Toric Mirror Symmetry for Homotopy Theorists

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Abstract

We construct a symmetric monoidal functor relating 'torus-equivariant' quasi-coherent sheaves on toric varieties over the sphere spectrum to constructible sheaves of spectra on real vector spaces. This provides a spectral lift of a theorem of Fang-Liu-Treumann-Zaslow. We also apply techniques from higher algebra to derive some formal consequences.

Bendz, Wilhelm. *A young artist (Ditlev Blunck) considers a sketch in a mirror.*
1826, painting. Statens Museum for Kunst, København.

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1 Introduction

In the classcial study of smooth projective toric varieties over $\mathbb{C}$, there is a dictionary between ample line bundles and their moment polytopes as explained in [10, Section 3.4]. It was observed by Robert Morelli that vector bundles also fit into this dictionary. He proved in [26, Theorem 7] that there is an injective map from the torus-equivariant Grothendieck K-group of an n-dimensional smooth projective toric variety X to the set of $\mathbb{Z}$-valued constructible functions on the real vector space $\mathbb{R}^n$ spanned by the character lattice of the torus T:

$$K_0^T(X) \longrightarrow \mathrm{Fun}^{\mathrm{cons}}(\mathbb{R}^n; \mathbb{Z}).$$

It becomes a map of commutative rings if one equips the set of constructible functions with pointwise addition and convolution product. This map generalizes the original dictionary: it takes the class of an ample line bundle to the characteristic function on the moment polytope.

In Morelli's theorem, each side admits a natural categorification. On the left hand side, one replaces $K_0^T(X)$ by $D_T^b(X)$, the bounded derived category of T-equivariant coherent sheaves on X. On the right hand side, one replaces the ring of constructible functions on $\mathbb{R}^n$ by $D_{cc}^b(\mathbb{R}^n; \mathcal{S}_\Sigma)$, the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on $\mathbb{R}^n$ which are compactly supported and constructible (in the strong sense: the stalks have to be perfect) for a stratification $\mathcal{S}_\Sigma$. This stratification $\mathcal{S}_\Sigma$ comes from an affine hyperplane arrangement determined by toric fan Σ for X (see Definition 4.4.9). The work of Fang-Liu-Treumann-Zaslow [9, Theorem 1.1] constructs an fully faithful functor between dg-categories (named as coherent-constructible correspondence)

$$\kappa : D_T^b(X) \longrightarrow D_{cc}^b(\mathbb{R}^n; \mathcal{S}_\Sigma)$$

which recovers Morelli's theorem upon taking K_0. Furthermore, they provided a description of the image of κ in terms of singular support.

Turning to the other side of the story, one can expect to remove the additional data of 'T-equivariance'. Alexey Bondal has independently proposed in [4] that under mild assumptions, there should be a fully faithful functor from bounded derived category of coherent sheaves on X to bounded derived category of sheaves of $\mathbb{C}$-vector spaces on the topological torus

$$\overline{\kappa} : D^b(X) \longrightarrow D^b(\mathbb{R}^n/\mathbb{Z}^n),$$

whose image is constructible for specific stratification. As it turns out in [35], one can define a functor $\overline{\kappa}$ in a similar way as κ, and describe the image of $\overline{\kappa}$ in terms of singular support. This line of work was pursued further by [32, 40, 21].

In this note, we provide an exposition of this story in the context of spectral algebraic geometry. We carefully construct the functors in the play and explain how to extract formal consequences out of the equivalences, taking advantage of available technologies in higher algebra.

1.1 What is done in this note?

This note was initiated with the observation that on the 'constructible' side of the story there is an obvious lift to the sphere spectrum: instead of the bounded derived category of sheaves of

$\mathbb{C}$-vector spaces, we might work with the large categories of sheaves of spectra on a real vector space:

$$\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Sp})$$

and the convolution product is naturally defined on this category, thanks to the new advances in the yoga of six-functor. On the 'coherent' side, it is generally difficult to lift varieties to sphere spectrum. It is however straightforward to write down lifts of toric varieties since they are Zariski locally monoid schemes glued together along maps induced by maps of monoid. In fact given a toric fan Σ, one may define the flat toric scheme X_Σ over sphere spectrum equipped with action by flat torus $\mathbb{T}$. The main purpose of this note is to supply the following construction:

Theorem A. Let N be a lattice and Σ be a smooth projective fan in $N_\mathbb{R} := N \otimes \mathbb{R}$ (see Notation 3.1.1). Let M and $M_\mathbb{R}$ be the dual lattice and vector space. There exists a fully faithful, symmetric monoidal functor

$$\kappa : \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}),$$

where X_Σ is the flat toric scheme associated to Σ and $\mathbb{T} = \mathrm{Sp\acute{e}t}(\mathbb{S}[M])$ is a flat torus, both defined over the sphere spectrum. One can explicitly describe the image of this functor:

$$\mathrm{Im}(\kappa) = \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) \subseteq \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}).$$

On the right hand side is the subcategory of sheaves characterized by the following two conditions[1]:

- It is constructible[2] for the stratification $\mathcal{S}_\Sigma$ given by the affine hyperplane arrangement H_Σ, which is determined by 1-cones in the fan Σ

$$H_\Sigma := \{m + \sigma^\perp : m \in M, \sigma \in \Sigma(1)\}$$

- It has singular support contained in the conic Lagrangian Λ_Σ:

$$\Lambda_\Sigma := \bigsqcup_{m \in M; \sigma \in \Sigma} m + \sigma^\vee \times -\sigma \subseteq M_\mathbb{R} \times N_\mathbb{R} = T^* M_\mathbb{R}.$$

Remark 1.1.1. This note treats in particular the case of coefficient ring being $\mathbb{S}$, but we have used nothing about $\mathbb{S}$ other than $\mathbb{S} \in \mathrm{CAlg}(\mathrm{Sp})$ is a connective commutative ring spectrum. One can similarly develop a construction over other connective commutative ring spectrum, and if one works with $\mathbb{C}$ this recovers a large category version of [9]. On the other hand, such a lift to spectral coefficient is already hinted at implicitly in [9] and explicitly in [38]. However, construction of the symmetric monoidal structure on the functor seems new - even over the complex numbers. Note though that the compatibility of κ with convolution operation was formulated and used in [9] in the context of dg-categories.

We also provide compatibility of the functor κ with action of $\mathrm{QCoh}(B\mathbb{T})$ on both sides.

[1] It is possible to remove the assumption on constructibility once one has a good understanding of singular support in greater generality, see Warning 5.1.10.

[2] Unless specified, we always mean constructible in the weak sense: there will be no constraints on the size of the stalk.

Theorem B. The functor κ fits into a diagram of symmetric monoidal categories and symmetric monoidal functors:

$$\begin{tikzcd} \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \arrow[r, "\kappa"'] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_\mathbb{R};\mathrm{Sp}) \\ \mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\pi^*"] \arrow[r, "\cong"'] & \mathcal{S}\mathrm{hv}(M;\mathrm{Sp}). \arrow[u, "i_!"'] \end{tikzcd}$$

The functor π^* is $*$-pullback along the projection $\pi : [X_\Sigma/\mathbb{T}] \to B\mathbb{T}$. The functor $i_!$ is !-pushforward along the inclusion of the topological group $i : M \to M_\mathbb{R}$. We used implicitly the identification of symmetric monoidal categories

$$\mathrm{QCoh}(B\mathbb{T}) \cong \mathrm{Fun}(M;\mathrm{Sp}) \cong \mathcal{S}\mathrm{hv}(M;\mathrm{Sp})$$

coming from Theorem 3.3.10 and proof of Lemma 6.1.4.

The above Theorem A and B could be found in the note as a combination of Proposition 3.3.1, Proposition 4.3.4, Remark 3.3.9, Remark 4.3.7 and Corollary 5.3.4. From this, one may deduce some formal consequences. First of all, we apply the technique of de-equivariantization Corollary 6.1.9 and obtain the following:

Theorem C. There is a symmetric monoidal fully faithful functor

$$\overline{\kappa} : \mathrm{QCoh}(X_\Sigma) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}/M;\mathrm{Sp})$$

whose image is described by constructibility and singular support similar to above:

$$\mathrm{Im}(\overline{\kappa}) = \mathcal{S}\mathrm{hv}_{\overline{\Lambda}_\Sigma}(M_\mathbb{R}/M;\mathrm{Sp})$$

where the right hand side is the subcategory of sheaves constructible for $\overline{\mathcal{S}}_\Sigma$ and has singular support contained in $\overline{\Lambda}_\Sigma$, where $\overline{\mathcal{S}}_\Sigma$ ($\overline{\Lambda}_\Sigma$) is the image of $\mathcal{S}_\Sigma$ (Λ_Σ) under projection map $\pi : M_\mathbb{R} \to M_\mathbb{R}/M$ (see Corollary 6.1.8).

As an application, one obtains Beilinson's theorem on projective line (we also include proof for projective spaces) as in Example 6.2.1.

Theorem D. There is an equivalences of categories:

$$\mathrm{QCoh}(\mathbb{P}^1_\mathbb{S}) \cong \mathrm{Fun}(\bullet \rightrightarrows \bullet;\mathrm{Sp}).$$

The de-equivariantization in Theorem C could be thought of as performing base-change along the symmetric monoidal functor

$$\mathrm{colim} : \mathrm{Fun}(\mathbb{Z}^n;\mathrm{Sp}) \longrightarrow \mathrm{Sp}.$$

More generally, one can obtain a relative version of toric construction as in Definition 6.3.1 by base changing along other colimit-preserving symmetric monoidal functors out of $\mathrm{Fun}(\mathbb{Z}^n;\mathrm{Sp})$: in particular this recovers the result of the second named author with Pyongwon Suh [17] relating quasi-coherent sheaves on a toric fibration to a category of sheaves on the torus with twisted coefficient category, see Example 6.3.2.

We also provide a conceptual approach to the 'log-perfectoid mirror symmetry' of Dmitry Vaintrob, so that [37, Theorem 2] would hold over $\mathbb{S}$ with symmetric monoidal structure. See Remark 4.5.8 for the connection to his work on log quasi-coherent sheaves. This may serve as a motivation for Sasha Efimov's computation with continuous K-theory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ in [8]. See [2] for an expository account of these materials, where the first named author made some computation of Picard groupoid out of this with Robert Burklund.

1.2 Inspirations and technicalities

Needless to say, there have been numerous papers on this story and we could only mention an incomplete list of references in this introduction. We will now list some of them that inspired our project. Then we provide some justifications for our (unfortunately, long) writing here. Finally we briefly mention some of the technical details, which should be interesting to devoted readers.

Remark 1.2.1 (Proof ideas from the literature)**.** Most ideas of this paper have appeared in one way or another in the literature: The main proof method is rephrasing constructions of [9] in the context of large categories, $\mathbb{S}$-coefficient and with symmetric monoidal structures. The method of localization along idempotent algebras was used in [21] in the disguise of Tamarkin projector. The proof we presented for characterization of the image in terms of singular support is taken from [40]. Finally, the idea of applying de-equivariantization in this story was spelled out in [33].

Remark 1.2.2 (Dropping assumptions on smooth and projective)**.** The restriction on fan being smooth and projective is removed in [21]. But we don't pursue the generality as in there.

Remark 1.2.3 (Necessity of higher algebra)**.** It is clear that in this story of coherent-constructible correspondence, higher categorical techniques were needed in constructing the functors and characterizing images. Here we give a presentation without directly using model categories or dg-categories, of all the functors and categories. For comparison, it would be difficult to articulate the convolution product on the category of sheaves on a real vector space as a symmetric monoidal structure in terms of derived category of sheaves. This kind of difficulties would only add up when one goes to spectral coefficient. It appears to us that applying the language of higher algebra is the most convenient way to spell out details.

Remark 1.2.4 (Large categories)**.** In this note we systematically work with large (presentable stable) categories. This makes several constructions with 'generators' easier, as their counterparts in small categories are more subtle. Another reason to stick to this generality is due to our curiosity about $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})$: since it is not compactly generated, there is no obvious reason to hope for an algebro-geometric mirror object Y such that

$$\mathrm{QCoh}(Y) \xrightarrow{\cong} \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp}).$$

The sheaf category is however dualizable in the sense of [8] with a presentably symmetric monoidal structure of convolution. Inspired by utility of such categories in analytic geometry, one would hope to get better understanding of them. For example, Dmitry Vaintrob's result [37] constructed an almost mathematics object Y as a (symmetric monoidal) mirror for $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})$. In other words, his 'log-perfectoid' construction provides such Y with $\mathrm{QCoh}(Y) \cong \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})$. This should be thought of as algebraization of the sheaf category.

Remark 1.2.5 (Mirror symmetry over sphere spectrum)**.** It is widely expected that one can define a version of Fukaya categories over the sphere spectrum (see for example [1, 25, 29, 19] for various perspectives of works towards a definition). The $\mathbb{Z}$-linear equivalence between Fukaya category and (microlocal) sheaf category supplied by [11] should carry over to this new setting. In particular, modeling an $\mathbb{S}$-linear Fukaya category of $T^*M_{\mathbb{R}}$ (with stop given by Λ_Σ) as $\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp})$, our result may be interpested as an instance of $\mathbb{S}$-linear mirror symmetry.

Remark 1.2.6 (Higher structures from mirror symmetry)**.** Another reason for us to implement mirror symmetry over $\mathbb{S}$ is the hope that it would motivate constructions in category theory and homotopy theory. A wonderful example of such adventure is provided in [23] where Jacob Lurie made the observation that Waldhausen S-construction is corepresented by cosimplicial objects $\mathrm{Quiv}^\bullet$ and this family of objects has certain coparacyclic structure. In fact, symplectic geometry provides a motivation of such observation (see [34, Section 1.2] for more on this): each Quiv^n could be seen (after 2-periodization) as $\mathbb{S}$-linear topological Fukaya categories on the 2-dimensional disc with $n+1$ stoppings on the boundary, and the (para)cyclic symmetry comes from rotations of the disc. The actual construction of $\mathrm{Quiv}^\bullet$ however, runs on the 'mirror' side, i.e., with the category of matrix factorization in spectral algebraic geometry. It is possible to relate the content in this note to the above story in the following way: it was explained in [28, 11] that topological Fukaya category could be modeled locally, on the (microlocalization of) category of sheaves with prescribed singular support. We hope the description of such category in terms of algebraic geometry might help with construction in higher structures such as suggested in [34]. Due to our ignorance of symplectic geometry we cannot say more.

Now we hightlight some technicalities in the paper that might be interesting.

Remark 1.2.7 (Strategy for construction of κ)**.** The idea of construction of the functor κ comes in two parts. First we construct κ for affine toric variety indexed by $\sigma \in \Sigma$. This is implemented by the following correspondence:

$$\mathrm{QCoh}([X_\sigma/\mathbb{T}]) \xleftarrow{\cong} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

where the functor on the right is lax symmetric monoidal and fully faithful. The middle category is presheaf category on a symmetric monoidal 1-category (which is combinatorial in nature). With the help of universal property of Day convolution, it suffices to construct symmetric functors out of $\Theta(\sigma)$ - which is still a laborious work: see the following remarks for a quick idea of how to write down these functors. With the functors at hand, one can follow the arguments from [27] to prove the left hand side functor is an equivalence.

Second step involves gluing: for inclusion of cones $\sigma \subseteq \tau$, one obtains symmetric monoidal functor of restriction

$$\mathrm{QCoh}([X_\tau/\mathbb{T}]) \longrightarrow \mathrm{QCoh}([X_\sigma/\mathbb{T}]).$$

One can think of this as a diagram indexed by $\sigma \in \Sigma^{\mathrm{op}}$ and Zariski descent implies that the limit of this diagram is the category of $\mathrm{QCoh}([X_\Sigma/\mathbb{T}])$. The construction in the first step is compatible with the restriction functor, thus allows us take limit on the sheaf category side to obtain the functor κ.

Remark 1.2.8 (Constructing functors into QCoh)**.** A typical case of the functor we construct mapping into $\mathrm{QCoh}([X_\sigma/\mathbb{T}])$ is the symmetric monoidal functor

$$\mathrm{Fun}(\mathbb{Z}_{\leqslant}; \mathrm{Sp}) \longrightarrow \mathrm{QCoh}([\mathbb{A}^1/\mathbb{G}_m])$$

which classifies the universal line bundle $\mathcal{O}(1)$ and the universal section $\cdot x : \mathcal{O} \to \mathcal{O}(1)$ (see [27]). Note that this says in particular that the line bundle $\mathcal{O}(1)$ is a strict Picard element as in [5]. See [SAG, Warning 5.4.3.3] for more on this notion of strictness. Our method of construction passes through an unstable (set-valued, actually) model for such data, which supplies an alternative construction of the functor in the proof of [27, Theorem 4.1]. We also constructed a slightly generalized version of this with target being $\mathrm{QCoh}([\mathbb{A}^n/\mathbb{G}_m^n])$.

Remark 1.2.9 (Constructing functors into $\mathcal{S}\mathrm{hv}$)**.** A typical case of the functor we construct mapping into $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$ (equipped with convolution) is a lax symmetric monoidal functor

$$\mathrm{Fun}(\mathbb{Z}_{\leqslant};\mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$$

which sends $n \in \mathbb{Z}$ to dualizing sheaf on the open half line $\omega_{(-\infty,n)}$. This is achieved by making a more general construction: given a commutative monoid M in LCH, we construct a lax symmetric monoidal structure on the relative homology functor taking a pair $(X, f : X \to M)$ to $f_! f^! \omega_M$. With this functor at hand, the problem is reduced to 1-categorical manipulation. The general construction is very much inspired by [12, Chapter 3], and we believe it has other interesting use.

Remark 1.2.10 (Gluing in $\mathcal{S}\mathrm{hv}$)**.** To make the gluing procedure precise, we prove a sheaf-theoretic counterpart of Zariski descent in $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})$. This is implemented with idempotent algebras as in [HA, Definition 4.8.2.8]. For example the dualizing sheaf $\omega_{(-\infty,0)}$ is an idempotent algebra in $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ for the convolution product, and this phenomenon generalizes to other cones. Given a smooth projective fan Σ, we produce a collection of idempotent algebras in $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})$ and show that their meet is the unit $\mathbb{1}_{\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Sp})}$ as an idempotent algebra.

Remark 1.2.11 (Singular support for polyhedral sheaf)**.** To characterize the image of κ, we make use of the recent advances [7, 13] of exodromy equivalence with large category of constructible sheaves. Following [9], we supply a definition of singular support for sheaves constructible for affine hyperplane arrangement - via Fourier-Sato transform (compare the general definition laid out in [19]). We demonstrate how one makes use of this definition in practice - by applying the non-characteristic deformation lemma [31]. The proof presented here supplies some details in [40] though the idea is same as there.

1.3 Thanks

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1.4 Conventions

Notation 1.4.1 (Category theory)**.** We don't touch on set-theoretic issue in this note. We write Cat for the $(\infty, 1)$-category of quasicategories, functors, natural isomorphisms and so on. We refer to objects in Cat as 'categories' to avoid putting ∞ in front of everything. This however makes us write 'stable category' for more established name 'stable ∞-category'. We identify a 1-category with its nerve in Cat and stress that it is 1-category when we have one. We write Spc for the category of spaces (or homotopy types, or anima) and Sp for the stable category of spectra. We write Map for mapping space in a category and map for mapping spectra in a stable category.

Notation 1.4.2 (Simplicial stuff)**.** By Δ we mean (a skeleton of) the (1-)category of nonempty ordered finite sets and order preserving maps between them. A (co)simplicial diagram in $\mathcal{C}$ is a functor from $(\Delta)\Delta^{\mathrm{op}}$ to $\mathcal{C}$. We only draw face maps when visualizing a (co)simplicial diagram. We write d^i for the structure (face) maps in a cosimplicial diagrams.

Notation 1.4.3 (Symmetric monoidal categories)**.** We write $(\mathcal{C}, \otimes)$ for a symmetric monoidal category and often refer to $\mathcal{C}$ as a symmetric monoidal category, omitting the monoidal structure. We write $\mathcal{C}^{\otimes}$ for the underlying operad of $(\mathcal{C}, \otimes)$. We write $\mathrm{CAlg}(\mathcal{C}, \otimes) := \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^{\otimes})$ for the category of $\mathbb{E}_\infty$-algebras in $\mathcal{C}$. And when there is no danger of confusion, we will omit the monoidal structure and write $\mathrm{CAlg}(C)$. For example, $\mathrm{CAlg}(\mathrm{Sp})$ would refer to the category of $\mathbb{E}_\infty$-ring spectra. In the special case for Set or Spc equipped with Cartesian symmetric monoidal structure, we also write CMon for the category of commutative monoids and CGrp for the category of commutative groups. All the functor (presheaf) categories are assumed to carry Day convolution structure when considered as a symmetric monoidal category.

Notation 1.4.4 ((Lax) symmetric monoidal functors)**.** For two symmetric monoidal category $\mathcal{C}$ and $\mathcal{D}$, we write $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ for the category of symmetric monoidal functor from $\mathcal{C}$ to $\mathcal{D}$. We write $\mathrm{Fun}^{\mathrm{lax}\otimes}(\mathcal{C}, \mathcal{D})$ for the category of symmetric monoidal functor from $\mathcal{C}$ to $\mathcal{D}$. We write SMCat for the category of symmetric monoidal categories and (strongly) symmetric monoidal functors between them. We also use the very nonstandard notation $\mathrm{SMCat}^{\mathrm{lax}}$ for the category of symmetric monoidal categories and lax symmetric monoidal functors between them.

Notation 1.4.5 (Algebraic geometry)**.** We approach spectral algebraic geometry through functor of points. We write Stk for the full subcategory of fpqc sheaves inside $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathrm{Spc})$ (what's better, the objects we are dealing with in this note are all geometric stacks in the sense of [SAG, Definition 9.3.0.1]), and we write $\mathrm{Spét}(-)$ for the Yoneda functor $\mathrm{CAlg}^{\mathrm{cn,op}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathrm{Spc})$ which factors through Stk (In SAG, Spét was used for another construction, but Lurie provided comparison with this Yoneda point of view in [SAG, Proposition 1.6.4.2]). We will pretend that Stk is a topos despite the size issue, and one may get around the inconveniences via cardinal truncation.

Notation 1.4.6 (Topological spaces)**.** We write LCH for the (1-)category of locally compact Hausdorff space and continuous maps between them. But we actually only care about finite dimensional manifolds. We often write $j_U : U \to X$ for the inclusion of an open subset and $i_Z : Z \to X$ for the inclusion of a closed subset.

Notation 1.4.7 (Sheaf theory)**.** It will be very convenient for us to extract a 'six-functor formalism' out of [39] on the category of locally compact Hausdorff topological spaces. We write $\mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$ for the category of sheaves of spectra on a locally compact Hausdorff topological space X, and we write $f^* \dashv f_*$, $f_! \dashv f^!$ and $\otimes \dashv \mathrm{Hom}$ for the six functors that comes with the whole package of formalism. For an open $U \subseteq X$, we write $\underline{\mathbb{S}}_U \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$ for the sheafification of the $\mathbb{S}$-linearized representable presheaf on U. In other words, if we write $j_U : U \to X$ for the inclusion map and $\underline{\mathbb{S}} \in \mathcal{S}\mathrm{hv}(U; \mathrm{Sp})$ for the constant sheaf valued at $\mathbb{S}$, $\underline{\mathbb{S}}_U$ is equivalently

$$\underline{\mathbb{S}}_U := j_{U!}\underline{\mathbb{S}} \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$$

and we abusively call it representable sheaf on U. Note that $\underline{\mathbb{S}}_X$ is just constant sheaf valued at $\mathbb{S}$ on X. Similarly for a closed subset $Z \subseteq X$ we write

$$\underline{\mathbb{S}}_Z := i_{Z*}\underline{\mathbb{S}} \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp}).$$

We reserve the symbol ω for dualizing sheaves. Let $p : X \to *$ be the canonical map to the final object. The dualizing sheaf of X is defined to be

$$\omega_X := p^!\underline{\mathbb{S}} \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp}).$$

Similarly, when we work with an open subset U or closed subset Z in X, we write

$$\omega_U := j_{U!}j_U^!\omega_X \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$$

and

$$\omega_Z := i_{Z!}i_Z^!\omega_X \in \mathcal{S}\mathrm{hv}(X; \mathrm{Sp}).$$

2 Combinatorial model

In [9, Section 3] the authors defined a poset $\Gamma(\Sigma, M)$ that interpolates between the category of quasi-coherent sheaves and the category of constructible sheaves. In this section we recall the definition and present functoriality of the definition. Go to Notation 3.1.1 for definitions of lattices, cones, fans and related stuff if you have never seen them before.

Definition 2.0.1 (Poset of cones)**.** Given a pair of lattice and fan (N, Σ), one might consider Σ as a poset as follows: the objects of Σ are cones $\sigma \in \Sigma$ and morphisms between two cones are inclusions.

Definition 2.0.2. Let $\mathrm{Poly}(M_{\mathbb{R}})$ be the poset of closed polyhedral subsets in $M_{\mathbb{R}}$ (the subsets which can be written as a Minkowski sum of a polytope and a polyhedral cone) with morphisms being inclusions. This is a symmetric monoidal 1-category if one takes the Minkowski sum $+$.

Definition 2.0.3 (The Θ category)**.** Fix a cone $\sigma \subset N_{\mathbb{R}}$, there is a (1-)category $\Theta(\sigma)$ defined as the full subcategory of posets of closed subsets in $M_{\mathbb{R}}$:

$$\Theta(\sigma) \subseteq \mathrm{Poly}(M_{\mathbb{R}}).$$

It is spanned on objects of the form $m + \sigma^\vee$ for $m \in M$.

Observe that this association $\sigma \mapsto \Theta(\sigma)$ is functorial in σ that it assembles into a functor

$$\Theta(-) : \Sigma^{\mathrm{op}} \to \mathrm{Cat}.$$

Given an inclusion $i : \sigma \to \tau \in \sigma$ of cones, the induced functor is

$$\Theta(i) : \Theta(\tau) \to \Theta(\sigma), \Theta(i)(m + \tau^\vee) := m + \sigma^\vee.$$

Remark 2.0.4 (Symmetric monoidal structure on $\Theta(-)$)**.** We make the following observations:

1. Each $\Theta(\sigma)$ has a structure of symmetric monoidal (1-)category. This could be obtained by observing that as a full subcategory, $\Theta(\sigma)$ inherits a (non-unital) symmetric monoidal structure from the symmetric monoidal category $(\mathrm{Poly}(M_{\mathbb{R}}), +)$. To make it unital, it suffices to note that $\sigma^\vee \in \Theta(\sigma)$ acts as a tensor unit.

2. We might as well observe that $\sigma^\vee$ is an idempotent algebra in $(\mathrm{Poly}(M_{\mathbb{R}}), +)$ and define $\Theta(\sigma)$ to be a full subcategory of $\mathrm{Mod}_{\sigma^\vee}(\mathrm{Poly}(M_{\mathbb{R}}))$, and it follows directly that $\Theta(\sigma)$ inherits a symmetric monoidal structure.

3. For each inclusion $i : \sigma \to \tau$, $\Theta(i)$ has a structure of symmetric monoidal functor which can be observed directly since we are working with posets: there is no coherence issue. *In conlusion*, $\Theta(-)$ lifts to a functor $\Sigma^{\mathrm{op}} \to \mathrm{SMCat}$.

4. For later use, consider the discrete category of M with symmetric monoidal structure given by addition. There is a symmetric monoidal functor

$$p_\sigma : M \longrightarrow \Theta(\sigma) : m \mapsto m + \sigma^\vee$$

   and this assembles into a natural transformation between diagrams in SMCat indexed by σ where the source is thought of as a constant diagram.

