Kinematics and Forces in a Reciprocating Engine

Introduction

In a reciprocating engine consisting of a piston, connecting rod and crankshaft, forces induced by pressure drive the linear motion of a piston and then transform into rotatary motion of a crankshaft. It is a typically process of converting chemical energy into mechanical energy, then finally drives a vehicle to move.

The continual motion constantly subjects stresses on the connecting rod which is caused by the force that becomes larger with increasing engine speed. The impact is crucial and harmful to the entire system that might give rise to undesirable vibration and fatigue failure of mechanism. Kojima (n.d.) proposed in his project on MotoIQ website, that a longer rod slows piston acceleration in the region of top dead center but has a slightly greater propensity to detonate.

The most common of reciprocating engine is a four-stroke cycle engine that uses four distinct piston strokes to complete one working cycle which includes intake, compression, power and exhaust stokes. The piston undergoes up and down movement inside the cylinder for twice in one cycle which requires two revolutions of crankshaft.

Due to an expansion of air-fuel mixture at a relative high pressure, power is generated at the start of the second revolution of the four stoke cycle. The piston force and subsequent motion drive the piston from TDC to BDC which is transmitted through connecting rod to apply torque that available on the crankshaft. The flywheel connects with at the end of crankshaft which has a crucial role of storing energy from periodic excitation so that allows the engine to run more smoothly.

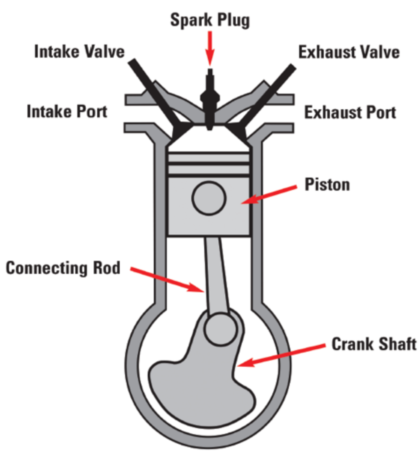


Figure 1. Reciprocating Engine Structure

The dynamic movement of a reciprocating engine could be analyzed by modelling as a system of rigid bodies. The flywheel and the crankshaft undergo rotation-only motion about a fixed axis. The piston moves back-and-forward inside the cylinder which indicates it undergoes rectilinear translation-only motion. However, the connecting rod is joining the cylinder to the crankshaft and undergoing general planar motion that both translation and rotation motion are applied at the same time.

In the next few pages, the angular and linear accelerations and velocities of crank mechanism are calculated by utilizing the principle of relative motion. However, Ranjbarkohan and Rasekh (2010) proposed in their latest research on Nissan Z24 four-cylinder engine, that all accelerations and velocities increased with increasing of engine velocity.

System Example

The system model is based on Subara EJ205 which is a horizontally opposed four-cylinder engine.
Crankshaft stroke: 87mm ($r = 0.0435m$)
Connecting rod: 100.43mm ($L = 0.1m$)
Crankshaft rotatory motion: 8000RPM ($\omega_{OB} = 8000 \times \frac{2\pi}{60}$)

The velocity of the crank pin B as a point on connecting rod AB, so that B will be used as the reference point for determining the velocity of piston A. By applying the principle of relative motion, the velocity equation may now be presented as $\boldsymbol{V_A} = \boldsymbol{V_B} + \boldsymbol{V_{AB}}$.

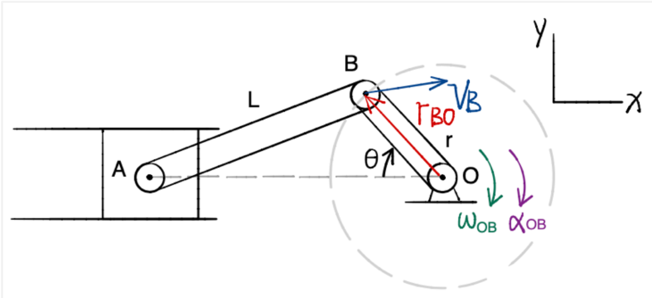


Figure 2. FBD1

Since the rotational speed of crank OB is in the clockwise direction. Hence, $\boldsymbol{\omega_{OB}} = -\omega_{OB}\boldsymbol{k}$.

