STAT3035/8035
Tutorial 7

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Outline

Aggregated Claims Modelling

- Model total amount, S, made on entire portfolio for some fixed period
    - Need model for claim sizes, X_i
    - Need model for claim numbers, N
    - Assumptions:
        - Claim sizes and rate constant over time period
        - Claim sizes and number independent
- Collective Risk Model
    - S is a random sum

$$S = \sum_{i=1}^{N} X_i$$

    - $X_i \overset{iid}{\sim} f_X(x)$, "portfolio-wide" distribution
    - $N \sim p_N(n)$, typically choose Poisson (λ), Binomial (m, q) or Negative Binomial (k, q)
    - If $N = 0$, define $S = 0$
    - Some notation:

$$\mu_k = E\left(X_i^k\right)$$

$$\nu = E(N)$$

$$\tau^2 = \mathrm{Var}(N)$$

Collective Risk Model

- Expected value of S

$$
\begin{aligned}
E(S) = E\{E(S|N)\} &= E\left\{E\left(\sum_{i=1}^{N} X_i|N\right)\right\} \\
&= E\left\{\sum_{i=1}^{N} E\left(X_i|N\right)\right\} = E\left\{\sum_{i=1}^{N} E\left(X_i\right)\right\} \\
&= E\left(N\mu_1\right) = \mu_1\nu
\end{aligned}
$$

So, expected total = (expected number) $\times$ (expected size of each)

- Variance of S

$$
\begin{aligned}
\mathrm{Var}(S) &= E\{\mathrm{Var}(S|N)\} + \mathrm{Var}\{E(S|N)\} \\
&= E\left\{\mathrm{Var}\left(\sum_{i=1}^{N} X_i|N\right)\right\} + \mathrm{Var}\left(N\mu_1\right) = E\left\{\sum_{i=1}^{N} \mathrm{Var}\left(X_i|N\right)\right\} + \mu_1^2\tau^2 \\
&= E\left\{\sum_{i=1}^{N} \mathrm{Var}\left(X_i\right)\right\} + \mu_1^2\tau^2 = E\left\{N\left(\mu_2 - \mu_1^2\right)\right\} + \mu_1^2\tau^2 \\
&= \nu\left(\mu_2 - \mu_1^2\right) + \mu_1^2\tau^2 = \nu\mu_2 + \mu_1^2\left(\tau^2 - \nu\right)
\end{aligned}
$$

Collective Risk Model

- Moment generating function of S

$$
\begin{aligned}
m_S(t) = E\left(e^{tS}\right) = E\left\{E\left(e^{tS}|N\right)\right\} &= E\left[E\left\{\exp\left(t\sum_{i=1}^{N} X_i\right)|N\right\}\right] \\
&= E\left[E\left\{\prod_{i=1}^{N}\exp\left(tX_i\right)|N\right\}\right] \\
&= E\left[\prod_{i=1}^{N} E\left\{\exp\left(tX_i\right)|N\right\}\right] \\
&= E\left[\prod_{i=1}^{N} E\left\{\exp\left(tX_i\right)\right\}\right] \\
&= E\left\{\prod_{i=1}^{N} m_X(t)\right\} \\
&= E\left[\left\{m_X(t)\right\}^N\right] \\
&= E\left(\exp\left[N\ln\left\{m_X(t)\right\}\right]\right) \\
&= m_N\left[\ln\left\{m_X(t)\right\}\right]
\end{aligned}
$$

Compound Poisson Distribution

- $N \sim \text{Poi}(\lambda)$: $p_N(n;\lambda) = \Pr_\lambda(N=n) = \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, 2, 3, \ldots$
- So,

$$\nu = E(N) = \lambda$$
$$\tau^2 = \text{Var}(N) = \lambda$$
$$m_N(t) = \exp\left\{\lambda\left(e^t - 1\right)\right\}$$

- Thus,

$$E(S) = \lambda\mu_1$$
$$\text{Var}(S) = \lambda\mu_2 + \mu_1^2(\lambda - \lambda) = \lambda\mu_2$$
$$\text{Skew}(S) = E\left[\{S - E(S)\}^3\right] = \lambda\mu_3$$
$$m_S(t) = \exp\left(\lambda\left[e^{\ln\{m_X(t)\}} - 1\right]\right) = \exp\left[\lambda\left\{m_X(t) - 1\right\}\right]$$

