# NOTES ON TENSOR AND EXTERIOR PRODUCTS

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These notes are on tensor product and exterior product on vector spaces, which were originally written for the course MAS212 of Spring Semester 2022, KAIST. There are many good books on these topics. For example, see [Bou98] for a complete treatment. Professor Garrett's book [Gar08] is also very helpful. Tensor product and exterior product could be defined for modules over (at least commutative) rings. For simplicity, we only consider these constructions for vector spaces. Throughout the notes, $F$ is a fixed field and $\mathrm{Vect}_F$ denotes all vector spaces over $F$.

## 1. Quotient space

**Definition 1.1.** *Let $V$ be a vector space over $F$ and $W \subset V$ be a subspace. The quotient vector space $Q$ of $V$ by $W$ is a pair $(\pi, Q)$, where*

1. *$Q$ is a vector space over $Q$, and*
2. *$\pi : V \to Q$ is linear map satisfying $\pi(w) = 0, \forall w \in W$,*

*such that for any pair $(f, X)$ with a vector space $X$ over $F$ and linear map $f : V \to X$ satisfying $f(w) = 0$ for all $w \in W$, there is a* ***unique*** *linear map $\bar{f} : Q \to X$ such that*

$$f = \bar{f} \circ \pi,$$

*namely, the following diagram commutes*

$$\begin{tikzcd} V \arrow[rr, "\pi"] \arrow[dr, "f"'] & & Q \arrow[dl, dashed, "\exists ! \bar{f}"] \\ & X & \end{tikzcd}$$

In this definition, we don't specify what the quotient vector space $Q$ actually is. In contrast, we impose a condition such that $Q$ must satisfy and this condition uses an arbitrary "external" vector space $X$ and an arbitrary linear map $f : V \to X$ with a condition (namely $f(w) = 0, \forall w \in W$). This kind condition is called a "universal property" and later you will find a lot of objects in math are defined using certain universal properties. In fact, many properties of the defined object (here, it is the quotient space) can be derived from this universal property, without knowing what exactly the space $Q$ is.

In the above, we defined *the* quotient space of $V$ by $W$ rather than *a* quotient space, because it is unique as explained in the following.

**Theorem 1.2** (Uniqueness)**.** *The quotient space $(Q, \pi)$ is unique up to a unique isomorphism.*

*Proof.* Let $(Q_1, \pi_1)$ and $(Q_2, \pi_2)$ be two quotient spaces. Replace $(X, f)$ by $(Q_2, \pi_2)$ in the definition of quotient space, we see that there is a unique linear map $f : Q_1 \to Q_2$ such that $\pi_2 = f \circ \pi_1$. Similarly, there is a unique linear map $g : Q_2 \to Q_1$ such that $\pi_1 = g \circ \pi_2$. Then we get $\pi_2 = f \circ g \circ \pi_2$. Since $\pi_2 = \mathrm{Id}_{Q_2} \circ \pi_2$ and $\mathrm{Id}_{Q_2}$ is a unique linear map with this property, we get that $f \circ g = \mathrm{Id}_{Q_2}$. Similarly, $g \circ f = \mathrm{Id}_{Q_1}$. This shows that $f$ and $g$ are isomorphisms, and the uniqueness of $f : Q_1 \to Q_2$ was shown above. □

**Notation:** Since the quotient space of $V$ by $W$ is unique if it exists, we will denote it by $V/W$. The linear map $\pi : V \to V/W$ is called the quotient map.

**Theorem 1.3** (Existence)**.** *The quotient space $V/W$ exists.*

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*Date*: March 23, 2024.

*Proof.* Define equivalence relation on $V$ by $\alpha \sim \beta$ iff $\alpha - \beta \in W$. Let $V/W$ be the set of equivalence classes of this equivalence relation. It is easy to define a vector space structure on this set as we did in class. $\square$

*Remark* 1.4. Here is a different way to state the definition of quotient space. Temporarily, for a vector space $X$, we denote $\mathrm{Hom}(V, X; W)$ the set $\{f \in \mathrm{Hom}_F(V, X) |\ f(w) = 0, \forall w \in W\}$. For any $g \in \mathrm{Hom}_F(V/W, X)$, we consider the composition $g \circ \pi : V \to V/W \to X$. Then $g \circ f \in \mathrm{Hom}(V, X; W)$. The definition of quotient space says that the map

$$\mathrm{Hom}_F(V/W, X) \to \mathrm{Hom}(V, X; W)$$

$$g \mapsto g \circ \pi$$

is bijective for any vector space $X$ (the uniqueness in Definition 1.1 says the map is injective and the existence in Definition 1.1 says the above map is surjective). This bijection can be used to define the quotient space $(\pi, V/W)$. In more abstract language, the definition of quotient spaces says that the pair $(\pi, V/W)$ represents the functor $\mathrm{Vect}_F \to \mathrm{Sets}$ defined by $X \mapsto \mathrm{Hom}(V, X; W)$.

*Remark* 1.5. The following is another way to understand the definition of quotient space using universal properties. Given a vector space $V$ and a subspace $W$. We consider the set

$$\mathcal{Q}(V/W) = \{(f, X) | X \in \mathrm{Vect}_F, f \in \mathrm{Hom}_F(V, X) \text{ satisfying } f(w) = 0, \forall w \in W\}.$$

We can define an order on the set $\mathcal{Q}(V/W)$ in the following way. Given two elements $(f, X), (g, Y) \in \mathcal{Q}(V/W)$, we say $(f, X) \le (g, Y)$ if there is a linear map $h : X \to Y$ such that $g = h \circ f$. In this terminology, the quotient space $(\pi, V/W)$ is just the minimal element in $\mathcal{Q}(V/W)$. Note that the uniqueness result in Theorem 1.2 says that there is a unique smallest element in $\mathcal{Q}(V/W)$.

In the following, we will show several properties of the quotient space using only the universal property. We avoid using the explicit construction in Theorem 1.3 on purpose. Before that, let us recall the following basic fact about linear transformation

**Theorem 1.6.**
1. *Let $V, X$ be vector spaces over $F$ and let $\dim_F V = n$. Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of $V$. Then for any $n$-vectors $\beta_1, \dots, \beta_n \in X$, there exists a unique linear transformation $f : V \to X$ such that $f(\alpha_i) = \beta_i$.*
2. *Let $V$ be a vector space over $F$ and $W \subset V$ be a subspace. If $\alpha \in V$ and $\alpha \notin W$, then there exists a linear map $f : V \to F$ such that $f(\alpha) = 1$ and $f(w) = 0$ for all $w \in W$.*

*Proof.* (1) is a basic theorem from the textbook [HK, Theorem 1 of Chapter 3]. (2) To prove (2), we need to assume that every vector space (including infinite dimensional) has a basis. Let $S_1 = \{\beta_i\}_{i \in I}$ be a basis of $W$. Since $\alpha \notin \mathrm{Span}(S_1)$, $S_2 = S_1 \cup \{\alpha\}$ is also linearly independent. We then extend $S_2$ to a basis of $V$, say $S_3 = \{\alpha, \beta_i, \gamma_j\}$. Then we define $f : V \to F$ by $f(\alpha) = 1$ and $f(\beta_i) = f(\gamma_j) = 0$. This $f$ satisfies the condition. $\square$

We now will only use the universal property to prove some properties of $V/W$.

**Proposition 1.7.** *The quotient space $V/W$ is the zero vector space if and only if $V = W$.*

Note that the proposition is trivial if we know the construction of $V/W$ as in Theorem 1.3. But it is not clear from the definition of quotient space given by the universal property.

*Proof.* We first suppose that $V = W$ and we will show that $V/W$ is the 0 vector space, namely, we will show that $V/V = 0$. Let $\pi : V \to V/V$ be the linear map in the definition of the quotient space. By definition, we have $\pi(v) = 0$ for all $v \in V$. Pick $f : V \to X$ in the Definition 1.1 as $\pi : V \to V/V$. We have the following commutative diagram

$$\begin{tikzcd} V \arrow[rr, "\pi"] \arrow[dr, "\pi"'] & & V/V \arrow[dl, dashed, "\mathrm{id}_{V/V}"] \\ & V/V & \end{tikzcd}$$

On the other hand, from the fact that $\pi(v) = 0, \forall v \in V$, we also have the following commutative diagram

$$\begin{tikzcd} V \arrow[rr, "\pi"] \arrow[dr, "\pi"'] & & V/V \arrow[dl, dashed, "0_{V/V}"] \\ & V/V & \end{tikzcd}$$

where $0_{V/V}$ is the zero map $0_{V/V} : V/V \to V/V$ defined by $0_{V/V}(\alpha) = 0$ for all $\alpha \in V/V$. By the uniqueness part in the Definition 1.1, we must have $\mathrm{id}_{V/V} = 0_{V/V}$. Thus for any $\alpha \in V/V$, we get $\alpha = \mathrm{id}_{V/V}(\alpha) = 0_{V/V}(\alpha) = 0$. Thus $V/V = \{0\}$.

Conversely, we show that if $V/W$ is the zero vector space, then $W = V$. We prove it by contradiction. Assume that $W \neq V$, we then have an element $\alpha \in V$ but $\alpha \notin W$. By Theorem 1.6, we can find a linear map $f : V \to F$ such that $f(\alpha) = 1$ and $f(w) = 0$ for all $w \in W$. Thus by the Definition 1.1, there is a unique $\bar{f} : V/W \to F$ such that $f = \bar{f} \circ \pi$. By assumption, $\pi(\alpha) \in V/W = \{0\}$, we get $f(\alpha) = \bar{f}(0) = 0$, which contradicts to $f(\alpha) = 1$. □

Note that, in the above, $W$ and $V$ might be infinite dimensional and we used the fact that any vector space has a basis.

**Lemma 1.8.** *Suppose that the quotient map $\pi : V \to V/W$ is the zero map, then $V/W = 0$.*

*Proof.* Suppose that $\pi : V/W$ is the zero map. Then from the commutativity of the diagrams

$$\begin{tikzcd} V \arrow[rr, "\pi"] \arrow[dr, "\pi"'] & & V/W \arrow[dl, dashed, "0_{V/W}"] \\ & V/W & \end{tikzcd} \qquad \begin{tikzcd} V \arrow[rr, "\pi"] \arrow[dr, "\pi"'] & & V/W \arrow[dl, dashed, "\mathrm{Id}_{V/W}"] \\ & V/W & \end{tikzcd}$$

and the uniqueness of the maps in the Definition 1.1, we can get $0_{V/W} = \mathrm{id}_{V/W}$. Thus for any $\alpha \in V/W$, we have $\alpha = \mathrm{Id}_{V/W}(\alpha) = 0_{V/W}(\alpha) = 0$. Thus $V/W = \{0\}$. □

**Theorem 1.9.** *The quotient map $\pi : V \to V/W$ is surjective and* $\ker(\pi) = W$.

*Proof.* Let $X = \mathrm{Im}(\pi) \subset V/W$ and consider the quotient map $\pi' : V/W \to (V/W)/X$ and $f = \pi' \circ \pi : V \to (V/W)/X$. For any $\alpha \in V$, we have $\pi(\alpha) \in X$. Thus $f(\alpha) = \pi'(\pi(\alpha)) = 0$ since $\pi'(\beta) = 0$ for all $\beta \in X$. This means that $f(\alpha) = 0$ for all $\alpha \in V$. Thus we have the following two commutative diagrams

$$\begin{tikzcd} V \arrow[rr, "\pi"] \arrow[dr, "f"'] & & V/W \arrow[dl, dashed, "0"] \\ & (V/W)/X & \end{tikzcd} \qquad \begin{tikzcd} V \arrow[rr, "\pi"] \arrow[dr, "f"'] & & V/W \arrow[dl, dashed, "\pi'"] \\ & (V/W)/X & \end{tikzcd}$$

we get $\pi' = 0$. Thus by Lemma 1.8, we have $(V/W)/X = 0$ and thus $X = V/W$ by Proposition 1.7. This shows that $\pi$ is surjective.

