivalence to spectral coefficient with symmetric monoidal structure. The computation of Proposition 6 can be performed (actually more transparently) on the mirror side.

As a result we also obtain continuous K-theory for (an equivariant flavor of) universal compactification $\mathrm{QCoh}^a(\mathbb{P}^{1,\mathrm{nov}}/\mathbb{G}_m^{\mathrm{nov}})$. Another feature of this computation is being symmetric monoidal: the diagram in Proposition 5 is a diagram in $\mathrm{CAlg}(\mathrm{Pr}^L)$. (The sheaf category with convolution product is is only locally rigid monoidal.)

2 Picard group of sheaves on $\mathbb{R}^1$ with convolution

Elaborating the ideas above, we provide a case study of a question asked by Oscar Harr and Branko Juran in Copenhagen homotopy theory problem solving seminar. We are very grateful to them for the question and discussion.

Question 13. For a topological group G, what are tensor invertible objects in $\mathcal{S}\mathrm{hv}(G;\mathrm{Sp})$ equipped with convolution tensor product?

For each $g \in G$, the skyscraper sheaf $S_{\{g\}}$ provides such a tensor invertible object. Somehow surprisingly, there are many more of them in the case of $G = \mathbb{R}^1$, as predicted by the toric mirror symmetry equivalence. For example we expect the following :

Conjecture 14. The Picard groupoid[2] of sheaves on the real line could be computed as

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) = \mathbb{R} \times \mathbb{R} \times \mathrm{Pic}(\mathrm{Sp}).$$

We cannot prove this directly yet. This subsection explains some tentative ideas in resolving the problem. Then we will explain a proof of above conjecture with coefficient Sp replaced by Mod_k where k is a field (so we can do homological algebra).

Remark 15. One can explicitly provide a map of groups

$$\mathbb{R} \times \mathbb{R} \longrightarrow \pi_0(\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})))$$

as follows. If $a < b$, then (a, b) is sent to the representable sheaf on the open interval (a, b) shifted up by 1 homologically. If $a \geqslant b$, then (a, b) is sent to the constant sheaf on the closed interval $[b, a]$. To lift it to a group map before taking π_0 takes some more effort (but can be done).

The idea we have about this computation is utilizing the fact that Proposition 5 provides a pullback diagram of symmetric monoidal categories. Applying $\mathrm{Pic}(-)$ gives

$$\begin{tikzcd}
\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \arrow[r] \arrow[d] & \mathrm{Pic}(\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})) \arrow[d] \\
\mathrm{Pic}(\mathcal{S}\mathrm{hv}^-(\mathbb{R}^1;\mathrm{Sp})) \arrow[r] & \mathrm{Pic}(\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp}))
\end{tikzcd}$$

which is still a pullback diagram since $\mathrm{Pic}(-)$ preserves limit. As in computation for K-theory, the down right corner reads

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp})) = \mathrm{Pic}(\mathrm{Sp}).$$

Now the problem is reduced to showing that

Conjecture 16. The Picard group of $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})$ can be computed as

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})) \cong \mathbb{R} \times \mathrm{Pic}(\mathrm{Sp}).$$

Moreover, the map in the above diagram

$$\mathbb{R} \times \mathrm{Pic}(\mathrm{Sp}) \cong \mathrm{Pic}(\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})) \longrightarrow \mathrm{Pic}(\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp})) = \mathrm{Pic}(\mathrm{Sp})$$

can be identified as projection to the $\mathrm{Pic}(\mathrm{Sp})$ factor.

Remark 17. It is easy to cook up a map from RHS to LHS via adjunction. The work lies in showing that the map is an equivalence. Or rather, explicitly computing π_0 of LHS.

By symmetry the same statement hold for $\mathcal{S}\mathrm{hv}^-$. It's obvious that this would imply Conjecture 14. To prove Conjecture 16, we can inspect the Verdier sequence Diagram 1, or equivalently Diagram 2. Here is a related computation.

[2]By $\mathrm{Pic}(-)$ we mean the maximal groupoid on tensor invertible objects, it inherits the tensor structure and we take it as an $\mathbb{E}_\infty$-group.

