STAT3035/8035
Tutorial 10

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Outline

Ruin Theory

- Examine "long-term" behaviour of aggregate claim amounts
- Use stochastic jump processes to model behaviour
  - $N(t)$ = number of claims made by time t
  - $S(t)$ = total amount of claims made by time t
  - Let $\{X_i\}_{i=1,2,\ldots}$ be iid with pdf $f_X(x)$ and mgf $m_X(r)$
  - Then, the aggregate claims process is:

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

  with $S(t) = 0$ whenever $N(t) = 0$

- For fixed t_0, $S(t_0)$ behaves as in Section 4 - Aggregate Claims Modelling
- Assume premium income accrues linearly through time at rate c
- Define the surplus process:

$$U(t) = U_0 + ct - S(t)$$

  where U_0 is the initial surplus (i.e., initial capital investment)

- $U(t)$ is the "current net worth of the portfolio"

The Probability of Ruin

- The probability of ultimate ruin:

$$\psi\left(U_0\right) = \Pr\{U(\tau) < 0 \text{ at some time } 0 \leq \tau < \infty\}$$

- The probability of ruin by time t

$$\psi\left(U_0, t\right) = \Pr\{U(\tau) < 0 \text{ at some time } 0 \leq \tau \leq t\}$$

Specification of $N(t)$ and $S(t)$ process

- $N(t)$ process
  - Homogeneous Poisson Process (HPP) with rate λ
  - $V_i = T_i - T_{i-1}$ are iid exponential with mean parameter $\theta = \lambda^{-1}$
  - $N(t+h) - N(t)|N(t) \sim \text{Pois}(\lambda h)$. This implies:
    - $N(0) = 0$
    - for any $t_1 < t_2$,

$$N(t_2) - N(t_1) \sim Poisson\{\lambda(t_2 - t_1)\}$$

    - for any $t_1 < t_2 \leq t_3 < t_4$

$$N(t_4) - N(t_3) \perp N(t_2) - N(t_1)$$

- $S(t)$ process
  - We choose $N(t)$ to be HPP with rate λ
  - We choose X_i 's to be iid with mgf $m_X(r)$ [or pdf $f_X(x)$]
  - Assume X_i 's and $N(t)$ are independent
  - $S(t)$ is a Compound Poisson Process. Name comes from fact:

$$S(t_0) \sim CompPois\{\lambda t_0, m_X(r)\}$$

Specification of claim rate c

- Require $c > \lambda\mu_1$ so that $ct_0 > \lambda\mu_1 t_0 = \lambda t_0 E(X_i) = E\{S(t_0)\}$
  - Premiums accrued by t_0 greater than expected total claims
  - If not, eventual ruin is inevitable
- Define premium loading, θ, so that:

$$c = (1+\theta)\lambda\mu_1 \quad \Longrightarrow \quad \theta = \frac{c}{\lambda\mu_1} - 1$$

- Note $c > \lambda\mu_1$ implies $\theta > 0$

Calculating Ruin Probabilities

- We have compound Poisson surplus process

$$U(t) = U_0 + ct - S(t) = U_0 + (1+\theta)\lambda\mu_1 t - S(t)$$

where:

  - $U_0 =$ Initial Surplus
  - $\theta =$ Premium Loading (greater than 0)
  - $\lambda =$ Claim Rate [i.e., parameter of the HPP $N(t)$]
  - $\mu_1 = E\left(X_i\right) =$ Expected Claim Size
  - $S(t) =$ Compound Poisson Aggregate Claims Process

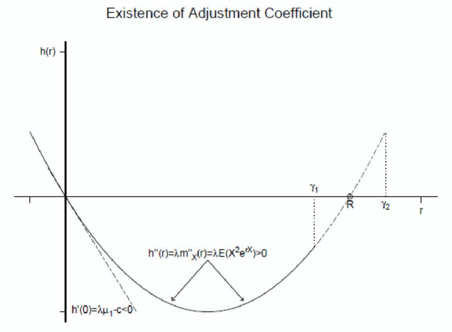
- Define T, the ruin time or time of ruin: $T = \min\{t > 0 : U(t) < 0\}$
- If there is a positive root, $R > 0$, for the function $h(r) = \lambda\left\{m_X(r) - 1\right\} - rc$, so that $\lambda\left\{m_X(R) - 1\right\} - Rc = h(R) = 0$, then,

$$\psi\left(U_0\right) = \frac{e^{-RU_0}}{E\left\{e^{-RU(T)} \mid T < \infty\right\}} \approx e^{-RU_0}$$

- Theorem only valid when $R > 0$ exists
  - need $m_X(r)$ to exist
  - need $h(r) = 0$ to have solution