By Definition 1.1, we know that $\pi(w) = 0, \forall w \in W$ and thus $W \subset \mathrm{Ker}(\pi)$. If $\mathrm{Ker}(\pi) \neq W$, we pick an element $\alpha \in \mathrm{Ker}(\pi)$ and $\alpha \notin W$. By Theorem 1.6 again, we can find a linear map $f : V \to F$ such that $f(\alpha) = 1, f(w) = 0$. Thus by the definition of quotient, there exists a linear map $\bar{f} : V/W \to F$ such that $f = \bar{f} \circ \pi$. Then $f(\alpha) = \bar{f}(\pi(\alpha)) = \bar{f}(0) = 0$, since $\pi(\alpha) = 0$ by assumption. This is a contradiction. □

**Theorem 1.10.** *If* $\dim V = n, \dim W = m$, *then* $\dim(V/W) = n - m$.

*Proof.* Let $\pi : V \to V/W$ be the quotient map. Assume that $\{\alpha_1, \ldots, \alpha_m\}$ is a basis of $W$ and extend it to a basis of $V$, say $\{\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n\}$. Since $\pi(w) = 0$ for all $w \in W$, we have $\pi(\alpha_i) = 0$ for all $i$ with $1 \leq i \leq m$. We will show that $S = \{\pi(\alpha_{m+1}), \ldots, \pi(\alpha_n)\}$ is a basis of of

$V/W$. By Theorem 1.9, we have $\mathrm{Span}(S) = V/W$. We now show that $S$ is linearly independent. Assume that

$$\sum_{i=m+1}^{n} c_i \pi(\alpha_i) = 0, c_i \in F. \tag{1.1}$$

We consider a vector space $X$ with $\dim X = n - m$ and $\mathcal{B} = \{e_{m+1}, \dots, e_n\}$ is a basis of $X$. By Theorem 1.6, there exists a linear map $f : V \to X$ such that $f(\alpha_i) = 0, i = 1, \dots, m$, and $f(\alpha_i) = e_i, i = m + 1, \dots, n$. By the definition of quotient, there is a unique map $\bar{f} : V/W \to X$ such that $f = \bar{f} \circ \pi$. By (1.1), we have

$$f(\sum_{i=m+1}^{n} c_i \alpha_i) = \bar{f}(\sum_{i=m+1}^{n} c_i \pi(\alpha_i)) = \bar{f}(0) = 0.$$

Thus we get

$$\sum_{i=m+1}^{n} c_i e_i = 0.$$

Since $\{e_i\}_{m+1 \le i \le n}$ is a basis of $X$, we get $c_i = 0$ for all $i = m + 1, \dots, n$. $\square$

## 2. Direct sum

We recall the following definition from [HK, §6.6]. Let $V$ be a vector space over $F$ and let $W_1, \dots, W_k$ be subspaces of $V$. Suppose that $V = W_1 + \cdots + W_k$ and the subspaces $W_1, \dots, W_k$ are independent, then we call $V$ a direct sum of $W_1, \dots, W_k$ and write

$$V = W_1 \oplus \cdots \oplus W_k.$$

Here the condition $W_1, \dots, W_k$ are independent means that if we have $\alpha_1 + \cdots + \alpha_k = 0$ for $\alpha_i \in W_i$, we then have $\alpha_i = 0$ for each $i$. Recall the following

**Theorem 2.1.** [HK, Theorem 9, page 212] *Suppose $V = W_1 \oplus \cdots \oplus W_k$, then there exists $k$ linear operators $E_1, \dots, E_k \in \mathrm{End}(V)$ such that*

(1) $E_i^2 = E_i$;
(2) $E_i \circ E_j = 0$, *if* $i \neq j$;
(3) $I = E_1 + \cdots + E_k$;
(4) $\mathrm{Im}(E_i) = W_i$.

*Conversely, suppose that there exists $k$ linear operators $E_1, \dots, E_k \in \mathrm{End}(V)$ satisfies* (2) *and* (3) *above, then*

$$V = W_1 \oplus \cdots \oplus W_k,$$

*with $W_i = \mathrm{Im}(E_i)$.*

Direct sum can also be characterized as follows.

**Proposition 2.2.** *Let $W_i, 1 \le i \le k$ be subspaces of $V$. Let $\iota_i : W_i \to W$ be the linear map defined by $\iota_i(\alpha_i) = \alpha_i$. This makes sense because $W_i$ is a subspace of $V$.*

(1) *Suppose*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

*Given any vector space $X$ and any linear map $f_i : W_i \to X$, then there is a unique linear map $f : V \to X$ such that $f_i = f \circ \iota_i$ for each $i$ with $1 \le i \le k$. In other words, there exists a commutative diagram*

$$\begin{tikzcd} W_i \arrow[rr, "\iota_i"] \arrow[dr, "f_i"'] & & V \arrow[dl, "\exists! f"] \\ & X & \end{tikzcd}$$

(2) *Given any vector space $X$ and any linear map $f_i : W_i \to X$ for all $i$ with $1 \le i \le m$, suppose there is a* ***unique*** *linear map $f : V \to X$ such that $f_i = f \circ \iota_i$ for each $i$ with $1 \le i \le k$. In other words, there exists a commutative diagram*

$$\begin{tikzcd} W_i \arrow[rr, "\iota_i"] \arrow[dr, "f_i"'] & & V \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

*Then*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

*Proof.* Part (1) is easy, we only prove part (2). Let $V' = W_1 + \cdots + W_k$ be the subspace of $V$. Let $\pi : V \to V/V'$ be the projection map. Let $f_i = \pi \circ \iota_i$. We consider the following diagrams

$$\begin{tikzcd} W_i \arrow[rr, "\iota_i"] \arrow[dr, "f_i"'] & & V \arrow[dl, "\pi"] \\ & V/V' & \end{tikzcd} \qquad \begin{tikzcd} W_i \arrow[rr, "\iota_i"] \arrow[dr, "f_i"'] & & V \arrow[dl, "0"] \\ & V/V' & \end{tikzcd}$$

The left side diagram is commutative by definition. The right side integral is commutative because $W_i \subset V'$ and thus $\pi \circ \iota = 0$. Thus the uniqueness assumption implies that $\pi = 0$. Thus $V = V' = W_1 + \cdots + W_k$.

Next, fix an integer $m$ with $1 \le m \le k$, and we consider the map $f_i^m : W_i \to W_m$ defined by

$$f_i^m = \mathrm{id}_{W_m}, \text{ if } i = m; \qquad f_i^m = 0, \text{ if } i \neq m.$$

Then by assumption, there is a unique linear map $E'_m : V \to W_m$ such that $f_i^m = E'_m \circ \iota_i$. We now define $E_m : V \to V$ by $E_m = \iota_m \circ E'_m$. Then we have

$$E_m \circ E_i = \iota_m \circ E'_m \circ \iota_i \circ E'_i = 0, \text{ if } m \neq i,$$

since $E'_m \circ \iota_i = 0$. To prove the assertion, we need to check that $I = E_1 + \cdots + E_k$, by [HK, Theorem 9, page 212] quoted above. First, for $\alpha \in V$, we can write $\alpha = \alpha_1 + \cdots + \alpha_k$ since $V = W_1 + \cdots + W_k$. It suffices to show that $E_m(\alpha) = \alpha_m$. First, for $\alpha_i \in W_i$ with $i \neq m$, we have

$$E_m(\alpha_i) = E_m(\iota_i(\alpha_i)) = \iota_m \circ E'_m \circ \iota_i(\alpha_i) = 0,$$

since $E'_m \circ \iota_i = 0$. Now apply $E_m$ to the decomposition $\alpha = \alpha_1 + \cdots + \alpha_k$, we get

$$E_m(\alpha) = E_m(\alpha_m).$$

Since $\alpha_m = \iota_m(\alpha_m)$, we get that

$$E_m(\alpha) = E_m(\alpha_m) = E_m \circ \iota_m(\alpha_m) = \iota_m \circ E'_m \circ \iota_m(\alpha_m) = \iota_m(\alpha_m) = \alpha_m,$$

where we used $E'_m \circ \iota_m = \mathrm{id}_{W_m}$. We then have

$$\alpha = \alpha_1 + \cdots + \alpha_k = E_1(\alpha) + \cdots + E_k(\alpha) = (E_1 + \cdots + E_k)(\alpha).$$

Thus $I = E_1 + \cdots + E_k$. We are done. ☐

The above defined direct sum is called the internal direct sum because $W_i$ are chosen to be subspaces of $V$. We can drop this condition and use universal property in Proposition 2.2 to define the external direct sum. Moreover, we can also consider direct sum of infinitely number of vector spaces.

**Definition 2.3.** *Let $I$ be an index set. Suppose that we are given a vector space $W_i$ over $F$ for each $i \in I$. The direct sum of $W_i$ is a pair $(\oplus_{i\in I} W_i, (\iota_i)_{i\in I})$, where*

(1) *$\oplus_{i\in I} W_i$ is a vector space over $F$; and*
(2) *$(\iota_i)_{i\in I}$ is a family of linear maps $\iota_i \in \mathrm{Hom}_F(W_i, \oplus_{i\in I} W_i)$,*

*such that for any other vector space $X$, and for any other family of linear maps $f_i \in \mathrm{Hom}_F(W_i, X)$ for each $i \in I$, there is a unique linear map $f : \oplus_{i\in I} W_i \to X$ such that $f_i = f \circ \iota_i$ for each $i \in I$. In other words, we have the following diagram*

$$\begin{tikzcd} W_i \arrow[rr, "\iota_i"] \arrow[dr, "f_i"'] & & \oplus_{i\in I} W_i \arrow[dl, dashed, "\exists! f"] \\ & X & \end{tikzcd}$$

**Proposition 2.4.** *Direct sum exists and is unique up to a unique isomorphism.*

*Proof.* The proof of the uniqueness part is the same with the proof in the quotient case and we omit the details. For the existence part, we give a sketch. It should be easy to fill the details. We first define

$$\prod_{i\in I} W_i.$$

As a set $\prod_{i\in I} W_i$ is the cartesian product of $W_i$, namely,

$$\prod_{i\in I} W_i = \{(\alpha_i)_{i\in I} | \alpha_i \in W_i\}.$$

Its vector space structure is defined by

$$c.(\alpha_i)_{i\in I} = (c\alpha_i)_{i\in I}, c \in F, \alpha_i \in W_i,$$

and

$$(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} = (\alpha_i + \beta_i)_{i\in I}, \alpha_i, \beta_i \in W_i.$$

Then we define

$$\oplus_{i\in I} W_i = \left\{ (\alpha_i)_{i\in I} \in \prod_{i\in I} W_i | \text{ there exists a finite set } S \subset I \text{ such that} \alpha_i = 0, \forall i \in I - S \right\}.$$

It is not hard to check that $\oplus_{i\in I} W_i$ is a subspace of $\prod_{i\in I} W_i$. Moreover, we consider the map

$$\iota_i : W_i \to \oplus_{i\in I} W_i$$

defined by $\iota_i(\alpha)$ is the element in $\oplus_{i\in I} W_i$ such that its $i$-th component is $\alpha_i$ and its $j$-th component is zero for any $j \in I, j \neq i$. It is not hard to check that $(\oplus_{i\in I} W_i, (\iota_i)_{i\in I})$ satisfies the universal property. □

*Remark* 2.5. The space $\prod_{i\in I} W_i$ defined in the above proof is called the **direct product**. For each $i$, we have linear map

$$\prod_{i\in I} W_i \to W_i$$

defined by $p((\alpha_i)_{i\in I}) = \alpha_i$. The pair $(\prod_{i\in I} W_i, (p_i)_{i\in I})$ satisfies the following universal property. Given any pair $(X, (f_i)_{i\in I})$, where $X$ is a vector space and $f_i \in \mathrm{Hom}_F(X, W_i)$, then there is a unique linear map $f : X \to \prod_{i\in I} W_i$ such that $f_i = p \circ f$.

$$\begin{tikzcd} W_i & & \prod_{i\in I} W_i \arrow[ll, "p_i"'] \\ & X \arrow[ul, "f_i"] \arrow[ur, dashed, "\exists! f"'] & \end{tikzcd}$$

This diagram is just obtained by reversing arrows in the direct sum commutative diagram. In this sense, the direct product is the dual object of the direct sum. From the construction, if $I$ is finite, then direct sum and direct product are the same. But when $I$ is infinite, direct product is larger than direct sum. Here is another example. As a vector space, we have

$$F[x] = \oplus_{i\in\mathbb{N}} F, \qquad F[[x]] = \prod_{i\in\mathbb{N}} F.$$

## 3. Free vector space

Let $S$ be any set.

