Aufgabe 1.a) MLI

$P(\text{error}) = \int P(\text{error} \mid x)\, p(x)\, dx$

$P(\text{error} \mid x) = \min[P(w_1 \mid X), P(w_2 \mid X)]$

$P(A \mid B) = \frac{P(A) \cdot P(B \mid A)}{P(B)}$

zu zeigen: $P(\text{error}) \leq \int \frac{2}{\frac{1}{P(w_1 \mid X)} + \frac{1}{P(w_2 \mid X)}} \cdot p(x) \cdot dx$

$P_1 = P(w_1 \mid X)$, $P_2 = P(w_2 \mid X)$

z.z. $\min\{P_1, P_2\} \leq \frac{2}{\frac{1}{P_1} + \frac{1}{P_2}}$

$\frac{2}{\frac{1}{P_1} + \frac{1}{P_2}} = \frac{2}{\frac{P_1 + P_2}{P_1 \cdot P_2}} = \frac{2P_1 \cdot P_2}{P_1 + P_2}$

Because it is a two class classification problem. $\Rightarrow P_1 + P_2 = 1 \Rightarrow \frac{2 \cdot P_1 \cdot P_2}{P_1 + P_2} = 2 \cdot P_1 \cdot P_2$

$\min\{P_1, P_2\} \leq \frac{2}{\frac{1}{P_1} + \frac{1}{P_2}} \Leftrightarrow \min\{P, (1-P)\} \leq 2 \cdot P \cdot (1-P)$

Fall 1: $\min\{P, (1-P)\} = P \Rightarrow P \leq \frac{1}{2}$

z.z. $P \leq 2P \cdot (1-P)$

$P = 2P \cdot (1-P) = 2P - 2P^2$

$-2P^2 + P = 0$, $P(-2P+1) = 0 \Rightarrow P = 0$ or $P = \frac{1}{2}$

therefote: it holds: $\min\{P, (1-P)\} \leq 2 \cdot P \cdot (1-P)$

Fall 2: $\min\{P, (1-P)\} = 1-P \Rightarrow \frac{1}{2} \leq P \leq 1$

z.z. $1-P = 2 \cdot P(1-P)$ has solution

$1-P = 2P - 2P^2 \Rightarrow -2P^2 + P - 1 = 0 \Rightarrow 2P^2 - P + 1 = 0$, $a=2$, $b=-1$, $c=1$

because $b^2 - 4 \cdot a \cdot c = (1 - 4 \times 2 \times 1) < 0 \Rightarrow$ ~~es gibt k~~ there is no solution

Summary: when $P(w_1 \mid X) = P(w_2 \mid X) = 0.5$, it holds

$P(\text{error}) \leq \int \frac{2}{\frac{1}{P(w_1 \mid X)} + \frac{1}{P(w_2 \mid X)}} p(x) \cdot dx$

□

Aufgabe 1.b) ML I

$$P(x \mid w_1) = \frac{\pi^{-1}}{1+(x-\mu)^2} \quad , \quad P(x \mid w_2) = \frac{\pi^{-1}}{1+(x+\mu)^2}$$

zu zeigen $P(\text{error}) \leq \dfrac{2P(w_1)\cdot P(w_2)}{\sqrt{1+4\mu^2 P(w_1)\cdot P(w_2)}}$, given: $\displaystyle\int \frac{1}{ax^2+bx+c}\,dx = \frac{2\pi}{\sqrt{4ac-b^2}}$

$b^2 < 4ac$

$\pi_1 = P(w_1)$, $\pi_2 = P(w_2)$

in Aufgabe 1.a) already proved:

$$P(\text{error}) \leq \int \frac{2}{\frac{1}{P(w_1 \mid x)} + \frac{1}{P(w_2 \mid x)}} \cdot p(x)\,dx$$

$$= \int \frac{2}{\frac{1}{\pi_1 \cdot P(x \mid w_1)} + \frac{1}{\pi_2 \cdot P(x \mid w_2)}}\,dx \qquad * \; P(w_i \mid x) = \frac{\pi_i \cdot P(x \mid w_i)}{P(x)}$$

$$A(x) = \cancel{\pi_1 \cdot P(x)}\ \pi_1 \cdot P(x \mid w_1) = \frac{\pi_1/\pi}{1+(x-\mu)^2} \quad , \quad B(x) = \pi_2 P(x \mid w_2) = \frac{\pi_2/\pi}{1+(x+\mu)^2}$$

