Math 104B HW 6

Qingze Lan 3046380

In [1]:
```
from numpy import zeros, array, dot, linspace, sin, pi, concatenate
from numpy.linalg import norm
import matplotlib.pyplot as plt
from pandas import DataFrame
```

Problem 1

(a)

In [3]:
```
# Solving tridiagonal systems of linear equations

# Solving upper diagonal system Ux = y
# INPUT: U (linear system matrix, upper diagonal),
# y (column vector)
# OUTPUT: x (unkowns column vector)
# Qingze Lan
# 03/02/2021
def linear_u_solver(U,y):
    x = zeros( len(y) )
    for i in range(len(y)-1,-1,-1):
        coef = [U[i][j]*x[j] for j in range(len(y)-1,i,-1)]
        x[i] = ( y[i]-sum(coef) )/U[i][i]
    return x

# Solving lower diagonal system Lx = y
# INPUT: L (linear system matrix, lower diagonal),
# y (column vector)
# OUTPUT: x (unkowns column vector)
# Qingze Lan
# 03/02/2021
def linear_l_solver(U,y):
    x = zeros( len(y) )
    for i in range(0,len(y)):
        coef = [U[i][j]*x[j] for j in range(0,i)]
        x[i] = ( y[i]-sum(coef) )/U[i][i]
    return x

# LU decomposition for tridiagonal system M = LU
# INPUT: M (tridiagonal matrix), L, U (zeros matrixes)
# OUTPUT: Filling L and U so that:
# L is lower diagonal
# U is upper diagonal
# Qingze Lan
# 03/02/2021
```

```python
def LU_dec_tri(M,L,U):
    U[0][0] = M[0][0]
    L[-1][-1] = 1
    for j in range(0,len(M)-1):
        L[j][j] = 1
        U[j][j+1] = M[j][j+1]
        L[j+1][j] = M[j+1][j]/U[j][j]
        U[j+1][j+1] = M[j+1][j+1] - L[j+1][j]*M[j][j+1]

# Solving a tridiagonal system Mx = y
# INPUT: A (tridiagonal matrix), y (column vector)
# OUTPUT: x (unknown column vector)
# Qingze Lan
# 03/02/2021
def linear_tri_solver(M,y):
    L = zeros( (len(M),len(M)) )
    U = zeros( (len(M),len(M)) )
    LU_dec_tri(M,L,U)
    u = linear_l_solver(L,y)
    x = linear_u_solver(U,u)
    return x
```

(b) We can test our code with

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} x = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

In [4]:

```python
m_1 = array([[1.,2.,0.,0.],[2.,1.,2.,0.],[0.,2.,1.,2.],[0.,0.,2.,1.
]])
y_1 = array([5.,0.,1.,0.])

x_1 = linear_tri_solver(m_1,y_1)
print('Solution to Ax = b')
print(x_1)

print('\nThe product Ax gives us')
print(dot(m_1,x_1))
```

```
Solution to Ax = b
[-6.2  5.6  3.4 -6.8]

The product Ax gives us
[5.00000000e+00 1.77635684e-15 1.00000000e+00 0.00000000e+00]
```

Problem 2

(a) Consider the boundry value problem:

$$-u'' + \pi^2 u = 2\pi^2 \sin(\pi x), \quad 0 < x < 1$$
$$u(0) = u(1) = 0$$

We can find a numerical approximation to the solution of this problem by employing the finite difference method.

We'll approximate the second derivative by a second order finite difference and use a grid of $N+1$ points.

Approximating $v_j \approx u(x_j)$ we need to solve the linear system:

$$\frac{-v_{j-1} + 2v_j - v_{j+1}}{h^2} + \pi^2 v_j = 2\pi^2 \sin(\pi x_j), \quad j = 1, \dots, N-1$$

and $v_0 = v_N = 0$.

