

HOMEWORK 1

Due date: Tuesday, Week 2,

Exercises: 1.6, 1.7, 1.8, 2.1, 2.2, 3.1, 3.3, 3.4, 3.6, 3.7, 3.9, 3.10, 3.11, 3.12, 3.13, 4.2, 4.3, pages 354-355, Artin's book.

All rings in our course is a commutative ring with an identity.

Let R be a ring. An ideal $\mathfrak{p} \subset R$ is called prime if for any $x, y \in R, xy \in \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Let $\text{Spec}(R)$ be the set of all prime ideals of R , which is called the spectrum of R .

Problem 1. Determine $\text{Spec}(\mathbb{Z})$ and $\text{Spec}(F[x])$, where F is a field.

Problem 2. Let $\phi : A \rightarrow B$ be a ring homomorphism. Let \mathfrak{p} be a prime ideal of B . Show that $\phi^{-1}(\mathfrak{p}) := \{x \in A : \phi(x) \in \mathfrak{p}\}$ is a prime ideal of A . Thus there is a map $\phi^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ defined by $\phi^*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$.

Problem 3. Let R be a ring. An element $a \in R$ is called nilpotent if $a^k = 0$ for some integer $k \geq 1$. Let \mathfrak{N} be the set of all nilpotent elements of R . Show that \mathfrak{N} is an ideal. This ideal \mathfrak{N} is called the nilradical of R . Moreover, show that \mathfrak{N} is contained in every prime ideal of R .

A ring is called reduced if its nilradical is zero.

Problem 4. Determine the nilradical of the following rings:

$$(1), \mathbb{Z}/4\mathbb{Z}; \quad (2), \mathbb{Z}/12\mathbb{Z}; \quad (3), \mathbb{Z}/n\mathbb{Z}.$$

Problem 5. Let R be a ring and let I be an ideal. Define

$$\sqrt{I} = \{x \in R : \exists n > 0 \text{ s.t. } x^n \in I\}.$$

Show that \sqrt{I} is an ideal of R . Let $\pi : R \rightarrow R/I$ be the projection map. Find an ideal $\mathfrak{b} \subset R/I$ such that $\sqrt{I} = \pi^{-1}(\mathfrak{b})$.

Problem 6. Let R be a ring and $x \in I$. Denote

$$\text{Ann}(x) = \{r \in R : rx = 0\}.$$

Show that $\text{Ann}(x)$ is an ideal of R . It is called the annihilator of x .