Market $\bar{S} = (S^0, S) = (S^0, S^1 \dots S^d)$

$S^i = (S^i_t)_{t\in[0,T]}$

trading strategies $\bar\theta = (\theta^0, \theta) = (\theta^0, \theta^1, \dots \theta^d)$

$\theta^i = (\theta^i_t)_{t\in[0,T]}$

self-financing $X^{\bar\theta}_t := \bar\theta_t \cdot \bar S_t := \sum_{i=0}^d \theta^i_t S^i_t$

$= \underbrace{\bar\theta_0 \cdot \bar S_0}_{\text{initial v.}} + \underbrace{\sum_{i=0}^d \int_0^t \theta^i_u \boxed{dS^i_u}}_{\text{trading gains.}}$

where

$\longrightarrow$ $dS^i_u = \mu^i_u\, du + \sum_{j=1}^n \sigma^{ij}_u\, dB^j_u$; $\bar S_0 \in \mathbb{R}_+^{d+1}$

$B = (B^1_t, \dots, B^n_t)_{t\in[0,T]}$ is a standard BM on $\mathbb{R}^n$.

Arbitrage $\bar\theta$ is an arbitrage opportunity if $X^{\bar\theta}_0 = 0$; $X^{\bar\theta}_T \geq 0$ $\mathbb{P}$-a.s ; $\mathbb{P}[X^{\bar\theta}_T > 0] > 0$.

standing assumption : $\bar S$ is arbitrage free

Price of G $\Pi^G = (\Pi^G_t)_{t\in[0,T]}$ is a no-arbitrage price of $G : \Omega \mapsto \mathbb{R}_+$ ($\mathcal{F}_T$- measurable r.v) if $(\bar S, \Pi^G)$ is arbitrage-free

Pricing by replication. if $\exists\ \bar\theta$ s.t

$\boxed{X^{\bar\theta}_T = G \quad \mathbb{P}\text{-a.s}}$

Law of one price. then $\boxed{\Pi^G_t := X^{\bar\theta}_t \quad , t\in[0,T]}$

is the unique no-arbitrage price of G.

1) Does a replicating strategy exist?
2) If it does, how to find it?

Black-Scholes Eq (d=1)

$\begin{cases} d S^0_t = S^0_t\, r_t\, dt \\ S^0_0 = 1 \end{cases}$ | Eg $r_t = r(t, S^0_t)$

$S^0_t = e^{\int_0^t r_u\, du}$ $t\in[0,T]$

$= e^{rt}$ (constant interest rate).

risky asset

$\begin{cases} dS_t = S_t \{\mu_t\, dt + \sigma_t\, dB_t\} \\ S_0 > 0 \end{cases}$

Goal: derive a self-financing and replicating strategy for $G : \Omega \to \mathbb{R}_+$

where $G = g(S_T)$

(e.g $g(x) = (x-K)^+$)

We make few "educated" success assumptions

- $\bar\theta$ that replicates G exists:

$$\underbrace{\theta^0_T S^0_T + \theta_T S_T}_{X^{\bar\theta}_T} = G(S_T) \quad \mathbb{P}\text{-a.s.}$$

- option payoff is a deterministic function of S_T
- dynamics of $\bar S$ are also deterministic functions of (t, S^0_t, S_t).

Then we suppose that $\Pi^G_t = \Pi(t, S_t)$. time t no arbitrage price of G.

$\Pi : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$

$\Pi(t, S_t) = \theta^0_t S^0_t + \theta_t S_t$

(Law of one price)

If we know $\Pi(\cdot,\cdot)$ and θ, then we know θ^0:

① $$\theta^0_t = \frac{\Pi(t, S_t) - \theta_t S_t}{S^0_t} \quad ; t \in [0, T]$$

$$\overset{\text{LHS}}{d\Pi(t, S_t)} = d(\theta^0_t S^0_t + \theta_t S_t) \overset{\text{RHS}}{=} d(X^{\bar\theta}_t)$$

RHS $dX^{\bar\theta}_t \overset{\text{self-financing}}{=} \theta^0_t dS^0_t + \theta_t dS_t$.

$\overset{①}{=} [\Pi(t, S_t) - \theta_t S_t] r_t dt + \theta_t dS_t$.

$= \{[\Pi(t, S_t) - \theta_t S_t] r_t + \theta_t S_t \mu_t\} dt + \theta_t S_t \sigma_t dB_t$

LHS $d\Pi(t, S_t) = \Pi_t(t, S_t) dt + \Pi_s(t, S_t) dS_t + \frac{1}{2}\Pi_{ss}(t, S_t) d[S, S]_t$.

$d[S,S]_t = (dS_t)^2$
$(dt)(dB) = (dt)^2 = 0$.
$(dB_t)^2 = dt$.

$= \{\Pi_t(t, S_t) + \Pi_s(t, S_t) S_t \mu_t + \frac{1}{2}\Pi_{ss}(t, S_t) S_t^2 \sigma_t^2\} dt + \Pi_s(t, S_t) S_t \sigma_t dB_t$

we need to find $\Pi(\cdot,\cdot)$ and θ_t.

$\{\Pi_t(t, S_t) + \Pi_s(t, S_t) S_t \mu_t + \frac{S_t^2 \sigma_t^2}{2}\Pi_{ss}(t, S_t)\} dt + \Pi_s(t, S_t) S_t \sigma_t dB_t$.

② $= \{[\Pi(t, S_t) - \theta_t S_t] r_t + \theta_t S_t \mu_t\} dt + \theta_t S_t \sigma_t dB_t$

Idea: equate "dB_t" & "dt"

"dB_t" ③ $\boxed{\theta_t = \Pi_s(t, S_t) \qquad t \in [0, T]}$.

Substitute ③ into ②

$[\Pi_t(t, S_t) + \underline{\Pi_s(t, S_t) S_t \mu_t} + \frac{S_t^2 \sigma_t^2}{2}\Pi_{ss}(t, S_t)] dt$

$= \{[\Pi(t, S_t) - \underline{\Pi_s(t, S_t) S_t}] r_t + \underline{\Pi_s(t, S_t) S_t \mu_t}\} dt$.

This gives us the black scholes PDE:

a) $\Pi_t(t,s) + \Pi_s(t,s) s r + \frac{s^2\sigma^2}{2}\Pi_{ss}(t,s) - \Pi(t,s) r = 0$ $\quad \forall (t,s,r) \in [0,T) \times \mathbb{R}_+ \times (-1,\infty)$

r_t is usually constant

b) Boundary condition: $\Pi(T,s) = G(s)$.

<u>proposition</u> Let Π be a smooth solution to the PDE "a)+b)". $\forall t \in [0,T]$
then

$$\theta_t := \Pi_s(t, S_t)$$
$$\theta_t^0 := \frac{\Pi(t,S_t) - \theta_t S_t}{S_t^0}$$

defines a self-financing trading strategy $\bar\theta = (\theta^0, \theta)$ that replicates a payoff G.

<u>proof:</u> By the def of $\bar\theta$, we have

$$X_t^{\bar\theta} = \boxed{\theta_t^0} S_t^0 + \theta_t S_t$$
$$= \Pi(t, S_t) - \theta_t S_t + \theta_t S_t$$
$$= \Pi(t, S_t). \qquad \forall t \in [0,T]$$

Choose $t=T$:

$$X_T^{\bar\theta} = \Pi(T, S_T) = G(S_T).$$

$\Rightarrow \bar\theta$ replicates G

<u>Self-financing property</u>:

i) Apply ito's formula to $\Pi(t,S_t)$

$d\Pi(t,S_t) = \{\ \}dt + \{\cdots\} dB_t$

ii) Plug in dS_t^0, dS_t and the forms of θ_t^0, θ_t.

iii) $d\Pi(t,S_t) = \theta_t^0 dS_t^0 + \theta_t dS_t + \{\cdots\} dt$

PDE=0.

But $d\Pi(t,S_t) = \boxed{dX_t^{\bar\theta}}$:

$$X_t^{\bar\theta} = X_0^{\bar\theta} + \int_0^t \theta_u^0 dS_u^0 + \int_0^t \theta_u dS_u.$$

self-financing.

<u>risk-neutral valuation</u>

Let Π be a smooth solution to "a)+b)" then

$$\frac{\Pi(t,S_t)}{S_t^0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{G(S_T)}{S_T^0} \,\middle|\, \mathcal{F}_t\right]$$

e.g. $t=0$,

time 0 price of G

$$\Pi(0, S_0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{G(S_T)}{S_T^0}\right]$$

- $S_0^0 = 1$
- $\mathcal{F}_0 = \{\emptyset, \Omega\}$

i) any r.v. X is independent of $\mathcal{F}_0$; $\sigma(X) \perp \mathcal{F}_0$

ii) $\mathbb{E}[Y|\mathcal{G}] = \mathbb{E}[Y]$, if Y is indep of $\mathcal{G}$.

What is $\mathbb{Q}$ here?

It is a prob. measure under which

$$\begin{cases} dS_t = S_t(r\,dt + \sigma\, dB_t^{\mathbb{Q}}) \\ S_0 > 0. \end{cases}$$

<u>under</u> $\mathbb{P}$. "original" dynamics of S:

$$\begin{cases} dS_t = S_t\{\mu_t dt + \sigma d B_t^{\mathbb{P}}\} \\ S_0 > 0. \end{cases}$$

question: is $\tilde B_t := B_t + t$, a BM under $\mathbb{P}$ provided B is a $\mathbb{P}$-Brownian motion?

3/9.

$dS_t = S_t(\mu_t dt + \sigma_t dB_t)$; $S_0 > 0$.
$dS_t^0 = S_t^0 (r\, dt)$; $S_0^0 = 1$

payoff at T : $G(S_T)$
pricing of G:
$\pi(t, S_t)$ where $\pi(t,s)$ solves certain PDE

$d(e^{-rt}\pi(t,S_t)) = \{\cdots\}dt + \{\cdots\}dB_t$
PDE $= 0$ + Boundary $\pi(T, S_T) = G(S_T)$.
so that $\pi(T, S) = G(S)$ $\forall S \in \mathbb{R}_+$.

a) we can replicate G:
$\Pi_t^G = X_t^{\bar{\theta}}$ $\forall t \in [0,T]$
no arbitrage price of G.
where $\bar{\theta}$ is a replicating portfolio.

b) $d\Pi_t^G = dX_t^{\bar{\theta}}$ use self-financing Condition

apply Itô's formula to $\Pi_t^G = \pi(t, S_t)$

· what if S pays dividends?
(there is a assumption process associated to $X^{\bar{\theta}}$?)

Dividends
Let $\bar{D} = (D^0, D) = (D^0, D^1, \cdots D^d)$ be adapted process. where for $i = 0 \cdots d$
$\forall i$ $D_0^i = 0$; D_t^i represents total dividends paid between time 0 and time t.

If D_t^i can be written as $\begin{cases} dD_t^i = \sigma_t^i dt \\ D_0^i = 0. \end{cases}$
then we called σ^i a dividend yield
In particular $D_t^i = \int_0^t \sigma_s^i ds$

Self financing without D

$X_t^{\bar{\theta}} = X_0^{\bar{\theta}} + \sum_{i=0}^d \left[\int_0^t \theta_u^i dS_u^i\right]$

Discrete time : $X_{t_k}^{\bar{\theta}} = X_0^{\bar{\theta}} + \sum_{i=0}^d \sum_{j=0}^{k-1} \theta_{t_{j+1}}^i (S_{t_{j+1}}^i - S_{t_j}^i)$
discrete at time t_j $[t_j, t_{j+1}]$.

Dividends between t_j and t_{j+1}:
$\theta_{t_{j+1}}^i (D_{t_{j+1}}^i - D_{t_j}^i)$
and hence

$X_t^{\bar{\theta}} = X_0^{\bar{\theta}} + \sum_{i=0}^d \sum_{j=0}^{k-1} \theta_{t_{j+1}}^i \{(S_{t_{j+1}}^i - S_{t_j}^i) + (D_{t_{j+1}}^i - D_{t_j}^i)\}$

self-financing with dividend cts-time case

$X_t^{\bar{\theta}} = X_0^{\bar{\theta}} + \sum_{i=0}^d \left(\int_0^t \theta_u^i dS_u^i + \int_0^t \theta_u^i dD_u^i\right)$
$\Updownarrow$
$dX_t^{\bar{\theta}} = \sum_{i=0}^d (\theta_t^i dS_t^i + \theta_t^i dD_t^i)$

Pricing PDE

a) $0=\Pi_t(t,s)+\Pi_s(t,s)rs+\frac{1}{2}\Pi_{ss}(t,s)\sigma^2 s^2-r\Pi(t,s)$ $\quad\forall(t,s)\in[0,T)\times\mathbb{R}_+$.

b) $\Pi(T,s)=G(s)$. $\quad s\in\mathbb{R}_+$.