Adjustment Coefficients

- Define $\gamma = \sup\{r : m_X(r) < \infty\}$, largest value where $m_X(r)$ exists
- $h(r) = \lambda\{m_X(r) - 1\} - rc$ shows:
    - $h(0) = 0$
    - $h'(0) = \lambda m'_X(0) - c = \lambda\mu_1 - c = -\theta\lambda\mu_1 < 0$
    - $h''(r) = \lambda m''_X(r) = \lambda E\left(X_i^2 e^{rX_i}\right) > 0$
- So, if $h(\gamma) > 0$ (more precisely, $\lim_{r\uparrow\gamma} h(r) > 0$), R exists
- If R exists, it is unique
- R does not depend on λ directly:



$$\lambda m_X(R) = \lambda + cR \Longrightarrow \lambda m_X(R) = \lambda + (1+\theta)\lambda\mu_1 R$$
$$\Longrightarrow m_X(R) = 1 + (1+\theta)\mu_1 R$$

Outline

Question 1

Let $U(t) = U_0 + ct - S(t)$ be the surplus process for a certain insurance portfolio, where $S(t)$ is a Compound Poisson aggregate claims process with rate λ and a claim amount distribution with mean μ and variance σ^2.
(a) Use a normal approximation to estimate $\mathbb{P}\{U(t) < 0\}$.
(b) Take the limit of your answer in part (a) as $t \to \infty$. Discuss the implications of this result.

Solution 1

Solution 1

(a) Since for any given value of t, $S(t)$ has a compound Poisson distribution with rate parameter λt, we know that $\mathbb{E}\{S(t)\} = \lambda t\mu$ and $\mathbb{V}\{S(t)\} = \lambda t\left(\sigma^2 + \mu^2\right)$. Therefore,

$$\mathbb{P}\{U(t) < 0\} = \mathbb{P}\left\{U_0 + ct - S(t) < 0\right\} = \mathbb{P}\left\{S(t) > U_0 + ct\right\} \approx 1 - \Phi\left\{\frac{U_0 + ct - \lambda t\mu}{\sqrt{\lambda t\left(\sigma^2 + \mu^2\right)}}\right\}$$

$$= 1 - \Phi\left\{\frac{U_0 + (1+\theta)\lambda\mu t - \lambda t\mu}{\sqrt{\lambda t\left(\sigma^2 + \mu^2\right)}}\right\} = 1 - \Phi\left\{\frac{U_0 + \theta\lambda\mu t}{\sqrt{\lambda t\left(\sigma^2 + \mu^2\right)}}\right\}$$

(b) As $t \to \infty$, we have:

$$\frac{U_0 + \theta\lambda\mu t}{\sqrt{\lambda t\left(\sigma^2 + \mu^2\right)}} = \frac{U_0}{\sqrt{\lambda t\left(\sigma^2 + \mu^2\right)}} + \frac{\theta\lambda\mu t}{\sqrt{\lambda t\left(\sigma^2 + \mu^2\right)}} = \frac{U_0}{\sqrt{\lambda t\left(\sigma^2 + \mu^2\right)}} + \frac{\theta\lambda\mu\sqrt{t}}{\sqrt{\lambda\left(\sigma^2 + \mu^2\right)}} \to \infty$$

Thus, the limit of the answer to part (a) is clearly $1 - \Phi(\infty) = 1 - 1 = 0$. Note that this means that for large values of t, we would estimate the probability that the surplus is negative to be extremely small. In other words, if ruin is going to occur, it is more likely to happen sooner rather than later. Think about this fact in relation to using $\psi\left(U_0\right)$ as an estimate of $\psi\left(U_0, t\right)$

Question 2

Let $N(t)$ be a Poisson process with rate λ, which tracks the timing of claims made on a certain insurance portfolio. Also, let T_i denote the time of the i th claim. Show that $T_1 \sim \text{Exp}\left(\lambda^{-1}\right)$. Use the fact that $\{T_1 > t\} = \{N(t) = 0\}$. Moreover recall that if we let $V_2 = T_2 - T_1$, then the V_2 is independent of T_1 and is exponentially distributed with mean parameter λ^{-1}.

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Solution: We have:

$$\mathbb{P}\left(T_1 \leq t\right) = 1 - \mathbb{P}\left(T_1 > t\right) = 1 - \mathbb{P}\{N(t) = 0\} = 1 - e^{-\lambda t} = 1 - e^{-t/\lambda^{-1}}$$

which is precisely the CDF of the exponential distribution with mean parameter λ^{-1}.

Question 3

Let $U(t) = U_0 + ct - S(t)$ be a surplus process with premium loading θ and Compound Poisson aggregate claims process with rate parameter λ and individual claim distribution which is exponential with mean parameter β. Define ${}_k\psi\left(U_0\right)$ to be the probability that ruin occurs at the kth claim. Show that ${}_1\psi\left(U_0\right) = \frac{1}{2+\theta}e^{-U_0/\beta}$.