Remark 2.0.5 (Comparison with other models)**.** Our definition of $\Theta(-)$ works cone by cone, while in [38, Section 5][9, Section 3] global categories were proposed. Later on we will see that one wants to compute

$$\lim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(-)^{\mathrm{op}}, \mathrm{Sp}).$$

It is still unclear to us how would one present the limit of such a diagram of presheaf categories with arrows given by left Kan extension of functors. But '(co)sheaves for Morelli topology' as in [38, Section 6] seems like a combinatorial presentaion of the limit.

3 Toric geometry

Classically, toric geometry builds on the linearization functor $\mathbb{Z}[-] : \mathrm{CMon}(\mathrm{Set}) \to \mathrm{CAlg}(\mathrm{Ab})$. For example, $\mathbb{Z}[\mathbb{N}] = \mathbb{Z}[X]$ is the one-variable polynomial ring. Toric schemes are constructed from $\mathrm{Spec}(-)$ of these monoid schemes by gluing along maps coming from $\mathrm{CMon}(\mathrm{Set})$. In this section we present some basic materials on flat toric geometry over $\mathbb{S}$ [3] . Although we will not prove it, the toric scheme we define here will be flat over the base ring $\mathbb{S}$. The adjective 'flat' is also reminiscent of the fact that after base changing to $\mathbb{Z}$, it recovers the classical construction of toric schemes, which are flat over $\mathbb{Z}$. The ideas of looking at flat toric scheme over $\mathbb{S}$ are certainly well-known, going back to [24] and [SAG, Remark 5.4.1.9]. Most of the discussion would be rather formal: we are mainly interested in the category of quasi-coherent sheaves and related categorical nonsense.

In the first part, we fix notation for toric construction and explain how the action diagram presents the quotient stack by the torus action. In the second part, we recall the functoriality of quasi-coherent sheaves and provide an unstable model for quasi-coherent sheaves on the quotient stack. This is used in the third part, where we construct combinatorial-coherent comparison functor. After that we follow the approach of [27] to show this functor is an equivalence.

3.1 Recollections on toric geometry

Notation 3.1.1. We recall the following notations useful in the combinatorics of toric varieties.

- A lattice is a finitely generated free abelian group $N \in \mathrm{Ab} = \mathrm{CGrp}(\mathrm{Set})$.
- The dual lattice M of N is $M := \mathrm{Hom}_{\mathrm{Ab}}(N, \mathbb{Z}) \in \mathrm{Ab}$.
- A cone $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ for us is a rational polyhedral cone in $N_{\mathbb{R}}$.
- The dual cone of $\sigma \subset N_{\mathbb{R}}$ is $\sigma^{\vee} := \{m \in M_{\mathbb{R}} : \langle m, n\rangle \geqslant 0, \forall n \in \sigma\} \subseteq M_{\mathbb{R}}$.
- A fan Σ in N is a collection of strongly convex cones in N closed under taking faces, such that every pair of cones either are disjoint or meet along a common face.
- A fan Σ is smooth [10, Section 2.1] if each of the cone σ is spanned by part of a basis of N.
- A fan Σ is projective [10, Section 3.4] if it admits an integral moment polytope $P \subset M_{\mathbb{R}}$: a polytope such that its faces are in bijection with cones in Σ and the cone σ corresponding to a face F is precisely the dual cone of the angle spanned by P along F.

Construction 3.1.2 (Flat toric scheme)**.** Given a pair (N, Σ) of lattice and fan. The assignment

$$\sigma \mapsto S_{\sigma} := \sigma^{\vee} \cap M \in \mathrm{CMon}(\mathrm{Set})$$

gives rise to a functor $\Sigma^{\mathrm{op}} \to \mathrm{CMon}(\mathrm{Set}) = \mathrm{CAlg}(\mathrm{Set})$. On the other hand, the symmetric monoidal functors

$$\mathrm{Set} \hookrightarrow \mathrm{Spc} \xrightarrow{\Sigma^{\infty}_{+}} \mathrm{Sp}$$

[3] While it's possible to make sense of, say, a non-flat $\mathbb{P}^1$ as in [SAG, Construction 19.2.6.1], how to develop the theory of non-flat toric varieties in full generality remains unclear to the authors.

induce a functor $\mathbb{S}[-] : \mathrm{CAlg}(\mathrm{Set}) \to \mathrm{CAlg}(\mathrm{Sp})$. Consider the image of σ under this composite functor

$$\mathcal{O}_\sigma := \mathbb{S}[\sigma^\vee \cap M] \in \mathrm{CAlg}(\mathrm{Sp})$$

which should be thought of as the ring of functions on the affine toric scheme X_σ associated to the cone σ. Postcomposing with Spét, we get a functor $\Sigma \longrightarrow \mathrm{Stk}$:

$$\sigma \mapsto \mathrm{Sp\acute{e}t}(\mathcal{O}_\sigma).$$

The flat toric scheme X_Σ associated to (N, Σ) is defined to be the colimit of this diagram

$$X_\Sigma := \operatorname*{colim}_{\Sigma} \mathrm{Sp\acute{e}t}\, \mathcal{O}_\sigma \in \mathrm{Stk}.$$

computed in the category of stacks.

Remark 3.1.3 (An alternative version of 'toric geometry')**.** Motivated by the fact that $\mathbb{N}^{\times k}$ is the free object on k points in CMon(Set) (and similarly $\mathbb{Z}^{\times k}$ is the free object on k points in CGrp(Set)), one might want to reimagine a toric geometry over the sphere spetrum building upon monoid algebra of free objects in CMon(Spc) (or CGrp(Spc)). We don't know how to pursue the construction, but only point out the following subtleties:

1. The natural numbers $\mathbb{N}$ (resp. $\mathbb{Z}$) is the free $\mathbb{E}_1$-monoid (resp. $\mathbb{E}_1$-group) on a point. However, when viewed as an $\mathbb{E}_\infty$-monoid, $\mathbb{N}$ is far from being a free object: a map in $\mathrm{CMon}(\mathrm{Spc})$ from $\mathbb{N}$ instead picks out a 'strictly commutative element' in the target.

2. The flat affine line $\mathrm{Sp\acute{e}t}(\mathbb{S}[\mathbb{N}])$ doesn't support the addition map, see [22, Section 3.5].

Example 3.1.4 (Flat torus over sphere)**.** If one picks the fan to consist only of the origin, the associated flat toric scheme (named $\mathbb{T}$) is the torus associated to M:

$$\mathbb{T} := \mathrm{Sp\acute{e}t}(\mathbb{S}[M]).$$

Note that $\mathbb{T}$ has the structure of group object (and we will call it a group scheme) given that M is a cogroup object in $\mathrm{CMon}(\mathrm{Spc})$.

Recall that a toric variety over a field k contains a torus as an open-dense subset and the torus action extends continuously to the whole variety. Now we explain the torus action in the setting of flat toric geometry.

Construction 3.1.5. Recall that given a category $\mathcal{C}$ with all limits, and considering $\mathcal{C}$ as a Cartesian symmetric monoidal category, every object $X \in \mathcal{C}$ acquires a canonical commutative coalgebra structure, informally specified by regarding the diagonal as the comultiplication map

$$\Delta : X \to X \times X.$$

In particular, every map $f : Y \to X$ exhibits Y as a comodule over X, with the coaction map informally specified by

$$\mu : Y \xrightarrow{\Delta} Y \times Y \xrightarrow{(f,\mathrm{id})} X \times Y.$$

In fact this map is induced by the lift of $f : Y \to X$ to a map of coalgebras. Specializing to the situtation $\mathcal{C} = \mathrm{CMon}(\mathrm{Set})$ [4], we see that every submonoid S_σ of M is canonically coacted on by M. Moreover, these coactions are compatible with inclusions among S_σ. Therefore, $\mathcal{O}_\sigma = \mathbb{S}[S_\sigma]$ acquires a canonical $\mathbb{S}[M]$-comodule structure. Further passing to Spét, this gives a compatible family of actions of the group scheme $\mathbb{T} = \mathrm{Sp\acute{e}t}\,\mathbb{S}[M]$ on $\mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma$, each encoded by a simplicial diagram

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma \times \mathbb{T} \times \mathbb{T} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma \times \mathbb{T} \rightrightarrows \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma.$$

Taking colimits along σ, we obtain the diagram

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} (\mathrm{colim}_\sigma\, \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma) \times \mathbb{T} \times \mathbb{T} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} (\mathrm{colim}_\sigma\, \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma) \times \mathbb{T} \rightrightarrows \mathrm{colim}_\sigma\, \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma,$$

because colimits are universal in Stk. [5] We therefore obtain an action of $\mathbb{T}$ on

$$X_\Sigma = \operatorname*{colim}_{\sigma\in\Sigma} \mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma,$$

to which we refer as the torus action of $\mathbb{T}$ on X_Σ, and the corresponding simplicial diagram $(X_\Sigma//\mathbb{T})_\bullet$ the action diagram of $\mathbb{T}$ on X_Σ. Very formally, one might think of each $\mathrm{Sp\acute{e}t}\,\mathcal{O}_\sigma$ as an object in $\mathrm{Mod}_{\mathrm{Sp\acute{e}t}(\mathbb{S}[M])}(\mathrm{Stk})$ and take colimit along Σ in the module category. Given that forgetful commutes with colimits, one sees that X_Σ acquires an action of $\mathbb{T}$, and this construction of the action might be identified with the above action diagram.

Definition 3.1.6. The quotient stack $[X_\Sigma/\mathbb{T}]$ is the geometric realization of the action diagram of $\mathbb{T}$ on X_Σ:

$$[X_\Sigma/\mathbb{T}] := \operatorname*{colim}_{\Delta^{op}} \left(\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X_\Sigma \times \mathbb{T} \times \mathbb{T} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X_\Sigma \times \mathbb{T} \rightrightarrows X_\Sigma \right) \in \mathrm{Stk}.$$

Remark 3.1.7. The Čech nerve of the projection $X_\Sigma \to [X_\Sigma/\mathbb{T}]$ is canonically identified with the action diagram of $\mathbb{T}$ on X_Σ. This is a direct consequence of Lemma 7.1.1 and the fact that every groupoid object in an ∞-topos is effective [HTT, Theorem 6.1.0.6].

Remark 3.1.8. Alternatively, one might take the quotient affine locally on each X_σ by defining

$$[X_\sigma/\mathbb{T}] := \operatorname*{colim}_{\Delta^{op}} \left(\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X_\sigma \times \mathbb{T} \times \mathbb{T} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X_\sigma \times \mathbb{T} \rightrightarrows X_\sigma \right) \in \mathrm{Stk}$$

via the action diagram. Then one might perform gluing

$$[X_\Sigma/\mathbb{T}] = \operatorname*{colim}_{\sigma\in\Sigma} [X_\sigma/\mathbb{T}]$$

and obtain the same stack, since colimit commutes with colimit.

[4] Note that CMon(Set) is preadditive.

[5] In particular, taking colimits commutes with taking finite products.

3.2 Quasi-coherent sheaves

There is a functor given in [SAG, Definition 6.2.2.1]

$$\mathrm{QCoh} : \mathrm{Stk}^{\mathrm{op}} \to \mathrm{Cat}$$

which is lax symmetric monoidal in view of [SAG, §6.2.6] and [HA, Theorem 2.4.3.18]. The point is that it assigns a symmetric monoidal category to each stack:

$$X \mapsto \mathrm{QCoh}(X) \in \mathrm{SMCat}$$

such that on affines $\mathrm{Sp\acute{e}t}(R)$ it assigns $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}(R)) \cong \mathrm{Mod}_R(\mathrm{Sp})$. In fact this functor preserves limit as in [SAG, Proposition 6.2.3.1], hence one gets a presentation of quasi-coherent sheaves on quotient stack as

$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \cong \lim_{\Sigma^{\mathrm{op}}} \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$

while in turn each piece is presented by

$$\mathrm{QCoh}([X_\sigma/\mathbb{T}]) \cong \lim_{\Delta}\left(\cdots \substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow} \mathrm{QCoh}(X_\sigma \times \mathbb{T} \times \mathbb{T}) \substack{\longleftarrow\\\longleftarrow\\\longleftarrow} \mathrm{QCoh}(X_\sigma \times \mathbb{T}) \substack{\longleftarrow\\\longleftarrow} \mathrm{QCoh}(X_\sigma) \right).$$

Note that this is actually a limit of symmetric monoidal categories [SAG, §6.2.6]. At first glance, it might seem difficult to write down objects explicitly in this category. Motivated by [SAG, Construction 5.4.2.1], we proceed by making the following unstable construction.

Construction 3.2.1 (Unstable analogue)**.** Fix a cone σ in a lattice N, recall Construction 3.1.5 provides an coaction of M on $S_\sigma = \sigma^\vee \cap M$. The coaction is presented by the following simplicial diagram in $\mathrm{CMon}(\mathrm{Spc})$:

$$\cdots \substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow} S_\sigma \times M \times M \substack{\longleftarrow\\\longleftarrow\\\longleftarrow} S_\sigma \times M \substack{\longleftarrow\\\longleftarrow} S_\sigma.$$

Passing to module category (with the extension-of-scalar functoriality), one obtains

$$\cdots \substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow} \mathrm{Mod}_{S_\sigma \times M \times M}(\mathrm{Spc}) \substack{\longleftarrow\\\longleftarrow\\\longleftarrow} \mathrm{Mod}_{S_\sigma \times M}(\mathrm{Spc}) \substack{\longleftarrow\\\longleftarrow} \mathrm{Mod}_{S_\sigma}(\mathrm{Spc}).$$

This is a cosimplicial diagram of symmetric monoidal categories, and we write $\mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M$ for the limit.

Remark 3.2.2 (1-categorical analogue and degeneracy)**.** One can replace Spc by Set in the above diagram and get 1-categorical constructions that we call $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$. As the categories involved are all 1-categories, the limit is canonically identified with the limit of the diagram restricted to $\Delta_{\leqslant 2}$ (see [15, Proposition A.1]. Note also that one can produce objects and morphisms in the limit with finite amount of data (actually very little is needed). More precisely, consider a cosimplicial diagram of 1-categories $\mathcal{C}_\bullet$, the limit is still a 1-category whose objects are pairs (x, f) where x is an object in $\mathcal{C}_0$, $f : d^1 x \to d^0 x$ is an isomorphism in $\mathcal{C}_1$ such that $d^0 f \circ d^2 f = d^1 f$ in $\mathcal{C}_2$. A map from (x, f) to (y, g) is a map $\varphi : x \to y$ in $\mathcal{C}_0$ that commutes with structure maps f and g.

Example 3.2.3 (How to write down an object in the unstable category)**.** Here is one concrete example of how one writes down objects in the category $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$. We supply a particular lift[6] of M to an object in this category, where M is an S_σ-module via the canonical inclusion. To provide the lift is to provide the structure map (note that the relative tensor products are induced by different maps $S_\sigma \to S_\sigma \times M$ where the left one is $(\mathrm{id}, 0)$ and the right one is $(\mathrm{id}, \mathrm{inclusion})$)

$$f : M \times_{S_\sigma} S_\sigma \times M \longrightarrow M \times_{S_\sigma} S_\sigma \times M \in \mathrm{Mod}_{S_\sigma \times M}(\mathrm{Set})$$

which is an isomorphism and we picked f such that

$$f(m, s, n) := (m, s, n - m) \in M \times_{S_\sigma} S_\sigma \times M.$$

It is a tedious exercise to check that f satisfies coherence (cocycle conditions) as above and we leave it to the reader. The point should be that this lift corresponds to coaction of M on itself.

Warning 3.2.4. Given a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ and $A \in \mathrm{CAlg}(\mathcal{C})$, it induces a functor $F_A : \mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_{F(A)}(\mathcal{D})$. If $\mathcal{C}$ and $\mathcal{D}$ have geometric realizations and tensor products in $\mathcal{C}$ and $\mathcal{D}$ commutes with geometric realizations, then both $\mathrm{Mod}_A(\mathcal{C})$ and $\mathrm{Mod}_{F(A)}(\mathcal{D})$ have symmetric monoidal structure given by relative tensor products. But(!) the functor F_A lifts to a symmetric monoidal functor only when F commutes with geometric realizations. The lift is functorial in the sense of [HA, Theorem 4.8.5.16] (see below). The example to keep in mind is the following:

$$\mathrm{Set} \to \mathrm{Spc} \to \mathrm{Sp}$$

which is a sequence of symmetric monoidal functors. The latter preserves geometric realization while the first one doesn't. For instance, the relative tensor product $X \times_{\mathbb{Z}} Y$ is in general not the same when computed in Spc compared to Set. When X and Y are both singleton, in Set the outcome is still a point while in Spc one gets $B\mathbb{Z}$.

Remark 3.2.5 (An antidote to the warning)**.** Limited by above warning, for a given monoid $S \in \mathrm{CMon}(\mathrm{Set})$, we don't have a symmetric monoidal structure on the inclusion functor $\mathrm{Mod}_S(\mathrm{Set}) \to \mathrm{Mod}_S(\mathrm{Spc})$. One can however, define a symmetric monoidal full subcategory sitting in both of them: take $\mathrm{Mod}_S(\mathrm{Spc})^{\mathrm{free}} \subset \mathrm{Mod}_S(\mathrm{Spc})$ to be the full subcategory on coproducts of S. This category inherits a symmetric monoidal structure and can be identified, symmetric monoidally, with the full subcategory on coproducts of S in $\mathrm{Mod}_S(\mathrm{Set})$. To be very rigorous with the construction that will follow, one should construct symmetric monoidal functor directly into $\mathrm{Mod}_S(\mathrm{Spc})$, but we will construct functor into $\mathrm{Mod}_S(\mathrm{Set})$ and observe that it lifts into $\mathrm{Mod}_S(\mathrm{Spc})$. We will also use the following fact unpacked from [HA].

Proposition 3.2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories which admit all geometric realizations. Given symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ such that:

1. Tensor products in $\mathcal{C}$ and $\mathcal{D}$ commute with geometric realization.
2. Functor F commutes with geometric realization.

[6] There are obviously others.

One can extract the following diagram

$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, bend left, "\mathrm{Mod}_{(-)}(\mathcal{C})"] \arrow[r, bend right, "\mathrm{Mod}_{F(-)}(\mathcal{D})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{SMCat} \end{tikzcd}$$

out of [HA, Theorem 4.8.5.16]. When evaluated on $A \to B \in \mathrm{CAlg}(\mathcal{C})$, the diagram reads

$$\begin{tikzcd} \mathrm{Mod}_A(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Mod}_B(\mathcal{C}) \arrow[d] \\ \mathrm{Mod}_{F(A)}(\mathcal{D}) \arrow[r] & \mathrm{Mod}_{F(B)}(\mathcal{D}) \end{tikzcd}.$$

Proof. See Proposition 7.2.1. □

The linearization functor $\mathbb{S}[-] : \mathrm{Spc} \to \mathrm{Sp}$ is symmetric monoidal and preserves geometric realization. So it induces, functorially, symmetric monoidal functors on module categories. This implies that there is a natural transformation from the cosimplicial diagram that presents $\mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M$ to the cosimplicial diagram that presents $\mathrm{QCoh}([X_\sigma/\mathbb{T}])$. We write

$$\mathcal{O}[-] : \mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M \to \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$

for the symmetric monoidal functor one obtains after taking limit along Δ. Note that both sides of above are indexed over $\sigma \in \Sigma^{\mathrm{op}}$, and for the same reason, $\mathcal{O}[-]$ assembles into a natural transformation of diagrams. It is this natural transformation that we would like to make use of in the next subsection to produce a comparison functor from combinatorial models.

3.3 Combinatorial v.s. quasi-coherent

The goal of this subsection is to provide the following construction.

Proposition 3.3.1. There exists a symmetric monoidal equivalence of categories

$$\Phi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\cong} \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$

where the left-hand side has the Day convolution tensor product of presheaves and right-hand side has the canonical tensor product of quasi-coherent sheaves. Moreover, these equivalences are functorial in $\sigma \in \Sigma^{\mathrm{op}}$ that they assemble into a natural transformation of diagrams in SMCat indexed by Σ^{op}. Hence taking limit produces

$$\lim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\cong} \lim_{\Sigma^{\mathrm{op}}} \mathrm{QCoh}([X_\sigma/\mathbb{T}]) \cong \mathrm{QCoh}([X_\Sigma/\mathbb{T}]).$$

Remark 3.3.2 (Compatibility with torus)**.** We will establish along the way an equivalence

$$\Phi_M : \mathrm{Fun}(M, \mathrm{Sp}) \cong \mathrm{QCoh}(B\mathbb{T})$$

and will also provide compatibility of Φ_M with above equivalence, i.e., the following diagram commutes

$$\begin{tikzcd}
\lim_{\Sigma^{op}} \mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \arrow[r, "\lim_{\Sigma^{op}} \Phi_\sigma"] & \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \\
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[u, "\lim_{\Sigma^{op}} (p_\sigma)_!"] \arrow[r, "\Phi_M"] & \mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\lim_{\Sigma^{op}} \pi_\sigma^*"']
\end{tikzcd}.$$

Remark 3.3.3 (The geometry of filtrations)**.** Take the pair of lattice and fan $N = \mathbb{Z}$ and $\Sigma = \{0, \mathbb{R}_{\geqslant 0}\}$. The theorem above reads

$$\mathrm{Fun}(\mathbb{Z}_{\leqslant}, \mathrm{Sp}) \cong \mathrm{QCoh}([\mathbb{A}^1/\mathbb{G}_m]).$$

which is [27, Theorem 1.1]. The proof presented in this subsection actually follows closely the approach in [27].

We begin by constructing the functor Φ_σ, then explain its naturality along $\sigma \in \Sigma^{op}$.

Construction 3.3.4. (Construction of the functor in the unstable case) Fix a cone σ in a lattice N, we define a functor

$$\phi_\sigma : \Theta(\sigma) \to \mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$$

as follows: for $V \in \Theta(\sigma)$, recall that V is an integral translation of $\sigma^\vee$. We define $\phi_\sigma(V)$ to have the underlying object

$$V \cap M \in \mathrm{Mod}_{S_\sigma}(\mathrm{Set})$$

which inherits the S_σ action from M. We take advantage of Example 3.2.3 to provide the structure map: it suffices to observe that the structure map for the lift of M as in Example 3.2.3 preserves $V \cap M$ because $V \cap M$ is closed under translation by S_σ. Since we are working with 1-categories, we learn from this that $V \cap M$ inherits a lift from M as in Example 3.2.3. Now we move on to morphisms. Given an inclusion $i : V \subset W \in \Theta(\sigma)$, $\phi_\sigma(i)$ is on the nose inclusion map

$$V \cap M \to W \cap M \in \mathrm{Mod}_{S_\sigma}(\mathrm{Set})$$

and since it is compatible with inclusion into M, we know that it lifts to a map in $\mathrm{Mod}_{S_\sigma}(\mathrm{Set})^M$. The symmetric monoidal structure on the functor can be supplied and checked directly as it is a functor between 1-categories. The construction lands in $\mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^{\mathrm{free}}$ in each degree and hence lifts to a symmetric monoidal functor to $\mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M$. To conclude, we explained how to obtain a symmetric monoidal functor

$$\phi_\sigma : \Theta(\sigma) \longrightarrow \mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M.$$

Remark 3.3.5. (Naturality along $\sigma \in \Sigma^{op}$) The functors ϕ_σ as above assembles into a natural transformation between diagrams in SMCat indexed by Σ^{op}:

$$\Theta(-) \to \mathrm{Mod}_{S_-}(\mathrm{Spc})^M.$$

Since we are working with 1-categories, the coherence could be inspected directly.

Definition 3.3.6. We define Φ_σ to be the left Kan extension of $\mathcal{O}[\phi_\sigma]$ along the stable Yoneda embedding:

$$\Phi_\sigma := \mathrm{Lan}_h(\mathcal{O}[\phi_\sigma]) : \mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \longrightarrow \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$

where we have used the linearization functor

$$\mathcal{O}[-] : \mathrm{Mod}_{S_\sigma}(\mathrm{Spc})^M \to \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$

from last paragraph of Section 3.2. Note that it is symmetric monoidal for Day convolution product on the domain. From the discussion in Remark 3.3.5 and functoriality of Day convolution (see Section 7.3) we learn that $\sigma \mapsto \Phi_\sigma$ is a natural transform

$$\begin{tikzcd} \Sigma^{\mathrm{op}} \arrow[r, bend left, "\mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp})"{name=U}] \arrow[r, bend right, "\mathrm{QCoh}([X_\sigma/\mathbb{T}])"'{name=D}] & \mathrm{SMCat} \arrow[from=U, to=D, phantom, "\Phi_\sigma\Downarrow"] \end{tikzcd}$$

bewteen diagrams in SMCat indexed by Σ.

Example 3.3.7 (Equivariant line bundles on affine line)**.** Take the pair of lattice and fan $N = \mathbb{Z}$ and $\Sigma = \{0, \mathbb{R}_{\geqslant 0}\}$. The construction above produces a family of line bundles from the following symmetric monoidal functor

$$\Phi_{\mathbb{R}_\geqslant} : \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}_{\leqslant}; \mathrm{Sp}) \longrightarrow \mathrm{QCoh}([\mathbb{A}^1/\mathbb{G}_m]).$$

Upon basechanging to $\mathbb{Z}$, it recovers the universal line bundles $\phi(n) = \mathcal{O}(n)$, universal sections $\cdot x : \mathcal{O}(n) \to \mathcal{O}(n+1)$, and isomorphisms $\mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(mn)$. One can globalize the construction and construct equivariant line bundles on more general toric schemes.

Now we move on to prove the main theorem of this section: to show that each Φ_σ is an equivalence. Before that we do some preparations.

Variant 3.3.8 (Compare with Construction 3.3.4)**.** We can define a symmetric monoidal functor

$$\phi_M : M \to \mathrm{Mod}_1(\mathrm{Set})^M$$

(where 1 is the initial monoid) as follows. On objects, $m \in M$ is taken to the pair $(\{*\}, m)$. Here $\{*\} \in \mathrm{Set}$ is the underlying object and $m : \{*\} \times M \to \{*\} \times M$ is the isomorphism of addition by m. Again one checks this satisfies cocycle condition as in Example 3.2.3 so $(\{*\}, M)$ defines an object in $\mathrm{Mod}_1(\mathrm{Set})^M$. This assignment lifts to a symmetric monoidal functor by direct inspection. Hence we get a symmetric monoidal functor $\Phi_M := \mathrm{Lan}_h\, \mathcal{O}[\phi_M]$ as

$$\Phi_M : \mathrm{Fun}(M, \mathrm{Sp}) \to \mathrm{QCoh}(B\mathbb{T}).$$

Remark 3.3.9 (Compatibility with lattice)**.** By the very explicit construction, the equivalence Φ_M above enjoys the following functoriality: it is compatible with Definition 3.3.6. There is a symmetric monoidal functor $p_\sigma : M \to \Theta(\sigma)$ which sends m to $m + \sigma^\vee$ that would make the diagram

$$\begin{tikzcd} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Phi_\sigma"] & \mathrm{QCoh}([X_\sigma/\mathbb{T}]) \\ \mathrm{Fun}(M, \mathrm{Sp}) \arrow[u, "(p_\sigma)_!"] \arrow[r, "\Phi_M"] & \mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\pi_\sigma^*"'] \end{tikzcd}$$

commutes, where $(p_\sigma)_!$ stands for left Kan extension of presheaf along p_σ and π_σ^* stands for pull-back of quasi-coherent sheaves along $\pi_\sigma : [X_\sigma/\mathbb{T}] \to B\mathbb{T}$. The coherence comes from 1-categorical inspection before linearization. Moreover, the maps above are natural in $\sigma \in \Sigma^{op}$ that one can interpret it as a square of natural transformations of diagrams in SMCat indexed by $\sigma \in \Sigma^{op}$.