Proposition 18. The Picard group of $\mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp})$ (with Day convolution) can be computed as

$$\mathrm{Pic}(\mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp})) \cong \mathbb{R} \times \mathrm{Pic}(\mathrm{Sp}).$$

Proof. It suffices to compute π_0 of Pic. The functor of taking left Kan extension to a point is symmetric monoidal, hence for any tensor invertible element F one must have that the colimit of F (which is a coproduct) is a shift of $\mathbb{S}$. This forces F to be supported at only one point with value a shift of $\mathbb{S}$. On the other hand, each skyscraper presheaf of a shift of $\mathbb{S}$ is tensor invertible. Thus we have classified isomorphism classes of tensor invertible objects in $\mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp})$. $\square$

Proposition 19. The Picard group of $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})$ (with Day convolution) can be computed as

$$\mathrm{Pic}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})) \cong \mathbb{R} \times \mathrm{Pic}(\mathrm{Sp}).$$

Proof. We will reduce to previous case. Recall that one already has a functor of taking associated graded: this functor is conservative on compact objects. Combining this with the fact that $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})$ is a rigid monoidal category, one learns that taking associated graded supplies an injection on π_0 of $\mathrm{Pic}(-)$, hence we have classified isomorphism classes of tensor invertible objects in $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})$. $\square$

It would be nice to provide further arguments along this line to prove the statement for $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Sp})$. However we run into the problem that $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Sp})$ is not rigid monoidal, so the dualizable objects need not be compact. Nevertheless one can be optimistic and hope for the following to be true: an object in the sheaf category is dualizable if and only if it lives in the smallest stable subcategory generated by representables.[3] We don't know how to prove this.

Now we take a turn to work with field coefficients and do (almost) homological algebra. The goal is to explain the following:

Proposition 20. Fix a field k. The Picard groupoid of $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ can be computed as

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)) \cong \mathbb{R} \times \mathrm{Pic}(\mathrm{Mod}_k).$$

Moreover, the map in the above diagram

$$\mathbb{R} \times \mathrm{Pic}(\mathrm{Mod}_k) \cong \mathrm{Pic}(\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)) \longrightarrow \mathrm{Pic}(\mathcal{S}\mathrm{hv}^{0}(\mathbb{R}^1; \mathrm{Mod}_k)) = \mathrm{Pic}(\mathrm{Mod}_k)$$

can be identified as projection to the $\mathrm{Pic}(\mathrm{Mod}_k)$ factor.

Strategy 21. We take the following steps to prove Proposition 20.

1. It suffices to compute $\pi_0\mathrm{Pic}$, the isomorphism classes of tensor-invertible objects.
2. There is a nice t-structure on the category $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$. One can classify tensor invertible objects which also sit in the heart.
3. Given an arbitrary tensor invertible object, we will argue using Kunneth spectral sequence that it has to sit in only one degree. This depends on the facts (1) the category has Tor-dimension 1 (2) one can classify subobjects of the unit (in the heart).

[3] It was suggested by Sasha Efimov.

4. Then we are done.

We start with some homological algebra preparations.

Definition 22. For future reference, we note that the category of presheaves

$$\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}};\mathrm{Mod}_k)$$

has a standard, pointwise t-structure which is left and right complete. As in Diagram 2 we think of $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ as a full subcategory of $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}};\mathrm{Mod}_k)$ and the inclusion has a right adjoint:

$$I \otimes - : \mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}},\mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k).$$

To define a t-structure on $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ we will use the following fact:

Lemma 23. *An object $\mathcal{F}$ of $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}};\mathrm{Mod}_k)$ is in $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ if and only if each of its homology object is in $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$.*

Proof. Note that an object $\mathcal{F}$ in $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}};\mathrm{Mod}_k)$ lies in $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ if and only if for each $a \in \mathbb{R}$ we have the following equivalence

$$\operatorname*{colim}_{b>a} \mathcal{F}(b) \xrightarrow{\cong} \mathcal{F}(a)$$

in Mod_k, which can be tested on taking homology. Since filtered colimits in Mod_k are exact, the above is an equivalence if and only if for each homology object $H_n(\mathcal{F})$ the same holds. Hence we are done. □

Definition 24. We define a t-structure on $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ as follows. An object in $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ is connective (coconnective) for the t-structure if and only it is connective (coconnective) as an object in $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}};\mathrm{Mod}_k)$ for the standard pointwise t-structure. It follows from the lemma above that this defines a t-structure on $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$.

Corollary 25. The following are true for the t-structure we specified:

1. The truncation functor is the same as the truncation functor for the standard t-structure on $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}};\mathrm{Mod}_k)$.

2. The t-structure on $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ is left and right complete.

3. The convolution product of two connective objects in $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ stays connective.

