STAT3035/8035
Tutorial 9

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Outline

1. Review
2. Questions

Individual Risk Model

- Idea:
  - Consider each policy individually; n policies in portfolio
  - Let Y_i = amount of claim(s) on i^{th} policy (during some period)
  - Let $N_i = I_{(Y_i>0)}; q_i = \Pr(N_i = 1) < 1$
  - Let $X_i = (Y_i | N_i = 1); F_i(x) = \Pr(X_i \le x) = \Pr(Y_i \le x | N_i = 1)$; $\mu_i = E(X_i) = E(Y_i | N_i = 1); \sigma_i^2 = \text{Var}(X_i) = \text{Var}(Y_i | N_i = 1)$
  - Aggregate claim amount is: $S = \sum_{i=1}^{n} Y_i$
- Basic characterisation of S:
  - $N_i \sim \text{Binomial}(1, q_i)$
  - We can write $Y_i = \sum_{i=1}^{N_i} X_i \sim$CompBin$\{1, q_i, F_i(x)\}$
  - So, $E(Y_i) = q_i \mu_i$; $\text{Var}(Y_i) = q_i \left\{\sigma_i^2 + (1 - q_i)\mu_i^2\right\}$
  - Unfortunately, $S = \sum_{i=1}^{n} Y_i$ is sum of independent *CompBin* distributed quantities with different q_i 's and $F_i(x)$'s. So, S not *CompBin* distributed.
  - However,

$$E(S) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} q_i \mu_i$$

$$\text{Var}(S) = \text{Var}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \text{Var}(Y_i) = \sum_{i=1}^{n} \left[q_i \left\{\sigma_i^2 + (1 - q_i)\mu_i^2\right\}\right]$$

Poisson Collective Risk Approximation

- Idea:
  - $N_i \sim \text{ Binomial}(1, q_i) \sim \text{Pois}(q_i)$
  - Let $\tilde{N}_i \sim \text{Pois}(q_i)$ and $\tilde{Y}_i = \sum_{j=1}^{\tilde{N}_i} X_{ij}$, where X_{i1}, X_{i2}, etc. are iid random variables with CDF $F_i(x)$ and are independent of $\tilde{N}_i$.
  - So, $\tilde{Y}_i \sim CompPois\ \{q_i, F_i(x)\}$ and:
  - Define $\tilde{S} = \sum_{i=1}^{n} \tilde{Y}_i \sim CompPois\ \{Q, F(x)\}$ where

$$Q = \sum_{i=1}^{n} q_i \quad \text{and} \quad F(x) = Q^{-1} \sum_{i=1}^{n} q_i F_i(x)$$

  - $\Pr(\tilde{S} \le s) = \Pr\left(\sum_{i=1}^{n} \tilde{Y}_i \le s\right) \approx \Pr\left(\sum_{i=1}^{n} Y_i \le s\right) = \Pr(S \le s)$
- Assessment of approximation accuracy:

$$E(\tilde{S}) = E\left(\sum_{i=1}^{n} \tilde{Y}_i\right) = \sum_{i=1}^{n} E\left(\tilde{Y}_i\right) = \sum_{i=1}^{n} q_i \mu_i = E(S)$$

$$\begin{aligned}\text{Var}(\tilde{S}) &= \text{Var}\left(\sum_{i=1}^{n} \tilde{Y}_i\right) = \sum_{i=1}^{n} \text{Var}\left(\tilde{Y}_i\right) = \sum_{i=1}^{n} q_i \left(\sigma_i^2 + \mu_i^2\right) \\ &= \sum_{i=1}^{n} q_i \left\{\sigma_i^2 + (1 - q_i)\,\mu_i^2 + q_i \mu_i^2\right\} = \text{Var}(S) + \sum_{i=1}^{n} q_i^2 \mu_i^2\end{aligned}$$

  Approximation will be good when $\sum_{i=1}^{n} q_i^2 \mu_i^2$ is small relative to the size of $\text{Var}(S)$