**Definition 3.1.** *The free vector space of $S$, is a pair $(\iota, F^S)$, where*

(1) *$F^S$ is vector space over $F$, and*
(2) *$\iota : S \to F^S$ is a map between two sets,*

*such that for any other pair $(\varphi, X)$, where $X$ is a vector space $X$ over $F$ and $\varphi : S \to F^S$ is a map, there is a unique linear map $\Phi : F^S \to X$ such that $\varphi = \Phi \circ \iota$, namely, the following diagram commutes:*

$$\begin{tikzcd} S \arrow[rr, "\iota"] \arrow[dr, "\varphi"'] & & F^S \arrow[dl, dashed, "\Phi"] \\ & X & \end{tikzcd}$$

*Remark* 3.2. In the above definition, $\iota, \varphi$ are maps between sets and $\Phi : F^S \to X$ is a linear map between two vector spaces. As in the quotient space case, the above definition says that there is a bijective map

$$\mathrm{Hom}_{\mathrm{Sets}}(S, X) \to \mathrm{Hom}_{\mathrm{Vect}_F}(F^S, X).$$

Here $\mathrm{Hom}_{\mathrm{Sets}}(S, X)$ denote the set of all maps from $S$ to $X$ (which has no other requirements) and $\mathrm{Hom}_{\mathrm{Vect}_F}(F^S, X)$ denotes the set of all linear maps from $F^S$ to $X$ (which are maps that respect the structure of vector spaces).

**Theorem 3.3** (Free vector space)**.** *The free vector space exists and is unique.*

*Proof.* The proof of the uniqueness is the same as the quotient space case. The proof of existence is easy. Take

$$F^S = \left\{ f : S \to F | f^{-1}(F - \{0\}) \text{ is finite} \right\}.$$

For $f, g \in F^S, c \in F$, define $cf + g : S \to F$ by

$$(cf + g)(x) = cf(x) + g(x), \forall x \in S.$$

In this way, $F^S$ becomes a vector space. The zero element is the zero function. For any $a \in S$, define $\Delta_a : S \to F$ by

$$\Delta_a(y) = \delta_{a,y},$$

where $\delta_{a,y} = 1$ if $a = y$ and $\delta_{a,y} = 0$ if $a \neq y$. We claim that $\mathcal{B} = \{\Delta_a : a \in S\}$ is a basis of the vector space $F^S$. First, for any $f \in F^S$, let $\mathrm{Supp}(f) = \{x \in S : f(x) \neq 0\}$. According the definition, we have $\mathrm{Supp}(f)$ is finite. Say $\mathrm{Supp}(f) = \{x_1, \ldots, x_n\}$. Then it is easy to check that

$$f = f(x_1)\Delta_{x_1} + \ldots f(x_n)\Delta_{x_n}.$$

This implies that $f \in \mathrm{Span}(\mathcal{B})$. Next, suppose that we have a linear combination relation

$$c_1\Delta_{x_1} + \cdots + c_n\Delta_{x_n} = 0.$$

Denote $f = c_1\Delta_{x_1} + \cdots + c_n\Delta_{x_n}$. Note that $f = 0$ means that $f$ is the zero function. Thus $f(x) = 0$ for any $x \in S$. From $f(x_i) = 0$, we get that $c_i = 0$. This implies that $\mathcal{B}$ is linearly independent. Thus $\mathcal{B}$ is a basis of $F^S$.

Consider the map $\iota : S \to F^S$ defined by

$$\iota(a) = \Delta_a.$$

We will show that for any vector space $X$ and any map $\varphi : S \to X$ (as a map of sets), there exists a linear transformation $\Phi : F^S \to X$ such that $\varphi = \Phi \circ \iota$, namely, $\varphi(a) = \Phi(\Delta_a)$ for all $a \in S$. Actually, since $\mathcal{B}$ is a basis of $F^S$, for any $f \in F^S$, we can write

$$f = \sum_{i=1}^{n} c_i \Delta_{a_i},$$

with uniquely determined $c_i$ (in fact, $c_i = f(a_i)$.) We just define $\Phi(f) = \sum_{i=1}^n c_i \Phi(\Delta_{a_i})$. Thus there is a unique such $\Phi$ satisfies $\Phi(\Delta_{a_i}) = \varphi(a_i)$. $\square$

*Remark* 3.4. Actually, any vector space is free. More precisely, let $V$ be a vector space and $\mathcal{B}$ be a basis of $V$. Then there is an isomorphism $F^{\mathcal{B}} \to V$.

## 4. Tensor product

### 4.1. **Tensor product of two vector spaces.**

**Definition 4.1.** *Let $V, W, X$ be vector spaces over $F$. A map $f : V \times W \to X$ is called **bilinear** if*

$$f(c\alpha + \alpha', \beta) = cf(\alpha, \beta) + f(\alpha', \beta),$$

*and*

$$f(\alpha, c\beta + \beta') = cf(\alpha, \beta) + f(\alpha, \beta'),$$

*for all $c \in F, \alpha, \alpha' \in V, \beta, \beta' \in W$.*

**Proposition 4.2.** *Assume that $\dim V = n, \dim W = m$ and $X$ is an arbitrary vector space over $F$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis of $V$ and $\{\beta_1, \ldots, \beta_m\}$ be a basis of $W$. For any set*

$$\{e_{ij} \in X | 1 \le i \le n, 1 \le j \le m\},$$

*there is a unique bilinear map $f : V \times W \to X$ such that*

$$f(\alpha_i, \beta_j) = e_{ij}, 1 \le i \le n, 1 \le j \le m.$$

*Proof.* This is an analogue of Theorem 1.6 and the proof is similar. □

**Definition 4.3.** *Given two vector spaces $V_1, V_2$ over $F$, the tensor product of $V_1$ with $V_2$ is an $F$ vector space, denoted by $V_1 \otimes_F V_2$, together with an $F$-bilinear map $\tau : V_1 \times V_2 \to V_1 \otimes_F V_2$ such that for every vector space $X$ over $F$ and for every $F$-bilinear map $\varphi : V_1 \times V_2 \to X$, there exists a unique linear map (linear transformation) $\Phi : V_1 \otimes_F V_2 \to X$ such that $\varphi = \Phi \circ \tau$, namely, we have the following commutative diagram*

$$\begin{tikzcd}
V_1 \times V_2 \arrow[rr, "\tau"] \arrow[dr, "\varphi"'] & & V_1 \otimes_F V_2 \arrow[dl, dashed, "\Phi"] \\
& X &
\end{tikzcd}$$

**Proposition 4.4.** *The tensor product, $V_1 \otimes_F V_2$ is unique up to unique isomorphism, if it exists.*

*Proof.* The proof is similar as in the quotient space case, Theorem 1.2. Try to fill the details by yourself. □

**Theorem 4.5.** *Tensor product exists.*

*Proof.* Let $F^{V_1 \times V_2}$ be the free vector space of the *set* $V_1 \times V_2$. Recall that there is an associated set map $\iota : V_1 \times V_2 \to F^{V_1 \times V_2}$ such that, for any vector space $X$ over $F$, and for any set map $\varphi : V_1 \times V_2 \to X$, there exists a unique linear map $\Phi : F^{V_1 \times V_2} \to X$ such that the following diagram is commutative

$$\begin{tikzcd}
V_1 \times V_2 \arrow[rr, "\iota"] \arrow[dr, "\varphi"'] & & F^{V_1 \times V_2} \arrow[dl, dashed, "\Phi"] \\
& X &
\end{tikzcd}$$

Let $Y \subset F^{V_1 \times V_2}$ be the subspace spanned by all elements

$$\iota(c\alpha + \alpha', \beta) - c\iota(\alpha, \beta) - \iota(\alpha', \beta)$$
$$\iota(\alpha, c\beta + \beta') - c\iota(\alpha, \beta) - \iota(\alpha, \beta')$$

for all $\alpha, \alpha' \in V_1, \beta, \beta' \in V_2, c \in F$. Let $V_1 \otimes_F V_2 := F^{V_1 \times V_2}/Y$ and let $\pi : F^{V_1 \times V_2} \to V_1 \otimes_F V_2$ be the quotient map. Let $\tau = \pi \circ \iota$, which is a map $V_1 \times V_2 \to V_1 \otimes_F V_2$. We will check that $(V_1 \otimes_F V_2, \tau)$ satisfies the conditions given in Definition 4.3. We first need to check that $\tau$ is bilinear. Thus for any $\alpha, \alpha' \in V_1, \beta, \beta' \in V_2$, $c \in F$, we need to check that

$$\tau(c\alpha + \alpha', \beta) = c\tau(\alpha, \beta) + \tau(\alpha', \beta), \tau(\alpha, c\beta + \beta') = c\tau(\alpha, \beta) + \tau(\alpha, \beta').$$

Note that

$$\begin{aligned}\tau(c\alpha+\alpha',\beta)-c\tau(\alpha,\beta)-\tau(\alpha',\beta) &= \pi(\iota(c\alpha+\alpha',\beta))-c\pi(\iota(\alpha,\beta))-\pi(\iota(\alpha',\beta))\\ &= \pi\left(\iota(c\alpha+\alpha',\beta)-c\iota(\alpha,\beta)-\iota(\alpha',\beta)\right)\\ &= 0.\end{aligned}$$

Here we used that $\pi$ is linear and $\iota(c\alpha+\alpha',\beta)-c\iota(\alpha,\beta)-\iota(\alpha',\beta)\in Y$ and thus $\pi(\iota(c\alpha+\alpha',\beta)-c\iota(\alpha,\beta)-\iota(\alpha',\beta))=0$. This shows the first equation. The second equation can be proved similarly.

Next, we check that $(V_1\otimes_F V_2,\tau)$ satisfies the universal property. Namely, for any bilinear map $\varphi: V_1\times V_2\to X$, we will show that there is a unique linear map $\Phi: V_1\otimes_F V_2\to X$ such that $\varphi=\Phi\circ\tau$.

$$\begin{tikzcd} V_1\times V_2 \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & F^{V_1\times V_2} \arrow[r, "\pi"] \arrow[d, dashed, "\varphi_1"'] & V_1\otimes V_2 \arrow[dl, dashed, "\Phi"] \\ & X & \end{tikzcd}$$

We first view $\varphi: V_1\times V_2\to X$ as a set map. Then by the universal property of free vector space, there exists a unique linear map $\varphi_1: F^{V_1\times V_2}\to X$ such that $\varphi=\varphi_1\circ\iota$. Moreover, since $\varphi$ is bilinear, we have

$$\varphi_1(\iota(c\alpha+\alpha',\beta)-c\iota(\alpha,\beta)-\iota(\alpha',\beta))=\varphi(c\alpha+\alpha',\beta)-c\varphi(\alpha,\beta)-\varphi(\alpha',\beta)=0.$$

Similarly, we have

$$\varphi_1(\iota(\alpha,c\beta+\beta')-c\iota(\alpha,\beta)-\iota(\alpha,\beta'))=0.$$

This implies that $\varphi_1(y)=0$ for any $y\in Y$. Now by the universal properties of the quotient space, we know that there exists a unique linear map $\Phi: V_1\otimes V_2\to X$ such that $\varphi_1=\Phi\circ\pi$. Thus we get

$$\Phi\circ\tau=\Phi\circ\pi\circ\iota=\varphi_1\circ\iota=\varphi.$$

The uniqueness of $\Phi$ also follows from the above proof. □

**Notation:** In the following, for $\alpha\in V_1,\beta\in V_2$, we will write

$$\alpha\otimes\beta:=\tau(\alpha,\beta),$$

and write the map $\tau: V_1\times V_2\to V_1\otimes_F V_2$ as $\otimes: V_1\times V_2\to V_1\otimes_F V_2$. Using this new notation, the basic properties of $\tau$ can be written as

$$(c\alpha+\alpha')\otimes\beta=c\alpha\otimes\beta+\alpha'\otimes\beta,\qquad \alpha\otimes(c\beta+\beta')=c\alpha\otimes\beta+\alpha\otimes\beta'.$$

When the field $F$ is understood, we will omit it from the notations and write $V_1\otimes V_2$ instead of $V_1\otimes_F V_2$.

For vector spaces $V_1,V_2,X$, denote $\mathrm{Bil}(V_1,V_2;X)$ the space of all bilinear maps $V_1\times V_2\to X$. Then the universal property of tensor product is equivalent to saying that the map

$$\mathrm{Bil}(V_1,V_2;X)\to\mathrm{Hom}_F(V_1\otimes V_2,X), \tag{4.1}$$

defined by $\varphi\mapsto\Phi$ for $\Phi$ with $\varphi=\Phi\circ\tau$ is a bijection. Compare this with Remark 1.4.

