STAT3035/8035
Tutorial 11

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Outline

1. Review
2. Questions
3. Suggestions

Ruin Theory and Reinsurance

- Proportional reinsurance
  - Consider proportional reinsurance with retention proportion p
  - Assume insurer charges premium rate $c = (1+\theta)\lambda\mu_1$
  - Assume reinsurer charges premium loading ξ, so that reinsurer's premium rate is

$$c_r = (1+\xi)\lambda E\left(Z_i\right) = (1+\xi)\lambda(1-p)\mu_1$$

  - Net Premium Rate for insurer is:

$$c_n = \{(1+\theta)-(1+\xi)(1-p)\}\lambda\mu_1 = (\theta-\xi+p+p\xi)\lambda\mu_1 = \left(1+\theta_{p,n}\right)\lambda p\mu_1$$

    where $\theta_{p,n} = \xi + (\theta-\xi)p^{-1}$ is the Net Premium Loading
  - Assume $\xi \geq \theta$ (otherwise, $p = 0$ would eliminate risk but not profit)
  - [NOTE : Also, assume ξ does not depend on p]

- Optimal reinsurance level
  - Choose p to minimise "risk of ruin", as measured by R
  - So choose p to maximise R

An example of proportional reinsurance

- Suppose $U(t)$ is compound Poisson surplus process:
  Claim frequency λ, exponential claim amounts with mean δ
  Surplus process for insurer under proportional reinsurance:

$$U_p(t) = U_0 + c_n t - \sum_{i=1}^{N(t)} pX_i = U_0 + c_n t - \sum_{i=1}^{N(t)} Y_i$$

  Note that $U_1(t) = U(t), U_0(t) = U_0 - (\xi - \theta)\lambda\delta t$
- We require non-negative premium loadings. So, $U_0(t)$ is not allowable; Moreover:

$$\theta_{p,n} > 0 \Longrightarrow \xi + (\theta - \xi)p^{-1} > 0 \Longrightarrow p > 1 - \frac{\theta}{\xi}$$

- For $1 - (\theta/\xi) < p \leq 1$, $U_p(t)$ is compound Poisson surplus process:
  claim frequency λ; exponential claim amounts with mean $p\delta$
- Thus, the adjustment coefficient for $U_p(t)$ is

$$R_p = \frac{\theta_{p,n}}{p\delta\left(1 + \theta_{p,n}\right)} = \frac{\xi + (\theta - \xi)p^{-1}}{p\delta\left\{1 + \xi + (\theta - \xi)p^{-1}\right\}} = \frac{p\xi + (\theta - \xi)}{\delta\left\{p^2(1 + \xi) + p(\theta - \xi)\right\}}$$

- R_p is maximised (so "risk of ruin" is minimised) when

$$p = \frac{(\xi - \theta)}{\xi}\left(1 + \frac{1}{\sqrt{1 + \xi}}\right)$$

  provided this value is less than 1

Ruin Theory and Reinsurance

- Excess-of-loss reinsurance
  - Consider excess-of-loss reinsurance with retention level M
  - Assume insurer charges premium rate

$$c = (1+\theta)\lambda\mu_1 = (1+\theta)\lambda\left\{E\left(Y_i\right) + E\left(Z_i\right)\right\}$$

  - Assume reinsurer charges premium loading ξ, so that reinsurer's premium rate is $c_r = (1+\xi)\lambda E\left(Z_i\right)$
  - Net Premium Rate for insurer is:

$$\begin{aligned} c_n &= (1+\theta)\lambda\left\{E\left(Y_i\right) + E\left(Z_i\right)\right\} - (1+\xi)\lambda E\left(Z_i\right) \\ &= (1+\theta)\lambda E\left(Y_i\right) + (\theta-\xi)\lambda E\left(Z_i\right) \end{aligned}$$

  - Again, assume $\xi \geq \theta$ and ξ does not depend on M
- Optimal reinsurance level
  - Assuming compound Poisson surplus. Thus, insurer's adjustment coefficient, R, satisfies:

$$1 + \frac{c_n}{\lambda}R = m_Y(R) = m_X(R) - e^{RM}\int_0^\infty \left(e^{Ry} - 1\right) f_X(y+M)dy$$

