# A study of sheaves on real vector spaces

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In the first section we recall the computation of $\mathrm{K}^{\mathrm{cont}}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}))$ sketched by Sasha Efimov in his talk. No knowledge of singular support will be assumed. We also provide a motivation for this computation through the looking-glass of homological mirror symmetry. Note that Marc Hoyois already provided a computation of $\mathrm{K}^{\mathrm{cont}}$ for any locally compact Hausdorff spaces in [3].

In the second section we compute the Picard groupoid of sheaves (of $k$-modules for a field $k$) on $\mathbb{R}^1$ under convolution product.

In the third section we propose some speculations for Picard groupoid of sheaves on $\mathbb{R}^n$ under convolution product.

Finally, in the fourth section we sketch another approach to the computation for Picard groupoid of sheaves on $\mathbb{R}^n$ (of $k$-modules for a field $k$) under convolution product, which builds on the notion of Fourier transform for wild Betti sheaves. The general strategy in this section was explained to us by Peter Scholze (but the inaccuracies are of course our own faults).

## Contents



## 1 K-theory of sheaves on $\mathbb{R}^1$

We will explain the following computation:

**Theorem 1.** The continuous K-theory of the category of sheaves of spectra on the real line can be computed as

$$\mathrm{K}^{\mathrm{cont}}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \cong \Omega(\mathrm{K}(\mathbb{S})).$$

The key ingredient is the following category.

**Definition 2.** Let $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1;\mathrm{Sp})$ be the full subcategory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ spanned by

$$\{\mathcal{F} \in \mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}) : \mathcal{F}((-\infty,a)) \xrightarrow{\cong} \mathcal{F}((b,a)) \forall b < a\}.$$

And one defines $\mathcal{S}\mathrm{hv}^{-}(\mathbb{R}^1;\mathrm{Sp})$ similarly as the full subcategory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ spanned by

$$\{\mathcal{F} \in \mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}) : \mathcal{F}((b,+\infty)) \xrightarrow{\cong} \mathcal{F}((b,a)) \forall b < a\}.$$

We also write $\mathcal{S}\mathrm{hv}^{0}(\mathbb{R}^1;\mathrm{Sp}) := \mathcal{S}\mathrm{hv}^{\mathrm{lc}}(\mathbb{R}^1;\mathrm{Sp})$ for the full subcategory of locally constant (hence constant) sheaves on $\mathbb{R}^1$.

The fact that we will prove about $\mathcal{S}\mathrm{hv}^{+}$ translates verbatimly to $\mathcal{S}\mathrm{hv}^{-}$. To start with, there are many more equivalent descriptions for $\mathcal{S}\mathrm{hv}^{+}$.

**Proposition 3.** The following subcategories of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ are the same.

1. The subcategory $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1;\mathrm{Sp})$ as we described in Definition 2.
2. The subcategory spanned by the image of a fully faithful functor of pulling back sheaves
$$\pi^* : \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp}) \to \mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}).$$
The topological space $\mathbb{R}^1_+$ has underlying set $\mathbb{R}^1$ and a weaker topology, with opens specified by
$$\{U_a := (-\infty, a) \subseteq \mathbb{R}^1 : a \in [-\infty,+\infty]\}.$$
The continuous map $\pi : \mathbb{R}^1 \to \mathbb{R}^1_+$ is given by identity on the underlying set.
3. The cocomplete stable subcategory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ spanned by $h(U_a)^{\mathrm{sh}}$, the sheafification of the stable Yoneda image of $U_a = (-\infty, a) \in \mathrm{Open}(\mathbb{R}^1)$ (we will abusively call them *representable sheaves*).
4. The subcategory $\mathrm{Mod}_{\theta_+}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}))$. We equip the category $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ with a convolution symmetric monoidal structure, and there is an idempotent algebra $\theta_+ := h(U_0)^{\mathrm{sh}}[1]$.
5. (For fans of singular support, we won't use this) The subcategory of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$ spanned by sheaves with singular support contained in $\mathbb{R}^1 \times \mathbb{R}^1_{\geqslant 0} \subseteq T^*\mathbb{R}^1$.

*Proof.* We will focus on comparing the first 4 descriptions.

- 2 versus 3: it is not so obvious that $\pi^*$ is fully faithful. To prove this, one has an adjunction between the locales:
$$\begin{tikzcd} \mathrm{Open}(\mathbb{R}^1) \arrow[rr, bend left, "\tau : U \mapsto U + U_0"] \arrow[rr, phantom, "\perp"] && \mathrm{Open}(\mathbb{R}^1_+) \arrow[ll, bend left, "\pi^{-1}"] \end{tikzcd}$$
where the addition means Minkowski sum. Hence we get $\tau^*$ left adjoint to $\pi^*$ on sheaf category. Now it suffices to check that on representable sheaf $x$, the counit map $\tau^*\pi^*x \to x$ is an isomorphism: this is true from the definition. It follows that $\pi^*$ is fully faithful. Now we need to see that the functor $\pi^*$ lands into the subcategory given in 3 and is essentially surjective. This follows from the following two facts: $\pi^*$ takes representable sheaves to representable sheaves and they generate under colimits in both categories.

- 3 versus 4: we first explain a bit more about 4. The convolution product is given by

$$\mathcal{F} * \mathcal{G} := +_!(\mathcal{F} \boxtimes \mathcal{G})$$

where $+ : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is the addition map. As a motivation, one can directly compute

$$\theta_+ * \theta_+ \cong \theta_+.$$

The recollement sequence for $(-\infty, 0] = (-\infty, 0) \cup \{0\}$ provides a map $\mathbb{1}_{\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})} \to \theta_+$ and makes the target into an idempotent algebra. Now the module category, as a localization, is colimit generated by the image of representable sheaves under convolution with $\theta_+$: and one directly computes that the convolution of representable sheaves with $\theta_+$ gives representable sheaves on $U_a$.

- 4 versus 1. The module category as a localization can be equally described by the collection of local objects:

$$\mathrm{Mod}_{\theta_+}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) = \{Y : \mathrm{Map}(X \otimes \theta_+, Y) = \mathrm{Map}(X, Y)\}.$$

(we write Map for mapping space and map for mapping spectrum, but note in above one might as well use mapping spectrum for the local condition) The local condition is closed under colimit in testing object X so one might as well check on representable sheaves X. By our discussion above this is exactly what 1 says.

- (*) 1 versus 5: the reader might consult the definition of singular support from [4] or [1]. It follows from the definition that condition in 1 implies condition in 5. Conversely, use non-characteristic deformation as in [5, Theorem 4.1].

□

**Remark 4.** The history of this category dates back to [4, Section 3.5] where it was studied from point 2 above. The localization kernel $\theta_+$ was introduced by D.Tamarkin.

**Proposition 5.** The categories in Definition 2 fits in to a Cartesian diagram in Cat:

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}) \arrow[r] \arrow[d] & \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp}) \arrow[d] \\
\mathcal{S}\mathrm{hv}^-(\mathbb{R}^1;\mathrm{Sp}) \arrow[r] & \mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp})
\end{tikzcd}$$

More precisely, from the view point of 4 above, we have the following diagram

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{1}}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \arrow[r, "*\theta_+"] \arrow[d, "*\theta_-"'] & \mathrm{Mod}_{\theta_+}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \arrow[d, "*\theta_-"] \\
\mathrm{Mod}_{\theta_-}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \arrow[r, "*\theta_+"'] & \mathrm{Mod}_{\theta_+ * \theta_-}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}))
\end{tikzcd}$$

which after identification gives the above diagram.

*Proof.* We first need to supply an identification between the categories on the down right corner - $\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp}) := \mathcal{S}\mathrm{hv}^{\mathrm{lc}}(\mathbb{R}^1;\mathrm{Sp})$ and $\mathrm{Mod}_{\theta_+ * \theta_-}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp}))$. One can follow exactly the same argument as comparing 4 versus 1 in Proposition 3, after identifying $\theta_+ * \theta_- \cong \underline{\mathbb{S}}[1]$.

We also need to show that the diagram is Cartesian. This follows from the standard consequence of descent along idempotent algebra, and the Cartesian diagram of idempotent algebras

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d] & \theta_+ \arrow[d] \\ \theta_- \arrow[r] & \theta_+ * \theta_- \end{tikzcd}$$

which could be verified directly: it is the recollement sequence for $\mathbb{R}^1 = \{0\} \coprod \mathbb{R}^\times$.