Next, we prove some basic properties of $V_1\otimes V_2$.

**Proposition 4.6.** *We have* $V_1\otimes_F V_2=\mathrm{Span}\{\alpha\otimes\beta:\alpha\in V_1,\beta\in V_2\}$.

*Proof.* Let $X=\mathrm{Span}\{\alpha\otimes\beta:\alpha\in V_1,\beta\in V_2\}$, which is a subspace of $V_1\otimes V_2$. Consider $Q=V_1\otimes V_2/X$ and let $q: V_1\otimes V_2\to Q$ be the quotient map. Note that for any $\alpha\in V_1,\beta\in V_2$, we have $q(\alpha\otimes\beta)=0$ since $\alpha\otimes\beta\in X$. In other words, we have the following commutative diagram

$$\begin{tikzcd} V_1\times V_2 \arrow[rr, "\otimes=\tau"] \arrow[dr, "0"'] & & V_1\otimes V_2 \arrow[dl, dashed, "q"] \\ & Q & \end{tikzcd}$$

where $0 : V_1 \times V_2 \to Q$ is the zero bilinear map. On the other hand, we clearly have the following diagram

$$\begin{tikzcd} V_1 \times V_2 \arrow[rr, "\otimes=\tau"] \arrow[dr, "0"'] & & V_1 \otimes V_2 \arrow[dl, dashed, "0"] \\ & Q & \end{tikzcd}$$

Thus by the uniqueness of the map in the definition of tensor product, we get that the quotient map $q$ is actually the zero map. This implies that $Q = 0$ and thus $X = V_1 \otimes V_2$ following Theorem 1.9. $\square$

**Warning:** A typical element in $V_1 \otimes V_2$ is of the form

$$\sum_{i=1}^{n} \alpha_i \otimes \beta_i, \alpha_i \in V_1, \beta_i \in V_2.$$

In general, not every element in $V_1 \otimes V_2$ can be written as $\alpha \otimes \beta$ for $\alpha \in V_1, \beta \in V_2$. Elements in $V_1 \otimes V_2$ of the form $\alpha \otimes \beta$ are called pure tensors.

**Theorem 4.7.** *If $\dim V_1 = m, \dim V_2 = n$, then $\dim V_1 \otimes V_2 = mn$. More precisely, if $\{\alpha_i\}_{1\le i\le m}$ is a basis of $V_1$ and $\{\beta_j\}_{1\le j\le n}$ is a basis of $V_2$, then $\{\alpha_i \otimes \beta_j\}_{1\le i\le m, 1\le j\le n}$ is a basis of $V_1 \otimes V_2$.*

*Proof.* Denote $\mathcal{B} = \{\alpha_i \otimes \beta_j\}_{1\le i\le m,1\le j\le n}$. We first show that $\mathcal{B}$ spans $V_1 \otimes V_2$. This follows from Proposition 4.6 directly. Actually any pure tensor $\alpha \otimes \beta$ for $\alpha \in V_1, \beta \in V_2$ is a linear combination of $\{\alpha_i \otimes \beta_j\}_{1\le i\le m,1\le j\le n}$: if $\alpha = \sum c_i \alpha_i, \beta = \sum d_j \beta_j$, then $\alpha \otimes \beta = \sum c_i d_j \alpha_i \otimes \beta_j$. By Proposition 4.6, $V_1 \otimes V_2$ is spanned by all pure tensors, thus it is spanned by $\mathcal{B}$.

Next we show $\mathcal{B}$ is linearly independent. Suppose that

$$\sum_{i,j} c_{ij} \alpha_i \otimes \beta_j = 0 \tag{4.2}$$

for $c_{ij} \in F$. We will show that $c_{ij} = 0$ for all $i, j$. Consider a vector space $X$ with dimension $mn$ and basis

$$\{e_{ij} : 1 \le i \le m, 1 \le j \le n\}.$$

We define a bilinear map $\varphi : V_1 \times V_2 \to X$ by $\varphi(\alpha_i, \beta_j) = e_{ij}$. By Proposition 4.2, there is a unique such map. By the definition of tensor product, there is a unique linear map $\Phi : V_1 \otimes V_2 \to X$ such that $\varphi(\alpha, \beta) = \Phi(\alpha \otimes \beta)$ for any $\alpha \in V_1, \beta \in V_2$. Apply $\Phi$ to (4.2), we then get

$$0 = \sum_{i,j} c_{ij} \Phi(\alpha_i \otimes \beta_j) = \sum_{i,j} c_{ij} e_{ij}.$$

Since $\{e_{ij}\}$ are linearly independent in $X$, we get $c_{ij} = 0$ for all $i, j$. $\square$

*Remark* 4.8. In the above proof, we indeed used an external vector space $X$ to detect the properties of $V_1 \otimes_F V_2$, whose internal structure (its dimension in the above example) are not very clear from its definition. Actually, the universal property of $V_1 \otimes_F V_2$ in its definition tells how it interacts with other vector spaces (external spaces) and it turns out that all of its internal structure can be determined using this. It is very like the following situation. Suppose that we know there exists a person, and we don't know much about her/his internal properties, like her/his appearance/voices and so on; but we know how she/he interacts with everybody in the whole world, then we should be able to know everything about her/him. Usually, how does one interact with other people can tell you more about this person than how she/he looks. Similarly, in mathematical world, if one knows how one math object (like vector space, topological space, group,...) interacts with any other similar math objects (the interaction between math objects are just maps between them which respects the math structures, like linear maps for vector spaces, continuous maps for topological spaces, group homomorphism for groups,...), we should understand everything about this math object. This is actually one very important spirit in modern math (in particular in category theory), morphisms (certain maps, eg. linear transformations, continuous maps, group homomorphism,...) are at least as important as (if not more important than) the math objects themselves (eg. vector spaces). A great example to illustrate this is the famous Grothendieck-Riemann-Roch theorem, see [Jac04].

Now let $T_1 : V_1 \to W_1, T_2 : V_2 \to W_2$ be two linear transformations. Consider the map $\otimes\circ(T_1, T_2) : V_1 \times V_2 \to W_1 \otimes W_2$ defined by

$$(v_1, v_2) \mapsto T_1(v_1) \otimes T_2(v_2).$$

Then this map is bilinear. Thus by the universal property of tensor product, we have a linear map

$$T_1 \otimes T_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$$

such that

$$T_1 \otimes T_2(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2),$$

in other words, we have the following commutative diagram

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "\otimes"] \arrow[d, "(T_1,T_2)"'] & V_1 \otimes V_2 \arrow[d, dashed, "T_1\otimes T_2"] \\ W_1 \times W_2 \arrow[r, "\otimes"] & W_1 \otimes W_2 \end{tikzcd}$$

Here the left side map $(T_1, T_2)$ is the map $(T_1, T_2)(v_1, v_2) = (T_1(v_1), T_2(v_2))$.

The map $T_1 \otimes T_2$ is called the tensor product of $T_1$ with $T_2$.

Let us try a small example. Consider $V_1 = V_2 = W_1 = W_2 = F^2$ and a map $T_i \in \mathrm{End}(F^2)$ is given by a matrix $A_i \in \mathrm{Mat}_{2\times2}(F)$. What is the matrix of $T_1 \otimes T_2$? We fix $\epsilon_1 = (1, 0)^t, \epsilon_2 = (0, 1)^t$. Then $\mathcal{B}_1 = \{\epsilon_1, \epsilon_2\}$ is a standard basis of $V = F^2$. Let $e_1 = \epsilon_1 \otimes \epsilon_1, e_2 = \epsilon_1 \otimes \epsilon_2, e_3 = \epsilon_2 \otimes \epsilon_1$ and $e_4 = e_2 \otimes e_2$. Then by Theorem 4.7, $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ is a basis of $V \otimes V$. Assume that $[T_1]_{\mathcal{B}_1} = A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $[T_2]_{\mathcal{B}_1} = B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Then

$$T_1\epsilon_1 = a_{11}\epsilon_1 + a_{21}\epsilon_2, T_1\epsilon_2 = a_{12}\epsilon_1 + a_{22}\epsilon_2,$$

$$T_2\epsilon_1 = b_{11}\epsilon_1 + b_{21}\epsilon_2, T_2\epsilon_2 = b_{12}\epsilon_1 + b_{22}\epsilon_2.$$

Thus we can get

$$(T_1 \otimes T_2)(\epsilon_1 \otimes \epsilon_1) = (a_{11}\epsilon_1 + a_{21}\epsilon_2) \otimes (b_{11}\epsilon_1 + b_{21}\epsilon_2) = a_{11}b_{11}e_1 + a_{11}b_{21}e_2 + a_{21}b_{11}e_3 + a_{21}b_{21}e_4.$$

A simple calculation shows that

$$(T_1 \otimes T_2)[e_1, e_2, e_3, e_4] = [e_1, e_2, e_3, e_4] \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{22} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}.$$

The right hand matrix is usually denoted by $A \otimes B$, which can be viewed in block form

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}.$$

In general, for $A \in \mathrm{Mat}_{m\times m}, B \in \mathrm{Mat}_{n\times n}$, we define a matrix $A \otimes B \in \mathrm{Mat}_{mn\times mn}$ in the same way by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \dots & \dots & \dots & \dots \\ a_{m1}B & a_{m2}B & \dots & a_{mm}B \end{bmatrix}.$$

It represents the matrix of a tensor product of two linear operators in some basis.

**Exercise 4.9.** *How does tensor product of two linear operators behave under composition? Namely, suppose $T_i, U_i : V_i \to V_i$ are linear operators, and we can compose the $U_i \circ T_i : V_i \to V_i$. What is the relationship between $(U_1 \circ T_1) \otimes (U_2 \circ T_2)$ with, say, $(U_1 \otimes U_2) \circ (T_1 \otimes T_2)$?*

**Exercise 4.10.** *If $T_i : V_i \to V_i$ are invertible, is $T_1 \otimes T_2 : V_1 \otimes V_2 \to V_1 \otimes V_2$ invertible? Given $A \in \mathrm{Mat}_{m\times m}, B \in \mathrm{Mat}_{n\times n}$, what is $\det(A \otimes B)$ in terms of $\det(A)$ and $\det(B)$?*

The answer is yes for the first question, which either follows from Problem 4.9 or the second part of Problem 4.10. The answer for the second part of Problem 4.10 is

$$\det(A \otimes B) = \det(A)^n \det(B)^m.$$

You can try to figure out a proof on your own, which is not very hard. Also, see this link.

From the above two problems, we know that there is a group homomorphism

$$\mathrm{GL}_m(F) \times \mathrm{GL}_n(F) \to \mathrm{GL}_{mn}(F)$$

$$(A, B) \mapsto A \otimes B.$$

(A group homomorphism is a map between groups which preserves product of the groups.) This indeed give a *representation* of the group $\mathrm{GL}_m(F) \times \mathrm{GL}_n(F)$, which is called the tensor product representation.

**Exercise 4.11.** *What can you say about* $\mathrm{Tr}(A \otimes B)$*?*

**Exercise 4.12.** *For any vector spaces* $V_1, V_2$*, show that there is a natural isomorphism* $V_1 \otimes V_2 \to V_2 \otimes V_1$*.*

**Exercise 4.13.** *Let* $V_1, V_2$ *be finite dimensional vector spaces and let* $V_1^*$ *be the dual space of* $V_1$*. Show that there is a linear map* $\Theta : V_1^* \otimes V_2 \to \mathrm{Hom}(V_1, V_2)$ *such that* $\Theta_{f\otimes\beta} \in \mathrm{Hom}(V_1, V_2)$ *is defined by*

$$\Theta_{f\otimes\beta}(\alpha) = f(\alpha)\beta,$$

*for all* $f \otimes \beta \in V_1^* \otimes V_2$*. Show that* $\Theta$ *is an isomorphism.*

This gives another proof that $\dim(V_1^* \otimes V_2) = \dim(V_1)\dim(V_2)$.

**Exercise 4.14.** *Given any vector spaces* $V_1, V_2, X$ *over* $F$*. Show that there is a natural isomorphism*

$$\mathrm{Hom}_F(V_1 \otimes V_2, X) \to \mathrm{Hom}_F(V_1, \mathrm{Hom}_F(V_2, X)).$$

Actually, both sides can be identified with $\mathrm{Bil}_F(V_1, V_2; X)$.