- Notation: $S \sim CompPois\left\{\lambda, F_X(x)\right\}$ or $S \sim CompPois\left\{\lambda, m_X(t)\right\}$
- Property: If $S_1, \ldots, S_K$ are independent, $S_i \sim CompPois\left\{\lambda_i, F_i(x)\right\}$, then

$$S = \sum_{i=1}^{K} S_i \sim \text{CompPois}\{\Lambda, F(x)\}$$

where $\Lambda = \sum_{i=1}^{K} \lambda_i, F(x) = \Lambda^{-1} \sum_{i=1}^{K} \lambda_i F_i(x)$

Compound Distributions and Reinsurance

- Definitions
  $S_Y = \sum_{i=1}^{N} Y_i$
  $S_Z = \sum_{i=1}^{N} Z_i$
- Suppose $S_X = \sum_{i=1}^{N} X_i \sim CompDist\{\theta, F(x)\}$, where

$$\theta = \lambda \text{ if } CompDist = CompPois;$$
$$\theta = (m, q) \text{ if } CompDist = CompBinomial;$$
$$\theta = (k, q) \text{ if } CompDist = CompNeqBin.$$

  [NOTE: Recall θ -part of parameters deals with N only, while reinsurance generally deals with X_i's]

- Then, $S_Y \sim CompDist\,\{\theta, F_Y(y)\}$ and $S_Z \sim CompDist\,\{\theta, F_Z(z)\}$ where $F_Y(y) = \Pr\{Y_i \leq y\}\,; F_Z(z) = \Pr\{Z_i \leq z\}$

Outline

Question 1

(Compound Poisson Distribution) Let S_1 and S_2 be independent random variables having compound Poisson distributions with parameters $\lambda_1 = 3$ and $F_1(x)$ and $\lambda_2 = 1$ and $F_2(x)$, respectively. Suppose that the support of both $F_1(x)$ and $F_2(x)$ is the set {1,2,3,4,5} and the CDF s are given in the following table:

x	1	2	3	4	5
$F_1(x)$	0.1	0.3	0.3	0.7	1.0
$F_2(x)$	0.2	0.4	0.6	0.8	1.0

Note the *support* of a random variable, and therefore its CDF, is the set of its possible outcomes.

(a) What is the distribution of $S = S_1 + S_2$? Find $\mathbb{E}S$, $\mathbb{V}S$ and the mgf of S.

Solution 1

Solution 1

Using Theorem 4.1, we know that S has a compound Poisson distribution with parameters $\Lambda = \lambda_1 + \lambda_2 = 4$ and $F(x) = \frac{3}{4}F_1(x) + \frac{1}{4}F_2(x)$, so that $F(x)$ is given by the table:

x	1	2	3	4	5
$F(x)$	0.125	0.325	0.375	0.725	1.0

From this information, we can calculate the *pmf* of a random variable with a *CDF* of $F(x)$ as $p(x) = F(x) - F(x-1)$

x	1	2	3	4	5
$p(x)$	0.125	0.2	0.05	0.35	0.275

So, the mean and raw second moment of a random variable X with *pmf* $p(x)$ are:

$$\mu_1 = \mathbb{E}X = 1(0.125) + 2(0.2) + 3(0.05) + 4(0.35) + 5(0.275) = 3.45$$
$$\mu_2 = \mathbb{E}X^2 = 1^2(0.125) + 2^2(0.2) + 3^2(0.05) + 4^2(0.35) + 5^2(0.275) = 13.85$$

Therefore, we have $\mathbb{E}S = \Lambda\mu_1 = 4(3.45) = 13.8$ and $\mathrm{V}S = \Lambda\mu_2 = 4(13.85) = 55.4$. Finally, we know that the mgf of S is given by $m_S(t) = \exp\left[\Lambda\left\{m_X(t) - 1\right\}\right]$ where $m_X(t) = \mathbb{E}e^{tX}$ $= 0.125e^t + 0.2e^{2t} + 0.05e^{3t} + 0.35e^{4t} + 0.275e^{5t}$. So,

$$m_S(t) = \exp\left(0.5e^t + 0.8e^{2t} + 0.2e^{3t} + 1.4e^{4t} + 1.1e^{5t} - 4\right)$$