$\square$

Note that the diagram lifts to $\mathrm{Cat}^{\mathrm{dual}}$ and all edges are moreover localizations. Applying excision (as in [3, Corollary 13], but for dualizable categories) we learn that after applying $\mathrm{K}^{\mathrm{cont}}$, the above diagram becomes Cartesian:

$$\begin{tikzcd} \mathrm{K}^{\mathrm{cont}}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \arrow[r] \arrow[d] & \mathrm{K}^{\mathrm{cont}}(\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})) \arrow[d] \\ \mathrm{K}^{\mathrm{cont}}(\mathcal{S}\mathrm{hv}^-(\mathbb{R}^1;\mathrm{Sp})) \arrow[r] & \mathrm{K}^{\mathrm{cont}}(\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp})) \end{tikzcd}$$

where we immediately note that

$$\mathrm{K}^{\mathrm{cont}}(\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp})) = \mathrm{K}^{\mathrm{cont}}(\mathcal{S}\mathrm{hv}^{\mathrm{lc}}(\mathbb{R}^1;\mathrm{Sp})) = \mathrm{K}^{\mathrm{cont}}(\mathrm{Sp}) = \mathrm{K}(\mathbb{S}).$$

This reduces the computation to the following fact:

**Proposition 6.** The continuous K-theory of $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})$ vanishes (and similarly for $\mathcal{S}\mathrm{hv}^-$).

To show this we provide a localization sequence for $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})$. For that we switchfoot and take the view of point 2 from Proposition 3.

**Proposition 7.** There is a Verdier sequence of the form

$$\begin{tikzcd} \prod\limits_{\mathbb{R}} \mathrm{Sp} \arrow[r, "i"] & \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) \arrow[l, bend right, "L_0"', "\perp"] \arrow[l, bend left, "R_0", "\perp"'] \arrow[r, "(-)^{\mathrm{sh}}"] & \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp}) \arrow[l, bend right, "L_1"', "\perp"] \arrow[l, bend left, "R_1", "\perp"'] \end{tikzcd} \tag{1}$$

with

$$i : \prod_{\mathbb{R}} \mathrm{Sp} \to \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) : a \mapsto \operatorname*{colim}_{b<a} h(b)$$

and

$$(-)^{\mathrm{sh}} : \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) \to \mathcal{S}\mathrm{hv}(\mathbb{R}_{\leqslant}, \mathrm{Sp}) = \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp})$$

being sheafification, using identification in Lemma 9 below. The other functors will be made explicit later, see Diagram 2.

Before we provide the construction, here is a definition and a technical fact (you should skip it and fast-forward to the proof of Proposition 7 on the first reading):

**Definition 8.** We define $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})^{+} \subseteq \mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})$ to be the subcategory of semi-continuous presheaves:

$$\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})^{+} := \{\mathcal{F} : \forall a \in \mathbb{R}, \mathcal{F}(a) \xrightarrow{\cong} \lim_{b<a} \mathcal{F}(b)\}.$$

This could be equivalently phrased as a sheaf condition for the induced topology on $\mathbb{R}_{\leqslant}$, if we include $\mathbb{R}_{\leqslant}$ into $[-\infty, +\infty] = \mathrm{Open}(\mathbb{R}_{+}^{1})$. A subtlety is that sheaf condition seems to be a bit stronger than semi-continuity. They are actually equivalent because every covering of $a \in \mathbb{R}$ which doesn't contain $a$ will be a cofinal subposet of $\{b \in \mathbb{R} : b < a\}$.

**Lemma 9.** *Following above identification, restriction of sheaves induces an equivalence*

$$\pi_* : \mathcal{S}\mathrm{hv}(\mathbb{R}_{+}^{1}; \mathrm{Sp}) \xrightarrow{\cong} \mathcal{S}\mathrm{hv}(\mathbb{R}_{\leqslant}; \mathrm{Sp}) = \mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})^{+}.$$

*Proof.* This is almost a direct consequence of $\mathbb{R}_{\leqslant}$ being a basis of $\mathrm{Open}(\mathbb{R}_{+}^{1})$. More precisely: a presheaf on $\mathbb{R}_{\leqslant}$ is a sheaf if and only if its right Kan extension to $\mathrm{Open}(\mathbb{R}_{+}^{1})$ is a sheaf. From this we obtain a geometric embedding: (where Res is restriction and Ran is right Kan extension)

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(\mathbb{R}_{\leqslant}, \mathrm{Sp}) \arrow[rr, bend right, "\mathrm{Ran}"'] \arrow[rr, phantom, "\perp"] && \mathcal{S}\mathrm{hv}(\mathbb{R}_{+}^{1}; \mathrm{Sp}) \arrow[ll, bend right, "\mathrm{Res}"'] \end{tikzcd} .$$

Now one can directly inspect the functor Res and show it's fully faithful and essentially surjective. We leave the details to the reader. ☐

*Proof of Proposition 7.* We will present another Verdier localization sequence then identify the terms with Diagram 1. The category $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})$ carries a symmetric monoidal structure $\otimes$ called Day convolution: this structure is inherited from the fact that $\mathbb{R}_{\leqslant}$ is a symmetric monoidal category, and it makes the stable Yoneda embedding:

$$h : \mathbb{R}_{\leqslant} \longrightarrow \mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})$$

carry a structure of symmetric monoidal functor. Now look at the following cofiber sequence in $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Sp})$:

$$\operatorname*{colim}_{b<0} h(b) \longrightarrow h(0) \longrightarrow \mathcal{S}_{\{0\}}$$

where the later one is 'skyscraper' presheaf at $0 \in \mathbb{R}_{\leqslant}$. Note that $h(0) = \mathbb{1}_{\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}, \mathrm{Sp})}$ and we claim this map $h(0) \to \mathcal{S}_{\{0\}}$ presents $\mathcal{S}_{\{0\}}$ as an idempotent algebra. This equivalent to the following

$$\operatorname*{colim}_{b<0} h(b) \otimes \operatorname*{colim}_{b<0} h(b) \xrightarrow{\cong} \operatorname*{colim}_{b<0} h(b)$$

which is true by virtue that $\otimes$ commutes with colimit in each variable and one knows $h(a) \otimes h(b) = h(a+b)$, so one can directly evaluate both side and compare. (Note that the fiber $I =$

$\operatorname{colim}_{b<0} h(b)$ of $h(0) \to \mathbb{S}_{\{0\}}$ 'sits in homological degree 0', as an incarnation of almost mathematics situation.) One writes down the Verdier sequence for localization at this idempotent algebra:

$$\begin{tikzcd}[column sep=huge]
\mathrm{Mod}_{\mathbb{S}_{\{0\}}}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})) \arrow[r, "\mathrm{inclusion}" description] & \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) \arrow[l, bend right, "\mathbb{S}_{\{0\}} \otimes -"', "\perp"] \arrow[l, bend left, "\underline{\mathrm{Map}}(\mathbb{S}_{\{0\}}, -)", "\perp"'] \arrow[r, "\mathrm{I} \otimes -" description] & \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}} \arrow[l, bend right, "\mathrm{inclusion}"', "\perp"] \arrow[l, bend left, "\underline{\mathrm{Map}}(\mathrm{I}, -)", "\perp"']
\end{tikzcd} \qquad (2)$$

To compare with Diagram 1, let's start by identifying the first term

$$\mathrm{Mod}_{\mathbb{S}_{\{0\}}}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})) \cong \prod_{\mathbb{R}} \mathrm{Sp}.$$

To do so, let's examine the condition on $\mathcal{F} \in \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})$ to be a module over $\mathbb{S}_{\{0\}}$. This is requiring:

$$\mathcal{F} \xrightarrow{\cong} \mathcal{F} \otimes h(0) \to \mathcal{F} \otimes \mathbb{S}_{\{0\}} \xrightarrow{\cong} \mathrm{cofib}[\operatorname*{colim}_{b<0} h(b) \otimes \mathcal{F} \to h(0) \otimes \mathcal{F}]$$

to be an equivalence, which is the same as

$$\operatorname*{colim}_{b<0} h(b) \otimes \mathcal{F} = 0.$$

Evaluating above on each $a \in \mathbb{R}$ we learn that

$$\left(\operatorname*{colim}_{b<0} h(b) \otimes \mathcal{F}\right)(a) = \operatorname*{colim}_{b<0} \mathcal{F}(a-b) = 0.$$