We will follow the approach taken in [27, Theorem 4.1] to prove the following:

Theorem 3.3.10. There is an equivalence of symmetric monoidal categories

$$\Phi_M : \mathrm{Fun}(M, \mathrm{Sp}) \cong \mathrm{QCoh}(B\mathbb{T})$$

where left-hand side comes with the Day convolution tensor product and right-hand side comes with the standard tensor product of quasi-coherent sheaves.

Proof. We interpret Φ_M as an augmentation of the cosimplicial diagram presenting $\mathrm{QCoh}(B\mathbb{T})$:

$$\cdots \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; \mathrm{QCoh}(* \times \mathbb{T} \times \mathbb{T}) \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; \mathrm{QCoh}(* \times \mathbb{T}) \;\substack{\longleftarrow\\[-2pt]\longleftarrow}\; \mathrm{QCoh}(*) \longleftarrow \mathrm{Fun}(M, \mathrm{Sp})$$

then this follows from a direct application of [HA, Corollary 4.7.5.3] in its comonadic form (as used in the proof of [SAG, Theorem 5.6.6.1]). So we want to check the following:

1. The functor $d^0 : \mathrm{Fun}(M, \mathrm{Sp}) \to \mathrm{QCoh}(*) = \mathrm{Sp}$ is comonadic.
2. The Beck-Chevalley condition holds: for each $\alpha : [m] \to [n]$ in Δ_+, the diagram

$$\begin{tikzcd} \mathcal{C}^m \arrow[r, "d^0"] \arrow[d, "\alpha"'] & \mathcal{C}^{m+1} \arrow[d, "\alpha+1"] \\ \mathcal{C}^n \arrow[r, "d^0"] & \mathcal{C}^{n+1} \end{tikzcd}$$

   is right adjointable (for horizontal maps).

We first show $d^0 : \mathrm{Fun}(M, \mathrm{Sp}) \to \mathrm{Sp}$ is comonadic. By construction, d^0 takes an M-family of spectra $\{X_m\}$ to the coproduct $\oplus X_m$. The crucial observation is that each X_m is a retract of $\oplus X_m$. If $\oplus X_m \cong 0$, then each of X_m is a retract of 0, hence we know that the family $\{X_m\}$ is 0. This shows d^0 is conservative. It remains to show that d^0 preserves limits of cosimplicial diagrams in $\mathrm{Fun}(M, \mathrm{Sp})$ with split images in Sp.

A cosimplicial diagram $X^\bullet$ in $\mathrm{Fun}(M, \mathrm{Sp})$ is just an M-family of cosimplicial diagrams $\{X_m^\bullet\}$ in Sp. Under this identification $d^0(X^\bullet) = \oplus X_m^\bullet$. Denote by $X^{-\infty} = \{X_m^{-\infty}\}$ a limit of $X^\bullet$. Note that each $X_m^\bullet$ is a retract of the split cosimplicial object $\oplus X_m^\bullet$, which itself must be split by [HA, Corollary 4.7.2.13]. Therefore each of the augmented cosimplicial object $X_m^{-\infty} \to X_m^\bullet$ is split. It follows that the coproduct

$$d^0(X^{-\infty}) \simeq \oplus_m X_m^{-\infty} \to \oplus_m X_m^\bullet \simeq d^0(X_m^\bullet)$$

is also split, thus a limit diagram, as desired.

Now we move on to check adjointability. When $\alpha : [m] \to [n]$ doesn't involve $[-1]$-term, one can look at the corresponding groupoid object in Stk:

$$\begin{tikzcd}
* \times \mathbb{T}^{\times n+1} \arrow[r, "\alpha+1"] \arrow[d, "d^0"] & * \times \mathbb{T}^{\times m+1} \arrow[r, "\{0,1\}"] \arrow[d, "d^0"] & * \times \mathbb{T} \arrow[d, "d^0"] \\
* \times \mathbb{T}^{\times n} \arrow[r, "\alpha"] & * \times \mathbb{T}^{\times m} \arrow[r, "\{0\}"] & *
\end{tikzcd}$$

By Segal condition [HTT, Proposition 6.1.2.6], both right square and the total rectangle are pullback squares, so the left square is also a pullback in Stk. Then we apply [SAG, Lemma D.3.5.6] and get right adjointability on QCoh. For diagrams that involves $[-1]$, we first check

$$\begin{tikzcd}
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "d^0"] \arrow[d, "\alpha=d^0"'] & \mathrm{Sp} \arrow[d, "\alpha+1=d^1"'] \\
\mathrm{Sp} \arrow[r, "d^0"] & \mathrm{QCoh}(\mathbb{T})
\end{tikzcd}$$

is right adjointable. We make some change in notations: put $p : M \to *$ to be the projection of set M to a point, and we write $p_! \dashv p^*$ for adjunction between left Kan extension and pullback of presheaves. Put $\pi : \mathbb{T} \to *$ to be the projection of stack $\mathbb{T}$ to a point, and we write $\pi^* \dashv \pi_*$ for adjunction between pullback and pushforward of quasi-coherent sheaves. Under this notation, the diagram above reads:

$$\begin{tikzcd}
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "p_!"] \arrow[d, "p_!"'] & \mathrm{Sp} \arrow[d, "\pi^*"'] \\
\mathrm{Sp} \arrow[r, "\pi^*"] & \mathrm{QCoh}(\mathbb{T})
\end{tikzcd}$$

and the coherence comes from the construction above. Warning: the coherence isomorphism is not the 'trivial' one (and the trivial one won't be right adjointable). We need to show

$$p_!p^* \to \pi_*\pi^*p_!p^* \to \pi_*\pi^*p_!p^* \to \pi_*\pi^*$$

is an equivalence of functors. We in turn used unit for $\pi^* \dashv \pi_*$, coherence of the diagram $\pi^*p_! \cong \pi^*p_!$ and counit for $p_! \dashv p^*$. Note that both $p_!p^*$ and $\pi_*\pi^*$ are colimit preserving, so we may check on $\mathbb{S} \in \mathrm{Sp}$. Once one unwinds the definition, the map reads

$$\begin{tikzcd}
p_!p^*\mathbb{S} \arrow[r] \arrow[d, equal] & \pi_*\pi^*p_!p^*\mathbb{S} \arrow[r] \arrow[d, equal] & \pi_*\pi^*p_!p^*\mathbb{S} \arrow[r] \arrow[d, equal] & \pi_*\pi^*\mathbb{S} \arrow[d, equal] \\
\bigoplus_M \mathbb{S} \arrow[r] & \bigoplus_M \mathbb{S}[M] \arrow[r] & \bigoplus_M \mathbb{S}[M] \arrow[r] & \mathbb{S}[M]
\end{tikzcd}.$$

The first map is coproduct of unit map $\mathbb{S} \to \mathbb{S}[M]$ for the algebra $\mathbb{S}[M]$. The second map is coproduct of maps $\cdot m : \mathbb{S}[M] \to \mathbb{S}[M]$ on each direct summand $m \in M$. The third map is induced by identity map $\mathrm{id} : \mathbb{S}[M] \to \mathbb{S}[M]$ on each summand. The composition, which is $\cdot m : \mathbb{S} \to \mathbb{S}[M]$ on each summand, is an equivalence of spectra. One way to see this is that this map might be identified with $\mathbb{S}[-]$ of the map $\amalg_M * \to M$ in Spc which is an equivalence.

Wait, we are not yet done. For a general map $\alpha : [-1] \to [n]$, observe that one can factorize (unfortunately the diagram is flipped to fit in)

$$\begin{tikzcd} {[-1]} \arrow[r, "\alpha'"'] \arrow[d, "d^0"] & {[0]} \arrow[r, "\beta"'] \arrow[d, "d^0"] & {[n]} \arrow[d, "d^0"'] \\ {[0]} \arrow[r, "\alpha'+1=d^1"] & {[1]} \arrow[r, "\beta+1"] & {[n+1]} \end{tikzcd}$$

in Δ_+. This is taken to a diagram of categories where both of the small diagrams are right adjointable (now along the vertical edges), we hence conclude that the big rectangle is also right adjointable as desired. ☐

Remark 3.3.11. This is a further technical claim about adjointability that we will use in proving Proposition 3.3.1. The proof will be offered later. We claim that the diagram in Remark 3.3.9 is right adjointable for taking right adjoints of $(p_\sigma)_!$ and π_σ^*. In other words, we would like to have the diagram

$$\begin{tikzcd} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Phi_\sigma"] \arrow[d, "(p_\sigma)^*"'] & \mathrm{QCoh}([X_\sigma/\mathbb{T}]) \arrow[d, "\pi_{\sigma*}"'] \\ \mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "\Phi_M"] & \mathrm{QCoh}(B\mathbb{T}) \end{tikzcd}$$

commute, with the homotopy specified by

$$\Phi_M p_\sigma^* \to \pi_{\sigma*}\pi_\sigma^*\Phi_M p_\sigma^* \to \pi_{\sigma*}\Phi_\sigma p_{\sigma!} p_\sigma^* \to \pi_{\sigma*}\Phi_\sigma$$

where we used the unit for $\pi_\sigma^* \dashv \pi_{\sigma*}$, coherence $\pi_\sigma^*\Phi_M \cong \Phi_\sigma p_{\sigma!}$ and counit for $p_{\sigma!} \dashv p_\sigma^*$.

Now we are ready to prove the main theorem of the section.

Proof of Proposition 3.3.1. Naturality of the mentioned functors has been explained in Remark 3.3.9. What's left to check is that for each σ, Φ_σ is an equivalence of categories. Given Remark 3.3.11 we are in the situation of comparing monadic adjunction [HA, Proposition 4.7.3.16]: each of the category sits over another category that they are monadic over. We claim that the condition to check to apply [HA, Proposition 4.7.3.16] is readily obvious in our case: (1) is true as our diagram is obtained by taking right adjoints of a right adjointable diagram. (2) and (3) follows from both p_σ^* and $\pi_{\sigma*}$ are colimit preserving. (4) is true because π is affine, note that p^* is also conservative since p is essentially surjective. And (5) requires essentially to check if the diagram is itself left adjointable: this should follow again from the fact that the diagram itself comes from taking right adjoints of a right adjointable diagram, see [HTT, Remark 7.3.1.3]. ☐

Proof of Remark 3.3.11. One can prove this adjointability along the following line. We look at the map between augmented action diagrams which presents the map $[X_\sigma/\mathbb{T}] \to B\mathbb{T}$

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_\sigma \times \mathbb{T} \times \mathbb{T} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & X_\sigma \times \mathbb{T} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X_\sigma \arrow[r] \arrow[d] & {[X_\sigma/\mathbb{T}]} \arrow[d] \\ \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbb{T} \times \mathbb{T} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{T} \arrow[r, shift left] \arrow[r, shift right] & * \arrow[r] & B\mathbb{T} \end{tikzcd}.$$

For each $\alpha : [m] \to [n]$ in simplex category, we have the diagram

$$\begin{tikzcd} X_\sigma \times \mathbb{T}^{\times n} \arrow[r, "\alpha"] \arrow[d] & X_\sigma \times \mathbb{T}^{\times m} \arrow[r] \arrow[d] & {[X_\sigma/\mathbb{T}]} \arrow[d] \\ \mathbb{T}^{\times n} \arrow[r, "\alpha"] & \mathbb{T}^{\times m} \arrow[r] & B\mathbb{T} \end{tikzcd}.$$

where both the big rectangle and right square are pullbacks, so the left square is also a pullback (see Lemma 7.1.1). Hence from [SAG, Lemma D.3.5.6] we learn that after taking QCoh, the left square becomes

$$\begin{tikzcd} \mathrm{QCoh}(X_\sigma \times \mathbb{T}^{\times n}) & \mathrm{QCoh}(X_\sigma \times \mathbb{T}^{\times m}) \arrow[l, "\alpha"'] \\ \mathrm{QCoh}(\mathbb{T}^{\times n}) \arrow[u] & \mathrm{QCoh}(\mathbb{T}^{\times m}) \arrow[l, "\alpha"'] \arrow[u] \end{tikzcd}$$

which is right adjointable (for vertical maps). By [HA, Corollary 4.7.4.18] this implies that the action diagram, viewed as $[n] \mapsto [\mathrm{QCoh}(\mathbb{T}^{\times n}) \to \mathrm{QCoh}(X_\sigma \times \mathbb{T}^{\times n})]$, lifts to a simplicial object in $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat})$, and the augmented action diagram is a limit diagram in $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat})$. Now one can similarly view the diagram

$$\begin{tikzcd} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Phi_\sigma"] & \mathrm{QCoh}([X_\sigma/\mathbb{T}]) \\ \mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "\Phi_M"] \arrow[u, "(p_\sigma)_!"] & \mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\pi_\sigma^*"'] \end{tikzcd}$$

as an augmentation to the simplicial object $[n] \mapsto [\mathrm{QCoh}(\mathbb{T}^{\times n}) \to \mathrm{QCoh}(X_\sigma \times \mathbb{T}^{\times n})]$ in $\mathrm{Fun}(\Delta^1, \mathrm{Cat})$ and the question of its right adjointability reduces to asking if this augmentation lifts to $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Cat})$. The only thing left to check is right adjointability of the diagram (for taking right adjoints of the vertical arrows)

$$\begin{tikzcd} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Phi_\sigma"] & \mathrm{QCoh}(X_\sigma) \\ \mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, "\Phi_M"] \arrow[u, "p_{\sigma!}"] & \mathrm{QCoh}(*) \arrow[u, "\pi^*"'] \end{tikzcd}.$$

This is readily true once one unwinds the definition as in the proof of Theorem 3.3.10.

$\square$

4 Constructible sheaves

Since the seminal book of [HTT], it became obvious that the convenient generality for the study of sheaves on manifolds (or more generally, locally compact Hausdorff topological spaces) is the presentable category

$$\mathcal{S}\mathrm{hv}(X;\mathrm{Spc})$$

and its stablization $\mathcal{S}\mathrm{hv}(X;\mathrm{Sp})$. We will adopt this convention and setup various functors involved in the coherent-constructible correspondence. This approach makes several constructions easier. First of all, the yoga of six-functor provides a neat way to write down convolution products defined on the category of sheaves on a (locally compact Hausdorff) topological group and related functors. Secondly, the recent advances in exodromy [13, 7] makes it gracefully simple to work with (large categories of) constructible sheaves.

The main goal of this section is to write down a symmetric monoidal functor from the combinatorial model to the category of sheaves on a real vector space. To do so, we first recall some generalities on convolution products for sheaves on real vector spaces. Then we move onto a digression on lax symmetric monoidal structure on taking relative homology. This is used in the next part to provide a combinatorial-constructible comparison functor along with its lax symmetric monoidal structure. After that we take a turn to recall some generalities on constructible sheaves and pin down a tratification following [9]. As a consequence we show the comparison functor is fully faithful for a smooth fan and its image are all constructible for the stratification we introduced. Finally we take a detour to collect a technical fact about descent along idempotent algebras in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$. Putting all these together, we conclude that for a smooth projective fan, the combinatorial-constructible comparison functor we constructed is fully faithful and symmetric monoidal. We leave the characterization of the image to the next section.

4.1 Convolution product for sheaves on real vector spaces

Remark 4.1.1 (Hypercompleteness)**.** One needs not to worry about hypercompleteness in our situation, as we will only care about sheaves on finite dimensional manifolds. In particular, equivalence of sheaves can be detected stalk-wise.

Take a finite dimensional real vector space $V \cong \mathbb{R}^{\oplus n}$. It acquires the structure of a commutative algebra in $(\mathrm{LCH}, \times)$ via addition of vectors

$$+ : V \times V \to V.$$

This equips $\mathcal{S}\mathrm{hv}(V;\mathrm{Sp})$ with a binary operation

$$* : \mathcal{S}\mathrm{hv}(V;\mathrm{Sp}) \times \mathcal{S}\mathrm{hv}(V;\mathrm{Sp}) \to \mathcal{S}\mathrm{hv}(V;\mathrm{Sp})$$

defined by

$$\mathcal{F} * \mathcal{G} := +_!(\mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{pr}_2^*\mathcal{G}).$$

This operation could be made coherently into a symmetric monoidal structure as described in the following.

Construction 4.1.2 (Convolution product)**.** Recall that the 'six-functor formalism' on LCH is a lax symmetric monoidal functor

$$\mathcal{D} : \mathrm{Corr}(\mathrm{LCH}, \mathrm{all}) \longrightarrow \mathrm{Cat}$$

and we have another symmetric monoidal functor ('Reg' for right leg)

$$\mathrm{Reg} : \mathrm{LCH} \to \mathrm{Corr}(\mathrm{LCH}, \mathrm{all})$$

which on objects acts as $X \mapsto X$ and on morphisms acts as

$$[X \xrightarrow{f} Y] \mapsto \left[\begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \right]$$

We define the composition

$$D_!(-) := \mathcal{D} \circ \mathrm{Reg} : \mathrm{LCH} \to \mathrm{Cat}$$

which is again a lax symmetric monoidal functor. This implies that for every commutative algebra $A \in \mathrm{CAlg}(\mathrm{LCH})$, the category $D_!(A) = \mathcal{S}\mathrm{hv}(A; \mathrm{Sp})$ acquires a symmetric monoidal structure through the functoriality of $D_!$. We name the monoidal product convolution and write as $*$.

Proposition 4.1.3. We will use the following properties of the convolution product:

1. The convolution product $*$ is cocontinuous in each variable.
2. Let $X, Y \subseteq V$ be polyhedral open subsets of a real vector space. We can compute very explicitly
   $$\underline{\mathbb{S}}_X * \underline{\mathbb{S}}_Y \cong \underline{\mathbb{S}}_{X+Y}[-\dim(V)]$$
   where
   $$X + Y := \{x + y : x \in X, y \in Y\}$$
   is the Minkowski sum of the subsets.

Proof. Point 1 follows from the fact that $*$-pullback, $\otimes$ of sheaves and !-pushforward all preserve colimits. For the second point, we apply proper base change and learn that

$$\underline{\mathbb{S}}_X * \underline{\mathbb{S}}_Y \cong +_{|X \times Y!} \underline{\mathbb{S}}_{X \times Y}$$

where $+$ is restricted to a map $X \times Y \to \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. By the fact that X and Y are polyhedral open subsets, one can prove this map $+$ is a smooth $\mathbb{R}^n$ bundle over its image $X + Y \subseteq \mathbb{R}^n$. It follows that the !-pushforward of $\underline{\mathbb{S}}_{X \times Y}$ along the addition map is locally constant. And the computation reduces to the fact that for a projection $p : Z \times \mathbb{R}^n \to Z$ one has

$$p_! \underline{\mathbb{S}} = \underline{\mathbb{S}}[-n].$$

and that $X + Y$ is contractible. □

Remark 4.1.4. As a side remark, polyhedral opens form a basis for the topology. In principle one can compute the convolution of any two sheaves using the above facts.

4.2 Digression: multiplicative structures on Betti homology

As we have seen above, the addition operation on the finite dimensional real vector space $M_{\mathbb{R}}$ makes it into a commutative monoid in the 1-category LCH. Thus the slice category $\mathrm{LCH}_{/M_{\mathbb{R}}}$ acquires a symmetric monoidal structure which can be informally defined as follows:

$$(X, f) \otimes (Y, g) := (X \times Y, f + g)$$

(see [HA, Theorem 2.2.2.4] for a general construction). We denote by $(\mathrm{LCH}_{/M_{\mathbb{R}}}, \otimes)$ this symmetric monoidal category. The structure of commutative monoid of $M_{\mathbb{R}}$ was also used to provide a convolution product on the category of sheaves on $M_{\mathbb{R}}$, and these two categories are indeed related. The goal of this digression is to explain the following construction.

Construction 4.2.1 (Taking homology is symmetric monoidal)**.** There is a lax symmetric monoidal functor

$$\Gamma_{M_{\mathbb{R}}} : (\mathrm{LCH}_{/M_{\mathbb{R}}}, \otimes) \longrightarrow (\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}), *)$$

which on objects acts by

$$(X, f) \longmapsto f_! f^! \omega_{M_{\mathbb{R}}}$$

where $\omega_{M_{\mathbb{R}}}$ is the dualizing sheaf on $M_{\mathbb{R}}$.

Remark 4.2.2 (A similar construction in the literature)**.** Let us immediately point out that, a very similar and more flexible construction was carried out (in ℓ-adic context) by Gaitsgory-Lurie in [12, Chapter 3]. An elaboration (in Betti context) of the ideas in that paper would produce a more general construction that easily provides the functor as above (for example, one could allow the base groups G to vary). We however decided to give an ad-hoc and cheap construction of the functor that we need in this note to minimize recollection of general theory (also because the situation we are dealing with here is extremely simple). We will return to this construction elsewhere.

The construction is technical in contrast to the simple application we have in mind. The reader is advised to skip the rest of this section and come back later. Before we go into the construction, here is a rough plan.

Remark 4.2.3 (Preview of strategy)**.** We will define a symmetric monoidal category $\mathcal{S}\mathrm{hv}_!$ which comes with a symmetric monoidal functor

$$p : \mathcal{S}\mathrm{hv}_! \to \mathrm{LCH}_{/M_{\mathbb{R}}}.$$

We will then produce a lax symmetric monoidal functor as a section of p:

$$s : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}\mathrm{hv}_!,$$

and another symmetric monoidal functor

$$t : \mathcal{S}\mathrm{hv}_! \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

So that the composition

$$t \circ s : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

is what we want.

Remark 4.2.4 (A rough description of the players)**.** Here is a heuristic description of the categories and functors appearing in the previous remark. One can describe the category $\mathcal{S}hv_!$ as follows. An object in $\mathcal{S}hv_!$ is a pair $(X, f, \mathcal{F})$ where (X, f) is an object of $\mathrm{LCH}_{/M_{\mathbb{R}}}$ and $\mathcal{F} \in \mathcal{S}hv(X; \mathrm{Sp})$. A map (h, ϕ) from $(X, f, \mathcal{F})$ to $(Y, g, \mathcal{G})$ consists of a map $h : (X, f) \to (Y, g)$ in $\mathrm{LCH}_{/M_{\mathbb{R}}}$ and a map $\phi : h_!\mathcal{F} \to \mathcal{G}$ in $\mathcal{S}hv(Y; \mathrm{Sp})$. The symmetric monoidal structure is a mixture of tensor product in $\mathrm{LCH}_{/M_{\mathbb{R}}}$ and exterior product of sheaves: $(X, f, \mathcal{F}) \otimes (Y, g, \mathcal{G}) = (X \times Y, f + g, \mathcal{F} \boxtimes \mathcal{G})$. With this we can also roughly describe the functors. The functor

$$p : \mathcal{S}hv_! \to \mathrm{LCH}_{/M_{\mathbb{R}}}$$

is the forgetful functor taking $(X, f, \mathcal{F})$ to (X, f). The functor

$$s : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}hv_!$$

takes (X, f) to $(X, f, f^!\omega_{M_{\mathbb{R}}}) \in \mathcal{S}hv_!$. The functor

$$t : \mathcal{S}hv_! \to \mathcal{S}hv(M_{\mathbb{R}}; \mathrm{Sp})$$

takes $(X, f, \mathcal{F})$ to $f_!\mathcal{F} \in \mathcal{S}hv(M_{\mathbb{R}}; \mathrm{Sp})$. This casual description suggests that $t \circ s$ supplies the construction we need. Note that we are not even mentioning what these functor does to maps or higher coherences, nor multiplicative structure. This is what makes the construction technical.

We start by constructing $\mathcal{S}hv_!$.

Notation 4.2.5. The forgetful functor forgetful : $\mathrm{LCH}_{/M_{\mathbb{R}}} \longrightarrow \mathrm{LCH}$ is symmetric monoidal and we have a composition of functors

$$\mathrm{LCH}_{/M_{\mathbb{R}}} \xrightarrow{\text{forgetful}} \mathrm{LCH} \xrightarrow{D_!} \mathrm{Cat}$$

where the later functor comes from Construction 4.1.2. We abuse notation and again write the composition as

$$D_! : \mathrm{LCH}_{/M_{\mathbb{R}}} \longrightarrow \mathrm{Cat}$$

when there is no danger of confusion. Note that this composition is also a lax symmetric monoidal functor.

The category $\mathcal{S}hv_!$ is just the unstraightening (i.e. Grothendieck construction) of the functor $D_! : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathrm{Cat}$, and the symmetric monoidal structure actually comes with unstraightening - using a symmetric monoidal version of the Grothendieck construction that we recall as follows. See [16, A.2.1][12, Proposition 3.3.4.11] [30, Theorem 2.1] for a history of the theorem.

Theorem 4.2.6 (Symmetric monoidal Grothendieck construction)**.** Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. There is an equivalence of categories

$$\mathrm{coCart}_{\mathcal{C}}^{\mathbb{E}_\infty} \simeq \mathrm{Fun}^{\mathrm{lax}\otimes}(\mathcal{C}, \mathrm{Cat})$$

which is compatible with the straightening-unstraightening equivalence

$$\mathrm{coCart}_{\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}).$$

Let's immediately recall the definition of the objects appearing in the theorem.

1. For a category $\mathcal{C}$, the category $\mathrm{coCart}_{\mathcal{C}}$ is defined to be the category of coCartesian fibrations over $\mathcal{C}$ with coCartesian edges preserving functors over $\mathcal{C}$ as morphisms.
2. If $(\mathcal{C}, \otimes)$ is a symmetric monoidal category with $\mathcal{C}^{\otimes} \to \mathbb{E}_{\infty}^{\otimes}$[7] being the underlying operad, the category $\mathrm{coCart}_{\mathcal{C}}^{\mathbb{E}_{\infty}^{\otimes}}$ is the category of $\mathbb{E}_{\infty}^{\otimes}$-monoidal coCartesian fibrations over $\mathcal{C}$ of [30, Definition 1.11]. It is defined to be the full subcategory of $\mathrm{coCart}_{\mathcal{C}^{\otimes}}$ spanned by those coCartesian fibrations $\mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ such that the underlying $\mathcal{D} \to \mathcal{C}$ is a coCartesian fibration and $\mathbb{E}_{\infty}^{\otimes}$-monoidal operations preserves coCartesian edges.

Definition 4.2.7. Applying the symmetric monoidal Grothendieck construction to lax symmetric monoidal functor $D_! : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathrm{Cat}$ produces an $\mathbb{E}_{\infty}^{\otimes}$-monoidal coCartesian fibration

$$p^{\otimes} : \mathcal{S}\mathrm{hv}_!^{\otimes} \longrightarrow \mathrm{LCH}^{\otimes}_{/M_{\mathbb{R}}}$$

and $\mathcal{S}\mathrm{hv}_!$ is defined to be the underlying category of the operad $\mathcal{S}\mathrm{hv}_!^{\otimes}$. We write

$$p : \mathcal{S}\mathrm{hv}_! \longrightarrow \mathrm{LCH}_{/M_{\mathbb{R}}}$$

for the underlying structure map making $\mathcal{S}\mathrm{hv}_!$ into a coCartesian fibration over $\mathrm{LCH}_{/M_{\mathbb{R}}}$.

