STAT3035/8035
Tutorial 3

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Outline

1. Review

2. Questions

Standard Procedure to Find MLE

⋆ Likelihood Function: Assuming independence,

$$L(\boldsymbol{\theta}; \mathbf{x}) = \prod_i f(x_i; \boldsymbol{\theta})$$

⋆ Log-Likelihood Function: Assuming independence,

$$\ell(\boldsymbol{\theta}; \mathbf{x}) = \sum_i \log f(x_i; \boldsymbol{\theta})$$

⋆ Score Function:

$$\mathrm{sc}(\boldsymbol{\theta}) = \frac{\partial \ell}{\partial \boldsymbol{\theta}}$$

⋆ Hessian:

$$\mathrm{hess}(\boldsymbol{\theta}) = \frac{\partial^2 \ell}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T}$$

⋆ Information Function:

$$\mathrm{info}(\boldsymbol{\theta}) = -\,\mathrm{hess}(\boldsymbol{\theta})$$

⋆ Expected Information Function:

$$I(\widehat{\boldsymbol{\theta}}) = \mathbb{E}[\mathrm{info}(\widehat{\boldsymbol{\theta}})]$$

⋆ Variance of $\widehat{\boldsymbol{\theta}}$:

$$\mathrm{Var}(\widehat{\boldsymbol{\theta}}) = I^{-1}(\widehat{\boldsymbol{\theta}})$$

Outline

Question 1

(Re-parametrisation and MLE) Let $X_1, \ldots, X_n$ be an *i.i.d.* sample from a log-normal distribution with parameters μ and σ^2.
(a) Write the log-likelihood function for the sample $X_1, \ldots, X_n$
(b) Define the quantity $\psi = e^{\mu + \frac{\sigma^2}{2}}$. Suppose we re-parameterise the log-normal distribution in terms of the parameters ψ and σ^2. Rewrite the log-likelihood function from part (a) using this new parametrisation.

Lognormal Distribution
If $Y = \ln X \sim N\left(\mu, \sigma^2\right)$, then $X \sim LN\left(\mu, \sigma^2\right)$

$$f_X\left(x; \mu, \sigma^2\right) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(\ln x - \mu)^2\right\}, \quad x > 0$$

$$E_{\mu,\sigma^2}(X) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \quad E_{\mu,\sigma^2}\left(X^2\right) = \exp\left(2\mu + 2\sigma^2\right)$$

Solution 1

Solution 1

(a) By independence, one can write down the sample likelihood function as follows.

$$L\left(\mu, \sigma^2; \mathbf{x}\right) = \prod_{i=1}^{n} f\left(x_i; \mu, \sigma^2\right) = \prod_{i=1}^{n} \frac{1}{x_i \sigma \sqrt{2\pi}} e^{-\frac{(\ln x_i - \mu)^2}{2\sigma^2}}$$

Then one can easily find the log-likelihood function as follows. Note that here we treat σ^2 as a whole instead of treating it as the square of σ

$$\begin{aligned}
\ell\left(\mu, \sigma^2; \mathbf{x}\right) &= \sum_{i=1}^{n} \ln f\left(x_i; \mu, \sigma^2\right) \\
&= \sum_{i=1}^{n} -\ln x_i - \frac{1}{2}\ln \sigma^2 - \frac{1}{2}\ln 2\pi - \frac{(\ln x_i - \mu)^2}{2\sigma^2} \\
&= -\sum_{i=1}^{n} \ln x_i - \frac{n}{2}\ln \sigma^2 - \frac{n}{2}\ln 2\pi - \sum_{i=1}^{n} \frac{(\ln x_i - \mu)^2}{2\sigma^2}
\end{aligned}$$

(b) Re-parametrisation is like giving a different name to a distribution. In this case, one simply needs to replace μ by a function of the two new parameters, namely, $\mu = \ln \psi - \frac{\sigma^2}{2}$. That is,

$$\ell\left(\psi, \sigma^2; \mathbf{x}\right) = -\sum_{i=1}^{n} \ln x_i - \frac{n}{2}\ln \sigma^2 - \frac{n}{2}\ln 2\pi - \sum_{i=1}^{n} \frac{\left(\ln x_i - \ln \psi + \frac{\sigma^2}{2}\right)^2}{2\sigma^2}$$

Question 1

(c) Using the log-likelihood you wrote in part (b), and recalling that the MLE of σ^2 is

$$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(\ln x_i)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\ln x_i\right)^2$$

calculate the MLE of ψ in terms of x_i' s and $\overline{\sigma^2}$. Furthermore, recall that

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\ln x_i$$

Does the MLE of ψ seem sensible? Why or why not?