  - Maximising over M requires iterative (computer-based) solution.
  - Only want M's for which net premium loading is positive:

$$c_n > \lambda E\left\{Y_i\right\} \quad \Longrightarrow \quad \frac{E\left(Y_i\right)}{E\left(X_i\right)} > 1 - \frac{\theta}{\xi}$$

Optimal Form of Reinsurance

- For reinsurance scheme defined by $Y_{1,i} = g_1(X_i)$ adjustment coefficient, R_1, satisfies

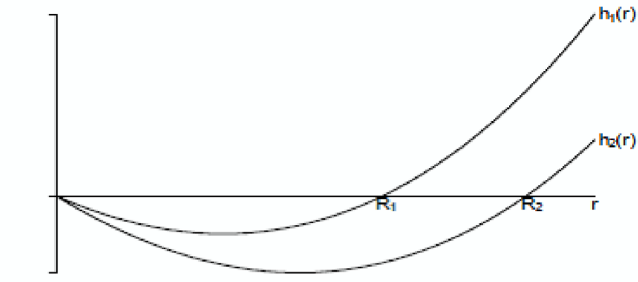
$$h_1(R_1) = \lambda m_{Y_1}(R_1) - \lambda - c_1 R_1 = 0$$

where $c_1 = (1+\theta_1)\lambda E(Y_{1,i})$

- For reinsurance scheme defined by $Y_{2,i} = g_2(X_i)$ adjustment coefficient, R_2, satisfies

$$h_2(R_2) = \lambda m_{Y_2}(R_2) - \lambda - c_2 R_2 = 0$$

where $c_2 = (1+\theta_2)\lambda E(Y_{2,i})$



- $h_1(R_2) > 0$ implies $R_2 > R_1$
- $h_2(R_1) < 0$ implies $R_1 < R_2$

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Question 1

Suppose that we have two independent risk portfolios. For each of the portfolios ($i = 1, 2$), the surplus process is $U_i(t) = U_{0,i} + c_i t - S_i(t)$, where $S_i(t) = \sum_{j=1}^{N_i(t)} X_{j,i}$ is a compound Poisson aggregate claims process with $N_i(t)$ being a Poisson process with rate λ_i and individual claim amounts with mgf $m_i(r)$ and $c_i = (1+\theta_i)\,\lambda_i\mu_i$ where $\mu_i = E\left(X_{j,i}\right)$ and θ_i is a premium loading. Suppose that we combine these two portfolios into a single, larger portfolio with surplus process $U(t) = U_0 + ct - S(t)$, where $U_0 = U_{0,1} + U_{0,2},\ c = c_1 + c_2$ and $S(t) = S_1(t) + S_2(t)$. Find the effective premium loading for the new portfolio, i.e., find the value θ such that $ct = (1+\theta)E\{S(t)\}$.

Solution 1

Solution 1

First, we note that $E\{S(t)\} = E\{S_1(t)\} + E\{S_2(t)\} = \lambda_1\mu_1 t + \lambda_2\mu_2 t$. Thus,

$$\begin{aligned} ct &= c_1 t + c_2 t = (1+\theta_1)\lambda_1\mu_1 t + (1+\theta_2)\lambda_2\mu_2 t \\ &= \lambda_1\mu_1 t + \lambda_2\mu_2 t + \theta_1\lambda_1\mu_1 t + \theta_2\lambda_2\mu_2 t \\ &= (\lambda_1\mu_1 t + \lambda_2\mu_2 t)\left(1 + \frac{\theta_1\lambda_1\mu_1 + \theta_2\lambda_2\mu_2}{\lambda_1\mu_1 + \lambda_2\mu_2}\right) \\ &= E\{S(t)\}\left(1 + \frac{\theta_1\lambda_1\mu_1 + \theta_2\lambda_2\mu_2}{\lambda_1\mu_1 + \lambda_2\mu_2}\right) \end{aligned}$$

Therefore, the effective premium loading, which is just $[ct/E\{S(t)\}] - 1$, is

$$\theta = \frac{\theta_1\lambda_1\mu_1 + \theta_2\lambda_2\mu_2}{\lambda_1\mu_1 + \lambda_2\mu_2}$$

Question 2

2. A portfolio has compound Poisson surplus process with claim rate λ and individual claim amount pdf:

$$f_X(x) = \frac{3}{2}e^{-3x} + \frac{7}{2}e^{-7x}$$

For what values of the premium loading factor θ does the adjustment coefficient, R, exist?

