HOMEWORK 1

Due date: Tuesday, Week 2,

Exercises: 1.6, 1.7, 1.8, 2.1, 2.2, 3.1, 3.3, 3.4, 3.6, 3.7, 3.9, 3.10, 3.11, 3.12, 3.13, 4.2, 4.3, pages 354-355, Artin's book.

All rings in our course is a commutative ring with an identity.

Let R be a ring. An ideal $\mathfrak{p} \subset R$ with $\mathfrak{p} \neq R$ is called prime if for any $x, y \in R, xy \in \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Let $\mathrm{Spec}(R)$ be the set of all prime ideals of R, which is called the spectrum of R.

Problem 1. *Let R be a ring and $\mathfrak{p}$ be a prime ideal of R. Let I, J be any two ideals of R such that $IJ \subset \mathfrak{p}$, show that either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.*

Here the product ideal IJ is defined in Ex.3.13, page 355.

Problem 2. *Determine* $\mathrm{Spec}(\mathbb{Z})$ *and* $\mathrm{Spec}(F[x])$*, where F is a field.*

Problem 3. *Let $\phi : A \to B$ be a ring homomorphism. Let $\mathfrak{p}$ be a prime ideal of B. Show that $\phi^{-1}(\mathfrak{p}) := \{x \in A : \phi(x) \in \mathfrak{p}\}$ is a prime ideal of A. Thus there is a map $\phi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ defined by $\phi^*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$.*

Problem 4. *Let R be a ring. An element $a \in R$ is called nilpotent if $a^k = 0$ for some integer $k \geq 1$. Let $\mathfrak{N}$ be the set of all nilpotent elements of R. Show that R is an ideal. This ideal $\mathfrak{N}$ is called the nilradical of R. Moreover, show that $\mathfrak{N}$ is contained in every prime ideal of R.*

A ring is called reduced if its nilradical is zero.

Problem 5. *Determine the nilradical of the following rings:*

$$(1),\ \mathbb{Z}/4\mathbb{Z}; \quad (2),\ \mathbb{Z}/12\mathbb{Z}; \quad (3),\ \mathbb{Z}/n\mathbb{Z}.$$

Problem 6. *Let R be a ring and let I be an ideal. Define*

$$\sqrt{I} = \{x \in R : \exists n > 0 \text{ s.t. } x^n \in I\}.$$

Show that $\sqrt{I}$ is an ideal of R. Let $\pi : R \to R/I$ be the projection map. Find an ideal $\mathfrak{b} \subset R/I$ such that $\sqrt{I} = \pi^{-1}(\mathfrak{b})$.

Problem 7. *Let R be a ring and $x \in I$. Denote*

$$\mathrm{Ann}(x) = \{r \in R : rx = 0\}.$$

Show that $\mathrm{Ann}(x)$ *is an ideal of R. It is called the annihilator of x.*