Parameter Variability

- Let S be aggregate claim amount based on Individual Risk Model
- S has parameters q_i's and $F_i(x)$
- Assume $F_i(x)$ known, but $q_i \overset{iid}{\sim} H(q)$
- Define $\eta_1 = E\left(q_i\right)$ and $\eta_2 = E\left(q_i^2\right)$ (NOTE: Assuming known η 's rather than known q_i 's reduces number of necessary parameters from n to 2)
- Examine how this structure effects characteristics of $\tilde{S}$, the Poisson Collective Risk Model approximation to S
- Previously: $E(\tilde{S}) = \sum_{i=1}^{n} q_i \mu_i$ and $\operatorname{Var}(\tilde{S}) = \sum_{i=1}^{n} q_i \left(\sigma_i^2 + \mu_i^2\right)$
- Now:

$$E(\tilde{S}) = E\left(\sum_{i=1}^{n} \tilde{Y}_i\right) = \sum_{i=1}^{n} E\left(\tilde{Y}_i\right) = \sum_{i=1}^{n} E\left\{E\left(\tilde{Y}_i | q_i\right)\right\} = \sum_{i=1}^{n} E\left(q_i \mu_i\right) = \eta_1 \sum_{i=1}^{n} \mu_i$$

$$\begin{aligned}
\operatorname{Var}(\tilde{S}) &= \operatorname{Var}\left(\sum_{i=1}^{n} \tilde{Y}_i\right) = \sum_{i=1}^{n} \operatorname{Var}\left(\tilde{Y}_i\right) = \sum_{i=1}^{n} \left[E\left\{\operatorname{Var}\left(\tilde{Y}_i | q_i\right)\right\} + \operatorname{Var}\left\{E\left(\tilde{Y}_i | q_i\right)\right\}\right] \\
&= \sum_{i=1}^{n} \left[E\left\{q_i \left(\sigma_i^2 + \mu_i^2\right)\right\} + \operatorname{Var}\left\{q_i \mu_i\right\}\right] = \sum_{i=1}^{n} \left\{\eta_1 \left(\sigma_i^2 + \mu_i^2\right) + \mu_i^2 \left(\eta_2 - \eta_1^2\right)\right\} \\
&= \eta_1 \sum_{i=1}^{n} \sigma_i^2 + \left(\eta_2 + \eta_1 - \eta_1^2\right) \sum_{i=1}^{n} \mu_i^2
\end{aligned}$$

Outline

1. Review

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Question 1

(Individual and Collective Risk Models) A group one-year term-life insurance portfolio contains policies of two types. On one type of policy, a benefit of \$5000 is paid if the policyholder dies within the year of coverage, while on the other a benefit of \$10000 is paid. Moreover, suppose that the probability of death during the year of coverage among those holding the first type of policy is 0.002, while for those holding the second type of policy the probability of death is 0.012. Suppose that the portfolio is composed of 500 type-1 and 80 type- 2 policies, and that the policies are independent with respect to their chance of mortality.

(a) Using the individual risk model, calculate the mean and variance of the aggregate claim amount, S, for this portfolio.

(b) Using the normal approximation, estimate the probability that the aggregate claim amount over the year for this portfolio exceeds \$11000.

Solution 1

Solution 1

(a) We have 580 policies, with $q_i = 0.002, X_i = 5000 = \mu_i$ for $i = 1, \ldots, 500$ and $q_i = 0.012$ $X_i = 10000 = \mu_i$ for $i = 501, \ldots, 580$. Further, we have $\sigma_i^2 = 0$ for $i = 1, \ldots, 580$. So

$$\mathbb{E}S = \sum_{i=1}^{580} q_i \mu_i = 500(0.002)(5000) + 80(0.012)(10000) = 14600$$

and

$$\begin{aligned}\mathbb{V}S &= \sum_{i=1}^{580} q_i \left\{\sigma_i^2 + (1 - q_i)\,\mu_i^2\right\} = 500(0.002)(0.998)\left(5000^2\right) + 80(0.012)(0.988)\left(10000^2\right) \\ &= 119798000\end{aligned}$$

(b) From the calculations of part a, the normal approximation yields:

$$\mathbb{P}(S > 11000) \approx 1 - \Phi\left(\frac{11000 - 14600}{\sqrt{119798000}}\right) = 1 - \Phi(-0.329) = 0.6288$$

Question 1

(c) Calculate the probability asked for in part b exactly. Do you think the normal approximation is reasonable here?