Equivalently:

$$\frac{1}{2\pi^2 h^2}\begin{bmatrix} 2+h^2\pi^2 & -1 & 0 & \dots & 0 \\ -1 & 2+h^2\pi^2 & -1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \vdots \\ & & -1 & 2+h^2\pi^2 & -1 \\ 0 & & 0 & -1 & 2+h^2\pi^2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ \\ \vdots \\ v_{N-1} \end{bmatrix} = \begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi x_2) \\ \vdots \\ \\ \vdots \\ \sin(\pi x_{N-1}) \end{bmatrix}$$

We'll use our tridiagonal solver

In [5]:

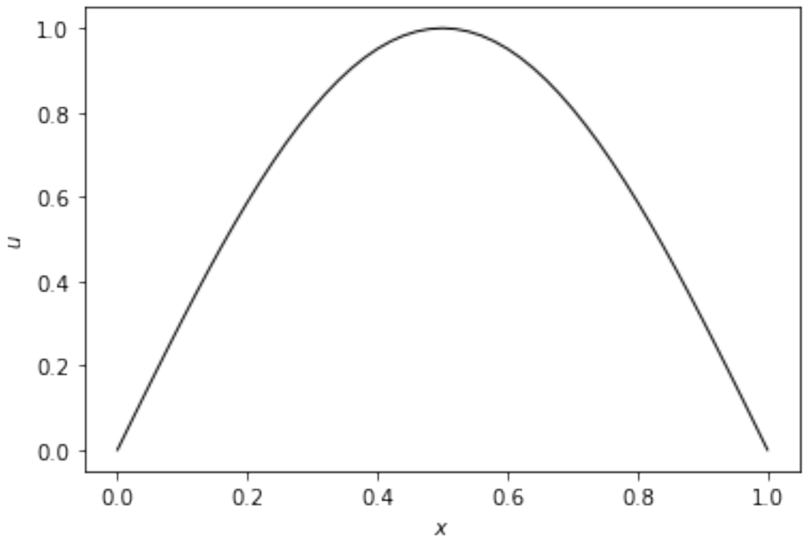
```
# Solving the boundry value problem
# INPUT: N (grid dim)
# OUTPUT: u (approximated solution)
# Qingze Lan
# 03/02/2021

def bvp_solver(N):
    x = linspace(0,1,N + 1)
    h = 1/N
    A = zeros((N - 1, N - 1))
    for i in range(0, N - 1):
        A[i,i] = 2 + h**2*pi**2
        if i > 0:
            A[i, i - 1] = -1
            A[i - 1, i] = -1
    return linear_tri_solver(1/2/h**2/pi**2*A, sin(pi*x)[1:-1])

# N = 50
N1 = 50
u1 = concatenate([[0],bvp_solver(N1),[0]])
x1 = linspace(0,1,N1 + 1)

# Plotting the results

plt.plot(x1,u1, color = 'black', linewidth = 1)
plt.xlabel('$x$')
plt.ylabel('$u$')
plt.show()
```



(b) The exact solution is $u(x) = \sin(\pi x)$.

Since

$$u(0) = \sin(0) = u(1) = \sin(\pi) = 0$$
$$u''(x) = -\pi^2 u(x) \implies -u'' + \pi^2 u = 2\pi^2 u = 2\pi^2 \sin(\pi x)$$

(c) Computing error in the 2-norm. We'll solve again the linear system and we'll compare the errors for $N = 50, 100$.

Since expected numerical error is $O(h^2) = O(N^{-2})$, by doubling N we expect the error to become 4 times smaller.

In [6]:
```
# N = 100
N2 = 100
u2 = concatenate([[0],bvp_solver(N2),[0]])

# Exact solutions
u1r = sin(pi*linspace(0,1,N1 + 1))
u2r = sin(pi*linspace(0,1,N2 + 1))

# Errors
data = {'$N$':[N1,N2], 'Error': [norm(u1r-u1),norm(u2r-u2)] }
DataFrame(data)
```

Out[6]:

	N	**Error**
0	50	0.000822
1	100	0.000291

(d) In real applications we don't know the exact solution, but if we can compute the approximated solution for different N, we can estimate when we are close to the exact solution.