Note that for all $a < c$, the restriction map $\mathcal{F}(c) \to \mathcal{F}(a)$ factorizes through $\operatorname*{colim}_{b<0} \mathcal{F}(a-b) = 0$, so we conclude that for such $\mathcal{F}$ all restriction maps have to be zero map. Conversely, if $\mathcal{F}$ is a presheaf such that all the restriction maps are zero, one can directly supply an equivalence

$$\bigoplus_{a \in \mathbb{R}} \mathcal{F}(a)_{\{a\}} \xrightarrow{\cong} \mathcal{F}.$$

On the left hand side, $\mathcal{F}(a)_{\{a\}}$ is the 'skyscraper' presheaf at $a \in \mathbb{R}$ whose value is $\mathcal{F}(a)$ at $a$ and 0 otherwise. Each $\mathcal{F}(a)_{\{a\}}$ is a module over $\mathbb{S}_{\{0\}}$ and it follows that $\mathcal{F}$ is a module over $\mathbb{S}_{\{0\}}$. To conclude, we have an identification:

$$\mathrm{Mod}_{\mathbb{S}_{\{0\}}}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})) = \{\mathcal{F} : \text{all restriction maps are zero}\} \xrightarrow{\cong} \mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp}) = \prod_{\mathbb{R}} \mathrm{Sp}.$$

For the middle equivalence, one can look at the functor of restriction of presheaves

$$\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp}) \to \mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp})$$

and check directly it is fully-faithful and essentially surjective on the subcategory.

Our next mission is to provide an identification of the categories

$$\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}} \cong \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+; \mathrm{Sp}).$$

First off Lemma 9 already provides an equivalence

$$\mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp}) \xrightarrow{\cong} \mathrm{Fun}(\mathbb{R}_{\leqslant},\mathrm{Sp})^+.$$

So it suffices to identify $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}}$ with $\mathrm{Fun}(\mathbb{R}_{\leqslant},\mathrm{Sp})^+$. Let's examine the condition on $\mathcal{F} \in \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})$ to be semi-continuous. For such a presheaf we must have for all $a \in \mathbb{R}$

$$\mathcal{F}(a) \xrightarrow{\cong} \lim_{b<a} \mathcal{F}(b).$$

On the other hand, with the lower localization sequence, we can identify $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}}$ with the kernel of the internal hom functor

$$\underline{\mathrm{Map}}(\mathbb{S}_{\{0\}},-) : \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp}) \to \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp}).$$

And we have

$$\begin{aligned}
\mathcal{F} \in \ker(\underline{\mathrm{Map}}(\mathbb{S}_{\{0\}},-)) &\Leftrightarrow \forall a \in \mathbb{R}, \mathrm{map}(h(a),\underline{\mathrm{Map}}(\mathbb{S}_{\{0\}},\mathcal{F}) = 0 \\
&\Leftrightarrow \forall a \in \mathbb{R}, \mathrm{map}(h(a)\otimes \mathbb{S}_{\{0\}},\mathcal{F}) = 0 \\
&\Leftrightarrow \forall a \in \mathbb{R}, \mathrm{map}(\mathbb{S}_{\{a\}},\mathcal{F}) = 0 \\
&\Leftrightarrow \forall a \in \mathbb{R}, \mathcal{F}(a) \xrightarrow{\cong} \lim_{b<a} \mathcal{F}(b)
\end{aligned}$$

which is exactly the condition on $\mathcal{F}$ to be semi-continuous. Hence we conclude that

$$\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}} \cong \ker(\underline{\mathrm{Map}}(\mathbb{S}_{\{0\}},-)) \cong \mathrm{Fun}(\mathbb{R}_{\leqslant},\mathrm{Sp})^+ \cong \mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp}).$$

□

Finally we can prove the promised Proposition 6.

*Proof of Proposition 6.* Let's look at the upper localization sequence of Diagram 1. Under the identification with later Diagram 2 it looks like

$$\mathrm{Mod}_{\mathbb{S}_{\{0\}}}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})) \xleftarrow{\mathbb{S}_{\{0\}}\otimes -} \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp}) \xleftarrow{\text{inclusion}} \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\mathbb{S}_{\{0\}}\text{-tors}} .$$

So we learn that

$$\mathcal{S}\mathrm{hv}(\mathbb{R}^1_+;\mathrm{Sp}) \cong \ker\left[L_0 : \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp}) \to \mathrm{Fun}(\mathbb{R}^{\mathrm{disc}},\mathrm{Sp}) = \prod_{\mathbb{R}} \mathrm{Sp}\right]$$

where $L_0$ is determined by

$$L_0(h(a)) = h(a)$$

and is strongly cocontinuous (more concretely, $L_0(\mathcal{F})(b) = \mathrm{cofib}[\mathrm{colim}_{a>b}\,\mathcal{F}(a) \to \mathcal{F}(b)]$). Now both categories are compactly generated and $L_0 = \mathrm{Ind}(l_0)$ where

$$l_0 : \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant},\mathrm{Sp})^{\omega} \to (\prod_{\mathbb{R}} \mathrm{Sp})^{\omega}$$

is the functor between compact objects (note that $l_0$ takes representables to representables, but doesn't come from left Kan extension of a functor between the posets!). To show $K^{\mathrm{cont}}(\mathcal{S}hv(\mathbb{R}^1_+;\mathrm{Sp}))$ vanishes it suffices to prove that $L_0$ (equivalently, $l_0$) induces an isomorphism on $K(-)$. So we've reduced to studying the map induced on $K(-)$ by a map in $\mathrm{Cat}^{\mathrm{perf}}$. Moreover, each of the categories involved have an 'infinite full exceptional collection'(IFEC): they are given by representables in both categories, and the functor $l_0$ takes IFEC of one to another. So we can apply the following technical lemma and win. $\square$

**Definition 10.** Let's recall that for a category $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$, a full exceptional collection (FEC) is a finite set of objects $E = \{X_\alpha : 0 \leqslant \alpha \leqslant n\}$ such that

- $\mathrm{map}(X_\alpha, X_\beta) = 0$ for all $\alpha > \beta$.
- $\mathrm{map}(X_\alpha, X_\alpha) = \mathbb{S}$.
- The inclusion of the span on each object $\langle X_\alpha \rangle$ into $\mathcal{C}$ admits a right adjoint.
- The span $\langle X_\alpha : 0 \leqslant \alpha \leqslant n\rangle$ is all of $\mathcal{C}$.

where by span we mean the idempotent completion of the stable subcategory spanned by a collection of objects.[1] An infinite full exceptional collection (IFEC) on $\mathcal{C}$ is a set of objects $E = \{X_\alpha : \alpha \in \mathbb{R}\}$ indexed by real numbers such that the span $\langle X_\alpha : \alpha \in \mathbb{R}\rangle = \mathcal{C}$ and for each finite subset $S \subseteq \mathbb{R}$, $\{X_\alpha : \alpha \in S\}$ is a full exceptional collection of $\langle X_\alpha : \alpha \in S\rangle$.

**Lemma 11.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor in $\mathrm{Cat}^{\mathrm{perf}}$. If $\mathcal{C}$ has an IFEC and $F$ takes this IFEC to an IFEC in $\mathcal{D}$, then $F$ induces isomorphism on $K(-)$.*

*Proof.* Using that $K(-)$ commutes with filterted colimit in $\mathrm{Cat}^{\mathrm{perf}}$, one reduces to the following claim: If $G : \mathcal{E} \to \mathcal{F}$ is in $\mathrm{Cat}^{\mathrm{perf}}$ and $G$ takes a finite full exceptional collection of $\mathcal{E}$ to a finite full exceptional collection of $\mathcal{F}$. Then $G$ induces isomorphism on $K(-)$. This is left to the reader as an exercise. $\square$

This finishes the proof. Now we take advantage of reader's attention and speculate a little bit:

**Remark 12.** Two of the ideas in the proof could be motivated through homological mirror symmetry of toric varieties. We remind the reader about homological mirror symmetry for projective line in point 0 below, then present these motivations.