Proof. They follow from the definition. □

We can talk about the Whitehead filtration on an object $X \in \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$: it is the filtered object

$$\{\tau_{\geqslant n}X \xrightarrow{f_n} \tau_{\geqslant n-1}X\}_{n\in\mathbb{Z}}$$

that filters X. In other words, the limit along n is 0 and the colimit along n is X: this follows from the fact that the t-structure is left and right complete. For two such objects X and Y in $\mathcal{S}hv^+(\mathbb{R}^1;\mathrm{Mod}_k)$, one can tensor the Whitehead filtrations and get a filtration on $X\otimes Y$. Again by the left and right completeness, this filtration has limit 0 and colimit $X\otimes Y$. We call the associated spectral sequence as Künneth spectral sequence.

Corollary 26. The Künneth spectral sequence for the tensor products on $\mathcal{S}hv^+(\mathbb{R}^1;\mathrm{Mod}_k)$ always converges. Moreover, if two objects X and Y are in the heart $\mathcal{S}hv^+(\mathbb{R}^1;\mathrm{Mod}_k)^\heartsuit$, then their tensor product lies in homological degree $[0,1]$. It follows that the Künneth spectral sequence degenerates at page 2.

Proof. It remains to explain the homological bound $\leqslant 1$. Let's take two objects X and Y in the heart $\mathcal{S}hv^+(\mathbb{R}^1;\mathrm{Mod}_k)^\heartsuit$ and try to evaluate

$$X\otimes Y(0)=\operatorname*{colim}_{a+b\geqslant 0} X(a)\otimes Y(b)$$

where the later $\otimes$ is the tensor product in Mod_k. Now the poset $\{(a,b): a+b\geqslant 0\}$ we take colimit over can be presented as the following filtered colimit of posets:

$$\{(a,b): a+b\geqslant 0\}=\operatorname*{colim}_{S\subseteq\mathbb{R},\text{finite}} \mathrm{Stair}(S)$$

where we define the staircase shaped poset associated to S to be

$$\mathrm{Stair}(S):=\bigcup_{s\in S}\{(a,b): a\geqslant s; b\geqslant -s\}.$$

By descent of colimits, we can compute $X\otimes Y(0)$ as a filtered colimit over objects of the form

$$\operatorname*{colim}_{\mathrm{Stair}(S)} X(a)\otimes Y(b).$$

Each of these can be explicitly computed as a pushout of objects in $\mathrm{Mod}_k^\heartsuit$, so in particular lives in homological degree $[0,1]$, and a filtered colimit of these still lives in homological degree $[0,1]$. The same argument works when evaluating each $X*Y(r)$, so we learn that $X\otimes Y$ lives in homological degree $[0,1]$. $\square$

We will also need the following classification:

Corollary 27. Recall that $\mathcal{S}hv^+(\mathbb{R}^1;\mathrm{Mod}_k)$ has $\mathbb{1}:=I$ as the tensor unit. It lies in the heart and each subobject of $\mathbb{1}$ is either 0 or of the form $I\otimes h_a$ for some $a\in\mathbb{R}$ where h_a is (k-linearized) representable presheaf on $a\in\mathbb{R}$.

This finishes all the homological algebra preparation. A crucial consequence is the following:

Proposition 28. Every convolution invertible object in $\mathcal{S}hv^+(\mathbb{R}^1;\mathrm{Mod}_k)$ lives in single homological degree.

Proof. Fix a convolution invertible object X, let's try to compare $\tau_{\geqslant 0}X$ to X. One can tensor up the canonical map

$$\tau_{\geqslant 0}X \longrightarrow X$$

with X^{-1} and get a map

$$\tau_{\geqslant 0}X \otimes X^{-1} \longrightarrow \mathbb{1}.$$

One can inspect the Künneth spectral sequence for the tensor products $\tau_{\geqslant 0}X \otimes X^{-1}$ and $X \otimes X^{-1}$. By the fact that it degenerates at page 2, we learn that the map

$$\tau_{\geqslant 0}X \otimes X^{-1} \longrightarrow \mathbb{1}$$

is injective on each homology object, which in particular implies that $\tau_{\geqslant 0}X \otimes X^{-1}$ lies in the heart and is a subobject of $\mathbb{1}$. Now by the classification, it is either 0 or $I \otimes h_a$ for some $a \in \mathbb{R}$. In the first case we learn that

$$\tau_{\geqslant 0}X = 0,$$

and in the latter case we learn that

$$\tau_{\geqslant 0}X = X \otimes h_a.$$

Note that tensoring with h_a only makes a shift in the $\mathbb{R}$-grading, so this in particular implies that X is connective, or

$$\tau_{<0}X = 0.$$

From this discussion, we learn that X either sits in homological degree $[0, +\infty)$ or $(-\infty, -1]$. But the same argument works verbatimly when one compares $\tau_{\geqslant n}X$ and X. Hence we learn that for each $n \in \mathbb{Z}$, X either sits in homological degree $[n, +\infty)$ or $(-\infty, n-1]$. This forces X to sit in only one homological degree. $\square$

From this we already learn that up to a homological shift every convolution invertible object lives in the heart. So it remains to classify the convolution invertible objects in the heart of $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)$.