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Solution: Since ${}_1\psi(U_0)$ is the probability that the portfolio is ruined at the first claim, we have:

$$
\begin{aligned}
{}_1\psi(U_0) &= \mathbb{P}\{U_0 + (1+\theta)\lambda\beta T_1 - X_1 < 0\} \\
&= \int_0^\infty \mathbb{P}\{U_0 + (1+\theta)\lambda\beta T_1 - X_1 < 0 | T_1 = t\} f_{T_1}(t)dt \\
&= \int_0^\infty \mathbb{P}\{X_1 > U_0 + (1+\theta)\lambda\beta t | T_1 = t\} \lambda e^{-\lambda t}dt \\
&= \lambda \int_0^\infty \mathbb{P}\{X_1 > U_0 + (1+\theta)\lambda\beta t\} e^{-\lambda t}dt \\
&= \lambda \int_0^\infty e^{-\{U_0 + (1+\theta)\lambda\beta t\}/\beta} e^{-\lambda t}dt = \lambda e^{-U_0/\beta} \int_0^\infty e^{-(1+\theta)\lambda t} e^{-\lambda t}dt \\
&= \lambda e^{-U_0/\beta} \int_0^\infty e^{-(2+\theta)\lambda t}dt = \lambda e^{-U_0/\beta} \frac{1}{\lambda(2+\theta)} = \frac{1}{(2+\theta)}e^{-U_0/\beta}
\end{aligned}
$$

where we have used the fact that X_1 and T_1 are independent. Recall that one of the fundamental assumptions we must make is that the timing of claims is independent of the size of claims.

Question 4

Let $U(t) = U_0 + ct - S(t)$ be the surplus process for a risk portfolio having compound Poisson aggregate claims process, $S(t)$, with rate λ and individual claim amounts, X_i, which are Gamma distributed with shape parameter $\alpha = 2$ and unit scale parameter ($i.e., X_i \sim G(2, 1)$).

(a) Suppose that the premium loading is $\theta = 0.875$. Calculate the adjustment coefficient, R.

(b) Show that $\lim_{\theta \to 0} R = 0$. Interpret this result.

Solution 4

Solution 4

(a) The mgf of $G(2,1)$ is $m_X(r) = (1-r)^{-2}$ for $r < 1$. Also, the mean of $G(2,1)$ is $\mu_1 = 2$ Therefore, the adjustment coefficient is the positive solution to:

$$\begin{aligned}
(1-R)^{-2} = 1 + 2(1+\theta)R &\Longrightarrow 1 = (1-R)^2 + 2(1+\theta)R(1-R)^2 \\
&\Longrightarrow 0 = 2\theta R - (4\theta+3)R^2 + 2(1+\theta)R^3 \\
&\Longrightarrow 0 = 2\theta - (4\theta+3)R + 2(1+\theta)R^2 \\
&\Longrightarrow R = \frac{(4\theta+3) \pm \sqrt{8\theta+9}}{4\theta+4}
\end{aligned}$$

Now, since$(4\theta+3) + \sqrt{8\theta+9} > (4\theta+3) + 3 > 4\theta + 4$, we must have

$$R = \frac{(4\theta+3) - \sqrt{8\theta+9}}{4\theta+4}$$

since $\frac{(4\theta+3)+\sqrt{8\theta+9}}{4\theta+4} > 1$, and is therefore not an allowable value of R, the mgf only being defined for argument values less than unity. Thus, when $\theta = 0.875$, we have:
$R = \frac{6.5-\sqrt{16}}{7.5} = \frac{1}{3}$.
(b)

$$\lim_{\theta \to 0} R = \lim_{\theta \to 0} \frac{(4\theta+3) - \sqrt{8\theta+9}}{4\theta+4} = \frac{3-\sqrt{9}}{4} = 0$$

So, the “riskiness” is “largest” (recall that small values of the adjustment coefficient equate to “riskier” portfolios) when the premium loading is small.

Question 5

Suppose that $U(t) = U_0 + ct - S(t)$ is the surplus process for a certain risk portfolio, and that $S(t) = \sum_{i=1}^{N(t)} X_i$ is a compound Poisson aggregate claims process where $N(t)$ is a homogeneous Poisson process with rate λ and individual claim amounts, X_i, have mgf $m_X(r)$.