0. The torus-equivariant homological mirror symmetry of projective line provides an equivalence of the following categories:

$$\mathrm{QCoh}(\mathbb{P}^1/\mathbb{G}_m) \xrightarrow{\cong} \mathrm{Cons}_P(\mathbb{R}^1;\mathrm{Sp})$$

   where left hand side is quasi-coherent modules over the flat toric scheme defined over $\mathbb{S}$ and right hand side is the full subcategory of $\mathcal{S}hv(\mathbb{R}^1;\mathrm{Sp})$ spanned by sheaves locally constant away from $\mathbb{Z} \subseteq \mathbb{R}^1$. This subcategory inherits the symmetric monoidal structure of convolution product and the equivalence upgrades to a symmetric monoidal one.

[1] Note that our definition is weaker than the usual non-commutative geometry terminology.

1. The idempotent algebra $\theta_+$ is the image of idempotent algebra

$$j_* \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m} \in \mathrm{QCoh}(\mathbb{P}^1/\mathbb{G}_m)$$

under the equivalence above. The diagram in Proposition 5 is the mirror picture (albeit in the bigger category $\mathcal{S}\mathrm{hv}$ instead of Cons) of the Zariski descent diagram for QCoh, with $\mathbb{P}^1/\mathbb{G}_m$ being covered by two pieces of $\mathbb{A}^1/\mathbb{G}_m$ .

2. One can ask what is the algebro-geometric mirror of $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$, and Dmitry Vaintrob provided an answer in [7]: it is (an equivariant flavor of) the universal compactification of the torus $\mathbb{G}_m$, implemented by almost mathematics. To be more precise, given a toric fan, instead of producing a scheme glued from pieces of monoid algebra, one can define directly, for each cone, a category of almost modules then glue them. One declares the gluing outcome to be the category of quasicoherent sheaves of the universal (partial) compactification corresponding to the fan. If we apply this construction to the fan of $\mathbb{A}^1$ we arrive at an almost mathematics situation where the Verdier localisation sequence for the almost module category is the mirror picture of Proposition 7. If we do this to the fan of $\mathbb{P}^1$ we get an algebro-geometric mirror to $\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$:

$$\mathrm{QCoh}^a(\mathbb{P}^{1,\mathrm{nov}}/\mathbb{G}_m^{\mathrm{nov}}) \cong \mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})$$

which is even a symmetric monoidal equivalence where we use tensor product of quasicoherent sheaves on the left and convolution product of sheaves on the right. For actual definition (with coefficient in ordinary rings), read around [7, Theorem 4]. The construction as in point 0 will lift this equivalence to spectral coefficient with symmetric monoidal structure. The computation of Proposition 6 can be performed (actually more transparently) on the mirror side.

As a result we also obtain continuous K-theory for (an equivariant flavor of) universal compactification $\mathrm{QCoh}^a(\mathbb{P}^{1,\mathrm{nov}}/\mathbb{G}_m^{\mathrm{nov}})$. Another feature of this computation is being symmetric monoidal: the diagram in Proposition 5 is a diagram in $\mathrm{CAlg}(\mathrm{Pr}^L)$. (The sheaf category with convolution product is is only locally rigid monoidal.)

## 2 Picard group of sheaves on $\mathbb{R}^1$ with convolution

Elaborating the ideas above, we provide a case study of a question asked by Oscar Harr and Branko Juran in Copenhagen homotopy theory problem solving seminar. We are very grateful to them for the question and discussion.

**Question 13.** For a topological group G, what are tensor invertible objects in $\mathcal{S}\mathrm{hv}(G;\mathrm{Sp})$ equipped with convolution tensor product?

For each $g \in G$, the skyscraper sheaf $S_{\{g\}}$ provides such a tensor invertible object. Somehow surprisingly, there are many more of them in the case of $G = \mathbb{R}^1$, as predicted by the toric mirror symmetry equivalence. For example we expect the following :

**Conjecture 14.** The Picard groupoid[2] of sheaves on the real line could be computed as

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) = \mathbb{R}\times\mathbb{R}\times\mathrm{Pic}(\mathrm{Sp}).$$

We cannot prove this directly yet. This subsection explains some tentative ideas in resolving the problem. Then we will explain a proof of above conjecture with coefficient Sp replaced by $\mathrm{Mod}_k$ where $k$ is a field (so we can do homological algebra).

**Remark 15.** One can explicitly provide a map of groups

$$\mathbb{R}\times\mathbb{R}\longrightarrow \pi_0(\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})))$$

as follows. If $a<b$, then $(a,b)$ is sent to the representable sheaf on the open interval $(a,b)$ shifted up by 1 homologically. If $a\geqslant b$, then $(a,b)$ is sent to the constant sheaf on the closed interval $[b,a]$. To lift it to a group map before taking $\pi_0$ takes some more effort (but can be done).

The idea we have about this computation is utilizing the fact that Proposition 5 provides a pullback diagram of symmetric monoidal categories. Applying $\mathrm{Pic}(-)$ gives

$$\begin{tikzcd}
\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1;\mathrm{Sp})) \arrow[r] \arrow[d] & \mathrm{Pic}(\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})) \arrow[d] \\
\mathrm{Pic}(\mathcal{S}\mathrm{hv}^-(\mathbb{R}^1;\mathrm{Sp})) \arrow[r] & \mathrm{Pic}(\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp}))
\end{tikzcd}$$

which is still a pullback diagram since $\mathrm{Pic}(-)$ preserves limit. As in computation for K-theory, the down right corner reads

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp})) = \mathrm{Pic}(\mathrm{Sp}).$$

Now the problem is reduced to showing that

**Conjecture 16.** The Picard group of $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp})$ can be computed as

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp}))\cong\mathbb{R}\times\mathrm{Pic}(\mathrm{Sp}).$$

Moreover, the map in the above diagram

$$\mathbb{R}\times\mathrm{Pic}(\mathrm{Sp})\cong\mathrm{Pic}(\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Sp}))\longrightarrow\mathrm{Pic}(\mathcal{S}\mathrm{hv}^0(\mathbb{R}^1;\mathrm{Sp}))=\mathrm{Pic}(\mathrm{Sp})$$

can be identified as projection to the $\mathrm{Pic}(\mathrm{Sp})$ factor.

**Remark 17.** It is easy to cook up a map from RHS to LHS via adjunction. The work lies in showing that the map is an equivalence. Or rather, explicitly computing $\pi_0$ of LHS.

By symmetry the same statement hold for $\mathcal{S}\mathrm{hv}^-$. It's obvious that this would imply Conjecture 14. To prove Conjecture 16, we can inspect the Verdier sequence Diagram 1, or equivalently Diagram 2. Here is a related computation.

[2] By $\mathrm{Pic}(-)$ we mean the maximal groupoid on tensor invertible objects, it inherits the tensor structure and we take it as an $\mathbb{E}_\infty$-group.

**Proposition 18.** The Picard group of $\mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp})$ (with Day convolution) can be computed as

$$\mathrm{Pic}(\mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp})) \cong \mathbb{R} \times \mathrm{Pic}(\mathrm{Sp}).$$

*Proof.* It suffices to compute $\pi_0$ of Pic. The functor of taking left Kan extension to a point is symmetric monoidal, hence for any tensor invertible element $F$ one must have that the colimit of $F$ (which is a coproduct) is a shift of $\mathbb{S}$. This forces $F$ to be supported at only one point with value a shift of $\mathbb{S}$. On the other hand, each skyscraper presheaf of a shift of $\mathbb{S}$ is tensor invertible. Thus we have classified isomorphism classes of tensor invertible objects in $\mathrm{Fun}(\mathbb{R}^{\mathrm{disc}}, \mathrm{Sp})$. ☐

**Proposition 19.** The Picard group of $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})$ (with Day convolution) can be computed as

$$\mathrm{Pic}(\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})) \cong \mathbb{R} \times \mathrm{Pic}(\mathrm{Sp}).$$

*Proof.* We will reduce to previous case. Recall that one already has a functor of taking associated graded: this functor is conservative on compact objects. Combining this with the fact that $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})$ is a rigid monoidal category, one learns that taking associated graded supplies an injection on $\pi_0$ of $\mathrm{Pic}(-)$, hence we have classified isomorphism classes of tensor invertible objects in $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Sp})$. ☐

It would be nice to provide further arguments along this line to prove the statement for $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Sp})$. However we run into the problem that $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Sp})$ is not rigid monoidal, so the dualizable objects need not be compact. Nevertheless one can be optimistic and hope for the following to be true: an object in the sheaf category is dualizable if and only if it lives in the smallest stable subcategory generated by representables.[3] We don't know how to prove this.