F-C

a) $0=-rF(t,x)+F_t(t,x)+F_x(t,x)\tilde{\mu}(t,x)+\frac{1}{2}F_{xx}(t,x)(\tilde{\sigma}(t,x))^2$

b) $F(T,x)=\Phi(x)$

$\Downarrow$

$F(t,x)=\mathbb{E}[e^{-r(T-t)}\Phi(X_T)|F_t]$

$\begin{cases} X_t=x \leftarrow \text{"initial position".} \\ dX_s=\tilde{\mu}(s,X_s)ds+\tilde{\sigma}(s,X_s)d\tilde{B}_s \end{cases}$

If $\begin{cases} dS_u=S_u(r\,du+\sigma d\tilde{B}_u) \\ S_t=s \end{cases}$

Then

$\Pi_t^G=\Pi(t,s)$

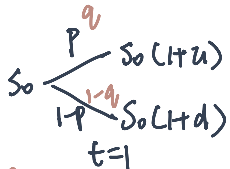
$=\tilde{\mathbb{E}}[e^{-r(T-t)}G(S_T)]$

original model

$\begin{cases} dS_u=S_u(\mu du+\sigma dB_u) \\ S_t=s \end{cases}$

Binomial model

$d<r<u$



$S_1=S_0(1+R_1)$

$\boxed{q=\frac{r-d}{u-d}}\neq p.$

$\mathbb{E}^{Q}\left[\frac{S_1-S_0}{S_0}\right]=\mathbb{E}^{Q}[R_1]$

$=uq+d(1-q)$

$=r \quad \neq \mathbb{E}^{P}[R_1]$

How to change measure?

$\mathbb{E}^{Q}[R_1]=u\frac{q}{p}p+d\frac{(1-q)}{1-p}(1-p)$

$=u(\frac{q}{p})\cdot p+d(\frac{1-q}{1-p})(1-p)$

$\Omega=\{w_u,w_d\}$

$R_1(w_u)=u;\ R_1(w_d)=d.$

$Z(w_u)=q/p.\ Z(w_d)=\frac{1-q}{1-p}.$

$\mathbb{E}^{Q}[R_1]=R_1(w_u)Z(w_u)\mathbb{P}[w_u]+R_1(w_d)Z(w_d)\mathbb{P}[w_d]$

$=\mathbb{E}^{P}[R_1Z].$

properties of Z?

a) $Z>0$ $\mathbb{P}$-a.s

b) $\mathbb{E}^{P}[Z]=\frac{q}{p}\cdot p+\frac{1-q}{1-p}(1-p)=1.$

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is given.

Let $Z \geq 0$ $\mathbb{P}$-a.s be a r.v. we can define a new prob measure $\tilde{\mathbb{P}}$ by

$$A \in \mathcal{F} \quad \tilde{\mathbb{P}}[A] := \int_A Z(\omega) d\mathbb{P}(\omega)$$
$$= \mathbb{E}^{\mathbb{P}}[I_A Z]$$

Binomial : $Q[\omega_n] = Z(\omega_n) \mathbb{P}[\omega_n]$
$= \frac{q}{p} \cdot p = q$

provided $\mathbb{E}^{\mathbb{P}}[Z] = 1$

Then for any r.v. X,
$$\mathbb{E}^{\tilde{\mathbb{P}}}[X] = \mathbb{E}^{\mathbb{P}}[XZ]$$
If $Z > 0$ $\mathbb{P}$-a.s, then
$$\mathbb{E}^{\mathbb{P}}[X] = \mathbb{E}^{\tilde{\mathbb{P}}}[X/Z]$$

$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ is the Rodon-Nikodym derivative of $\tilde{\mathbb{P}}$ and $\mathbb{P}$

theorem : If $\mathbb{P}[A] = 0 \Rightarrow \tilde{\mathbb{P}}[A] = 0$ for two prob. measures on $(\Omega, \mathcal{F})$. $(\tilde{\mathbb{P}} \ll \mathbb{P})$
then $\exists$ $Z \geq 0$ $\mathbb{P}$-a.s with $\mathbb{E}^{\mathbb{P}}[Z] = 1$. s.t
$$\tilde{\mathbb{P}}[A] = \mathbb{E}^{\mathbb{P}}[I_A Z]$$

If also $\mathbb{P} \ll \tilde{\mathbb{P}}$. then $Z > 0$ $\mathbb{P}$-a.s ($\tilde{\mathbb{P}}$-a.s).

Two measures are equivalent if $\tilde{\mathbb{P}} \ll \mathbb{P}$ and $\mathbb{P} \ll \tilde{\mathbb{P}}$.

Let $(Z_t)_{t \in [0,T]}$ be given by

$Z_t := \mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{F}_t] \quad t \in [0,T]$.

where $Z > 0$ $\mathbb{P}$-a.s and $\mathbb{E}^{\mathbb{P}}[Z] = 1$.

R-N process.

Claim : $(Z_t)_{t \in [0,T]}$ is a $\mathbb{P}$-martingale.

Jensen's :
$\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$

proof : · tower property.

$$\mathbb{E}[|Z_t|] = \mathbb{E}[|\mathbb{E}[Z \mid \mathcal{F}_t]|]$$
$$\leq \mathbb{E}[\mathbb{E}[|Z| \mid \mathcal{F}_t]].$$
$$= \mathbb{E}[|Z|].$$
$$< +\infty.$$

Radon-nikodym theorem

RADON - NIKODYM THEOREM

On $(\Omega, \mathcal{F})$, two measures $\mathbb{P}$ and $\mathbb{Q}$

are equivalent if and only if

$\exists\ Z > 0$ $\mathbb{P}$-a.s. with $\mathbb{E}^{\mathbb{P}}[Z] = 1$ s.t.

$$\forall A \in \mathcal{F}; \quad \mathbb{Q}[A] = \mathbb{E}^{\mathbb{Q}}[I_A] = \mathbb{E}^{\mathbb{P}}[I_A Z]$$

GOAL: for a stochastic process $S = (S_t)$ we will have two dynamics

$\mathbb{P}$-dynamics $\quad dS_t = S_t \{ \mu(t, S_t)\,dt + \sigma(t, S_t)\,dB_t \}$

⊛

$\mathbb{Q}$-dynamics $\quad dS_t = S_t \{ \tilde{\mu}(t, S_t)\,dt + \tilde{\sigma}(t, S_t)\,d\tilde{B}_t \}$

(Remark: $\mathbb{P}[S_t \in A] \neq \mathbb{Q}[S_t \in A]$

IDEA: construct Z that can be used in R-N theorem and that "does the job" ⊛

TOOL: GIRSANOV'S THEOREM

3/14/2023

$\mathbb{P}[B_t - B_s \in A] = \int_A f_N(z)\,dz \qquad N \sim \mathcal{N}(0, t-s)$
$\mathbb{Q}[B_t - B_s \in A] \neq \mathbb{P}[B_t - B_s \in A] \quad$ if $\mathbb{Q} \neq \mathbb{P}$

Take Z as in R-N thm. Define a martingale $(Z_t)_{t\in[0,T]}$ by $Z_t = \mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{F}_t] \quad t\in[0,T]$
Note that (using that $\mathcal{F} = \mathcal{F}_T$), $Z_T = Z$
$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ s.t $\mathcal{F} = \mathcal{F}_T$
is $\mathcal{F}_t$-measurable.

Lemma 1 suppose y is $\mathcal{F}_t$-measurable ($t \in [0,T]$).
then $\mathbb{E}^{\mathbb{Q}}[y] = \mathbb{E}^{\mathbb{P}}[y Z_t]$

proof: From R-N thm.
$\mathbb{E}^{\mathbb{Q}}[y] = \mathbb{E}^{\mathbb{P}}[y Z]$

$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. $\mathbb{E}^{\mathbb{P}}[yZ] = \int yZ\,d\mathbb{P} = \int y \frac{d\mathbb{Q}}{d\mathbb{P}}\,d\mathbb{P} = \int y\,d\mathbb{Q}$

tower property

$\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[yZ \mid \mathcal{F}_t]] = \mathbb{E}^{\mathbb{P}}[y \underbrace{\mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{F}_t]}_{Z_t}]$

Lemma 2: Again y is $\mathcal{F}_t$-measurable. Let $0 \le s \le t$. $\mathbb{E}^{\mathbb{Q}}[y \mid \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}^{\mathbb{P}}[y Z_t \mid \mathcal{F}_s]$

proof: partial averaging!
$\forall A \in \mathcal{F}_s \qquad \int_A y\,d\mathbb{Q} = \int_A \mathbb{E}^{\mathbb{Q}}[y \mid \mathcal{F}_s]\,d\mathbb{Q}$

for any version of "$\mathbb{E}^{\mathbb{Q}}[y \mid \mathcal{F}_s]$"

$\frac{1}{Z_s}\mathbb{E}^{\mathbb{P}}[yZ_t \mid \mathcal{F}_s]\,d\mathbb{Q} = \mathbb{E}^{\mathbb{Q}}[\underbrace{\mathbb{1}_A \cdot \frac{1}{Z_s}\mathbb{E}^{\mathbb{P}}[yZ_t \mid \mathcal{F}_s]}_{\mathcal{F}_s\text{-measurable}}]$

Lemma 1. $\mathcal{F}_s$
$= \mathbb{E}^{\mathbb{P}}[\mathbb{1}_A \mathbb{E}^{\mathbb{P}}[yZ_t \mid \mathcal{F}_s]]$

$= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\mathbb{1}_A y Z_t \mid \mathcal{F}_s]] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_A y Z_t]$

Lemma 1. $\mathcal{F}_t$
$\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_A y] = \int_A y\,d\mathbb{Q}$

girsanov's thm

Let $B = (B_t)_{t\in[0,T]}$ be a standard 1-dim BM defined on $(\Omega, \mathcal{F}, \mathbb{F}^B, \mathbb{P})$
1) Let $M = (M_t)_{t\in[0,T]}$ be a $\mathbb{F}^B$-adapted.
2) Define stochastic exponential / "exponential martingale".

$\begin{cases} dZ_t = -M_t Z_t\,dB_t \\ Z_0 = 1 \end{cases}$
$\rightarrow Z_t = \exp\left(-\int_0^t M_s\,dB_s - \frac{1}{2}\int_0^t M_s^2\,ds\right)$

$Z_t = 1 - \int_0^t M_s Z_s\,dB_s$

when is Z a martingale.
a) $(M_t Z_t)$ is "nice".
b) Novikov $\mathbb{E}^{\mathbb{P}}[e^{\frac{1}{2}\int_0^T M_s^2\,ds}] < \infty$
c) Suppose $(Z_t)_{t\in[0,T]}$ is a $\mathbb{P}$-martingale. then $\boxed{\mathbb{E}^{\mathbb{P}}[Z_T] = Z_0 = 1}$
Also $\boxed{Z_T > 0}$ $\mathbb{P}$-a.s

we can define, using R-N thm $\mathbb{Q}[A] := \mathbb{E}^{\mathbb{P}}[Z_T \mathbb{1}_A]$.

Then $W_t = B_t + \int_0^t M_s\,ds$ is a B-M.

Finance :

B-S model under $\mathbb{P}$ $\begin{cases} dS_t = S_t(\mu dt + \sigma dB_t) \\ S_0 > 0. \end{cases}$

price of $G(S_t)$ at time-0 is given by $\mathbb{E}^{\mathbb{Q}}\left[\frac{G(S_T)}{S_T^0}\right]$ $S_t^0 = e^{rt}$

where $\begin{cases} dS_t = S_t(r dt + \sigma dW_t) \\ S_0 > 0. \end{cases}$ for a $\mathbb{Q}$-BM W.
under $\mathbb{Q}$

what should M be? $dS_t = S_t\{\mu dt + \sigma dB_t\}$
$= S_t\{\mu dt + \sigma(dW_t - M_t dt)\}$
$= S_t\{(\mu - \sigma M_t)dt + \sigma dW_t\}$.
choose M s.t. $\mu - \sigma M_t = r \quad \forall t$.
$\Rightarrow M_t = \frac{\mu - r}{\sigma} \quad \forall t$.
$\rightarrow dS_t = S_t(r dt + \sigma dW_t)$ under $\mathbb{Q}$

proof : we need to check that $W_t = B_t + \int_0^t M_s ds$. is a BM under $\mathbb{Q}$.
we will need : Levy's characterization of BM.

- Suppose $M = (M_t)_{t\in[0,T]}$ is a martingale
- Suppose $M_0 = 0$
- Suppose M has cts trajectories
- Suppose $[M, M]_t = t$

then M is a B-M

$\mathbb{E}^{\mathbb{Q}}[W_t | \mathcal{F}_s] = W_s$.