If we combine the above two problems, we get the following

**Proposition 4.15.** *Given two vector spaces* $V_1, V_2/F$*, we have a natural isomorphism*

$$V_1^* \otimes V_2^* \to (V_1 \otimes V_2)^* = \mathrm{Hom}_F(V_1 \otimes V_2, F).$$

*Proof.* This follows from the above two problems directly. This isomorphism can also be directly described as follows. Given $f \in V_1^*, g \in V_2^*$, we consider the map $\Phi_{f,g} : V_1 \times V_2 \to F$ defined by

$$\Phi_{f,g}(\alpha, \beta) = f(\alpha)g(\beta), \alpha \in V_1, \beta \in V_2.$$

Then it is clear that $\Phi_{f,g}$ is bilinear. Thus by the universal property of tensor product, there is a unique linear map $\Psi_{f,g} : V_1 \otimes V_2 \to F$ such that

$$\Psi_{f,g}(\alpha \otimes \beta) = f(\alpha)g(\beta), \alpha \in V_1, \beta \in V_2.$$

Thus we get an element $\Psi_{f,g} \in \mathrm{Hom}_F(V_1 \otimes V_2, F) = (V_1 \otimes V_2)^*$. Thus we get a map

$$V_1^* \times V_2^* \to (V_1 \otimes V_2)^*$$

defined by

$$(f, g) \mapsto \Psi_{f,g}.$$

This map is also bilinear, and thus we get a linear map $\Psi : V_1^* \otimes V_2^* \to (V_1 \otimes V_2)^*$ defined by

$$\Psi(f \otimes g) \to \Psi_{f,g}.$$

One can check that this map is an isomorphism. Actually, the inverse map of $\Psi$ is easy to describe. We omit the details. □

4.2. **General tensor product.** Given a finite number of vector spaces $V_1, V_2, \dots, V_n$, we can consider the vector space

$$((V_1 \otimes V_2) \otimes V_3) \otimes V_4 ...$$

by tensoring two of them each time. It turns out that no matter how you choose the order to take tensors, you will get the same (up to unique isomorphism) linear vector space, which will be simply denoted by

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n.$$

If $V_1 = V_2 = \cdots = V_n = V$, we denote the above space by $\otimes^n V$ or $V^{\otimes n}$. It can be characterized using a universal property of *multi-linear* maps. You can try to fill the details.

Here is an explanation of the notations from the textbook [HK, Section 5.6]. In the textbook, $M^r(V)$ denotes the set of multilinear functions on $V^r$. In particular, $M^2(V) = \mathrm{Bil}(V, V; F) \cong \mathrm{Hom}_F(V \otimes V, F)$. Thus $M^2(V)$ indeed denotes the dual space of $V \otimes V$ (in our notation). Similarly, $M^r(V)$ denotes the dual space of $(\otimes^r V)^*$, which is in fact isomorphic to $\otimes^r V^*$. This is a generalization of Proposition 4.15.

Given $f \in (V^r)^*, g \in (V^s)^*$, the function $f \otimes g$ denotes the function on $V^{r+s}$ by

$$f \otimes g(\alpha, \beta) = f(\alpha)g(\beta), f \in V^r, g \in V^s.$$

In our terminology, this $f \otimes g$ is in fact an element in $\mathrm{Bil}(V^r, V^s; F)$. Since $\mathrm{Bil}(V^r, V^s; F) \cong (V^r \otimes V^s)^* = (V^r)^* \otimes (V^s)^*$, $f \otimes g$ indeed can be viewed as an element in $(V^r)^* \otimes (V^s)^*$, which agrees with our notation under some identifications.

## 5. Field extension

Let $K$ be a field and $F$ be a subfield of $K$. Then $K$ is called a field extension of $F$. Some examples: $F = \mathbb{R}, K = \mathbb{C}$; $F = \mathbb{Q}, F = \mathbb{Q}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + \sqrt[3]{4} : a, b, c \in \mathbb{Q}\}$; $F = \mathbb{Q}, K = \mathbb{R}$; $F = \mathbb{Q}, K = \mathbb{Q}[\sqrt{-1}] = \{a + b\sqrt{-1} : a, b \in \mathbb{Q}\}$. The field extension $K/F$ is called *finite* if $K$ is a finite dimensional vector space over $F$. For example, $\mathbb{Q}[\sqrt[3]{2}]/\mathbb{Q}$ is a finite field extension, while $\mathbb{R}/\mathbb{Q}$ is an infinite extension.

We care about how linear algebra behaves under field extension. For example, in our HW, we were asked to prove the following: if $A, B \in \mathrm{Mat}_{n\times n}(F)$ are similar over $K$, then they are also similar over $F$. Such problems can be studied using tensor product. The most basic fact is

**Lemma 5.1.** *We have $F \otimes_F K \cong K$.*

Here the notation $\otimes_F$ means tensor product over the field $F$, in other words, both variables $K$ and $F$ in $F \otimes_F K$ are viewed as $F$-vector space.

The proof of the above lemma is easy and can be checked directly. Moreover, we have the following

**Lemma 5.2.** (1) *We have $F^n \otimes_F K \cong K^n$.*
(2) *We have $F[x] \otimes_F K \cong K[x]$.*
(3) *Let $d \in F[x]$, we have*

$$(F[x]/dF[x]) \otimes_F K \cong K[x]/dK[x].$$

Note here, we don't require that $K/F$ is finite.

**Exercise 5.3.** *Prove (1) and (2) of Lemma 5.2.*

As an example of Lemma 5.2, we have $\mathrm{Mat}_{m\times n}(F) \otimes_F K \cong \mathrm{Mat}_{m\times n}(K)$.

Given a matrix $A \in \mathrm{Mat}_{m\times n}(F)$, we can view $A$ as a linear operator $T_A : F^n \to F^m$ defined by $T_A(\alpha) = A\alpha$. Let $K$ be a field extension of $F$. We can also view $A$ as an element of $\mathrm{Mat}_{m\times n}(K)$ and thus we get a linear operator $T'_A : K^n \to K^m$ defined by $T'_A(\alpha) = A\alpha$.

Now suppose that $V, W$ are finite dimensional vector spaces over $F$ and let $T : V \to W$ be a linear operator. We assume that $\dim_F V = n, \dim_F W = m$. After fixing basis of $V, W$, we have isomorphisms $V \cong F^n, W \cong F^m$ and thus $T$ can be identified with an element $A \in \mathrm{Mat}_{m\times n}(F)$. As discussed above, we should be able to define a linear map $T'$ from an $n$-dimensional vector space over $K$ to an $m$-dimensional vector space over $K$, which corresponds to $A$ when we view $A$ as an element of $\mathrm{Mat}_{m\times n}(K)$. This can be done as above if we identify $V$ with $F^n$ and $W$ with $F^m$. But this process depends on choices of basis of $V$ and $W$ and it is not intrinsic. The isomorphism in Lemma 5.2 reminds us that we can define $T'$ intrinsically as

$$T' = T \otimes \mathrm{Id}_K : V \otimes_F K \to W \otimes_F K.$$

By Lemma 5.2, we have $\dim_K(V \otimes_F K) = \dim_F V = n$ and $\dim_K(W \otimes_F K) = \dim_F W = m$. If we fix a basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ as a basis of $V$ over $F$, then $\mathcal{B} \otimes_F K = \{\alpha_1 \otimes 1, \ldots, \alpha_n \otimes 1\}$ is a basis of $V \times_F K$ over $K$. Similarly, if $\mathcal{B}'$ is a basis of $W$ over $F$, then $\mathcal{B}' \otimes_F K$ is a basis of $W \otimes_F K$ over $K$. In these basis, we can see that the matrix of $A = [T]_{\mathcal{B},\mathcal{B}'}$ is exactly the same as the matrix $A' = [T']_{\mathcal{B}\otimes K, \mathcal{B}'\otimes K}$. The only difference is: $A$ is viewed as a matrix in $\mathrm{Mat}_{m\times n}(F)$ and $A'$ is viewed as a matrix in $\mathrm{Mat}_{m\times n}(K)$. Under the isomorphism $\mathrm{Mat}_{m\times n}(K) \cong \mathrm{Mat}_{m\times n}(F) \otimes_F K$, the matrix $A'$ is just $A \otimes 1$, where $1 \in K$ is the identity element.

Now we suppose that $m = n$. Then we have

$$\det(T) = \det(T \otimes \mathrm{Id}_K), \mathrm{Tr}(T) = \mathrm{Tr}(T \otimes \mathrm{Id}_K), \tag{5.1}$$

since $T$ and $T \otimes \mathrm{Id}_K$ are represented by the same matrix.

These simple formulae (5.1) could be useful. We give one simple example. Consider the vector space $V = \mathbb{R}[x]/(x^2+1)$ over $\mathbb{R}$ and the linear map $T : V \to V$ defined by $T(\overline{f}) = \overline{xf}$, where $f \in \mathbb{R}[x]$ and $\overline{f}$ denotes its equivalence class. Thus $T$ defines an element $\mathrm{GL}_2(\mathbb{R})$ since $\dim_{\mathbb{R}} \mathbb{R}[x]/(x^2+1) = 2$. We want to compute $\det(T)$. Of course, this can be simply done by writing down explicitly the matrix of $T$ after fixing an $\mathbb{R}$ basis of $V$. But we consider the other way. We consider

$$T' = T \otimes \mathbb{C} : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}.$$

We have

$$V \otimes_{\mathbb{R}} \mathbb{C} = \big(\mathbb{R}[x]/(x^2+1)\big) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x]/(x^2+1) \cong \mathbb{C}[x]/(x+i) \otimes \mathbb{C}[x]/(x-i).$$

Note that the map $T'$ is still induced by multiplication by $x$ and thus $\mathbb{C}[x]/(x+i)$ and $\mathbb{C}[x]/(x-i)$ are $T'$-invariant. But note that $\dim \mathbb{C}[x]/(x-i) = 1$ and the restriction of $T'$ on $\mathbb{C}[x]/(x-i)$ is just multiplication by $i$. Similarly, the restriction of $T'$ on $\mathbb{C}[x]/(x+i)$ is just multiplication by $-i$. In certain basis, $T'$ can be represented by the matrix

$$\begin{bmatrix} i & \\ & -i \end{bmatrix}.$$

Thus $\det(T) = \det(T') = 1$. Note that $T$ is not diagonalizable over $\mathbb{R}$.

In the following, we give a generalization of the above example, which is useful in number theory. Let $F$ be a subfield of $\mathbb{C}$. Let $\alpha \in \mathbb{C}$ be an element such that it satisfies an irreducible polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n, a_i \in F.$$

Let $K = F(\alpha) = \{g(\alpha) : g \in F[x]\} = \left\{c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1} : c_i \in F\right\}$. We can check that the map

$$\Phi : F[x] \to K$$

defined by $\Phi(g) = g(\alpha)$ is surjective and its kernel is the ideal $fF[x]$. Thus we get an isomorphism

$$K \cong F[x]/(f).$$

This construction is indeed parallel to cyclic vector space as we discussed in [HK, §7.1]. The difference is that, in the construction of cyclic vector space we don't require $f$ is irreducible. Consider the $F$-linear map $T_\alpha : K \to K$ defined by $T_\alpha(x) = \alpha x, x \in K$. When we view $K$ as an $n$-dimensional vector space over $F$, $T_\alpha$ defines an element in $\mathrm{Mat}_{n\times n}(F)$. We have defined

$$\mathrm{Nm}_{K/F}(\alpha) = \det(T_\alpha) \in F$$

in a homework problem. Note that, after we identity $K$ with $F[x]/f$, the map $T_\alpha : K \to K$ can be identified with the map $T_x : F[x]/f \to F[x]/f$, where $T_x(\overline{g}) = \overline{xg}$. One example of the above construction is when $F = \mathbb{Q}$ and $K = \mathbb{Q}(\alpha)$ with $\alpha^3 = 2$. We considered this example many times. Let $\hat{F}$ be another (possibly infinite) extension of $F$. Then we have

$$K \otimes_F \hat{F} = F[x]/(f) \otimes_F \hat{F} = \hat{F}[x]/(f).$$

Note that $f \in \hat{F}[x]$ in general is not irreducible anymore. We can factor $f$ as

$$f = f_1 f_2 \dots f_k$$

with irreducible $f_i \in \hat{F}[x]$. By [HK, page 266, Lemma], $f_i$ are distinct. Thus by Chinese remainder theorem, we have

$$K \otimes_F \hat{F} = \hat{F}[x]/(f_1 \dots f_k) = \prod_j \hat{F}[x]/(f_j).$$

Now each factor $\hat{F}[x]/(f_i)$ is a field and we denote it by $K_i$. Then $K_i$ is a finite field extension of $\hat{F}$. Moreover, the space $\hat{F}[x]/(f_i)$ is invariant under the map $T_x \otimes_F K$. Thus (5.1), we have

$$\mathrm{Nm}_{K/F}(\alpha) = \det(T_x) = \det(T_x \otimes_F K) = \prod_{1 \le j \le k} \mathrm{Nm}_{K_j/\hat{F}}(\alpha). \tag{5.2}$$

Here in the $j$-th factor on the right side, $\alpha$ should be interpreted as the image of $\alpha \otimes 1_k$ under the map $K \otimes_F \hat{F} \to \hat{F}[x]/(f_j)$.