Now we take a turn to work with field coefficients and do (almost) homological algebra. The goal is to explain the following:

**Proposition 20.** Fix a field $k$. The Picard groupoid of $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ can be computed as

$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)) \cong \mathbb{R} \times \mathrm{Pic}(\mathrm{Mod}_k).$$

Moreover, the map in the above diagram

$$\mathbb{R} \times \mathrm{Pic}(\mathrm{Mod}_k) \cong \mathrm{Pic}(\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)) \longrightarrow \mathrm{Pic}(\mathcal{S}\mathrm{hv}^{0}(\mathbb{R}^1; \mathrm{Mod}_k)) = \mathrm{Pic}(\mathrm{Mod}_k)$$

can be identified as projection to the $\mathrm{Pic}(\mathrm{Mod}_k)$ factor.

**Strategy 21.** We take the following steps to prove Proposition 20.

1. It suffices to compute $\pi_0\mathrm{Pic}$, the isomorphism classes of tensor-invertible objects.
2. There is a nice t-structure on the category $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$. One can classify tensor invertible objects which also sit in the heart.
3. Given an arbitrary tensor invertible object, we will argue using Kunneth spectral sequence that it has to sit in only one degree. This depends on the facts (1) the category has Tor-dimension 1 (2) one can classify subobjects of the unit (in the heart).

[3] It was suggested by Sasha Efimov.

4. Then we are done.

We start with some homological algebra preparations.

**Definition 22.** For future reference, we note that the category of presheaves

$$\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}; \mathrm{Mod}_k)$$

has a standard, pointwise t-structure which is left and right complete. As in Diagram 2 we think of $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ as a full subcategory of $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}; \mathrm{Mod}_k)$ and the inclusion has a right adjoint:

$$I \otimes - : \mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}, \mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k).$$

To define a t-structure on $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ we will use the following fact:

**Lemma 23.** *An object $\mathcal{F}$ of $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}; \mathrm{Mod}_k)$ is in $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ if and only if each of its homology object is in $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$.*

*Proof.* Note that an object $\mathcal{F}$ in $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}; \mathrm{Mod}_k)$ lies in $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ if and only if for each $a \in \mathbb{R}$ we have the following equivalence

$$\operatorname*{colim}_{b > a} \mathcal{F}(b) \xrightarrow{\cong} \mathcal{F}(a)$$

in $\mathrm{Mod}_k$, which can be tested on taking homology. Since filtered colimits in $\mathrm{Mod}_k$ are exact, the above is an equivalence if and only if for each homology object $H_n(\mathcal{F})$ the same holds. Hence we are done. □

**Definition 24.** We define a t-structure on $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ as follows. An object in $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ is connective (coconnective) for the t-structure if and only it is connective (coconnective) as an object in $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}; \mathrm{Mod}_k)$ for the standard pointwise t-structure. It follows from the lemma above that this defines a t-structure on $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$.

**Corollary 25.** The following are true for the t-structure we specified:

1. The truncation functor is the same as the truncation functor for the standard t-structure on $\mathrm{Fun}(\mathbb{R}_{\leqslant}^{\mathrm{op}}; \mathrm{Mod}_k)$.
2. The t-structure on $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ is left and right complete.
3. The convolution product of two connective objects in $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$ stays connective.

*Proof.* They follow from the definition. □

We can talk about the Whitehead filtration on an object $X \in \mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$: it is the filtered object

$$\{\tau_{\geqslant n} X \xrightarrow{f_n} \tau_{\geqslant n-1} X\}_{n \in \mathbb{Z}}$$

that filters X. In other words, the limit along $n$ is 0 and the colimit along $n$ is X: this follows from the fact that the t-structure is left and right complete. For two such objects X and Y in $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$, one can tensor the Whitehead filtrations and get a filtration on $X \otimes Y$. Again by the left and right completeness, this filtration has limit 0 and colimit $X \otimes Y$. We call the associated spectral sequence as Künneth spectral sequence.

**Corollary 26.** The Künneth spectral sequence for the tensor products on $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ always converges. Moreover, if two objects X and Y are in the heart $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)^\heartsuit$, then their tensor product lies in homological degree $[0,1]$. It follows that the Künneth spectral sequence degenerates at page 2.

*Proof.* It remains to explain the homological bound $\leqslant 1$. Let's take two objects X and Y in the heart $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)^\heartsuit$ and try to evaluate

$$X \otimes Y(0) = \operatorname*{colim}_{a+b\geqslant 0} X(a) \otimes Y(b)$$

where the later $\otimes$ is the tensor product in $\mathrm{Mod}_k$. Now the poset $\{(a,b) : a+b \geqslant 0\}$ we take colimit over can be presented as the following filtered colimit of posets:

$$\{(a,b) : a+b \geqslant 0\} = \operatorname*{colim}_{S\subseteq\mathbb{R},\text{finite}} \mathrm{Stair}(S)$$

where we define the staircase shaped poset associated to S to be

$$\mathrm{Stair}(S) := \bigcup_{s\in S} \{(a,b) : a \geqslant s; b \geqslant -s\}.$$

By descent of colimits, we can compute $X \otimes Y(0)$ as a filtered colimit over objects of the form

$$\operatorname*{colim}_{\mathrm{Stair}(S)} X(a) \otimes Y(b).$$

Each of these can be explicitly computed as a pushout of objects in $\mathrm{Mod}_k^\heartsuit$, so in particular lives in homological degree $[0,1]$, and a filtered colimit of these still lives in homological degree $[0,1]$. The same argument works when evaluating each $X * Y(r)$, so we learn that $X \otimes Y$ lives in homological degree $[0,1]$. $\square$

We will also need the following classification:

**Corollary 27.** Recall that $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ has $\mathbb{1} := I$ as the tensor unit. It lies in the heart and each subobject of $\mathbb{1}$ is either 0 or of the form $I \otimes h_a$ for some $a \in \mathbb{R}$ where $h_a$ is ($k$-linearized) representable presheaf on $a \in \mathbb{R}$.

This finishes all the homological algebra preparation. A crucial consequence is the following:

**Proposition 28.** Every convolution invertible object in $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1;\mathrm{Mod}_k)$ lives in single homological degree.

*Proof.* Fix a convolution invertible object $X$, let's try to compare $\tau_{\geqslant 0}X$ to $X$. One can tensor up the canonical map

$$\tau_{\geqslant 0}X \longrightarrow X$$

with $X^{-1}$ and get a map

$$\tau_{\geqslant 0}X \otimes X^{-1} \longrightarrow \mathbb{1}.$$

One can inspect the Künneth spectral sequence for the tensor products $\tau_{\geqslant 0}X \otimes X^{-1}$ and $X \otimes X^{-1}$. By the fact that it degenerates at page 2, we learn that the map

$$\tau_{\geqslant 0}X \otimes X^{-1} \longrightarrow \mathbb{1}$$

is injective on each homology object, which in particular implies that $\tau_{\geqslant 0}X \otimes X^{-1}$ lies in the heart and is a subobject of $\mathbb{1}$. Now by the classification, it is either $0$ or $I \otimes h_a$ for some $a \in \mathbb{R}$. In the first case we learn that

$$\tau_{\geqslant 0}X = 0,$$

and in the latter case we learn that

$$\tau_{\geqslant 0}X = X \otimes h_a.$$

Note that tensoring with $h_a$ only makes a shift in the $\mathbb{R}$-grading, so this in particular implies that $X$ is connective, or

$$\tau_{<0}X = 0.$$

From this discussion, we learn that $X$ either sits in homological degree $[0, +\infty)$ or $(-\infty, -1]$. But the same argument works verbatimly when one compares $\tau_{\geqslant n}X$ and $X$. Hence we learn that for each $n \in \mathbb{Z}$, $X$ either sits in homological degree $[n, +\infty)$ or $(-\infty, n-1]$. This forces $X$ to sit in only one homological degree. □

From this we already learn that up to a homological shift every convolution invertible object lives in the heart. So it remains to classify the convolution invertible objects in the heart of $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)$.

