An Error Estimate for Euler's Method

An approximate solution to the initial value problem

$$\begin{aligned}\frac{dy}{dt} &= f(t, y)\\ y(t_0) &= y_0\end{aligned}$$

on $t \in [t_0, t_N]$ is given by Euler's method

$$y_{n+1} = y_n + \Delta t f(t_n, y_n),$$

in which $t_n = t_0 + n\Delta t$. We would like to bound the error, the absolute difference between the approximate solution values and the true solution values, $E_n = |y_n - y(t_n)|$.

We begin by forming a Taylor series approximation in the variable Δt for $y(t_{n+1})$:

$$\begin{aligned}y(t_{n+1}) &= y(t_n + \Delta t)\\ &= y(t_n) + \Delta t y'(t_n) + \frac{(\Delta t)^2}{2} y''(\xi_n)\\ &= y(t_n) + \Delta t f(t_n, y(t_n)) + \frac{(\Delta t)^2}{2} y''(\xi_n)\end{aligned}$$

for some $\xi_n \in [t_n, t_{n+1}]$. We have used the Lagrange form of the error from Taylor's theorem, which means y needs to be twice differentiable. Since $y''(t) = \partial_t f(t, y(t)) + \partial_y f(t, y(t)) f(t, y(t))$, we assume f, $\partial_t f$, and $\partial_y f$ are continuous. This assumption on f also allows us to bound f, $\partial_t f$, and $\partial_y f$ on $[t_0, t_N]$: there exists an $M > 0$ so that $|f| \leq M$, $|\partial_t f| \leq M$, and $|\partial_y f| \leq M$.

The error changes from step n to step $n+1$ as

$$\begin{aligned}&E_{n+1}\\ &= |y_{n+1} - y(t_{n+1})|, \quad \text{definition}\\ &= |y_n + \Delta t f(t_n, y_n) - y(t_{n+1})|, \quad \text{Euler's method}\\ &= \left| y_n + \Delta t f(t_n, y_n) - y(t_n) - f(t_n, y(t_n))\Delta t - \frac{(\Delta t)^2}{2} y''(\xi_n) \right|, \quad \text{Taylor series}\\ &\leq |y_n - y(t_n)| + \Delta t |f(t_n, y_n) - f(t_n, y(t_n))| + \frac{(\Delta t)^2}{2} |y''(\xi_n)|, \quad \text{triangle inequality}\\ &\leq E_n + \Delta t M E_n + \frac{(\Delta t)^2}{2}(M + M^2), \quad \text{mean value theorem, bounds}\\ &= (1 + \Delta t M) E_n + \frac{(\Delta t)^2}{2}(M + M^2).\end{aligned}$$

Note that we have used the mean value theorem in the y variable as

$$|f(t_n, y_n) - f(t_n, y(t_n))| = |\partial_y f(t_n, c)||y_n - y(t_n)| \leq M |y_n - y(t_n)|,$$

for c between y_n and $y(t_n)$. Also,

$$|y''(\xi_n)| = |\partial_t f(\xi_n, y(\xi_n) + \partial_y f(\xi_n, y(\xi_n)) f(\xi_n, y(\xi_n))| \leq M + M^2.$$

We now use the recursive formula for E_n beginning from $E_0 = 0$:

$$\begin{aligned}
E_1 &= \frac{M+M^2}{2}(\Delta t)^2 \\
E_2 &= (1+(1+M\Delta t))\,\frac{M+M^2}{2}(\Delta t)^2 \\
E_3 &= \left(1+(1+M\Delta t)+(1+M\Delta t)^2\right)\frac{M+M^2}{2}(\Delta t)^2 \\
E_n &= \left(1+(1+M\Delta t)+(1+M\Delta t)^2+\ldots+(1+M\Delta t)^{n-1}\right)\frac{M+M^2}{2}(\Delta t)^2 \\
&\le \frac{(1+M)\Delta t}{2}\left(e^{M\Delta t n}-1\right).
\end{aligned}$$

The finite geometric sum is bounded as

$$\sum_{i=0}^{n-1}(1+M\Delta t)^i = \frac{(1+M\Delta t)^n-1}{M\Delta t} \le \frac{1}{M\Delta t}\left(e^{M\Delta t n}-1\right),$$

using $1+M\Delta t \le e^{M\Delta t}$ since the exponential function is convex (a tangent line is entirely below the curve).

It has been shown that $E_n \sim \frac{1+M}{2}\Delta t(M\Delta t n)$ as $\Delta t \to 0$. Thus, for small n, the error decreases like $(\Delta t)^2$ as $\Delta t \to 0$. For large n around $1/\Delta t$, near the end of the interval, E_n decreases like $(\Delta t)^1$ as $\Delta t \to 0$. Euler's method is first order.