STAT3035/8035
Tutorial 6

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Outline

1. Review
2. Questions

Reinsurance

- Excess-of-Loss
  - Retention level, M
  - $Y = \begin{cases} X & \text{if } X \leq M \\ M & \text{if } X > M \end{cases} = XI_{(X \leq M)} + MI_{(X > M)}$
  - $Z = \begin{cases} 0 & \text{if } X \leq M \\ X - M & \text{if } X > M \end{cases} = (X - M)I_{(X > M)}$
- Expected Claim Size
  - If $X \sim f_X(x;\theta)$:

$$
\begin{aligned}
E_\theta(Y) &= E_\theta\left\{XI_{(X \leq M)}\right\} + E_\theta\left\{MI_{(X > M)}\right\} \\
&= \int_0^\infty xI_{(x \leq M)} f_X(x;\theta)dx + M \Pr_\theta(X > M) \\
&= \int_0^M x f_X(x;\theta)dx + M \int_M^\infty f_X(x;\theta)dx \\
&= \int_0^\infty x f_X(x;\theta)dx - \int_M^\infty x f_X(x;\theta)dx + M \int_M^\infty f_X(x;\theta)dx \\
&= E_\theta(X) - \int_M^\infty (x - M) f_X(x;\theta)dx \\
&= E_\theta(X) - \int_0^\infty y f_X(y + M;\theta)dy
\end{aligned}
$$

Insurer's perspective - model fitting with Y data

- Only know the claim amount paid by insurer (i.e., Y_i)
- Leads to right-censored datasets: observe uncensored data if $X_i \leq M$ (i.e., $Y_i = X_i < M$), and censored data if $X_i > M$ (i.e., $Y_i = M$)
- Likelihood concept:

$$L(\theta) = \prod_i \Pr_\theta \left(i^{\text{th}} \text{ data point observed} \right)$$

- For n uncensored and m censored observations:
- Contribution to likelihood of n uncensored observations

$$L(\theta; x_1, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i; \theta)$$

where $X_i \sim f_X(x; \theta)$ is underlying *pdf* for claim model

- Contribution to likelihood of m censored observations $\{\Pr_\theta (Y_i = M)\}^m = \{\Pr_\theta(X > M)\}^m = \{1 - F_X(M; \theta)\}^m$

Insurer's perspective - model fitting with Y data

- Likelihood of θ based on Y_i 's:

$$L_1\left(\theta; y_1, \ldots, y_{n+m}\right) = \prod_{i=1}^{n} f_X\left(x_i; \theta\right) \prod_{j=1}^{m} \left\{1 - F_X(M; \theta)\right\}$$
$$= L\left(\theta; x_1, \ldots, x_n\right) \left\{1 - F_X(M; \theta)\right\}^m$$

- Log-likelihood:

$$l_1(\theta) = \ln\left\{L_1\left(\theta; y_1, \ldots, y_{n+m}\right)\right\}$$
$$= \ln\left\{L\left(\theta; x_1, \ldots, x_n\right)\right\} + m \ln\left\{1 - F_X(M; \theta)\right\}$$

- MLE Theorem still holds - can create confidence intervals as:

$$\hat{\theta}_1 \pm 1.96 \sqrt{I_1^{-1}\left(\hat{\theta}_1\right)}$$

where $\hat{\theta}_1$ solves $\frac{\partial}{\partial \theta} l_1(\theta) = 0$ and

$$I_1(\theta) = -E_\theta\left\{\frac{\partial^2}{\partial \theta \partial \theta^T} l_1(\theta)\right\}$$

[NOTE: n and m are random, though $n+m$ is considered fixed]

Reinsurer's perspective

- Total number of claims unknown
- Leads to truncated datasets: only observe Z if $Z > 0$
- Estimate distribution parameters of $f_X(x;\theta)$ based observed Z_i 's:
  - Use conditional distribution, which has CDF :

$$\begin{aligned}\Pr_\theta(Z \leq z | Z > 0) &= \Pr_\theta(X \leq z + M | X > M) \\ &= \frac{\Pr_\theta(M < X \leq z + M)}{\Pr_\theta(X > M)} \\ &= \frac{F_X(z + M;\theta) - F_X(M;\theta)}{1 - F_X(M;\theta)}\end{aligned}$$