**Lemma 29.** *Let $X \in \mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)^{\heartsuit}$ be convolution invertible, then $X^{-1}$ is also in the heart. Moreover, $X \otimes (-)$ takes $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)^{\heartsuit}$ to $\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)^{\heartsuit}$.*

*Proof.* Taking colimit along $\mathbb{R}^{\mathrm{op}}_{\leqslant}$ gives a symmetric monoidal functor

$$\mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k) \longrightarrow \mathrm{Sp}$$

and we learn that

$$\operatorname*{colim}_{\mathbb{R}^{\mathrm{op}}_{\leqslant}} X \otimes \operatorname*{colim}_{\mathbb{R}^{\mathrm{op}}_{\leqslant}} X^{-1} \cong k.$$

In particular this implies $H_0(X^{-1}) \neq 0$, and hence $X^{-1}$ is in the heart. Now for some object $Z \in \mathcal{S}\mathrm{hv}^{+}(\mathbb{R}^1; \mathrm{Mod}_k)^{\heartsuit}$ we want to argue that

$$H_1(X \otimes Z) = 0.$$

For this one notes that the only term that contributes to $H_0$ of $X^{-1} \otimes (X \otimes Z)$ is $X^{-1} \otimes H_0(X \otimes Z)$. Hence we must have $X^{-1} \otimes H_1(X \otimes Z) = 0$ which implies $H_1(X \otimes Z) = 0$. □

**Proposition 30.** Let $X \in \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)^\heartsuit$ be a convolution invertible object, then X is of the form $I \otimes h_a$ for some k-linearized representable presheaf $h_a$.

*Proof.* For an object
$$Y \in \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)^\heartsuit,$$
let sub(Y) be the category of subobjects of Y. More precisely, it is the full subcategory of
$$\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)^\heartsuit_{/Y}$$
spanned by monomorphisms. Note that this is a poset. Making use of previous classification of subobjects of $\mathbb{1}$, one learns that $\mathrm{sub}(\mathbb{1})$ is a totally ordered poset. Since $X \otimes (-)$ is an auto-equivalence of $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)^\heartsuit$, we must have
$$\mathrm{sub}(X) \cong \mathrm{sub}(\mathbb{1}).$$
Now we learn that for each tensor invertible X, its category of subobjects is also a totally ordered poset. From this one can easily deduce that it has to be of the form $I \otimes h_a$. □

The proof of Proposition 20 is now complete. One immediately gets:

**Corollary 31.** The Picard groupoid of sheaves on $\mathbb{R}^1$ valued in $\mathrm{Mod}_k$ with convolution is
$$\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^1; \mathrm{Mod}_k)) = \mathbb{R} \times \mathbb{R} \times \mathrm{Pic}(\mathrm{Mod}_k).$$

# 3 Picard group of sheaves on $\mathbb{R}^n$ with convolution

**Remark 32.** Here is a speculation: one can ask what happens for $\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k)$. The set of closed convex bodies in $\mathbb{R}^n$ form a commutative monoid under Minkowski sum:
$$M_n := (\{\text{closed convex bodies in } \mathbb{R}^n\}, +)$$
and one can make a map (of symmetric monoidal categories)
$$M_n \longrightarrow \mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k)$$
sending each closed convex body to lower shriek of its dualizing sheaf. We expect such sheaf to be convolution invertible and thus there is an induced map on group completion
$$M_n^{\mathrm{gp}} \times \mathrm{Pic}(\mathrm{Mod}_k) \to \mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^n; \mathrm{Mod}_k)).$$
This map is injective and provides many convolution invertible objects. Note that for $n = 1$ we have
$$M_n^{\mathrm{gp}} = \mathbb{R} \times \mathbb{R}.$$

**Remark 33.** We formulate a more precise guess for $\mathrm{Pic}(\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k))$. Be warned that this part contains no proof! To do so we need a functor of Radon transform (compare the work of Honghao Gao [2]). Consider the following correspondence

$$\begin{tikzcd} & K \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{R}^n & & S^{n-1}\times\mathbb{R}^1 \end{tikzcd}$$

where $K$ is the Radon kernel (Beware that we are using a different kernel than [2].)

$$K := \{(x,n,r) : \langle x,n\rangle < r\} \subseteq \mathbb{R}^n \times S^{n-1} \times \mathbb{R}^1$$

where we think of $S^{n-1}$ as unit sphere in the dual vector space of $\mathbb{R}^n$. One gets a functor

$$R := q_!p^* : \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k).$$

that we will call Radon transform. Moreover, the functor $R$ lands in the full subcategory

$$\mathcal{S}\mathrm{hv}^+(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k) \subseteq \mathcal{S}\mathrm{hv}(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k)$$

on the objects whose singular support along $\mathbb{R}^1$ direction are non-negative (this could also be formulated as being a module for some idempotent algebra). We will take $R$ as a functor

$$R : \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}^+(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k).$$

It's helpful to take some sheaf $F$ on $\mathbb{R}^n$ and compute the restriction of $R(F)$ along some $\{n\}\times\mathbb{R}^1$. From this perspective the functor $R$ is a family version of localization at idempotent algebra of dualizing sheaf on $\{x : \langle x,n\rangle < 0\} \subseteq \mathbb{R}^n$. Hopefully the following claims make sense:

1. The functor $R$ could be endowed with a symmetric monoidal structure, when we equip $\mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k)$ with convolution product and $\mathcal{S}\mathrm{hv}^+(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k)$ with fiberwise convolution product.

2. There is a pullback square of symmetric monoidal categories (in particular all functors appearing in the following square are symmetric monoidal)

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(\mathbb{R}^n;\mathrm{Mod}_k) \arrow[r, "R"] \arrow[d, "\Gamma_c"] & \mathcal{S}\mathrm{hv}^+(S^{n-1}\times\mathbb{R}^1;\mathrm{Mod}_k) \arrow[d, "\pi_!"] \\ \mathcal{S}\mathrm{hv}(\mathrm{pt};\mathrm{Mod}_k) \arrow[r, "p^*"] & \mathcal{S}\mathrm{hv}(S^{n-1};\mathrm{Mod}_k) \end{tikzcd}.$$

   Let's explain a bit more about the notation. Both sheaf categories on the bottom are equipped with pointwise tensor product, with the functor $\tau^*$ being the pullback functor along projection

   $$\tau : S^{n-1} \to \mathrm{pt}.$$

   The functor $\Gamma_c$ takes an sheaf $F$ to its compactly supported global section (i.e. shriek-pushforward to the point). The functor $\pi_!$ is lower-shriek along the projection

   $$\pi : S^{n-1}\times\mathbb{R}^1 \to S^{n-1}.$$

   Compare Proposition 5 when $n = 1$.

3. The Picard groupoid of $\mathcal{S}hv^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k)$ equipped with fiberwise convolution product could be computed as

$$\mathrm{Pic}(\mathcal{S}hv^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k)) \cong \mathrm{Func}^{\mathrm{cts}}(S^{n-1}; \mathbb{R}) \times \mathrm{Pic}(\mathcal{S}hv(S^{n-1}; \mathrm{Mod}_k)).$$

Here we take $\mathrm{Func}^{\mathrm{cts}}$ as the group of continuous function under addition. Putting all these together, we learn that

$$\mathrm{Pic}(\mathcal{S}hv(\mathbb{R}^n; \mathrm{Mod}_k)) \cong \mathrm{Func}^{\mathrm{cts}}(S^{n-1}; \mathbb{R}) \times \mathrm{Pic}(\mathrm{Mod}_k).$$

Under this identification, the invertible objects indexed by convex bodies are sent to their support function. Note that not every continuous function on $S^{n-1}$ can be realized as difference of support functions of convex body.

**Remark 34.** Here are some thoughts on how to produce proofs for the above claims (but none of these are actual proofs):

1. The construction of the symmetric monoidal structure should be some six-functor nonsense.

2. The coherence of the diagram should again come from six-functor nonsense. Note that we abused notation for $R$ since the functor $R$ lands in $\mathcal{S}hv^+$ by previous pointwise observation. We observe that each functor in this diagram has an explicit right adjoint. In particular for $R$ we can write down another kernel

$$\overline{K} := \{(x, n, r) : \langle x, n \rangle \leqslant r\} \subseteq \mathbb{R}^n \times S^{n-1} \times \mathbb{R}^1$$

and the corresponding functor

$$\overline{R} : \mathcal{S}hv^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k) \longrightarrow \mathcal{S}hv(\mathbb{R}^n; \mathrm{Mod}_k).$$

We claim that $\overline{R}$ is a right adjoint of $R$. Moreover the composition is

$$R \circ \overline{R} \cong \mathrm{Id}$$

on $\mathcal{S}hv^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k)$ via direct computation. For the other direction, use the explicit right adjoints to compute.