$V_B = V_O + V_{BO} = 0 + \boldsymbol{\omega_{OB}} \times \boldsymbol{r_{BO}} = -\omega_{OB}\boldsymbol{k} \times (-rcos\theta\boldsymbol{i} + rsin\theta\boldsymbol{j}) = \omega_{OB}rsin\theta\boldsymbol{i} + \omega_{OB}rcos\theta\boldsymbol{j}$

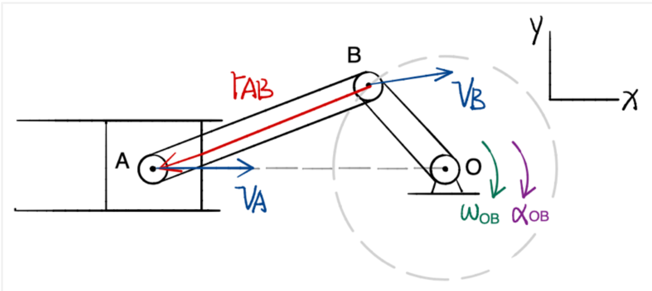


Figure 3. FBD2

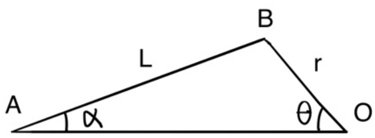
Assume the rotational speed of crank AB is in the counterclockwise direction. $\boldsymbol{\omega_{AB}} = \omega_{AB}k$.

$V_A = V_B + V_{AB}$

$\quad = \omega_{OB}rsin\theta\boldsymbol{i} + \omega_{OB}rcos\theta\boldsymbol{j} + \boldsymbol{\omega_{AB}} \times \boldsymbol{r_{AB}}$

$\quad = \omega_{OB}rsin\theta\boldsymbol{i} + \omega_{OB}rcos\theta\boldsymbol{j} + \omega_{AB}k \times (Lcos\alpha\boldsymbol{i} + Lsin\alpha\boldsymbol{j})$

$V_A i = \omega_{OB}rsin\theta\boldsymbol{i} + \omega_{OB}rcos\theta\boldsymbol{j} - \omega_{AB}Lsin\alpha\boldsymbol{i} + \omega_{AB}Lcos\alpha\boldsymbol{j}$



$$\frac{sin\theta}{L} = \frac{sin\alpha}{r}$$

$$\alpha = \sin^{-1}(\frac{sin\theta}{L}r)$$

Solve for $\boldsymbol{j}$-component:

$\omega_{OB}rcos\theta + \omega_{AB}Lcos\alpha = 0$ …… *(1)* $\quad \omega_{AB} = \frac{-\omega_{OB}rcos\theta}{Lcos\alpha}$ …… *(2)*

Solve for $\boldsymbol{i}$-component: $V_A = \omega_{OB}rsin\theta - \omega_{AB}Lsin\alpha$ …… *(3)*

Substitute *(2)* into *(3)*,

$V_A = \omega_{OB}rsin\theta + \frac{\omega_{OB}rcos\theta}{cos\alpha}sin\alpha = \omega_{OB}r(sin\theta + cos\theta tan\alpha)$

The acceleration of piston A may be expressed in terms of the acceleration of the crank pin B. Thus, $\boldsymbol{a_A} = \boldsymbol{a_B} + (\boldsymbol{a_{AB}})_{\boldsymbol{n}} + (\boldsymbol{a_{AB}})_{\boldsymbol{t}}$.