(a) Suppose we increase the rate at which claims are made by a factor k, and we also increase the rate at which premium in come is collected by the same factor (perhaps by increasing the number of policies in the portfolio). In other words, suppose that we examine a new surplus process $U_k(t) = U_0 + c_k t - S_k(t)$, where $c_k = kc,\ S_k(t) = \sum_{i=1}^{N_k(t)} X_i$, and $N_k(t)$ is a homogeneous Poisson process with rate $\lambda_k = k\lambda$. Calculate the adjustment coefficient of $U_k(t)$ in terms of R, the adjustment coefficient of the original surplus process $U(t)$. Interpret your result.

Question 5

Suppose that $U(t) = U_0 + ct - S(t)$ is the surplus process for a certain risk portfolio, and that $S(t) = \sum_{i=1}^{N(t)} X_i$ is a compound Poisson aggregate claims process where $N(t)$ is a homogeneous Poisson process with rate λ and individual claim amounts, X_i, have mgf $m_X(r)$.

(a) Suppose we increase the rate at which claims are made by a factor k, and we also increase the rate at which premium in come is collected by the same factor (perhaps by increasing the number of policies in the portfolio). In other words, suppose that we examine a new surplus process $U_k(t) = U_0 + c_k t - S_k(t)$, where $c_k = kc,\ S_k(t) = \sum_{i=1}^{N_k(t)} X_i$, and $N_k(t)$ is a homogeneous Poisson process with rate $\lambda_k = k\lambda$. Calculate the adjustment coefficient of $U_k(t)$ in terms of R, the adjustment coefficient of the original surplus process $U(t)$. Interpret your result.

Solution: Letting R_k be the adjustment coefficient for the portfolio with surplus process $U_k(t)$, we see that R_k satisfies:

$$\lambda_k m_X(R_k) = \lambda_k + c_k R_k \Longrightarrow k\lambda m_X(R_k) = k\lambda + kcR_k \Longrightarrow \lambda m_X(R_k) = \lambda + cR_k$$

which is precisely the defining relationship for R. Thus, by uniqueness, we have $R_k = R$. The interpretation of this result is that if we increase the rate of claims (and increase the premium rate accordingly) we do not change the "riskiness" of the portfolio. The reason for this is that changing the rate at which things occur only affects the "timing" of ruin, and not whether it ever occurs or not (i.e., if ruin would have occurred under the original structure, then it would occur sooner under the new structure, but if ruin would not have occurred under the original structure, then it would still not occur under the new one).

Question 5

(b) Alternatively, suppose we increase the size of claims by a factor k, and we also increase the rate at which premium income is collected by the same factor. In other words, suppose that we examine a new surplus process $U_k(t) = U_0 + c_k t - S_k(t)$, where $c_k = kc$, $S_k(t) = \sum_{i=1}^{N(t)} kX_i$. Calculate the adjustment coefficient of $U_k(t)$ in terms of R, the adjustment coefficient of the original surplus process $U(t)$. Interpret your result. In particular, discuss its relationship with the result of part a. You may find it useful to calculate $\mathbb{E}U_k(t)$ and $\mathbb{V}U_k(t)$ in each case and compare them to $\mathbb{E}U(t)$ and $\mathbb{V}U(t)$.

Solution 5

Solution 5

(b) Again letting R_k be the adjustment coefficient for the portfolio with surplus process $U_k(t)$, we see that R_k satisfies: $\lambda m_{kX}(R_k) = \lambda + kcR_k$, where $m_{kX}(r)$ is the mgf of the $kX_i's$. In other words,

$$m_{kX}(r) = E\left\{e^{r(kX)}\right\} = \mathbb{E}\left(e^{rkX}\right) = m_X(rk)$$

Therefore, R_k satisfies

$$\lambda m_X(kR_k) = \lambda + c(kR_k)$$

which is precisely the defining relationship for R, but with R replaced by kR_k. In other words, we have $R_k = k^{-1}R$. Note that for $k > 1$, we see that $R_k < R$, implying that the new structure is "riskier".
Note that under the new structure in part (a), we could calculate $\mathbb{E}U_k(t) = U_0 + kct - \mathbb{E}S_k(t) = U_0 + kct - k\lambda\mu_1 t$ and $\mathbb{V}U_k(t) = \mathbb{V}S_k(t) = k\lambda\mu_2 t$, and thus $\mathbb{E}U_k(t)$ and $\mathbb{V}U_k(t)$ have the same relative relationship as $\mathbb{E}U(t) = U_0 + ct - \lambda\mu_1 t$ and $VU(t) = \lambda\mu_2 t$.
By contrast, under the new structure in part (b), we have $\mathbb{E}U_k(t) = U_0 + kct - \mathbb{E}S_k(t) = U_0 + kct - k\lambda\mu_1 t$, which is the same as before, but $\mathbb{V}U_k(t) = \mathbb{V}S_k(t) = k^2\lambda\mu_2 t$ (since multiplying the claims by a factor k means that the second raw moment is multiplied by a factor k^2) so that the variability is now larger than before.