3. Take a convolution invertible object $\mathcal{F}$ in $\mathcal{S}hv^+(S^{n-1} \times \mathbb{R}^1; \mathrm{Mod}_k)$. First off we know that there exists a function $f : S^{n-1} \to \mathbb{R}$ such that

$$\{p \in S^{n-1} \times \mathbb{R}^1 : \mathcal{F}_p \neq 0\} = \{(s, r) : r < f(s)\}.$$

Why should this $f$ be continuous? We might assume its lower-shriek is the constant sheaf $\underline{k}$ on $S^{n-1}$. Now by adjunction and pointwise observation, we should learn that $\mathcal{F}$ is a subobject of a shift of constant sheaf $\underline{k}$, in particular the set on the left should be open. This should force $f$ to be lower-semi-continuous. Now the same argument applied to $\mathcal{F}^{-1}$ implies that $f$ is continuous. And all this forces $\mathcal{F}$ to be (a shift of) lower shriek of the constant sheaf on the right side.

## 4 Why wild Betti sheaves are cool

The paradigm of (categorical) Fourier transform aims to produce a symmetric monoidal equivalence between categories of the form

$$\mathcal{F} : (\mathcal{C}, *) \longrightarrow (\mathcal{C}^\vee, \otimes)$$

where $\mathcal{C}$ and $\mathcal{C}^\vee$ are categories constructed out of dual geometric data. Moreover, the $*$-'convolution' tensor product on $\mathcal{C}$ is intertwined under $\mathcal{F}$ with the $\otimes$-'ordinary' tensor product. For our purpose, it would be certainly helpful to have the following symmetric monoidal equivalence for a finite dimensional real vector space $V$:

$$\mathcal{F} : (\mathcal{S}\mathrm{hv}(V; \mathrm{Mod}_k), *) \longrightarrow (\mathcal{S}\mathrm{hv}(V^\vee; \mathrm{Mod}_k), \otimes).$$

Such equivalence immediately teaches us how to compute the Picard groupoid on the left, since we know every tensor invertible object on the right is locally constant. However, as the previous computation with $V = \mathbb{R}^1$ suggests, this is too good to be true. We will see that wild Betti sheaves come to rescue.

In [6], Peter Scholze explained the following universal way to fix the discrepancy: one can enlarge the coefficient category $\mathrm{Mod}_k$ to something exotic (i.e. non-compactly-generated) - the category $W$[4] of complete continuous presheaves on $\mathbb{R}_\leqslant$ valued in $\mathrm{Mod}_k$ [6, Definition 2.2]. The assignment

$$X \longmapsto \mathcal{S}\mathrm{hv}(X; W)$$

inherits a structure of six operations. Now one indeed has a Fourier transform for wild Betti sheaves as an equivalence between symmetric monoidal[5] categories [6, Theorem 4.1]

$$\mathcal{F} : (\mathcal{S}\mathrm{hv}(V; W), *) \longrightarrow (\mathcal{S}\mathrm{hv}(V^\vee; W), \otimes).$$

One can now pre-compose with the inclusion of ordinary sheaves (which is symmetric monoidal for the convolution product)

$$\iota : (\mathcal{S}\mathrm{hv}(V; \mathrm{Mod}_k), *) \longrightarrow (\mathcal{S}\mathrm{hv}(V; W), *)$$

to obtain a symmetric monoidal and fully faithful functor

$$\mathcal{F} \circ \iota : (\mathcal{S}\mathrm{hv}(V; \mathrm{Mod}_k), *) \longrightarrow (\mathcal{S}\mathrm{hv}(V^\vee; W), \otimes).$$

We are now faced with the following two questions:

1. Compute the Picard group of the symmetric monoidal category

$$(\mathcal{S}\mathrm{hv}(V^\vee; W), \otimes)$$

on the right.

[4] The careful reader will note that our orientation of $\mathbb{R}$ as a poset is opposite to the cited definition.

[5] In *loc. cit.* it was noted that the functor $\mathcal{F}$ intertwines the tensor products on the source and the target. It takes more effort to equip the functor with a symmetric monoidal structure - either via explicitly presenting the symmetric monoidal structure, or using the parametrized Fourier transform formalism. We refrain from carrying out the hard work in this note.

2. Describe the essential image of the functor

$$\mathcal{F} \circ \iota : (\mathcal{S}\mathrm{hv}(V; \mathrm{Mod}_k), *) \longrightarrow (\mathcal{S}\mathrm{hv}(V^\vee; \mathcal{W}), \otimes).$$

Our goal is to explain that each of the questions has a good answer. In the previous section, we have essentially supplied all the ingredients for answering point 1 - so we start from here.

**Proposition 35.** Let X be a locally compact Hausdorff space and $\mathcal{S}\mathrm{hv}(X; \mathcal{W})$ be the category of wild Betti sheaves (with base field k) on X equipped with the pointwise tensor product. We have

$$\pi_0\mathrm{Pic}(\mathcal{S}\mathrm{hv}(X; \mathcal{W})) \cong \mathrm{Func}^{\mathrm{cts}}(X) \times \pi_0\mathrm{Pic}(\mathcal{S}\mathrm{hv}(X; \mathrm{Mod}_k))$$

where the category of ordinary sheaves on the right is also equipped with pointwise tensor product.

Let's treat the special case when X is a point first. We will explain how the proof for Proposition 20 works here up to small modification. To start with, here is a useful way to think about $\mathcal{W}$.

**Lemma 36.** *There are two idempotent algebras $\underline{k}_{\{0\}}$ (the skyscraper presheaf supported at 0) and $\underline{k}$ in $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}; \mathrm{Mod}_k)$ (the constant presheaf). Moreover:*

1. *The tensor product of $\underline{k}_{\{0\}}$ with $\underline{k}$ is 0. In particular, their union is the Cartesian product $A := \underline{k}_{\{0\}} \times \underline{k}$.*
2. *An object $\mathcal{F} \in \mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}; \mathrm{Mod}_k)$ lies in the full subcategory $\mathcal{W}$ if and only if $\underline{\mathrm{map}}(A, \mathcal{F}) = 0$.*
3. *The ideal $I_A := \mathrm{fib}(\mathbb{1} \to A)$ can be identified with the constant presheaf $\underline{k}_{[0,+\infty)}[-1]$ supported on $[0, +\infty)$.*

*Proof.* We leave the details to the reader: these are direct computations. □

It follows that one can embed the category $\mathcal{W}$ differently - as the full subcategory of $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Mod}_k)$ consisting of objects which are killed by $- \otimes A$. The tensor unit of $\mathcal{W}$ is identified with $I_A$ and the tensor product on $\mathcal{W}$ is identified with the tensor product on $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Mod}_k)$.[6]

**Claim 37.** Each of the statements from Lemma 23 to Proposition 30 remains true, if one replaces $\mathcal{S}\mathrm{hv}^+$ by $\mathcal{W}$ and I by $I_A$ there.

Some of the statements needs further clarification. For Corollary 27, the unit $I_A$ of $\mathcal{W}$ lives in $-1$ shift of the heart. Every subobject of $I_A$ is a constant presheaf with value $k[-1]$ on a half open interval $[0, r)$. For Lemma 29, the statement should be that if a convolution invertible object X lives in $-1$ shift of the heart, then so does its inverse. Moreover, tensoring with X takes heart to heart.

We will not repeat all the arguments, the point is that all these statements are about objects in $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Mod}_k)$ and their convolution products and the standard t-structure. The only point

[6] The reader is advised to compare this identification with what happens in the proof of Proposition 20: there we identified the category of $\mathcal{S}\mathrm{hv}^+$ with the subcategory of $\mathrm{Fun}(\mathbb{R}^{\mathrm{op}}_{\leqslant}, \mathrm{Mod}_k)$ killed by $- \otimes \underline{k}_0$. The upshot is that, under such presentation, the tensor product is underlying - there is no need for further completion.

where we have used something about $\mathcal{S}\mathrm{hv}^+$ is the classification of subobjects of the unit in the heart stated in Corollary 27 whose proof depends on the continuity of objects in $\mathcal{S}\mathrm{hv}^+$ (the characterization used in the proof of Lemma 23). Such property also holds for objects in $W$. The proof of Lemma 29 also needs to be modified. The strategy is to make use of the following claim: if $A$ and $B$ lives in the heart and also in $W$, then $\pi_1(A \otimes B)$ has to be nonzero.