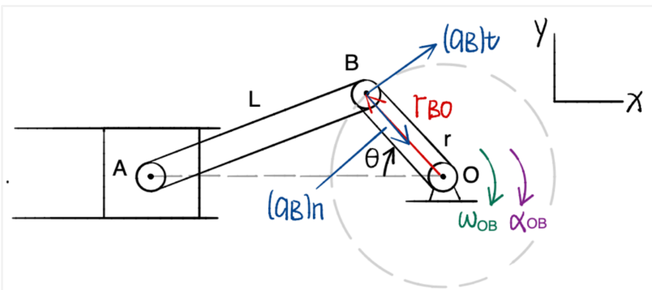


Figure 4. FBD3

Pin B moves in a circle with a constant speed so that it has only a normal component of acceleration.
Hence, $a_B = (a_B)_n = \omega_{OB} \times (\omega_{OB} \times r_{BO})$

$$= -\omega_{OB}\boldsymbol{k} \times (\omega_{OB} r sin\theta \boldsymbol{i} + \omega_{OB} r cos\theta \boldsymbol{j})$$
$$= (\omega_{OB})^2 r cos\theta \boldsymbol{i} - (\omega_{OB})^2 r sin\theta \boldsymbol{j}$$

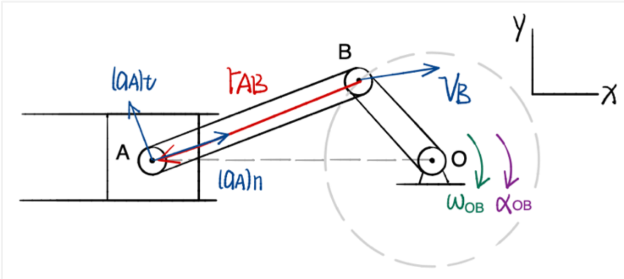


Figure 5. FBD4

$$a_A \boldsymbol{i} = a_B + (a_{AB})_n + (a_{AB})_t$$
$$= a_B + \alpha_{AB} \times r_{AB} + (\omega_{AB})^2 r_{AB}$$
$$= (\omega_{OB})^2 r cos\theta \boldsymbol{i} - (\omega_{OB})^2 r sin\theta \boldsymbol{j} + \alpha_{AB}\boldsymbol{k} \times (Lcos\alpha \boldsymbol{i} + Lsin\alpha \boldsymbol{j}) + (\omega_{AB})^2 (Lcos\alpha \boldsymbol{i} + Lsin\alpha \boldsymbol{j})$$
$$= (\omega_{OB})^2 r cos\theta \boldsymbol{i} - (\omega_{OB})^2 r sin\theta \boldsymbol{j} - \alpha_{AB} Lsin\alpha \boldsymbol{i} + \alpha_{AB} Lcos\alpha \boldsymbol{j} + (\omega_{AB})^2 Lcos\alpha \boldsymbol{i} + (\omega_{AB})^2 Lsin\alpha \boldsymbol{j}$$

Solve for $\boldsymbol{j}$-component:

$-(\omega_{OB})^2 r sin\theta + \alpha_{AB} Lcos\alpha + (\omega_{AB})^2 Lsin\alpha = 0$ …… *(4)* $\alpha_{AB} = \frac{(\omega_{OB})^2 r sin\theta - (\omega_{AB})^2 Lsin\alpha}{Lcos\alpha}$…… *(5)*

Solve for $\boldsymbol{i}$-component: $a_A = (\omega_{OB})^2 r cos\theta - \alpha_{AB} Lsin\alpha + (\omega_{AB})^2 Lcos\alpha$ …… *(6)*

Substitute *(2)* into *(3)*,

$a_A = (\omega_{OB})^2 r cos\theta + (\omega_{AB})^2 Lcos\alpha - tan\alpha[(\omega_{OB})^2 r sin\theta - (\omega_{AB})^2 Lsin\alpha]$

Results

In order to make the results more reliable, evaluate specific motion at 40º crank angle.

$\vec{\omega_{OB}} = -\omega_{OB} k = -\frac{800}{3}\pi k$

$v_B = \omega_{OB} \times r_{BO}$

$= -\frac{800}{3}\pi k \times (-0.0435\cos40° i + 0.0435 \sin 40° j)$

$= \frac{800}{3}\pi 0.0435 \sin 40° i + \frac{800}{3}\pi 0.0435 \cos 40° j$

$v_A i = v_B + \omega_{AB} \times r_{AB}$

$= v_B + \omega_{AB} k \times (0.1\cos 16.237° i + 0.1 \sin 16.237° j)$ $\alpha = \sin^{-1}(\frac{\sin 40° \cdot 0.0435}{0.1}) = 16.237°$