Question 2

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Solution: The mgf associate with $f_X(x)$ is

$$m_X(r) = \int_0^\infty e^{rx} f_X(x)dx = \frac{3}{2}\int_0^\infty e^{x(r-3)}dx + \frac{7}{2}\int_0^\infty e^{x(r-7)}dx = \frac{3}{2}(3-r)^{-1} + \frac{7}{2}(7-r)^{-1}$$

for $r < 3$ (since the first integral is infinite for $r \geq 3$ and the second integral is infinite for $r \geq 7$). Thus, $\gamma = 3$ in this case, and we have $\lim_{r\to 3} m_X(r) = \infty$. Therefore, the adjustment coefficient, R, will exist for any value of $\theta > 0$, since $\lim_{r\to 3} m_X(r) > 1 + 3(1+\theta)\mu_1$ (it is not hard to show that $\mu_1 = 5/21$ in this case).

Question 3

A certain insurance portfolio has a compound Poisson surplus process with claim rate λ and individual claim amounts with first two moments μ_1 and μ_2. Assume that the insurer charges premiums at a rate $c = (1+\theta)\lambda\mu_1$, where θ is the premium loading. Recall that the value

$$\hat{R} = \frac{2\theta\mu_1}{\mu_2}$$

is a reasonable approximation to the adjustment coefficient R. The insurer is considering proportional reinsurance with retention proportion p and reinsurer's premium loading is ϕ.
(a) Find the net premium rate, c_n, as well as the appropriate approximation $\hat{R}_p$ to the adjustment coefficient net of reinsurance.

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is a reasonable approximation to the adjustment coefficient R. The insurer is considering proportional reinsurance with retention proportion p and reinsurer's premium loading is ϕ.
(a) Find the net premium rate, c_n, as well as the appropriate approximation $\hat{R}_p$ to the adjustment coefficient net of reinsurance.

Solution: Under proportional reinsurance, we know that the net premium rate is $c_n = (\theta - \phi + p + p\phi)\lambda\mu_1 = (1 + \theta_{p,n})\, \lambda p \mu_1$, where $\theta_{p,n} = \phi + (\theta - \phi)p^{-1}$. Thus, the approximate adjustment coefficient is:

$$\hat{R}_p = \frac{2\theta_{p,n} p \mu_1}{p^2 \mu_2} = \frac{2\mu_1\{\theta + \phi(p-1)\}}{p^2\mu_2}$$

since the claim amounts for which the insurer is liable now have first two moments $p\mu_1$ and $p^2\mu_2$

Question 3

(b) Find the value of p, in terms of θ and ϕ, which maximises the value of $\hat{R}_p$.
(c) Discuss the implications of this result with regard to changes in ϕ. In particular, discuss the cases where $\phi > 2\theta$ and $\phi \leq \theta$.

Question 3

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Solution:
(b) Differentiating shows that

$$\frac{d}{dp}\hat{R}_p = \frac{2\mu_1\phi}{p^2\mu_2} - \frac{4\mu_1\{\theta + \phi(p-1)\}}{p^3\mu_2}$$

Setting the derivative equal to zero shows that

$$2p\mu_1\phi - 4\mu_1\{\theta + \phi(p-1)\} = 0 \quad \Longrightarrow \quad p = 2\left(1 - \frac{\theta}{\phi}\right)$$

(c) Note that when $\phi \leq \theta$ the above formula would suggest $p \leq 0$, which means that in this case, the best strategy would be to choose $p = 0$ and "offload" all risk to the re-insurer. On the other hand, when $\phi > 2\theta$ the above formula would suggest $p > 1$, which means that the best strategy in this case would be to choose $p = 1$ and retain all the risk. For $\theta < \phi < 2\theta$, we see that as ϕ increases, so does the optimal value of p, which indicates that as it becomes more and more "expensive" to purchase reinsurance, we should "offload" less and less risk.