$dX_t = \mu_t dt + \sigma_t dB_t$. $d[X, X]_t = \sigma_t^2 dt$. $(dW_t)^2 = dt$.

proof of levy's thm :
$f(t, B_t) = f(0, B_0) + \int_0^t f_t(s, B_s) ds + \int_0^t f_x(s, B_s) dB_s + \frac{1}{2}\int_0^t f_{xx}(s, B_s)\, d[B, B]_s$

To prove this, we only need that B is "cts" process with $[B, B]_t = t$.
Hence, $f(t, M_t) = f(0, M_0) + \int_0^t f_t(s, M_s) + \frac{1}{2} f_{xx}(s, M_s) ds + \int_0^t f_x(s, M_s) dM_s$

Choose $f(t, x) = \exp(ux - \frac{1}{2}u^2 t)$ for $u \in \mathbb{R}$.

$f_t(t, x) = -\frac{1}{2}u^2 f(t, x)$
$\frac{1}{2} f_{xx}(t, x) = \frac{1}{2} u^2 f(t, x)$.
$\rightarrow df(t, M_t) = f_x(t, M_t) dM_t$

- no "dt" term
- M is a martingale

$\Rightarrow f(t, M_t)$ is a martingale.

$\mathbb{E}^{\mathbb{Q}}[f(t, M_t)] = f(0, M_0)$
$\mathbb{E}^{\mathbb{Q}}[\exp(uM_t - \frac{1}{2}u^2 t)] = 1$

$\Rightarrow \mathbb{E}^{\mathbb{Q}}[e^{uM_t}] = e^{\frac{1}{2}u^2 t}$.

$\mathbb{E}^{\mathbb{Q}}[e^{u(M_t - M_s)} | \mathcal{F}_s] <$
- $M_t - M_s$ is indep of $\mathcal{F}_s$
- $M_t - M_s$ dist only depends on $(t-s)$

Back to W: need to show that it is $\mathbb{Q}$-mart.

1) $(W_t Z_t)_{t\in[0,T]}$ is a $\mathbb{P}$-martingale.

a) M

b) $dZ_t = -H_t Z_t \, dB_t$

$d(W_t Z_t) =$ exercise 1

no "dt" term.

2) $\mathbb{E}^{\mathbb{Q}}[W_t \mid \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}^{\mathbb{P}}[W_t Z_t \mid \mathcal{F}_s] = \frac{1}{Z_s} \underbrace{W_s Z_s}$

exercise 2.

W_t is a $\mathbb{P}$-martingale.

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$d=1$ 1. Assume $\bar{S}=(S^0, S)$, "living" on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, is arbitrage free.

$\nexists$ $\bar{\theta}=(\theta^0, \theta)$ s.t

i) $X_0^{\bar{\theta}} = \theta_0^0 S_0^0 + \theta_0 S_0 = 0$

ii) $\mathbb{P}[X_T^{\bar{\theta}} \geq 0] = 1$

iii) $\mathbb{P}[X_T^{\bar{\theta}} > 0] > 0$.

2. Generalized B-S model

$$\begin{cases} dS_t = S_t \{\mu(t, S_t)\, dt + \sigma(t, S_t)\, dB_t\} \\ S_0 > 0 \end{cases}$$

$\uparrow$ $\mathbb{P}$-Brownian motion.

$$\begin{cases} dS_t^0 = S_t^0 r_t\, dt \\ S_0^0 = 1 \end{cases}$$

3. How to price a derivative security $G(S_T)$?

The pricing rule Π_t^G should not introduce arbitrage opportunities: $(\bar{S}, \Pi^G)$ remains arbitrage-free.

F-K

a) solve Black-Scholes PDE:

$$\begin{cases} (\Pi_t + rS\Pi_s + \frac{\sigma^2 s^2}{2}\Pi_{ss} - r\Pi)(t,s) = 0. & \forall (t,s) \in (0,T) \times \mathbb{R}_+ \\ \Pi(T,s) = G(s) & \forall s \in \mathbb{R}_+ \end{cases}$$

solution $\Pi(t,s)$ provides a self-financing and replicating strategy:

$\theta_t = \Pi_s(t, S_t)$

$\theta_t^0 = (\Pi(t, S_t) - \theta_t S_t)/S_t^0$

b) Pricing via EMM:

i) use Girsanov's thm to construct a measure $\mathbb{Q}$, that's equivalent to $\mathbb{P}$, s.t.

$$\begin{cases} dS_t = S_t\{r_t\, dt + \sigma(t, S_t)\, dW_t\} \\ S_0 > 0 \end{cases}$$

where W is a $\mathbb{Q}$-BM.

ii) No-arbitrage price is then given by.

$$\mathbb{E}^{\mathbb{Q}}\left[S_t^0 \cdot \frac{G(S_T)}{S_T^0} \,\middle|\, \mathcal{F}_t\right]$$

Why $\mathbb{Q}$ is an EMM?

a) Pick $(M_t)_{t\in[0,T]}$

b) Define

$$Z_t = \exp\left(-\int_0^t M_s\, dB_s - \frac{1}{2}\int_0^t M_s^2\, ds\right)$$

$\mathbb{E}[Z_T] = \mathbb{E}[Z_T | \mathcal{F}_0] = Z_0 = 1$

c) If $(Z_t)_{t\in[0,T]}$ is a $\mathbb{P}$-martingale.

(e.g. Novikov's condition.
$\mathbb{E}^{\mathbb{P}}[\exp(\frac{1}{2}\int_0^T M_s^2\, ds)] < \infty$)

then $\cdot\, \mathbb{P}[Z_T] = 1$

$\cdot\, \mathbb{P}[Z_T > 0] = 1$

R-N thm
pick $Z > 0$, & $\mathbb{E}[Z] = 1$.

$\mathbb{Q}[A] = \mathbb{E}^{\mathbb{Q}}[I_A] := \mathbb{E}^{\mathbb{P}}[I_A Z_T]$

$W_t = B_t + \int_0^t M_s\, ds$ is a $\mathbb{Q}$ martingale.

Under $\mathbb{Q}$ we have

$$dS_t = S_t(r_t\,dt + \sigma(t,S_t)\,dW_t)$$

$f(\frac{x}{y})$ $\quad d\left(\frac{S_t}{S_t^0}\right) = \frac{1}{S_t^0}dS_t - \frac{S_t}{(S_t^0)^2}dS_t^0$

$$dS_t^0 = S_t^0 r_t\,dt \longrightarrow d[S^0,S^0]_t = (dS_t^0)^2 = (S_t^0)^2 r_t^2 \underbrace{(dt)^2}_{=0} = 0.$$

$$= \frac{S_t}{S_t^0}(r_t\,dt + \sigma(t,S_t)dW_t) - \frac{S_t}{S_t^0}\left\{\frac{S_t^0 r_t}{S_t^0}dt\right\}$$

$$= \left(\frac{S_t}{S_t^0}\right)\sigma(t,S_t)\,dW_t.$$

$\Longrightarrow$ $\hat{S}_t = \frac{S_t}{S_t^0}$ is a $\mathbb{Q}$ martingale.

Consider the black-scholes model

$$\mu_t = \mu(t,S_t) = \mu \in \mathbb{R}.$$
$$\sigma_t = \sigma(t,S_t) = \sigma > 0. \qquad \forall t \in [0,T]$$
$$r_t = r > -1.$$

$dW_t = dB_t + M_t dt$

under $\mathbb{P}$ $\quad dS_t = S_t(\mu\,dt + \sigma\,dB_t)$

under $\mathbb{Q}$ $\quad = S_t(\mu\,dt + \sigma(dW_t - M_t\,dt))$

$$= S_t(\underbrace{(\mu - \sigma M_t)}_{=r}dt + \sigma\,dW_t)$$

$\Longrightarrow$ $M_t = \frac{\mu - r}{\sigma} \in \mathbb{R} \quad \forall t \in [0,T]$ $\longleftarrow$ this is the unique choice.

Novikov's condition:

$$\exp\left(\tfrac{1}{2}\int_0^T M_s^2\,ds\right) = \exp\left(\left(\frac{\mu-r}{\sqrt{2}\sigma}\right)^2 T\right) < +\infty.$$

In fact this will work for any deterministic M with $\int_0^T M_s^2\,ds < \infty$.

For any $G(S_T)$ we know that

$$\Pi_t^G = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}G(S_T)\,\middle|\,\mathcal{F}_t\right]$$

$$= e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}[G(S_T)\,|\,\mathcal{F}_t]$$

$S_t^0 = e^{\int_0^t r_s\,ds}$ $\longrightarrow$ $\frac{S_t^0}{S_T^0} = e^{-\int_t^T r_s\,ds}$

apply Itô's formula

$dS_t^0 = S_t^0 r_t\,dt.$

European call: $\quad G(S) = (S-K)^+ = \begin{cases} S-K & \text{if } S \geq 0 \\ 0 & \text{oth} \end{cases}$ $\quad$ for some $K > 0$ strike.

Claim: time t price of a call $(S_T - K)^+$ is

$$\Pi_t^G = S_t\,\Phi(d_1) - e^{-r(T-t)}K\,\Phi(d_2).$$

have $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy$

$$d_1 = d_1(t,S_t) = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\,;\quad d_2 = d_2(t,S_t) = d_1(t,S_t) - \sigma\sqrt{T-t}.$$

Furthermore, the delta-hedge $\Delta_t = \frac{d\Pi^G}{dS}(t, S_t) = \Phi(d_1) \in [0,1]$.

$$\begin{cases} dS_t = S_t(r\,dt + \sigma\, dW_t) \\ S_0 > 0 \end{cases}$$

$\downarrow$ solution is a gbM

$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma W_t\right)$.

plug in $t=T$.

$$S_T = S_0 \exp\left((r-\frac{\sigma^2}{2})T + \sigma W_T\right).$$

$$= S_0 \exp\left((r-\frac{\sigma^2}{2})t + \sigma W_t\right) \times \exp\left(-(r-\frac{\sigma^2}{2})t - \sigma W_t\right) \times \exp\left((r-\frac{\sigma^2}{2})T + \sigma W_T\right)$$

$$= S_t \cdot \frac{S_T}{S_t}$$

$$= \underbrace{S_t}_{F_t\text{-measurable}} \exp\left((r-\frac{\sigma^2}{2})(T-t) + \sigma\underbrace{(W_T - W_t)}_{\text{indep of } F_t}\right)$$

If X is G-measurable and Y is indep of G.

$\mathbb{E}[f(X,Y) \mid G] = g(X)$. where $g(z) = \mathbb{E}[f(z,Y)]$ $\leftarrow$ independence Lemma.

<u>back to pricing of G.</u>

$$\Pi_t^G = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T-K)^+ \mid F_t]$$

$$= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_t\frac{S_T}{S_t} - K)^+ \mid F_t]$$

$$= e^{-r(T-t)} g(S_t)$$

where $g(s) = \mathbb{E}^{\mathbb{Q}}[(s\frac{S_T}{S_t} - K)^+]$

What's the law of $\frac{S_T}{S_t}$ given F_t?

Given F_t, $\log(\frac{S_T}{S_t}) \sim N\left((r-\frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right)$

$$\Pi_t^G = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(se^X - K)^+]\Big|_{s=S_t} \quad \text{where } X \sim N\left((r-\frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right).$$

$$= \int_{\mathbb{R}} e^{-r(T-t)}(se^x - K)^+ f_X(x)\,dx \Big|_{s=S_t}$$

$$\left\{ se^x - K \geq 0 \rightarrow e^x \geq \frac{K}{s} \rightarrow x \geq \log(\frac{K}{s}). \right\}$$

$$= \int_{\log(\frac{K}{s})}^{+\infty} e^{-r(T-t)}(se^x - K)^+ f_X(x)\,dx \Big|_{s=S_t}$$

$$= s\int_{\log(\frac{K}{s})}^{+\infty} e^{-r(T-t)+x} f_X(x)\,dx - e^{-r(T-t)}K\int_{\log(\frac{K}{s})}^{+\infty} f_X(x)\,dx \Big|_{s=S_t}$$

need to show $\cdot \int_{\log(\frac{K}{S})}^{+\infty} e^{-r(T-t)+x} f_X(x)\,dx = \Phi(d_1)$

$\cdot \int_{\log(\frac{K}{S})}^{+\infty} f_X(x)\,dx = \Phi(d_2)$

$$\int_{\log(\frac{K}{S})}^{+\infty} f_X(x)\,dx = \mathbb{Q}[X \geq \log\tfrac{K}{S}].$$

$\rightarrow \mathbb{Q}[X \leq \log(\frac{S}{K})]$ by symmetry.

$$\int_{\log\frac{K}{S}}^{+\infty} f_X(x)\,dx = \mathbb{Q}[\underbrace{(r-\tfrac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,Z}_{X} \leq \log(\tfrac{S}{K})]$$

$Z \sim N(0,1)$ (under $\mathbb{Q}$)

$$= \mathbb{Q}\left[Z \leq \frac{\log(\frac{S}{K}) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right]$$

$$= \Phi(d_2).$$

Exercise: $\int_{\log(\frac{K}{S})}^{+\infty} e^{-r(T-t)+x} f_X(x)\,dx = \Phi(d_1)$ use change of variables.