If $v^{(N)}$ is the approximated solution for a grid of N nodes, we can increase sequentially this N to N' - by a factor of 2, 10 for instance - and check if the relative error is smaller than a fixed tolerance ϵ, this is:

$$\frac{\|v^{(N)} - v^{(N')}\|_\infty}{v^{(N')}} \leq \epsilon$$

If we increase N always by the same factor, we can easily estimate the rate of convergence.

One issue can come up from this, because the discretized approximations $v^{(N)}$, $v^{(N')}$ don't have the same dimension. However, one could consider only the components of each solution corresponding to common points. A better solution would be, give the approximated solution in the form of a spline and then calculate this relative error.

Problem 3

Consider the linear system:

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

(a) First two iterationsof Jacobi. The Jacobi iterations are defined by:

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \Leftrightarrow \mathbf{x}^{(k+1)} = N\mathbf{x}^{(k)} + \mathbf{b}$$

Starting with an initial guess $\mathbf{x}^{(0)} = \mathbf{0}$ we have $\mathbf{x}^{(2)} = N\mathbf{x}^{(1)} + \mathbf{b} = N\mathbf{b} + \mathbf{b}$:

$$\mathbf{x}^{(2)} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

(b) First two Gauss-Seidel iterations. Gauss-Seidel iterations are defined by:

$$\begin{aligned} x_1^{(k+1)} &= 2x_2^{(k)} - x_3^{(k)} - 1 \\ x_2^{(k+1)} &= -2x_1^{(k+1)} + 3x_3^{(k)} + 3 \\ x_3^{(k+1)} &= -x_1^{(k+1)} + x_2^{(k+1)} \end{aligned}$$

Using the same initial guess:

$$\begin{aligned} x_1^{(1)} &= -1 \\ x_2^{(1)} &= 2 + 3 = 5 \\ x_3^{(1)} &= 1 + 5 = 6 \end{aligned}$$

and the second iteration gives us:

$$\begin{aligned} x_1^{(2)} &= 10 - 6 - 1 = 3 \\ x_2^{(2)} &= -6 + 18 + 3 = 15 \\ x_3^{(2)} &= -3 + 15 = 12 \end{aligned}$$

(c) We'll use the 2-norm to estimate which one is closer.

$$\begin{aligned} \|\mathbf{x} - \mathbf{x}_j\| &= 4\sqrt{3} \\ \|\mathbf{x} - \mathbf{x}_{gs}\| &= \sqrt{344} \end{aligned}$$

So after two iterations, the approximation given by the Jacobi iteration method is closer to the exact solution $(1, 1, 0)$.

Problem 4

Consider the system:

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 2x_1 + 2x_2 + 2x_3 &= 4 \\ -x_1 - x_2 + 2x_3 &= -5 \end{aligned}$$

The matrices $M\mathbf{x}^{(k+1)} = N\mathbf{x}^{(k)} + b$ for the Jacobi and Gauss-Seidel methods are:

$$M_j = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad N_j = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{gs} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix} \quad N_{gs} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The inverses of the M matrices can be easily found:

$$M_j^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad M_{gs}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

And now:

$$T_{gs} = M_{gs}^{-1} N_{gs} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

which is an upper triangular matrix. Its eigenvalues are in the diagonal and hence $\rho(T_{gs}) = \frac{1}{2} < 1$, proving that the Gauss-Seidel method converges for this linear system.

On the other hand:

$$T_j = M_j^{-1} N_j = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Its characteristic polynomial is: $x^3 + \frac{5}{4}x \implies \rho(T_j) = \frac{\sqrt{5}}{2} > 1$, which proves that the Jacobi method diverges.