In view of [HA, Remark 2.1.2.14] and Lemma 4.2.13, the structure map $p^{\otimes}$ is a map of $\mathbb{E}_{\infty}^{\otimes}$-monoidal category. In other words, it presents p as a symmetric monoidal functor. This functor p won't appear in the final construction, but we will introduce other players that revolve around $\mathcal{S}\mathrm{hv}_!$ and p. We start with introducing the following diagram

$$\begin{tikzcd} \mathrm{LCH}_{/M_{\mathbb{R}}} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\underline{M_{\mathbb{R}}}"'] \arrow[r, phantom, "\Downarrow h"] & \mathrm{LCH}_{/M_{\mathbb{R}}} \arrow[r, "D_!"] & \mathrm{Cat} \end{tikzcd}$$

where $\underline{M_{\mathbb{R}}}$ is the constant functor at $(M_{\mathbb{R}}, \mathrm{id}) \in \mathrm{LCH}_{/M_{\mathbb{R}}}$ and h is the natural transformation to the constant functor on terminal object. Note that h is actually a natural transformation between lax symmetric monoidal functors. Now we apply Grothendieck construction to $D_!(h) : D_! \circ \mathrm{id} \to D_! \circ \underline{M_{\mathbb{R}}}$ and get the following diagram

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}_! \arrow[rr, "\mathrm{Un}(D_!(h))"] \arrow[dr, "p"'] & & \mathrm{LCH}_{/M_{\mathbb{R}}} \times \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \arrow[dl, "q"] \\ & \mathrm{LCH}_{/M_{\mathbb{R}}} & \end{tikzcd}$$

[7] Note that $\mathbb{E}_{\infty}^{\otimes}$ is just a fancy name for Fin_*.

underlying the diagram of operads supplied by the symmetric monoidal Grothendieck construction

$$\begin{tikzcd}
\mathcal{S}hv_!^{\otimes} \arrow[rr, "\mathrm{Un}(D_!(h))^{\otimes}"] \arrow[dr, "p^{\otimes}"] \arrow[ddr, bend right, "\pi_2^{\otimes}"'] & & (\mathrm{LCH}_{/M_{\mathbb{R}}} \times \mathcal{S}hv(M_{\mathbb{R}};\mathrm{Sp}))^{\otimes} \arrow[dl, "q^{\otimes}"'] \arrow[ddl, bend left, "\pi_3^{\otimes}"] \\
& \mathrm{LCH}^{\otimes}_{/M_{\mathbb{R}}} \arrow[d, "\pi_1^{\otimes}"] & \\
& \mathbb{E}_{\infty}^{\otimes} &
\end{tikzcd}.$$

In the diagram, $\pi_i^{\otimes}$ are the structure maps of the operads. Our first goal is to produce the right adjoint r of $\mathrm{Un}(D_!(h))$ along with the lax symmetric monoidal structure on it.

Proposition 4.2.8. The functor $\mathrm{Un}(D_!(h)) : \mathcal{S}hv_! \to \mathrm{LCH}_{/M_{\mathbb{R}}} \times \mathcal{S}hv(M_{\mathbb{R}};\mathrm{Sp})$ admits a right adjoint r. Moreover, r admits a lax symmetric monoidal structure.

Proof. To begin with, we want to show that $\mathrm{Un}(D_!(h))$ has a right adjoint functor r. We know the following facts about $\mathrm{Un}(D_!(h))$: that the restriction of $\mathrm{Un}(D_!(h))$ to each fiber over $\mathrm{LCH}_{/M_{\mathbb{R}}}$ has a right adjoint and that $\mathrm{Un}(D_!(h))$ preserves coCartesian edges since it is unstraightened from a natural transformation. Knowing these one can apply [HA, Proposition 7.3.2.6] and learn that it has a right adjoint (even relative to $\mathrm{LCH}_{/M_{\mathbb{R}}}$). By construction, r restricts to fiberwise right adjoint. Now we explain the lax symmetric monoidal structure on r. From Lemma 4.2.13 we learn that $\mathrm{Un}(D_!(h)^{\otimes})$ is a map of $\mathbb{E}_{\infty}^{\otimes}$-monoidal categories, i.e. $\mathrm{Un}(D_!(h))$ is a symmetric monoidal functor. Now one can invoke [HA, Corollary 7.3.2.7] and learn that r has a structure of lax symmetirc monoidal functor. ∎

We have achieved our first goal. Our next player is the functor

$$\mathrm{id} \times \underline{\omega_{M_{\mathbb{R}}}} : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathrm{LCH}_{/M_{\mathbb{R}}} \times \mathcal{S}hv(M_{\mathbb{R}};\mathrm{Sp}).$$

As the name suggests, it is induced by $\mathrm{id} : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathrm{LCH}_{/M_{\mathbb{R}}}$ and the constant functor $\underline{\omega_{M_{\mathbb{R}}}} : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}hv(M_{\mathbb{R}};\mathrm{Sp})$. Recall that we have the dualizing sheaf $\omega_{M_{\mathbb{R}}}$ defined to be

$$\omega_{M_{\mathbb{R}}} := \pi^! \mathbb{1}_{\mathcal{S}hv(*;\mathrm{Sp})} \in \mathcal{S}hv(M_{\mathbb{R}};\mathrm{Sp})$$

where $\pi : M_{\mathbb{R}} \to *$ is the map from $M_{\mathbb{R}}$ to final object $*$. Let's make an observation on $\omega_{M_{\mathbb{R}}}$.

Proposition 4.2.9. The *dualizing sheaf* $\omega_{M_{\mathbb{R}}}$ acquires a structure of commutative algebra for the convolution product.

Proof. This follows from the fact that $\pi^! : \mathcal{S}hv(*;\mathrm{Sp}) \to \mathcal{S}hv(M_{\mathbb{R}};\mathrm{Sp})$ has the structure of a lax symmetric monoidal functor where both side has the convolution symmetric monoidal structure. In addition, the convolution product on $\mathcal{S}hv(*;\mathrm{Sp})$ is the same as the point-wise tensor product that is usually used. The lax symmetric monoidal structure on $\pi^!$ is acquired by the (strong) symmetric monoidal structure on its left adjoint $\pi_!$. To be more precise: the map π is actually a map of commutative monoids in LCH. Hence by construction of convolution tensor product, π induces a symmetric monoidal functor

$$\pi_! : \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(*; \mathrm{Sp}).$$

We again take advantage of [HA, Corollary 7.3.2.7] and get a lax symmetric monoidal structure on its right adjoint

$$\pi^! : \mathcal{S}\mathrm{hv}(*; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

In particular it takes $\mathbb{1}_{\mathcal{S}\mathrm{hv}(*;\mathrm{Sp})}$ to a commutative algebra as desired. □

The commutative algebra structure on $\omega_{M_{\mathbb{R}}}$ furnishes the constant functor

$$\underline{\omega_{M_{\mathbb{R}}}} : \mathrm{LCH}_{M_{\mathbb{R}}} \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

with a lax symmetric monoidal structure. From this discussion, one learns that

Proposition 4.2.10. The functor $\mathrm{id} \times \underline{\omega_{\mathbb{R}}}$ has a structure of lax symmetric monoidal functors.

Proof. By previous discussion, it is a product of two lax symmetric monoidal functors, hence has a lax symmetric monoidal structure. □

We arrive at the following diagram

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}_! \arrow[rr, "\mathrm{Un}(D_!(h))"] \arrow[dr, "p"'] & & \mathrm{LCH}_{/M_{\mathbb{R}}} \times \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \arrow[ll, bend right, "r"'] \arrow[dl, "q"] \arrow[r, "p_2"] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \\
& \mathrm{LCH}_{/M_{\mathbb{R}}} \arrow[ur, bend right, "\mathrm{id} \times \underline{\omega_{M_{\mathbb{R}}}}"'] & &
\end{tikzcd}$$

where we are going to make use of the red-colored functors, which are lax symmetric monoidal. We conclude the construction by a composition of these four functors: according to the plan, we constructed the following lax symmetric monoidal functors

$$s = r \circ (\mathrm{id} \times \underline{\omega_{M_{\mathbb{R}}}}) : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}\mathrm{hv}_!$$

and

$$t = p_2 \circ \mathrm{Un}(D_!(h)) : \mathcal{S}\mathrm{hv}_! \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

so that the composition

$$t \circ s : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

is what we aimed for.

Definition 4.2.11 (Sheaf of relative homology)**.** We define the lax symmetric monoidal functor

$$\Gamma_{M_{\mathbb{R}}} = t \circ s : \mathrm{LCH}_{/M_{\mathbb{R}}} \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

as the output of the construction. And we call $\Gamma_{M_{\mathbb{R}}}(X, f)$ the sheaf of homology of X relative to $M_{\mathbb{R}}$. The naming follows from the fact that after further !-pushfoward to a point, the sheaf $\Gamma_{M_{\mathbb{R}}}(X, f)$ is taken to the homology of X:

$$\pi_! \Gamma_{M_{\mathbb{R}}}(X, f) \cong C_c(X, \omega_X).$$

Note however it might be misleading since the stalk of the sheaf $\Gamma_{M_{\mathbb{R}}}(X, f)$ needs not to be the homology of the fiber.

Variant 4.2.12. For later purpose, we also abusively write the restriction of the functor $\Gamma_{M_{\mathbb{R}}}$ to the full subcategory of polyhedral subsets as

$$\Gamma_{M_{\mathbb{R}}} : \mathrm{Poly}(M_{\mathbb{R}}) \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

Moreover, the category of $\mathrm{Poly}(M_{\mathbb{R}})$ carries a symmetric monoidal structure of Minkowski sum that makes the inclusion functor

$$\mathrm{Poly}(M_{\mathbb{R}}) \longrightarrow \mathrm{LCH}_{/M_{\mathbb{R}}}$$

lax symmetric monoidal, we hence conclude that the functor

$$\Gamma_{M_{\mathbb{R}}} : \mathrm{Poly}(M_{\mathbb{R}}) \to \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

is also lax symmetric monoidal.

We end the section by collecting the following elaboration of the argument in [HA, Proposition 2.1.2.12]. See also [Kerodon, 01UL].

Lemma 4.2.13. We have the following concerning coCartesian edges and coCartesian fibrations:

1. Consider the following commuting diagram of categories:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "p"] \arrow[dr, "q \circ p"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{E} & \end{tikzcd}$$

   If both q and p are coCartesian fibraitons, then so is $q \circ p$. Moreover, given an edge $f \in \mathcal{E}$ and a $q \circ p$-coCartesian lift $f' \in \mathcal{C}$ of f, there exists an edge $f'' \in \mathcal{D}$ which is a q-coCartesian lift of f and $p(f')$ is equivalent to f''. Consequently, p preserves coCartesian lifts from $\mathcal{E}$.

2. Consider the following commuting diagram of categories:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "p"] \arrow[dr, "q \circ p"] \arrow[ddr, "\pi_2"'] & & \mathcal{D} \arrow[dl, "q"'] \arrow[ddl, "\pi_3"] \\ & \mathcal{E} \arrow[d, "\pi_1"] & \\ & \mathcal{O} & \end{tikzcd} .$$

   Assume that q, $q \circ p$ and π_1 are coCartesian fibrations. Assume further that p preserves coCartesian lifts from $\mathcal{E}$. Then p preserves coCartesian lifts from $\mathcal{O}$.

Proof. 1. That a composition of coCartesian fibration is coCartesian fibration was proved in [HTT, Proposition 2.4.2.3]. For the second part, given $f \in \mathcal{E}$ and a $q \circ p$ coCartesian lift $f' \in \mathcal{C}$ of f, one can choose $f'' \in \mathcal{D}$ to be a q-coCartesian lift of f. Let $\bar{f}' \in \mathcal{C}$ be a p-coCartesian lift of f'', then $\bar{f}'$ would also be a $q \circ p$-coCartesian lift of f using [HTT, Proposition 2.4.1.3]. We conclude that $\bar{f}'$ is equivalent to f' and hence $p(f')$ is equivalent to $p(\bar{f}') = f''$. The last claim about p preserves coCartesian lifts from $\mathcal{E}$ follows.

2. Let $f' \in \mathcal{C}$ be a π_2-coCartesian lift of $f \in \mathcal{O}$. By previous item, we might assume f' is a $q \circ p$-coCartesian lift of $q \circ p(f')$. Then by assumption on p, the image $p(f') \in \mathcal{D}$ is a q-coCartesian lift of $q \circ p(f')$, hence is a π_3-coCartesian lift of $f \in \mathcal{O}$ as desired.

□

4.3 Combinatorial v.s. constructible

Now we take advantage of the functor $\Gamma_{M_{\mathbb{R}}}$ from previous section to write down the combinatorial-constructible comparison functor. First we give a quick idea of the construction.

Fix a toric data (N, Σ) and pick a cone σ in the fan Σ. Recall that we defined the combinatorial category $\Theta(\sigma)$ to be a full subcategory of $\mathrm{Poly}(M_{\mathbb{R}})$. The category of $\mathrm{Poly}(M_{\mathbb{R}})$ has a symmetric monoidal structure given by Minkowski sum and one can think of the symmetric monoidal structure on $\Theta(\sigma)$ as inherited from the inclusion (to be very precise, $\Theta(\sigma)$ includes into the full subcategory $\mathrm{Mod}_{\sigma^\vee}\mathrm{Poly}(M_{\mathbb{R}})$ over the idempotent algebra $\sigma^\vee \in \mathrm{Poly}(M_{\mathbb{R}})$ and this inclusion is symmetric monoidal). Post-composing this inclusion with $\Gamma_{M_{\mathbb{R}}}$ that we have defined earlier, we get a combinatorial-to-constructible comparison functor. The goal of this section is to construct this functor and present its functoriality along Σ.

We start with constructing a family of idempotent algebras in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$. Here is a technical observation of the interaction of $\Gamma_{M_{\mathbb{R}}}$ with polytopes which is conceptually helpful, though not necessarily needed.

Lemma 4.3.1 (Relative homology sheaf for closed and open polytopes agree)**.** For a closed polyhedral subset (of top dimension) $\overline{U}$ and its interior U, the map of sheaves

$$\Gamma_{M_{\mathbb{R}}}(U) \to \Gamma_{M_{\mathbb{R}}}(\overline{U})$$

induced from $U \to \overline{U}$ is an equivalence. Note that left hand side is a more familiar object: the extension-by-zero of a shift of constant sheaf on an open subset.

Proof. This could be proved by comparing the recollement sequence for U and $\overline{U}$. Here we supply a more direct proof. In this case, one can check equivalence on stalks. By proper base-change, it is easy to check for $x \notin \partial\overline{U}$ the map is an equivalence on stalk at x. It remains to check that at $x \in \partial\overline{U}$ the stalk of right hand side vanishes (again by proper base-change it vanishes on the left hand side). To compute the stalk, one can pick a family of open balls D_i of shrinking radius centered at x and compute

$$\Gamma_{M_{\mathbb{R}}}(\overline{U})_x \cong \mathrm{colim}\, \Gamma_{M_{\mathbb{R}}}(\overline{U})(D_i).$$

To compute the right hand side, one makes identification $\omega_{M_{\mathbb{R}}} \cong \underline{\mathbb{S}}[n]$ and apply proper base-change to get

$$\Gamma_{M_{\mathbb{R}}}(\overline{U})(D_i) \cong (i_{\overline{U}!} i^!_{\overline{U}} \underline{\mathbb{S}}[n])(D_i) \cong \mathrm{fib}[(\underline{\mathbb{S}}(D_i) \to \underline{\mathbb{S}}(D_i \setminus \overline{U})][n]$$

and since $\overline{U}$ is a polyhedral, for sufficiently small ball $D_i \to D_i \setminus \overline{U}$ is a homotopy equivalence and hence we win. □

Proposition 4.3.2 (Dualizing sheaf of a cone is an idempotent algebra)**.** For each $\sigma \in \Sigma$, the object $\sigma^\vee \in \mathrm{Poly}(M_{\mathbb{R}})$ has the structure of an idempotent algebra. Thus, we might think of $\sigma^\vee$ as a diagram of idempotent algebras in $\mathrm{Poly}(M_{\mathbb{R}})$ indexed by $\sigma \in \Sigma^{\mathrm{op}}$. Moreover, the image of each $\sigma^\vee$

under $\Gamma_{M_{\mathbb{R}}}$ is also an idempotent algebra. So we get $\Gamma_{M_{\mathbb{R}}}(\sigma^\vee) = \omega_{\sigma^\vee}$ as a diagram of idempotent algebras in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$ indexed by Σ^{op}.

Proof. The first observation is direct, using that $\sigma^\vee + \sigma^\vee = \sigma^\vee$ since it's a cone. For the second assertion, one needs to compute that the multiplication map of the algebra $\Gamma_{M_{\mathbb{R}}}(\sigma^\vee)$ is an isomorphism

$$\Gamma_{M_{\mathbb{R}}}(\sigma^\vee) * \Gamma_{M_{\mathbb{R}}}(\sigma^\vee) \xrightarrow{\cong} \Gamma_{M_{\mathbb{R}}}(\sigma^\vee).$$

By previous lemma, it is equivalent to showing that $\Gamma_{M_{\mathbb{R}}}(\sigma^{\vee,\circ})$ is an idempotent algebra. Now that we are working with a polyhedral open subset we can unpack the definition of multiplication map and this reduces to the same computation as in Proposition 4.1.3 up to a shift. ☐

Corollary 4.3.3. There is a diagram in SMCat indexed by Σ^{op} given by

$$\sigma \mapsto \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \in \mathrm{SMCat}.$$

Furthermore, there is a symmetric monoidal left adjoint functor

$$L : \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

given by tensoring with $\omega_{\sigma^\vee}$ in each component. In particular, this functor has a lax symmetric monoidal right adjoint

$$R : \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

Note that since each $\omega_{\sigma^\vee}$ is an idempotent algebra, the forgetful

$$\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

is a fully faithful functor, hence R is also fully faithful. One can describe the functor R explicitly as follows: given an object in the limit, one applies forgetful functor to $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$ pointwise to get a diagram in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$ and then take the limit. See Proposition 4.5.4 for more on this functor R and that it is always an equivalence for a smooth projective fan, hence in particular symmetric monoidal.

We move on to the main construction. The following proposition sketches our goal and the construction will be provided right after.

Proposition 4.3.4. There is a symmetric monoidal functor

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

where the left-hand side has the Day convolution tensor product and right-hand side has the convolution product of sheaves. Moreover, these functors are natural in $\sigma \in \Sigma^{\mathrm{op}}$ that they assemble into a natural transformation of diagrams in SMCat indexed by $\sigma \in \Sigma^{\mathrm{op}}$. Hence taking limit produces

$$\lim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\lim \Psi_\sigma} \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \xrightarrow{R} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

The first functor is symmetric monoidal. It is fully faithful when the fan is smooth, as shown in Corollary 4.4.16. The latter functor is the right adjoint functor R in Corollary 4.3.3 which is lax symmetric monoidal and fully faithful. It is symmetric monoidal when the fan is smooth and projective, as shown in Proposition 4.5.4. In conclusion, when the fan Σ is smooth and projective we have a symmetric monoidal fully faithful functor

$$\Psi_\Sigma : \lim_{\Sigma^{op}} \mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \xrightarrow{\lim \Psi_\sigma} \lim_{\Sigma^{op}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}).$$

We first construct Ψ_σ pointwise. Although this will not be necessarily needed, since later we will construct it again with full functoriality, it should offer a general feeling of the construction.

Construction 4.3.5 (Construction of Ψ_σ pointwise)**.** Fix $\sigma \in \Sigma$, consider the composition of lax symmetric monoidal functors:

$$\Theta(\sigma) \longrightarrow \mathrm{Poly}(M_\mathbb{R}) \xrightarrow{\Gamma_{M_\mathbb{R}}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

where the first functor is the canonical inclusion (recall that $\Theta(\sigma)$ is by definition a full subcategory of $\mathrm{Poly}(M_\mathbb{R})$) and the second functor is $\Gamma_{M_\mathbb{R}}$ as we constructed in Definition 4.2.11. Now we observe that the image of $\Theta(\sigma)$ all lies in the full subcatgory of $\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$. It follows that we have a lax symmetric monoidal functor

$$\psi_\sigma : \Theta(\sigma) \to \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

which is readily checked to be symmetric monoidal functor by Proposition 4.1.3. Now one can left Kan extend this to a symmetric monoidal functor

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \to \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

which is what we want.

Now we construct Ψ_σ with functoriality along σ.

Remark 4.3.6 (Functoriality of Ψ_σ along σ)**.** To provide functoriality of symmetric monoidal functors

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \to \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

along $\sigma \in \Sigma^{op}$, one can take the following steps. Consider the subcategory $\mathrm{Poly}^*(M_\mathbb{R}) \subseteq \mathrm{Poly}(M_\mathbb{R})$ spanned by the origin and top dimensional polyhedral subsets. The category inherits a symmetric monoidal structure and Proposition 4.1.3 implies that the restriction

$$\Gamma_{M_\mathbb{R}} : \mathrm{Poly}^*(M_\mathbb{R}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

is symmetric monoidal[8]. Now we consider the presheaf category [9]

$$\mathrm{Fun}(\mathrm{Poly}^*(M_\mathbb{R})^{op}, \mathrm{Spc})$$

[8]It is true that $\Gamma_{M_\mathbb{R}}$ is symmetric monoidal on $\mathrm{Poly}(M_\mathbb{R})$, but it takes more effort to show. We restrict to $\mathrm{Poly}^*(M_\mathbb{R})$ as it suffices for our purpose here.

[9]Note that we have played the same trick of passing to presheaf category in Definition 3.3.6.

equipped with the Day convolution product. We have a family of idempotent commutative algebras in $\mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc})$ given by

$$\sigma^\vee \in \mathrm{CAlg}(\mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc}))$$

coming from Corollary 4.3.3 and we abusively identify $\sigma^\vee$ with its Yoneda image. Now $\Gamma_{M_{\mathbb{R}}}$ induces a symmetric monoidal colimit preserving functor

$$\mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

so we can apply Proposition 7.2.1 and obtain a diagram in SMCat indexed by Σ:

$$\mathrm{Mod}_{\sigma^\vee} \mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

It remains to write down a natural transformation in SMCat of diagrams indexed by Σ^{op}

$$\psi_\sigma : \Theta(\sigma) \longrightarrow \mathrm{Mod}_{\sigma^\vee} \mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc}).$$

This reduces readily to 1-categorical consideration: for example one way to do this is to identify (as a symmetric monoidal category) $\Theta(\sigma)$ with the full subcategory of $\mathrm{Mod}_{\sigma^\vee} \mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc})$ spanned by Yoneda image of integral translations of $\sigma^\vee$ as in Remark 2.0.4. Given $\tau \subseteq \sigma$ in Σ, the symmetric monoidal functors of base change

$$\mathrm{Mod}_{\sigma^\vee} \mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc}) \longrightarrow \mathrm{Mod}_{\tau^\vee} \mathrm{Fun}(\mathrm{Poly}^*(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc})$$

restricts to structure maps

$$\Theta(\sigma) \longrightarrow \Theta(\tau).$$

So we obtain the natural transformaiton between diagrams in SMCat indexed by Σ:

$$\Theta(\sigma) \longrightarrow \mathrm{Mod}_{\sigma^\vee} \mathrm{Fun}(\mathrm{Poly}(M_{\mathbb{R}})^{\mathrm{op}}, \mathrm{Spc}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

Now one can left Kan extend as in Section 7.3 and obtain the symmetric monoidal functors naturally along Σ^{op}

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

Remark 4.3.7 (Compatibility with lattice)**.** For convenience, we assume the fan to be smooth and projective. For each σ, recall that we have a symmetric monoidal inclusion $p_\sigma : M \to \Theta(\sigma) : m \mapsto m + \sigma^\vee$ and one obtains its left Kan extension as a symmetric monoidal functor:

$$p_{\sigma!} : \mathrm{Fun}(M, \mathrm{Sp}) \to \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}).$$

This functor is natural in σ when we take $\mathrm{Fun}(M, \mathrm{Sp})$ as a constant diagram indexed by $\sigma \in \Sigma^{\mathrm{op}}$. It follows that we have the following diagram in after taking limit:

$$\begin{tikzcd}
\lim_\sigma \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Psi_\Sigma"] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \\
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[u, "\lim_\sigma p_{\sigma!}"] \arrow[ur, "\Psi_\Sigma \circ \lim_\sigma p_{\sigma!}"'] &
\end{tikzcd}.$$

It turns out that one can identify the diagonal functor with a more familiar one when working with a smooth projective fan Σ: (on the bottom right we take M as a discrete topological group, and $i_!$ is the !-pushforward along inclusion of topological groups which is symmetric monoidal for sheaf categories with convolution product)

$$\begin{tikzcd}
\lim_\sigma \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "\Psi_\Sigma"] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \\
\mathrm{Fun}(M, \mathrm{Sp}) \arrow[u, "\lim_\sigma p_{\sigma !}"] \arrow[r, "\cong"'] & \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \arrow[u, "i_!"']
\end{tikzcd}.$$

As we don't need it for now, we defer it to the proof of Lemma 6.1.4.

We summarize the constructions so far by making the following definition.

Definition 4.3.8. Let Σ be a smooth projective fan. Combining Proposition 3.3.1 and Proposition 4.3.4, we arrive at

$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}] \xleftarrow{\cong} \lim_{\sigma \in \Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}; \mathrm{Sp}) \longrightarrow \lim_{\sigma \in \Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}),$$

where the first two functors are symmetric monoidal functors supplied by $\lim \Phi_\sigma$ and $\lim \Psi_\sigma$. We take the inverse of the left and obtain the coherent-constructible correspondence functor

$$\kappa : \mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

which is symmetric monoidal and fully faithful when Σ is smooth projective, in view of Proposition 4.3.4. Given Remark 3.3.9 and Remark 4.3.7, we also have the following diagram in SMCat:

$$\begin{tikzcd}
\mathrm{QCoh}(X_\Sigma/\mathbb{T}) \arrow[r, "\kappa"'] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \\
\mathrm{QCoh}(B\mathbb{T}) \arrow[u, "\pi^*"] \arrow[r, "\cong"'] & \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \arrow[u, "i_!"]
\end{tikzcd}.$$

4.4 Polyhedral stratification

The goal of this subsection is twofold: on the one hand, we show that the functors Ψ_σ previously constructed are fully-faithful, on the other hand, we pin down a first-order approximation of the characterization of the image of κ. That is to say, we will not actually work with the whole (gigantic) category of sheaves, but only a subcategory: those constructible for some fixed stratification. Moreover, the stratification has an elementary description in terms of the fan data. We first do a quick review on constructible sheaves following [7].

Definition 4.4.1. A poset P is said to satisfy the ascending chain condition if every strictly increasing chain in P stops after finitely many steps. A poset P is said to be locally finite if each $P_{\geqslant q} := \{p : p \geqslant q\}$ is finite. Note that locally finite implies the ascending chain condition but not vice versa.

Definition 4.4.2. A stratified topological space is a continuous map $\pi : X \to P$ where X is a topological space and P is a poset equipped with the Alexandroff topology [10]. We often write (X, P) for a stratified topological space and omit the map π. For each $p \in P$, the preimage $\pi^{-1}(p) \subset X$ is called its p-stratum X_p. The stratum X_p is closed subspace of $U_p := \pi^{-1}\{q : p \leqslant q\} \subset X$, the open star around p.

Definition 4.4.3. A map of stratified topological space $f : (X, P) \to (Y, Q)$ is a stratified homotopy equivalence if there is a map g going in the other direction, such that both of their compositions are homotopic to identity in a stratified manner: for example, the homotopy $X \times [0, 1] \to X$ should be a map of stratified topological space, where $X \times [0, 1]$ is stratified by the stratification of X.

Definition 4.4.4. Fix a compactly generated category $\mathcal{C}$ (we will only care about Spc or Sp) as coefficient and a stratified topological space $\pi : X \to P$. A sheaf on X valued in $\mathcal{C}$ is P-constructible [11] if its restriction to each stratum X_p is locally constant. We write $\mathrm{Cons}_P(X; \mathcal{C})$ for the full subcategory of P-construcitible sheaves.

We want to take advantage of the exodromy equivalence to identify a family of compact generators for the category of constructible sheaves. We start by importing the following theorem which realizes exodromy equivalence for a class of particularly simple stratified topological spaces.