$$\frac{2}{\frac{1}{A}+\frac{1}{B}} = \frac{2A\cdot B}{A+B} = \frac{2(\pi_1/\pi)(\pi_2/\pi)}{\frac{\pi_1}{\pi}(1+(x+\mu)^2) + \frac{\pi_2}{\pi}(1+(x-\mu)^2)}$$

$$= \frac{2\pi_1\pi_2}{\pi} \cdot \frac{1}{x^2+2\mu(\pi_1-\pi_2)x+(1+\mu^2)}$$

zu zeigen

$$\Rightarrow P(\text{error}) \leq \frac{2\pi_1\pi_2}{\pi}\int_{-\infty}^{\infty} \frac{dx}{x^2+2\mu(\pi_1-\pi_2)x+(1+\mu^2)}$$

use the given formula: $\displaystyle\int \frac{1}{ax^2+bx+c}\,dx = \frac{2\pi}{\sqrt{4ac-b^2}}$, $b^2 < 4ac$

$a = 1$, $b = 2\mu(\pi_1-\pi_2)$, $c = (1+\mu^2)$

$(\pi_1-\pi_2)^2 = (\pi_1+\pi_2)^2 - 4\pi_1\pi_2 = 1 - 4\pi_1\pi_2$

$$4ac-b^2 = 4(1+\mu^2) - 4\mu^2(\pi_1-\pi_2)^2 = 4\cdot[1+\mu^2(1-(\pi_1-\pi_2)^2)] = 4\cdot[1+4\mu^2\pi_1\pi_2]$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\cdots\cdots} = \frac{2\pi}{2\sqrt{1+4\mu^2\pi_1\pi_2}} = \frac{\pi}{\sqrt{1+4\mu^2\pi_1\pi_2}}.$$

$$\Rightarrow P(\text{error}) \leq \frac{2\pi_1\pi_2}{\sqrt{1+4\mu^2\pi_1\pi_2}}$$

Aufgabe 1. c)

low-dimensional data:

1) Direct numerical integration . if densities are known
2) Tail-aware quadrature / transformations
3) Monte-Carlo (MC) integration

high dimensional data

1) MC with variance reduction
2) reduce dimension, then integrate
3) plug-in estimation + bracketing
4) classifier-based proxy

ML sheet 1

Aufgabe 2.a)

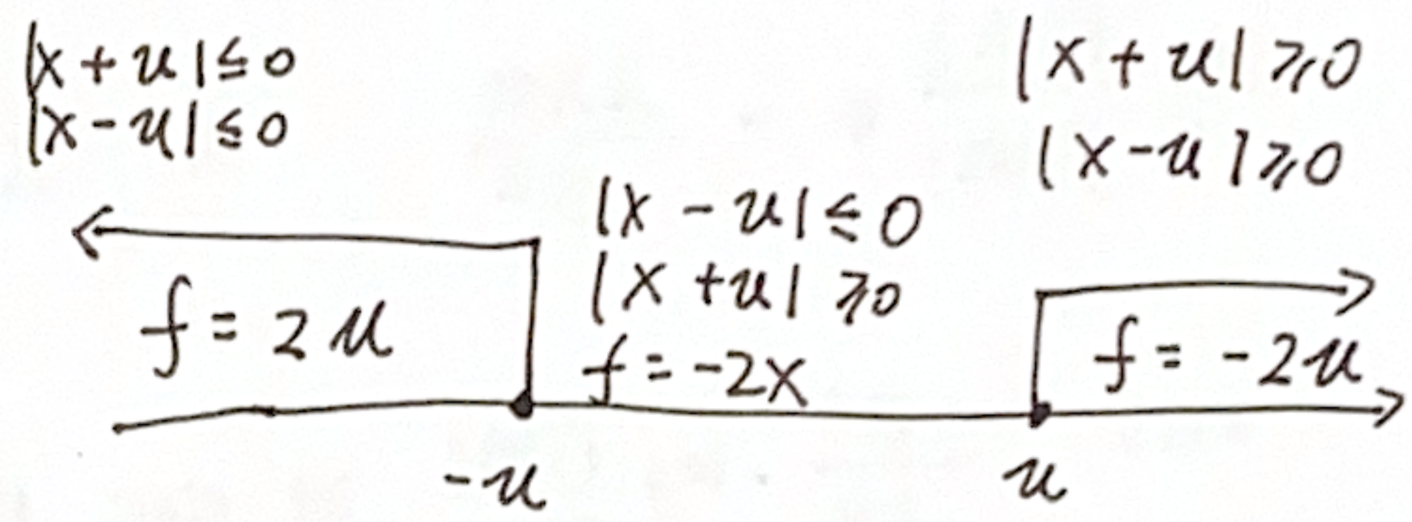
$$P(X|w_1) = \frac{1}{2\sigma}\exp\left(-\frac{|x-\mu|}{\sigma}\right) \quad P(X|w_2) = \frac{1}{2\sigma}\exp\left(-\frac{|x+\mu|}{\sigma}\right), \quad \mu, \sigma > 0$$