The above process and (5.2) could be explained in matrix language in a simple way. Given a matrix $A \in \mathrm{Mat}_{n\times n}(F)$. It's determinant $\det(A)$ might be hard to compute. But if we view $A$ as an element in $\mathrm{Mat}_{n\times n}(\hat{F})$ for a field extension $\hat{F}$ of $F$, the matrix $A$ is similar to a block diagonal matrix

$$\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{bmatrix}$$

Then we have

$$\det(A) = \prod_{1 \le j \le k} \det(A_j).$$

Note that, each $\det(A_j)$ is just an element in $\hat{F}$ but the product $\prod_j \det(A_j)$ is in $F$. In our first example for $\mathbb{R}[x]/(x^2+1)$, the map $T$ is given by the matrix

$$A = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$$

in some basis, which is not diagonalizable. But after we view it as an element in $\mathrm{GL}_2(\mathbb{C})$, it is similar to the diagonal matrix

$$\begin{bmatrix} i & \\ & -i \end{bmatrix},$$

whose determinant is $i \cdot (-i) = 1$. Try to analyze (5.2) when $F = \mathbb{Q}, K = \mathbb{Q}(\alpha)$ and $\hat{F} = \mathbb{R}$, where $\alpha^3 = 2$.

The next proposition gives another useful fact about $T \otimes_F \mathrm{Id}_K$.

**Proposition 5.4.** *Let $T : V \to W$ be a linear map over $F$ and $K$ be a field extension of $F$. Then we have* $\mathrm{Ker}(T \otimes_F \mathrm{Id}_K) = \mathrm{Ker}(T) \otimes_F K$.

This is essentially proved in solutions of Ex 12, Section 7.2. Try to figure out a detailed proof on your own in this language.

## 6. Exterior powers

6.1. **Exterior square.** Let $V, X$ be two vector spaces over $F$. A **bilinear** map $f : V \times V \to X$ is called **alternating** if $f(\alpha, \alpha) = 0$ for all $\alpha \in V$.

**Proposition 6.1.** *Suppose that $f : V \times V \to X$ is alternating, then $f(\alpha, \beta) = -f(\beta, \alpha)$ for all $\alpha, \beta \in V$.*

*Proof.* Since $f$ is bilinear, we have

$$\begin{aligned} 0 &= f(\alpha+\beta, \alpha+\beta) \\ &= f(\alpha,\alpha) + f(\alpha,\beta) + f(\beta,\alpha) + f(\beta,\beta) \\ &= f(\alpha,\beta) + f(\beta,\alpha). \end{aligned}$$

The result follows. □

**Proposition 6.2.** *Suppose that $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ is a basis of $V$ and $\{e_{ij}\}_{1\le i<j\le n} \in X$ be $n(n-1)/2$ elements of $X$, then there is a unique alternating map $f : V \times V \to X$ such that*

$$f(\alpha_i, \alpha_j) = e_{ij}, 1 \le i < j \le n.$$

*Proof.* Suppose that $f : V \times V \to X$ is an alternating map. For $\alpha = \sum c_i\alpha_i, \beta = \sum d_j\beta_j$, since $f$ is bilinear, we get

$$f(\alpha,\beta) = \sum_{i,j} c_i d_j f(\alpha_i, \alpha_j).$$

Since $f$ is alternating, we get $f(\alpha_i, \alpha_i) = 0, \forall i$, and $f(\alpha_j, \alpha_i) = -f(\alpha_i, \alpha_j)$. Thus we get

$$f(\alpha,\beta) = \sum_{1\le i<j\le n} (c_i d_j - c_j d_i) f(\alpha_i, \alpha_j). \tag{6.1}$$

Thus $f$ is uniquely determined by the values $f(\alpha_i, \alpha_j), 1 \le i < j \le n$. This proves the uniqueness. To prove the existence, one need to check the above formula indeed defines an alternating bilinear function on $V \times V$. This is straightforward.

Here is another way to write the above formula. Let $\mathcal{B}^* = \{f_1, \dots, f_n\}$ be the dual basis of $\mathcal{B}$. Then define

$$f(\alpha,\beta) = \sum_{1\le i<j\le n} (f_i(\alpha)f_j(\beta) - f_j(\alpha)f_i(\beta))e_{ij}.$$

□

Section 5.7 of the textbook contains more information about the above construction when $X = F$. Here we need to use any vector space $X$, not only $F$.

**Definition 6.3.** *Let $V$ be a vector space over $F$. The exterior square of $V$ is a vector space $\bigwedge^2 V$ together with an alternating map $\tau : V \times V \to \bigwedge^2 V$, such that for any vector space $X$ over $F$ and for any alternating map $\varphi : V \times V \to X$, there is a unique linear map (linear transformation) $\Phi : \bigwedge^2 V \to X$ such that $\varphi = \Phi \circ \tau$, namely, the following diagram is commutative*

$$\begin{tikzcd} V \times V \arrow[rr, "\tau"] \arrow[dr, "\varphi"'] & & \bigwedge^2 V \arrow[dl, dashed, "\Phi"] \\ & X & \end{tikzcd}$$

The above definition can be restated in the following way. The alternating map $\tau$ induces a bijection

$$\begin{gathered} \mathrm{Hom}_F(\bigwedge^2 V, X) \to \mathrm{Alt}(V \times V, X) \\ \Phi \mapsto \Phi \circ \tau. \end{gathered} \tag{6.2}$$

Here $\mathrm{Alt}(V \times V, X)$ denotes the set of all alternation maps $V \times V \to X$.

**Theorem 6.4.** *Exterior square exists and is unique up to a unique isomorphism.*

*Proof.* The uniqueness can be proved in the same way as other cases discussed above. To prove the existence, we consider the subspace $Y \subset V \otimes V$ spanned by $\alpha \otimes \alpha$. Denote by $\bigwedge^2 V = V \otimes V/Y$ $\pi : V \otimes V \to \bigwedge^2 V$ the quotient map. Let $\tau = \pi \circ \otimes : V \times V \to \bigwedge^2(V)$ be the composition of $\pi$ with the tensor product map $\otimes : V \times V \to V \otimes V$. We show that $(\bigwedge^2 V, \tau)$ is the exterior square. It is clear that $\tau$ is bilinear and $\tau(\alpha,\alpha) = 0$. Thus $\tau$ is alternating. We now show that it satisfies the universal property. The proof is indeed similar with the proof of existence of tensor product. Let

$\varphi : V \times V \to X$ be an alternating map for an arbitrary vector space $X$. We will show that there is a unique linear map $\Phi : \bigwedge^2 V \to X$ such that $\varphi = \Phi \circ \tau$.

$$\begin{tikzcd}
V \times V \arrow[r, "\otimes"] \arrow[dr, "\varphi"'] & V \otimes V \arrow[r, "\pi"] \arrow[d, dashed, "\varphi_1"'] & \bigwedge^2(V) \arrow[dl, dashed, "\Phi"] \\
& X &
\end{tikzcd}$$

Since $\varphi$ is bilinear (recall an alternating map is bilinear), there exists a unique map $\varphi_1 : V \otimes V \to X$ such that $\varphi = \varphi_1 \circ \otimes$. Since $\varphi$ is alternating, we get $\varphi_1(\alpha \otimes \alpha) = \varphi_1 \circ \otimes(\alpha, \alpha) = \varphi(\alpha, \alpha) = 0$. This shows that $\varphi_1(y) = 0$ for any $y \in Y$. Thus by the universal property of quotient, there exists a unique map $\Phi : \bigwedge^2(V) = (V \otimes V)/Y \to X$ such that $\varphi_1 = \Phi \circ \pi$. Thus $\varphi = \Phi \circ \tau$. ☐

**Notation:** In the future, we will write $\alpha \wedge \beta = \tau(\alpha, \beta) \in \bigwedge^2(V)$, and write the map $\tau$ as $\wedge : V \times V \to \bigwedge^2(V)$. Since $\tau$ is alternating, we have

$$\alpha \wedge \alpha = 0, \quad \alpha \wedge \beta = -\beta \wedge \alpha, \forall \alpha, \beta \in V. \tag{6.3}$$

The notation $\wedge$ is read as wedge and $\alpha \wedge \beta$ is called the wedge product of $\alpha$ with $\beta$, or $\alpha$ wedge $\beta$.

**Theorem 6.5.** *Suppose that $\{\alpha_i\}_{1 \le i \le n}$ is a basis of $V$, then $\{\alpha_i \wedge \alpha_j\}_{1 \le i < j \le n}$ is a basis of $\bigwedge^2(V)$. In particular, if $\dim V = n$, then $\dim \bigwedge^2(V) = \frac{1}{2}n(n-1)$.*

*Proof.* The proof is the same as in the tensor product case, see Theorem 4.7. Try to fill the details on your own. ☐

Next, we will consider the wedge product of linear maps.

**Lemma 6.6.** *Let $T : V \to V$ be a linear map. Then there exists a unique linear map $\wedge^2 T : \bigwedge^2 V \to \bigwedge^2 V$ such that*

$$(\wedge^2 T)(\alpha \wedge \beta) = T(\alpha) \wedge T(\beta), \forall \alpha, \beta \in V.$$

*Proof.* Consider the map $\varphi : V \times V \to \bigwedge^2(V)$ defined by

$$(\alpha, \beta) \mapsto T(\alpha) \wedge T(\beta).$$

Then $\varphi$ is alternating and thus there exists a unique map $\wedge^2 T : \bigwedge^2(V) \to \bigwedge^2(V)$ such that

$$\varphi = \wedge^2 T \circ \wedge.$$

$$\begin{tikzcd}
V \times V \arrow[rr, "\wedge"] \arrow[dr, "\varphi"'] & & \bigwedge^2 V \arrow[dl, dashed, "\wedge^2 T"] \\
& \bigwedge^2 V &
\end{tikzcd}$$

This $\wedge^2 T$ satisfies the required property. ☐

Here are some small examples. Suppose that $V = F^2$ and $T : V \to V$ is given by a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Denote the standard basis of $V$ by $\epsilon_1, \epsilon_2$ as usual. Then $\bigwedge^2(V)$ has dimension 1 and is spanned by $e = \epsilon_1 \wedge \epsilon_2$.

**Lemma 6.7.** *We have $\wedge^2 T(e) = \det(A)e$.*

*Proof.* This follows from the general theory of determinant. But we can do the explicit calculation as follows. We have $T\epsilon_1 = a_{11}\epsilon_1 + a_{12}\epsilon_2, T(\epsilon_2) = a_{21}\epsilon_1 + a_{22}\epsilon_2$. Thus

$$\begin{aligned}
\wedge^2 T(e) &= T(\epsilon_1) \wedge T(\epsilon_2) \\
&= (a_{11}\epsilon_1 + a_{12}\epsilon_2) \wedge (a_{21}\epsilon_1 + a_{22}\epsilon_2) \\
&= a_{11}a_{22}\epsilon_1 \wedge \epsilon_2 + a_{12}a_{21}\epsilon_2 \wedge \epsilon_1 \\
&= (a_{11}a_{22} - a_{21}a_{12})\epsilon_1 \wedge \epsilon_2 \\
&= \det(A)e.
\end{aligned}$$

$\square$

Next we consider a 3 dimensional case. Let $V = F^3$ and let $T : V \to V$ be given by the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Note that $\dim \bigwedge^2(V) = 3$ and we fix an ordered basis $\mathcal{B} = \{e_1, e_2, e_3\}$ of it, with $e_1 = \epsilon_2 \wedge \epsilon_3, e_2 = \epsilon_3 \wedge \epsilon_1, e_3 = \epsilon_1 \wedge \epsilon_2$.

