STAT3035/8035
Tutorial 4

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Slides can be downloaded from: https://qingyue-li.github.io/

Outline

1. Review
2. Questions

Ideas of mixture distribution

- Each policy has distribution in same family, $f(x;\theta)$
- However, i^{th} policy has $\theta = \theta_i$
- Distribution of θ_i 's in portfolio: $\theta \sim g(t;\eta)$
- Claim generation from portfolio perspective:
  Choose Random Policy $\rightarrow$ Random Claim Amount from Chosen Policy
  $[\theta_i \sim g(t;\eta)]$ $[X|\theta_i \sim f(x;\theta_i)]$
- "Portfolio-Wide" (Mixture Distribution) pdf:
  - Idea:
  $$\Pr(\text{Claim} = x) = \sum_i \Pr(\text{Claim} = x|\text{Policy } i) \Pr(\text{Policy } i)$$
  - Formally:
  $$f_X(x;\eta) = \int_\Theta f(x;t)g(t;\eta)dt$$
  where Θ is set of possible θ values; usually $(0,\infty)$.

Pareto and Negative Binomial distribution

- Pareto distribution
  - Claims for policy i follows exponential distribution with (mean) parameter θ_i
  - θ_i's distributed in portfolio according to the inverse Gamma distribution:

$$g(t;\alpha,\delta) = \frac{\delta^\alpha}{\Gamma(\alpha)} t^{-(\alpha+1)} \exp\left(-\frac{\delta}{t}\right)$$

  - Mixture distribution pdf:

$$f_X(x;\alpha,\delta) = \frac{\alpha\delta^\alpha}{(x+\delta)^{\alpha+1}}$$

- Negative Binomial distribution
  - Model for number of claims per policy
  - Number of claims from Policy i has Poisson distribution with rate λ_i
  - λ_i's distributed in portfolio according to the Gamma distribution: $\lambda \sim G(\alpha,\theta)$
  - Mixture Distribution pmf:

$$p_N(n;\alpha,\theta) = \frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)} \left(\frac{1}{1+\theta}\right)^\alpha \left(\frac{\theta}{1+\theta}\right)^n$$

Some useful formula in conditional distribution

- $E(X) = E(E(X|\theta))$
- $E\left(X^2\right) = E\left(E\left(X^2|\theta\right)\right)$

Outline

Question 1

(Beta-Binomial Model) Suppose that Q is a random variable with density function:

$$f_Q(q) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} q^{\alpha-1}(1-q)^{\beta-1}, \quad 0 \leq q \leq 1$$

The distributions with this density function belong to the Beta family.
Remark 1. Beta-Binomial Model is used when one investigates the probability of certain events.
(a) Find $\mathbb{E}Q$.

Solution 1

Solution 1

(a)

$$
\begin{aligned}
\mathbb{E}Q &= \int_0^1 q f(q) dq = \int_0^1 q \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} q^{\alpha-1}(1-q)^{\beta-1} dq \\
&= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} q^{\alpha+1-1}(1-q)^{\beta-1} dq \\
&= \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+1+\beta)} \int_0^1 \frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} q^{\alpha+1-1}(1-q)^{\beta-1} dq \\
&= \frac{\alpha}{\alpha+\beta}
\end{aligned}
$$

Question 1

(b) Suppose that $(X|Q)$ has a binomial distribution with parameters m and Q. Find the unconditional (mixture) density function of X.

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Solution: This is a typical mixing distribution question. To solve it we make use of the well-known law of total probability. For $x = 0, 1, \ldots, m$, we have

$$\begin{aligned}
\mathbb{P}(X = x) &= \int_0^1 \mathbb{P}(X = x|q) f(q) dq \\
&= \int_0^1 \binom{m}{x} q^x (1-q)^{(m-x)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} q^{\alpha-1} (1-q)^{\beta-1} dq \\
&= \int_0^1 \binom{m}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} q^{\alpha+x-1} (1-q)^{\beta+m-x-1} dq \\
&= \text{ Constant } \int_0^1 \frac{\Gamma(\alpha+m+\beta) q^{\alpha+x-1} (1-q)^{\beta+m-x-1}}{\Gamma(\alpha+x)\Gamma(\beta+m-x)} dq \\
&= \binom{m}{x} \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+x)\Gamma(\beta+m-x)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+m+\beta)}
\end{aligned}$$

Question 1

(c) Suppose that $m = 3, \alpha = 2$ and $\beta = 2$. Calculate the probability mass function of X you found in part b using these values. Compare this probability mass function to the binomial probability mass function with parameters $m = 3$ and $q = 0.5$. Why is this particular comparison sensible?