  - And pdf:

$$f_{Z|Z>0}(z;\theta) = \frac{f_X(z + M;\theta)}{1 - F_X(M;\theta)}$$

Outline

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2. Questions

Part (a)

(Reinsurance) Suppose claims amounts, X_i, made on a portfolio have a distribution with density function:

$$f(x) = \sqrt{\frac{2}{\pi\tau}}e^{-x^2/2\tau}$$

(a) Find $\mathbb{E}X_i$ and $\mathrm{V}X_i$

Solution (a)

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Solution: We have:

$$\mathbb{E}X_i = \int_0^\infty x\sqrt{\frac{2}{\pi\tau}}e^{-x^2/2\tau}\,dx = -\sqrt{\frac{2\tau}{\pi}}e^{-x^2/2\tau}\bigg|_{x=0}^{\infty} = \sqrt{\frac{2\tau}{\pi}}$$

and

$$\begin{aligned}\mathbb{E}X_i^2 &= \int_0^\infty x^2\sqrt{\frac{2}{\pi\tau}}e^{-x^2/2\tau}\,dx \\ &= 2\int_0^\infty x^2\sqrt{\frac{1}{2\pi\tau}}e^{-x^2/2\tau}\,dx \\ &= \int_0^\infty x^2\sqrt{\frac{1}{2\pi\tau}}e^{-x^2/2\tau}\,dx + \int_0^\infty x^2\sqrt{\frac{1}{2\pi\tau}}e^{-x^2/2\tau}\,dx \\ &= \int_0^\infty x^2\sqrt{\frac{1}{2\pi\tau}}e^{-x^2/2\tau}\,dx + \int_{-\infty}^0 x^2\sqrt{\frac{1}{2\pi\tau}}e^{-x^2/2\tau}\,dx \\ &= \int_{-\infty}^\infty x^2\sqrt{\frac{1}{2\pi\tau}}e^{-x^2/2\tau}\,dx \\ &= \tau\end{aligned}$$

where the final equality follows upon recognising the integral as the second moment of a normal distribution with mean 0 and variance τ. Therefore, $\mathrm{V}X_i = \tau - 2\tau\pi^{-1}$.

Part (b)

Suppose claims amounts, X_i, made on a portfolio have a distribution with density function:

$$f(x) = \sqrt{\frac{2}{\pi\tau}} e^{-x^2/2\tau}$$

(b) Suppose that the insurer takes out an excess-of-loss reinsurance policy with retention level M. Calculate the drop in the insurer's mean claim liability resulting from the reinsurance. [HINT: Your answer will involve $\Phi(\cdot)$, the distribution function of the standard normal distribution.]

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Solution: We calculate $\mathbb{E}X_i - \mathbb{E}Y_i$ as follows.

$$\begin{aligned}
&\sqrt{\frac{2\tau}{\pi}} - \int_0^M x\sqrt{\frac{2}{\pi\tau}} e^{-x^2/2\tau}\,dx - M\int_M^\infty \sqrt{\frac{2}{\pi\tau}} e^{-x^2/2\tau}\,dx \\
=&\sqrt{\frac{2\tau}{\pi}} + \sqrt{\frac{2\tau}{\pi}} e^{-x^2/2\tau}\Big|_{x=0}^{M} - 2M\int_M^\infty \sqrt{\frac{1}{2\pi\tau}} e^{-x^2/2\tau}\,dx \\
=&\sqrt{\frac{2\tau}{\pi}} - \sqrt{\frac{2\tau}{\pi}}\left(1 - e^{-M^2/2\tau}\right) - 2M\int_{M/\sqrt{\tau}}^\infty \sqrt{\frac{1}{2\pi}} e^{-y^2/2}\,dy \\
=&\sqrt{\frac{2\tau}{\pi}} e^{-M^2/2\tau} - 2M\{1 - \Phi(M/\sqrt{\tau})\}
\end{aligned}$$

Part (c)

(c) Assume $\tau = 1$ and calculate the drop in mean claim liability for the retention levels $M = 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$.

