**HOMEWORK 8**

Due date: Monday of Week 9

Exercises: 2, 4, 5, 6, 7, 9, 10, 11, pages 366-367
Exercises: 3, 6, 7, 8, 17, pages 373-375.

For problem 7 of page 373, go through the proof of Theorem 3, page 369. Of course you can orthogonal diagonolize the corresponding symmetric matrix. But here, try to practice the procedure given in the proof of Theorem 3.

Let $V$ be a finite dimensional inner product space over $\mathbb{R}$. Note that the normal operator $T$ on $V$ is more complicate because $\chi_T$ is not necessary a product of linear factors, and thus $T$ is not (orthogonally) diagonalizable over $\mathbb{R}$ in general. Actually, we have seen that a normal linear operator $T$ is orthogonally diagonalizable if and only if it is self-adjoint. But many other properties of normal vectors still hold. Actually, the following are equivalent

(1) $T$ is normal, i.e., $TT^* = T^*T$;
(2) $(T\alpha|T\beta) = (T^*\alpha|T^*\beta)$, for all $\alpha, \beta \in V$;
(3) $||T\alpha|| = ||\alpha||$, for every $\alpha \in V$;
(4) $T_1$ commutes with $T_2$, where $T_1 = \frac{T+T^*}{2}, T_2 = \frac{T-T^*}{2}$;
(5) $T^* = TU$ for some orthogonal operator $U \in \mathrm{End}(V)$, where $U$ is said to be orthogonal if $UU^* = I$ and $U$ is a linear operator on real vector space;
(6) $U$ commutes with $N$, where $T = UN$ is the polar decomposition of $T$ with $N$ non-negative and $U$ orthogonal;
(7) there exists a polynomial $f \in \mathbb{R}[x]$ such that $T^* = f(T)$.

Many of the above equivalences were proved in class; the proofs of the rest are similar to the complex case. Notice the difference between real and complex case. Over the complex field, for a normal operator $T$, there exists a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal. Over the real field $\mathbb{R}$, this statement is no longer true. The simplest form of $[T]_{\mathcal{B}}$ is given in Theorem 17 and 18.

**Problem 1.** *Let $V$ be a finite dimensional inner product space over $\mathbb{R}$. Let $T \in \mathrm{End}(V)$.*

(1) *Show that $T$ is normal if and only if there exists a polynomial $f \in \mathbb{R}[x]$ such that $T^* = f(T)$.*
(2) *Consider the matrix*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{R}).$$

*Chek that $A$ is normal. Moreover, find a polynomial $f \in \mathbb{R}[x]$ such that $A^t = f(A)$.*

(3) *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by the matrix $A$. Find an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^3$ such that $[T]_{\mathcal{B}}$ is of the form*

$$\begin{bmatrix} c & & \\ & a & -b \\ & b & a \end{bmatrix},$$

*for $a, b, c \in \mathbb{R}$.*

Hint for (1), translate this to a problem on matrix and view the corresponding matrix in $\mathrm{Mat}_{n\times n}(\mathbb{C})$, then use the conclusion in the complex case. Then show the corresponding polynomial is in fact in $\mathbb{R}[x]$. For (2), just go through the proof of (1) and specialize everything to this example. Note that part (1) gives a different proof of Theorem 19, page 354.

## 1. Various decompositions

It is almost the end of linear algebra part of this course. It is a good time to review what we have learned about various decompositions of matrices, which are important parts of linear algebra. I will remind you those decompositions in the following, and it is helpful to keep a record of a proof for each decomposition. We learned all of these in class or from HW problems.

### 1.1. **Bruhat decomposition.**

Let $F$ be a general field. Denote by $B_n$ the subset of $\mathrm{GL}_n(F)$ consisting of upper triangular invertible matrices with entries in $F$. Let $W \subset \mathrm{GL}_n(F)$ be the subset of permutation matrices. The set $W$ was denoted by $P$ in Recall that a matrix $g \in \mathrm{GL}_n(F)$ is called a permutation matrix if in each row and each column of $g$, there is only one nonzero term and that nonzero term is 1. Also recall that, we have a map