Solution 1

Solution 1

(c) To find the MLE of ψ, first calculate the score functions as follows.

$$\mathrm{sc}(\psi) = \frac{\partial \ell}{\partial \psi} = \sum_{i=1}^{n} \frac{\left(\ln x_i - \ln \psi + \frac{\sigma^2}{2}\right)}{\psi \sigma^2} = \frac{\sum_{i=1}^{n} \ln x_i - n \ln \psi + \frac{n\sigma^2}{2}}{\psi \sigma^2}$$

Now, set $\mathrm{sc}(\psi)$ to zero and solve for $\widehat{\psi}_{MLE}$

$$0 = \sum_{i=1}^{n} \ln x_i - n \ln \widehat{\psi}_{MLE} + \frac{n\widehat{\sigma^2}}{2}$$

$$\widehat{\psi}_{MLE} = e^{\frac{\sum_{i=1}^{n} \ln x_i}{n} + \frac{\widehat{\sigma^2}}{2}} = e^{\widehat{\mu} + \frac{\widehat{\sigma^2}}{2}}$$

Note that in fact, one needs to find the other score function for σ^2 and solve the system of two equations. However, in the current case, the results for $\widehat{\sigma^2}$ is given, which saves us some effort. However, since we still need both score functions in the next part, I simply list the other score function below.

$$\mathrm{sc}\left(\sigma^2\right) = \frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \sum_{i=1}^{n} \frac{\sigma^2 \left(\ln x_i - \ln \psi + \frac{\sigma^2}{2}\right) - \left(\ln x_i - \ln \psi + \frac{\sigma^2}{2}\right)^2}{2\sigma^4}$$

Question 1

(d) Estimate the variance of $\widehat{\psi}$.

Solution 1

Solution 1

(d) Please be warned that this part is rather tedious.

$$\text{info}\left(\psi,\sigma^2\right) = -\,\text{Hessian}\left(\psi,\sigma^2\right) = \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \psi^2} & -\frac{\partial^2 \ell}{\partial \psi \partial \sigma^2} \\ -\frac{\partial^2 \ell}{\partial \sigma^2 \psi} & -\frac{\partial^2 \ell}{\partial \left(\sigma^2\right)^2} \end{bmatrix}$$

$$-\frac{\partial^2 \ell}{\partial \psi^2} = \frac{n + \sum_{i=1}^n \ln x_i - n \ln \psi + \frac{n\sigma^2}{2}}{\sigma^2 \psi^2}$$

$$-\frac{\partial^2 \ell}{\partial \psi \partial \sigma^2} = -\frac{\partial^2 \ell}{\partial \sigma^2 \partial \psi} = \frac{\sum_{i=1}^n \ln x_i - n \ln \psi}{\sigma^4 \psi}$$

$$-\frac{\partial^2 \ell}{\partial \left(\sigma^2\right)^2} = -\frac{n}{2\sigma^4} + \frac{\sum_{i=1}^n \left(\ln x_i - \ln \psi + \frac{\sigma^2}{2}\right)^2}{\sigma^6} - \frac{\sum_{i=1}^n \left(\ln x_i - \ln \psi + \frac{\sigma^2}{2}\right)}{\sigma^4} + \frac{n}{4\sigma^2}$$

Solution 1

(d) Now one can find the expected value of the above items as follows. Please be aware that some smart observations might make the calculations easier.

$$\mathbb{E}\left[-\frac{\partial^2 \ell}{\partial \psi^2}\right] = \frac{n}{\sigma^2 \psi^2}$$

