HOMEWORK 11

Due date: Monday of Week 16

Exercises: 3, 6, 9, 14, pages 162-164
Exercises: 3, 4, 5, 6, 7, pages 189-190

Keep in mind the assertion of Problem 9, page 163. That gives a different characterization of rank of a matrix. Actually, this is the definition of rank in many other books. **In the future HWs and exams, you can freely use the equivalence between *determinant rank* and rank.** A related terminology is "minor". A minor is just the determinant of a submatrix.

Problem 1. *Consider a matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathrm{Mat}_3(F),$$

and

$$xI_3 - A = \begin{bmatrix} x-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & x-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & x-a_{33} \end{bmatrix} \in \mathrm{Mat}_3(F[x]).$$

Write $\det(xI_3 - A) = c_0 + c_1x + c_2x^2 + c_3x^3$. *Show that* $c_3 = 1, c_0 = -\det(A)$. *What are* c_2 *and* c_1*?*

You might recognize that c_2 is related to $\mathrm{tr}(A)$. But what about c_1? Find the expression of c_1 even it is complicate. At the very end of this course (next spring), we will see how c_1 is related to A (for general A over general field. Over an algebraically closed field, it might be easier to connect c_1 with A.)

Problem 2. *Let* $A_i \in \mathrm{Mat}_{n_i\times n_i}$ *be a square matrix for* $1 \le i \le k$. *We consider the following matrix in block form*

$$A = \begin{bmatrix} A_1 & * & * & \dots & * \\ 0 & A_2 & * & \dots & * \\ 0 & 0 & A_3 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_k \end{bmatrix}.$$

Show that $\det(A) = \det(A_1)\det(A_2)\dots\det(A_k)$.

This is a slight generalization of Exercise 7, page 155.

The next two problems might be hard.

Problem 3. *Let* $V = \mathbb{C}^2$*, which can be viewed as a dimension 4 vector space over* $\mathbb{R}$*. Fix a basis* $\mathcal{B}$ *when viewed as a vector space over* $\mathbb{R}$*. Given an element* $A \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$*, we can consider the (*$\mathbb{R}$*-)linear operator* $T_A : V \to V$ *given by* $T_A(\alpha) = A\alpha$*. Thus we can consider the matrix* $[T_A]_{\mathcal{B}} \in \mathrm{Mat}_{4\times 4}(\mathbb{R})$*. Take* $A = \begin{bmatrix} a+bi & x+yi \\ 0 & c+di \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$*. Compute* $\det([T_A]_{\mathcal{B}}) \in \mathbb{R}$ *and compare it with* $\det(A) \in \mathbb{C}$*.*

We can consider the same question for $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. The result is not hard to guess. But its proof seems complicate. We will prove this after Chapter 6. One could summarize the result of the general case in the following commutative diagram

$$\begin{tikzcd}
\mathrm{Mat}_{n\times n}(\mathbb{C}) \arrow[r] \arrow[d, "\det"] & \mathrm{Mat}_{(2n)\times(2n)}(\mathbb{R}) \arrow[d, "\det"] \\
\mathbb{C} \arrow[r, "\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}"] & \mathbb{R},
\end{tikzcd}$$

where the top map is that defined by $A \mapsto [T_A]_{\mathcal{B}}$, and the $\mathrm{Nm}_{\mathbb{C}/\mathbb{R}} : \mathbb{C} \to \mathbb{R}$ map is $\mathrm{Nm}(z) = z\overline{z}$. One consequence of this result is $\det([T_A]_{\mathcal{B}}) \geq 0$. Note that if $n = 1$, the above gives a way to compute the (familiar) norm map using determinant of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

The following problem is similar to the above one, but in a different situation. You only have to do part (1) of the next problem. Try part (2) for one specific example.

Problem 4. *Denote $\alpha = \sqrt[3]{2}$. Consider the field $F = \{a + b\alpha + c\alpha^2 : a, b, c\}$. We also view F as a vector space over $\mathbb{Q}$ of dimension 3.*

(1) *For $x = a + b\alpha + c\alpha^2$. We define the linear map $T_x : F \to F$ by $T_x(y) = xy$, which is viewed a linear map between $\mathbb{Q}$-vector spaces. Fix a basis $\mathcal{B}$ of F over $\mathbb{Q}$, we can get the matrix $[T_x]_{\mathcal{B}} \in \mathrm{Mat}_{3\times 3}(\mathbb{Q})$. We define a map $\mathrm{Nm}_{F/\mathbb{Q}} : F \to \mathbb{Q}$ by $\mathrm{Nm}_{F/\mathbb{Q}}(x) = \det([T_x]_{\mathcal{B}})$. Compute $\mathrm{Nm}_{F/\mathbb{Q}}$ explicitly. Show that $\mathrm{Nm}_{F/\mathbb{Q}}(xy) = \mathrm{Nm}_{F/\mathbb{Q}}(x)\mathrm{Nm}_{F/\mathbb{Q}}(y)$, and $\mathrm{Nm}_{F/\mathbb{Q}}(x) \neq 0$ unless $x = 0$.*

