

HOMEWORK 7

Due date: Tuesday of Week 8

Exercises: 7.1, 7.2, 7.5, 7.8, 7.9, 8.6, page 439-440 of Artin's book; Exercises: 1.1, 1.2, 2.1, 2.3, page 472.

Problem 1. Consider the abelian group

$$A = C_{30} \oplus C_{49} \oplus C_{12} \oplus C_{25} \oplus C_{40}.$$

Find the invariant divisors of A .

Problem 2. Let R be a PID and M be a free R -module of rank m . Let N be a submodule of M . We know that N is a free module of rank n with $n \leq m$. Show that there exists a basis $\mathcal{B} = \{e_1, \dots, e_m\}$ of M and non-zero elements $a_1, \dots, a_n \in R$ such that:

- (1) the elements $a_1e_1, a_2e_2, \dots, a_ne_n$ form a basis of N over R ;
- (2) we have $a_i | a_{i+1}$ for $i = 1, \dots, n-1$.

The sequence of ideas $(a_1), \dots, (a_n)$ is uniquely determined by the above conditions.

Problem 3. Let $G = \text{GL}_2(\mathbb{Q})$ and $H = \text{GL}_2(\mathbb{Z})$. Determine the double coset

$$H \backslash G / H.$$

Problem 4. Let K/F be a field extension and $\alpha \in K$ is algebraic over F with $\deg(\alpha) = d$. Show that $\{1, \alpha, \dots, \alpha^{d-1}\}$ is a basis of $F[\alpha]/F$.

1. LINEAR OPERATORS AND F.G. MODULES OVER PID

In this section, let F be a field and V be a finite dimensional vector space over F . Let $T : V \rightarrow V$ be a linear operator. We can view V as an $F[x]$ -module by $f(x) \cdot v := f(T)v$ for any $f \in F[x]$.

Problem 5. (1) Show that a subspace $W \subset V$ is T -invariant iff W is a submodule of V ;
 (2) Show that V has a cyclic vector iff V can be generated by a single element as an $F[x]$ -module.

Recall that a subspace $W \subset V$ is called T -admissible if (1) W is T -invariant; and (2) if $f(T)\beta \in W$ for $\beta \in V, f \in F[x]$, then there exists a vector $\gamma \in W$ such that $f(T)\beta = f(T)\gamma$. See Section 7.2 of Hoffman-Kunze. The cyclic decomposition theorem (Theorem 7.3 and its corollary of Hoffman-Kunze) said that W is T -admissible iff there exists another T -invariant subspace W' such that $V = W \oplus W'$.

The following are some generalizations of the above terminology into more general modules. Let R be a general ring and let M be an R -module. A submodule N of M is called a **direct summand** of M if there exists another submodule N' of M such that $M = N \oplus N'$. This is a generalization that there exists another T -invariant subspace W' such that $V = W \oplus W'$.

A submodule N of M is called **pure** if for any $m \times n$ matrix $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \in \text{Mat}_{m \times n}(R)$, and any element $Y = (y_1, \dots, y_m)^t$ with $y_i \in N$, if there exist $X = (x_1, \dots, x_n)^t$ with $x_i \in M$ such that

$$AX = Y,$$

then there exists $X' = (x'_1, \dots, x'_n)^t$ with $x'_i \in N$ such that

$$AX' = Y.$$

The definition of pure submodule looks complicate. Here is a digression.

Problem 6. Suppose that N is a pure submodule of M and there is a commutative diagram of R -modules

$$\begin{array}{ccccc} & R^n & \xrightarrow{f} & R^m & \\ & \downarrow u & & \downarrow v & \\ 0 & \longrightarrow & N & \xrightarrow{i} & M \end{array}$$

Here m, n are positive integers and $i : N \rightarrow M$ denotes the inclusion. Show that there is homomorphism $\phi : R^m \rightarrow N$ such that $u = \phi \circ f$. (We don't require $v = i \circ \phi$.)

Let M be an R -module, a submodule $N < M$ is called **admissible** if for any $r \in R$ and $x \in M$ if $rx \in N$, then there exists an $n \in N$ such that $rx = rn$. This agrees with the notation defined in Hoffman-Kunze when $R = F[x]$ and $M = V$. Note that, a pure submodule is admissible (since pure requires a condition for any $m \times n$).

Problem 7. Let R be a PID. Let M be an R -module and $N < M$ be a submodule. Show that N is a pure submodule iff it is an admissible submodule.

Hint: You need to show any admissible submodule is pure. Use diagonalization. This is not hard.

Problem 8. Let R be a ring and M be an R -module. Let N be a submodule of M . Consider the short exact sequence

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{\pi} M/N \longrightarrow 0.$$

Show that the following are equivalent

- (1) N is a direct summand of M ;
- (2) there exists a homomorphism $s \in \text{Hom}_R(M, N)$ such that $s(x) = x$ for all $x \in N$ (namely, $s \circ i = \text{id}_N$);
- (3) for each R -module P , the sequence

$$0 \rightarrow \text{Hom}_R(M/N, P) \rightarrow \text{Hom}_R(M, P) \rightarrow \text{Hom}_R(N, P) \rightarrow 0$$

is exact;

- (4) there exists a homomorphism $u \in \text{Hom}_R(M/N, M)$ such that $\pi \circ u = \text{id}_{M/N}$;
- (5) for each R -module P , the sequence

$$0 \rightarrow \text{Hom}_R(P, N) \rightarrow \text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, M/N) \rightarrow 0$$

is exact.