**Lemma 6.8.** *Under the ordered basis $\mathcal{B}$, defined above, the linear map $\wedge^2 : \bigwedge^2(F^3) \to \bigwedge^2(F^3)$ is given by the following matrix*

$$\wedge^2 A := \begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{31}a_{22} \\ a_{32}a_{13} - a_{12}a_{23} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{31} - a_{11}a_{32} \\ a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{23}a_{11} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

*In particular, we have*

$$\mathrm{tr}(\wedge^2 T) = a_{22}a_{33} - a_{32}a_{23} + a_{11}a_{33} - a_{13}a_{31} + a_{11}a_{22} - a_{12}a_{21}.$$

*Proof.* This follows from a tedious calculation. You can check that each entry $(\wedge^2 A)_{ij}$ is indeed equals to $(-1)^{i+j}\det(A(i|j))$, where $A(i|j)$ is the submatrix of $A$ by deleting its $i$-th row and $j$-th column. $\square$

**Exercise 6.9.** *Explain the cross product on $\mathbb{R}^3$ using exterior square.*

One can observe that each entry of $\wedge^2(A)$ is a minor of $A$ (determinant of a sub-matrix). This is indeed a general phenomenon.

**Exercise 6.10.** *How does $\wedge^2(T)$ behaves under composition? Is it true that $\wedge^2(T \circ U) = \wedge^2(T) \circ \wedge^2(U)$? Is it true that $\wedge^2(\mathrm{id}_V) = \mathrm{id}_{\bigwedge^2(V)}$? What is $\det(\wedge^2(T))$ in terms of $\det(T)$?*

Actually, if $T$ is invertible, then $\wedge^2 T$ is also invertible. Moreover, $\wedge^2$ preserves the composition and thus matrix product. Thus we have a group homomorphism

$$\mathrm{GL}_n(F) \to \mathrm{GL}_N(F)$$
$$A \mapsto \wedge^2(A),$$

where $N = n(n-1)/2$. This is called the exterior square representation of the group $\mathrm{GL}_n(F)$.

6.2. **exterior powers.** Let $V$ be a vector space over $F$ and $r \geq 2$ be a positive integer. Denote $V^r = V \times V \cdots \times V$ ($r$ times of $V$). Let $X$ be another vector space over $F$. A *multilinear* map $\varphi : V^r \to X$ is called alternating if

$$\varphi(\alpha_1, \alpha_2, \ldots, \alpha_r) = 0,$$

whenever $\alpha_i = \alpha_j$ for some $i \neq j$.

**Definition 6.11.** *Let $V$ be a vector space over $F$. The exterior $r$-th power of $V$, is a vector space $\bigwedge^r(V)$, together with an alternating map $\tau : V^r \to \bigwedge^r(V)$ such that for any vector space $X$ and any alternating map $\varphi : V^r \to X$, there exists a unique linear map $\Phi : \bigwedge^r(V) \to X$ such that $\varphi = \Phi \circ \tau$; namely, such that the following diagram is commutative*

$$\begin{tikzcd} V^r \arrow[r, "\tau"] \arrow[dr, "\varphi"'] & \bigwedge^r V \arrow[d, dashed, "\Phi"] \\ & X \end{tikzcd}$$

For $\alpha_1, \ldots, \alpha_r \in V$, denote

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r = \tau(\alpha_1, \ldots, \alpha_r) \in \bigwedge^r(V). \tag{6.4}$$

**Theorem 6.12.** *The exterior $r$-th power exists and is unique up to unique isomorphisms.*

*Proof.* The proof is the same as in the exterior square case. □

**Theorem 6.13.** *Let $V$ be a vector space of dimension $n$, then $\dim \bigwedge^r(V) = \binom{n}{r}$ for $r \geq 2$. More precisely, suppose $\{\alpha_1, \dots, \alpha_n\}$ is a basis of $V$. Let $I = \{i_1, \dots, i_r\} \subset \{1, 2, \dots, n\}$ be a subset with $r$ elements with $i_1 < \cdots < i_r$. We define*

$$\alpha_I = \alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_r} \in \wedge^r(V).$$

*Then the set*

$$\{\alpha_I\}_{I \subset \{1, \dots, n\}, |I| = r}$$

*is a basis of $\bigwedge^r(V)$. In particular, we have $\dim \bigwedge^n(V) = 1$, which is spanned by $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n$, and $\bigwedge^r(V) = 0$ if $r > n$.*

The proof is similar to the tensor product and exterior square case. Try to fill the details by yourself.

For $\beta_1, ..., \beta_r \in V$, we have an element $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_r \in \bigwedge^r(V)$. Such an element in $\bigwedge^r(V)$ is called a pure wedge. A general element in $\bigwedge^r(V)$ is a sum (or linear combination) of pure wedges. For example, if $V = F^4$ with standard basis $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$, we can consider an element $\epsilon_1 \wedge \epsilon_2 + \epsilon_3 \wedge \epsilon_4 \in \bigwedge^2(V)$.

In the above, we only defined $\bigwedge^r(V)$ for $r \geq 2$, because to define $r$-linear map, we need $r \geq 2$. It is easy to extend the above to the case when $r = 1$. Actually, $r$-linear map when $r = 1$ just means linear maps. From this, we can check that $\bigwedge^1(V) = V$ (the alternating condition is empty when $r = 1$). As a convention, we denote $\bigwedge^0(V) = F$. Note that under this convention, we get

$$\dim \bigwedge^r(V) = \dim \bigwedge^{n-r}(V), 0 \leq r \leq n = \dim V.$$

In particular, we have an isomorphism

$$\bigwedge^n(V) \cong F.$$

This isomorphism is not unique. Fix a basis $\{\alpha_1, \dots, \alpha_n\} \in V$, we know that $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n$ is a basis of $\bigwedge^n(V)$. The isomorphism $\bigwedge^n(V) \cong F$ can be chosen such that $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n \mapsto 1$.

**Proposition 6.14.** *Let $r, s \geq 0$ be two integers, the natural alternating map $V^{r+s} \to \bigwedge^{r+s}(V)$*

$$(\alpha_1, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_{r+s}) \to \alpha_1 \wedge \cdots \wedge \alpha_{r+s}$$

*induces a natural bilinear map*

$$\bigwedge^r(V) \times \bigwedge^s(V) \to \bigwedge^{r+s}(V),$$

*determined by*

$$(\xi, \eta) \mapsto \xi \wedge \eta, \quad \xi \in \bigwedge^r(V), \eta \in \bigwedge^s(V).$$

*Proof.* For fixed $(\alpha_{r+1}, \dots, \alpha_{r+s}) \in V^s$, the map

$$V^r \to \bigwedge^{r+s}(V)$$

defined by

$$(\alpha_1, \dots, \alpha_r) \mapsto \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{r+s}$$

is clearly alternating. Thus by the universal property of wedge product, we get a linear map

$$f_{\alpha_{r+1}, \dots, \alpha_{r+s}} : \bigwedge^r(V) \to \bigwedge^{r+s}(V)$$

such that

$$f_{\alpha_{r+1}, \dots, \alpha_{r+s}}(\alpha_1 \wedge \cdots \wedge \alpha_r) = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{r+s}.$$

Since $f$ is linear, we can check that

$$f_{\alpha_{r+1}, \dots, \alpha_{r+s}}(\xi) = \xi \wedge \alpha_{r+1} \wedge \cdots \wedge \alpha_{r+s}, \forall \xi \in \bigwedge^r(V).$$

For a fixed $\xi \in \bigwedge^r(V)$, the multi-linear map

$$V^s \to \bigwedge^{r+s}(V)$$
$$(\alpha_{r+1}, \ldots, \alpha_{r+s}) \mapsto f_{\alpha_{r+1},\ldots,\alpha_{r+s}}(\xi) = \xi \wedge \alpha_{r+1} \wedge \cdots \wedge \alpha_{r+s}$$

is alternating. By the universal property again, we get a linear map

$$g_\xi : \bigwedge^s(V) \to \bigwedge^{r+s}(V)$$

determined by

$$g_\xi(\alpha_{r+1} \wedge \cdots \wedge \alpha_{r+s}) = \xi \wedge \alpha_{r+1} \wedge \cdots \wedge \alpha_{r+s}.$$

Since the above map is linear, we see that

$$g_\xi(\eta) = \xi \wedge \eta, \forall \eta \in \bigwedge^s(V).$$

It is not easy to see that the map

$$\wedge^r(V) \times \wedge^s(V) \to \wedge^{r+s}(V)$$
$$(\xi, \eta) \mapsto g_\xi(\eta) = \xi \wedge \eta$$

is bilinear. □

The element $\xi \wedge \eta$ in the above Proposition can be computed using the usual rules of wedge product. For example, if $V = F^4$, and

$$\xi = \epsilon_1 \wedge \epsilon_2 + \epsilon_3 \wedge \epsilon_4 \in \bigwedge^2(V), \eta = \epsilon_1 + \epsilon_3 \in V = \bigwedge^1(V),$$

we have

$$\begin{aligned}
\xi \wedge \eta &= (\epsilon_1 \wedge \epsilon_2 + \epsilon_3 \wedge \epsilon_4) \wedge (\epsilon_1 + \epsilon_3) \\
&= \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_1 + \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 + \epsilon_3 \wedge \epsilon_4 \wedge \epsilon_1 + \epsilon_3 \wedge \epsilon_4 \wedge \epsilon_3 \\
&= \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 + \epsilon_3 \wedge \epsilon_4 \wedge \epsilon_1.
\end{aligned}$$

As a particular case of Proposition 6.14, there is a map $\bigwedge^r(V) \times \bigwedge^{n-r}(V) \to \bigwedge^n(V) \cong F$. We fix an isomorphism $t : \bigwedge^n(V) \to F$.

**Theorem 6.15.** *For any $\xi \in \bigwedge^{n-r}(V)$, define a map $f_\xi \in (\bigwedge^r(V))^*$ by $f_\xi(\eta) = t(\xi \wedge \eta), \eta \in \bigwedge^r(V)$. Then $f_\xi$ is well-defined, and the map $\xi \mapsto f_\xi$ defines an isomorphism*

$$\bigwedge^{n-r}(V) \to (\bigwedge^r(V))^*.$$

*Proof.* The map $f_\xi$ is the composition of $t : \bigwedge^n \to F$ with the map $g_\xi : \bigwedge^r(V) \to \bigwedge^n(V)$ used in the Proof of Proposition 6.14. It is well-defined and linear. To show the map $\xi \mapsto f_\xi$ is an isomorphism, we only need to show it is injective since $\dim \bigwedge^{n-r}(V) = \dim(\bigwedge^r(V))^*$. Suppose that $f_\xi = 0$, which is equivalent to $\xi \wedge \eta = 0, \forall \eta \in \bigwedge^{n-r}(V)$, we need to show that $\xi = 0$. We fix a basis $\{\alpha_1, \ldots, \alpha_n\}$ of $V$ and use the basis $\{e_I\}_{I \subset \{1,\ldots,n\}, |I|=r}$ of $\bigwedge^r(V)$, see Theorem 6.13. Write $\xi = \sum_I c_I e_I$. Note that $\xi = 0$ is equivalent to $c_I = 0$ for all $I$. We proceed by contradiction and assume that $c_I \neq 0$ for one fixed $I$ and hence $\xi \neq 0$. We need to construct one $\eta \in \bigwedge^{n-r}(V)$ such that $\xi \wedge \eta \neq 0$. Let $J = \{1, \ldots, n\} - I$ be the complement of $I$ and we consider $e_J$ as defined in Theorem 6.13. Then if $I' \neq I$ be a subset of $\{1, \ldots, n\}$ with $|I'| = r$. Then $I' \cap J \neq \emptyset$. Thus $e_{I'} \wedge e_J = 0$. Thus we get

$$\xi \wedge e_J = c_I e_I \wedge e_J \neq 0.$$

This concludes the proof. □

**Theorem 6.16.** *Given a set of vectors $S = \{\beta_1, \ldots, \beta_r\} \subset V$. Then $S$ is linearly dependent if and only if*

$$\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_r = 0 \in \bigwedge^r(V).$$

*Proof.* If $\beta_1, \dots, \beta_r$ are linearly independent, we can extend it to a basis of $V$, say,

$$\mathcal{B} = \{\beta_1, \dots, \beta_r, \dots, \beta_n\}.$$

Then by Theorem 6.13, $\beta_1 \wedge \cdots \wedge \beta_r \neq 0$. Since the proof of Theorem 6.13 is omitted, here is a sketch of the proof. Let $\mathcal{B}^* = \{f_1, \dots, f_r, \dots, f_n\}$ be the dual basis of $\mathcal{B}$. We can construct an $r$-linear alternating map $\varphi : V^r \to F$ such that $\varphi(\beta_1, \dots, \beta_r) \neq 0$ by an analogue of Proposition 6.2. In fact, we can define

$$\varphi(\alpha_1, \dots, \alpha_r) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \sum_{\sigma \in S_r} (-1)^{\mathrm{Sgn}(\sigma)} f_{\sigma(i_1)}(\alpha_1) \dots f_{\sigma(i_r)}(\alpha_r).$$

We can check that $\varphi$ is alternating, and $\varphi(\beta_1, \dots, \beta_r) = 1$. By the universal property, there exists a map $\Phi : \bigwedge^r(V) \to F$ such that $\Phi \circ \wedge = \varphi$, namely,

$$\Phi(\beta_1 \wedge \cdots \wedge \beta_r) = \varphi(\beta_1, \dots, \beta_r) \neq 0.$$

Thus $\beta_1 \wedge \cdots \wedge \beta_r \neq 0$.