Lemma 29. *Let $X \in \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)^\heartsuit$ be convolution invertible, then X^{-1} is also in the heart. Moreover, $X \otimes (-)$ takes $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)^\heartsuit$ to $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)^\heartsuit$.*

Proof. Taking colimit along $\mathbb{R}^{\mathrm{op}}_{\leqslant}$ gives a symmetric monoidal functor

$$\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k) \longrightarrow \mathrm{Sp}$$

and we learn that

$$\operatorname*{colim}_{\mathbb{R}^{\mathrm{op}}_{\leqslant}} X \otimes \operatorname*{colim}_{\mathbb{R}^{\mathrm{op}}_{\leqslant}} X^{-1} \cong k.$$

In particular this implies $H_0(X^{-1}) \neq 0$, and hence X^{-1} is in the heart. Now for some object $Z \in \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)^\heartsuit$ we want to argue that

$$H_1(X \otimes Z) = 0.$$

For this one notes that the only term that contributes to H_0 of $X^{-1} \otimes (X \otimes Z)$ is $X^{-1} \otimes H_0(X \otimes Z)$. Hence we must have $X^{-1} \otimes H_1(X \otimes Z) = 0$ which implies $H_1(X \otimes Z) = 0$. $\square$

Proposition 30. Let $X \in \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)^\heartsuit$ be a convolution invertible object, then X is of the form $I \otimes h_a$ for some k-linearized representable presheaf h_a.

Proof. For an object

$$Y \in \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)^\heartsuit,$$

let $\mathrm{sub}(Y)$ be the category of subobjects of Y. More precisely, it is the full subcategory of

$$\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)^\heartsuit_{/Y}$$

spanned by monomorphisms. Note that this is a poset. Making use of previous classification of subobjects of $\mathbb{1}$, one learns that $\mathrm{sub}(\mathbb{1})$ is a totally ordered poset. Since $X \otimes (-)$ is an auto-equivalence of $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)^\heartsuit$, we must have

$$\mathrm{sub}(X) \cong \mathrm{sub}(\mathbb{1}).$$

Now we learn that for each tensor invertible X, its category of subobjects is also a totally ordered poset. From this one can easily deduce that it has to be of the form $I \otimes h_a$. □

The proof of Proposition 20 is now complete. One immediately gets:

Corollary 31. The Picard groupoid of sheaves on $\mathbb{R}^1$ valued in Mod_k with convolution is

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Mod}_k)) = \mathbb{R} \times \mathbb{R} \times \mathrm{Pic}(\mathrm{Mod}_k).$$

3 Picard group of sheaves on $\mathbb{R}^n$ with convolution

Remark 32. Here is a speculation: one can ask what happens for $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k)$. The set of closed convex bodies in $\mathbb{R}^n$ form a commutative monoid under Minkowski sum:

$$M_n := (\{\text{closed convex bodies in } \mathbb{R}^n\}, +)$$

and one can make a map (of symmetric monoidal categories)

$$M_n \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k)$$

sending each closed convex body to lower shriek of its dualizing sheaf. We expect such sheaf to be convolution invertible and thus there is an induced map on group completion

$$M_n^{\mathrm{gp}} \times \mathrm{Pic}(\mathrm{Mod}_k) \to \mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k)).$$

This map is injective and provides many convolution invertible objects. Note that for $n = 1$ we have

$$M_n^{\mathrm{gp}} = \mathbb{R} \times \mathbb{R}.$$

Remark 33. We formulate a more precise guess for $\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k))$. Be warned that this part contains no proof! To do so we need a functor of Radon transform (compare the work of Honghao Gao [2]). Consider the following correspondence

$$\begin{tikzcd} & K \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{R}^n & & S^{n-1}\times\mathbb{R}^1 \end{tikzcd}$$

where K is the Radon kernel (Beware that we are using a different kernel than [2].)

$$K := \{(x,n,r) : \langle x,n\rangle < r\} \subseteq \mathbb{R}^n \times S^{n-1} \times \mathbb{R}^1$$

where we think of S^{n-1} as unit sphere in the dual vector space of $\mathbb{R}^n$. One gets a functor

$$R := q_!p^* : \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k).$$

that we will call Radon transform. Moreover, the functor R lands in the full subcategory

$$\mathcal{S}\mathrm{hv}^+(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k) \subseteq \mathcal{S}\mathrm{hv}(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k)$$

on the objects whose singular support along $\mathbb{R}^1$ direction are non-negative (this could also be formulated as being a module for some idempotent algebra). We will take R as a functor

$$R : \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}^+(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k).$$