$= v_B - 0.1 \sin 16.237° \omega_{AB} i + 0.1\cos 16.237° \omega_{AB} j$

$j \rightarrow$: $0 = \frac{800}{3}\pi 0.0435 \cos 40° + 0.1 \cos 16.237° \omega_{AB}$

$i \rightarrow$: $v_A = \frac{800}{3}\pi 0.0435 \sin 40° - 0.1 \sin 16.237° \omega_{AB} = 31.55 m/s$

$\omega_{AB} = \frac{-\frac{800}{3}\pi 0.0435 \cos 40°}{0.1\cos 16.237°} = -290.76 rad/s$

$\boxed{\text{At } \theta = 40°;\ \omega_{AB} = -290.76 rad/s\ \ v_A = 31.55 m/s\ \ \alpha_{AB} = 179774 rad/s^2\ \ a_A = 26478 m/s^2}$

$a_B = \omega_{OB}^2 r_{BO} = (-\frac{800}{3}\pi)^2(-0.0435\cos40° i + 0.0435 \sin 40° j)$

$a_A = a_B + \alpha_{AB} \times r_{AB} + (\omega_{AB})^2 r_{AB}$

$= a_B - 0.1 \sin 16.237° \alpha_{AB} i + 0.1 \cos 16.237° \alpha_{AB} j + (-290.76)^2 (0.1\cos 16.237° i + 0.1 \sin 16.237° j)$

$j \rightarrow$: $(-\frac{800}{3}\pi)^2 0.0435 \sin 40° + 0.1 \cos 16.237° \omega_{AB} + (-290.76)^2 0.1 \sin 16.237° = 0$ $\alpha_{AB} = 179774 rad/s^2$

$i \rightarrow$: $a_A = (-\frac{800}{3}\pi)^2(-0.0435\cos 40°) - 0.1 \sin 16.237° \alpha_{AB} + (-290.76)^2(0.1 \cos 16.237°) = 26478 m/s^2$

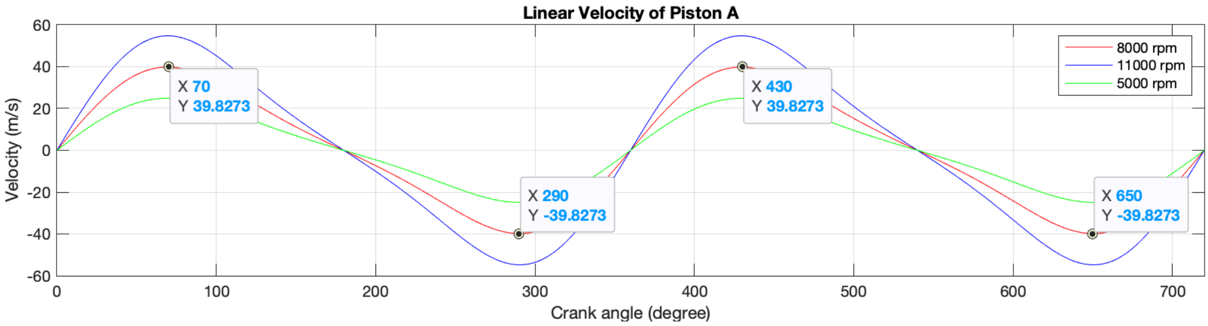


Figure 6. Linear acceleration of piston in different rpms (a_A)

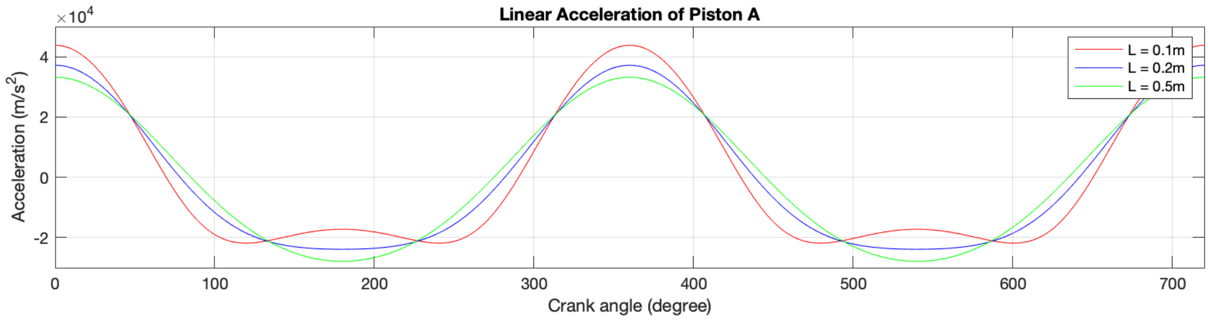


Figure 7. Linear acceleration of piston in different rod lengths (a_A)