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BS model

$$\begin{cases} dS_t = S_t\{\mu dt + \sigma dB_t\} \\ S_0 > 0 \end{cases}$$

$$\begin{cases} dS_t^0 = S_t^0 \, r dt \\ S_0^0 = 1 \end{cases}$$

The time t no arbitrage price

$G(S) = (S-K)^+$

i) ASS 1 : $\exists$ a self-financing, replicating portfolio $\bar\theta = (\theta^0, \theta)$

(S-F) : $X_t^{\bar\theta} := \theta_t^0 S_t^0 + \theta_t S_t$ ✗ Dividends $+ \int_0^t \theta_r d D_r$.

$= X_0^{\bar\theta} + \int_0^t \theta_r^0 dS_r^0 + \int_0^t \theta_r dS_r$

Replication : $X_T^{\bar\theta} = G(S_T)$ $\mathbb{P}$-a.s. $G(S_T, Z_T)$ ✓

replication $\Rightarrow$ Law of one price.

If π_T^G represents the no-arbitrage price of G at time t.

$\pi_t^G = X_t^{\bar\theta}$ $\forall t \in [0,T]$

ii) ASS2 : $\exists\, \pi : [0,T] \times \mathbb{R}_+ \longmapsto \mathbb{R}_+$ s.t. $\pi_t^G = \pi(t, S_t)$ $\forall t \in [0,T]$. $\pi(t, S_t, Z_t)$

ASS 1 + ASS 2 $\Rightarrow$

$\pi(t, S_t) = X_t^{\bar\theta}$ $\forall t \in [0,T]$

$\uparrow$

this will hold if $d\pi(t, S_t) = dX_t^{\bar\theta}$

(S-F) $dX_t^{\bar\theta} = \theta_t^0 dS_t^0 + \theta_t dS_t + \theta_t dD_t$

iii) ASS 3 : $\pi(\cdot,\cdot)$ is smooth enough (for Itô's). $\pi(\cdot,\cdot,\cdot)$ $d\pi(t, S_t, Z_t)$

iv) Consider $\underset{\text{LHS}}{d\pi(t, S_t)} = \underset{\text{RHS}}{dX_t^{\bar\theta}}$ and try to determine for which $\bar\theta$ and π it holds

From $\pi(t, S_t) = X_t^{\bar\theta}$ we can determine $\theta_t^0 = \dfrac{\pi(t, S_t) - \theta_t S_t}{S_t^0}$

- Equating "dB_t" parts of LHS & RHS, fixes $\theta_t = \pi_S(t, S_t)$
- π will be determined by equating "dt" terms of LHS & RHS.

  $\Downarrow$ PDE (will be different with dividends).

(iii) is a solution to (pricing) PDE indeed a no-arbitrage price of G?

$\pi(t, S_t)$ is a no-arbitrage price of G at time t if $\pi(t, S_t) = X_t^{\bar\theta}$ $\forall t \in [0,T]$.

Provided !

(this part is hard with D_t) $\leftarrow$ a) $\bar\theta$ is self-financing

b) $X_T^{\bar\theta} = G(S_T)$.

Find one such $\bar\theta$.

(derivation of PDE suggests a candidate $\bar\theta$).

1.

iii) F-K: $F_t + F_s \tilde{\mu}(t,s) + \frac{1}{2}\tilde{\sigma}^2(t,s)F_{ss} - \tilde{r}F$

$F(T,s) = \Phi(s)$

$\Updownarrow$

$F(t,S_t) = e^{-\tilde{r}(T-t)} E^{\tilde{P}}[\Phi(S_T)|F_t]$

where

$S_T \in \mathbb{R}$

$\tilde{P}$ $\quad dS_r = \tilde{\mu}(r, S_r)\,dr + \tilde{\sigma}(s, S_r)\,d\tilde{W}_r.$

Black-Scholes formula

$\pi(t,S_t) = e^{-r(T-t)} E^{Q}[G(S_T)|F_t]$ where $dS_r = S_r(r\,dt + \sigma\,dB_t^{Q})$

can we compute $\pi(t,s)$?

$\pi(t,S_t) = e^{-r(T-t)} E^{Q}[G(S_T)|F_t]$

$= e^{-r(T-t)} E^{Q}[G(S_t \cdot \frac{S_T}{S_t})|F_t].$

$\forall t:$ $S_t = S_0 \exp((r-\frac{\sigma^2}{2})t + \sigma B_t)$

$\Rightarrow e^{-r(T-t)}[E^{Q}[G[se^x]]]_{s=S_t}$

$\tilde{G}(s) := E^{Q}[G(se^x)]$

European call:

$\pi(t,S_t) = S_t\,\Phi(d_1(t,S_t)) - e^{-r(T-t)}K\Phi(d_2(t,S_t))$ (with s written above each S_t)

$\theta_t = \pi_s(t,S_t)$

Claim: $\pi_s(t,s) = \Phi(d_1(t,s))$

proof: $\pi_s(t,s) = \Phi(d_1(t,s)) + \underbrace{s\Phi'(d_1(t,s)) \times d_1'(t,s) - e^{-r(T-t)}K\Phi'(d_2(t,s))\,d_2'(t,s)}_{=0\ ?}$

$d_2 = d_1 - \sigma\sqrt{T-t}$. $\therefore d_2'(t,s) = d_1'(t,s)$

need to have that $s\Phi'(d_1(t,s)) = e^{-r(T-t)}K\Phi'(d_2)$

$\uparrow$ exercise HW.

useful traits.

- $\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$
- $(\frac{s}{K}) = e^{\log(\frac{s}{K})}$.

HW 3 Q1 (iv):

$G(s) = \begin{cases} K & \text{if } s \in [\alpha, \beta]. \\ 0 & \text{otw.} \end{cases}$

$\Updownarrow$

$\pi(t,s) = e^{-r(T-t)} E^{Q}[G(S_T)|F_t]$

$= e^{-r(T-t)} K E^{Q}[I_{[\alpha,\beta]}(S_T)|F_t]$

$= e^{-r(T-t)} K E^{Q}[I_{[\alpha,\beta]}(S_t \cdot \frac{S_T}{S_t})|F_t]$

$= e^{-r(T-t)} K E^{Q}[I_{[\alpha,\beta]}(se^x)]$

$\forall t,$

$S_t = S_0 \exp((r+c) - \frac{\sigma^2}{2})t + \sigma B_t)$

only change here.

HW3. Q5 ≈ Q1.

$Z_t = \int_0^t h(u, S_u)\,du = \int_0^t S_u\,du$ (Asian option).

HW3. 2.

$$\underset{\text{Put}}{(K-S)^+} = \underset{\substack{\uparrow \\ \text{bank}}}{K} - \underset{\substack{\uparrow \\ \text{stock}}}{S} + \underset{\text{Call}}{(S-K)^+}$$

(K>S) ... (K<S) ... ∀s

if at time 0 we take the following actions.

i) buy $e^{-r(T-t)}K$ of S^0.

ii) short 1 unit of S.

iii) buy 1 call with strike K.

a) Put price at time 0.

$$e^{-rT}\,\mathbb{E}^{Q}[(K-S_T)^+]$$

if we sell a put, then we follow $\theta_t = \alpha(t, S_t)$

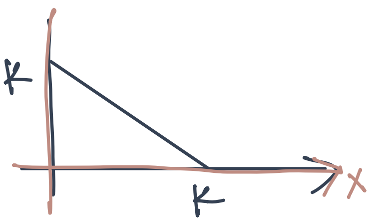
$$\theta_t^0 = \frac{\pi(t, S_t) - \theta_t S_t}{S_t^0} \qquad \forall t \in [0, T].$$

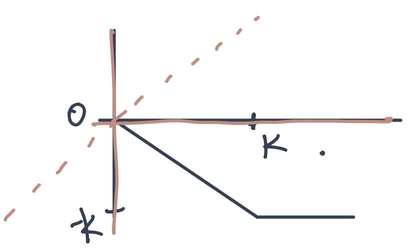
b) $e^{-rT}\,\mathbb{E}^{Q}[(K-S_T)^+] = e^{-rT}\,\mathbb{E}^{Q}[K - S_T + (S_T-K)^+]$

$= e^{-rT}K - \mathbb{E}^{Q}[e^{-rT}S_T] + e^{-rT}\,\mathbb{E}^{Q}[(S_T-K)^+].$

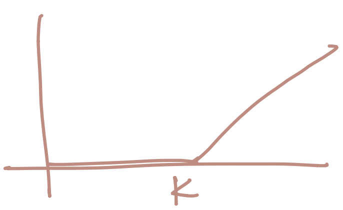
$=$

$(K-S)^+$





$[(K-S)^+ - K]$



$[(K-S)^+ - K] + S.$

Tracting with a Cost
(transation cost)

- bid-ask spread
- brokers charge fees

(short-selling)

Less frequent rebalancing of a portfolio is beneficial!

Can we replicate G with $\boxed{\text{static / buy-and-hold}}$ strategies?

What can we achieve if we trade once (at time 0) S^0 and S? (at time T)

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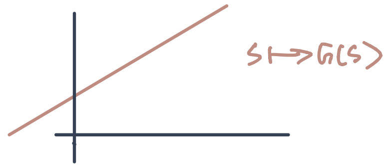
$\alpha = \theta_t^0 \quad \forall t \in [0,T]$

$\beta = \theta_t \quad \forall t \in [0,T]$

time T (payoffs) $X_T^\theta = \theta_T^0 S_T^0 + \theta_T S_T = \alpha S_T^0 + \beta S_T.$

time 0 (price) $X_0^\theta = \theta_0^0 S_0^0 + \theta_0 S_0 = \alpha + \beta S_0$



- Note that $\alpha S_T^0 + \beta S_T$ is given in S_T
- $S \longmapsto G(S)$ must be linear then

$G(S_T) = \alpha^G S_T^0 + \beta^G S_T$ for some α^G, β^G

this won't work if $G(S) = (S-K)^+$

What can we do if we are also trade at time 0 European calls/puts?

$G^{P,K}(S) = (K-S)^+$

$G^0(S) = 1$

$G^1(S) = S$

(Consider black scholes model)

$\pi^G(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[G(S_T) | \mathcal{F}_t]$ ANY G.

a) $\pi^{G^0}(t, S_t) = e^{-r(T-t)}$

b) $\pi^{G^1}(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[S_T | \mathcal{F}_t] = e^{rt} \mathbb{E}^{\mathbb{Q}}[\frac{S_T}{e^{rT}} | \mathcal{F}_t]$

$= e^{rt} \frac{S_t}{e^{rt}} = S_t.$

c) $\pi^{G^{P,K}}(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[(K-S_T)^+ | \mathcal{F}_t]$

$= P(t, S_t, K, T) \leftarrow$ given to us!

d) Put-Call Parity $\pi^{G^{C,K}}(t, S_t) = C(t, S_t, K, T)$

What M can we achieve by trading in S^0, S, puts/calls?

① $M(S) = \alpha G^0(S) + \beta G^1(S) + \sum_{n=1}^{N} \rho_n G^{C,K_n}(S)$

ρ_n: # of calls with strike K_n.
(maturity is common T)

Proposition: let $F^1(S_T)$ and $F^2(S_T)$ be 2 derivatives. then

time t price of derivative $\alpha F^1(S_T) + \beta F^2(S_T)$ $\rightarrow \pi(t, S_t; \alpha F^1 + \beta F^2) = \alpha \pi(t, S_t, F^1) + \beta \pi(t, S_t, F^2)$

proof: time t price of αF^1, βF^2

e.g. $\pi(t, S_t; \alpha F^1) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\alpha F^1(S_T) | \mathcal{F}_t]$

$\pi(t, S_t; \alpha F^1 + \beta F^2) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\alpha F^1(S_T) + \beta F^2(S_T) | \mathcal{F}_t]$

$= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\alpha F^1(S_T) | \mathcal{F}_t] + e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\beta F^2(S_T) | \mathcal{F}_t]$

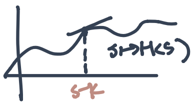
$\uparrow$ linearity of $\mathbb{E}^{\mathbb{Q}}[\cdot | \mathcal{F}_t]$.