the optimal decision is always predict the first class, if and only if

$$P(X|w_1)\cdot P(w_1) \geq P(X|w_2)\cdot P(w_2), \text{ for all } X$$

$$\Rightarrow \frac{P(w_1)}{P(w_2)} \geq \frac{P(X|w_2)}{P(X|w_1)} = \frac{\frac{1}{2\sigma}\exp\left(-\frac{|x+\mu|}{\sigma}\right)}{\frac{1}{2\sigma}\exp\left(-\frac{|x-\mu|}{\sigma}\right)} = \exp\left(\frac{|x-\mu|-|x+\mu|}{\sigma}\right), \text{ for all } X$$

$$f = |x-\mu| - |x+\mu|$$



$$|x-\mu| \geq 0 \iff x-\mu \geq 0 \iff x \geq \mu$$

$$|x+\mu| \geq 0 \iff x+\mu \geq 0 \iff x \geq -\mu$$

$$f = \begin{cases} 2\mu, & x \leq -\mu \\ -2x, & -\mu < x < \mu \\ -2\mu, & x \geq \mu \end{cases}$$

$$\Rightarrow \sup_{x\in\mathbb{R}} f = 2\mu$$

$$\Rightarrow \frac{P(w_1)}{P(w_2)} \geq e^{2\mu/\sigma}$$

Because $P(w_1) + P(w_2) = 1 \Rightarrow P(w_2) = 1 - P(w_1)$

$$\Rightarrow P(w_1) \geq P(w_2)\cdot e^{2\mu/\sigma} \qquad \Rightarrow P(w_1) \geq (1-P(w_1))\cdot e^{2\mu/\sigma}$$

note $P(w_1)$ as P_1, $P_1 \geq (1-P_1)\cdot e^{2\mu/\sigma}$, $0 \geq e^{2\mu/\sigma} - P_1(e^{2\mu/\sigma}+1)$

$$\Rightarrow P_1 \geq \frac{1}{1+e^{-2\mu/\sigma}}$$

M μ: Mu

Δ δ: delta

σ: sigma

Ω, ω: omiga

Aufgabe 2b)

$$P(X|W_1) = \frac{1}{6\sqrt{2\pi}} \exp\left(-\frac{(X-u)^2}{26^2}\right) \quad P(X|W_2) = \frac{1}{6\sqrt{2\pi}} \exp\left(-\frac{(X+u)^2}{26^2}\right), \quad u, 6 > 0$$

to let optimal decision always predict the first class

$$\Leftrightarrow \quad P(W_1) \cdot P(X|W_1) \geq P(W_2) \cdot P(X|W_2)$$

$$\frac{P(W_1)}{P(W_2)} \geq \frac{P(X|W_2)}{P(X|W_1)} = \exp\left(\frac{(x-u)^2 - (X+u)^2}{26^2}\right) = \exp\left(\frac{\cancel{x^2} - 2xu + \cancel{u^2} - (\cancel{x^2} + 2xu + \cancel{u^2})}{26^2}\right)$$

$$= \exp\left(\frac{-4xu}{26^2}\right) = \exp\left(\frac{-2xu}{6^2}\right)$$

$$\log \frac{P(W_1)}{P(W_2)} \geq -\frac{2Xu}{6^2} \quad \Rightarrow \quad X \geq -\frac{6^2}{2u} \cdot \log \frac{P(W_1)}{P(W_2)} \quad ①$$

weil $-\frac{6^2}{2u} \cdot \log \frac{P(W_1)}{P(W_2)}$ ist eine Konstante

wenn $X \to -\infty$, kann die Formel ① nicht mehr gelten.

therefore, the formel can't hold.

$\square$