$$S_n \to W$$

$$\sigma \mapsto g_\sigma = \left[e_{\sigma(1)}, \ldots, e_{\sigma(n)}\right],$$

where $S_n$ is the symmetric group on $n$-elements which consists of bijections $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, $e_i$ is the column vector whose only nonzero entry is 1 and it is at the $i$-th position. See HW 3 and HW 10 of last year. Recall that we have

$$g_{\sigma\tau} = g_\sigma g_\tau.$$

We also consider the special element $w_\ell \in W$ defined by

$$w_\ell = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}$$

**Problem 2.** *Let $A$ be a upper triangular matrix. Show that $w_\ell A w_\ell$ is lower triangular.*

**Proposition 1** (Bruhat decomposition)**.** *For any element $g \in \mathrm{GL}_n(F)$, there exists $b_1, b_2 \in B$ and $w \in W$ such that $g = b_1 w b_2$. The elements $b_1, b_2$ are not unique in general, but $w \in W$ is uniquely determined by $g$. In other words, we have the decomposition*

$$\mathrm{GL}_n(F) = \coprod_{w \in W} BwB,$$

*where $\coprod$ denotes disjoint union (which means $BwB \cap Bw'B = \emptyset$ if $w \neq w'$).*

The decomposition $g = b_1 w b_2$ is equivalent to the $LUP$ decomposition, which was in HW 3 of last year. We did not check the uniqueness in our HW.

**Problem 3.**
1. *Show that the Bruhat decomposition in Proposition 1 is equivalent to the $LUP$ decomposition in Problem 5, HW3 of last year.*
2. *Prove the above Bruhat decomposition for $n = 2, 3, 4$ by proving the $LUP$ decomposition first. Also check the uniqueness part for $w \in W$ in the decomposition for $n = 2, 3, 4$.*
3. *Given $g \in \mathrm{GL}_n(F)$. Show that $g$ has an $LU$ decomposition (which means $g = g_1 g_2$ for $g_1$ lower triangular and $g_2$ upper triangular) if and only if all of its principle minors are all different from zero.*
4. *Given $g \in \mathrm{GL}_n(F)$. Find a condition on $g$ such that $g$ has a decomposition $g = b_1 w_\ell b_2$ for $b_1, b_2 \in B$.*

Part (3) is a result from our textbook (Lemma, page 326). You don't need to submit a solution of this but you should know how to prove it. Part (3) is here because it gives you a hint for (4).

### 1.2. **C-R decomposition.**

**Proposition 2** (C-R decomposition)**.** *Let $A \in \mathrm{Mat}_{m\times n}(F)$ be a matrix of rank $r$. Then there exists a matrix $C \in \mathrm{Mat}_{m\times r}(F)$ and a matrix $R \in \mathrm{Mat}_{r\times n}(F)$ such that $A = CR$.*

A special case of the above decomposition is when $A$ has rank 1, then $A = uv$ for $u \in \mathrm{Mat}_{m\times 1}(F)$ and $v \in \mathrm{Mat}_{1\times n}(F)$. If $m = n$, then from $A = uv$, we can get that $A^2 = \mathrm{tr}(A)A$. The existence of the above C-R (which means column-row) decomposition was given in HW 5 of last year. Here is another related fact. Let $k$ be a position integer with $k < r$, then there does not exist matrices $C \in \mathrm{Mat}_{m\times k}(F), R \in \mathrm{Mat}_{k\times n}(F)$ such that $A = CR$.

**Problem 4.** *Let $A \in \mathrm{Mat}_{m\times n}(F)$ be a matrix of rank $r$ and let $A = CR$ be a C-R decomposition with $C \in \mathrm{Mat}_{m\times r}$ and $R = \mathrm{Mat}_{r\times n}$. For any $P \in \mathrm{GL}_r(F)$, if we denote $C' = CP \in \mathrm{Mat}_{m\times r}, R' = P^{-1}R \in \mathrm{Mat}_{r\times n}$, we $A = C'R'$ is another C-R decomposition. The question is: do we know all C-R decomposition has the above form? In other words, suppose that $A = CR = C'R'$ with $C, C' \in \mathrm{Mat}_{m\times r}, R, R' \in \mathrm{Mat}_{r\times n}$ such that*

$$A = CR = C'R'.$$

*Is there a matrix $P \in \mathrm{GL}_r(F)$ such that $C' = CP$ and $R' = P^{-1}R$? If so, prove it. If not, find a counter-example.*

This is certain uniqueness of C-R decomposition. If you think this hard, try to consider some examples with small $m, n, r$, for example, when $m = n = 3$ and $r = 2$.