Theorem 4.4.5. [7, Theorem 3.4] Let $\pi : X \to P$ be a stratified topological space with π surjective and P satisfying the ascending chain condition. Suppose there is a collection $\mathcal{B}$ of open subsets of X such that

1. the representable sheaves h_U for $U \in \mathcal{B}$ generate the topos $\mathcal{S}\mathrm{hv}(X; \mathrm{Spc})$.
2. for all $U \in \mathcal{B}$, there is a $p \in P$ such that U includes into U_p by a stratified homotopy equivalence.

Then the pullback map

$$\pi^* : \mathrm{Fun}(P, \mathrm{Spc}) \to \mathcal{S}\mathrm{hv}(X; \mathrm{Spc})$$

preserves all limits and colimits and is fully faithful with essential image $\mathrm{Cons}_P(X; \mathrm{Spc})$.

Remark 4.4.6. The theorem in [7] was stated and proved for sheaves valued in Spc. The proof works verbatim for Sp coefficient. It is also true for other compactly generated coefficient categories, which isn't needed for our exposition.

This gives, for a locally finite poset P and stratification $X \to P$ as above, an explicit realization of the exodromy equivalence

$$\pi^* : \mathrm{Fun}(P, \mathrm{Sp}) \to \mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$$

which is the left adjoint of $\mathcal{S}\mathrm{hv}(X; \mathrm{Sp}) \to \mathrm{Fun}(P, \mathrm{Sp})$ sending $\mathcal{F}$ to $[q \mapsto \mathcal{F}(U_q)]$. Tracing through the equivalences, one sees that for $q \in P$, the image of q under stable Yoneda embedding (i.e. $\mathbb{S}[\mathrm{Map}_P(q, -)]$) is taken to $i_{U_q!}(\underline{\mathbb{S}})$ where i_{U_q} is the inclusion of U_q into X.

[10] Recall that a subset $U \subseteq P$ is open in the Alexandroff topology if and only if for $p \in U$, $p \leqslant q$ implies $q \in U$. In other words, U is a 'cosieve': a subset that is upward closed for the partial order of P.

[11] These are sometimes called quasi-constructible in the literature, where the word constructible is reserved for objects also satisfying a finiteness condition which we don't impose here.

Corollary 4.4.7. Let $\pi : X \to P$ be as in Theorem 4.4.5. Then $\mathrm{Cons}_P(X; \mathrm{Sp})$ is generated by compact objects $\{i_{U_q !}(\underline{\mathbb{S}})\}_{q \in P}$ in the following sense: the smallest cocomplete stable subcategory of $\mathrm{Cons}_P(X; \mathrm{Sp})$ that contains these objects is itself.

For future use we record a description of compact objects in functor category here:

Lemma 4.4.8. Let P be a locally finite poset. A functor $X \in \mathrm{Fun}(P, \mathrm{Sp})$ is compact if and only if its value is none-zero on finitely many objects and each of its value is a finite spectrum.

Proof. This is the same as [23, Proposition 2.2.6] which proves the case where P is finite. If a functor F is compact, then as in [23, Proposition 2.2.5] it is a retract of an object in the smallest stable subcategory of $\mathrm{Fun}(P, \mathrm{Sp})$ containing the image of stable Yoneda embedding. By the assumption that P is locally finite, each of the Yoneda functor $\mathbb{S}[\mathrm{Map}(x, -)]$ is non-zero on finitely many objects and each of its value is a finite spectrum. Such condition cuts out a stable subcategory $\mathcal{D}$ of $\mathrm{Fun}(P, \mathrm{Sp})$ which is closed under retract, so we know that F is in $\mathcal{D}$. On the other hand, if F satisfies such assumption, we may take the subposet $\mathrm{supp}^\star(F) := \{y \in P : \exists x \leqslant y, F(x) \neq 0\}$. By assumption this is a finite poset and the restriction of F to $\mathrm{supp}^\star(F)$ is thus compact. Now note that F is left Kan extension of its restriction to $\mathrm{supp}^\star(F)$ and left Kan extension preserves compact objects. $\square$

Now we specialize to the case of interest:

Definition 4.4.9 (FLTZ stratification)**.** (See also [35, Definition 4.3]) Fix a pair (N, Σ) of lattice and fan, and assume further that Σ spans $N_{\mathbb{R}}$ as an $\mathbb{R}$-vector space. We define a stratification $\mathcal{S}_\Sigma$ on $M_{\mathbb{R}}$. To start with, one has an affine hyperplane arrangement in $M_{\mathbb{R}}$ given by

$$H_\Sigma := \{m + \sigma^\perp : m \in M, \sigma \in \Sigma(1)\}$$

where $\sigma^\perp := \{m \in M : (m, n) = 0 \,\forall\, n \in \sigma\}$. One has the following induction procedure to specify strata of a stratification: first look at the complement

$$V := M_{\mathbb{R}} \setminus \bigcup_{h \in H_\Sigma} h$$

and each of the connected component of V should be considered as a single stratum. For each $h \in H_\Sigma$, intersecting $h' \in H_\Sigma$ with h produces an affine hyperplane arrangement on h. Thus one can work inductively and define a poset of strata $\mathcal{S}_\Sigma$ of $M_{\mathbb{R}}$ (note they are locally closed). The closure of each stratum is a union of strata and one specify a poset structure by closure-inclusion. The map sending each point in $M_{\mathbb{R}}$ to the stratum it belongs to in $\mathcal{S}_\Sigma$ would be a continuous map and this gives a stratification on $M_{\mathbb{R}}$. We refer to this stratification $\mathcal{S}_\Sigma$ as the FLTZ stratification for Σ and we will often omit mentioning Σ when it is clear from the context.

Remark 4.4.10. Note that the FLTZ stratification only depends on the collection of 1-cones in Σ.

We wish to use exodromy equivalence Theorem 4.4.5 to get a better control of category of $\mathcal{S}_\Sigma$-constructible sheaves. For that we need:

Proposition 4.4.11. The FLTZ stratification $\mathcal{S}_\Sigma$ on $M_{\mathbb{R}}$ meets the assumption of Theorem 4.4.5 above.

Proof. We need to provide a basis of opens for $M_{\mathbb{R}}$ with desired properties. Consider the standard basis

$$\mathcal{B} := \{D(x,r) : \text{open ball of radius } r \text{ centered at } x \in M_{\mathbb{R}}\}$$

and a subset of it.

$$\mathcal{B}(\mathcal{S}_\Sigma) := \{D(x,r) \in \mathcal{B} : D(x,r) \text{ is stratified homotopy equivalent to the open star at } x\}$$

By definition each $D(x,r) \in \mathcal{B}(\mathcal{S}_\Sigma)$ would go through point 2. It suffices to check point 1, that it is a basis (or at the very least, nonempty). We claim that: for each $x \in M_{\mathbb{R}}$ there exists $r_x > 0$ such that $r < r_x$ implies $D(x,r) \in \mathcal{B}(\mathcal{S}_\Sigma)$. This directly implies that $\mathcal{B}(\mathcal{S}_\Sigma)$ is a basis of opens for $M_{\mathbb{R}}$. To prove the claim, a first observation is that for sufficiently small r, $D(x,r)$ with restricted stratification of $\mathcal{S}_\Sigma$ is (stratified) isomorphic to a real vector space with stratification given by a family of hyperplane arrangements. There is no other stratum coming into the picture than those passing through x. Fix such small r_x, then for all $r \leqslant r_x$, each $D(x,r)$ includes into each other as a stratified homotopy equivalence. It remains to prove that $D(x,r_x)$ is stratified homotopy equivalent to the open star at x. For this a straight-line linear homotopy should do the work. Note this works because the open star is convex and the linear scaling towards x respects the stratification. □

Remark 4.4.12. Note also that for a smooth projective fan Σ the poset P underlying the stratification $\mathcal{S}_\Sigma$ is locally finite, as each stratum is only in the closure of finitely many other strata, and each exit-path will enter a higher dimensional stratum, so must stop after finitely many steps.

The reason to introduce $\mathcal{S}_\Sigma$ is the following:

Proposition 4.4.13. Fix a pair (N,Σ) of lattice and fan. One might post-compose the functor

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

in Proposition 4.3.4 with forgetful into $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$, then its image all lands into the subcategory $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp})$ of sheaves constructible for the FLTZ stratification. As a consequence, the functor

$$\lim_{\Sigma^{op}} \mathrm{Fun}(\Theta(\sigma)^{op}, \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

of Proposition 4.3.4 also lands into $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp})$.

Proof. It suffices to note that each $U \in \Theta(\sigma)$ is given by a cone bound by the hyperplane arrangement H_Σ. Any stratum of the stratification would be either contained in it or be disjoint from it. Using Lemma 4.3.1 and proper base-change, it follows that $\Gamma_{M_{\mathbb{R}}}(U)$ is constructible for the FLTZ stratification. Now the image of $\Theta(\sigma)$ is colimit generated by these objects as a stable category, and constructible sheaf category is also closed under colimit, so we are done. □

We give a standard example to illustrate the ideas of the definitions so far.

Example 4.4.14. Take the fan spanned by $\{e_1, \ldots, e_n\} \subset \mathbb{Z}^n = N$. To be more precise, $\Sigma = \{\mathrm{span}(S) : S \subseteq \{e_1, \ldots, e_n\}\}$. This is the fan corresponding to $\mathbb{A}^n$ in toric geometry. It specifies the standard grid in $M_{\mathbb{R}} \cong \mathbb{R}^n$ as the FLTZ stratification. The strata of $M_{\mathbb{R}} \to \mathcal{S}_\Sigma$ are faces of the unit hypercubes whose vertices have integer coordinates. More precisely, each stratum is cut out by equalities $\{x_i = n_i : i \in I\}$ and inequalities $\{x_j \in (n_j, n_j + 1) : j \in J\}$ where n_i and n_j are integers and the pair (I, J) is a decomposition of $\{1, \ldots, n\}$. The open stars in this case are also very explicit: they are certain hyperrectangles whose vertices have integer coordinates. Using Proposition 4.1.3 one can compute the convolution product of representable sheaves on these open stars and it turns out to be again $\mathcal{S}_\Sigma$-constructible. It follows that in this case $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ is closed under convolution product.

Warning 4.4.15. The convolution product usually doesn't interact well with the FLTZ stratification $\mathcal{S}_\Sigma$. More precisely, for a fixed Σ, the convolution product of two $\mathcal{S}_\Sigma$-constructible sheaves needs not to stay $\mathcal{S}_\Sigma$-constructible. We will see later how to fix this.

Corollary 4.4.16. For a pair (N, Σ) with the fan Σ being smooth, the functor Ψ_Σ constructed in Proposition 4.3.4 is fully faithful. More precisely, for each $\sigma \in \Sigma$, the functor Ψ_σ is fully faithful.

Proof. Fix such σ, by assumption on the smoothness, one can perform a linear transform in $SL(n, \mathbb{Z})$ which takes σ to the cone $\{e_1, \ldots, e_k\}$ in the standard fan $\{e_1, \ldots, e_n\} \subset N = \mathbb{Z}^n$ as in the previous example. So without loss of generality, we will prove for this standard case the functor Ψ_σ is fully faithful. Recall that Ψ_σ is of the form

$$\Psi_\sigma : \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$

and we note that it first of all factors through the full subcategory $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) \cap \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$ of FLTZ constructible sheaves inside $\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$ (for the standard fan Σ spanned by $\{e_1, \ldots, e_n\}$ as above). The domain category is a compactly generated presentable stable category, with a set of compact generators supplied by the stable Yoneda image of representables. By construction of the functor Ψ_σ, it is fully faithful on this set of compact generators. Let's try to give an explicit description of the intersection $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) \cap \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$. We make the following observations:

1. The image of $\Psi_\sigma(\sigma^\vee)$ is an idempotent algebra for the constructible sheaf category $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ equipped with convolution product. As before we denote $\omega_{\sigma^\vee}$ for this algebra and consider the category $\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$. This is a category compactly generated by convolution of representable sheaves on open stars with $\omega_{\sigma^\vee}$. From previous example we know explicitly these open stars are integral hyper-rectangles, and the convolution products are (shifts of) representable sheaves on $\sigma^{\vee,\circ} + m$ for $m \in M$. Note that these are precisely image of $\sigma^{\vee,\circ} + m$ under Ψ_σ.

2. It follows that we can describe

$$\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \cap \mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) = \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$$

   and the functor Ψ_σ lands in this full subcategory. Moreover Ψ_σ takes a set of compact generators (representable presheaves on $\sigma^{\vee,\circ} + m$) to compact objects in the target $\mathrm{Mod}_{\omega_{\sigma^\vee}} \mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$, and is fully faithful on these compact generators.

We apply the following Lemma 4.4.17 and learn that Ψ_σ is fully faithful. It follows that $\lim_{\Sigma^{\mathrm{op}}} \Psi_\sigma$ is also fully faithful. Now Ψ_Σ is a composition of two fully faithful functors, and hence is itself fully faithful. $\square$

We used the following lemma:

Lemma 4.4.17. Let $\mathcal{C}$ be a compactly generated presentable stable category, with a chosen set of compact generators S (in other words, the smallest stable cocomplete full subcategory of $\mathcal{C}$ that contains S is $\mathcal{C}$ itself). Given a cocontinuous functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ with $\mathcal{D}$ a presentable stable category. Assume that F is fully faithful on S, and it takes S to compact obejcts in $\mathcal{D}$. Then F is fully faithful on all of $\mathcal{C}$.

4.5 Digression: Gluing of idempotents in sheaf category

This subsection is meant to answer the following question: can one give a description of sheaf category like the limit diagram provided by Zariski descent for QCoh category? For that we recall how descent works in a presentable symmetric monoidal category with idempotent algebras. The following material is taken from [6, Lecture 8].

Definition 4.5.1. [HA, Definition 4.8.2.1] Fix a presentably symmetric monoidal category $\mathcal{C}$. The category of idempotent objects $\mathcal{C}^{\mathrm{idem}} \subset \mathrm{Fun}([1], \mathcal{C})$ is the full subcategory of pairs $(A, f : 1_{\mathcal{C}} \to A)$ such that $f \otimes A : A \to A \otimes A$ is an equivalence.

We also recall the following facts:

1. [HA, Proposition 4.8.2.9] Take $\mathrm{CAlg}(\mathcal{C})^{\mathrm{idem}}$ to be the full subcategory of $\mathrm{CAlg}(\mathcal{C})$ spanned by $A \in \mathrm{CAlg}(\mathcal{C})$ such that the unit map makes A into an idempotent object of $\mathcal{C}$. The forgetful functor $\mathrm{CAlg}(\mathcal{C})^{\mathrm{idem}} \to \mathcal{C}^{\mathrm{idem}}$ is an equivalence. In particular every idempotent object acquires uniquely a commutative algebra structure.
2. [HA, Proposition 4.8.2.4] Take $A \in \mathcal{C}^{\mathrm{idem}}$. The functor $\mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ is a localization. In particular the forgetful $\mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ is fully faithful, with image those $X \in \mathcal{C}$ such that $X \to X \otimes A$ is an equivalence.
3. (You can read about the following points from Theorem 4 and Lemma 5 in of [6, Lecture 8]) The category $\mathcal{C}^{\mathrm{idem}}$ is a poset.
4. As a poset $\mathcal{C}^{\mathrm{idem}}$ has all joins (unions) and finite meets (intersections). The join of A and B is computed as $A \otimes B$, and join of an infinite family $\{A_i : i \in I\}$ is computed as filtered colimit over the join of finite subsets (in the underlying category).
$$A \vee B = A \otimes B$$
$$\vee_{i \in I} A_i = \underset{J \subset I,\ \text{finite}}{\mathrm{colim}} \bigotimes_{j \in J} A_j$$
The meet of A and B is computed as fiber of $A \times B \to A \otimes B$ and meet of *finite* family of $\{A_i : i \in I\}$ is computed as a limit over the poset of nonempty subsets $J \subset I$ of the functor $J \mapsto \otimes_{j \in J} A_j$ (in the underlying category). Note that the limit diagram would be a cubical diagram.
$$A \wedge B = A \underset{A \otimes B}{\times} B$$
$$\wedge_{i \in I} A_i = \lim_{J \subset I,\ \text{nonempty}} \bigotimes_{j \in J} A_j$$

5. One can put a Grothendieck topology on $\mathcal{C}^{\mathrm{idem,op}}$ by specifying covers are those which contain a finite family of maps $\{f_i : A \to A_i \in \mathcal{C}^{\mathrm{idem}}\}$ such that it presents A as a meet for $\{A_i\}$.

Theorem 4.5.2. The presheaf $\mathrm{Mod}_{(-)}(\mathcal{C}) : \mathcal{C}^{\mathrm{idem}} \to \mathrm{SMCat}$ which takes A to $\mathrm{Mod}_A(\mathcal{C})$ is a sheaf for above topology.

Proof. This is the same as Theorem 4 in of [6, Lecture 8]. □

Now we run this machine in practice, the most important example is the following:

Example 4.5.3 (Zariski descent in algebraic geometry)**.** For a scheme X and an open $U \subset X$, $*$-pushforward of the structure sheaf $i_*\mathcal{O}_U$ is an idempotent algebra in $\mathrm{QCoh}(X)$ (equipped with standard tensor product of quasi-coherent sheaves). If a finite family $\{U_i\}$ form a Zariski cover of X, one can show that the family $\mathbb{1}_{\mathrm{QCoh}(X)} \to i_*\mathcal{O}_U$ is a cover and evaluating $\mathrm{Mod}_{(-)}(\mathcal{C})$ on this cover recovers the limit diagram of categories for Zariski descent.

The game we are going to play is to formulate a convolution-of-sheaf version of such phenomenon. First of all let's consider a family of idempotent algebras. We fix a smooth projective fan Σ on N until the end of the subsection.

Proposition 4.5.4. Let Σ be a smooth projective fan. Take subset $\Sigma(n) \subset \Sigma$ to be the top dimensional cones, then $\{\mathbb{1}_{\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})} \to \omega_{\sigma^\vee} : \sigma \in \Sigma(n)\}$ is a cover of $\mathbb{1}_{\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})}$. More explicitly one has the following equivalences

$$\mathbb{1}_{\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})} \xrightarrow{\cong} \lim_{\sigma \in \Sigma^{\mathrm{op}}} \omega_{\sigma^\vee} \xrightarrow{\cong} \lim_{S \in \mathcal{P}_{\neq \emptyset}(\Sigma(n))} \bigotimes_{\tau \in S} \omega_{\tau^\vee}$$

where the second map is always an equivalence by the following observation with limit-equivalence.

Remark 4.5.5. We make the following observations about the diagrams.

- Fix a smooth projective fan Σ. There exits an adjunction $l : \mathcal{P}_{\neq\emptyset}(\Sigma(n)) \rightleftharpoons \Sigma^{\mathrm{op}} : r$ between the poset of nonempty subsets of top dimensional cones $\Sigma(n)$ and the opposite poset of all cones in the fan Σ. The map l sends a subset $S \subseteq \Sigma(n)$ to its intersection

  $$l(S) := \bigcap_{\sigma \in S} \sigma \in \Sigma^{\mathrm{op}}.$$

  The map r sends a cone $\tau \in \Sigma^{\mathrm{op}}$ to all the top dimensional cones containing it

  $$r(\tau) := \{\sigma \in \Sigma(n) : \tau \subseteq \sigma\}.$$

  The adjunction reduces to the following observation: the intersection of cones in a subset S contains τ if and only if S is contained in $r(\tau)$ which is the subset of all the cones containing τ. In particular the map l is a final functor, or a limit-equivalence. Note also that the composition $l \circ r$ is the identity map on Σ^{op}.

- The composite of functors

$$\mathcal{P}_{\neq\emptyset}(\Sigma(n)) \xrightarrow{l} \Sigma^{op} \longrightarrow \mathrm{CAlg}(\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}))^{\mathrm{idem}}$$

$$S \mapsto l(S) \mapsto \omega_{l(S)^\vee}$$

can be identified with the Čech diagram

$$\mathcal{P}_{\neq\emptyset}(\Sigma(n)) \longrightarrow \mathrm{CAlg}(\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}))^{\mathrm{idem}}$$

$$S \mapsto \bigotimes_{\tau\in S} \omega_{\tau^\vee}.$$

This follows from that $\omega_{\sigma^\vee} * \omega_{\tau^\vee} \cong \omega_{(\sigma\cap\tau)^\vee}$, which is a consequence of the combinatorial fact $\sigma^\vee + \tau^\vee = (\sigma\cap\tau)^\vee$ plus the computation of convolution Proposition 4.1.3. Note that by our assumption on the fan, each pair of cones meets along a common face and one can apply separation lemma as in [10, (11) and (12) of Section 1.2] to obtain the above combinatorial fact.

Proof. By finality, we switch to the diagram indexed by Σ^{op}. We need to show that

$$\mathbb{1}_{\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})} \to \lim_{\sigma\in\Sigma^{op}} \omega_{\sigma^\vee}$$

is an equivalence. Let's compute the stalk of the limit. At the origin, the stalk is

$$\lim_{\sigma\in\Sigma^{op}} (\omega_{\sigma^\vee})_0 \cong \lim_{\sigma\in\Sigma^{op}} \mathbb{S}_{\{0\}}[n] \in \mathrm{Sp}$$

where $\mathbb{S}_{\{0\}} : \Sigma^{op} \to \mathrm{Sp}$ is the presheaf on Σ that takes value $\mathbb{S}$ at the origin and zero otherwise. To evaluate the limit, note that there is a fiber sequence in $\mathrm{Fun}(\Sigma^{op}, \mathrm{Sp})$

$$\mathbb{S}_{\{0\}} \to \underline{\mathbb{S}} \to \mathbb{S}_{\Sigma^{op}\setminus\{0\}}$$

where $\underline{\mathbb{S}}$ is the constant presheaf and $\mathbb{S}_{\Sigma^{op}\setminus\{0\}}$ is the right Kan extension of the constant presheaf on $\Sigma\setminus\{0\}$. Taking global sections, we get the fiber sequence

$$\lim_{\sigma\in\Sigma^{op}} \mathbb{S}_{\{0\}} \to \mathbb{S} \to \lim_{\sigma\in\Sigma^{op}} \mathbb{S}_{\Sigma^{op}\setminus\{0\}},$$

where the last term can be further computed by

$$\begin{aligned}\lim_{\sigma\in\Sigma^{op}} \mathbb{S}_{\Sigma^{op}\setminus\{0\}} &\simeq \lim_{\sigma\in\Sigma^{op}\setminus\{0\}} \underline{\mathbb{S}} \\ &\simeq \mathbb{S}\oplus\mathbb{S}[-n+1].\end{aligned}$$

Indeed, $\Sigma^{op}\setminus\{0\}$ could be identified with opposite of exit path category on S^{n-1} with the stratification induced by the fan, and taking global sections of the constant presheaf $\underline{\mathbb{S}}$ thus computes the cotensor

$$\mathbb{S}^{S^{n-1}} \simeq \mathbb{S}\oplus\mathbb{S}[-n+1].$$

Under this identification, $\mathbb{S} \to \mathbb{S} \oplus \mathbb{S}[-n+1]$ is the inclusion of the first factor. Consequently,

$$\lim_{\sigma \in \Sigma^{op}} \mathbb{S}_{\{0\}} \simeq \mathbb{S}[-n]$$

and thus

$$\lim_{\sigma \in \Sigma^{op}} (\omega_{\sigma^\vee})_0 \simeq \lim_{\sigma \in \Sigma^{op}} \mathbb{S}_{\{0\}}[n] \simeq \mathbb{S}$$

as desired.

Next we compute the stalk of the limit at $m \in M_{\mathbb{R}}$ (which is away from the origin). Similarly we look at the limit

$$\lim_{\sigma \in \Sigma^{op}} \mathbb{S}_{m,+}[n]$$

where $\mathbb{S}_{m,+}$ is the functor which evaluates on σ to be $\mathbb{S}$ if $m \in \sigma^{\vee,\circ}$ and 0 otherwise. To be precise one can put it in a fiber sequence of functors

$$\mathbb{S}_{m,+} \to \underline{\mathbb{S}} \to \mathbb{S}_{m,-}$$

in $\mathrm{Fun}(\Sigma^{op}, \mathrm{Sp})$. Here $\underline{\mathbb{S}}$ is the constant functor and $\mathbb{S}_{m,-}$ is right Kan extended from the constant presheaf on the sub-poset

$$\Sigma_{m,-} := \{\sigma : m \notin \sigma^{\vee,\circ}\} \subseteq \Sigma$$

(one can check that the right Kan extension takes everything outside of $\Sigma^{op}_{m,-}$ to 0). We claim that the limit along Σ^{op} of the map

$$\underline{\mathbb{S}} \to \mathbb{S}_{m,-}$$

is an isomorphism $\mathbb{S} \to \mathbb{S}$. It suffices to show that the poset $\Sigma^{op}_{m,-}$ is contractible. For that we make the following combinatorial argument.

We will adapt the proof of [9, Proposition 3.7] to our situation. We fix a moment polytope P for the fan Σ. Consider the poset $F(P)$ of faces of P under inclusion, then there is an (inclusion) order reversing bijection between $F(P)$ and Σ. For example, the codimension 0 face of P (which is P itself) corresponds to the 0 dimensional cone of the origin. Now we consider the following subposet: (informally, the subset of $F(P)$ that's visible from ∞ through the direction m)

$$F(P)_{m,-} := \{C \in F(P) : \forall c \in C, \text{the ray } c + \mathbb{R}_{>0} \cdot m \text{ doesn't meet } P^\circ\}.$$

We claim that the canonical bijection between $F(P)$ and Σ^{op} induces a bijection between $F(P)_{m,-}$ and $\Sigma^{op}_{m,-}$. This follows readily from the definition: if $m \notin \sigma^{\vee,\circ}$, then m is also not in $\tau^{\vee,\circ}$ for $\sigma \subseteq \tau$. Consider the corresponding face C_σ in P, at each point $c \in C$, the angle spanned by P is $\tau^\vee$ for some $\sigma \subseteq \tau$, which means that the ray $c + t \cdot m$ will not pass through P°. Conversely, let $m \in \sigma^{\vee,\circ}$ for some σ (so $\sigma \notin \Sigma^{op}_{m,-}$) and C_σ be the corresponding face in P, it follows that at an relative interior point c of C_σ, the angle spanned by P is precisely $\sigma^\vee$, and that $m \in \sigma^{\vee,\circ}$ means that the ray $c + t \cdot m$ will pass through P°. Now the topological space $P_{m,-}$ of union of faces in $F(P)_{m,-}$ is contractible because if one fixes a hyperplane H perpendicular to m and consider projection to H along m, the image of $P_{m,-}$ is the same as P, which is convex. But the map from $P_{m,-}$ to its image is a homotopy equivalence, hence we conclude that $P_{m,-}$ is contractible. Now $F(P)_{m,-}$ is the exit-path category for the stratification on $P_{m,-}$ by the faces, hence it is also contractible. We conclude that $\Sigma^{op}_{m,-}$ is also contractible as desired. □

Remark 4.5.6. For the fans corresponding to $\mathbb{P}^n$, one can give a slick proof by noting that the limit diagram for Σ^{op} is the same as the Čech diagram for the open cover of $M_{\mathbb{R}}$ as a topological space by $\{\sigma^{\vee,\circ} \to M_{\mathbb{R}} : \sigma \in \Sigma(1)\}$ and use [HA, Proposition 1.2.4.13]. But it is not true in general that the diagram as above is the Čech diagram for an open cover of $M_{\mathbb{R}}$. We opt for a different proof as above instead.

Corollary 4.5.7. For smooth projective fan the functor Ψ_Σ assembled in Proposition 4.3.4 is symmetric monoidal.