It's helpful to take some sheaf F on $\mathbb{R}^n$ and compute the restriction of $R(F)$ along some $\{n\}\times\mathbb{R}^1$. From this perspective the functor R is a family version of localization at idempotent algebra of dualizing sheaf on $\{x : \langle x,n\rangle < 0\} \subseteq \mathbb{R}^n$. Hopefully the following claims make sense:

1. The functor R could be endowed with a symmetric monoidal structure, when we equip $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k)$ with convolution product and $\mathcal{S}\mathrm{hv}^+(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k)$ with fiberwise convolution product.

2. There is a pullback square of symmetric monoidal categories (in particular all functors appearing in the following square are symmetric monoidal)

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k) \arrow[r, "R"] \arrow[d, "\Gamma_c"] & \mathcal{S}\mathrm{hv}^+(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k) \arrow[d, "\pi_!"] \\ \mathcal{S}\mathrm{hv}(\mathrm{pt};\mathrm{Mod}_k) \arrow[r, "p^*"] & \mathcal{S}\mathrm{hv}(S^{n-1};\mathrm{Mod}_k) \end{tikzcd}.$$

   Let's explain a bit more about the notation. Both sheaf categories on the bottom are equipped with pointwise tensor product, with the functor τ^* being the pullback functor along projection

   $$\tau : S^{n-1} \to \mathrm{pt}.$$

   The functor Γ_c takes an sheaf F to its compactly supported global section (i.e. shriek-pushforward to the point). The functor $\pi_!$ is lower-shriek along the projection

   $$\pi : S^{n-1}\times\mathbb{R}^1 \to S^{n-1}.$$

   Compare Proposition 5 when $n = 1$.

3. The Picard groupoid of $\mathcal{S}\mathrm{hv}^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k)$ equipped with fiberwise convolution product could be computed as

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k)) \cong \mathrm{Func}^{\mathrm{cts}}(S^{n-1}; \mathbb{R}) \times \mathrm{Pic}(\mathcal{S}\mathrm{hv}(S^{n-1}; \mathrm{Mod}_k)).$$

Here we take $\mathrm{Func}^{\mathrm{cts}}$ as the group of continuous function under addition. Putting all these together, we learn that

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k)) \cong \mathrm{Func}^{\mathrm{cts}}(S^{n-1}; \mathbb{R}) \times \mathrm{Pic}(\mathrm{Mod}_k).$$

Under this identification, the invertible objects indexed by convex bodies are sent to their support function. Note that not every continuous function on S^{n-1} can be realized as difference of support functions of convex body.

Remark 34. Here are some thoughts on how to produce proofs for the above claims (but none of these are actual proofs):

1. The construction of the symmetric monoidal structure should be some six-functor nonsense.
2. The coherence of the diagram should again come from six-functor nonsense. Note that we abused notation for R since the functor R lands in $\mathcal{S}\mathrm{hv}^+$ by previous pointwise observation. We observe that each functor in this diagram has an explicit right adjoint. In particular for R we can write down another kernel

$$\overline{K} := \{(x, n, r) : \langle x, n \rangle \leqslant r\} \subseteq \mathbb{R}^n \times S^{n-1} \times \mathbb{R}^1$$

and the corresponding functor

$$\overline{R} : \mathcal{S}\mathrm{hv}^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k).$$

We claim that $\overline{R}$ is a right adjoint of R. Moreover the composition is

$$R \circ \overline{R} \cong \mathrm{Id}$$

on $\mathcal{S}\mathrm{hv}^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k)$ via direct computation. For the other direction, use the explicit right adjoints to compute.

3. Take a convolution invertible object $\mathcal{F}$ in $\mathcal{S}\mathrm{hv}^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k)$. First off we know that there exists a function $f : S^{n-1} \to \mathbb{R}$ such that

$$\{p \in S^{n-1} \times \mathbb{R}^1 : \mathcal{F}_p \neq 0\} = \{(s, r) : r < f(s)\}.$$

Why should this f be continuous? We might assume its lower-shriek is the constant sheaf $\underline{k}$ on S^{n-1}. Now by adjunction and pointwise observation, we should learn that $\mathcal{F}$ is a subobject of a shift of constant sheaf $\underline{k}$, in particular the set on the left should be open. This should force f to be lower-semi-continuous. Now the same argument applied to $\mathcal{F}^{-1}$ implies that f is continuous. And all this forces $\mathcal{F}$ to be (a shift of) lower shriek of the constant sheaf on the right side.