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$$\mathbb{E}\left[-\frac{\partial^2 \ell}{\partial \left(\sigma^2\right)^2}\right] = \frac{n}{2\sigma^4} + \frac{n}{4\sigma^2}$$

$$I(\widehat{\psi}, \widehat{\sigma^2}) = \begin{bmatrix} \frac{n}{\widehat{\sigma^2}\,\widehat{\psi}^2} & -\frac{n}{2\widehat{\sigma^2}\,\widehat{\psi}} \\ -\frac{n}{2\widehat{\sigma^2}\,\widehat{\psi}} & \frac{n}{2\widehat{\sigma^2}^2} + \frac{n}{4\widehat{\sigma^2}} \end{bmatrix}$$

$$I^{-1}(\widehat{\psi}, \widehat{\sigma^2}) = \frac{1}{\frac{n}{\widehat{\sigma^2}\,\widehat{\psi}^2} \times \left(\frac{n}{2\widehat{\sigma^2}^2} + \frac{n}{4\widehat{\sigma^2}}\right) - \frac{n}{2\widehat{\sigma^2}\,\widehat{\psi}} \times \frac{n}{2\widehat{\sigma^2}\,\widehat{\psi}}} \begin{bmatrix} \frac{n}{2\widehat{\sigma^2}^2} + \frac{n}{4\widehat{\sigma^2}} & \frac{n}{2\widehat{\sigma^2}\,\widehat{\psi}} \\ \frac{n}{2\widehat{\sigma^2}\,\widehat{\psi}} & \frac{n}{\widehat{\sigma^2}\,\widehat{\psi}^2} \end{bmatrix}$$

$$\mathbb{V}(\widehat{\psi}) = \frac{\frac{n}{2\widehat{\sigma^2}^2} + \frac{n}{4\widehat{\sigma^2}}}{\frac{n}{\widehat{\sigma^2}\,\widehat{\psi}^2} \times \left(\frac{n}{2\widehat{\sigma^2}^2} + \frac{n}{4\widehat{\sigma^2}}\right) - \frac{n}{2\widehat{\sigma^2}\,\widehat{\psi}} \times \frac{n}{2\widehat{\sigma^2}\,\widehat{\psi}}}$$

Question 2

(MOP) Let Y be a random variable having Pareto distribution with parameters α and δ. Define $X = Y^{1/\gamma}$, for some parameter γ. The distribution of X belongs to the Burr family.
(a)* Find the *pdf* of X.

Pareto Distribution

$$f_Y(y;\alpha,\delta) = \frac{\alpha\delta^\alpha}{(y+\delta)^{\alpha+1}}, y \geq 0 \quad F_Y(y;\alpha,\delta) = 1 - \frac{\delta^\alpha}{(y+\delta)^\alpha}$$
$$E_{\alpha,\beta}(Y) = \frac{\delta}{\alpha-1}, \alpha > 1 \quad E_{\alpha,\delta}\left(Y^2\right) = \frac{2\delta^2}{(\alpha-1)(\alpha-2)}, \alpha > 2$$

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Solution: Using the CDF of the Pareto distribution, we have:

$$\Pr(X \leq x) = \Pr\left(X^\gamma \leq x^\gamma\right) = \Pr\left(Y \leq x^\gamma\right) = 1 - \left(\frac{\delta}{\delta + x^\gamma}\right)^\alpha, \quad x \geq 0$$

Taking the derivative with respect to x yields the pdf of X as:

$$f_X(x) = -\alpha\left(\frac{\delta}{\delta + x^\gamma}\right)^{\alpha-1}\left\{\frac{-\delta\gamma x^{\gamma-1}}{(\delta + x^\gamma)^2}\right\} = \frac{\alpha\gamma\delta^\alpha x^{\gamma-1}}{(\delta + x^\gamma)^{\alpha+1}}, \quad x \geq 0$$

Question 2

(b)* Find the the expectation of X. [HINT: Use a change of integration variable $y = \delta / (\delta + x^\gamma)$, and the "beta integral", which shows $\int_0^1 y^a (1-y)^b dy = \Gamma(a+1)\Gamma(b+1)/\Gamma(a+b+2)$.]