Proof. Recall from Proposition 4.3.4 that Ψ_Σ is a composition:

$$\Psi_\Sigma : \lim_{\Sigma^{\mathrm{op}}} \mathrm{Fun}(\Theta(\sigma)^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\lim \Psi_\sigma} \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}).$$

The first functor is always symmetric monoidal, and we are concerned with the second functor. Note that it is defined as a right adjoint to the functor

$$\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \lim_{\Sigma^{\mathrm{op}}} \mathrm{Mod}_{\omega_{\sigma^\vee}} \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

which is an equivalence when the fan is smooth and projective, given Proposition 4.5.4. So we conclude that the second functor is also symmetric monoidal, and so is Ψ_Σ. $\square$

Remark 4.5.8. More generally, the result of Dmitry Vaintrob in [37] could be interpreted to suggest that the limit of the family of idempotent algebras in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$ as in Proposition 4.5.4 should only depend on the support, but not a particular fan. This is very related to his construction [36] of log quasi-coherent category of toroidal compactifications. A direct adaptation of the construction of our comparison functor to Dmitry Vaintrob's setting will produce a symmetric monoidal equivalence in the setting of sheaf category without constructibility.

5 Singular support

The aim of this section is to characterize $\mathrm{Im}(\kappa)$ for smooth projective fan Σ in terms of a notion of singular support as elegantly constructed in [9]. We write Λ_Σ for the conic Lagrangian subset of the cotangent bundle $T^*M_{\mathbb{R}}$ given in Definition 5.1.17 and define a full subcategory of $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$ containing $\mathrm{Im}(\kappa)$ using singular support:

$$\mathrm{Im}(\kappa) \subseteq \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}).$$

We follow the idea of [40] to show that the inclusion is an equality. The benefit of this approach is that along the way we construct an explicit family of compact generators of $\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp})$.

We will first take a quick tour of singular support for polyhedral sheaves. This is particularly simple, since locally we are working with conic sheaves on a real vector space. Then we revisit the interplay between twisted polyhedra and sheaves. Eventually we invoke non-characteristic deformation lemma from [31] to prove the characterization.

The reason why our proof is less straightforward as opposed to what appears in [40] is the following. We find that there is a lack of a general theory of singular supports for sheaves of spectra, so that arguments one can make in its classical counterpart [20] would carry over without much modification (see, however, [19] for an exposition in this direction.) We hope that this section provides an invitation to homotopy theorists to revisit the notion of singular support in greater generality and investigate questions like Remark 5.3.1.

5.1 Singular support for polyhedral sheaves

Following [9, Section 4], we define the notion of singular support for polyhedrally constructible sheaves on real vector spaces (and also torus). 'Polyhedral' means that we fix a stratification P on a real vector space V, specified (as in Definition 4.4.9) by an affine hyperplane arrangement. We will consider sheaves which are constructible for such 'polyhedral' stratification. Locally, these sheaves are modeled on conic sheaves F on a real vector space V. So we first consider the case for conic sheaves on a vector space. (All vector spaces appearing here will be finite dimensional.)

Remark 5.1.1. We will make use of results in [20], but the reader should be warned that the book was written with the classical language of bounded derived category of sheaves. So it is not directly applicable in our situation. However, the results we make use of could be verified with the same proof from there: the reason is that it comes down to computation with explicit kernels, and the coherences come from adjunction. We will revisit these facts somewhere else.

Definition 5.1.2. Recall that the topological group $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$ acts continuously on a real vector space V via multiplication. We define the category of conic sheaves on V to be the full subcategory of sheaves which are constant when restricted to each orbit, and write it as

$$\mathcal{S}\mathrm{hv}^{\mathrm{conic}}(V;\mathrm{Sp}) \subseteq \mathcal{S}\mathrm{hv}(V;\mathrm{Sp}).$$

Definition 5.1.3 (Fourier-Sato transform)**.** Let V be a real vector space with dual V^*. The functor of Fourier-Sato transform is defined to be

$$\mathcal{FS} : \mathcal{S}\mathrm{hv}^{\mathrm{conic}}(V;\mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}^{\mathrm{conic}}(V^*;\mathrm{Sp})$$

$$F \mapsto p_! q^* F$$

where $p : K \to V^*$ and $q : K \to V$ are projections from the kernel:

$$K := \{(x, y) \in V \times V^* : \langle x, y \rangle \leqslant 0\} \subset V \times V^*.$$

We define the singular support at the origin of a conic sheaf F to be the support (closure of the points where stalk doesn't vanish) of $\mathcal{FS}(F) \subseteq V^*$ which could be identified with the cotangent space of V at the origin:

$$\mu\mathrm{supp}_0(F) := \mathrm{supp}(\mathcal{FS}(F)) \subset V^*.$$

Proposition 5.1.4 ([20, Theorem 3.7.9])**.** The Fourier-Sato transform is an equivalence of categories between conic sheaves on V and conic sheaves on V^* :

$$\mathcal{FS} : \mathcal{S}\mathrm{hv}^{\mathrm{conic}}(V; \mathrm{Sp}) \xrightarrow{\cong} \mathcal{S}\mathrm{hv}^{\mathrm{conic}}(V^*; \mathrm{Sp}).$$

Remark 5.1.5 (An alternative defintion)**.** One can also define a notion of singular support using Morse-type construction as in [31, Definition 4.5]. It coincides with this definition, but we will not use it here.

One particular feature of such definition we will use is that it interacts nicely with cones.

Lemma 5.1.6 ([20, Lemma 3.7.10])**.** Let V be a real vector space with V^* its dual. Let $\tau \subseteq V$ be an open convex cone and $-\tau^\vee \subseteq V^*$ be negative of its dual cone. Then

$$\mathcal{FS}(\omega_\tau) = \underline{S}_{-\tau^\vee}.$$

In particular the singular support at the origin of ω_τ is

$$\mu\mathrm{supp}(\omega_\tau)_0 = -\tau^\vee.$$

Now we globalize above definition:

Definition 5.1.7 (Singular support)**.** Let V be a vector space equipped with a stratification P specified by an affine hyperplane arrangement as in Definition 4.4.9. For a constructible sheaf $F \in \mathrm{Cons}_P(V; \mathrm{Sp})$ one can specify a subset of the cotangent bundle of V:

$$\mu\mathrm{supp}(F) \subseteq T^*V \cong V \times V^*$$

to be the (global) singular support of F. Its fiber at a point $\nu \in V$, denoted by $\mu\mathrm{supp}_\nu(F)$ is determined as follows: pick an open ball U centered at ν that only meets the hyperplanes passing through ν. Pick an exponential map from the tangent space:

$$\exp : V \xrightarrow{\cong} U$$

and it pulls F back to a conic sheaf $\exp^* F \in \mathcal{S}\mathrm{hv}^{\mathrm{conic}}(V; \mathrm{Sp})$. We define $\mu\mathrm{supp}(F)_\nu := \mu\mathrm{supp}_0(\exp^* F) \subseteq V^*$ and we identify canonically V^* with $T^*_\nu V$.

Remark 5.1.8 (Singular support is well-defined)**.** We remark that at each point ν this subset $\mu\mathrm{supp}_\nu(F)$ doesn't depend on the choice of the open ball U nor the exponential map exp. To compare different choices we end up with a transition map

$$V \to V$$

which is given by multiplication of a continuous function in $\mathbb{R}_+$ on V. Since all the orbits are contractible and the sheaf involved is conic, one can produce an equivalence between sheaves $\exp^*(F)$ under different choices. We don't spell out the details here.

Definition 5.1.9 (Sheaves with prescribed singular support)**.** Following the notation as Definition 5.1.7. Let $\Lambda \subset T^*V \cong V \times V^*$ be a subset. We define a full subcategory $\mathcal{S}\mathrm{hv}_\Lambda(V;\mathrm{Sp})$ of $\mathrm{Cons}_P(V;\mathrm{Sp})$ to be

$$\mathcal{S}\mathrm{hv}_\Lambda(V;\mathrm{Sp}) := \{F : \mu\mathrm{supp}(F) \subseteq \Lambda\}.$$

This is the subcategory of P-constructible sheaves with singular support contained in Λ.

Warning 5.1.10. Note that the notation didn't make explicit the dependence on P, but we would always fix such a stratification and work inside the full subcategory of P-constructible sheaves. This should not cause confusion as we will work with a single fixed stratification at a time. It is true that $\mu\mathrm{supp}(F)$ doesn't depend on the ambient stratification - and in fact one can define singular support of a sheaf without the help of constructibility and arrive at the same notion. But beware that, given Λ, the category of P-constructible sheaves with singular support contained in Λ can vary as P changes. It is also true that they will be the same as long as conormal variety of P contains Λ. We will not prove these facts nor use them.

Variant 5.1.11. The definition makes sense also for a quotient of a vector space by a lattice V/Γ, in particular for tori $\mathbb{R}^n/\mathbb{Z}^n$: fix a polyhedral stratification P on V/Γ and a constructible sheaf F for $(V/\Gamma, P)$, one can define a subset $\mu\mathrm{supp}(F) \subseteq T^*V/\Gamma$, and thus talk about subcategory of P-constructible sheaves with prescribed singular support. We will make use of this notion in the final section.

Then we make several quick observations with the definition.

Remark 5.1.12 (Locality)**.** The definition is local in nature. This in particular implies that one can check if a constructible sheaf F on V/Γ has the prescribed singular support by pulling back and checking on V, since the projection map is a local homeomorphism preserving the linear structure.

Remark 5.1.13 (Closed under colimits)**.** Given polyhedral stratification P on V and a subset Λ in T^*V. The subcategory $\mathcal{S}\mathrm{hv}_\Lambda(V;\mathrm{Sp})$ is a stable subcategory closed under colimit in $\mathrm{Cons}_P(V;\mathrm{Sp})$ and hence also in $\mathcal{S}\mathrm{hv}(V;\mathrm{Sp})$. This follows from that restriction and Fourier-Sato preserves colimits, and also support condition is closed under colimits.

The most important example of computation with global singular support is the following:

Lemma 5.1.14. [9, Proposition 5.1] Take a smooth projective fan Σ and work with $\mathcal{S}_\Sigma$-constructible sheaves. We can estimate singular support of the sheaf $\omega_{m+\sigma^\vee}$ for $\sigma \in \Sigma$:

$$\mu\mathrm{supp}(\omega_{m+\sigma^\vee}) \subseteq \bigsqcup_{\tau \subset \sigma} m + \tau^\perp \times -\tau \subset M_\mathbb{R} \times N_\mathbb{R} \cong T^*M_\mathbb{R}.$$

We refer to the original treatment for the proof: it is a direct application of Lemma 5.1.6.

One feature of the notion of singular support is that it supports Morse theory. In our context, the foundational non-characteristic deformation lemma is supplied by [31, Theorem 4.1]:

Proposition 5.1.15. Let $M \in \mathrm{LCH}$ and $F \in \mathcal{S}hv^{\mathrm{hyp}}(M;\mathrm{Sp})$ be hypercomplete. Let $\{U_s\}_{s\in\mathbb{R}}$ be a family of open subsets of M. Assume:

1. For all $t \in \mathbb{R}$, $U_t = \cup_{s<t} U_s$.
2. For all pairs $s \leqslant t$, the set $\overline{U_t \setminus U_s} \cap \mathrm{supp}(F)$ is compact.
3. Setting $Z_s := \cap_{t>s}\overline{U_t \setminus U_s}$, we have for all pairs $s \leqslant t$ and all $x \in Z_s$:

   $$i^!(F)_x = 0$$

   where $i : X \setminus U_t \to X$ is the inclusion. Note that by the recollement sequence where $j : U_t \to X$ is the inclusion

   $$i_! i^!(F) \longrightarrow F \longrightarrow j_* j^*(F)$$

   this is the same as asking $F_x \to j_* j^*(F)_x$ be an isomorphism for each $x \in Z_s$.

Then we have for all $t \in \mathbb{R}$:

$$F(\bigcup_s U_s) \xrightarrow{\cong} F(U_t).$$

Remark 5.1.16. As we will be working with a finite dimensional real vector space, every sheaf is automatically hypercomplete. Beware that it is crucial that the coefficient category Sp is compactly generated presentable - otherwise one needs to change the definition of singular support. See [8, Remark 4.24].

So much for the abstract nonsense. Here is the crucial part of this subsection: we will provide a refinement of the $\mathcal{S}_\Sigma$-constructible sheaf category such that the image of κ lies in it:

Definition 5.1.17 (FLTZ skeleton)**.** [12] Take a smooth projective fan Σ. We define a conic Lagrangian subset of T^*M as follows:

$$\Lambda_\Sigma := \bigsqcup_{m\in M, \sigma\in\Sigma} m + \sigma^\vee \times -\sigma \subseteq M_\mathbb{R} \times N_\mathbb{R} \cong T^*M_\mathbb{R}.$$

From now on we will focus on the category $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R};\mathrm{Sp})$ of $\mathcal{S}_\Sigma$-constructible sheaves with singular support in Λ_Σ.

Lemma 5.1.18. The image of κ lies in $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R};\mathrm{Sp})$.

Proof. The category $\mathrm{Im}(\kappa)$ is generated as a stable category under colimit and shifts by the objects of the form $\omega_{m+\sigma^\vee}$, and each of them has singular support contained in Λ_Σ by Lemma 5.1.14. Since the category of sheaves with prescribed singular support is closed under colimit Remark 5.1.13 we are done. $\square$

[12] The name 'FLTZ skeleton' is borrowed from symplectic geometry.

5.2 Combinatorics of smooth projective fan

One distinguishing feature of a smooth projective fan Σ in $N_{\mathbb{R}}$ is that it can be presented as the dual fan of an integral polytope P. See [10, Section 1.5] for the construction. Such polytope P has the following properties:

1. The Minkowski sum of P with any dual cone of $\sigma \in \Sigma$ is an integral translation of the dual cone of σ.
2. Each of the dual cone $\sigma^\vee$ could be written as an increasing union of translations of polytopes of the form $n \cdot P$, where each $n \cdot P$ is an integral multiple of the polytope P.

We will see that these properties imply that after fixing one such P, the objects $\{\omega_{m+n\cdot P}\}$ for varying n and translation along $m \in M$ supply an explicit collection of compact generators for $\mathrm{Im}(\kappa)$. On the mirror side, this is reminiscent of the familiar fact from algebraic geometry: tensor powers of ample line bundle generate the category of quasi-coherent sheaves under colimits.

We will explain the association $P \mapsto \omega_P$ generalizes to a bigger collection of combinatorial objects, namely, twisted polytopes. To start with, we will make use of the following description of $\mathrm{Im}(\kappa)$.

Proposition 5.2.1. The category $\mathrm{Im}(\kappa)$ enjoys the following properties and characterizations.

1. The category $\mathrm{Im}(\kappa)$ is closed under colimits and shifts in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$.
2. The category $\mathrm{Im}(\kappa)$ could be characterized explicitly as
$$\{\mathcal{F} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) : \mathcal{F} * \omega_{\sigma^\vee} \in \langle \omega_{m+\sigma^\vee} : m \in M \rangle\}.$$
3. The category $\mathrm{Im}(\kappa)$ is generated under colimits and shifts of the following collection of objects:
$$\{\omega_{m+\sigma^\vee} : \sigma \in \Sigma, m \in M\}.$$
4. The category $\mathrm{Im}(\kappa)$ is closed under convolution product in $\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$.

Proof. The first point comes from the fact that κ is a fully faithful, colimit preserving functor from a presentable stable category, as κ is constructed from taking limit of a diagram in Pr^L. The second point follows directly from the limit description of κ. For the third point, using descent along idempotent algebras, every object $X \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$ is a finite limit of terms like $X * \omega_{\sigma^\vee}$, and each of them lies in the category spanned by $\omega_{m+\sigma^\vee}$ as in point two, so we are OK. Finally since κ is symmetric monoidal, its image is closed under tensor product. $\square$

With this knowledge at hand, let's try to write down some objects in the category $\mathrm{Im}(\kappa)$.

Proposition 5.2.2. For a smooth projective fan (N, Σ), there exist (in fact, many) polytopes P in $M_{\mathbb{R}}$ with integral vertices such that Σ could be realized as the dual fan of P. Coversely P might be called a moment polytope of Σ (actually, associated to some line bundle). More precisely, P has the following properties:

- The Minkowski sum of P with any dual cone $\sigma^\vee$ of $\sigma \in \Sigma$ is an integral translation of the dual cone of σ. Concretely this says for each $\sigma \in \Sigma$, there exists some $m \in M$ such that
$$P + \sigma^\vee = m + \sigma^\vee.$$

- Each of the dual cone $\sigma^\vee$ could be written as an increasing union of integral translations of polytopes of the form nP, where each nP is an integral multiple of the polytope P. Concretely this says for each $\sigma \in \Sigma$, one can pick a collection of $m_i \in M$ and form a increasing union

$$\bigcup_{i \geqslant 0} m_i + i \cdot P = \sigma^\vee.$$

For the existence, a polytope as in [10, Section 1.5] would do the job - both claims above are direct combinatorics. We will consider the object $\omega_P \in \mathcal{S}hv(M_\mathbb{R}; Sp)$.

Proposition 5.2.3. For such polytope P as above:

1. The object ω_P lies in $\mathrm{Im}(\kappa)$.
2. The object ω_P is an compact object in $\mathrm{Cons}_{\mathcal{S}_\Sigma}(M_\mathbb{R}; Sp)$ and hence also compact in $\mathrm{Im}(\kappa)$.
3. The same is true for $\omega_{m+n\cdot P}$ for each $m \in M$ and $n \in \mathbb{Z}_{>0}$. Moreover, these objects supply a collection of compact generators of the category $\mathrm{Im}(\kappa)$.

Proof. The first point comes from the characterization of $\mathrm{Im}(\kappa)$ above via

$$\omega_P * \omega_{\sigma^\vee} \cong \omega_{P+\sigma^\vee} \cong \omega_{m+\omega_{\sigma^\vee}}$$

using the first property of P as a dual polytope. The second point comes from an application of exodromy equivalence and the description of compact objects in presheaf category Lemma 4.4.8, noting that such polytope P is assumed to be bounded. For the final point, since one can write each $\sigma^\vee$ as increasing union of polytopes of the form $m + n \cdot P$, one can form a filtered colimit

$$\operatorname*{colim}_{m+n\cdot P \subseteq \sigma^\vee} \omega_{m+n\cdot P} \cong \omega_{\sigma^\vee}.$$

Up to translation, this shows that every $\omega_{m+\sigma^\vee}$ can be written as a colimit of $\omega_{m+n\cdot P}$, hence $\mathrm{Im}(\kappa)$ is generated by $\omega_{m+n\cdot P}$ for varying $m \in M$ and $n > 0$. $\square$

Remark 5.2.4 (Divisors and piecewise linear functions)**.** Here we give two more combinatorial ways to present the data of such polytope P. Firstly as 'divisors': the polytope P is the intersection of several half-spaces in $M_\mathbb{R}$, indexed by the 1-cones $\eta \in \Sigma(1)$. Let us fix primitive integral vectors $v_\eta \in N$ for each $\eta \in \Sigma(1)$, then we can write

$$P = \bigcap_{\eta \in \Sigma(1)} \{m \in M_\mathbb{R} : \langle m, v_\eta \rangle \geqslant -n_\eta \in \mathbb{Z}\}.$$

Hence we can recover the polytope P from the collection of integers $\{n_\eta : \eta \in \Sigma(1)\}$. More generally by a divisor we would mean such a sequence of integers $\{n_\eta : \eta \in \Sigma(1)\}$ and we write D for a divisor. In case of a moment polytope P we write D_P for the associated divisor as above. Note that one can make sense of addition of divisors as pointwise addition.

Secondly as piecewise linear functions: given a divisor $D_P = \{n_\eta : \eta \in \Sigma(1)\}$ coming from a moment polytope P, one may extend the assignment $v_\eta \mapsto -n_\eta$ $\mathbb{R}$-linearly on each cone to obtain a $\mathbb{R}$-valued function f_P on $N_\mathbb{R}$ (here we use the fan is smooth and projective). For each top dimensional cone σ, there is a unique $m_\sigma \in M$ such that when restricted to σ

$$\langle m, - \rangle = f_P.$$

Such $\{m_\sigma\}$ is precisely the collection of vertices of P, see [10, Section 3.4]. So one might recover the polytope P from the data of f_P. This is part of the beautiful connection between line bundles, divisors and piecewise linear functions, as explained in Fulton's book.

Variant 5.2.5 (Twisted polytopes)**.** It is not true that every divisor $D = \{n_\eta\}$ or every integral piecewise linear function f corresponds to a polytope. However, one can still write down an object in $\mathrm{Im}(\kappa)$ starting from such data. Let us explain the idea here: fix a collection of integers $\{n_\eta\}$ as a divisor D. We may look at the corresponding piecewise linear function f constructed same way as above. As above, this function f determines and is determined by a collection of elements $\{m_\sigma \in M : \sigma \in \Sigma(n)\}$. We might consider the collection of closed subsets

$$\{m_\sigma + \sigma^\vee \subseteq M_\mathbb{R} : \sigma \in \Sigma(n)\}.$$

The fact that m_σ and m_τ agrees as function on $\sigma \cap \tau$ (as they are both given by f, or the divisor D) implies that

$$m_\sigma + (\sigma \cap \tau)^\vee = m_\tau + (\sigma \cap \tau)^\vee.$$

In fact the function f (or the divisor D) determines an integral element

$$m_\sigma \in M/\sigma^\perp$$

for each $\sigma \in \Sigma$. Thus the subset

$$m_\sigma + \sigma^\vee \subseteq M_\mathbb{R}$$

is well defined. Note per definition one has for $\tau \subseteq \sigma$

$$(m_\sigma + \sigma^\vee) + \tau^\vee = m_\tau + \tau^\vee.$$

Now we claim that the collection of objects

$$\{\omega_{m_\sigma + \sigma^\vee} \in \mathrm{Mod}_{\omega_{\sigma^\vee}} : \sigma \in \Sigma\}$$

underlies an object in $\mathrm{Im}(\kappa)$ using descent along idempotent algebra Section 4.5. In other words, we claim there exists an object $\omega(D) \in \mathrm{Im}(\kappa)$ such that

$$\omega(D) * \omega_{\sigma^\vee} \cong \omega_{m_\sigma + \sigma^\vee}.$$

To do so, it suffices to provide isomorphisms for $\tau \subset \sigma$

$$\omega_{m_\sigma + \sigma^\vee} * \omega_{\tau^\vee} \xrightarrow{\cong} \omega_{m_\tau + \tau^\vee},$$

and the homotopies between compositions and so on. Such isomorphism should follow from the equality

$$(m_\sigma + \sigma^\vee) + \tau^\vee = m_\tau + \tau^\vee$$

above. And to seriously supply them, one could apply the $\Gamma_{M_\mathbb{R}}$ functor to the collection of subset $\{m_\sigma + \sigma^\vee\}$ and inclusions between them. If the divisor D comes from an polytope P, this construction will recover ω_P. We call a divisor twisted polytope as it needs not to come from a polytope and the assignment $D \mapsto \omega(D)$ generalizes $P \mapsto \omega_P$.

Remark 5.2.6. The passage from moment polytopes to divisors is additive in the sense that it takes Minkowski sum of moment polytopes to component-wise addition of divisors. In the similar way, the passage from divisors to sheaf is additive: it takes component-wise addition of divisors to convolution product of sheaves

$$\omega(D_1 + D_2) \cong \omega(D_1) * \omega(D_2).$$

This could be observed after convolution with each $\omega_{\sigma^\vee}$: one has

$$\omega_{m_1+\sigma^\vee} * \omega_{m_2+\sigma^\vee} \cong \omega_{m_1+m_2+\sigma^\vee}.$$

One can carefully phrase this as a symmetric monoidal functor, but we will not do so.

Remark 5.2.7 (Every divisor could be dominated by an ample one)**.** Even though not every divisor D comes from a polytope, it is true that after adding a large multiple of a divisor D_p coming from a polytope, the divisor $D + n \cdot D_P$ corresponds to a polytope. To see this, use the characterization of such divisor in terms of strictly convex function, as in [10, Section 3.4]. For algebraic geometers, this is similar to the fact that a line bundle would become ample after tensoring with a large multiple of ample line bundle.

Variant 5.2.8 (Sheaves and polytopes from $\mathbb{R}$-coefficient divisors)**.** The assumption on $\{n_\eta\}$ being a collection of integers or f_P being integral on each cone is not essential in this construction: one can write down objects in $\mathcal{S}hv(M_\mathbb{R}; Sp)$ from the data of an $\mathbb{R}$-coefficient 'divisor' $\{r_\eta\}$, or equivalently, a piecewise linear function f on $N_\mathbb{R}$. We leave the details to the reader as we will not use them explicitly here.

5.3 Microlocal characterization of image

In this section we prove the promised characterization of $\mathrm{Im}(\kappa)$. Before presenting details of the proof, here is a quick idea: we are going to show the right orthogonal of the image $\mathrm{Im}(\kappa)$ in $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; Sp)$ is zero. This would follow from the following explicit construction: for each $x \in M_\mathbb{R}$, we are going to write down an object $\omega(D_x) \in \mathrm{Im}(\kappa)$ in the image of κ, such that it corepresents the functor of taking stalk at x in $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; Sp)$

$$\forall \mathcal{F} \in \mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; Sp) : \mathrm{map}(\omega(D_x), \mathcal{F})[n] = \mathcal{F}_x.$$

This would imply that the right orthogonal to $\mathrm{Im}(\kappa)$ in $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; Sp)$ vanishes (as they would have vanishing stalk everywhere), and hence the two categories coincide (with the help of adjoint functor theorem). To prove such a statement about $\omega(D_x)$, we are going to apply the technique of non-characteristic deformation, after convolution with a large enough multiple of ω_P for a moment polytope P for Σ. Some complication arises in the convolution procedure - as we don't know a priori if the category $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; Sp)$ is closed under convolution (we will prove it nonetheless is, a fortiori.) This is where our narrative diverges from [40]: we play a trick to get around this issue. Note that in [40] this complication was not explicitly addressed.

Remark 5.3.1. We wish to highlight that we show that the category $\mathcal{S}hv_{\Lambda_\Sigma}(M_\mathbb{R}; Sp)$ is compactly generated, and pick out an explicitly collection of generators. Now, in general, for each conic Lagrangian L in the cotangent bundle of a manifold X, one could define (as in [19]) a category of sheaves (of spectra) with singular support lying inside L. We are curious if said category is always compactly generated and if there is a natural procedure to pick out compact generators in that category. Specifically, as the functor of taking microlocal stalk is one profitable perspective offered by the microlocal analysis of sheaves, we are curious if there is any natural way to write down corepresenting objects for the functor of taking microlocal stalk and compute mapping spectra between them.

We begin by defining the object $\omega(D_x)$ mentioned above.

Definition 5.3.2. [40, Definition 4.1] For a point $x \in M_{\mathbb{R}}$, we define the probing sheaf at x

$$\omega(D_x) \in \mathcal{S}hv(M_{\mathbb{R}}; Sp)$$

to be the object associated to the divisor

$$D_x = \{n_\eta(D_x) = \lfloor -\langle x, v_\eta \rangle \rfloor + 1 : \eta \in \Sigma(1)\}$$

via the construction of Variant 5.2.5. The integer $\lfloor -\langle x, v_\eta \rangle \rfloor + 1$ is the smallest integer that's strictly bigger than $-\langle x, v_\eta \rangle$. Note that by Proposition 5.2.1 we know $D_x \in \mathrm{Im}(\kappa)$.