Apply the proposition to Eq ①:

$$\Pi(t, S_t; H) = \alpha_0 e^{-r(T-t)} + \beta S_t + \sum_{n=1}^{N} \rho_n C(t, S_t, K, T)$$

How big is the class of $S \mapsto H(S)$ that has representation ①?

<u>thm</u>: Suppose $S \mapsto H(S)$ is smooth (enough), then for any $\bar{K} > 0$, the "Carr-Madan representation" holds.



$$H(S) = \underline{H(\bar{K})} + \overline{H'(\bar{K})}(S-\bar{K}) + \int_0^{\bar{K}} H''(K)\underline{(K-S)^+}\, dK + \int_{\bar{K}}^{+\infty} H''(K)\underline{(S-K)^+}\, dK.$$

$H'(\bar{K})S = H'(\bar{K})G'(S)$

and $$\pi(t, S_t; H) = e^{-r(T-t)}\{H(\bar{K}) - H'(\bar{K})\bar{K}\} + H'(\bar{K}) S_t$$

$\mathbb{E}[\int_0^T X_s ds] = \int_0^T \mathbb{E}[X_s]\, ds$

$$+ \int_0^{\bar{K}} H''(K) P(t, S_t, K, T)\, \underline{dK} + \int_{\bar{K}}^{+\infty} H''(K) C(t, S_t, K, T)\, \underline{dK}.$$

<u>proof</u>: Suppose i) $\bar{K} \le S$
ii) $\bar{K} > S$

if (i): Consider

$$\int_0^{\bar{K}} H''(K)\underbrace{(K-S)^+}_{=0} dK + \int_{\bar{K}}^{+\infty} H''(K)(S-K)^+ dK$$

$$= \int_{\bar{K}}^{+\infty} H''(K)(S-K)^+ dK.$$

$$= \int_{\bar{K}}^{S} H''(K)(S-K)\, dK.$$

$$= S\int_{\bar{K}}^{S} H''(K)dK - \int_{\bar{K}}^{S} H''(K)K dK. \quad \text{integration by parts.}$$

$$= S[H'(S) - H'(\bar{K})] - \left\{ H'(K)K\Big|_{K=\bar{K}}^{K=S} - \int_{\bar{K}}^{S} H'(K)\, dK \right\}$$

$$= \underline{S[H'(S)} - H'(\bar{K})] - \{\underline{SH'(S)} - \bar{K}H'(\bar{K}) - [H(S) - H(\bar{K})]\}$$

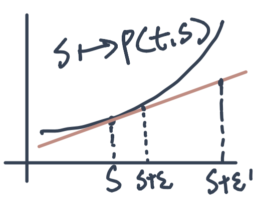
$$= H(S) - H(\bar{K}) - H'(\bar{K})(S - \bar{K}).$$

<u>The Greeks</u>: Suppose we have a portfolio that has a "pricing" function $P(t, S)$

i.e. time-t value of the portfolio is $\underline{P(t, S_t)}$ given that $S_t = S$.

<u>How $P(t, S_t)$ is sensitive w.r.t S_t?</u>

From time 0 perspective $P(t, S_t)$ is random
S_t $\quad$ $S_t + \varepsilon$



when $\varepsilon > 0$ is small. $P(t, S+\varepsilon) \approx P(t, S) + P_S(t, S)\varepsilon$

delta $\Delta(t, S) = \dfrac{\partial P(t, S)}{\partial S}$ $\qquad$ $\theta = \dfrac{\partial P(t, S)}{\partial t}$ theta

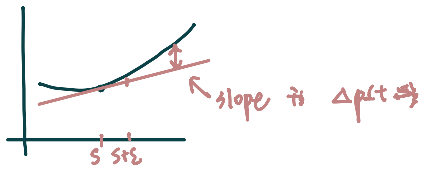
Gam $\Gamma(t, S) = \dfrac{\partial \Delta(t, S)}{\partial S}$ $\qquad$ $\nu = \dfrac{\partial P(t, S)}{\partial \sigma}$ vega.

Rho $\rho = \dfrac{\partial P(t, S)}{\partial r}$

3/28/2023

Consider a portfolio that has a pricing func $\pi(t,s)$.
i.e. $p(t, S_t)$ is a no-arbitrage price of this port
"will be"

$\Delta p(t,s) := \frac{\partial p}{\partial s}(t,s)$



A portfolio is "delta neutral" if $\Delta p(t,s)=0$] same definition for other greeks

How to value a portfolio Δ-neutral?
Recall: BS hedging strategy :

$\theta_t = \frac{\partial \pi^G}{\partial s}(t, S_t)$; θ_t^0.

1) : if $\Delta p(t,s)=0 \Rightarrow$ do nothing
2) : if $\Delta p(t,s) \neq 0$; $\boxed{\begin{array}{l}\cdot \text{ sell portfolio} \\ \cdot \text{ receive } p(t,s) \\ \cdot \text{ invest in } S^0.\end{array}}$ $\leftarrow$ conservative !

$G^0(s)=1$ (s is a possible value for S_T).

$\pi^{G^0}(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[G^0(S_T) \mid \mathcal{F}_t) = \boxed{e^{-r(T-t)}}$

$\Rightarrow \pi^{G^0}(t,s)$ does not depend on s.
$\Delta \pi^{G^0}(t,s) = 0$.

3) Add to the old portfolio some assets (additional portfolio)
$\downarrow$
total portfolio : $p(t,s) + x\tilde{p}(t,s)$ pricing function $\tilde{p}(t,s)$

$p_T(t,s) = p(t,s) + x\tilde{p}(t,s)$.
$\Delta p_T(t,s) = \Delta p(t,s) + x \, \Delta\tilde{p}(t,s)$

Choose x s.t. $\Delta p_T(t,s) = 0$.

$\Rightarrow x = -\frac{\Delta p(t,s)}{\Delta \tilde{p}(t,s)}$

If $\tilde{p}$ is just s:
$G'(s) = s$. (time T payoff).

$\tilde{p}(t, S_t) = \pi^{G'}(t, S_t)$
$= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[G'(S_T) \mid \mathcal{F}_t]$.
$= S_t$ $(e^{-rt}S_t)_{t \geq 0}$ is a $\mathbb{Q}$ martingale

$\tilde{p}(t,s) = s$ (time-t price).
$\Delta\tilde{p}(t,s) = 1$.
$\Rightarrow x^* = -\Delta p(t,s)$.

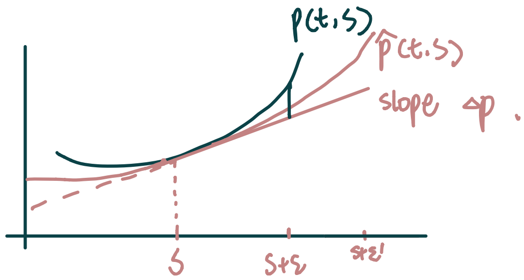
Furthermore, if $p(t,S) = -\pi^{G}(t,S)$ 　　an agent sold 1 unit of option that pays G at maturity.

$$\Rightarrow \Delta p(t,S) = -\frac{\partial \pi^{G}}{\partial S}(t,S)$$
$$\Rightarrow X^* = \frac{\partial \pi^{G}}{\partial S}(t,S).$$

warning: this needs to be implemented at each t.

But trading costs.

	frequent rebalancing	less frequent rebalancing.
Hedging E	small	Large.
transaction C	Large	small



$\Delta p(t,S) = \Delta \tilde{p}(t,S)$

$$\Gamma p(t,S) = \frac{\partial \Delta p(t,S)}{\partial S}.$$

sensitivity of $\Delta p(t,S)$ w.r.t. S.

Small Γ_p implies that some value of $\Delta p(t,S)$ work for longer period of time.

How to value P Γ-neutral?

$\Delta \pi^{G^0}(t,S) = 0$ 　　$\pi^{G^0}(t,S) = e^{-r(T-t)}$

$\Delta \pi^{G^1}(t,S) = 1.$ 　　$\pi^{G^1}(t,S) = S.$

$\rightarrow \Gamma_{\pi^{G^0}}(t,S) = \Gamma_{\pi^{G^1}}(t,S) = 0.$

$\rightarrow S^0, S$ cannot be used to make a portfolio Γ-neutral.

we need option!

Δ-Γ - neutrality.

① Γ- neutral.

Suppose $\Gamma p(t,S) \neq 0$.

$\Rightarrow$ suppose we also have $\tilde{p}$ with $\Gamma \tilde{p}(t,S) \neq 0$. ★

choose y s.t

$$\Gamma p(t,S) + y\,\Gamma \tilde{p}(t,S) = 0.$$

$$\Rightarrow y^* = -\frac{\Gamma p(t,S)}{\Gamma \tilde{p}(t,S)}.$$

now, make an adjusted portfolio

$$p(t,S) + y^* \tilde{p}(t,S).$$

Δ-neutral.

ADD x units of S: find x^* s.t.

$p(t,s) + y^* \tilde{p}(t,s) + x^* s$

26.

$\Delta p(t,s) + y^* \Delta \tilde{p}(t,s) + x^* = 0.$

Exotic Option

- Asian option: usually depends on the average process.

$$A_t := \frac{1}{T}\int_0^t S_u du.$$

e.g. floating strike Asian call.

$G = (S_T - A_T)^+.$

pricing: 1) Suppose $\exists\ \bar{\theta} = (\theta^0, \theta)$ s.t $\bar{\theta}$ is a) self-financing b) $\theta_T^0 S_T^0 + \theta_T S_T = G$.

Law of 1 price

2) $\pi_t^G = \theta_t^0 S_t^0 + \theta_t S_t.$

time t no arbitrage price of G.

self-financing

3) $d\pi_t^G = \theta_t^0 dS_t^0 + \theta_t dS_t.$

what about $d\pi_t^G$?

Assume: $\pi_t^G = \pi(t, S_t)$ apply itô's

won't work.

- $G = G(S_T)$
- $dS_t = S_t\{\mu(t,S_t)dt + \sigma(t,S_t)dB_t\}.$

Try $\pi_t^G = \pi(t, S_t, A_t)$

$(dt)(dA_t)$

$(dS_t)(dA_t)$

$dA_t = \frac{1}{T} S_t dt$

$(dA_t)^2$

PDE will have an extra term. $\underbrace{\pi_a(t,s,a)}_{\frac{\partial \pi(t,s,a)}{\partial a}} \frac{1}{T} \cdot s$

$e^{-rt}\pi_t = \underbrace{E^Q\left[\frac{G}{S_T^0} \mid F_t\right]}_{\text{martingale}}$

$d(e^{-rt}\pi_t) = \{ \quad \}dt + \{ \quad \}dB_t$

$\pi_t = \pi(t, S_t, A_t, V_t)\cdots$

Barrier Option.

Up and Out call option

$(S_T - K)^+ I\{M_T < B\}$

- B is a fixed barrier level (at time 0).
- $M_t := \sup_{0 \le u \le t} S_u \quad \forall t \in [0,T].$

running maximum

$\to (S_T - K) I\{S_T \ge K, M_T < B\}$

$\pi_t^G = e^{-r(T-t)} E^Q[(S_T - K) I\{S_T \ge K, M_T < B\} \mid F_t]$

We used to know the joint distribution. of (S_T, M_T)

How to price.

$$G = \max_{0 \le t \le T} S_t - S_T \quad ?$$

4/4/2023

1) Suppose $\exists$ a self-financing and replicating strategy $\bar\theta = (\theta^0, \theta)$

$\Downarrow$ Law of one price

$\pi_t^G = X_t^{\bar\theta}$ ← time-t value of portfolio $\bar\theta$.

↑ no-arbitrage price

$\pi_t^G = X_t^{\bar\theta} = \theta_t^0 S_t^0 + \theta_t S_t \qquad \theta^0 = \dfrac{\pi_t^G - \theta_t S_t}{S_t^0}$.

2) Suppose $\pi_t^G = \Pi(t, S_t, M_t)$

3) To determine $\bar\theta$ and Π, we let $d\pi(t, S_t, M_t) = dX_t^{\bar\theta}$

↑ Itô's formula ↑ self-financing condition

$d\pi(t,S_t,M_t)$
$= \pi_t(t,S_t,M_t) + \pi_S(t,S_t,M_t)\mu_t S_t dt + \pi_S(t,S_t,M_t)\sigma_t S_t dB_t + \frac{1}{2}\pi_{SS}(\sigma_t S_t)^2 dt + \boxed{\Pi_M dM_t}$.