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Solution: Using the definition of expectation and following the hint:

$$\begin{aligned}
E(X) &= \int_0^\infty x \frac{\alpha\gamma\delta^\alpha x^{\gamma-1}}{(\delta + x^\gamma)^{\alpha+1}} dx = \int_0^\infty \frac{\alpha\gamma\delta^\alpha x^\gamma}{(\delta + x^\gamma)^{\alpha+1}} dx \\
&= \int_0^\infty \alpha\gamma \left(\frac{\delta}{\delta + x^\gamma}\right)^\alpha \frac{x^\gamma}{\delta + x^\gamma} dx = \int_0^\infty \alpha\gamma \left(\frac{\delta}{\delta + x^\gamma}\right)^\alpha \left(1 - \frac{\delta}{\delta + x^\gamma}\right) dx \\
&= \int_1^0 \alpha\gamma y^\alpha (1-y) \left\{-\gamma^{-1}\delta^{1/\gamma} y^{-1-1/\gamma} (1-y)^{-1+1/\gamma}\right\} dy \\
&= \int_0^1 \alpha\delta^{1/\gamma} y^{\alpha-1-1/\gamma} (1-y)^{1/\gamma} dy = \alpha\delta^{1/\gamma} \left\{\frac{\Gamma(\alpha - 1/\gamma)\Gamma(1+1/\gamma)}{\Gamma(\alpha+1)}\right\} \\
&= \delta^{1/\gamma} \left\{\frac{\Gamma(\alpha - 1/\gamma)\Gamma(1+1/\gamma)}{\Gamma(\alpha)}\right\}
\end{aligned}$$

where the fifth equality follows from the change of variable, which implies that $dx = -\gamma^{-1}\delta^{1/\gamma} y^{-1-1/\gamma}(1- y)^{-1+1/\gamma} dy$ (and the negative sign will cause a reverse in the limits of integration), and the seventh equality follows from the beta integral formula.

Question 2

(c)* Find the quartiles of the Burr distribution.

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Solution: Using the CDF in part a, the quartiles, $x_{0.25}$ and $x_{0.75}$, are the solutions to the equations:

$$0.25 = \Pr\left(X \leq x_{0.25}\right) = 1 - \left(\frac{\delta}{\delta + x_{0.25}^\gamma}\right)^\alpha$$
$$0.75 = \Pr\left(X \leq x_{0.75}\right) = 1 - \left(\frac{\delta}{\delta + x_{0.75}^\gamma}\right)^\alpha$$

Therefore, the quartiles are:

$$x_{0.25} = \left(\frac{\delta}{0.75^{1/\alpha}} - \delta\right)^{1/\gamma}$$

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Question 2

(d)* Suppose that we know that $\alpha = 1$. Use the quartiles to find an MOP estimate of δ and γ.

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(d)* Suppose that we know that $\alpha = 1$. Use the quartiles to find an MOP estimate of δ and γ.
Solution: If $\alpha = 1$, then the quartiles of the Burr distribution reduce to:

$$\begin{aligned} x_{0.25} &= \left(\tfrac{\delta}{0.75} - \delta\right)^{1/\gamma} = \left(\tfrac{1}{3}\delta\right)^{1/\gamma} \\ x_{0.75} &= \left(\tfrac{\delta}{0.25} - \delta\right)^{1/\gamma} = (3\delta)^{1/\gamma} \end{aligned}$$

Thus, solving the equations

$$\hat{x}_{0.25} = \left(\frac{1}{3}\delta\right)^{1/\gamma}, \quad \hat{x}_{0.75} = (3\delta)^{1/\gamma}$$

shows that

$$\frac{\hat{x}_{0.75}}{\hat{x}_{0.25}} = 3^{2/\gamma} \quad \Longrightarrow \quad \hat{\gamma} = \frac{2\ln 3}{\ln \hat{x}_{0.75} - \ln \hat{x}_{0.25}}$$

and $\hat{\delta} = (\hat{x}_{0.25}\hat{x}_{0.75})^{\hat{\gamma}/2}$