Discussion

Based on the calculation and graphical analysation above, the results matched the research consistently and the motion of the piston resembles a simple harmonic motion. The velocity of piston is shown in the Figure 6. With increasing rotational speed of the crank OB, the peak amplitude gets larger and the piston moves in a larger linear velocity.

Also shown in Figure 6, consider under the 8000 RPM angular velocity. The velocity of the piston is equal to zero when the piston is located at either TDC (Top Dead Center) or BDC (Bottom Dead Center) inside cylinder. At that instant, the piston changes its movement direction. When the piston is travelling from TDC to BDC (0º~180º), it reaches its maximum velocity (39.8273m/s) in positive direction before the middle position of the stroke where the crank angle is 70º. Inversely (180º~360º), it reaches its maximum velocity (−39.8273m/s) in negative position after the middle position of stroke where the crank angle is 290º.

To access whether the rod length has a potential affection on the piston acceleration, hold other variables unchanged and the results have depicted through three different rod lengths in Figure 7. The graph is exactly the same as Kojima (n.d.) depicted in earlier article and shows that a longer rod could slow down the acceleration of piston at TDC. Piston acceleration is a major source of vibration in a reciprocating engine, so the longer rod can reduce undesirable stress on the engine reciprocating parts.

Nevertheless, there is an observation shown in Firgure7 that the acceleration and deceleration of the piston are greater in the upper half of the crankshaft rotation than in the lower half. The shorter connecting rod moves significantly faster and has unstable amplitude of vibration when undergoing the lower half of the crankshaft rotation.

