

Exercise 1

Sheet 02 ML1

$$(a) P(x,y) = \lambda y e^{-\lambda x - \eta y}, \lambda, \eta > 0, x, y \geq 0$$

zu zeigen: x, y sind unabhängig

x, y independent $\Leftrightarrow P_{x,y}(x,y) = P_x(x) \cdot P_y(y)$

$$P_x(x) = \int_0^\infty p(x,y) dy$$

$$= \int_0^\infty \lambda y e^{-\lambda x - \eta y} dy$$

$$= \int_0^\infty \lambda e^{-\lambda x} \cdot \eta e^{-\eta y} dy$$

$$= \lambda e^{-\lambda x} \int_0^\infty \eta e^{-\eta y} dy$$

$$\star (-e^{-\eta y})' = -e^{-\eta y} \cdot (-\eta) = \eta e^{-\eta y}$$

$$\cancel{\int_0^\infty \eta e^{-\eta y} dy}$$

$$= \lambda e^{-\lambda x} \cdot \left[-e^{\eta y} \Big|_0^\infty \right]$$

$$= \lambda e^{-\lambda x} \cdot (-e^0 - (-e^\infty))$$

$$= \lambda e^{-\lambda x}$$

$$\text{analog dazu } P_y(y) = \int_0^\infty p(x,y) dx = \eta e^{-\eta y}$$

$$\text{because } P_{x,y}(x,y) = \lambda y \cdot e^{-\lambda x - \eta y} = \cancel{\lambda y} \lambda e^{-\lambda x} \cdot \eta e^{-\eta y} = P_x(x) \cdot P_y(y)$$

$\Rightarrow x, y$ are independent

$$\cancel{\int_0^\infty \lambda y e^{-\lambda x - \eta y} dy} = \cancel{\lambda x - \eta y} = \cancel{(0, \infty)} \cancel{\frac{y}{\eta}} = \cancel{(0, \infty)} \cancel{\frac{1}{\eta}} = \cancel{(0, \infty)}$$

$$\cancel{x \frac{1}{\lambda} - y \frac{1}{\eta}} = \cancel{(0, \infty)} \cancel{\frac{1}{\lambda}} = \cancel{(0, \infty)}$$

$$\cancel{x \frac{1}{\lambda} + y \frac{1}{\eta}} = \cancel{\frac{16}{16}} = \cancel{\frac{16}{16}}$$

$$\text{such that } \cancel{\frac{1}{\lambda} + \frac{1}{\eta}} = 1 \Leftrightarrow 0 = \cancel{x \frac{1}{\lambda} + y \frac{1}{\eta}} - 1 \Leftrightarrow 0 = \cancel{\frac{16}{16}} - 1 \Leftrightarrow 0 = \cancel{\frac{16}{16}} - 1$$

$$\cancel{\frac{1}{\lambda} + \frac{1}{\eta}} = 1 \Leftrightarrow$$

Exercise 1.b)

$$L(\lambda, \eta) = \prod_{i=1}^N P(x_i, y_i) = \prod_{i=1}^N (\lambda \eta e^{-\lambda x_i - \eta y_i}) = (\lambda \eta)^N \cdot e^{-\sum_{i=1}^N (\lambda x_i + \eta y_i)}$$

$$= (\lambda \eta)^N \cdot e^{-\lambda \sum_{i=1}^N x_i} \cdot e^{-\eta \sum_{i=1}^N y_i}$$

$$l(\lambda, \eta) = \ln(L(\lambda, \eta)) = \ln((\lambda \eta)^N \cdot e^{-\lambda \sum_{i=1}^N x_i} \cdot e^{-\eta \sum_{i=1}^N y_i})$$

$$= N \ln(\lambda) + N \ln(\eta) - \lambda \sum_{i=1}^N x_i - \eta \sum_{i=1}^N y_i$$

$$\frac{\partial l}{\partial \lambda} = N \cdot \frac{1}{\lambda} - 1 \cdot \sum_{i=1}^N x_i$$

make Maximum-Likelihood-Schätzers equal to 0.

$$\frac{\partial l}{\partial \lambda} = 0 \Leftrightarrow N \cdot \frac{1}{\lambda} - \sum_{i=1}^N x_i = 0$$

$$\lambda = \frac{N}{\sum_{i=1}^N x_i}$$

Exercise 1.c)

$$\eta = \frac{1}{\lambda}, \quad P(x, y) = \lambda \cdot \eta \cdot e^{-\lambda x - \eta y} = e^{-\lambda x - \frac{1}{\lambda} y}, \quad x, y \geq 0$$

$$L(\lambda; D) = \prod_{i=1}^N P(x_i, y_i) = \prod_{i=1}^N e^{-\lambda x_i - \frac{1}{\lambda} y_i} = \exp(-\lambda \cdot \sum_{i=1}^N x_i - \frac{1}{\lambda} \sum_{i=1}^N y_i)$$

$$l(\lambda, \eta) = \ln(L(\lambda; D)) = -\lambda \sum_{i=1}^N x_i - \frac{1}{\lambda} \sum_{i=1}^N y_i$$

$$\frac{\partial l}{\partial \lambda} = -\sum_{i=1}^N x_i + \frac{1}{\lambda^2} \sum_{i=1}^N y_i$$

$$\text{make } \frac{\partial l}{\partial \lambda} = 0 \Leftrightarrow -\sum_{i=1}^N x_i + \frac{1}{\lambda^2} \sum_{i=1}^N y_i = 0 \Rightarrow \lambda^2 = \frac{\sum_{i=1}^N y_i}{\sum_{i=1}^N x_i}, \quad \lambda > 0$$

$$\Rightarrow \lambda = \sqrt{\frac{\sum_{i=1}^N y_i}{\sum_{i=1}^N x_i}}$$

exercise 1.d)

$$\eta = 1 - \lambda \quad , \quad P(X, Y) = \lambda \eta e^{-\lambda X - \eta y} = \lambda(1-\lambda) \cdot e^{-\lambda X - (1-\lambda)y}$$

$$L(\lambda; D) = \prod_{i=1}^N \lambda(1-\lambda) e^{-\lambda x_i - (1-\lambda)y_i}$$

$$= [\lambda(1-\lambda)]^N \cdot \exp(-\lambda \sum_{i=1}^N x_i - (1-\lambda) \sum_{i=1}^N y_i)$$

$$l(\lambda) = \ln(L(\lambda; D)) = N \cdot \ln(\lambda) + N \cdot \ln(1-\lambda) - \lambda \cdot \sum_{i=1}^N x_i - (1-\lambda) \sum_{i=1}^N y_i$$

$$\frac{\partial l}{\partial \lambda} = \frac{N}{\lambda} - \frac{N}{1-\lambda} - \sum_{i=1}^N x_i + \sum_{i=1}^N y_i = 0$$

$$\frac{N \cdot (1-\lambda) - N \lambda}{\lambda(1-\lambda)} = \sum_{i=1}^N x_i - \sum_{i=1}^N y_i$$

$$\sum_{i=1}^N x_i - \sum_{i=1}^N y_i = A \quad \Rightarrow \quad \frac{-2N\lambda + N}{\lambda(1-\lambda)} = A \quad \Rightarrow \quad A\lambda^2 - \lambda(A+2N) + N = 0$$

Because $(A+2N)^2 - 4AN = (A^2 + 4N^2) \geq 0 \Rightarrow$ the equation is solvable.

$$\lambda = \begin{cases} \frac{A+2N - \sqrt{A^2 + 4N^2}}{2A}, & A \neq 0 \\ \frac{1}{2}, & A = 0 \end{cases}, \quad A = \sum_{i=1}^N x_i - \sum_{i=1}^N y_i$$

$$\lambda = \begin{cases} \frac{\sum_{i=1}^N x_i - \sum_{i=1}^N y_i - \sqrt{(\sum_{i=1}^N x_i - \sum_{i=1}^N y_i)^2 + 4N^2}}{2(\sum_{i=1}^N x_i - \sum_{i=1}^N y_i)}, & \sum_{i=1}^N x_i \neq \sum_{i=1}^N y_i \\ \frac{1}{2}, & \sum_{i=1}^N x_i = \sum_{i=1}^N y_i \end{cases}$$

Exercise 2.a)

$$P(X|\theta) = \begin{cases} \theta & \text{if } x = \text{head} \\ 1-\theta & \text{if } x = \text{tail} \end{cases}$$

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$$\frac{1}{\theta^5} + \frac{n}{(1-\theta)^2} = \frac{1}{\theta^5} \quad (d)$$

a) $P(D|\theta) = \theta^5 \cdot (1-\theta)^2$

$$L(\theta) = \theta^5 \cdot (1-\theta)^2$$

$$\frac{1}{\theta^5} + \frac{n}{(1-\theta)^2} = \frac{1}{\theta^5} \quad \text{no}$$

b) ~~$L(\theta)$~~ $L(\theta) = 5\ln(\theta) + 2\ln(1-\theta)$

$$\begin{aligned} L'(\theta) &= \frac{5}{\theta} - \frac{2}{1-\theta} = 0 \Rightarrow \theta = \frac{5}{7} \text{ or d.o. No root} \\ \frac{5}{\theta} &= \frac{2}{1-\theta} \quad 5(1-\theta) = 2\theta \quad , \quad 7\theta = 5 \quad , \quad \theta = \frac{5}{7} \quad , \quad \frac{n}{\theta} = 5 \quad \text{do} \end{aligned}$$

$$P(X_8 = \text{head}, X_9 = \text{head} | \theta) = \left(\frac{5}{7}\right)^2 = \frac{25}{49} \approx 0.5102 = \frac{1}{\theta^2}$$

c)

$$\left\{ \frac{1}{\theta}, \frac{n}{\theta} \right\} \text{ min} \geq \frac{1}{\theta}$$

$$\sum_{i=1}^n \frac{1}{\theta} = n \frac{1}{\theta}, \quad \frac{\partial L}{\partial \theta} + n \frac{1}{\theta} \frac{n}{\theta} = \frac{n^2}{\theta} \quad , \quad \frac{1}{\theta} + \frac{n}{\theta} = \frac{1}{\theta} \quad (d)$$

$$(n \frac{1}{\theta}, n \frac{1}{\theta}) \times \text{min} \geq n \frac{1}{\theta} \geq (n \frac{1}{\theta}, n \frac{1}{\theta}) \text{ min} : \text{using w}$$

$$n \left(\frac{1}{\theta} + n \frac{1}{\theta} \frac{n}{\theta} \right) \cdot \frac{1}{\theta} = n \frac{1}{\theta} \Rightarrow \frac{\partial L}{\partial \theta} + n \frac{1}{\theta} \frac{n}{\theta} = \frac{n^2}{\theta}$$

$$\left(n \frac{1}{\theta}, n \frac{1}{\theta} \frac{n}{\theta} \right) \frac{1}{1 + \frac{n}{\theta}} = n \frac{1}{\theta} \Rightarrow \frac{1}{1 + \frac{n}{\theta}} = \frac{1}{\theta}$$

$$n \frac{1}{\theta} \cdot \frac{1}{\frac{1}{\theta} + \frac{n}{\theta}} + n \frac{1}{\theta} \frac{n}{\theta} \cdot \frac{1}{\frac{1}{\theta} + \frac{n}{\theta}} = n \frac{1}{\theta} + n \frac{n}{\theta} \cdot \frac{1}{\theta} = n \frac{1}{\theta} + n \frac{n}{\theta} = n \frac{1+n}{\theta}$$

$$[1, \theta] \ni w \mapsto n \frac{1}{\theta} (w-1) + n \frac{n}{\theta} w = n \frac{1}{\theta} \quad \frac{1}{\frac{1}{\theta} + \frac{n}{\theta}} = w-1 \quad \frac{n}{\frac{1}{\theta} + \frac{n}{\theta}} = w$$

$$(n \frac{1}{\theta}, n \frac{1}{\theta}) \times \text{min} \geq n \frac{1}{\theta} \geq (n \frac{1}{\theta}, n \frac{1}{\theta}) \text{ min} \quad \text{E}$$

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Exercise 3

$$\begin{aligned} \text{Joint} &= X \text{ f.f. } \theta \\ \text{Joint} &= X \text{ f.f. } (\theta-1) \end{aligned}$$

(a,b) 2019/20

$$a) \frac{1}{6_n^2} = \frac{n}{6^2} + \frac{1}{6_0^2}$$

$$\cancel{6_n^2} \quad 6_n^2 = \frac{1}{\frac{n}{6^2} + \frac{1}{6_0^2}}$$

$$(6-1) \cdot 2\theta = (6)J$$

because for all $a, b > 0$, it holds $\frac{1}{a+b} \leq \frac{1}{a}$ 且 $\frac{1}{a+b} \leq \frac{1}{b}$

$$\text{let } a = \frac{n}{6^2}, b = \frac{1}{6_0^2}, \theta = 6, \theta_0 = (6-1)2, \frac{2}{\theta-1} = \frac{2}{\theta}$$

$$6_n^2 = \frac{1}{a+b} \leq \frac{1}{a} = \frac{6^2}{n}, \left(\frac{2}{\theta} \right)^2 \leq \frac{1}{b} = 6_0^2$$

$$\Rightarrow 6_n^2 \leq \min \left\{ \frac{6^2}{n}, 6_0^2 \right\}$$

$$b) \frac{1}{6_n^2} = \frac{n}{6^2} + \frac{1}{6_0^2}, \frac{u_n}{6_n^2} = \frac{n}{6^2} \hat{u}_n + \frac{m_0}{6_0^2}, \hat{u}_n = \frac{1}{n} \sum_{k=1}^n x_k$$

$$\text{to prove: } \min(\hat{u}_n, m_0) \leq u_n \leq \max(\hat{u}_n, m_0)$$

$$\frac{u_n}{6_n^2} = \frac{n}{6^2} \hat{u}_n + \frac{m_0}{6_0^2} \Rightarrow u_n = 6_n^2 \cdot \left(\frac{n}{6^2} \hat{u}_n + \frac{m_0}{6_0^2} \right)$$

$$6_n^2 = \frac{1}{\frac{n}{6^2} + \frac{1}{6_0^2}} \Rightarrow u_n = \frac{1}{\frac{n}{6^2} + \frac{1}{6_0^2}} \left(\frac{n}{6^2} \hat{u}_n + \frac{m_0}{6_0^2} \right)$$

$$= \frac{\frac{n}{6^2}}{\frac{n}{6^2} + \frac{1}{6_0^2}} \hat{u}_n + \frac{\frac{1}{6_0^2}}{\frac{n}{6^2} + \frac{1}{6_0^2}} m_0$$

$$w = \frac{\frac{n}{6^2}}{\frac{n}{6^2} + \frac{1}{6_0^2}}, 1-w = \frac{\frac{1}{6_0^2}}{\frac{n}{6^2} + \frac{1}{6_0^2}} \Rightarrow u_n = w \hat{u}_n + (1-w) m_0, w \in [0,1]$$

$$\Rightarrow \min(\hat{u}_n, m_0) \leq u_n \leq \max(\hat{u}_n, m_0)$$

□