4 Why wild Betti sheaves are cool

The paradigm of (categorical) Fourier transform aims to produce a symmetric monoidal equivalence between categories of the form

$$\mathcal{F} : (\mathcal{C}, *) \longrightarrow (\mathcal{C}^{\vee}, \otimes)$$

where $\mathcal{C}$ and $\mathcal{C}^{\vee}$ are categories constructed out of dual geometric data. Moreover, the $*$-'convolution' tensor product on $\mathcal{C}$ is intertwined under $\mathcal{F}$ with the $\otimes$-'ordinary' tensor product. For our purpose, it would be certainly helpful to have the following symmetric monoidal equivalence for a finite dimensional real vector space V:

$$\mathcal{F} : (\mathcal{S}\mathrm{hv}(V; \mathrm{Mod}_k), *) \longrightarrow (\mathcal{S}\mathrm{hv}(V^{\vee}; \mathrm{Mod}_k), \otimes).$$

Such equivalence immediately teaches us how to compute the Picard groupoid on the left, since we know every tensor invertible object on the right is locally constant. However, as the previous computation with $V = \mathbb{R}^1$ suggests, this is too good to be true. We will see that wild Betti sheaves come to rescue.

In [6], Peter Scholze explained the following universal way to fix the discrepancy: one can enlarge the coefficient category Mod_k to something exotic (i.e. non-compactly-generated) - the category W[4] of complete continuous presheaves on $\mathbb{R}_{\leqslant}$ valued in Mod_k [6, Definition 2.2]. The assignment

$$X \longmapsto \mathcal{S}\mathrm{hv}(X; W)$$

inherits a structure of six operations. Now one indeed has a Fourier transform for wild Betti sheaves as an equivalence between symmetric monoidal categories [6, Theorem 4.1]

$$\mathcal{F} : (\mathcal{S}\mathrm{hv}(V; W), *) \longrightarrow (\mathcal{S}\mathrm{hv}(V^{\vee}; W), \otimes).$$

One can now pre-compose with the inclusion of ordinary sheaves

$$\iota : (\mathcal{S}\mathrm{hv}(V; \mathrm{Mod}_k), *) \longrightarrow (\mathcal{S}\mathrm{hv}(V; W), *)$$

to obtain a symmetric monoidal and fully faithful functor

$$F \circ \iota : (\mathcal{S}\mathrm{hv}(V; \mathrm{Mod}_k), *) \longrightarrow (\mathcal{S}\mathrm{hv}(V^{\vee}; W), \otimes).$$

We are now faced with the following two questions:

1. Compute the Picard groupoid of the symmetric monoidal category on the right.
2. Describe the essential image of the above functor.

Our goal is to explain that each of the questions has a good answer. In the previous section, we have essentially supplied all the ingredients for answering point 1 - so we start from here.

[4]The careful reader will note that our orientation of $\mathbb{R}$ as a poset is opposite to the cited definition.

Proposition 35. Let X be a locally compact Hausdorff space and $\mathcal{S}\mathrm{hv}(X; \mathcal{W})$ be the category of wild Betti sheaves (with base field k) on X equipped with the ordinary tensor product. We have

$$\pi_0\mathrm{Pic}(\mathcal{S}\mathrm{hv}(X; \mathcal{W})) \cong \mathrm{Func}^{\mathrm{cts}}(X) \times \pi_0\mathrm{Pic}(\mathcal{S}\mathrm{hv}(X; \mathrm{Mod}_k)).$$

Let's treat the special case when X is a point first. We will explain how the proof for Proposition 20 works here up to small modification. To start with, here is a useful way to think about $\mathcal{W}$.

Lemma 36. *There are two idempotent algebras $\underline{k}_{\{0\}}$ (the skyscraper presheaf supported at 0) and $\underline{k}$ in $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}; \mathrm{Mod}_k)$ (the constant presheaf). Moreover:*

1. *The tensor product of $\underline{k}_{\{0\}}$ with $\underline{k}$ is 0. In particular, their union is the Cartesian product $A := \underline{k}_{\{0\}} \times \underline{k}$.*
2. *An object $\mathcal{F} \in \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}; \mathrm{Mod}_k)$ lies in the full subcategory $\mathcal{W}$ if and only if $\underline{\mathrm{map}}(A, \mathcal{F}) = 0$.*
3. *The ideal $I_A := \mathrm{fib}(\mathbb{1} \to A)$ can be identified with the constant presheaf $\underline{k}_{[0,+\infty)}[-1]$ supported on $[0, +\infty)$.*

Proof. We leave the details to the reader: these are direct computations. □

It follows that one can embed the category $\mathcal{W}$ differently - as the full subcategory of $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Mod}_k)$ consisting of objects which are killed by $- \otimes A$. The tensor unit of $\mathcal{W}$ is identified with I_A and the tensor product on $\mathcal{W}$ is identified with the tensor product on $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Mod}_k)$.[5]

Claim 37. Each of the statements from Lemma 23 to Proposition 30 remains true, if one replaces $\mathcal{S}\mathrm{hv}^+$ by $\mathcal{W}$ and I by I_A there.