$\Downarrow$ 0

$dX_t^{\bar\theta} = \theta_t^0 dS_t^0 + \theta_t dS_t$.
$= [\Pi(t, S_t, M_t) - \theta_t S_t] r_t dt + \theta_t \mu_t S_t dt + \underline{\theta_t \sigma_t S_t dB_t}$.

a) Equating "dB_t" terms gives, as before, $\theta_t = \Pi_S(t, S_t, M_t)$,

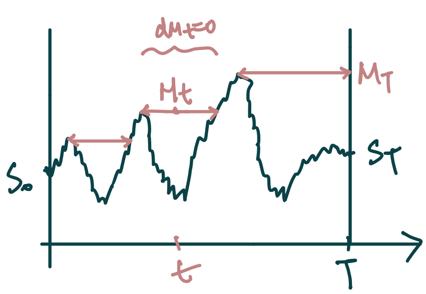
b) Equating "dt" terms gives, as before, the PDE
$\pi_t + \pi_S rS + \frac{1}{2}\pi_{SS}\sigma^2 S^2 - r\Pi = 0$. ✓.

! a) and b) holds, provided dM_t in neither "dt" nor "dB_t" term
Finally, equate "dM_t" term: $\underline{\Pi_M(t, S_t, M_t) dM_t = 0}$.

Verification: $t \in \{u : S_u < M_u\}$ then $dM_t = 0$.

Apply Itô's $\to e^{-rt}\Pi(t, S_t, M_t)$ and show that it is a martingale.

- Recall, $[M, M]_t = 0$.
  But if $dM_t = g_s dB_s$, then $[M, M]_t = \int_0^t g_s^2 ds$.
  a contradiction (unless $g_s \equiv 0$)
- what about
  $M_t = M_0 + \int_0^t \phi_u du$
  i.e. $dM_t = \phi_t dt$. ?



$[0, T] = \underline{\{u \in [0,T] : S_u < M_u\}} \cup \{u \in [0,T] : S_u = M_u\}$.

Countably many intervals

- if $dM_t = \phi_t dt$, we must have that $\phi_t = 0$ if $t \in \{u \in [0,T] : S_u < M_u\}$

- What about $t \in \{u \in [0,T] : S_u = M_u\}$? It seems that "→" contains intervals. wrong!

Suppose $S_t = M_t$ for $t \in [\alpha, \beta]$. This cannot happen since $(dS_t)^2 = \sigma^2 dt$, $(dM_t)^2 = 0$

→ $\{u \in [0,T] : S_u = M_u\}$ consists of points. would be unctb.

One can show $Leb(\{u \in [0,T] : S_u(\omega) = M_u(\omega)\}) = 0$ for a.e. $\omega \in \Omega$.

$$M_t = M_0 + \int_0^t \phi_u du = M_0 + \underbrace{\int_0^t \mathbb{1}_{\{u : M_u > S_u\}} \phi_u du}_{\phi_u = 0} + \int_0^t \mathbb{1}_{\{u : M_u = S_u\}} \phi_u du$$
$$= M_0 + \int_0^t \mathbb{1}_{\{u : M_u = S_u\}} \phi_u du.$$
$$= M_0 \quad \text{since } Leb(u : M_u = S_u) = 0.$$

$Leb((a,b)) = b-a$.
$\int f(x)dx = \int_a^b dx = b-a$
$f(x) = \mathbb{1}_{(a,b)}(x)$.

So if $dM_t = \phi_t dt$, then $M_t = M_0$. But this cannot happen!

? at singleton where $t \in (u \in [0,T] : S_u = M_u)$ times when M increases!

Hence to assume that $\pi_m(t, S_t, M_t) dM_t = 0$ $\forall t$.

we require $\pi_m(t, m, m) = 0$. $\forall t \in [0,T)$, $\forall m \in \mathbb{R}_+$

Of course, don't forget the boundary condition $\pi(T, s, m) = m - s$. $\forall m \geq s$

Can we price a payoff $G \in \mathcal{F}_T$, even if G is not a function of $S_T, A_T, M_T \cdots$?

↓

what if a PDE / hedging argument fails?

Recall: we need to determine $\boxed{(\pi_t^G)_{t \in [0,T]}}$ s.t. $(\bar{S}_t, \pi_t^G)_{t \in [0,T]}$ is arbitrage-free, provided $\boxed{\bar{S} \text{ is arbitrage-free.}}$

EMM:

1. $\bar{S}$ is arbitrage-free if $\exists$ a measure $\mathbb{Q}$ s.t.
   a) $\mathbb{Q}$ is equivalent to $\mathbb{P}$ (physical measure)
   $\mathbb{P}[A] = 0 \Leftrightarrow \mathbb{Q}[A] = 0$.
   b) $\frac{\bar{S}}{S^0}$ is a $\mathbb{Q}$-martingale.

$\frac{S^0}{S^0}, \frac{S^1}{S^0}, \ldots \frac{S^d}{S^0}$

2. Suppose $\bar{S}$ is arbitrage-free, i.e. $\exists$ at least one EMM for $\frac{\bar{S}}{S^0}$.
   Then setting
   $$\pi_t^G = S_t^0 \mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0} \middle| \mathcal{F}_t\right] \quad \forall t \in [0,T].$$
   does not introduce arbitrage into extended market. $(\bar{S}, \pi^G)$

For 1 we will need multidimensional Girsanov's thm.

Warning: there will be a case when we cannot find an EMM $\mathbb{Q}$ via Girsanov's thm

$\mathbb{Q}$ is Girsanov's thm the only method that allows to construct an EMM.

There is no other way of $\mathbb{F}=\mathbb{F}^B$. $\Rightarrow$

$\mathbb{Q}[A]=\mathbb{E}^{\mathbb{P}}[I_A Z]$.

- $\mathbb{E}^{\mathbb{P}}[Z]=1$
- $Z>0$ $\mathbb{P}$-a.s.

8 fail

- H_t
- $dZ_t = -Z_t H_t \, dB_t$.
- check if Z is a $\mathbb{P}$-martingale
- $Z=Z_T$

$(\Omega, F, \mathbb{F}, \mathbb{P})$

4/6/2023 $\bar{S} = (S^0, S^1, \cdots S^d)$. Are there any arbitrage opportunities?

a) $S^0 \equiv 1$ and suppose $S = (S^1, \cdots S^d)$ are martingales under some $\mathbb{Q} \sim \mathbb{P}$.
Then if $\bar{\theta}$ is self-financing:

$$X_t^{\bar{\theta}} = X_0^{\bar{\theta}} + \sum_{i=1}^{d} \int_0^t \theta_u^i \, dS_u^i$$

each $\int_0^t \theta_u^i \, dS^i$ is an integral w.r.t a $\mathbb{Q}$-martingale
we expect $t \mapsto \int_0^t \theta_u^i dS_u^i$ and thus that $t \mapsto X_t^{\bar{\theta}}$ to be a martingale
But then can $\bar{\theta}$ be an arbitrage opportunity?

Let $X_0^{\bar{\theta}} = 0$ and $X_T^{\bar{\theta}} \geq 0$ a.s.

$0 \leq \mathbb{E}[X_T^{\bar{\theta}}]$

(super)-martingal $\leq$ $= X_0^{\bar{\theta}} = 0$

$\Rightarrow X_T^{\bar{\theta}} = 0$ a.s

Note that $\mathbb{E}[X_T^{\bar{\theta}}] = 0$ & $X_T^{\bar{\theta}} \geq 0$. Hence if $\mathbb{Q}[X_T^{\bar{\theta}} > 0] > 0$, then we cannot have $\mathbb{E}[X_T^{\bar{\theta}}] = 0$.

If an EMM for $\bar{S}$ exists then $\nexists$ an arbitrage opportunity among sufficiently integrable trading strategies.

(For practical ... ?

b) $S^0 \neq 1$. e.g $S_t^0 = e^{\int_0^t r_u du}$ and suppose $\exists$ an EMM for $\bar{S}$. i.e, $\bar{S}/S^0$ is a $\mathbb{Q}$-martingale.
if $\bar{\theta}$ self-financing,

$$\hat{X}_t^{\bar{\theta}} := \frac{X_t^{\bar{\theta}}}{S_t^0} = \hat{X}_0^{\bar{\theta}} + \sum_{i=1}^{d} \int_0^t \theta_u^i \, d\underbrace{\hat{S}_u^i}_{\mathbb{Q}\text{-martingale.}}$$

Again, for "sufficiently nice" $\bar{\theta}$, $\hat{X}_t^{\bar{\theta}}$ is a $\mathbb{Q}$-martingale.

Take $\bar{\theta}$ s.t. $X_0^{\bar{\theta}} = 0$ & $X_T^{\bar{\theta}} \geq 0$ a.s.

$\hat{X}_0^{\bar{\theta}} = \frac{X_0^{\bar{\theta}}}{S_0^0} = 0$ & $\hat{X}_T^{\bar{\theta}} = \frac{X_T^{\bar{\theta}}}{S_T^0} \geq 0$ a.s

$0 = \mathbb{E}^{\mathbb{Q}}[\hat{X}_T^{\bar{\theta}}] = \hat{X}_0^{\bar{\theta}} = 0$

$\Rightarrow \mathbb{E}^{\mathbb{Q}}[\hat{X}_T^{\bar{\theta}}] = 0$ "+" $\hat{X}_T^{\bar{\theta}} \geq 0$

$\Rightarrow \hat{X}_T^{\bar{\theta}} = 0$ $\mathbb{Q}$-a.s

$\mathbb{P}$-a.s.

$X_T^{\bar{\theta}} = S_T^0 \hat{X}_T^{\bar{\theta}} = 0$ $\mathbb{P}$-a.s.

$\Rightarrow \mathbb{P}[X_T^{\bar{\theta}} > 0] = 0$.

FTAP I

<u>Thm 1</u>: if $\exists$ an EMM for $\bar{S}$, then for any "sufficiently integrable" (e.g) bounded $\bar{\theta}$ is NOT an arbitrage opportunity.

<u>Thm 2</u>. No arbitrage $\Rightarrow$ $\exists$ an EMM.

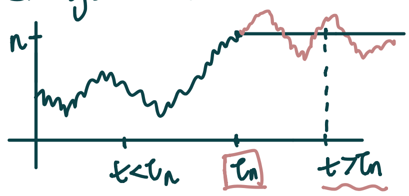
Bjorko book: $\bar{\theta}$ admissible if $\exists$ $\alpha \in \mathbb{R}_+$ s.t $\forall t \in [0,T]$, $X_t^{\bar{\theta}} \geq -\alpha$ a.s

E.g. <u>Doubling strategy</u>:

Suppose there is a game where you can get a either "red" or "black". If you set 1 on "red and it is red indeed. then you receive 2.
If you lost then borrow 2×1 and bet on "red" again
⋮
at each round, given that you lost all the precious rounds, double the bet and keep betting on "red" (In discrete time, this is NOT an arbitrage!)

We have that $\tilde{X}_t^{\tilde{\varphi}} = \tilde{X}_0^{\tilde{\varphi}} + \sum_{i=1}^{d} \int_0^t \varphi_u^i \, d\tilde{S}_u^i$ $\geq -\alpha$

- $\tilde{S}$ is a $\mathbb{Q}$-martingale
- In general, each $\int_0^t \varphi_u^i \, d\tilde{S}_u^i$ is only a local martingale.



$\exists \, (\tau_n)_{n \geq 1}$ s.t $\tau_n \leq \tau_{n+1}$ and $\tau_n \to +\infty$ as $n \to \infty$

M is a local martingale if $t \mapsto M_{t \wedge \tau_n}$ is a martingale

thm 1 is more important.

a) It allows "easily" to check wether $\bar{S}$ is arbitrage-free.

b) we can now price derivative contracts G:

Let Π_t^G be given by

$$\Pi_t^G := \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0} \cdot G \,\middle|\, \mathcal{F}_t\right] = S_t^0 \, \mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0} \,\middle|\, \mathcal{F}_t\right].$$

G is any $\mathcal{F}_t$-measurable r.v.

Is the extended market $(\bar{S}, \Pi^G)$ arbitrage-free?

By thm 1. we need to find $\tilde{\mathbb{Q}} \sim \mathbb{P}$ s.t $\frac{S_t^1}{S_t^0}, \frac{S_t^2}{S_t^0} \dots \frac{S_t^d}{S_t^0}, \frac{\Pi_t^G}{S_t^0}$, are all $\tilde{\mathbb{Q}}$-martingale

Let $\tilde{\mathbb{Q}} := \mathbb{Q}$?

- $\frac{\bar{S}}{S^0}$ are already $\mathbb{Q}$-martingales.
- $\frac{\Pi_t^G}{S_t^0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0} \,\middle|\, \mathcal{F}_t\right]$ $t \in [0,T]$ which is a $\mathbb{Q}$-martingale provided $\frac{G}{S_T^0}$ is integrable.