Question 1

(c) Suppose that $m = 3, \alpha = 2$ and $\beta = 2$. Calculate the probability mass function of X you found in part b using these values. Compare this probability mass function to the binomial probability mass function with parameters $m = 3$ and $q = 0.5$. Why is this particular comparison sensible?
Solution: For $m = 3, \alpha = \beta = 2$, we have:

$$p(x) = \mathbb{P}(X = x) = \frac{\Gamma(x+2)\Gamma(5-x)\Gamma(4)3!}{\Gamma(7)\Gamma(2)\Gamma(2)x!(3-x)!} = \frac{(x+1)!(4-x)!3!3!}{6!1!1!x!(3-x)!} = \frac{1}{20}(x+1)(4-x)$$

Thus, $p(0) = 0.2$, $p(1) = 0.3$, $p(2) = 0.3$ and $p(3) = 0.2$. For comparison, the probability mass function of a binomial distribution with parameters 3 and 0.5 is $p(0) = 0.125$, $p(1) = 0.375$, $p(2) = 0.375$ and $p(3) = 0.125$.

Model	$\mathbb{P}(X = 0)$	$\mathbb{P}(X = 1)$	$\mathbb{P}(X = 2)$	$\mathbb{P}(X = 3)$
Beta-Binomial	0.2	0.3	0.3	0.2
Binomial	0.125	0.375	0.375	0.125

Note that the binomial with parameters 3 and 0.5 has the same mean as the mixture probability mass function (which is a result of the fact that $\mathbb{E}Q = 0.5$ in this case, so that the given binomial is indeed an appropriate benchmark), but it has less spread (i.e., the probability of the extreme values is lower).

Question 2

2. (Poisson-Gamma Model) Recall that if $(N|\Lambda = \lambda)$ has a conditional Poisson distribution with rate parameter λ, and $\Lambda \sim G(\alpha, \theta)$, then N has an unconditional negative binomial distribution with probability mass function:

$$p_N(n) = \mathbb{P}(N = n) = \frac{\Gamma(\alpha + n)}{n!\Gamma(\alpha)} \left(\frac{1}{1+\theta}\right)^{\alpha} \left(\frac{\theta}{1+\theta}\right)^{n}$$

Remark 2. Another commonly used mixing model.

(a) Calculate $\mathbb{E}N$ and $\mathbb{V}N$

Solution 2

Solution 2

(a) We can use the law of the iterated expectation to immediately see that $\mathbb{E}N = \mathbb{E}[\mathbb{E}(N|\Lambda)] = \mathbb{E}(\Lambda) = \alpha\theta$, Similarly we have:

$$\mathbb{V}N = \mathbb{E}[\mathbb{V}(N|\Lambda)] + \mathbb{V}[\mathbb{E}(N|\Lambda)] = \mathbb{E}(\Lambda) + \mathbb{V}(\Lambda) = \alpha\theta + \alpha\theta^2 = \alpha\theta(1+\theta)$$

Note that these solutions are very easy. Alternatively, we can find the required quantities from the first principle as follows. However, it is much more involved.

$$\begin{aligned}
\mathbb{E}N &= \sum_{n=0}^{\infty} n\frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)}\left(\frac{1}{1+\theta}\right)^{\alpha}\left(\frac{\theta}{1+\theta}\right)^{n} \\
&= \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)}{(n-1)!\Gamma(\alpha)}\left(\frac{1}{1+\theta}\right)^{\alpha}\left(\frac{\theta}{1+\theta}\right)^{n} \\
&= \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m+1)}{m!\Gamma(\alpha)}\left(\frac{1}{1+\theta}\right)^{\alpha}\left(\frac{\theta}{1+\theta}\right)^{m+1} \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\left(\frac{1}{1+\theta}\right)^{-1}\left(\frac{\theta}{1+\theta}\right)\sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m+1)}{m!\Gamma(\alpha+1)}\left(\frac{1}{1+\theta}\right)^{\alpha+1}\left(\frac{\theta}{1+\theta}\right)^{m} \\
&= \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)}\theta = \alpha\theta
\end{aligned}$$

where the final summation is seen to be equal to one since it is the sum over the full range of the probability mass function of a negative binomial distribution with parameters $\alpha+1$ and $(1+\theta)^{-1}$.