The naming comes from the following theorem, whose proof takes up the rest of the section:

Theorem 5.3.3. For arbitrary sheaf $\mathcal{F} \in \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; Sp)$, there exists an isomorphism (which would be spelled out explicitly in the proof)

$$\mathrm{map}(\omega(D_x), \mathcal{F})[n] \xrightarrow{\cong} \mathcal{F}_x \in Sp.$$

Given this, one can look at the inclusion $\mathrm{Im}(\kappa) \to \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; Sp)$: the right orthogonal of $\mathrm{Im}(\kappa)$ vanishes because any object in there would have vanishing stalk everywhere. Applying adjoint functor theorem, one obtains a right adjoint $\mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; Sp) \to \mathrm{Im}(\kappa)$ such that the composition with inclusion is identity on $\mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; Sp)$ - which proves that the inclusion is essentially surjective: we have obtained the following:

Corollary 5.3.4. There is an identification of full subcategories in $\mathcal{S}hv(M_{\mathbb{R}}; Sp)$:

$$\mathrm{Im}(\kappa) = \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; Sp).$$

Notation 5.3.5 (Convention of moment polytope)**.** From now on we fix a moment polytope P for Σ, and we assume the origin is contained in the interior of P. The polytope P is given by the combinatorial data of a divisor (Remark 5.2.4) as collection of integers $\{n_\eta(P) : \eta \in \Sigma(1)\}$. In particular this corresponds to a presentation of P as intersection of half spaces

$$P = \bigcap_{\eta \in \Sigma(1)} \{m : \langle m, v_\eta \rangle \geqslant -n_\eta(P)\}$$

(recall that v is a fixed primitive element of η). Note also from the fact that the origin is in the interior of P, we know that

$$n_\eta(P) > 0$$

for each η. Moreover, we fix a fundamental domain $W \subset M_{\mathbb{R}}$ for $M_{\mathbb{R}}/M$: pick a basis $\{m_i\}$ for the lattice M and take the half-closed hypercube

$$W := \{\Sigma_i r_i m_i : m_i \in M; r_i \in [0,1)\}.$$

By replacing P with some large multiple $n \cdot P$, we might assume for each $x \in W$, the divisor

$$D_x + D_P$$

also comes from a moment polytope which we call P_x. One can achieve this by observing that there are only finitely many different divisors D_x for $x \in W$ while for each fixed D_x one can dominate it by a large multiple of P as in Remark 5.2.7.

Remark 5.3.6. We will prove that $\omega(D_x)$ corepresents taking stalk for $x \in W$, but the same would follow for every point $x \in M_{\mathbb{R}}$, by observing that for $m \in M$

$$\omega(D_{x+m}) \cong \omega_m * \omega(D_x)$$

while convolution with ω_m is just induced by translation along m. So we might translate other points into the fundamental domain and obtain the statement for other points. Another way to see this is that such P as above would actually dominate D_x for all points x.

Now we supply a family of polytopes deforming P_x.

Definition 5.3.7 (Non-Characteristic deformation of probing sheaf)**.** Fix $x \in W$ and a small positive real number $\epsilon \ll 1$ so that

$$-\langle x, \nu_\eta\rangle + \epsilon \cdot n_\eta < \lfloor -\langle x, \nu_\eta\rangle \rfloor + 1$$

for all $\eta \in \Sigma(1)$. Consider the following increasing family of polytopes indexed by $s \in [0,1]$:

$$P_{x,s} := s \cdot P_x + (1-s) \cdot (x + (1+\epsilon) \cdot P).$$

It grows from (when $s = 0$) $x + (1+\epsilon) \cdot P$ to (when $s = 1$) P_x. Note this definition of $P_{x,s}$ depends on ϵ though we do not make it explicit.

We will apply non-characteristic deformation lemma to this family. To do so, we start with an observation about its interaction with Λ_Σ.

Lemma 5.3.8. If we write $P_{x,s}$ as an intersection of half-planes

$$P_{x,s} = \bigcap_{\eta \in \Sigma(1)} \{m \in M_{\mathbb{R}} : \langle m, \nu_\eta\rangle \geqslant -n_{\eta,x,s} \in \mathbb{R}\}.$$

Then for $s \in [0,1)$, none of the real numbers $-n_{\eta,x,s}$ will be an integer. (In terms of Variant 5.2.8, these $\{n_{\eta,x,s}\}$ give the real coeffient divisors for $P_{x,s}$.)

Proof. Since the assignment from polytopes to divisors is linear, we might look at the two ends of the interpolation for the coefficients in the divisor:

$$n_{\eta,x,1} = n_\eta(P_x) = n_\eta(P) + \lfloor -\langle x, \nu_\eta\rangle \rfloor + 1,$$
$$n_{\eta,x,0} = n_\eta(P) - \langle x, \nu_\eta\rangle + \epsilon \cdot n_\eta(P).$$

As long as

$$-\langle x, \nu_\eta\rangle + \epsilon \cdot n_\eta < \lfloor -\langle x, \nu_\eta\rangle \rfloor + 1$$

for each $\eta \in \Sigma(1)$, there will be no integer between $n_{\eta,x,0}$ and $n_{\eta,x,1}$, hence the claim. ∎

Lemma 5.3.9. [40, Lemma 3.13] Let $s \in [0,1)$. Let $y \in \partial P_{x,s}$ be on the boundary of the polytope $P_{x,s}$, then we have the following estimate on the fiber of Λ_Σ at y:

$$\Lambda_{\Sigma,y} \cap -\sigma(y) = 0 \subseteq N_{\mathbb{R}} \cong T^*_y(M_{\mathbb{R}}).$$

Here $\sigma(y) \in \Sigma$ is the cone dual to the angle spanned by $P_{x,s}$ at y, formally determined as follows: the vectors $\{p - y : p \in P_{x,s}\} \subseteq M_{\mathbb{R}}$ span a cone $\sigma(y)^\vee$ in $M_{\mathbb{R}}$, then take its dual cone $\sigma(y) \in \Sigma$.

Proof. That $\sigma(y) \in \Sigma$ follows from $P_{x,s}$ is also a moment polytope but with non-integral vertices (since it is a convex linear combination of moment polytopes). Now if

$$0 \neq u \in -\sigma(y) \cap \Lambda_{\Sigma,y},$$

one can find some $\tau \in \Sigma$ (note that τ cannot be the origin, so τ contains some 1-cone) and $m \in M$ such that

$$(y,u) \in m + \tau^\perp \times -\tau$$

and thus $u \in -\sigma(y) \cap -\tau$. This implies $\sigma(y) \cap \tau \neq \{0\}$. Now pick an 1-cone $\rho \subseteq \sigma(y) \cap \tau$, it follows that $\langle y, v_\rho \rangle = \langle m, v_\rho \rangle$ is an integer. On the other hand, $\rho \subseteq \sigma(y)$ implies that v_ρ attains minimum at y on $P_{x,s}$, which means $-n_{\rho,x,s} = \langle y, v_\rho \rangle$ is an integer. This contradicts previous Lemma 5.3.8. □

With this we can contemplate the family of open polytopes given by the interiors $P^\circ_{x,s}$ for $s \in [0,1)$.

Lemma 5.3.10. Consider a sheaf $F \in \mathcal{S}hv_{\Lambda_\Sigma}(M_{\mathbb{R}}; Sp)$ and the family of open polytopes given by the interior $P^\circ_{x,s}$ for $s \in (-1,1) \cong \mathbb{R}$, where we extend the original family over $[0,1)$ by constant to the left: $P_{x,s} := P_{x,0}$ for $s < 0$. Then the assumption of the non-characteristic deformation lemma Proposition 5.1.15 is met for the sheaf F and this family of open subsets $P^\circ_{x,s}$.

Proof. The point 1 and 2 in assumption of Proposition 5.1.15 follows directly from the definition of $P^\circ_{x,s}$. Unpacking the final point, we see that Z_s is empty for $s \in (-1,0)$ and $Z_s = \partial P_{x,s}$ for $s \in [0,1)$. Following the notations there, we write

$$i : M_{\mathbb{R}} \setminus U_s \longrightarrow M_{\mathbb{R}}$$

and

$$j : U_s \longrightarrow M_{\mathbb{R}}$$

for the inclusion maps. Now the goal is to show $i_! i^! F_y = 0$. Applying recollement sequence for $P_{x,s}$, we wish to show that

$$F_y \to j_* j^*(F)_y$$

is an isomorphism for $y \in \partial P_{x,s}$ and $s \in [0,1)$, with $j : P^\circ_{x,s}$ is the inclusion. Since the determination of stalk is local, we might work locally and apply an exponential map as in Definition 5.1.7, to reduce to the case of a sheaf $\mathcal{F}$ on a vector space $M_{\mathbb{R}}$ constructible for a stratfication by linear subspace (hence in particular, conic). The sheaf $\mathcal{F}$ has the same singular support at origin as F at y. We are asking if the comparison of stalks at origin is an isomorphism:

$$\mathcal{F}_0 \to j_* j^*(\mathcal{F})_0$$

where $j : \sigma^{\vee,\circ}(y) \to M_{\mathbb{R}}$ is inclusion of an open cone $\sigma^{\vee,\circ}(y)$ determined as in Lemma 5.3.9 (whose dual is named $\sigma(y) \subseteq N_{\mathbb{R}}$). By stratified homotopy invariance [7, Corollary 3.3] (or [20, Corollary 3.7.3]), one may identify this map with

$$\mathcal{F}(M_{\mathbb{R}}) \to \mathcal{F}(\sigma^{\vee,\circ}(y)).$$

Now one can apply Fourier-Sato transform: the map becomes

$$\mathrm{map}(\mathcal{FS}(\underline{\mathbb{S}}_{M_{\mathbb{R}}}), \mathcal{FS}(\mathcal{F})) \longrightarrow \mathrm{map}(\mathcal{FS}(\underline{\mathbb{S}}_{\sigma^{\vee,\circ}(y)}), \mathcal{FS}(\mathcal{F})).$$

To show the map is an isomorphism, it suffices to show

$$\mathrm{map}(\mathrm{cofib}(\mathcal{FS}(\underline{\mathbb{S}}_{\sigma^{\vee,\circ}(y)}) \to \mathcal{FS}(\underline{\mathbb{S}}_{M_{\mathbb{R}}})), \mathcal{FS}(\mathcal{F})) = 0.$$

By Lemma 5.1.6, we know that

$$\mathcal{FS}(\underline{\mathbb{S}}_{M_{\mathbb{R}}}) \cong \underline{\mathbb{S}}_0[-n] \in \mathcal{S}\mathrm{hv}(N_{\mathbb{R}}; \mathrm{Sp})$$
$$\mathcal{FS}(\underline{\mathbb{S}}_{\sigma^{\vee,\circ}(y)}) \cong \underline{\mathbb{S}}_{-\sigma(y)}[-n] \in \mathcal{S}\mathrm{hv}(N_{\mathbb{R}}; \mathrm{Sp})$$

and the map between them is induced by inclusion, thus one can identify

$$\mathrm{cofib}(\mathcal{FS}(\underline{\mathbb{S}}_{\sigma^{\vee,\circ}(y)}) \to \mathcal{FS}(\underline{\mathbb{S}}_{M_{\mathbb{R}}})) \cong \mathrm{cofib}(\underline{\mathbb{S}}_{-\sigma(y)} \longrightarrow \underline{\mathbb{S}}_0)[-n] \cong h_!\underline{\mathbb{S}}[1-n]$$

for $h : -\sigma(y) \setminus \{0\} \to N_{\mathbb{R}}$. Now the assumption on $\mu\mathrm{supp}(\mathcal{F}) \subset \Lambda_\Sigma$ says

$$\mathrm{supp}(\mathcal{FS}(\mathcal{F})) \subseteq \Lambda_{\Sigma,y} \subseteq N_{\mathbb{R}}.$$

Moreover, from Lemma 5.3.9 we learn that $\mathrm{supp}(\mathcal{FS}(\mathcal{F})) \cap -\sigma(y) \subseteq \{0\}$. This implies the map h above factorizes through the open subset of complement of support of $\mathcal{FS}(\mathcal{F})$, thus we must have

$$\mathrm{map}(\mathrm{cofib}(\mathcal{FS}(\mathbb{S}_{\sigma^{\vee,\circ}(y)}) \to \mathcal{FS}(\mathbb{S}_{M_{\mathbb{R}}})), \mathcal{FS}(\mathcal{F})) = \mathrm{map}(h!\underline{\mathbb{S}}[1-n], \mathcal{FS}(\mathcal{F})) = 0.$$

This concludes the proof. □

Corollary 5.3.11. For $F \in \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp})$ and ϵ sufficiently small as above, the restriction map

$$F(P_x^\circ) \longrightarrow F(x + (1+\epsilon) \cdot P^\circ)$$

is an isomorphism.

We are going to deduce from this that $\omega(D_x)$ corepresents taking stalk.

Proposition 5.3.12. If $\mathcal{G} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$ satisfies that

$$\mathcal{G} * \omega_P \in \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}),$$

then for sufficiently small ϵ as above and $x \in W$, we have

$$\mathcal{G}(x + \epsilon \cdot P^\circ) \xrightarrow{\cong} \mathrm{map}(\omega(D_x), \mathcal{G})[n].$$

Taking colimit along shrinking ϵ, one learns that for $x \in W$

$$\mathcal{G}_x \xrightarrow{\cong} \mathrm{map}(D_x, \mathcal{G})[n].$$

The same is true for all $x \in M_{\mathbb{R}}$.

Proof. Given that ω_P is a convolution invertible object, one can identify

$$\mathcal{G}(x+\epsilon\cdot P^\circ) \cong \mathrm{map}(\underline{\mathbb{S}}_{x+\epsilon\cdot P^\circ}, \mathcal{G}) \xrightarrow{\cong} \mathrm{map}(\underline{\mathbb{S}}_{x+\epsilon\cdot P^\circ} * \omega_P, \mathcal{G} * \omega_P) \cong (\mathcal{G} * \omega_P)(x+(1+\epsilon)\cdot P^\circ)$$

Now by assumption that $\mathcal{G} * \omega_P$ lies in $\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$, we can apply Corollary 5.3.11 and learn that the restriction map

$$(\mathcal{G} * \omega_P)(x+(1+\epsilon)\cdot P^\circ) \xleftarrow{\cong} (\mathcal{G} * \omega_P)(P_x^\circ)$$

is an isomorphism. Finally again using ω_P is convolution invertible, we have (recall that P_x is associated to the divisor $D_x + D_P$)

$$\mathrm{map}(\omega(D_x), \mathcal{G}) \xrightarrow{\cong} \mathrm{map}(\omega(D_x) * \omega_P, \mathcal{G} * \omega_P) \cong \mathrm{map}(\omega_{P_x}, \mathcal{G} * \omega_P) \cong (\mathcal{G} * \omega_P(P_X^\circ))[-n].$$

Putting above equivalences together we arrive at

$$\mathcal{G}(x+\epsilon\cdot P^\circ) \cong \mathrm{map}(\omega(D_x), \mathcal{G})[n]$$

by the explicit construction. This isomorphism is compatible with restriction maps along shrinking ϵ, and hence we get

$$\mathcal{G}_x \cong \mathrm{map}(\omega(D_x), \mathcal{G})[n]$$

as promised, for $x \in W$. As explained in Remark 5.3.6, the same result holds for any $x \in M_\mathbb{R}$. □

Warning 5.3.13. Beware that this does not conclude the proof: the missing point is that we don't know if $(-) * \omega_P$ preserves the subcategory

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) \subseteq \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}).$$

To circumvent the above disadvantage, we consider the following subcategory of $\mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$:

$$\mathcal{C} := \{\mathcal{G} \in \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) : \mathcal{G} * \omega_P \in \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})\}.$$

A quick observation is that, since $\mathrm{Im}(\kappa)$ is contained in $\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp})$ and closed under convolution, we have $\mathrm{Im}(\kappa) \subseteq \mathcal{C}$. The above argument effectively shows the following.

Proposition 5.3.14. The functor of taking stalk at x is corepresented by D_x (up to a shift) in $\mathcal{C}$.

A second observation we will need is that the category $\mathcal{C}$ is closed under colimits and limits in $\mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$, and in particular presentable (but we actually only need cocompleteness).

Proposition 5.3.15. The inclusion $\mathrm{Im}(\kappa) \subseteq \mathcal{C}$ is an equality.

Proof. The same proof as in the argument following Theorem 5.3.3 does the job here. □

A final observation we will use is that, since ω_P is a convolution-invertible object in $\mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$, we have a functor

$$(-) * \omega_P^{-1} : \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_\mathbb{R}; \mathrm{Sp}) \to \mathcal{C}.$$

Applying above proposition, one learns that for each $\mathcal{F} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$,

$$\mathcal{F} * \omega_P^{-1} \in \mathcal{C} = \mathrm{Im}(\kappa).$$

But now that $\mathrm{Im}(\kappa)$ is closed under convolution, one learns that

$$\mathcal{F} = \mathcal{F} * \omega_P^{-1} * \omega_P \in \mathrm{Im}(\kappa).$$

At this point we already obtain our goal (!)

$$\mathrm{Im}(\kappa) = \mathcal{S}\mathrm{hv}_{\Lambda}(M_{\mathbb{R}}; \mathrm{Sp})$$

and Theorem 5.3.3 follows easily (beware the flip of logic here).

6 Epilogue

In the final section, we exploit the results developed thus far to derive some tangible ramifications. Firstly, we apply the folklore method of de-equivariantization to obtain the 'non-equivariant' version of the equivalence. Next, as a concrete consequence, we provide a (certainly over-complicated) proof of Beilinson's equivalence for flat $\mathbb{P}^1$ over $\mathbb{S}$ (and also $\mathbb{P}^n$). More generally, we introduce a definition of the toric construction in an abstract setting and explain how the equivalence fits into this framework. As an example, we demonstrate how this method recovers a family version of the equivalence as in [17].

Throughout the section we always work with a smooth projective fan.

6.1 De-equivariantization

One of the most basic notions in the theory of stacks is that of quotient stacks. The fundamental insight is that the quotient $[X/G]$ of X by G encodes all the G-equivariant information about X. In this regard, $\mathrm{QCoh}([X/G])$ is just the category of objects in $\mathrm{QCoh}(X)$ together with a G-action, i.e., the category of G-modules in $\mathrm{QCoh}(X)$. Therefore, $\mathrm{QCoh}([X/G])$ is completely determined by $\mathrm{QCoh}(X)$, along with the action of G on $\mathrm{QCoh}(X)$.

This process of determining $F([X/G])$ from $F(X)$, together with the information of a G-action on $F(X)$, is colloquially referred to as equivariantization, where F is a sheaf, with $F = \mathrm{QCoh}(-)$ in the previous example.

A less-exploited point of view, dubbed de-equivariantization, allows us to sometimes go in the other direction. When $F = \mathrm{QCoh}(-)$, we often have

$$\mathrm{QCoh}(X) \simeq \mathrm{QCoh}([X/G]) \otimes_{\mathrm{QCoh}(BG)} \mathrm{QCoh}(*),$$

where the relative tensor product is taken in Pr^{L}. A typical situation where one can make such move could be found in [3, Proposition 4.6][SAG, Corollary 9.4.2.3]. Now we apply this method to the case which is interesting for us. We first make some preparations.

Lemma 6.1.1. For each $\sigma \in \Sigma$, the stack $[X_\sigma/\mathbb{T}]$ is a perfect stack in the sense of [SAG, Definition 9.4.4.1]. Similarly, the stack $B\mathbb{T}$ is also a perfect stack.

Proof. We only present the proof for $[X_\sigma/\mathbb{T}]$, the other case could be proved along the same line. We need to check three things:

- That $[X_\sigma/\mathbb{T}]$ is a quasi-geometric stack. It is in fact geometric. Given [SAG, Corollary 9.3.1.4], this follows (in the same way as [27, Remark 2.1]) from the fact that it is presented as a colimit of an action diagram, where the degree 0 term X_σ is affine and the map $d_0 : X_\sigma \times \mathbb{T} \to X_\sigma$ is representable, affine and faithfully flat.
- That the structure sheaf $\mathcal{O}$ is a compact object in $\mathrm{QCoh}([X_\sigma/\mathbb{T}])$. Via Proposition 3.3.1, the structure sheaf is sent to a representable presheaf, which is certainly compact.
- That the category $\mathrm{QCoh}([X_\sigma/\mathbb{T}])$ is generated by compact objects. Via Proposition 3.3.1 this again translates to the fact that presheaf category is compactly generated.

□

Corollary 6.1.2 (De-equivariantization for QCoh)**.** The method of de-equivariantization applies to the following stacks:

- For each $\sigma \in \Sigma$, we have a symmetric monoidal equivalence
$$\mathrm{QCoh}([X_\sigma/\mathbb{T}]) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}(*) \xrightarrow{\cong} \mathrm{QCoh}(X_\sigma).$$

- We have a symmetric monoidal equivalence
$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}]) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}(*) \xrightarrow{\cong} \mathrm{QCoh}(X_\Sigma).$$

Proof. The first point is a direct application of [SAG, Corollary 9.4.2.3] given that both $[X_\sigma/\mathbb{T}]$ and $B\mathbb{T}$ are perfect stacks. For the second point, note that by the colimit presentation of $[X_\Sigma/\mathbb{T}]$ one has
$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}] \cong \lim_{\Sigma^{op}} \mathrm{QCoh}([X_\sigma/\mathbb{T}])$$
hence the relative tensor product gives
$$\begin{aligned}\mathrm{QCoh}([X_\Sigma/\mathbb{T}] \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}(*) &\cong \lim_{\Sigma^{op}} \mathrm{QCoh}([X_\sigma/\mathbb{T}]) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{QCoh}(*) \\ &\cong \lim_{\Sigma^{op}} \mathrm{QCoh}(X_\sigma) \\ &\cong \mathrm{QCoh}(X_\Sigma)\end{aligned}$$
where we have used that $\mathrm{QCoh}(*)$ is dualizable over $\mathrm{QCoh}(B\mathbb{T})$ hence the relative tensor product commutes with limits. □

Remark 6.1.3. For the second point, one can directly show that $[X_\Sigma/\mathbb{T}]$ is a perfect stack and apply de-equivariantization.

Now we move on to work with sheaves. Note that on the mirror side, the de-equivariantization is reflected as equivariantization, as we explain now.

Lemma 6.1.4. The fully faithful symmetric monoidal functor
$$\mathrm{Fun}(M, \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp})$$
constructed in Remark 4.3.7 identifies with the symmetric monoidal functor (where $i : M \to M_\mathbb{R}$ is the inclusion of the topological groups)
$$i_! : \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}).$$
Moreover, the relative tensor product can be identified as
$$\mathcal{S}\mathrm{hv}(M_\mathbb{R}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \cong \mathcal{S}\mathrm{hv}(M_\mathbb{R}/M; \mathrm{Sp}) \in \mathrm{CAlg}(\mathrm{Pr}^L).$$

Proof sketch. Recall that the functor is a composition

$$\mathrm{Fun}(M, \mathrm{Sp}) \longrightarrow \lim_{\sigma} \mathrm{Fun}(\Theta(\sigma), \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

and each of the two arrows is fully faithful by unwinding the definitions, so we know the composition is also fully faithful. By Proposition 4.5.4, we know that it takes $m \in M$ to the skyscraper $\underline{\mathbb{S}}_{\{m\}} \in \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$. Note also that it commutes with colimits, so its image lands in the image of the fully faithful functor $i_!$. Hence we get a symmetric monoidal factorization

$$\mathrm{Fun}(M, \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M; \mathrm{Sp})$$

and this functor is readily checked to be an equivalence. For the second point, we may now replace $\mathrm{Fun}(M, \mathrm{Sp})$ by $\mathcal{S}\mathrm{hv}(M; \mathrm{Sp})$ and Sp by $\mathcal{S}\mathrm{hv}(*; \mathrm{Sp})$. We are now looking at the relative tensor product

$$\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathcal{S}\mathrm{hv}(M;\mathrm{Sp})} \mathcal{S}\mathrm{hv}(*; \mathrm{Sp})$$

formed along the symmetric monoidal functors

$$i_! : \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

and

$$p_! : \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(*; \mathrm{Sp})$$

thought of as morphisms in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. To compute the tensor product, one can look at the colimit of the simplicial diagram in Pr^{L}:

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \otimes \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \otimes \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \mathcal{S}\mathrm{hv}(M; \mathrm{Sp}) \otimes \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

given by the Bar complex presentation of relative tensor product. By Künneth formula [39, Proposition 2.30], one might identify each term in above with

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \mathcal{S}\mathrm{hv}(M \times M \times M_{\mathbb{R}}; \mathrm{Sp}) \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \mathcal{S}\mathrm{hv}(M \times M_{\mathbb{R}}; \mathrm{Sp}) \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

where all the functors are now given by !-pushforward. In other words, this diagram is the outcome of applying $D(-)_!$ to the diagram of Čech nerve of the map

$$M_{\mathbb{R}} \longrightarrow M_{\mathbb{R}}/M.$$

Now one can take right adjoints and compute the limit of the following diagram in Cat

$$\cdots \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; \mathcal{S}\mathrm{hv}(M \times M \times M_{\mathbb{R}}; \mathrm{Sp}) \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; \mathcal{S}\mathrm{hv}(M \times M_{\mathbb{R}}; \mathrm{Sp}) \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})$$

where all the functors are now !-pullback, but since all the maps are étale, they are canonically identified with $*$-pullback. Thus the diagram is identified with the outcome of taking $\mathcal{S}\mathrm{hv}(-)$ and star-pullback of the Čech nerve of the covering map $M_{\mathbb{R}} \to M_{\mathbb{R}}/M$. So we might conclude that the limit

$$\lim_{\Delta} \mathcal{S}\mathrm{hv}(M^{\times n} \times M_{\mathbb{R}}; \mathrm{Sp}) \cong \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M; \mathrm{Sp})$$

by étale descent of taking $\mathcal{S}\mathrm{hv}(-)$ and $*$-pullback. In the above argument, we only explained the identification of the underlying category, and also have been very careless with coherences. A careful reader should read the following remark for technical details. □

Remark 6.1.5 (Coherences and symmetric monoidal structures in the proof)**.** Here we supply technical details of the proof. To start with, we have a symmetric monoidal functor

$$D_!(-) : \mathrm{LCH} \longrightarrow \mathrm{Pr}^{\mathrm{L}}$$

and one can left Kan extend it to a symmetric monoidal colimit preserving functor on the category of presheaves [13] on LCH (ignoring size issues)

$$D_!(-) : \mathrm{Fun}(\mathrm{LCH}^{\mathrm{op}}, \mathrm{Spc}) \longrightarrow \mathrm{Pr}^{\mathrm{L}}$$

and apply Proposition 7.2.1 to know that it is compatible with relative tensor product. In particular we get an identification

$$D_!(M_{\mathbb{R}}) \otimes_{D_!(M)} D(\mathrm{pt}) \xrightarrow{\cong} D_!(h_{M_{\mathbb{R}}} \times_{h_M} h_{\mathrm{pt}}) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

where the underlying object of the RHS is computed as a colimit of simplicial diagram of $D_!(-)$ applied to the Bar complex of relative tensor product

$$h_{M_{\mathbb{R}}} \times_{h_M} h_* \in \mathrm{Fun}(\mathrm{LCH}^{\mathrm{op}}, \mathrm{Spc}).$$

It remains to understand this colimit as an object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. We have a map

$$h_{M_{\mathbb{R}}} \times_{h_M} h_* \longrightarrow h_{M_{\mathbb{R}}/M} \in \mathrm{CAlg}(\mathrm{Fun}(\mathrm{LCH}^{\mathrm{op}}, \mathrm{Spc}))$$

and we claim it becomes an equivalence once we apply $D_!(-)$. This essentially follows from étale descent for taking sheaf category. We supply an explanation as follows: given the map, it suffices to show that after applying $D_!(-)$ one gets an equivalence of categories. We can identify the simplical object of the Bar complex for with the Yoneda image of the simplicial object of Čech nerve of the covering map

$$M_{\mathbb{R}} \longrightarrow M_{\mathbb{R}}/M \in \mathrm{LCH},$$

since now we are actually comparing diagrams sitting inside in a sub-1-category in $\mathrm{Fun}(\mathrm{LCH}^{\mathrm{op}}, \mathrm{Spc})$ and the coherences can be readily checked. It follows that we have an identification of simplicial diagram of categories

$$[n \mapsto \mathcal{S}\mathrm{hv}(M; \mathrm{Sp})^{\otimes n} \otimes \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp})] \cong [n \mapsto \mathcal{S}\mathrm{hv}(M^{\times n} \times M_{\mathbb{R}})].$$

Now the question is reduced to checking

$$\cdots \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \mathcal{S}\mathrm{hv}(M \times M \times M_{\mathbb{R}}; \mathrm{Sp}) \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \mathcal{S}\mathrm{hv}(M \times M_{\mathbb{R}}; \mathrm{Sp}) \rightrightarrows \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M; \mathrm{Sp})$$

to be a colimit diagram in Pr^{L}, and we can argue as above by taking right adjoints and using that taking $\mathcal{S}\mathrm{hv}(-)$ with $*$-pullback has étale descent. It follows that

$$D_!(h_{M_{\mathbb{R}}} \times_{h_M} h_*) \xrightarrow{\cong} D_!(h_{M_{\mathbb{R}}/M}) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

so we have a symmetric monoidal equivalence

$$\mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M, \mathrm{Sp})} \mathrm{Sp} \cong \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M; \mathrm{Sp}) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}).$$

[13] Alternatively, one can go to étale sheaves and simplify some of the arguments below.