Warning: there may be many EMM!

if $\mathbb{Q}^1$ and $\mathbb{Q}^2$ are two EMMs. do we have $\mathbb{E}^{\mathbb{Q}^1}[$? 12:29

In order to apply thm 1. we need at least one EMM.

1) Market model (under $\mathbb{P}$)

d-dimensional

$$\frac{dS_t}{S_t} = \underset{d \times 1}{\begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^d \end{bmatrix}} dt + \underset{d \times n}{\begin{bmatrix} (\sigma_{11})_t & \cdots & (\sigma_{1n})_t \\ \vdots & & \vdots \\ (\sigma_{d1})_t & \cdots & (\sigma_{dn})_t \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} dB_t^1 \\ \vdots \\ dB_t^n \end{bmatrix}}$$

$$\frac{dS_t^0}{S_t^0} = r_t \, dt$$

How to find $\mathbb{Q} \sim \mathbb{P}$ s.t.

$$\frac{dS_t}{S_t} = \begin{bmatrix} r_t \\ \vdots \\ r_t \end{bmatrix} dt + \underbrace{\begin{bmatrix} (\sigma_{11})_t & \cdots & (\sigma_{1n})_t \\ \vdots & & \vdots \\ (\sigma_{d1})_t & \cdots & (\sigma_{dn})_t \end{bmatrix}}_{\text{Same volatility matrix}} \begin{bmatrix} dW_t^1 \\ \vdots \\ dW_t^n \end{bmatrix}$$

and $W = (W^1 \dots W^n)$ is a $\mathbb{Q}$-BM?

Way out! n-dimensional Girsanov's thm

Girsanov

- Let $B = (B^1, \dots B^n)$ be a standard BM. on $\mathbb{R}^n$ (under $\mathbb{P}$)

· Fix $\mathbb{R}^n$-valued process M:

$$M_t = (M_t^1 \cdots M_t^n).$$

· Define a stochastic exponential:

on $\mathbb{R}$ ⊛ $\begin{cases} dZ_t = -Z_t \underbrace{M_t}_{\text{row}} \underbrace{dB_t}_{\text{column}} \\ Z_0 = 1 \end{cases}$

$$M_t dB_t = (M_t^1 \cdots M_t^n)\begin{bmatrix} dB_t^1 \\ \vdots \\ dB_t^n \end{bmatrix} = \sum_{i=1}^n M_t^i dB_t^i$$

Exercise: use multi-dimensional Itô's formula to solve ⊛.

· If $(Z_t)_{t\in[0,T]}$ is a $\mathbb{P}$-martingale, then $\mathbb{E}^{\mathbb{P}}[Z_T] = Z_0 = 1$.

· Then define $\mathbb{Q}[A] := \mathbb{E}^{\mathbb{P}}[I_A Z_T]$

$\Rightarrow \mathbb{Q}$ is equivalent to $\mathbb{P}$.

Furthermore, $W_t = \begin{bmatrix} W_t^1 \\ \vdots \\ W_t^n \end{bmatrix} := \begin{bmatrix} B_t^1 + \int_0^t M_s^1 ds \\ \vdots \\ B_t^n + \int_0^t M_s^n ds \end{bmatrix}$ is a $\mathbb{Q}$-BM.

Check martingality of Z. use (in worst cases) Novikov's condition:

$$\mathbb{E}^{\mathbb{P}}[\exp(\tfrac{1}{2}\int_0^T M_s^{Tr} M_s\, ds)] < \infty$$

$$M_s^{Tr} M_s = [M_s^1 \cdots M_s^n]\begin{bmatrix} M_s^1 \\ \vdots \\ M_s^n \end{bmatrix} = \sum_{i=1}^n (M_s^i)^2.$$

E.g. this works if M is bounded.

From Girsanov:

$$\begin{bmatrix} dB_t^1 \\ \vdots \\ dB_t^n \end{bmatrix} = \begin{bmatrix} dW_t^1 - \overset{M_t^1 dt}{\cancel{\int_0^t M_s^1 ds}} \\ \vdots \\ dW_t^n - \overset{M_t^n dt}{\int_0^t M_s^n ds} \end{bmatrix}$$

$$\frac{dS_t}{S_t} = \underbrace{\begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^d \end{bmatrix} dt - \begin{bmatrix} (\sigma_{11})_t & \cdots & (\sigma_{1n})_t \\ \vdots & & \vdots \\ (\sigma_{d1})_t & \cdots & (\sigma_{dn})_t \end{bmatrix}\begin{bmatrix} M_t^1 dt \\ \vdots \\ M_t^n dt \end{bmatrix}}_{\begin{bmatrix} r_t \\ \vdots \\ r_t \end{bmatrix} dt.} + \underbrace{\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} dW_t^1 \\ \vdots \\ dW_t^n \end{bmatrix}}$$

$$\underset{d\times 1}{\mu_t} - \underset{1\times 1}{r_t} - \underset{d\times n}{\sigma_t} \underset{n\times 1}{M_t} = 0.$$

· $d>n$ no solution

· $d=n$ unique solution

· $d<n$ infinitely many.

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$$\frac{dS_t}{S_t} = \begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^d \end{bmatrix} dt + \begin{bmatrix} \sigma_t^{11} & \cdots & \sigma_t^{1n} \\ \vdots & & \vdots \\ \sigma_t^{d1} & \cdots & \sigma_t^{dn} \end{bmatrix}\begin{bmatrix} dB_t^1 \\ \\ dB_t^n \end{bmatrix}$$

$$dS_t^0 = S_t^0 r\, dt$$

To conclude that $\bar{S} = (S^0, S^1 \dots S^d)$ is arbitrage-free, we need to find at least 1 EMM.

· Girsanov's thm

· $H_t = (H_t^1, \cdots H_t^n)$ $\quad t \in [0,T]$

· $dZ_t = -Z_t H_t \, dB_t$; $Z_0 = 1$

$\Updownarrow$

$$Z_t := \exp\left(\sum_{i=1}^{n} \left\{ -\int_0^t H_s^i \, dB_s^i - \frac{1}{2}\int_0^t (H_s^i)^2 \, ds \right\}\right)$$

· If Z_t is a $\mathbb{P}$-martingale, then $\mathbb{E}^{\mathbb{P}}[Z_T] = 1 \Rightarrow \mathbb{Q}[A] = \mathbb{E}^{\mathbb{P}}[I_A Z_T] \quad \forall A \in \mathcal{F}_T$

· $$W_t = \underbrace{\begin{bmatrix} W_t^1 \\ \vdots \\ W_t^n \end{bmatrix}}_{\text{BM under } \mathbb{Q}} := \begin{bmatrix} B_t^1 + \int_0^t H_s^1 \, ds \\ \vdots \\ \underbrace{B_t^n}_{\text{BM under } \mathbb{P}} + \int_0^t H_s^n \, ds \end{bmatrix}$$

$u \le t$ $\quad \mathbb{E}[B_t^1 + \int_0^t H_s^1 \, ds \mid \mathcal{F}_u] = B_u^1 + \int_0^u H_s^1 \, ds + \mathbb{E}[\int_u^t H_s^1 \, ds \mid \mathcal{F}_u] + \cdots$

$+ \mathbb{E}[\int_u^t H_s^n \, ds \mid \mathcal{F}_u]$?

12:40.

using that $$\begin{bmatrix} B_t^1 \\ \vdots \\ B_t^n \end{bmatrix} = \begin{bmatrix} W_t^1 - \int_0^t H_s^1 \, ds \\ \vdots \\ W_t^n - \int_0^t H_s^n \, ds \end{bmatrix}$$

we have that under $\mathbb{Q}$

$$\frac{dS_t}{S_t} = \begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^d \end{bmatrix} dt + \sigma_t \begin{bmatrix} dW_t^1 - H_t^1 \, dt \\ \vdots \\ dW_t^n - H_t^n \, dt \end{bmatrix}$$

$$= \begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^d \end{bmatrix} dt - \sigma_t \begin{bmatrix} H_t^1 \\ \vdots \\ H_t^n \end{bmatrix} dt + \sigma_t \begin{bmatrix} dW_t^1 \\ \vdots \\ dW_t^n \end{bmatrix}$$

$$= \begin{bmatrix} \mu^1 - \sum_{j=1}^{n} \sigma_t^{1j} H_t^j \\ \vdots \\ \mu^d - \sum_{j=1}^{n} \sigma_t^{dj} H_t^j \end{bmatrix} dt + \sigma_t \begin{bmatrix} dW_t^1 \\ \vdots \\ dW_t^n \end{bmatrix}$$

we want $d\left(\frac{S}{S^0}\right)_t = \underbrace{\{ \quad \}}_{=0} dt + \{ \quad \} dW_t$ under $\mathbb{Q}$

H follows that $\mathbb{Q}^H$ is an EMM if

$$\begin{bmatrix} \mu_t^1 - \sum_{j=1}^{n} \sigma_t^{1j} H_t^j \\ \vdots \\ \mu_t^d - \sum_{j=1}^{n} \sigma_t^{dj} H_t^j \end{bmatrix} = \begin{bmatrix} r \\ \vdots \\ r \end{bmatrix}$$

need to choose $H_t^1 \dots H_t^n$ s.t. ⊛ $\mu_t^i - \sum_{j=1}^{n} \sigma_t^{ij} H_t^j - r = 0$ $\quad \forall i = 1 \dots d$

d-equations

1) Suppose there is "no" BM at all.

$\begin{cases} dS_t^0 = S_t^0 r\,dt \\ S_0^0 = 1 \end{cases}$; $dS_t = S_t \mu\,dt$, $S_0 > 0$

$n = 0 < 1 = d$

⊛ does not have a solution, unless $r = \mu$.

2) $n = 1 < 2 = d$; B is a 1-dim BM.

$\begin{cases} dS_t^1 = 2dt + dB_t & ; \ S_0^1 = 1 \\ dS_t^2 = dB_t & ; \ S_0^2 = 1 \end{cases}$

] exercise : find an arbitrage opportunity

Construct θ s.t. $X_0^\theta = 0$ and $dX_t^\theta = \tilde{\mu}_t dt$, $\tilde{\mu}_t > 0$

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⊛ : $2 - M_t - r = 0$, $M_t = 2 - r$

$0 - M_t - r = 0$ $M_t = -r$.

In general, there's no solution if $n < d$!

This implies that we cannot specify a R.V Z (via Z_T) using Girsanov's thm

maybe there is another way to choose Z. $\mathbb{Q}[A] = \mathbb{E}^{\mathbb{P}}[1_A Z]$ ← $\begin{cases} \cdot Z > 0 \quad \mathbb{P}\text{-a.s} \\ \cdot \mathbb{E}^{\mathbb{P}}[Z] = 1 \end{cases}$

and then $\mathbb{Q}$ s.t $\mathbb{Q}$ is an EMM.

If $\mathbb{F} = \mathbb{F}^b$, then Girsanov's thm is the only way to define/construct EMM.

If $n = d$. usually we have the unique solution:

$(\mu - r) - \sigma M = 0 \implies (\mu_t - r) = \sigma_t M_t$

$\mu = \begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^d \end{bmatrix}$ $\quad r = \begin{bmatrix} r \\ \vdots \\ r \end{bmatrix}$ $\quad \sigma_t^{-1}(\mu_t - r) = M_t$.

· If $n > d$ infinitely many solutions

example: Heston

$dS_t^0 = S_t^0 r\,dt$

$dS_t = S_t \mu_t\,dt + S_t \tilde{\sigma}_t\,dB_t^1$

$d\tilde{\sigma}_t = \alpha_t\,dt + \beta_t\{\rho_t^1 dB_t^1 + \rho_t^2 dB_t^2\}$

MRT (martingale representation thm)

Suppose $\mathbb{F} = \mathbb{F}^b$, then for every martingale $M = (M_t)_{t \in [0,T]}$

$\exists\ L = (L_t)_{t \in [0,T]}$ s.t. $M_t = M_0 + \int_0^t L_s\,dB_s$

M_0: $\mathcal{F}_0^b$-measurable, but $B_0 = 0$

$\sigma(B_0) = \{\emptyset, \Omega\}$

$\implies M_0$ is a constant.

Remark: If $B = (B_t^1 \cdots B_t^d)$, then MRT reads as $M_t = M_0 + \sum_{i=1}^d \int_0^t L_s^i\,dB_s^i$

· Suppose $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

$\implies \frac{d\mathbb{Q}}{d\mathbb{P}} = Z > 0$ $\mathbb{P}$-a.s.

· Define the Radon–Nikodym derivatives process

$$Z_t := \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \Big| \mathcal{F}_t^B\right] = \mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{F}_t^B]$$

Since $\mathbb{E}^{\mathbb{P}}[Z] = 1$, Z_t is a $\mathbb{P}$-martingale.