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Solution: Setting $\tau = 1$ and using the above formula as well as

$$\mathbb{P}(X_i > M) = \int_M^\infty \sqrt{\frac{2}{\pi\tau}} e^{-x^2/2\tau} dx = 2\int_{M/\sqrt{\tau}}^\infty \sqrt{\frac{1}{2\pi}} e^{-y^2/2} dy = 2\{1 - \Phi(M/\sqrt{\tau})\}$$

we can construct the following table of values.

M	Mean Drop	$\mathbb{P}(X_i > M)$
0.0	0.798	1.000
0.5	0.396	0.617
1.0	0.167	0.317
1.5	0.059	0.134
2.0	0.017	0.046
2.5	0.004	0.012
3.0	0.001	0.003

Part (d)

Suppose claims amounts, X_i, made on a portfolio have a distribution with density function:

$$f(x) = \sqrt{\frac{2}{\pi\tau}} e^{-x^2/2\tau}$$

(d) Suppose we have data on 100 claims from this portfolio, and that the reinsurance retention level is $M = 16.45$. Suppose the observed data had ten claims above the retention level and the remaining 90 claim values, $X_1, \ldots, X_{90}$, gave $\sum_{i=1}^{90} X_i^2 = 6372.5$. Estimate τ using maximum likelihood ignoring the reinsurance (i.e., assume the 10 values over M were actually equal to M). Calculate an approximate 95% confidence interval for τ based on your estimate.

Solution (d)

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Ignoring the reinsurance, the log-likelihood is:

$$l(\tau) = \sum_{i=1}^{n} \ln f_X\left(x_i; \tau\right) = \frac{n}{2}\ln(2/\pi) - \frac{n}{2}\ln\tau - \frac{1}{2\tau}\sum_{i=1}^{n} X_i^2$$

So, the score equation is:

$$l'(\tau) = -\frac{n}{2\tau} + \frac{1}{2\tau^2}\sum_{i=1}^{n} X_i^2 = 0 \quad \Longrightarrow \quad \widehat{\tau} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

For the data at hand, we have $n = 100$ (since in this case we are ignoring the reinsurance) and

$$\sum_{i=1}^{100} X_i^2 = \sum_{i=1}^{90} X_i^2 + \sum_{i=91}^{100} X_i^2 = 6372.5 + 10\left(16.45^2\right) = 9078.525$$

Thus, $\widehat{\tau} = 90.78$. Now, to calculate a 95% confidence interval we not that $\mathbb{V}(\widehat{\tau}) = 1/I(\tau)$ where

$$I(\tau) = -\mathbb{E}\left\{l''(\tau)\right\} = -\mathbb{E}\left\{\frac{n}{2\tau^2} - \frac{1}{\tau^3}\sum_{i=1}^{n} X_i^2\right\} = -\frac{n}{2\tau^2} + \frac{1}{\tau^3}\sum_{i=1}^{n}\mathbb{E}\left(X_i^2\right) = \frac{n}{2\tau^2}$$

since $\mathbb{E}\left(X_i^2\right) = \tau$ was shown in part (a). Therefore, the estimated variance of $\widehat{\tau}$ is $\mathrm{V}(\widehat{\tau}) = 2\hat{\tau}^2/n = 2\left(90.78^2\right)/100 = 164.84$. Finally, then, an approximate 95% confidence interval is $90.78 \pm 1.96\sqrt{164.84} = (65.62,\ 115.94)$.

Part (e)

Suppose claims amounts, X_i, made on a portfolio have a distribution with density function:

$$f(x) = \sqrt{\frac{2}{\pi\tau}} e^{-x^2/2\tau}$$

(e) Write down the appropriate equation for estimating τ using maximum likelihood and properly incorporating the reinsurance structure of the portfolio.

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Suppose claims amounts, X_i, made on a portfolio have a distribution with density function:

$$f(x) = \sqrt{\frac{2}{\pi\tau}} e^{-x^2/2\tau}$$

(e) Write down the appropriate equation for estimating τ using maximum likelihood and properly incorporating the reinsurance structure of the portfolio.