We have also used the following fact:

Lemma 6.1.6. The lax symmetric monoidal functor

$$D_!(-) : \mathrm{LCH} \longrightarrow \mathrm{Cat}$$

lifts to a symmetric monoidal functor (which we abusively give the same name)

$$D_!(-) : \mathrm{LCH} \longrightarrow \mathrm{Pr}^{\mathrm{L}}.$$

Proof. It follows from [HA, Remark 4.8.1.9] and

- On objects each X is taken to a presentable category $\mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$.
- On morphisms each f is taken to a colimit preserving functor $f_!$
- The box tensor product on $\mathcal{S}\mathrm{hv}(X; \mathrm{Sp})$ is colimit preserving in each variable.
- The Künneth formula holds [39, Proposition 2.30].

□

Finally, we can apply equivariantization to sheaves with singular support:

Remark 6.1.7. Let's quickly remind ourselves that the condition of being constructible and having prescribed singular support is preserved and can be checked after pullback along an étale cover map. This follows from the local nature of the definition Remark 5.1.12. See [18, Lemma 3.7] for a related account on locally constancy and constructibility.

Corollary 6.1.8 (Equivariantization for $\mathcal{S}\mathrm{hv}$)**.** One has the symmetric monoidal equivalence

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \cong \mathcal{S}\mathrm{hv}_{\overline{\Lambda}_\Sigma}(M_{\mathbb{R}}/M; \mathrm{Sp})$$

where right hand side is the subcategory of sheaves of spectra on $M_{\mathbb{R}}/M$ on objects that

- are constructible for the stratification $\overline{\mathcal{S}}_\Sigma := \pi(\mathcal{S}_\Sigma)$ inherited from the projection map π.
- have singular support lying in $\overline{\Lambda}_\Sigma := d\pi(\Lambda_\Sigma) \subset T^*M_{\mathbb{R}}/M$ inherited form the projection map π.

Proof. We make use of functoriality of the relative tensor product to get a functor

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \longrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \cong \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M; \mathrm{Sp}) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

We will show that this is fully faithful[14] and describe its image in terms of singular support. By functoriality of bar resolution, we have the following map between augmented simplicial diagrams that gives the above map after taking colimit:

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes \mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \arrow[r] \arrow[d] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes \mathrm{Fun}(M, \mathrm{Sp}) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \arrow[r] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}; \mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp}
\end{tikzcd}$$

[14]This is actually straightfoward, if one notes that tensoring with dualizable category preserves fully faithful functors.

where all the vertical maps are fully faithful (since tensoring with a dualizable category preserves fully faithful functors [8, Theorem 2.2]). We may identify each term in the top row with its image:

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \otimes \mathrm{Fun}(M,\mathrm{Sp})^{\otimes n} \cong \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}} \times M^{\times n};\mathrm{Sp})$$

where we have implicitly used Künneth formula for the bottom row, and the right hand side means sheaves on $M_{\mathbb{R}} \times M^{\times n}$ such that on each component of $M_{\mathbb{R}}$ it is constructible for $\mathcal{S}_\Sigma$ and has singular support contained in Λ_Σ. Now we observe the following: the right adjoint of each functor in the bottom row preserves the condition of constructibliy and singular support, as it is the $*$-pullback along an étale cover. Thus taking right adjoints of the bottom row restricts to taking right adjoints of the top row:

$$\begin{tikzcd} \cdots \arrow[r, shift left=2, leftarrow] \arrow[r, leftarrow] \arrow[r, shift right=2, leftarrow] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}} \times M;\mathrm{Sp}) \arrow[d] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1, leftarrow] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \arrow[d] \arrow[r, leftarrow] & \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \arrow[d] \\ \cdots \arrow[r, shift left=2, leftarrow] \arrow[r, leftarrow] \arrow[r, shift right=2, leftarrow] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}} \times M;\mathrm{Sp}) \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1, leftarrow] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp}) \arrow[r, leftarrow] & \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M;\mathrm{Sp}) \end{tikzcd}$$

Note that both rows are now limit diagrams in Cat. So we learn that the relative tensor product we look at sits inside

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp} \hookrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}}/M;\mathrm{Sp})$$

as a full subcategory, spanned by the objects which land into the full subcategory

$$\mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \hookrightarrow \mathcal{S}\mathrm{hv}(M_{\mathbb{R}};\mathrm{Sp})$$

along $*$-pullback. By the observation we made in the very beginning, this is precisely the category of sheaves on $M_{\mathbb{R}}/M$ which are constructible for $\overline{\mathcal{S}}_\Sigma$ and has prescribed singular support contained in $\overline{\Lambda}_\Sigma$. The proof is now done. □

Corollary 6.1.9. We have the 'non-equivariant' version of the equivalence. There is a symmetric monoidal functor

$$\overline{\kappa} : \mathrm{QCoh}(X_\Sigma) \xrightarrow{\cong} \mathcal{S}\mathrm{hv}_{\overline{\Lambda}_\Sigma}(M_{\mathbb{R}}/M;\mathrm{Sp})$$

where the right hand side is the category appeared in Corollary 6.1.8.

Proof. It follows from the commutative diagram Theorem B that the relative tensor products are identified in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\mathrm{QCoh}([X_\Sigma/\mathbb{T}) \otimes_{\mathrm{QCoh}(B\mathbb{T})} \mathrm{Sp} \cong \mathcal{S}\mathrm{hv}_{\Lambda_\Sigma}(M_{\mathbb{R}};\mathrm{Sp}) \otimes_{\mathrm{Fun}(M,\mathrm{Sp})} \mathrm{Sp}$$

Now we can apply Corollary 6.1.2 and Corollary 6.1.8 and win. □

6.2 Beilinson's theorem about projective space

As a concrete example of the abstract nonsense we have developed, we now give a overcomplicated explanation of Beilinson's linear algebraic description for quasi-coherent sheaves on $\mathbb{P}^1_{\mathbb{S}}$, the flat projective line over $\mathbb{S}$. Recall that the toric data corresponding to projective line is given by lattice $\mathbb{Z}$ and fan $\{\{0\}, \mathbb{R}_{\geqslant 0}, \mathbb{R}_{\leqslant 0}\}$ inside $\mathbb{R}^1$.

Example 6.2.1. There are equivalences of categories:

$$\mathrm{QCoh}(\mathbb{P}^1_{\mathbb{S}}) \simeq \mathrm{Cons}_{\overline{\mathcal{S}}_{\Sigma};\mathrm{Sp}}(S^1) \simeq \mathrm{Fun}(\bullet \rightrightarrows \bullet; \mathrm{Sp})$$

where the stratification $\overline{\mathcal{S}}_{\Sigma}$ has two strata: the origin and its complement. The first equivalence is given by $\overline{\kappa}$ and the second is given by exodromy [13].

Proof. The first functor is $\overline{\kappa}$ supplied by Corollary 6.1.9. More precisely, it embeds $\mathrm{QCoh}(\mathbb{P}^1_{\mathbb{S}})$ as a full subcategory, but one checks readily that the condition on singular support is vacuous. Away from the origin, every $\overline{\mathcal{S}}_{\Sigma}$ constructible sheaf becomes locally constant, hence the singular support is always contained in the 0-section. At the origin, the singular support asks for the support of some sheaf on $\mathbb{R}^1$ to have support contained in $\mathbb{R}^1$, which is again no restriction. We thus conclude that the first functor is an equivalence. The second functor is directly applying [13] (in fact, Theorem 4.4.5 is enough) and note that the exit path category of $(S^1, \overline{\mathcal{S}}_{\Sigma})$ is precisely the quiver $\bullet \rightrightarrows \bullet$. $\square$

Remark 6.2.2. It is possible to obtain the similar result for $\mathbb{P}^n_{\mathbb{S}}$ which states that the category $\mathrm{QCoh}(\mathbb{P}^n_{\mathbb{S}})$ is compactly generated by a collection of objects $\mathcal{O}(1), \cdots, \mathcal{O}(n+1)$, and they form an exceptional collection. This however is more involved since condition on singular support puts an actual constraint so one needs further arguments than applying exodromy equivalence. We only present a sketch of the proof idea here. Pick some equivariant lifts $\tilde{\mathcal{O}}(i) \in \mathrm{QCoh}(\mathbb{P}^n_{\mathbb{S}}/\mathbb{T}])$. The image of these $\tilde{\mathcal{O}}(i)$ under κ are dualizing sheaves on explicit moment polytopes as in Section 5.2. To show that they generated, one can run the argument in Section 5.3 to see that these images $\kappa(\tilde{\mathcal{O}}(i))$ corepresents taking stalks at each points in a fundamental domain for $\mathbb{R}^n/\mathbb{Z}^n$, so by adjunction $\overline{\kappa}(\mathcal{O}(i))$ also corepresents taking stalks at each point on $\mathbb{R}^n/\mathbb{Z}^n$. This shows they generate, and the mapping spectra can be directly computed by looking at intersections of these moment polytopes, which we omit. The computation could also recover the presentation of $\mathrm{QCoh}(\mathbb{P}^n_{\mathbb{S}})$ as presheaf of spectra on an explicit quiver with relation defined by Beilinson.

Remark 6.2.3. This suggests we might dream of exodromy for constructible sheaves with prescribed singular support: can one read off Beilinson's quiver directly from the singular support $\overline{\Lambda}_{\Sigma_n}$ where Σ_n is the fan for $\mathbb{P}^n$? We have no clue yet. See Figure 1 for the heuristic picture of $\mathbb{P}^2$.

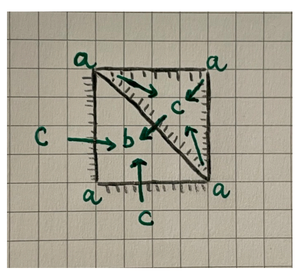

Figure 1: The hairy drawing indicates the singular support in a fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$. Three distinguished stalks and ways that they are allowed to exit were drawn in green.

Remark 6.2.4. When the fan Σ is zonotopal and unimodular[15] (see [35, Definition 4.2]), the conic Lagrangian Λ_Σ is identified with the conormal variety of the stratification $\mathcal{S}_\Sigma$ (similarly for $\overline{\Lambda}_\Sigma$) [35, Theorem 4.4]. From this one can argue that the singular support condition is automatically satisfied for all $\mathcal{S}_\Sigma$ ($\overline{\mathcal{S}}_\Sigma$, respectively) constructible sheaves. Thus Corollary 6.1.9 identifies $\mathrm{QCoh}(X_\Sigma)$ with a constructible sheaf category

$$\mathrm{Cons}_{\overline{\mathcal{S}}_\Sigma}(M_\mathbb{R}/M;\mathrm{Sp})$$

where exodromy [13] applies. How does one write down the exit path category from the combinatorics of the fan? We have no good idea.

6.3 Relative toric bundle

The proof of Corollary 6.1.9 depends on the base change functor

$$(-)\otimes_{\mathrm{QCoh}(B\mathbb{T})}\mathrm{QCoh}([X_\Sigma/\mathbb{T}]):\mathrm{CAlg}(\mathrm{Pr}^L)_{\mathrm{QCoh}(B\mathbb{T})/}\longrightarrow \mathrm{CAlg}(\mathrm{Pr}^L)$$

applied to the symmetric monoidal functor

$$\mathrm{QCoh}(B\mathbb{T})\cong \mathrm{Fun}(M,\mathrm{Sp})\longrightarrow \mathrm{Sp}\in \mathrm{CAlg}(\mathrm{Pr}^L).$$

There is no reason to stop at this case, so we make the formal defintion:

Definition 6.3.1 (Relative toric construction)**.** Fix a lattice N and fan Σ. Given a symmetric monoidal functor

$$f:M\longrightarrow \mathcal{C}$$

where $\mathcal{C}\in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$, it induces a map

$$F:\mathrm{QCoh}(B\mathbb{T})\cong \mathrm{Fun}(M;\mathrm{Sp})\longrightarrow \mathcal{C}\in \mathrm{CAlg}(\mathrm{Pr}^L)$$

and we define

$$\mathrm{Mod}_{X_{\Sigma,f}}\mathcal{C}:=\mathcal{C}\otimes_{\mathrm{QCoh}(B\mathbb{T})}\mathrm{QCoh}([X_\Sigma/\mathbb{T}])\in \mathrm{CAlg}(\mathrm{Pr}^L)$$

to be the relative toric bundle over $\mathcal{C}$ associated with Σ and f.

It follows from the definition that $\mathrm{QCoh}([X_\Sigma/\mathbb{T}])$ and $\mathrm{QCoh}(X_\Sigma)$ can be constructed in this way.

Example 6.3.2. In [17], the second named author with Pyongwon Suh considered the data of a classical scheme S and n line bundles $\{L_n\in \mathrm{Pic}(S)\}$ on S. Such collection of line bundles defines a symmetric monoidal functor

$$f:\mathbb{Z}^n\longrightarrow \mathrm{QCoh}(S)$$

and the relative toric bunlde over $\mathrm{QCoh}(S)$ associated with Σ and f could be identified with the category of quasi-coherent sheaves on an S-scheme $\mathcal{X}_{\Sigma,f}$:

$$\mathrm{Mod}_{X_{\Sigma,f}}\mathrm{QCoh}(S)\cong \mathrm{QCoh}(\mathcal{X}_{\Sigma,f}).$$

The relative toric scheme (or so-called toric fibration $\mathcal{X}_{\Sigma,f}$ is constructed affine locally on S, as a toric scheme with respect to the torus associated with $\oplus L_i$ over S. Equivalently, it could be

[15] Unfortunately these assumptions are quite restrictive.

identified with base change of $[X_\Sigma/\mathbb{T}] \to B\mathbb{T}$ along the map $S \to B\mathbb{T}$ classifying these line bundles $\{L_i\}$. On the mirror side, the base change could be interpreted as sheaves on the torus $\mathbb{R}^n/\mathbb{Z}^n$ with twisted-coefficient - roughly the stalk of the coefficient category is $\mathrm{QCoh}(S)$ and the monodromy is given by tensoring with L_i.

Remark 6.3.3. Heuristically, such $f : \mathbb{Z}^n \longrightarrow \mathcal{C}$ classifies n strict Picard elements that also strictly commutes with each other. Beware that such datum is rare in the wild, see [5].

7 Appendix

7.1 Modules over grouplike monoid

We find the following lemma straight forward, but could not locate a proof.

Lemma 7.1.1. Let T be an ∞-category admitting finite limits, $\mathsf{G} \in \mathrm{Mon}(\mathsf{T})$, and X a G-module. If G is grouplike, then $(\mathsf{X}//\mathsf{G})_\bullet$ is a groupoid object.

Proof. Unwinding the definitions, there is a canonical map

$$p : (\mathsf{X}//\mathsf{G})_\bullet \to (*//\mathsf{G})_\bullet,$$

where the latter can be identified with the underlying simplicial object of G, hence a groupoid object [HA, Remark 5.2.6.5]. Therefore it suffices to show that this map is a Cartesian natural transformation (see [HTT, Definition 6.1.3.1].)

In other words, we want to show that for every $\alpha : [m] \to [n]$, the diagram

$$\begin{tikzcd}
X \times G^n \arrow[r, "\simeq"] & (X//G)_n \arrow[r] \arrow[d] & (X//G)_m \arrow[d] & X \times G^m \arrow[l, "\simeq"'] \\
G^n \arrow[r, "\simeq"] & (*//G)_n \arrow[r] & (*//G)_m & G^m \arrow[l, "\simeq"']
\end{tikzcd}$$

is a pullback, i.e., $p(\alpha) : p([n]) \to p([m]) \in \mathrm{Fun}([1], \mathsf{T})$ is a Cartesian morphism.

We proceed by induction and show $p|_{\Delta^{op}_{\leqslant n}}$ is a Cartesian transformation for each n. For the base case $n = 0$, there is nothing to prove. For $n \geqslant 1$, note that every map in $\Delta_{\leqslant n}$ can be factored into a sequence of maps in which each is either in $\Delta_{\leqslant n-1}$ or one of the follows: the injective maps $\delta_k : [n-1] \to [n]$ and the surjective maps $\sigma_k : [n] \to [n-1]$. Therefore it suffices to show that $p(\delta_k)$ and $p(\sigma_k)$ are Cartesian morphisms.

For $p(\delta_k)$, we claim that it suffices to prove $p(\delta_0)$ and $p(\delta_n)$ are Cartesian: indeed for $0 < k < n$, consider the decomposition $[0, k] \cup [k, n] = [n]$ and the diagram

$$\begin{tikzcd}
p([n]) \arrow[r, "\shortmid" marking] \arrow[d, "\shortmid" marking] \arrow[dr, dashed] & p([0,k]) \arrow[d, rightsquigarrow] \arrow[dr, rightsquigarrow] & \\
p([k,n]) \arrow[r, rightsquigarrow] \arrow[dr, rightsquigarrow] & p(\{k\}) & \\
 & p(\{\cdots < k-1 < k+1 < \cdots\}) \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & p([0,k-1]) \\
 & p([k+1,n]) &
\end{tikzcd}.$$

By induction hypothesis, all the squiggly arrows are Cartesian. By the 2-out-of-3 property of Cartesian morphisms, to show the dashed arrow is Cartesian (and hence every arrow is Cartesian), it suffices to show either of the barred arrows is Cartesian. However $[0, k] \hookrightarrow [n]$ factors as a map in $\Delta_{\leqslant n-1}$ followed by δ_n.

Using the identifications
$$(X//G)_n \simeq X \times G^n,$$
and
$$\prod_i ([i < i+1] \hookrightarrow [n])^* : (*//G)_n \simeq G^n,$$
$p(\delta_0)$ is equivalent to
$$\begin{tikzcd} X \times G^n \arrow[r] \arrow[d] & G^n \arrow[d] \\ X \times G^{n-1} \arrow[r] & G^{n-1} \end{tikzcd},$$
where all the maps are projection, hence Cartesian.
Similarly, $p(\delta_n)$ is equivalent to the product of
$$\begin{tikzcd} X \times G \arrow[r] \arrow[d, "a"'] & G \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$
with G^{n-1}. Therefore it suffices to show the map $X \times G \xrightarrow{(a,pr)} X \times G$ is an equivalence, which is indeed true as it admits a homotopy inverse given by shearing.
To see $p(\sigma_k)$ is Cartesian, simply note that both its source and target are (induced by) diagonal maps. □

7.2 Functoriality of forming module categories

The following could be directly unpacked from [HA].

Proposition 7.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories which admit all geometric realizations. Given symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ such that:

1. Tensor products in $\mathcal{C}$ and $\mathcal{D}$ commute with geometric realization.
2. Functor F commutes with geometric realization.

One can extract the following diagram
$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, bend left, "\mathrm{Mod}_{(-)}(\mathcal{C})", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Mod}_{F(-)}(\mathcal{D})"', ""{name=D, above}] & \mathrm{CAlg}(\mathrm{Cat}) \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$
out of [HA, Theorem 4.8.5.16]. When evaluated on $A \to B$, the diagram reads
$$\begin{tikzcd} \mathrm{Mod}_A(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Mod}_B(\mathcal{C}) \arrow[d] \\ \mathrm{Mod}_{F(A)}(\mathcal{D}) \arrow[r] & \mathrm{Mod}_{F(B)}(\mathcal{D}) \end{tikzcd}.$$

Proof. We pick up notations in [HA, Theorem 4.8.5.16] and fix $\mathcal{K}$ to be just $\{\Delta^{op}\}$ (in particular the following terms refer to terms in there). The symmetric monoidal coCartesian fibrations in (1) and the functor $\Theta^{\otimes}$ in (3) there straighten to lax symmetric monoidal functors and natural transformations [16]

$$\begin{tikzcd} \mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat}) \arrow[r, bend left, "\mathrm{Alg}(-)"] \arrow[r, bend right, "\mathrm{Mod}_{(-)}(\mathrm{Cat})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Cat} \end{tikzcd} .$$

One applies further CAlg on both sides and obtain

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat})) \arrow[r, bend left, "\mathrm{Alg}(-)"] \arrow[r, bend right, "\mathrm{Mod}_{(-)}(\mathrm{Cat})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{CAlg}(\mathrm{Cat}) \end{tikzcd} .$$

The assumption on $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ ensures that it lifts to a map in $\mathrm{CAlg}(\mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat}))$. We evaluate the above natural transformation on F and obtain a commuting diagram in $\mathrm{CAlg}(\mathrm{Cat})$ as

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{C}) \arrow[r, "\mathrm{Mod}_{(-)}(\mathcal{C})"] \arrow[d, "\mathsf{F}"] & \mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}) \arrow[d, "(-)\otimes_{\mathcal{C}}\mathcal{D}"] \\ \mathrm{Alg}(\mathcal{D}) \arrow[r, "\mathrm{Mod}_{(-)}(\mathcal{D})"] & \mathrm{Mod}_{\mathcal{D}}(\mathrm{Cat}) \end{tikzcd}$$

Apply again CAlg everywhere

$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, "\mathrm{Mod}_{(-)}(\mathcal{C})"] \arrow[d, "\mathsf{F}"] & \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat})) \arrow[d, "(-)\otimes_{\mathcal{C}}\mathcal{D}"] \\ \mathrm{CAlg}(\mathcal{D}) \arrow[r, "\mathrm{Mod}_{(-)}(\mathcal{D})"] & \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{D}}(\mathrm{Cat})) \end{tikzcd}$$

and note that $(-) \otimes_{\mathcal{C}} \mathcal{D}$ being a symmetric monoidal left adjoint implies that there is an adjunction $(-) \otimes_{\mathcal{C}} \mathcal{D} \dashv \mathrm{fgt}$ between $\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}))$ and $\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{D}}(\mathrm{Cat}))$. Putting everything together we end up with a natural transformation

$$\begin{tikzcd} \mathrm{CAlg}(\mathcal{C}) \arrow[r, bend left, "\mathrm{Mod}_{(-)}(\mathcal{C})"] \arrow[r, bend right, "\mathrm{Mod}_{\mathsf{F}(-)}(\mathcal{D})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat})) \end{tikzcd} .$$

Post-composing with forgetful to $\mathrm{CAlg}(\mathrm{Cat})$ gives what we claimed. ☐

[16] Note that $\Theta^{\otimes}$ preserves coCartesian edges over $\mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat})^{\otimes}$. This follows from the following two facts: from (4) we know it is a symmetric monoidal functor hence preserves coCartesian lift from Fin_* and from [HA, Proposition 4.8.5.1] we know the underlying functor Θ preserves coCartesian edges over $\mathrm{Mon}^{\mathcal{K}}_{\mathrm{Assoc}}(\mathrm{Cat})$.

7.3 Reminders on Day convolutions

Remark 7.3.1 (Day convolution and its universal property)**.** Recall that given a small symmetric monoidal category $(\mathcal{C}, \otimes)$, there is a symmetric monoidal structure on spectral presheaf category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp})$ called 'Day convolution'. The stable Yoneda embedding h has a structure of symmetric monoidal functor and has the following universal property:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "h"] \arrow[dr, "F"'] & & \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp}) \arrow[dl, dashed, "\exists!"] \\ & \mathcal{D} & \end{tikzcd}$$

For any presentably symmetric monoidal stable category $\mathcal{D}$ with a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$, there exists unique symmetric monoidal, colimit preserving lift to $\mathrm{Fun}(\mathcal{C}^{op}, Sp)$. We write $\mathrm{Lan}_h F$ for the lift.

To be precise, one learns from [HA, Proposition 4.8.1.10] that for each small symmetric monoidal category $(\mathcal{C}, \otimes)$, the presheaf category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spc})$ has the structure of a presentably symmetric monoidal category, and the (unstable) Yoneda functor

$$h : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spc})$$

has a structure of symmetric monoidal functor. Moreover, the restriction map

$$\mathrm{Fun}^{lax,L}(\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spc}), \mathcal{D}) \xrightarrow{h^*} \mathrm{Fun}^{lax}(\mathcal{C}, \mathcal{D})$$

is an equivalence for any presentably symmetric monoidal category $\mathcal{D}$. The restriction of above functor to the full subcategory of symmetric monoidal functors

$$\mathrm{Fun}^{\otimes,L}(\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spc}), \mathcal{D}) \xrightarrow{h^*} \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$$

is also an equivalence. Using the symmetric monoidal adjunction

$$\begin{tikzcd} \mathrm{Pr}^L \arrow[r, bend left, "-\otimes \mathrm{Sp}"] & \mathrm{Pr}^L_{st} \arrow[l, bend left, "\text{forgetful}"] \end{tikzcd}$$

one learns that the stable analogues (we abuse notation by writing h for the stable Yoneda)

$$\mathrm{Fun}^{lax,L}(\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp}), \mathcal{D}) \xrightarrow{h^*} \mathrm{Fun}^{lax}(\mathcal{C}, \mathcal{D})$$

$$\mathrm{Fun}^{\otimes,L}(\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp}), \mathcal{D}) \xrightarrow{h^*} \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$$

also hold for any presentably symmetric monoidal stable category $\mathcal{D}$. These equivalences provide for us pointwise lifting constructions.

Remark 7.3.2 (Day convolution as a partial adjunction)**.** The equivalence above could be understood as a partial adjunction between forgetful and taking presheaf (and similarly for $\mathrm{CAlg}^{\mathrm{lax}}$):

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{CAT}) & & \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}) \arrow[ll, "\text{forgetful}"'] \\
\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{small}}) \arrow[u, "i"'] \arrow[urr, "\mathrm{Fun}(-^{\mathrm{op}},\mathrm{Sp})"']
\end{tikzcd}.$$

See, for example, [14, 1.32] on how to extract adjoint functorially. In particular, the equivalence

$$\mathrm{Fun}^{\mathrm{lax},\mathrm{L}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp}),\mathcal{D})^{\cong} \xrightarrow{h^*} \mathrm{Fun}^{\mathrm{lax}}(\mathcal{C},\mathcal{D})^{\cong}$$

$$\mathrm{Fun}^{\otimes,\mathrm{L}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp}),\mathcal{D})^{\cong} \xrightarrow{h^*} \mathrm{Fun}^{\otimes}(\mathcal{C},\mathcal{D})^{\cong}$$

is functorial in $\mathcal{C}$ and $\mathcal{D}$. This implies that when we deal with diagrams in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{small}})$ and $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ the pointwise liftings will be functorial, and we will freely use this fact without mentioning the explicit construction.

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