### 1.3. **Jordan decomposition.**

**Proposition 3** (Jordan decomposition)**.** *Let $A \in \mathrm{Mat}_{n\times n}(F)$ be a matrix such that $\mu_A$ is a product of linear factors. There exists a unique diagonalizable matrix $D \in \mathrm{Mat}_{n\times n}(F)$ and a unique nilpotent matrix $N \in \mathrm{Mat}_{n\times n}(F)$ such that $DN = ND$ and $A = D + N$.*

Moreover, we know that such $D, N$ are polynomials of $A$. This is Theorem 13, page 222.

**Proposition 4** (Jordan decomposition, semisimple version)**.** *Let $F$ be a field of characteristic zero. Let $A \in \mathrm{Mat}_{n\times n}(F)$ be a matrix. Then there exists a unique semi-simple matrix $S \in \mathrm{Mat}_{n\times n}(F)$ and a unique nilpotent matrix $N \in \mathrm{Mat}_{n\times n}(F)$ such that $SN = NS$ and $A = S + N$.*

This is Theorem 13, page 267.

### 1.4. **Iwasawa decomposition.**

Let $F = \mathbb{R}$ or $\mathbb{C}$. We consider the group $\mathrm{GL}_n(F)$. We still let $B_n$ be the upper triangular matrices in $\mathrm{GL}_n(F)$. Let $K_n = \mathrm{O}_n(\mathbb{R})$ if $F = \mathbb{R}$ and let $K_n = \mathrm{U}(n)$ if $F = \mathbb{C}$.

**Proposition 5** (Iwasawa decomposition)**.** *We have $\mathrm{GL}_n(F) = B_n \cdot K_n$. In other words, for any $g \in \mathrm{GL}_n(F)$, there exists an element $b \in B_n$ and an element $k \in K_n$ such that $g = bk$.*

This is equivalent to Theorem 14 of page 305. Explain the equivalence between the above Proposition and Theorem 14 of page 305.

**Problem 5.** *Consider the matrix*

$$g = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 9 \\ 4 & 7 & 11 \end{bmatrix} \in \mathrm{GL}_3(\mathbb{R}).$$

*Find a matrix $b \in B_3$ and $k \in \mathrm{O}_3(\mathbb{R})$ such that $g = bk$.*

### 1.5. **Singular value decomposition, polar decomposition and Cartan decomposition.**

Let $F = \mathbb{R}$ or $\mathbb{C}$. We consider the group $\mathrm{GL}_n(F)$. We still let $A_n$ be the set of all diagonal matrices in $\mathrm{GL}_n(F)$. Let $K_n = \mathrm{O}_n(\mathbb{R})$ if $F = \mathbb{R}$ and let $K_n = \mathrm{U}(n)$ if $F = \mathbb{C}$.

**Proposition 6** (Cartan decomposition)**.** *We have $\mathrm{GL}_n(F) = K_n \cdot A_n \cdot K_n$. In other words, for any $g \in \mathrm{GL}_n(F)$, there exists $k_1, k_2 \in K_n$ and $a \in A_n$ such that $g = k_1 a k_2$.*

This is just a slightly different way to say the singular value decomposition.

**Proposition 7** (Polar decomposition)**.** *For any $g \in \mathrm{GL}_n(F)$, there exists a matrix $k \in K_n$ and a positive matrix $p$ such that $g = kp$.*

This is Theorem 14, page 342. Singular value decomposition and polar decomposition are closely related.

1.6. **Schur decomposition.** Let $F = \mathbb{C}$ and let $K_n = \mathrm{U}(n)$. Let $B_n \subset \mathrm{GL}_n(F)$ be the subset consisting of upper triangular matrices.

**Proposition 8** (Schur decomposition)**.** *For any $g \in \mathrm{GL}_n(F)$, there exists an element $k \in K_n$ and an element $b \in B_n$ such that $g = kbk^{-1}$.*

This is Theorem 21, page 316. See also HW 6.