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(c) Calculate the probability asked for in part b exactly. Do you think the normal approximation is reasonable here?

Solution: Note that the only way for S to be less than 11000 is if: (1) no claims are made, (2) exactly one type-1 claim is made, (3) exactly one type-2 claim is made, or (4) exactly two type-1 claims are made. since the number of type- 1 claims made is clearly binomial with parameters 500 and 0.002, while the number of type- 2 claims made is binomial with parameters 80 and 0.012, we have:

$$
\begin{aligned}
\mathbb{P}(S > 11000) =& 1 - \mathbb{P}(S \leq 11000) \\
=& 1 - \mathbb{P}(\text{ no claims }) - \mathbb{P}(1 \text{ type } -1 \text{ claim, no type } -2 \text{ claims}) \\
& - \mathbb{P}(\text{ no type } -1 \text{ claims, } 1 \text{ type } -2 \text{ claim }) \\
& - \mathbb{P}(2 \text{ type } -1 \text{ claims, no type } -2 \text{ claims}) \\
=& 1 - \left(0.998^{500}\right)\left(0.988^{80}\right) - 500(0.002)\left(0.998^{499}\right)\left(0.988^{80}\right) \\
& - 80(0.012)\left(0.988^{79}\right)\left(0.998^{500}\right) - \frac{500(499)}{2}\left(0.002^{2}\right)\left(0.998^{498}\right)\left(0.988^{80}\right) \\
=& 0.5139
\end{aligned}
$$

Note that the normal approximation is rather poor in this case.

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(d) Approximate S using the collective risk model with an appropriate compound Poisson distributed quantity, $\tilde{S}$. Find the mean and variance of $\tilde{S}$. Is the approximation adequate?

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Solution: The appropriate compound Poisson quantity $\tilde{S}$ has rate parameter $Q = \sum_{i=1}^{580} q_i = 500(0.002) + 80(0.012) = 1.96$ and

$$\begin{aligned} F(x) &= (1.96)^{-1} \sum_{i=1}^{580} q_i F_i(x) = (1.96)^{-1} \{500(0.002) F_1(x) + 80(0.012) F_2(x)\} \\ &= (1.96)^{-1} \{F_1(x) + 0.96 F_2(x)\} \end{aligned}$$

where $F_1(x)$ is the CDF of a random variable which only takes the value 5000 and $F_2(x)$ is the CDF of a random variable which only takes the value 10000. A little thought shows that $F(x)$ is the CDF of a random variable which takes the value 5000 with probability 1/1.96 and takes the value 10000 with probability 0.96/1.96. Therefore, $\mu_1 = (1/1.96)5000 + (0.96/1.96)10000 = 7448.98$ and $\mathbb{E}\tilde{S} = Q\mu_1 = 1.96(7448.98) = 14600 = \mathbb{E}S$. Also, we know that $\mathbb{V}\tilde{S} = \mathbb{V}S + \sum_{i=1}^{580} q_i^2 \mu_i^2 = 121000000$.
Alternatively, we can calculate the second moment of a random variable with CDF $F(x)$ as $\mu_2 = (1/1.96)5000^2 + (0.96/1.96)10000^2 = 61734694$, and thus $\text{Var}(\tilde{S}) = Q\mu_2 = 1.96(61734694) = 121000000$. Note that $\frac{Var(\tilde{S})}{Var(S)} \approx 1.01$, so we might expect that the approximation would be adequate.

Question 1

(e) Calculate $Skew(\tilde{S})$ and use the translated Gamma approximation to the distribution of $\tilde{S}$ to estimate the probability that the aggregate claim amount exceeds \$11000.