Solution: The appropriate log-likelihood taking the reinsurance into account is:

$$l_1(\tau) = \sum_{i=1}^{n} \ln f_X(x_i; \tau) = \frac{n}{2}\ln(2/\pi) - \frac{n}{2}\ln\tau - \frac{1}{2\tau}\sum_{i=1}^{n} X_i^2 + m\ln[2\{1 - \Phi(M/\sqrt{\tau})\}]$$

where we have employed the result that $\mathbb{P}(X_i > M) = 2\{1 - \Phi(M/\sqrt{\tau})\}$ derived previously in part (c). Thus, the appropriate score equation for estimating τ is:

$$l_1'(\tau) = -\frac{n}{2\tau} + \frac{1}{2\tau^2}\sum_{i=1}^{n} X_i^2 + \frac{1}{2} m M \phi(M/\sqrt{\tau})\tau^{-3/2}\{1 - \Phi(M/\sqrt{\tau})\}^{-1} = 0$$

where $\phi(x) = \Phi'(x)$ is the standard normal density function.

Part (f)

(f) Verify, by simple substitution, that the solution to the equation from part (e) is $\widehat{\tau} = 109.03$ (which was derived using a computer). Compare this estimate to the one you calculated in part (d) and discuss the consequences of ignoring the reinsurance.

Part (f)

(f) Verify, by simple substitution, that the solution to the equation from part (e) is $\widehat{\tau} = 109.03$ (which was derived using a computer). Compare this estimate to the one you calculated in part (d) and discuss the consequences of ignoring the reinsurance.

Solution: In the above likelihood, we have $n = 90$ and $m = 10$, so simple substitution of the estimate $\hat{\tau} = 109.03$ yields

$$
\begin{aligned}
l_1'(\hat{\tau}) &= -\frac{90}{2(109.03)} + \frac{6372.5}{2(109.03)^2} \\
&\quad + \frac{10}{2}(16.45)\phi(16.45/\sqrt{109.03})(109.03)^{-3/2}\{1 - \Phi(16.45/\sqrt{109.03})\}^{-1} \\
&= -0.41273 + 0.26803 + 0.07225\phi(1.5754)\{1 - \Phi(1.5754)\}^{-1} \\
&= -0.14470 + 0.07225(0.11534)(1 - 0.94242)^{-1} \\
&= 0.00003
\end{aligned}
$$

which is essentially zero (up to rounding error). Clearly, ignoring the reinsurance has had a dramatic effect on the value of the estimate of τ. However, we do note that the "proper" estimate is within the 95% confidence interval found in part (d), so the two methods are not completely inconsistent with one another. [NOTE: Typically, finding the MLE under censoring will require an iterative computer algorithm. As these algorithms invariably need a reasonable initial guess for the MLE, it is generally reasonable to use the MLE calculated by ignoring the censoring (assuming that this MLE itself does not require a computer-based search for its own calculation, of course).]

Part (g)

(g) (Advanced) : Calculate an approximate 95% confidence interval for τ properly incorporating the reinsurance and based on the estimate given in part (f).

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Solution: This question is listed as "advanced" only because the calculations required are quite complicated. The concept involved is identical to that used in part (d). A rather tedious calculation shows that:

$$\begin{aligned} l_1''(\tau) = \frac{n}{2\tau^2} - \frac{1}{\tau^3}\sum_{i=1}^{n} X_i^2 + \frac{1}{4} m M^3 \tau^{-7/2} \phi(M/\sqrt{\tau})\{\Phi(-M/\sqrt{\tau})\}^{-1} \\ - \frac{1}{4} m M \tau^{-3} \phi(M/\sqrt{\tau})\{M\phi(M/\sqrt{\tau}) + 3\sqrt{\tau}\Phi(-M/\sqrt{\tau})\}\{\Phi(-M/\sqrt{\tau})\}^{-2} \end{aligned}$$

Solution (g)

