

### HOMEWORK 3

Due date: Monday of Week 8, Oct. 16

Exercise: 5, 8 (page 21);

Exercise: 1, 3, 6, 7, 10 (page 27).

Note that Exercise 10, page 27 tells us that if  $A \in \text{Mat}_{m \times n}(F)$  with  $n < m$ , then  $A$  has no right inverse in the following sense: there is no  $B \in \text{Mat}_{n \times m}$  such that  $AB = I_m$ . Find an example of  $A \in \text{Mat}_{m \times n}(F)$  with  $n < m$  such that  $A$  has a left inverse, namely, such that there exists a matrix  $C \in \text{Mat}_{n \times m}(F)$  with  $CA = I_n$ .

**Problem 1.** Let  $F$  be a field. For any  $A \in \text{Mat}_{m \times n}(F)$  and  $B \in \text{Mat}_{n \times m}(F)$ , show that  $\text{tr}(AB) = \text{tr}(BA)$ .

Here recall that for a square matrix  $C = (c_{ij})_{1 \leq i, j \leq n}$ ,  $\text{tr}(C) = \sum_{i=1}^n c_{ii}$ .

A square matrix  $A \in \text{Mat}_{n \times n}(F)$  is called **nilpotent** if  $A^k = 0$  for some integer  $k > 0$ . For example the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \text{Mat}_{3 \times 3}(F)$$

is nilpotent, because  $A^3 = 0$ .

**Problem 2.** Let  $B \in \text{Mat}_{n \times n}(F)$  be a nilpotent matrix. Show that  $I_n + B$  is invertible.

**Problem 3.** Let  $F$  be a field,  $A \in \text{Mat}_{m \times n}(F)$  and  $B \in \text{Mat}_{n \times m}(F)$ . Show that  $I_m - AB$  is invertible if and only if  $I_n - BA$  is invertible.

(Hint: Use the identity  $A(I_n - BA) = (I_m - AB)A$ . This is Problem M.10, page 36 of Artin's book "Algebra", edition 2.)

**Problem 4.** Let  $F$  be a field,  $A, B \in \text{Mat}_{n \times n}(F)$ . If  $AB = A + B$  show that  $AB = BA$ .

(Hint: consider  $(A - I_n)(B - I_n)$ .)

A matrix  $A \in \text{Mat}_{n \times n}(F)$  is called a permutation matrix if each row and each column has only one nonzero entry and that non-zero entry is 1. For example,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \text{Mat}_{3 \times 3}(F)$$

is a permutation matrix. A matrix  $A = (a_{ij})_{1 \leq i, j \leq n} \in \text{Mat}_{n \times n}(F)$  is called upper triangular if  $a_{ij} = 0$  for all  $i, j$  with  $i > j$ , namely, if each entry below the main diagonal is zero. Similarly, a matrix  $A = (a_{ij})_{1 \leq i, j \leq n} \in \text{Mat}_{n \times n}(F)$  is called lower triangular if  $a_{ij} = 0$  for all  $i, j$  with  $i < j$ . For example

$$\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \in \text{Mat}_{3 \times 3}(F)$$

is upper triangular. Here an entry with  $*$  means that the value of that entry is not important. Moreover, it is easy to see that if a matrix  $A \in \text{Mat}_{n \times n}(F)$  is a row reduced echelon matrix, then  $A$  is upper triangular. Following our book, an elementary row operation by interchanging two rows will be called **type III**. Similarly, one can define type III elementary matrix, which is just a matrix by applying a type III ERO to the identity matrix.

**Problem 5.** (1) Show that a matrix  $A \in \text{Mat}_{n \times n}(F)$  is a permutation matrix if and only if it is a product of type III elementary matrices.

- (2) Let  $A \in \text{Mat}_{n \times n}(F)$  be a matrix such that it can be reduced to Row-Reduced echelon matrix without using Type III elementary row operations. Show that one can write  $A = LU$ , where  $L \in \text{Mat}_{n \times n}(F)$  is a lower triangular matrix and  $U \in \text{Mat}_{n \times n}(F)$  is an upper triangular matrix.
- (3) Show that each matrix  $A \in \text{GL}_2(F)$  can be written as a product  $A = LPU$ , where  $P \in \text{Mat}_{2 \times 2}(F)$  is a permutation matrix,  $L \in \text{Mat}_{2 \times 2}(F)$  is lower triangular, and  $U \in \text{Mat}_{2 \times 2}(F)$  is upper triangular.
- (4) Show that each matrix  $A \in \text{GL}_3(F)$  can be written as a product  $A = LPU$ , where  $P \in \text{Mat}_{3 \times 3}(F)$  is a permutation matrix,  $L \in \text{Mat}_{3 \times 3}(F)$  is lower triangular, and  $U \in \text{Mat}_{3 \times 3}(F)$  is upper triangular.

(Actually, in part (3) and (4), there is no need to assume the matrix  $A$  is invertible. But the argument might be a little bit simpler when we add this condition.) Think about how to do part (3) and (4) for general  $n \times n$  matrix. Namely, for any  $A \in \text{GL}_n(F)$ , show that it can be written as a product  $A = LPU$ , where  $P \in \text{Mat}_{n \times n}(F)$  is a permutation matrix,  $L \in \text{Mat}_{n \times n}(F)$  is a lower triangular matrix, and  $U \in \text{Mat}_{n \times n}(F)$  is an upper triangular matrix. There is no need to submit a proof of this. But it is helpful to keep in mind this assertion. This decomposition of a matrix  $A \in \text{GL}_n(F)$  is called the Bruhat decomposition.