Graded Homework 4

Solutions (not just answers) to the following problems are due on Wednesday, March 6, by the end of the class period.

1. Consider the differential equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F_0 \sin{(at)}$$

   (a) Find the general solution of the corresponding homogeneous equation.

   (b) Show the particular solution of the differential equation is

$$x_p(t) = \frac{F_0(\omega^2 - a^2)}{(\omega^2 - a^2)^2 + 4\lambda^2 a^2}\sin{(at)} + \frac{F_0(-2\lambda a)}{(\omega^2 - a^2)^2 + 4\lambda^2 a^2}\cos{(at)}$$

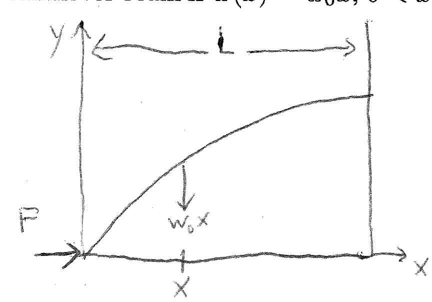
   (c) Inspection shows that the homogeneous solution is transient (it goes to zero as $t \to \infty$), and so for large values of t, the solution is approximated by $x_p(t)$. Using the fact mentioned on the in-class springs handout, that the amplitude A of $c_1 \cos{(\alpha t)} + c_2 \sin{(\alpha t)}$ is $A = \sqrt{c_1^2 + c_2^2}$, show the maximum amplitude of the system will occur at $a = \sqrt{\omega^2 - 2\lambda^2}$, and find the value of the maximum amplitude. The number $\sqrt{\omega^2 - 2\lambda^2}/(2\pi)$ is called the **resonance frequency** of the system.

   (d) Given $F_0 = 2$, $m = 1$, and $k = 4$, consider the amplitude of x_p as a function of a. On the same set of axes, graph the amplitude for $2\lambda = 2$, $2\lambda = 1$, $2\lambda = 1/2$, and $2\lambda = 1/4$. This family of graphs is called the resonance curve or frequency response curve of the system.

2. A cantilever beam of length L is embedded at its right end, and a horizontal compressive force of F newtons is applied to its free left end. When the origin is taken at its free end, as in the picture, the deflection $y(x)$ of the beam can be shown to satisfy the differential equation

$$EIy'' = -Fy - w(x)\frac{x}{2}$$

Find the deflection of the cantilever beam if $w(x) = w_0 x$, $0 < x < L$, and $y(0) = 0$, $y'(L) = 0$.



3. The temperature $u(r)$ in the circular ring shown below is modeled by the boundary-value problem

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0, \quad u(a) = u_0, \quad u(b) = u_1$$

where u_0 and u_1 are constants. Show that

$$u(r) = \frac{u_0 \ln{(r/b)} - u_1 \ln{(r/a)}}{\ln{(a/b)}}$$

(over for picture)

Picture for problem 3: