Exercise 1

Sheet 02 ML I

a)

$$p(x,y) = \lambda\eta e^{-\lambda x - \eta y}, \quad \lambda, \eta > 0, \; x, y \geq 0$$

zu zeigen: x, y sind unabhängig

x, y independent $\Leftrightarrow$ $P_{x,y}(x,y) = P_x(x) \cdot P_y(y)$

$$P_x(x) = \int_0^\infty p(x,y)\,dy$$

$$= \int_0^\infty \lambda\eta e^{-\lambda x - \eta y}\,dy$$

$$= \int_0^\infty \lambda e^{-\lambda x} \cdot \eta e^{-\eta y}\,dy$$

$$= \lambda e^{-\lambda x} \int_0^\infty \eta e^{-\eta y}\,dy \qquad * \; (-e^{-\eta y})' = -e^{-\eta y} \cdot (-\eta) = \eta e^{-\eta y}$$

$$= \lambda e^{-\lambda x} \cdot \left[-e^{\eta y} \Big|_0^\infty \right]$$

$$= \lambda e^{-\lambda x} \cdot (- \cdot (0 - 1))$$

$$= \lambda e^{-\lambda x}$$

analog dazu $P_y(y) = \int_0^\infty p(x,y)\,dx = \eta e^{-\eta y}$

because $P_{x,y}(x,y) = \lambda\eta \cdot e^{-\lambda x - \eta y} = \lambda e^{-\lambda x} \cdot \eta e^{-\eta y} = P_x(x) \cdot P_y(y)$

$\Rightarrow$ x, y are independent

Exercises will be graded from exercises 03 onwards xercise 1.b)

$$L(\lambda,\eta) = \prod_{i=1}^{N} P(x_i,y_i) = \prod_{i=1}^{N} (\lambda\eta e^{-\lambda x_i - \eta y_i}) = (\lambda\eta)^N \cdot e^{-\sum_{i=1}^{N}(\lambda x_i + \eta y_i)}$$

$$= (\lambda\eta)^N \cdot e^{-\lambda \sum_{i=1}^{N} x_i} \cdot e^{-\eta \sum_{i=1}^{N} y_i}$$

$$l(\lambda,\eta) = \ln(L(\lambda,\eta)) = \ln\left((\lambda\eta)^N \cdot e^{-\lambda \sum_{i=1}^{N} x_i} \cdot e^{-\eta \sum_{i=1}^{N} y_i}\right)$$

$$= N\ln(\lambda) + N\ln(\eta) - \lambda \sum_{i=1}^{N} x_i - \eta \sum_{i=1}^{N} y_i$$

$$\frac{\partial l}{\partial \lambda} = N \cdot \frac{1}{\lambda} - 1 \cdot \sum_{i=1}^{N} x_i$$

make Maximum-Likelihood-Schätzers equals to 0.

$$\frac{\partial l}{\partial \lambda} = 0 \iff N \cdot \frac{1}{\lambda} - \sum_{i=1}^{N} x_i = 0$$

$$\lambda = \frac{N}{\sum_{i=1}^{N} x_i}$$

Exercise 1.c)

$$\eta = \frac{1}{\lambda}, \quad P(x,y) = \lambda \cdot \eta \cdot e^{-\lambda x - \eta y} = e^{-\lambda x - \frac{1}{\lambda} y}, \quad x,y \geq 0$$

$$L(\lambda;D) = \prod_{i=1}^{N} P(x_i,y_i) = \prod_{i=1}^{N} e^{-\lambda x - \frac{1}{\lambda} y} = \exp\left(-\lambda \cdot \sum_{i=1}^{N} x_i - \frac{1}{\lambda}\sum_{i=1}^{N} y_i\right)$$

$$l(\lambda,\eta) = \ln(L(\lambda;D)) = -\lambda \sum_{i=1}^{N} x_i - \frac{1}{\lambda} \sum_{i=1}^{N} y_i$$

$$\frac{\partial l}{\partial \lambda} = -\sum_{i=1}^{N} x_i + \frac{1}{\lambda^2} \sum_{i=1}^{N} y_i$$

make $\frac{\partial l}{\partial \lambda} = 0 \iff -\sum_{i=1}^{N} x_i + \frac{1}{\lambda^2}\sum_{i=1}^{N} y_i = 0 \Rightarrow \lambda^2 = \frac{\sum_{i=1}^{N} y_i}{\sum_{i=1}^{N} x_i}$, $\lambda > 0$

$$\Rightarrow \lambda = \sqrt{\frac{\sum_{i=1}^{N} y_i}{\sum_{i=1}^{N} x_i}}$$

exercise 1.d)

$$\eta = 1-\lambda \quad , \quad P(x,y) = \lambda\eta e^{-\lambda x - \eta y} = \lambda(1-\lambda)\cdot e^{-\lambda x - (1-\lambda) y}$$

$$L(\lambda ; D) = \prod_{i=1}^{N} \lambda(1-\lambda) e^{-\lambda x_i - (1-\lambda) y_i}$$

$$= [\lambda\cdot(1-\lambda)]^N \cdot \exp\left(-\lambda \sum_{i=1}^{N} x_i - (1-\lambda)\sum_{i=1}^{N} y_i\right)$$

$$l(\lambda) = \ln(L(\lambda;D)) = N\cdot\ln(\lambda) + N\cdot\ln(1-\lambda) - \lambda\cdot\sum_{i=1}^{N} x_i - (1-\lambda)\sum_{i=1}^{N} y_i$$

$$\frac{\partial l}{\partial \lambda} = \frac{N}{\lambda} - \frac{N}{1-\lambda} - \sum_{i=1}^{N} x_i + \sum_{i=1}^{N} y_i = 0$$

$$\frac{N\cdot(1-\lambda) - N\lambda}{\lambda(1-\lambda)} = \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i$$

$$\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i = A \quad \Rightarrow \quad \frac{-2N\lambda + N}{\lambda(1-\lambda)} = A \quad \Rightarrow \quad A\lambda^2 - \lambda(A+2N) + N = 0$$

Because $(A+2N)^2 - 4AN = (A^2+4N^2) \geq 0$ $\Rightarrow$ the equation is solvable.

$$\lambda = \begin{cases} \dfrac{A+2N - \sqrt{A^2+4N^2}}{2A} & , A \neq 0 \\[2ex] \dfrac{1}{2} & , A = 0 \end{cases} \quad , A = \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i$$

$$\lambda = \begin{cases} \dfrac{\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i - \sqrt{\left(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i\right)^2 + 4N^2}}{2\left(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i\right)} & , \sum_{i=1}^{N} x_i \neq \sum_{i=1}^{N} y_i \\[3ex] \dfrac{1}{2} & , \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i \end{cases}$$

Exercise 2.a)

$$P(X|\theta) = \begin{cases} \theta & \text{, if } x = \text{head} \\ 1-\theta & \text{, if } x = \text{tail} \end{cases}$$

a) $\cancel{P(D|\theta) = \theta^5 \cdot (1-\theta)^2}$

$L(\theta) = \theta^5 \cdot (1-\theta)^2$

b) $l(\theta) = 5\ln(\theta) + 2\ln(1-\theta)$

$l'(\theta) = \frac{5}{\theta} - \frac{2}{1-\theta} = 0 \Rightarrow \theta = \frac{5}{7}$

$\frac{5}{\theta} = \frac{2}{1-\theta}$ $\quad 5(1-\theta) = 2\theta$, $7\theta = 5$, $\theta = \frac{5}{7}$

$P(X_8 = \text{head}, X_9 = \text{head} \mid \theta) = \left(\frac{5}{7}\right)^2 = \frac{25}{49} \approx 0.5102$

c)

Exercise 3

a) $$\frac{1}{\sigma_n^2} = \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}$$

$$\sigma_n^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}$$

because for all $a, b > 0$, it holds $\frac{1}{a+b} \le \frac{1}{a}$ 且 $\frac{1}{a+b} \le \frac{1}{b}$

let $a = \frac{n}{\sigma^2}$, $b = \frac{1}{\sigma_0^2}$

$$\sigma_n^2 = \frac{1}{a+b} \le \frac{1}{a} = \frac{\sigma^2}{n}, \quad \sigma_n^2 \le \frac{1}{b} = \sigma_0^2$$

$$\Rightarrow \sigma_n^2 \le \min\left\{\frac{\sigma^2}{n}, \sigma_0^2\right\}$$

b) $$\frac{1}{\sigma_n^2} = \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}, \quad \frac{\mu_n}{\sigma_n^2} = \frac{n}{\sigma^2}\hat{\mu}_n + \frac{\mu_0}{\sigma_0^2}, \quad \hat{\mu}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$$

to prove: $\min(\hat{\mu}_n, \mu_0) \le \mu_n \le \max(\hat{\mu}_n, \mu_0)$

$$\frac{\mu_n}{\sigma_n^2} = \frac{n}{\sigma^2}\hat{\mu}_n + \frac{\mu_0}{\sigma_0^2} \Rightarrow \mu_n = \sigma_n^2 \cdot \left(\frac{n}{\sigma^2}\hat{\mu}_n + \frac{\mu_0}{\sigma_0^2}\right)$$

$$\sigma_n^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}} \Rightarrow \mu_n = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}\left(\frac{n}{\sigma^2}\hat{\mu}_n + \frac{\mu_0}{\sigma_0^2}\right)$$

$$= \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}\hat{\mu}_n + \frac{\frac{1}{\sigma_0^2}}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}\mu_0$$

$$w = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}} \qquad 1-w = \frac{\frac{1}{\sigma_0^2}}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}} \qquad \Rightarrow \mu_n = w\hat{\mu}_n + (1-w)\mu_0, \quad w \in [0,1]$$

$$\Rightarrow \min(\hat{\mu}_n, \mu_0) \le \mu_n \le \max(\hat{\mu}_n, \mu_0)$$

□