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Now, to take the expectation of this quantity, we note that both m and n are clearly binomially distributed with $(n+m)$ "trials" and "success probabilities" $\mathbb{P}(X_i > M)$ and $\mathbb{P}(X_i \leq M)$, respectively. Thus,

$$\mathbb{E}m = (n+m)\mathbb{P}(X_i > M) = 2(n+m)\{1 - \Phi(M/\sqrt{\tau})\} = 2(n+m)\Phi(-M/\sqrt{\tau})$$
$$\mathbb{E}n = (n+m)\mathbb{P}(X_i \leq M) = (n+m)\{1 - \mathbb{P}(X_i > M)\} = (n+m)\{2\Phi(M/\sqrt{\tau}) - 1\}$$

Thus, we have:

$$\begin{aligned}
-\mathbb{E}\left\{l_1''(\tau)\right\} =& - \frac{\mathbb{E}(n)}{2\tau^2} + \frac{1}{\tau^3}\mathbb{E}\left(\sum_{i=1}^{n} X_i^2\right) - \frac{1}{4}\mathbb{E}mM^3\tau^{-7/2}\phi(M/\sqrt{\tau})\{\Phi(-M/\sqrt{\tau})\}^{-1} \\
&+ \frac{1}{4}\mathbb{E}mM\tau^{-3}\phi(M/\sqrt{\tau})\{M\phi(M/\sqrt{\tau}) + 3\sqrt{\tau}\Phi(-M/\sqrt{\tau})\}\{\Phi(-M/\sqrt{\tau})\}^{-2} \\
=& \frac{1}{\tau^3}\mathbb{E}\left(\sum_{i=1}^{n} X_i^2\right) - \frac{(n+m)}{2\tau^2}\{2\Phi(M/\sqrt{\tau}) - 1\} - \frac{1}{2}(n+m)M^3\tau^{-7/2}\phi(M/\sqrt{\tau}) \\
&+ \frac{1}{2}(n+m)M^2\tau^{-3}\{\phi(M/\sqrt{\tau})\}^2\{\Phi(-M/\sqrt{\tau})\}^{-1} \\
&+ \frac{3}{2}(n+m)M\tau^{-5/2}\phi(M/\sqrt{\tau})
\end{aligned}$$

Solution (g)

Now, to calculate $\mathbb{E}\left(\sum_{i=1}^{n} X_i^2\right)$ is tricky since n is not independent of the X_i 's. However, we can use the fact that $n+m$ is not random to write:

$$
\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} X_i^2\right) &= \mathbb{E}\left\{\sum_{i=1}^{n+m} X_i^2 I_{(X_i \le M)}\right\} \\
&= \sum_{i=1}^{n+m} \mathbb{E}\left\{X_i^2 I_{(X_i \le M)}\right\} \\
&= \sum_{i=1}^{n+m} \int_0^M x^2 \sqrt{\frac{2}{\pi\tau}} e^{-\frac{1}{2\tau}x^2}\, dx \\
&= (n+m)\tau \int_0^{M/\sqrt{\tau}} y^2 \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}y^2}\, dy, \quad (\text{ letting } x = y\sqrt{\tau}) \\
&= (n+m)\tau \sqrt{\frac{2}{\pi}} \left(-\, y e^{-\frac{1}{2}y^2}\Big|_0^{M/\sqrt{\tau}} + \int_0^{M/\sqrt{\tau}} e^{-\frac{1}{2}y^2}\, dy \right) \\
&= -2(n+m)M\sqrt{\tau}\sqrt{\frac{1}{2\pi}} e^{-\frac{1}{2\tau}M^2} + 2(n+m)\tau \int_0^{M/\sqrt{\tau}} \sqrt{\frac{1}{2\pi}} e^{-\frac{1}{2}y^2}\, dy \\
&= -2(n+m)M\sqrt{\tau}\phi(M/\sqrt{\tau}) + 2(n+m)\tau \left\{\Phi(M/\sqrt{\tau}) - \frac{1}{2}\right\} \\
&= (n+m)\tau\{2\Phi(M/\sqrt{\tau}) - 1\} - 2(n+m)M\sqrt{\tau}\phi(M/\sqrt{\tau})
\end{aligned}
$$