Solution 2

(a) Similarly, we can cal culate $\mathbb{E}N^2$ as:

$$
\begin{aligned}
\mathbb{E}N^2 &= \mathbb{E}N + \mathbb{E}[N(N-1)] \\
&= \alpha\theta + \sum_{n=0}^{\infty} n(n-1)\frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)}\left(\frac{1}{1+\theta}\right)^{\alpha}\left(\frac{\theta}{1+\theta}\right)^{n} \\
&= \alpha\theta + \sum_{n=2}^{\infty} \frac{\Gamma(\alpha+n)}{(n-2)!\Gamma(\alpha)}\left(\frac{1}{1+\theta}\right)^{\alpha}\left(\frac{\theta}{1+\theta}\right)^{n} \\
&= \alpha\theta + \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m+2)}{m!\Gamma(\alpha)}\left(\frac{1}{1+\theta}\right)^{\alpha}\left(\frac{\theta}{1+\theta}\right)^{m+2} \\
&= \alpha\theta + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\left(\frac{1}{1+\theta}\right)^{-2}\left(\frac{\theta}{1+\theta}\right)^{2}\sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m+2)}{m!\Gamma(\alpha+2)}\left(\frac{1}{1+\theta}\right)^{\alpha+2}\left(\frac{\theta}{1+\theta}\right)^{m} \\
&= \alpha\theta + \left\{(\alpha+1)\alpha\theta^2\right\} = \alpha\theta + \alpha\theta^2 + \alpha^2\theta^2
\end{aligned}
$$

Therefore, $\mathbb{V}N = \alpha\theta + \alpha\theta^2 + \alpha^2\theta^2 - (\alpha\theta)^2 = \alpha\theta(1+\theta)$

Question 2

(b) Find the moment generating function of N by using the identity $(1-x)^{-\alpha} = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+i)}{i!\Gamma(\alpha)} x^i$, which is the Taylor-expansion of $f(x) = (1-x)^{-\alpha}$ about $x = 0$.

Question 2

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Solution: Ignore the Taylor-expansion for the moment. Instead we use the iterated laws. Recalling that the moment generating function of a Poisson random variable with parameter λ has the form $m(t) = \exp\left\{\lambda\left(e^t - 1\right)\right\}$ and the moment generating function of a Gamma random variable with shape parameter α and scale parameter θ has the form $m(t) = (1-\theta t)^{-\alpha}$, we have:

$$
\begin{aligned}
m_N(t) = E\left(e^{tN}\right) &= E\left\{E\left(e^{tN}|\Lambda\right)\right\} = E\left[\exp\left\{\Lambda\left(e^t - 1\right)\right\}\right] \\
&= m_\Lambda\left(e^t - 1\right) = \left\{1 - \theta\left(e^t - 1\right)\right\}^{-\alpha} = \left(1 + \theta - \theta e^t\right)^{-\alpha}
\end{aligned}
$$

Alternatively, using the given formula in the question, we have:

$$
\begin{aligned}
\mathbb{E}e^{tN} &= \sum_{n=0}^{\infty} e^{tn} \frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)} \left(\frac{1}{1+\theta}\right)^{\alpha} \left(\frac{\theta}{1+\theta}\right)^{n} = \left(\frac{1}{1+\theta}\right)^{\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)} \left(\frac{\theta e^t}{1+\theta}\right)^{n} \\
&= \left(\frac{1}{1+\theta}\right)^{\alpha} \left(1 - \frac{\theta e^t}{1+\theta}\right)^{-\alpha} = \left(1 + \theta - \theta e^t\right)^{-\alpha}
\end{aligned}
$$

Question 2

(c) Use the moment generating function from part (b) to verify the values of $\mathbb{E}N$ and $\mathbb{V}N$ calculated in part (a).

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(c) Use the moment generating function from part (b) to verify the values of $\mathbb{E}N$ and $\mathbb{V}N$ calculated in part (a).
Solution: Taking derivatives of $m_N(t)$ calculated in part (b) gives:

$$\begin{aligned} m_N'(t) &= -\alpha\left(1+\theta-\theta e^t\right)^{-\alpha-1}\left(-\theta e^t\right) = \alpha\theta e^t\left(1+\theta-\theta e^t\right)^{-\alpha-1} \\ m_N''(t) &= \alpha\theta e^t\left(1+\theta-\theta e^t\right)^{-\alpha-1} - (\alpha+1)\alpha\theta e^t\left(1+\theta-\theta e^t\right)^{-\alpha-2}\left(-\theta e^t\right) \\ &= \alpha\theta e^t\left(1+\theta-\theta e^t\right)^{-\alpha-1} + (\alpha+1)\alpha\theta^2 e^{2t}\left(1+\theta-\theta e^t\right)^{-\alpha-2} \end{aligned}$$

Therefore, $\mathbb{E}N = m_N'(0) = \alpha\theta$ and $\mathbb{E}N^2 = \alpha\theta + (\alpha+1)\alpha\theta^2$, so that $\mathbb{V}N = \alpha\theta + (\alpha+1)\alpha\theta^2 - (\alpha\theta)^2 = \alpha\theta + \alpha\theta^2 = \alpha\theta(1+\theta)$.