[$\mathbb{E}^{\mathbb{P}}[|X|] < +\infty$, then $X_t = \mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_t]$ is a martingale]

by MRT. $dZ_t = L_t \, dB_t$

$$= Z_t\left(\frac{L_t}{Z_t}\right) dB_t$$

$$= -Z_t\left(\underbrace{-\frac{L_t}{Z_t}}_{H_t}\right) dB_t$$

$$dZ_t = -Z_t H_t \, dB_t$$

$\mathbb{E}^{\mathbb{P}}[Z_T] = 1$.

$\mathbb{E}^{\mathbb{P}}[Z_T] = \mathbb{E}^{\mathbb{P}}[Z]$

For any equivalent measure change, $\exists$ a girsanov-type measure change

If $n < d \Rightarrow$ we cannot solve $\circledast$

$\Downarrow$ MRT.

EMM $\nexists$

$\Downarrow$ FTAP I

$\exists$ an arbitrage opportunity

$\Downarrow$

Find it!

<u>$n=d$</u> : unique solution.

$\downarrow$

all claims can be replicated.

Def^n : $\mathfrak{Z}$ is complete if for every $G \geq 0$ ($\mathcal{F}_T$-measurable)

$\exists \ \Theta$ s.t. $G = X_T^{\Theta}$ a.s.

self-financing replication.

(G, S_T)
(G, S_T, A_T)
(G, S_T, M_T)

<u>Lemma</u> : Let $G \geq 0$ be an $\mathcal{F}_T$ measurable payoff. Let $\mathbb{Q}$ be an EMM. Suppose G is "nice"

$$M_t := \mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0} \Big| \mathcal{F}_t\right] \text{ is a martingale}$$

The following are equivalent:

i) M admits a stochastic integral representation.

$$M_t = x + \sum_{i=1}^{n} \int_0^t L_s^i \, dB_s^i$$

ii) G can be replicated

Proof:

i) $\Rightarrow$ ii) we need to construct $\theta_t^0, \theta_t^1 \cdots \theta_t^d$. s.t

$$G = \sum_{i=0}^{d} \theta_T^i S_T^i .$$

$$\frac{G}{S_T^0} = \sum_{i=0}^{d} \theta_T^i \hat{S}_T^i$$

$$\theta_t^0 := M_t - \sum_{i=0}^{d} \theta_t^i \hat{S}_t^i$$

$$\theta_t^i := \tilde{L}_t^i$$

Check that θ is
a) self-f
b) Replication.

$$\hat{X}_t^{\theta} \overset{?}{=} M_t .$$

$$M_t = x + \sum_{i=1}^{n} \int_0^t L_s^i \, dB_s^i \cdot \frac{\sigma_s \hat{S}_s^i}{\sigma_s \hat{S}_s^i}$$

$$= x + \sum_{i=1}^{d} \int_0^t \tilde{L}_s^i \, d\hat{S}_s^i$$

Note $d\hat{S}_s^i$ is ... so

$$M_t = x + \sum \int_0^t \tilde{L}_s^i \, d\hat{S}_s^i$$

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?

S is arbitrage free.

- Sell G for $\mathbb{E}^{\mathbb{Q}}[\frac{G}{S_T^0}]$ at time 0.
  Does there exists a self-financing, replicating θ?
  $$\hat{X}_T^{\theta} = \frac{G}{S_T^0}$$
- Market is complete if any G is replicable / attainable.
- FTAP II: the following are equivalent (provided S is arbitrage free)
  a) S is complete
  b) there's only 1 EMM $\mathbb{Q}$

?

$\mathbb{Q}$ (and $\tilde{\mathbb{Q}}$) is defined by setting $\mathbb{Q}[A] \in [0,1]$ $\forall A \in \mathcal{F}_T$.
we need to show that

$$\mathbb{Q}(A) = \tilde{\mathbb{Q}}(A) \quad \forall A \in \mathcal{F}_T .$$

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{Q}}[I_A]$$

time $\to 0$
price of G under $\mathbb{Q}$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T^0}{S_T^0} I_A\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0}\right] \quad \text{when } G_i = S_T^0 I_A.$$

we need to show that the no arbitrage price is unique (independent of EMM)
But by a) we can replicate any claim
↳ replicate G_A:

$\exists\ \bar\theta_A$ s.t. $X_T^{\bar\theta_A} = G_A$

↓ Law of 1 price

$X_t^{\bar\theta_A}$ is the unique no arbitrage price of G_A at time $t \in [0,T]$

↓ choose $t=0$

$X_0^{\bar\theta_A} = \sum_{i=0}^{d} \theta_0^{iA} S_0^i$ is the unique time 0 no arbitrage price of G_A

↓ $\mathbb{Q}(A) = X_0^{\bar\theta_A} = \tilde{\mathbb{Q}}(A) \quad \forall A \in \mathcal{F}_T.$

<u>Lemma</u>: Let G be an arbitrary claim. Let $\mathbb{Q}$ be an EMM.
Suppose G is $\mathbb{Q}$-integrable.
Define

$\mathbb{Q}$-martingale $\to M_t := \mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0} \middle| \mathcal{F}_t\right] \quad t \in [0,T].$

The following are equivalent:

i) $\exists\ L = (L_t)_{t\in[0,T]}$ where $L_t = (L_t^1, \cdots, L_t^d)$
s.t.

$$M_t = x + \sum_{i=1}^{d} \int_0^t L_s^i \, d\hat S_s^i \quad \text{for some } x \in \mathbb{R}.$$

ii) G is replicable $\iff$ $\boxed{\hat G = \frac{G}{S_T^0} \text{ is replicable}}$ | $X_T^{\bar\theta} = G$. $\hat X_T^{\bar\theta} = \frac{G}{S_T^0} = \hat G$.

proof: (i) ⇒ (ii):

$$L_t^0 := M_t - \sum_{i=0}^{d} L_t^i \hat S_t^i$$

Claim: $(L^0, L) = (L^0, L^1 \cdots L^d)$ is self-financing and it replicates G.

- $\hat X_t^L = L_t^0 \hat S_t^0 + \sum_{i=1}^{d} L_t^i \hat S_t^i$

$\hat S_t^0 = \frac{S_t^0}{S_t^0} = 1$

$$= L_t^0 + \sum_{i=1}^{d} L_t^i \hat S_t^i$$

$$= M_t.$$

$$= x + \underbrace{\sum_{i=1}^{d} \int_0^t L_s^i \, d\hat S_s^i}_{\text{self-financing}}.$$

- $\hat X_T^L = M_T = \mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0} \middle| \mathcal{F}_T\right] = \frac{G}{S_T^0}$ ∎

(ii) ⇒ (i):

now we assume that $\exists\ \bar\theta$ s.t. (self-financing.)

$$\hat X_T^{\bar\theta} = \frac{G}{S_T^0}$$

Then $M_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0} \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\hat X_T^{\bar\theta} \middle| \mathcal{F}_t\right]$

$= \hat{X}_t^{\theta}$

S-F: $= \hat{X}_0^{\theta} + \sum_{i=1}^{d} \int_0^t \theta_s^i \, d\hat{S}_s^i$

★ Back to complete market issue

WTS: $\mathbb{Q}$ is the unique EMM $\Rightarrow$ market is complete.

Consider $\mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0} \middle| \mathcal{F}_t\right]$

$\boxed{\text{If}}$ $\mathbb{E}^{\mathbb{Q}}\left[\frac{G}{S_T^0} \middle| \mathcal{F}_t\right] = x + \sum_{i=1}^{d} \int_0^t L_s^i \, d\hat{S}_s^i$. then apply previous result.

$d\hat{S}_s^i = \hat{S}_s^i \sigma_s^i \, dB_s$

Jacod's thm: Let $\mathbb{P}$ be a (convex) set of EMMs $\mathbb{Q}$.

$\mathbb{Q}_1, \mathbb{Q}_2 \in \mathbb{P}$, then $\lambda \mathbb{Q}_1 + (1-\lambda)\mathbb{Q}_2 \in \mathbb{P}$. $\lambda \in [0,1]$.

proof: a) $\lambda \mathbb{Q}_1 + (1-\lambda)\mathbb{Q}_2 = \mathbb{Q}^{\lambda}$ is a prob. measure

wts $\rightarrow$ b) $\mathbb{E}^{\mathbb{Q}^{\lambda}}\left[\hat{S}_t^i \middle| \mathcal{F}_s\right] = \hat{S}_s^i$.

use partial averaging:

$= \lambda \hat{S}_s^i + (1-\lambda)\hat{S}_s^i$

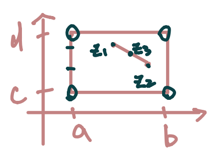
$= \lambda \mathbb{E}^{\mathbb{Q}_1}\left[\hat{S}_t^i \middle| \mathcal{F}_s\right] + (1-\lambda)\mathbb{E}^{\mathbb{Q}_2}\left[\hat{S}_t^i \middle| \mathcal{F}_s\right]$

The following are equivalent:

i) Every $\mathbb{Q}$-(local) martingale $M = (M_t)_{t \in [0,T]}$ has a stochastic integral representation

$M_t = \overset{x}{M_0} + \sum_{i=1}^{d} \int_0^t L_s^i \, d\hat{S}_s^i$ for some (x, L)

ii) $\mathbb{Q}$ is an extremal point of $\mathbb{P}$.



* $z_1, z_2 \neq z_3$, and $z_3 = \lambda z_1 + (1-\lambda) z_2$ for some $\lambda = [0,1]$.

- If a set consists of 1 element then it is convex set.
  $|C| = 1$. $x_1, x_2 \in C$. $\Rightarrow x_1 = x_2$.
  $x_1\lambda + (1-\lambda)x_2 = x_1 = x_2$.
- If $x \in C$ must be extremal.
  $C \ni x_1, x_2 \neq x \in C$.
  we cannot find elements in C that are not equal to x.

Since by assumption, P has only 1 element $\mathbb{Q}$, $\mathbb{Q}$ is extremal $\Leftarrow$ apply result Jacob.

$d = 1$, $n = 2$.

- $dS_t = S_t(\mu\, dt + \sqrt{\sigma_t}\, dB_t^1)$
  $S_0 > 0$.
- $dS_t^0 = S_t^0 r\, dt$; $S_0^0 = 1$
  $d\sigma_t = \lambda(\tilde{\sigma} - \sigma_t)\, dt + \beta\sqrt{\sigma_t}(\rho\, dB_t^1 + \sqrt{1-\rho^2}\, dB_t^2)$

$\lambda \geq 0$. mean-reversion speed
$\tilde{\sigma} \geq 0$ mean-reversion level
$\beta \geq 0$ vol-of-vol
$\rho \in [-1, 1]$.

$\tilde{B}_t = \rho B_t^1 + \sqrt{1-\rho^2} B_t^2$

$d\tilde{B}_t = \rho\, dB_t^1 + \sqrt{1-\rho^2}\, dB_t^2$ $(B^1, B^2$ independent$)$

$(d\tilde{B}_t)^2 = \rho^2 dt + (1-\rho^2) dt = dt$.

$\tilde{B}$ is a BM. $(dB_t^1)(dB_t^m) = \rho\, dt$

Girsanov's thm

- $(H_t^1, H_t^2)_{t \in [0,T]}$.
- $dZ_t = -Z_t H_t dB_t$, $Z_0 = 1$
- If Z_t is a $\mathbb{P}$-martingale, then $\mathbb{P}[Z_T] = Z_0 = 1$.

and $\mathbb{Q}[A] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_A Z_T]$ is equivalent to $\mathbb{P}$.

and $W = (W^1, W^2)$

$W_t^1 := B_t^1 + \int_0^t H_s^1 ds$

$W_t^2 := B_t^2 + \int_0^t H_s^2 ds$.

is a $\mathbb{Q}$-BM.

we need to choose $H = (H^1, H^2)$ to make $\mathbb{Q}$ an EMM

$$d\hat{S}_t = d\left(\frac{S_t}{S_t^0}\right) = \hat{S}_t\left((\mu - r)dt + \sqrt{\sigma_t}\, dB_t^1\right)$$

$$= \hat{S}_t\left((\mu - r)dt + \sqrt{\sigma_t}\, dW_t^1 - \sqrt{\sigma_t}\, H_t^1\, dt\right)$$

$$= \hat{S}_t\left(\underbrace{(\mu - r - \sqrt{\sigma_t}\, H_t^1)}_{=0 \text{ if } H_t^1 = \frac{\mu - r}{\sqrt{\sigma_t}}} dt + \sqrt{\sigma_t}\, dW_t^1\right).$$

EMM property is not affected by H^2!

infinitely many EMMs!

$\Downarrow$ market incomplete

$\Downarrow$ $\exists$ G s.t. we cannot find Φ with $G = X_T^{\Phi}$.