Question 1

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Solution: since the minimum value of the support of both $F_1(x)$ and $F_2(x)$ is 1, it must be the case that $S \geq N$. Therefore, $S \leq 3$ implies $N \leq 3$. hence, we can see that:

$$\begin{aligned}
\mathbb{P}(S \leq 3) &= \sum_{n=0}^{\infty} \mathbb{P}(S \leq 3|N=n)\mathbb{P}(N=n) \\
&= \sum_{n=0}^{3} \mathbb{P}(S \leq 3|N=n)\frac{e^{-4}4^n}{n!} \\
&= \mathbb{P}(S=0|N=0)e^{-4} + \sum_{n=1}^{3} \mathbb{P}\left(\sum_{i=1}^{n} X_i \leq 3|N=n\right)\frac{e^{-4}4^n}{n!} \\
&= e^{-4} + 4\mathbb{P}\left(X_1 \leq 3\right)e^{-4} + 8\mathbb{P}\left(X_1 + X_2 \leq 3\right)e^{-4} + \frac{32}{3}\mathbb{P}\left(X_1 + X_2 + X_3 \leq 3\right)e^{-4} \\
&= \frac{1}{3}e^{-4}\left\{3 + 12F(3) + 24F^{*(2)}(3) + 32F^{*(3)}(3)\right\}
\end{aligned}$$

Now, clearly $F(3) = 0.125 + 0.2 + 0.05 = 0.375$ and $F^{*(3)}(3) = \mathbb{P}\left(X_1 + X_2 + X_3 \leq 3\right) = \mathbb{P}\left(X_1 = 1, X_2 = 1, X_3 = 1\right) = (0.125)^3$ (since $X_i \geq 1$). Finally, we have:

$$\begin{aligned}
F^{*(2)}(3) &= \sum_{n=0}^{\infty} F(3-n)p(n) = \sum_{n=1}^{2} F(3-n)p(n) = F(2)p(1) + F(1)p(2) \\
&= (0.125 + 0.2)(0.125) + (0.125)(0.2) = 0.065625
\end{aligned}$$

Therefore,

$$\mathbb{P}(S \leq 3) = \frac{1}{3}e^{-4}\left\{3 + 12(0.375) + 24(0.065625) + 32(0.125)^3\right\} = 0.0558$$

Question 2

(Aggregate Model and Reinsurance) Let $S = \sum_{i=1}^{N} X_i$ be the aggregate claim amount for an insurance portfolio, and assume that N has a Poisson distribution with parameter λ and is independent of the i.i.d. claim amounts, X_i, which have a Pareto distribution with parameters $\alpha > 1$ and δ. Suppose that there is a fixed administrative charge of an amount a for every claim which must be handled by the company.
(a) What is the expected total cost (claims and administrative) for the insurance company?

Pareto Distribution

$$f_X(x;\alpha,\delta) = \frac{\alpha\delta^\alpha}{(x+\delta)^{\alpha+1}} \qquad F_X(x;\alpha,\delta) = 1 - \frac{\delta^\alpha}{(x+\delta)^\alpha}$$
$$E_{\alpha,\delta}(X) = \frac{\delta}{\alpha-1}, \quad \alpha > 1; \quad E_{\alpha,\delta}\left(X^2\right) = \frac{2\delta^2}{(\alpha-1)(\alpha-2)}, \quad \alpha > 2$$

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Solution: We expect to have λ claims, and each of these claims has an expected amount (provided $\alpha>1$) of $\delta(\alpha-1)^{-1}$ (since the claims are Pareto distributed) plus an administrative charge of size a. Thus, the expected total cost is just $\lambda\left\{a+\delta(\alpha-1)^{-1}\right\}$.

Question 2

(b) Suppose that the insurance company introduces a policy excess of size d. If we assume that any claim amount under the excess level goes unreported (and thus does not incur any administration charges), how much does the insurance company expect to save by the introduction of the excess?

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Solution: Let $Y_i = (X_i - d)\, I_{(X_i > d)}$ be the amount of the i th claim for which the insurer is liable. Then, using integration by parts and assuming that $\alpha > 1$:

$$\begin{aligned}
\mathrm{E}Y_i &= \int_d^\infty (x-d) f_X(x) dx = \int_d^\infty (x-d) \left\{ \frac{\alpha\delta^\alpha}{(\delta+x)^{\alpha+1}} \right\} dx \\
&= \int_d^\infty \frac{x\alpha\delta^\alpha}{(\delta+x)^{\alpha+1}} dx - d\int_d^\infty \frac{\alpha\delta^\alpha}{(\delta+x)^{\alpha+1}} dx \\
&= -\left.\frac{x\delta^\alpha}{(\delta+x)^\alpha}\right|_{x=d}^\infty + \int_d^\infty \frac{\delta^\alpha}{(\delta+x)^\alpha} dx - \left. d\frac{-\delta^\alpha}{(\delta+x)^\alpha}\right|_{x=d}^\infty \\
&= \frac{d\delta^\alpha}{(\delta+d)^\alpha} - \left.\frac{\delta^\alpha}{(\alpha-1)(\delta+x)^{\alpha-1}}\right|_{x=d}^\infty - \frac{d\delta^\alpha}{(\delta+d)^\alpha} \\
&= \frac{\delta^\alpha}{(\alpha-1)(\delta+d)^{\alpha-1}} \\
&= \left(\frac{\delta}{\alpha-1}\right)\left(\frac{\delta}{\delta+d}\right)^{\alpha-1}
\end{aligned}$$

Solution 2

So, the expected total cost to the insurer under the policy excess scheme is

$$\mathbb{E}S_Y = \lambda \mathbb{E}Y_i + a\lambda \mathbb{P}\left(X_i > d\right) = \lambda \left(\frac{\delta}{\alpha - 1}\right)\left(\frac{\delta}{\delta + d}\right)^{\alpha - 1} + a\lambda \left(\frac{\delta}{\delta + d}\right)^{\alpha}$$

[NOTE: We might also have calculated this expectation as:

$$\mathbb{E}S_Y = \lambda \mathbb{P}\left(X_i > d\right)\mathbb{E}\left(Y_i|Y_i > 0\right) + a\lambda \mathbb{P}\left(X_i > d\right) = \lambda P\left(X_i > d\right)\left\{\mathbb{E}\left(Y_i|Y_i > 0\right) + a\right\}$$

where $\lambda \mathbb{P}\left(X_i > d\right)$ is the expected number of claims that the insurer sees and $\mathbb{E}\left(Y_i|Y_i > 0\right)$ is the expected size of claims that the insurer sees. Of course, it is easily shown that $\mathbb{E}\left(Y_i|Y_i > 0\right) = \mathrm{E}Y_i/\mathrm{P}\left(Y_i > 0\right)$, which demonstrates the equivalence of this method with the one given above.]

Therefore, the expected savings is:

$$\lambda\left(\mathbb{E}X_i + a\right) - \mathbb{E}S_Y = \lambda\left(a + \frac{\delta}{\alpha - 1}\right) - \lambda\left(\frac{\delta}{\alpha - 1}\right)\left(\frac{\delta}{\delta + d}\right)^{\alpha - 1} - a\lambda\left(\frac{\delta}{\delta + d}\right)^{\alpha}$$

Question 3

(Moment Generating Function) Let $S = \sum_{i=1}^{N} X_i$ be a random sum with N having mgf $m_N(t)$. Also, assume that the $X_{i'}$ s are independent of N and are i.i.d. such that $\mathbb{P}\left(X_i = 0\right) = p > 0$ (i.e., the $X_{i'}$ s can take the value zero). Show that $\mathbb{P}(S = 0) = m_N(\ln p)$.

Solution 3

Solution 3

Since the $X_{i'}$ s are non-negative, the only way for $\sum_{i=1}^{n} X_i$ to be equal to zero is for all the X_i's$(i = 1, \ldots, n)$ to be 0. Therefore

$$
\begin{aligned}
\mathbb{P}(S=0) &= \mathbb{P}\left(\sum_{i=1}^{N} X_i = 0\right) = \sum_{n=0}^{\infty} \mathbb{P}\left(\sum_{i=1}^{N} X_i = 0 | N = n\right) p_N(n) \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} X_i = 0 | N = n\right) p_N(n) = \sum_{n=0}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} X_i = 0\right) p_N(n) \\
&= \mathbb{P}(0=0) p_N(0) + \sum_{n=1}^{\infty} \mathbb{P}\left(X_1 = 0, \ldots, X_n = 0\right) p_N(n) \\
&= p_N(0) + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \mathbb{P}\left(X_i = 0\right) p_N(n) \\
&= p^0 p_N(0) + \sum_{n=1}^{\infty} p^n p_N(n) \\
&= \sum_{n=0}^{\infty} p^n p_N(n) = \mathbb{E}\left(p^N\right) \\
&= \mathbb{E}\left(e^{N \ln p}\right) \\
&= m_N(\ln p)
\end{aligned}
$$