It follows that we have the following calculation.

**Proposition 38.** The tensor invertible objects in $W$ are classified by

$$\pi_0 \mathrm{Pic}(W) \cong \mathbb{R} \times \pi_0 \mathrm{Pic}(\mathrm{Mod}_k).$$

We turn back to the general case:

*Proof of Proposition 35.* To treat the general case, we shift back to the perspective in [6] viewing $W$ as continuous complete $\mathbb{R}$-filtered $k$-modules. Recall that $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)$ can be identified with continuous $\mathbb{R}$-filtered $k$-modules, and the subcategory $W$ of continuous complete $\mathbb{R}$-filtered $k$-modules receives localization functor $L$ from $\mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k)$. We might thus embed $W$ into $\mathcal{S}\mathrm{hv}(\mathbb{R}^1; \mathrm{Mod}_k)$ via the right adjoint to the following localization functor:

$$\mathcal{S}\mathrm{hv}(\mathbb{R}^1; \mathrm{Mod}_k) \xrightarrow{*\theta_+} \mathcal{S}\mathrm{hv}^+(\mathbb{R}^1; \mathrm{Mod}_k) \xrightarrow{L} W.$$

It follows that we can think of $\mathcal{S}\mathrm{hv}(X; W)$ as a full subcategory of $\mathcal{S}\mathrm{hv}(X \times \mathbb{R}^1; \mathrm{Mod}_k)$. At this point, we can run the argument from the very end of Section 3 and win. $\square$

We have fulfilled our promise about the first question. Now we move on to the second one. To do so, we need to characterize the image of embedding of ordinary sheaves into wild Betti sheaves. The crucial observation is the following.

**Observation 39.** *The embedding of* $\mathrm{Mod}_k$ *into* $W$ *sends every object* $M \in \mathrm{Mod}_k$ *to* $M \otimes \mathbb{1}$. *Every object of such form is invariant under the* $\mathbb{R}_{>0}$ *scaling action on* $W$ *defined in [6, Section 6]. By 'invariant' we mean that for each* $a \in \mathbb{R}_{>0}$ *we have the action map*

$$a \cdot : W \longrightarrow W$$

*and there always exists an isomorphism*

$$a \cdot (M \otimes \mathbb{1}) \cong M \otimes \mathbb{1}.$$

To make this idea precise, we take up the stacky approach.

**Lemma 40.** *Let* $\mathbb{R}$ *act additively on a locally compact Hausdorff space* $X$ *and we form the quotient stack* $X/\mathbb{R}$ *(in your favorite topology, e.g. open covering). Then the pullback functor*

$$\pi^* : \mathcal{S}\mathrm{hv}(X/\mathbb{R}; \mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}(X; \mathrm{Mod}_k)$$

*is fully faithful. Its image is precisely those sheaves on* $X$ *which are locally constant on each orbit of* $\mathbb{R}$ *action.*

**Remark 41.** As a side remark, such fact could be mind-twisting for people who think of objects living on the quotient stack as objects on the total space plus equivariant structures - the lemma says equivariance for $\mathbb{R}$ is merely a property (instead of a structure)! This fact is somehow well-known in the literature for geometric representation theory (in the form for perverse sheaves or $\mathcal{D}$-modules), and we take the opportunity to record the Betti form. The statement for other contractible group should also be true.

**Proposition 42.** Under the embedding (the right adjoint functor to the localization functor mentioned in the Proof of Proposition 35)

$$i : \mathcal{S}\mathrm{hv}(X; \mathcal{W}) \longrightarrow \mathcal{S}\mathrm{hv}(X \times \mathbb{R}^1; \mathrm{Mod}_k),$$

the image of the inclusion of ordinary sheaves

$$\mathcal{S}\mathrm{hv}(X; \mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}(X; \mathcal{W}) \xrightarrow{i} \mathcal{S}\mathrm{hv}(X \times \mathbb{R}^1; \mathrm{Mod}_k)$$

is precisely the collection of objects in $\mathcal{S}\mathrm{hv}(X \times \mathbb{R}^1; \mathrm{Mod}_k)$ which are

- in the image of $i$ and,
- equivariant under the $\mathbb{R}_{>0}$ action (by scaling) in the $\mathbb{R}^1$ direction.

It remains to study how the $\mathbb{R}_{>0}$ action interacts with Fourier transform.

**Proposition 43.** The Fourier transform intertwines the $\mathbb{R}_{>0}$ action through the coefficient on $\mathcal{S}\mathrm{hv}(V; \mathcal{W})$, and the $\mathbb{R}_{>0}$ action through diagonal on $\mathcal{S}\mathrm{hv}(V^\vee; \mathcal{W})$.

*Proof.* The span defining Fourier transform for the wild Betti sheaves can be presented as

$$V \times \mathbb{R}^1 \longleftarrow V \times V^\vee \times \mathbb{R}^1 \longrightarrow V^\vee \times \mathbb{R}^1.$$

We equip left hand side with the $\mathbb{R}_{>0}$ action in $\mathbb{R}^1$ direction and equip right hand side with the diagonal $\mathbb{R}_{>0}$ action. It suffices to observe that

- one can put an $\mathbb{R}_{>0}$ action on $V \times V^\vee \times \mathbb{R}^1$ making both of the maps $\mathbb{R}_{>0}$-equivariant and
- the exponential sheaf exp thought of as a sheaf on $V \times V^\vee \times \mathbb{R}^1$ is $\mathbb{R}_{>0}$-equivariant.

It follows that one can now produce a span of quotient stacks (by the $\mathbb{R}_{>0}$ action) refining the above span

$$V \times \mathbb{R}^1/\mathbb{R}_{>0} \longleftarrow V \times V^\vee \times \mathbb{R}^1/\mathbb{R}_{>0} \longrightarrow V^\vee \times \mathbb{R}^1/\mathbb{R}_{>0}.$$

And one can chase the diagram to show that the Fourier transform preserve $\mathbb{R}_{>0}$-equivariant objects. □

We can now answer the second question.

**Corollary 44.** Under Fourier transform, the image of the inclusion of ordinary sheaves

$$\mathcal{S}\mathrm{hv}(V; \mathrm{Mod}_k) \longrightarrow \mathcal{S}\mathrm{hv}(V; \mathcal{W}) \xrightarrow{\mathcal{F}} \mathcal{S}\mathrm{hv}(V^\vee; \mathcal{W}) \xrightarrow{i_{V^\vee}} \mathcal{S}\mathrm{hv}(V^\vee \times \mathbb{R}^1; \mathrm{Mod}_k)$$

is precisely those objects in $\mathcal{S}\mathrm{hv}(V^\vee \times \mathbb{R}^1; \mathrm{Mod}_k)$ which are

- in the image of $i_{V^\vee}$ and,
- equivariant under the $\mathbb{R}_{>0}$ action which acts diagonally in both $\mathbb{R}^1$ direction and $V^\vee$ direction.

From this, we learn that a convolution invertible ordinary sheaf on $V$ is taken under the Fourier transform to a tensor invertible wild Betti sheaf on $V^\vee$, which is $\mathbb{R}_{>0}$-equivariant through the diagonal action. This means that the corresponding function $f \in \mathrm{Func}^{\mathrm{cont}}(V^\vee, \mathbb{R})$ is homogeneous. Such functions are in bijection with continuous functions on $S^{n-1}$! The concludes our computation of the Picard group.

**Remark 45.** We have the following funny consequence: the Fourier-Sato transform for conic ordinary sheaves is now a direct corollary from the Fourier transform for wild Betti sheaves. To see this, it suffices to note that conic sheaves on $V$ are precisely those sheaves on $V$ equivariant for the scaling action of $\mathbb{R}_{>0}$ on $V$. Under Fourier transform for wild Betti sheaves, two of the $\mathbb{R}_{>0}$ equivariant structures cutting out conic sheaves on $V$ are exchanged with those for conic sheaves on $V^\vee$. We thus learn that the Fourier transform restricts to an equivalence between conic ordinary sheaves.

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