(2) *Consider the vector space $V = F^n$. We have $\dim_F V = n$ and $\dim_{\mathbb{Q}} V = 3n$. Given a matrix $A \in \mathrm{Mat}_{n\times n}(F)$, we can consider the linear map $T_A : F^n \to F^n$ defined by $T_A(\alpha) = A\alpha$. Fix an ordered basis $\mathcal{B}$ of V as a $\mathbb{Q}$-vector space and we can get a matrix $[T_A]_{\mathcal{B}} \in \mathrm{Mat}_{(3n)\times(3n)}(\mathbb{Q})$. What is the relationship between $\det(A) \in F$ and $\det([T_A]_{\mathcal{B}}) \in \mathbb{Q}$?*

The result is similar to the above one and it could be summarized using the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{Mat}_{n\times n}(F) \arrow[r] \arrow[d, "\det"] & \mathrm{Mat}_{(3n)\times(3n)}(\mathbb{Q}) \arrow[d, "\det"] \\
F \arrow[r, "\mathrm{Nm}_{F/\mathbb{Q}}"] & \mathbb{Q}.
\end{tikzcd}$$

We won't prove this result in this course. If you are interested, see [Bou98, Proposition 6, page 546] for a proof.

The next problem is important in some sense because it explains one (deep hidden) reason why the matrix $xI_n - A$ is so important in Chapters 6 and 7. It is better to keep in mind the assertion at least. We might go back to this problem again in a future course (abstract algebra, which you will learn in your sophomore year.)

Let F be a field. We consider $K = F[x]$ and K^n. An element $u \in K^n$ will be considered as a column vector and thus it has the form

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

and each $u_i \in F[x]$ can be written as $u_i = u_{i0} + u_{i1}x + u_{i2}x^2 + \cdots + u_{ik}t^k$ with $u_{ij} \in F$. Since u_{ik} can be zero, we can take a k such that it works for all i, namely each u_i has its last term of the form $u_{ik}x^k$. Thus we can write u as

$$u = \begin{bmatrix} u_{10} \\ u_{20} \\ \vdots \\ u_{n0} \end{bmatrix} + \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix} x + \cdots + \begin{bmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{bmatrix} x^k.$$

Write

$$\mathbf{u}_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{nj} \end{bmatrix} \in F^n,$$

then we can write $u = \mathbf{u}_0 + x\mathbf{u}_1 + \cdots + x^k\mathbf{u}_k$. Here we write x^j in front of $\mathbf{u}_j$ (so that it looks like a scaler times a column vector). Thus an element in $K^n = F[x]^n$ can be viewed as a polynomial with coefficients in F^n. Note that as an element in $\mathrm{Mat}_{n\times n}(K)$, the matrix $xI_n - A$ defines a linear map $T_{(xI_n-A)} : K^n \to K^n$ defined by

$$T_{(xI_n-A)}u = (xI_n - A)u,$$

as usual. We now consider the map $\phi : K^n \to F^n$ defines as follows. Given an element

$$u = \mathbf{u}_0 + x\mathbf{u}_1 + \cdots + x^k\mathbf{u}_k \in K^n,$$

we define

$$\phi(u) = \mathbf{u}_0 + A\mathbf{u}_1 + \cdots + A^k\mathbf{u}_k \in F^n.$$

Namely, we just replace the symbol x by the matrix A. The notation should be clear.

Problem 5. (1) *Show that ϕ is surjective. (This should be trivial).*

(2) *Show that* $\mathrm{Im}(T_{(xI_n-A)}) \subset \ker(\phi)$. *(This is also trivial).*

(3) *Show that* $\ker(\phi) \subset \mathrm{Im}(T_{(xI_n-A)})$. *(It needs some work, but not very hard).*

If you don't know how to do part (3), try the example when $n = 2$. Using notations and terminology you will learn later (in a different course), the assertions of this problem say that the sequence

$$K^n \xrightarrow{T_{(xI_n-A)}} K^n \xrightarrow{\phi} F^n \longrightarrow 0$$

is exact (as K-modules). Currently, you don't have to worry about the terminology.

References

[Bou98] Nicolas Bourbaki, *Algebra I. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation [MR0979982 (90d:00002)]. ↑2