This is called the splitting lemma. See Problems 3 and 4 of HW 5. Hint: Show $(1) \implies [(2) \iff (3)] \implies [(4) \iff (5)] \implies (1)$.

Problem 9. Let M be an R -module and let $N \subset M$ be a submodule. If N is a direct summand of M , show that N is a pure submodule.

Problem 10. Let R be a general ring and M be an R -module. Let $N < M$ be a pure submodule and X is a finitely presented R -modules. Show that the sequence the sequence

$$0 \rightarrow \text{Hom}_R(X, N) \rightarrow \text{Hom}_R(X, M) \rightarrow \text{Hom}_R(X, M/N) \rightarrow 0$$

is exact. As a consequence, show that if M/N is finitely presented, then N is a pure submodule of M iff it is a direct summand.

Hint: This one might be hard. One only needs to show that $\text{Hom}_R(X, M) \rightarrow \text{Hom}_R(X, M/N)$ is surjective. Given $w \in \text{Hom}_R(X, M/N)$, try to produce a commutative diagram

$$\begin{array}{ccccccc} R^n & \xrightarrow{f} & R^m & \longrightarrow & X & \longrightarrow & 0 \\ & \downarrow u & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & N & \xrightarrow{i} & M & \xrightarrow{\pi} & M/N \longrightarrow 0 \end{array}$$

and use Problem 6 to get a hom $\phi : R^m \rightarrow N$. Then consider $\tilde{w} \in \text{Hom}_R(R^m, M)$ defined by $\tilde{w} = v - \phi$.

- Problem 11.** (1) Let R be a Noetherian ring and M be a finitely generated R -module. Show that a submodule $N < M$ is pure iff it is a direct summand.
- (2) Let R be a PID and M be a finitely generated R -module. Show that a submodule $N < M$ is admissible iff it is a direct summand.

This problem together with the structure theorem of finite generated modules over PID fully covers Theorem 3 of page 233 of Hoffman-Kunze. In the general case, we have

$$(\text{direct summand submodules}) \subset (\text{pure submodules}) \subset (\text{admissible submodules}).$$

See [this link](#) for an example of pure submodule which is not a direct summand.

2. PRESENTATION OF LINEAR OPERATOR AS $F[x]$ -MODULES

This problem is from HW11, 2023. It is also Exercise 8.4, page 440 of Artin's book. Do it again.

Let F be a field. We consider $K = F[x]$ and K^n . An element $u \in K^n$ will be considered as a column vector and thus it has the form

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

and each $u_i \in F[x]$ can be written as $u_i = u_{i0} + u_{i1}x + u_{i2}x^2 + \cdots + u_{ik}x^k$ with $u_{ij} \in F$. Since u_{ik} can be zero, we can take a k such that it works for all i , namely each u_i has its last term of the form $u_{ik}x^k$. Thus we can write u as

$$u = \begin{bmatrix} u_{10} \\ u_{20} \\ \vdots \\ u_{n0} \end{bmatrix} + \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix} x + \cdots + \begin{bmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{bmatrix} x^k.$$

Write

$$\mathbf{u}_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{nj} \end{bmatrix} \in F^n,$$

then we can write $u = \mathbf{u}_0 + x\mathbf{u}_1 + \cdots + x^k\mathbf{u}_k$. Here we write x^j in front of \mathbf{u}_j (so that it looks like a scalar times a column vector). Thus an element in $K^n = F[x]^n$ can be viewed as a polynomial with coefficients in F^n .

Fix a matrix $A \in \text{Mat}_{n \times n}(F)$. Note that as an element in $\text{Mat}_{n \times n}(K)$, the matrix $xI_n - A$ defines a linear map $T_{(xI_n - A)} : K^n \rightarrow K^n$ defined by

$$T_{(xI_n - A)}u = (xI_n - A)u,$$

as usual. We now consider the map $\phi : K^n \rightarrow F^n$ defines as follows. Given an element

$$u = \mathbf{u}_0 + x\mathbf{u}_1 + \cdots + x^k\mathbf{u}_k \in K^n,$$

we define

$$\phi(u) = \mathbf{u}_0 + A\mathbf{u}_1 + \cdots + A^k\mathbf{u}_k \in F^n.$$

Namely, we just replace the symbol x by the matrix A . The notation should be clear.

- Problem 12.** (1) Show that ϕ is surjective. (This should be trivial).
- (2) Show that $\text{Im}(T_{(xI_n - A)}) \subset \ker(\phi)$. (This is also trivial).
- (3) Show that $\ker(\phi) \subset \text{Im}(T_{(xI_n - A)})$. (It needs some work, but not very hard).

The assertions of this problem say that the sequence

$$K^n \xrightarrow{T_{(xI_n - A)}} K^n \xrightarrow{\phi} F^n \longrightarrow 0$$

is exact (as K -modules), which gives a presentation of F^n as an $F[x]$ -module.