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Solution: We first note that the third raw moment of a random variable with $CDFF(x)$ is $\mu_3 = (1/1.96)5000^3 + (0.96/1.96)10000^3 = 5.5357 \times 10^{11}$. Thus

$$\rho_{\tilde{S}} = \frac{Q\mu_3}{(Q\mu_2)^{3/2}} = \frac{\mu_3}{\mu_2^{3/2}\sqrt{Q}} = \frac{5.5357 \times 10^{11}}{61734694^{3/2}\sqrt{1.96}} = 0.81517$$

So, the appropriate translated Gamma approximation has parameters which solve the equations:

$$\frac{2}{\sqrt{\alpha_g}} = 0.81517, \quad \alpha_g\theta_g^2 = 121000000, \quad k + \alpha_g\theta_g = 14600$$

which have solutions $\alpha_g = 6.019, \theta_g = 4483.47$ and $k = -12388.02$. Therefore,

$$\begin{aligned}
\mathbb{P}(S > 11000) &\approx \mathbb{P}(\tilde{S} > 11000) \\
&\approx \mathbb{P}(Y > 11000 + 12388.02) \\
&= \mathbb{P}\left\{2(4483.47)^{-1}Y > 2(4483.47)^{-1}(23388.02)\right\} \\
&\approx \mathbb{P}\left(\chi^2_{12} > 10.433\right) \\
&= 0.5780
\end{aligned}$$

where $Y \sim G(6.019, 4483.47)$.
(NB : The actual value of $\Pr(Y > 11000 + 12388.02)$ is 0.5813.)

Question 2

(Parameter Variability) An insurance portfolio consists of n policies. For each policy, the number of claims made during the term of coverage has a Poisson distribution with parameter q (i.e., the same for each policy). Furthermore, the individual claim amounts have mean μ and standard deviation σ (again, the same for each policy). Finally, suppose that q is random and comes from a distribution having mean η_1 and second raw moment η_2. Calculate the mean and variance of S, the aggregate claim amount for this portfolio. Compare your results with those of the case discussed in Section 4.4.2 of the course notes, in which each policy has a different (random) claim rate q_i each independently distributed with common mean η_1 and second raw moment η_2 (but still assume that the policies have the same claim amount distribution). Discuss the reasons for any differences.

Solution 2

Solution 2

From the description of the problem, we can consider the total claim amount arising from the i th policy, Y_i, to have a compound Poisson distribution with rate parameter q and individual claim distribution $F(x)$, where $F(x)$ has mean μ and variance σ^2 (the same for all policies). Therefore,

$$\mathbb{E}S = \mathbb{E}\{\mathbb{E}(S|q)\} = \mathbb{E}\left\{\mathbb{E}\left(\sum_{i=1}^{n} Y_i|q\right)\right\} = \mathbb{E}\left\{\sum_{i=1}^{n} \mathbb{E}\left(Y_i|q\right)\right\} = \mathbb{E}\left(\sum_{i=1}^{n} q\mu\right) = n\mu\mathbb{E}q = n\mu\eta_1$$

and

$$\begin{aligned}
\mathbb{V}S &= \mathbb{E}\{\mathbb{V}(S|q)\} + \mathbb{V}\{\mathbb{E}(S|q)\} = \mathbb{E}\left\{\operatorname{Var}\left(\sum_{i=1}^{n} Y_i|q\right)\right\} + \mathbb{V}\left\{\mathbb{E}\left(\sum_{i=1}^{n} Y_i|q\right)\right\} \\
&= \mathbb{E}\left\{\sum_{i=1}^{n} \mathbb{V}\left(Y_i|q\right)\right\} + \mathbb{V}\left\{\sum_{i=1}^{n} \mathbb{E}\left(Y_i|q\right)\right\} = \sum_{i=1}^{n} \mathbb{E}\left\{q\left(\sigma^2+\mu^2\right)\right\} + \mathbb{V}(nq\mu) \\
&= n\eta_1\left(\sigma^2+\mu^2\right) + n^2\mu^2\left(\eta_2-\eta_1^2\right)
\end{aligned}$$

In the case where q_i was different for each policy, we saw that $\mathbb{E}S = n\mu\eta_1$ which is the same as above, but $\mathrm{VS} = n\eta_1\left(\sigma^2+\mu^2\right) + n\mu^2\left(\eta_2-\eta_1^2\right)$ which is smaller than the above variance (since an n appears in place of an n^2). The reason for this essentially comes from the fact that if we simply choose a single random q for all the policies, than a “non-representative” realisation of q will affect all the policies, whereas when the q_i's are all independent, an “overly large” q_i for some policy will likely be “averaged out” by an "overly small" q_i for some other policy.