Some of the statements needs further clarification. For Corollary 27, the unit I_A of $\mathcal{W}$ lives in -1 shift of the heart. Every subobject of I_A is like a constant sheaf on a half open interval $[0, r)$. For Lemma 29, the statement should be that if a convolution invertible object X lives in -1 shift of the heart, then so does its inverse. Moreover, tensoring with X takes heart to heart.

We will not repeat all the arguments, the point is that all these statements are about objects in $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Mod}_k)$ and their convolution products and the standard t-structure. The only point where we have used something about $\mathcal{S}\mathrm{hv}^+$ is the classification of subobjects of the unit in the heart stated in Corollary 27 whose proof depends on the continuity of objects in $\mathcal{S}\mathrm{hv}^+$ (the characterization used in the proof of Lemma 23). Such property also holds for objects in $\mathcal{W}$. The proof of Lemma 29 also needs to be modified. The crucial claim to make is then if A and B lives in the heart and also in $\mathcal{W}$, then $\pi_1(A \otimes B)$ has to be none zero. We end up with the following proposition.

Proposition 38. The tensor invertible objects in $\mathcal{W}$ are classified by

$$\pi_0\mathrm{Pic}(\mathcal{W}) \cong \mathbb{R} \times \pi_0\mathrm{Pic}(\mathrm{Mod}_k).$$

[5]The reader is advised to compare this identification with what happens in the proof of Proposition 20: there we identified the category of $\mathcal{S}\mathrm{hv}^+$ with the subcategory of $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Mod}_k)$ killed by $- \otimes \underline{k}_0$. The upshot is that, under such presentation, the tensor product is underlying - there is no need for further completion.

We turn back to the general case:

Proof of Proposition 35. To treat the general case, we shift back to the perspective in [6] viewing W as continuous complete $\mathbb{R}$-filtered k-modules. We might embed W into $\mathcal{S}\mathrm{hv}(\mathbb{R}^1; \mathrm{Mod}_\mathrm{k})$ via the right adjoint functor to the localization. It follows that we can think of $\mathcal{S}\mathrm{hv}(X; W)$ as a full subcategory of $\mathcal{S}\mathrm{hv}(X \times \mathbb{R}^1; \mathrm{Mod}_\mathrm{k})$. At this point, we can run the argument from the very end of Section 3 and win. $\square$

We have fulfilled our promise about the first question. Now we move on to the second one. To do so, we need a useful way to characterize the image of embedding of ordinary sheaves into wild Betti sheaves. The crucial observation is the following.

Observation 39. *The embedding of* Mod_k *into* W *sends every object* $M \in \mathrm{Mod}_\mathrm{k}$ *to* $M \otimes h_0$*. Every object of such form is invariant under the* $\mathbb{R}_{>0}$ *scaling action on* W *defined in [6, Section 6]. By 'invariant' we mean that for each* $a \in \mathbb{R}_{>0}$ *we have the action map*

$$a\cdot : W \longrightarrow W$$

and there always exists an isomorphism

$$a \cdot (M \otimes h_0) \cong M \otimes h_0.$$

To make this idea precise, we take up the stacky approach.

Lemma 40. *Let* $\mathbb{R}$ *act additively on a locally compact Hausdorff space* X *and we form the quotient stack* $X/\mathbb{R}$ *(in your favorite topology, e.g. open covering). Then the pullback functor*

$$\pi^* : \mathcal{S}\mathrm{hv}(X/\mathbb{R}; \mathrm{Mod}_\mathrm{k}) \longrightarrow \mathcal{S}\mathrm{hv}(X; \mathrm{Mod}_\mathrm{k})$$

is fully faithful. Its image is precisely those sheaves on X *which are locally constant on each orbit of* $\mathbb{R}$ *action.*

Remark 41. As a side remark, such fact could be mind-twisting for people who think of objects living on the quotient stack as objects living on the total space plus equivariant structures - the lemma says equivariance for $\mathbb{R}$ is merely a property (instead of a structure)! This fact is somehow well-known in the literature for geometric representation theory (in the form for perverse sheaves or D-modules), and we take the opportunity to record the Betti form. The statement for other contractible group should also be true.