Reference

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Appendix: MATLAB code

For Figure 6:

```
L1 = 0.1; % connecting rod
r1 = 0.0435; % crankshaft diameter
omega_OB = 8000*(2*pi)/60; % rotational speed of crank OB
omega_OB2 = 11000*(2*pi)/60;
omega_OB3 = 5000*(2*pi)/60;
theta = 0:1.0:720; % angle vector

for i = 1:length(theta)

    A = [0 0 -omega_OB];
    A2 = [0 0 -omega_OB2];
    A3 = [0 0 -omega_OB3];

    B = [-r1*cosd(theta(i)) r1*sind(theta(i)) 0];

    Velocity_B = cross(A,B); % velocity B in vector form
    Velocity_B2 = cross(A2,B);
    Velocity_B3 = cross(A3,B);

    % linear velocity at Pin B
    vB = [(Velocity_B(1))^(2)+(Velocity_B(2))^(2)]^(1/2);
    vB2 = [(Velocity_B2(1))^(2)+(Velocity_B2(2))^(2)]^(1/2);
    vB3 = [(Velocity_B3(1))^(2)+(Velocity_B3(2))^(2)]^(1/2);

    Vector_vB(1,i) = vB;
    Vector_vB2(1,i) = vB2;
    Vector_vB3(1,i) = vB3;

    alpha = asind(sind(theta(i))*r1/L1); % aplpha in degree

    % rotational speed of connecting rod AB
    omega_AB = (-omega_OB *r1*cosd(theta(i)))/(L1*cosd(alpha));
    omega_AB2 = (-omega_OB2 *r1*cosd(theta(i)))/(L1*cosd(alpha));
    omega_AB3 = (-omega_OB3 *r1*cosd(theta(i)))/(L1*cosd(alpha));

    Vector_omega_AB(1,i) = omega_AB;
    Vector_omega_AB2(1,i) = omega_AB2;
    Vector_omega_AB3(1,i) = omega_AB3;

    % linear velocity of piston A
    vA = omega_OB*r1*(sind(theta(i))+cosd(theta(i))*tand(alpha));
    vA2 = omega_OB2*r1*(sind(theta(i))+cosd(theta(i))*tand(alpha));
    vA3 = omega_OB3*r1*(sind(theta(i))+cosd(theta(i))*tand(alpha));

    Vector_vA(1,i) = vA;
    Vector_vA2(1,i) = vA2;
    Vector_vA3(1,i) = vA3;

    % linear acceleration at Pin B
    Acceleration_B = cross(A,Velocity_B);
    Acceleration_B2 = cross(A2,Velocity_B);
    Acceleration_B3 = cross(A3,Velocity_B);

    Vector_aB_i(1,i) = Acceleration_B(1);
    Vector_aB_j(1,i) = Acceleration_B(2);

    Vector_aB_i2(1,i) = Acceleration_B2(1);
    Vector_aB_j2(1,i) = Acceleration_B2(2);

    Vector_aB_i3(1,i) = Acceleration_B3(1);
    Vector_aB_j3(1,i) = Acceleration_B3(2);

    % angular acceleration of connecting rod AB
    alpha_AB = ((omega_OB^2*r1*sind(theta(i)))-(omega_AB^2*L1*sind(alpha)))/(L1*cosd(alpha));
    alpha_AB2 = ((omega_OB2^2*r1*sind(theta(i)))-(omega_AB2^2*L1*sind(alpha)))/(L1*cosd(alpha));
    alpha_AB3 = ((omega_OB3^2*r1*sind(theta(i)))-(omega_AB3^2*L1*sind(alpha)))/(L1*cosd(alpha));

    Vector_alpha_AB(1,i) = round(alpha_AB);
    Vector_alpha_AB2(1,i) = round(alpha_AB2);
    Vector_alpha_AB3(1,i) = round(alpha_AB3);
```

```
    % linear acceleration of piston A
    aA = ((omega_OB)^(2)*r1*cosd(theta(i)))-(alpha_AB*L1*sind(alpha))+((omega_AB)^(2)*L1*cosd(alpha));
    aA2 = ((omega_OB2)^(2)*r1*cosd(theta(i)))-(alpha_AB2*L1*sind(alpha))+((omega_AB2)^(2)*L1*cosd(alpha));
    aA3 = ((omega_OB3)^(2)*r1*cosd(theta(i)))-(alpha_AB3*L1*sind(alpha))+((omega_AB3)^(2)*L1*cosd(alpha));

    Vector_aA(1,i) = aA;
    Vector_aA2(1,i) = aA2;
    Vector_aA3(1,i) = aA3;

end

% plotting
figure

% plot1: linear velocity of piston A
subplot(3,1,1);
TF = islocalmin(Vector_vA);
TA = islocalmax(Vector_vA);
plot(theta,Vector_vA,'r',theta,Vector_vA2,'b',theta,Vector_vA3,'g',theta(TF),Vector_vA(TF),'ko',theta(TA),Vector_vA(TA),'ko');
axis([0 720 -60 60]);
title('Linear Velocity of Piston A');
xlabel('Crank angle (degree)');
ylabel('Velocity (m/s)');
legend('8000 rpm','11000 rpm','5000 rpm');
grid on;

% plot2: linear acceleration of piston A
subplot(3,1,2);
plot(theta,Vector_aA,'r');
axis([0 720 -30000 50000]);
title('Linear Acceleration of Piston A');
xlabel('Crank angle (degree)');
ylabel('Acceleration (m/s^2)');
grid on;
```


```
% plot2: linear acceleration of piston A
subplot(3,1,2);
plot(theta,Vector_aA,'r');
axis([0 720 -30000 50000]);
title('Linear Acceleration of Piston A');
xlabel('Crank angle (degree)');
ylabel('Acceleration (m/s^2)');
grid on;
```