Question 4

Suppose that the aggregate claims process, $S(t)$, for a particular insurance portfolio is compound Poisson with rate parameter $\lambda = 100$ and an individual claim amount distribution having pdf:

$$f_X(x) = 0.2e^{-0.2(x-5)}, \quad x > 5$$

The premium charged by the insurer is calculated using a loading factor of 15% [i.e., $ct = 1.15E\{S(t)\}$]. Suppose that the insurer is considering excess-of-loss reinsurance for this portfolio, and the reinsurance company of choice charges premiums using a loading factor of 30%. For various choices of retention level, M, the table below shows the insurer's expected profit in one year (i.e., the total net premium income minus the expected total amount of claims for which the insurer is liable) as well as the insurer's adjustment coefficient (net of reinsurance), although some values have been lost and replaced by "$*$"s.

Retention Level M	Expected Annual Profit	Adjustment Coefficient
7.5	59.02	0.0227
8.75	$*$	0.0246
10	$*$	$*$
12.5	$*$	0.0248
15	$*$	$*$
20	$*$	0.0228
$*$	147.25	0.0220
50	$*$	0.0213
∞	$*$	0.0213

Question 4

(a) Calculate the insurer's premium rate, c.
(b) Find the reinsurer's permium rate, c_r, as a function of the retention level, M.

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Solution:
(a) The insurer's premium rate is $c=(1+\theta)\lambda\mu_1=(1.15)(100)\mu_1$. Now,

$$\begin{aligned}\mu_1=E\left(X_i\right)=\int_5^{\infty} 0.2 x e^{-0.2(x-5)} d x=0.2 \int_0^{\infty}(y+5) e^{-0.2 y} d y \\ =0.2\left[-5 e^{-0.2 y}(y+10)\right]_0^{\infty}=0.2[0+5(10)]=10\end{aligned}$$

Thus, $c=1150$.
(b) The reinsurer's premium rate is $c_r=(1+\xi)\lambda E\left(Z_i\right)$, where $Z_i=\left(X_i-M\right) I_{\left(X_i>M\right)}$ is the amount of the i th claim for which the reinsurer is liable. Now,

$$\begin{aligned}E\left(Z_i\right)=0.2 \int_M^{\infty}(x-M) e^{-0.2(x-5)} d x=0.2 \int_0^{\infty} y e^{-0.2(y+M-5)} d y \\ =0.2 e^{-0.2(M-5)} \int_0^{\infty} y e^{-0.2 y} d y=5 e^{-0.2(M-5)}\end{aligned}$$

Therefore, $c_r=(1.3)(100) E\left(Z_i\right)=650 e^{-0.2(M-5)}$.

Question 4

(c) Find the mgf of the Y_i's, the amount of claims for which the insurer is liable.

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Solution: The desired mgf is

$$
\begin{aligned}
m_Y(r) &= \int_5^M 0.2e^{rx}e^{-0.2(x-5)}dx + \int_M^\infty 0.2e^{rM}e^{-0.2(x-5)}dx \\
&= 0.2e\int_5^M e^{x(r-0.2)}dx + 0.2e^{rM+1}\int_M^\infty e^{-0.2x}dx \\
&= 0.2e\left[\frac{1}{r-0.2}e^{x(r-0.2)}\right]_5^M + 0.2e^{rM+1}\left[-\frac{1}{0.2}e^{-0.2x}\right]_M^\infty \\
&= 0.2e\left\{\frac{1}{r-0.2}e^{rM-0.2M} - \frac{1}{r-0.2}e^{5r-1}\right\} + 0.2e^{rM+1}\left\{(0.2)^{-1}e^{-0.2M}\right\}. \\
&= \frac{0.2}{r-0.2}e^{rM-0.2M+1} - \frac{0.2}{r-0.2}e^{5r} + e^{rM-0.2M+1} \\
&= \frac{0.2}{0.2-r}e^{5r} + \left(\frac{0.2}{r-0.2}+1\right)e^{rM-0.2M+1} \\
&= \frac{0.2}{0.2-r}e^{5r} + \frac{r}{r-0.2}e^{rM-0.2M+1}
\end{aligned}
$$

Question 4

(d) Write the defining equation for the insurer's adjustment coefficient net of reinsurance. Also, confirm that the two missing adjustment coefficient values in the table for $M = 10$ and $M = 15$ are $R = 0.0252$ and $R = 0.0241$, respectively.