Proposition 42. Under the embedding

$$i : \mathcal{S}\mathrm{hv}(X; W) \longrightarrow \mathcal{S}\mathrm{hv}(X \times \mathbb{R}^1; \mathrm{Mod}_\mathrm{k}),$$

the image of the inclusion of ordinary sheaves

$$\mathcal{S}\mathrm{hv}(X; \mathrm{Mod}_\mathrm{k}) \longrightarrow \mathcal{S}\mathrm{hv}(X; W) \longrightarrow \mathcal{S}\mathrm{hv}(X \times \mathbb{R}^1; \mathrm{Mod}_\mathrm{k})$$

is precisely the collection of objects in $\mathcal{S}\mathrm{hv}(X \times \mathbb{R}^1; \mathrm{Mod}_\mathrm{k})$ which are

- in the image of i and,

- equivariant under the $\mathbb{R}_{>0}$ scaling action in the $\mathbb{R}^1$ direction.

It remains to study how the $\mathbb{R}_{>0}$-action interacts with Fourier transform.

Proposition 43. The Fourier transform intertwines the $\mathbb{R}_{>0}$ action through the coefficient on $\mathcal{S}\mathrm{hv}(V; W)$, and the $\mathbb{R}_{>0}$ action through diagonal on $\mathcal{S}\mathrm{hv}(V^\vee; W)$.

Proof. The span defining Fourier transform can be presented as

$$V \times \mathbb{R}^1 \longleftarrow V \times V^\vee \times \mathbb{R}^1 \longrightarrow V^\vee \times \mathbb{R}^1.$$

We equip left hand side with the $\mathbb{R}_{>0}$ action in $\mathbb{R}^1$ direction and equip right hand side with the diagonal $\mathbb{R}_{>0}$ action. It suffices to observe that

- one can put an $\mathbb{R}_{>0}$ action on $V \times V^\vee \times \mathbb{R}^1$ making both of the maps $\mathbb{R}_{>0}$ equivariant and
- the exponential sheaf exp thought of as a sheaf on $V \times V^\vee \times \mathbb{R}^1$ is $\mathbb{R}_{>0}$ equivariant.

It follows that the Fourier transform respect the $\mathbb{R}_{>0}$ equivariant objects. To be more precise, one can now produce a span of quotient stacks (by the $\mathbb{R}_{>0}$ action) refining the above span, and one can chase the push-pull diagram to prove that $\mathbb{R}_{>0}$ equivariant object is taken to $\mathbb{R}_{>0}$ equivariant object by Fourier transform. $\square$

We can now answer the second question.

Corollary 44. Under Fourier transform, the image of the inclusion of ordinary sheaves

$$\mathcal{S}\mathrm{hv}(V; \mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}(V; W) \xrightarrow{\mathcal{F}} \mathcal{S}\mathrm{hv}(V^\vee; W) \xrightarrow{i_{V^\vee}} \mathcal{S}\mathrm{hv}(V^\vee \times \mathbb{R}^1; \mathrm{Mod}_k)$$

is precisely those objects in $\mathcal{S}\mathrm{hv}(V^\vee \times \mathbb{R}^1; \mathrm{Mod}_k)$ which are

- in the image of $i_{V^\vee}$ and,
- equivariant under the $\mathbb{R}_{>0}$ scaling action, which acts diagonally in both $\mathbb{R}^1$ direction and $V^\vee$ direction.

From this, we learn that a convolution invertible ordinary sheaf on V is taken under the Fourier transform to a tensor invertible wild Betti sheaf on $V^\vee$, which is $\mathbb{R}_{>0}$ equivariant through the diagonal action. This means that the corresponding function $f \in \mathrm{Func}^{\mathrm{cont}}(V^\vee, \mathbb{R})$ is homogeneous. Such functions are in bijection with continuous functions on S^{n-1}! The concludes our computation of the Picard group.

Remark 45. We have the following funny consequence: the Fourier-Sato transform for conic ordinary sheaves is now a direct corollary from the Fourier transform for wild Betti sheaves. To see this, it suffices to note that conic sheaves on V are precisely those sheaves on V equivariant for the scaling action of $\mathbb{R}_{>0}$ on V. Under Fourier transform for wild Betti sheaves, two of the $\mathbb{R}_{>0}$ equivariant structures cutting out conic sheaves on V are exchanged with those for conic sheaves on $V^\vee$. We thus learn that the Fourier transform restricts to an equivalence between conic ordinary sheaves.

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