For Figure 7:

```
L1 = 0.1; % connecting rod
L2 = 0.2;
L3 = 0.5;
r1 = 0.0435; % crankshaft diameter
omega_OB = 8000*(2*pi)/60; % rotational speed of crank OB
theta = 0:1.0:720; % angle vector

for i = 1:length(theta)

    A = [0 0 -omega_OB];
    B = [-r1*cosd(theta(i)) r1*sind(theta(i)) 0];
    Velocity_B = cross(A,B); % velocity B in vector form

    % linear velocity at Pin B
    vB = [(Velocity_B(1))^(2)+(Velocity_B(2))^(2)]^(1/2);
    Vector_vB(1,i) = vB;

    alpha = asind(sind(theta(i))*r1/L1); % aplpha in degree
    alpha2 = asind(sind(theta(i))*r1/L2);
    alpha3 = asind(sind(theta(i))*r1/L3);

    % rotational speed of connecting rod AB
    omega_AB = (-omega_OB *r1*cosd(theta(i)))/(L1*cosd(alpha));
    omega_AB2 = (-omega_OB *r1*cosd(theta(i)))/(L2*cosd(alpha2));
    omega_AB3 = (-omega_OB *r1*cosd(theta(i)))/(L3*cosd(alpha3));

    Vector_omega_AB(1,i) = omega_AB;
    Vector_omega_AB2(1,i) = omega_AB2;
    Vector_omega_AB3(1,i) = omega_AB3;

    % linear velocity of piston A
    vA = omega_OB*r1*(sind(theta(i))+cosd(theta(i))*tand(alpha));
    vA2 = omega_OB*r1*(sind(theta(i))+cosd(theta(i))*tand(alpha2));
    vA3 = omega_OB*r1*(sind(theta(i))+cosd(theta(i))*tand(alpha3));
```

```
    Vector_vA(1,i) = vA;
    Vector_vA2(1,i) = vA2;
    Vector_vA3(1,i) = vA3;

    % linear acceleration at Pin B
    Acceleration_B = cross(A,Velocity_B);
    Vector_aB_i(1,i) = Acceleration_B(1);
    Vector_aB_j(1,i) = Acceleration_B(2);

    % angular acceleration of connecting rod AB
    alpha_AB = ((omega_OB^2*r1*sind(theta(i)))-(omega_AB^2*L1*sind(alpha)))/(L1*cosd(alpha));
    alpha_AB2 = ((omega_OB^2*r1*sind(theta(i)))-(omega_AB2^2*L2*sind(alpha2)))/(L2*cosd(alpha2));
    alpha_AB3 = ((omega_OB^2*r1*sind(theta(i)))-(omega_AB3^2*L3*sind(alpha3)))/(L3*cosd(alpha3));

    Vector_alpha_AB(1,i) = round(alpha_AB);
    Vector_alpha_AB2(1,i) = round(alpha_AB2);
    Vector_alpha_AB3(1,i) = round(alpha_AB3);

    % linear acceleration of piston A
    aA = ((omega_OB)^(2)*r1*cosd(theta(i)))-(alpha_AB*L1*sind(alpha))+((omega_AB)^(2)*L1*cosd(alpha));
    aA2 = ((omega_OB)^(2)*r1*cosd(theta(i)))-(alpha_AB2*L2*sind(alpha2))+((omega_AB2)^(2)*L2*cosd(alpha2));
    aA3 = ((omega_OB)^(2)*r1*cosd(theta(i)))-(alpha_AB3*L3*sind(alpha3))+((omega_AB3)^(2)*L3*cosd(alpha3));

    Vector_aA(1,i) = aA;
    Vector_aA2(1,i) = aA2;
    Vector_aA3(1,i) = aA3;

end

% plotting
figure
```

```
% plot1: linear velocity of piston A
subplot(3,1,1);
plot(theta,Vector_vA);
axis([0 720 -50 50]);
title('Linear Velocity of Piston A');
xlabel('Crank angle (degree)');
ylabel('Velocity (m/s)');
grid on;

% plot2: linear acceleration of piston A
subplot(3,1,2);
plot(theta,Vector_aA,'r',theta,Vector_aA2,'b',theta,Vector_aA3,'g');
axis([0 720 -30000 50000]);
title('Linear Acceleration of Piston A');
xlabel('Crank angle (degree)');
ylabel('Acceleration (m/s^2)');
legend('L = 0.1m','L = 0.2m','L = 0.5m');
grid on;
```