Question 4

(d) Write the defining equation for the insurer's adjustment coefficient net of reinsurance. Also, confirm that the two missing adjustment coefficient values in the table for $M = 10$ and $M = 15$ are $R = 0.0252$ and $R = 0.0241$, respectively.

Solution: The defining equation for the insurer's adjustment coefficient net of reinsurance is $m_Y(R) - 1 - \frac{c_n}{\lambda} R = 0$, which is:

$$\frac{0.2}{0.2 - R} e^{5R} + \frac{R}{R - 0.2} e^{RM - 0.2M + 1} - 1 - 0.01R\left\{1150 - 650e^{-0.2(M-5)}\right\} = 0$$

where we have used the fact that $c_n = c - c_r = 1150 - 650e^{-0.2(M-5)}$, So, if $M = 10$, we can substitute $R = 0.0252$ into the above equation and get

$$\begin{aligned}\frac{0.2}{0.2 - 0.0252} e^{5(0.0252)} + \frac{0.0252}{0.0252 - 0.2} e^{10(0.0252) - 0.2(10) + 1} - 1 \\ - 0.01(0.0252)\left\{1150 - 650e^{-0.2(10-5)}\right\} \\ =& 1.1442e^{0.126} - 0.1442e^{-0.748} - 1 - 0.000252\left\{1150 - 650e^{-1}\right\} \\ =& 1.2978 - 0.0683 - 1 - 0.000252(910.8784) \\ =& 0.2295 - 0.2295 \\ =& 0\end{aligned}$$

Similarly, for $M = 15$ and $R = 0.0241$, a substitution shows the validity of the adjustment coefficient value.

Question 4

(e) Find the other missing table values. Discuss the implications of the table values. If the insurer were planning to take out reinsurance with $M = 8.75$, what would you advise?

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Solution: The expected annual profit is just $\rho = c_n - \lambda E(Y_i)$. Now, $E(Y_i) = E(X_i) - E(Z_i) = 10 - 5e^{-0.2(M-5)}$, so that the expected annual profit is:

$$\rho = 1150 - 650e^{-0.2(M-5)} - 100\left\{10 - 5e^{-0.2(M-5)}\right\} = 150 - 150e^{-0.2(M-5)}$$

Substitution of the values $M = 8.75, 10, 12.5, 15, 20, 50$ and ∞ into this equation yield the expected annual profits listed in the completed table given below. Moreover, when $\rho = 147.25$, we can find the associated retention level by solving the equation $147.25 = 150 - 150e^{-0.2(M-5)}$ which implies that $M = 25$.

Solution 4

Retention Level M	Expected Annual Profit	Adjustment Coefficient
7.5	59.02	0.0227
8.75	79.15	0.0246
10	94.82	0.0252
12.5	116.53	0.0248
15	129.70	0.0241
20	142.53	0.0228
25	147.25	0.0220
50	149.98	0.0213
∞	150.00	0.0213

The conclusions that can be drawn from this table are that reducing M (i.e., taking on greater reinsurance protection) decreases the expected annual profit. On the other hand, reducing M increases the adjustment coefficient (and thus reduces "risk") until $M = 10$, beyond which point the adjustment coefficient decreases. In other words, it makes no sense to take out reinsurance with $M = 8.75$ since we can increase expected profit and decrease "risk" by increasing the retention level to $M = 10$ (assuming that this is an available option from the reinsurer).

Outline

How to prepare for final exam

- Use the Class summary provided by Daning to review course materials - pay attention: there are non-examinable topics
- Review tutorial and assignment questions, especially the ones you got wrong. Make sure you understand every question
- Do the Sample exam questions under exam condition: limited time, no interruptions. Mark yourself and study the questions you find difficult
- There will be customized numbers in final exam - read the Customized Numbers Instruction and Example on Wattle, make sure you understand how to deal with them during the exam
- Familiarise yourself with the procedure of scanning and uploading answers - use the mock exam on Wattle to practice
- Leave enough time for scanning and uploading your answers - this could be time consuming, and late submission will not be accepted nor marked
- Lastly, thanks for watching and good luck with your exam!