Conversely, suppose that $\beta_1, \dots, \beta_r$ are linearly dependent. Without loss of generality, we assume that

$$\beta_r = \sum_{i=1}^{r-1} c_i \beta_i.$$

Thus

$$\begin{aligned}
\beta_1 \wedge \cdots \wedge \beta_r &= \beta_1 \wedge \cdots \wedge \beta_{r-1} \wedge (\sum_{i=1}^{r-1} c_i \beta_i) \\
&= \sum_{i=1}^{r-1} c_i \beta_1 \wedge \cdots \wedge \beta_{r-1} \wedge \beta_i \\
&= 0.
\end{aligned}$$

The last expression is zero because $\beta_i$ appeared twice in each expression. □

Given a linear map $T : V \to V$, we can define a map $\wedge^r T : \bigwedge^r V \to \bigwedge^r V$ by

$$(\wedge^r T)(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r) = T(\alpha_1) \wedge \cdots \wedge T(\alpha_r), \forall \alpha_1, \dots, \alpha_r \in V.$$

One can check that if $T$ is invertible then $\wedge^r T$ is also invertible and $\wedge^r(T_1 \circ T_2) = \wedge^r(T_1) \circ \wedge^r(T_2)$ for $r \leq n$. Moreover, we denote

$$\wedge^0 (T) = \mathrm{Id}_F \in \mathrm{Hom}_F(F, F); \qquad \wedge^1(T) = T. \tag{6.5}$$

Suppose that $V = F^n$ and $T$ is given by a matrix $A$, we denote by $\wedge^r(A)$ the matrix of $\wedge^r(T)$ with respect to a fixed ordered basis. Thus this gives a map

$$\wedge^r : \mathrm{Mat}_{n \times n}(F) \to \mathrm{Mat}_{N \times N}(F)$$

$$A \mapsto \wedge^r A,$$

such that $\wedge^r(AB) = \wedge^r(A) \wedge^r(B)$, where $N = \binom{n}{r}$. In particular, for $r = n$, $\wedge^n(A) \in \mathrm{Mat}_{1 \times 1}(F) = F$ is a scaler which satisfies

$$A\alpha_1 \wedge \cdots \wedge A\alpha_n = \wedge^n(A) \alpha_1 \wedge \cdots \wedge \alpha_n,$$

for all $\alpha_1, \dots, \alpha_n \in V = F^n$.

**Theorem 6.17.** *We have*

$$\wedge^n (A) = \det(A), \tag{6.6}$$

*namely, we have*

$$A\epsilon_1 \wedge A\epsilon_2 \wedge \cdots \wedge A\epsilon_n = \det(A) \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n. \tag{6.7}$$

*Proof.* For $V = F^n$, $\bigwedge^n(V)$ is spanned by $\epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n$. Thus

$$A\epsilon_1 \wedge A\epsilon_2 \wedge \cdots \wedge A\epsilon_n = D_A \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n,$$

for some $D_A \in F$. Assume $\epsilon_i$ are column vectors and $A\epsilon_i$ is the $i$-th column $A_i$ of $A$. It is clear that if $A_i = A_j$ for $i \neq j$, we have $D_A = 0$. This shows that the function $A \mapsto D_A$ is alternating. Moreover, $D_{I_n} = 1$. Thus $D_A = \det(A)$ by the uniqueness of determinant function. □

Using this description, it is easy to show many properties of det function. For example,

**Theorem 6.18.** *we have* $\det(AB) = \det(A)\det(B)$.

*Proof.* This follows from $\wedge^n(AB) = \wedge^n(A) \circ \wedge^n(B)$. □

**Exercise 6.19.** *Let $A \in \mathrm{Mat}_{n\times n}(F)$ be a diagonal matrix. Try to work out the matrix $\wedge^r(A)$ after fixing some basis.*

6.3. **characteristic polynomial and exterior power.** We prepare two lemmas.

**Lemma 6.20.** *Let $V$ be a finite dimensional vector space over a field $F$. Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a basis of $V$ and let $\mathcal{B}^* = \{f_1, \dots, f_n\}$ be the dual basis of $\mathcal{B}$. Given $T \in \mathrm{End}(V)$, we have*

$$\mathrm{Tr}(T) = \sum_{i=1}^n \langle Te_i, f_i \rangle,$$

*where $\langle Te_i, f_i \rangle$ denotes $f_i(Te_i)$.*

*Proof.* Suppose $[T]_{\mathcal{B}} = \{a_{ij}\}$. Then $\mathrm{Tr}(T) = \sum_{i=1}^n a_{ii}$. Moreover, we have

$$Te_i = \sum_{j=1}^n a_{ij} e_j.$$

Thus $f_i(Te_i) = \sum_{j=1}^n a_{ij} f_i(e_j) = a_{ii}$. This shows that $\mathrm{Tr}(T) = \sum_{i=1}^n f_i(Te_i)$. □

Let $V$ be a finite dimensional vector space over a field $F$. Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a basis of $V$. Let $I = \{i_1, \dots, i_r\}$ be a subset of $\{1, \dots, n\}$ with $|I| = r$. We have defined

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_n} \in \wedge^r(V).$$

Note that $\mathcal{B}_r = \{e_I |\ I \subset \{1, \dots, n\}, |I| = r\}$ is a basis of $\wedge^r(V)$ as $I$ runs over all subset of $\{1, \dots, r\}$ with $r$ elements. For such an $I$, we define $f^I \in (\wedge^r(V))^*$ as follows. Let $J = \{j_1, \dots, j_{n-r}\}$ be the complement of $I$ in $\{1, 2, \dots, n\}$ with $j_1 < \cdots < j_{n-r}$. Then for $\alpha \in \wedge^r(V)$, we define $f^I(\alpha) \in F$ by

$$f^I(\alpha) = \mathrm{sgn}(\sigma_I) f_\alpha(e_J) = t(\alpha \wedge e_J),$$

where $f_{e_I}$ is the map in Theorem 6.15, and $\sigma_I : \{1, \dots, n\} \to \{1, 2, \dots, n\}$ is the bijection $\sigma(k) = i_k, k \leq r$ and $\sigma(k) = j_{k-r}$ if $k > r$. Denote $z = e_1 \wedge \cdots \wedge e_n \in \bigwedge^n(F)$ and we fix the isomorphism $t : \bigwedge^n(F) \to F$ by $xz \mapsto z$. Then the definition of $f^I$ can be written as

$$\mathrm{sgn}(\sigma_I) f^I(\alpha) z = \alpha \wedge e_J.$$

**Lemma 6.21.** *The set $\left\{f^I : I \subset \{1, \dots, n\}, |I| = r\right\}$ is the dual basis of $\mathcal{B}_r$.*

*Proof.* Let $I' \subset \{1, \dots, n\}$, we need to show that $f^I(e_{I'}) = \delta_{I,I'}$. Let $J$ be the complement of $I$ in $\{1, 2, \dots, n\}$. If $I' \neq I$, then $I \cap I' \neq \emptyset$ and thus $e_{I'} \wedge e_J = 0$ and thus $f^I(e_{I'}) = 0$. On the other hand, $e_I \wedge e_J = \mathrm{sgn}(\sigma_I) z$, and thus $f^I(e_I) = 1$. □

The next theorem gives a formula of the coefficients of the characteristic polynomial of a matrix $A$ in terms of exterior power.

**Theorem 6.22.** *For $A \in \mathrm{Mat}_{n\times n}(F)$, we have*

$$\chi_A = \det(xI_n - A) = \sum_{k=0}^n (-1)^k \mathrm{tr}(\wedge^k(A)) x^{n-k}.$$

*Proof.* For a complete proof of this equation, see [Bou98, page 529 and page 540]. Here we give a sketch of a proof of this formula following [Bou98], which will demonstrate how the general proof goes.

Let $\{\epsilon_i, 1 \le i \le n\}$ be the standard basis of $F^n$. Write $z = \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n$, which is a basis element of $\wedge^n(F^n)$. From Theorem 6.17, we know that

$$\det(xI_n - A)z = (xI_n - A)\epsilon_1 \wedge (xI_n - A)\epsilon_2 \wedge \cdots \wedge (xI_n - A)\epsilon_n. \tag{6.8}$$

The right hand side of the above formula is a polynomial in $x$. Its highest term is $x^n\epsilon_1 \wedge \cdots \wedge \epsilon_n = x^n z$. The coefficient of $x^{n-1}$ is

$$-(A\epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n + \epsilon_1 \wedge A\epsilon_2 \wedge \cdots \wedge \epsilon_n + \cdots + \epsilon_1 \wedge \ldots \epsilon_{n-1} \wedge A\epsilon_n). \tag{6.9}$$

We claim that the above expression (6.9) is exactly $-\mathrm{tr}(A)z$. One way to see this is to write $A = (a_{ij})_{1 \le i,j \le n}$ explicitly. Then $A\epsilon_1 = a_{11}\epsilon_1 + a_{21}\epsilon_2 + \cdots + a_{n1}\epsilon_n$. Then the first term in (6.9) is

$$(a_{11}\epsilon_1 + a_{21}\epsilon_2 + \cdots + a_{n1}\epsilon_n) \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n = a_{11}z,$$

because $\epsilon_i \wedge \epsilon_2 \wedge \cdots \wedge \ldots \epsilon_n = 0$ if $i > 1$. Similarly, the general term in (6.9) is exactly $a_{ii}z$. Thus (6.9) is $-\mathrm{tr}(A)z$. This works well for the coefficient of $x^{n-1}$ but it is complicate to generalize it to the computation of coefficients of $x^{n-k}$. The above computation can be explained using the above two Lemmas. In fact, $\{\epsilon_1, \ldots, \epsilon_n\}$ is a basis of $V = \wedge^1(V)$ and its dual basis $\{f_1, \ldots, f_n\}$ can be characterized using Lemma 6.21. By Lemma 6.20, we have

$$\mathrm{tr}(A) = \sum_{i=1}^n \langle A\epsilon_i, f_i\rangle.$$

By Lemma 6.21, we have

$$\langle A\epsilon_i, f_i\rangle z = \mathrm{sgn}(\sigma_i)A\epsilon_i \wedge \epsilon_1 \wedge \cdots \wedge \epsilon_{i-1} \wedge \epsilon_{i+1} \wedge \cdots \wedge \epsilon_n = \epsilon_1 \wedge \cdots \wedge \epsilon_{i-1} \wedge A\epsilon_i \wedge \epsilon_{i+1} \wedge \cdots \wedge \epsilon_n.$$

Thus (6.9) is exactly $-\mathrm{tr}(A)z$.

Next, we check the coefficient of $x^{n-2}$ in the right hand side of (6.8). This coefficient is of the form

$$\sum_{1 \le i_1 < i_2 \le n} \epsilon_1 \wedge \cdots \wedge \epsilon_{i_1 - 1} \wedge A\epsilon_{i_1} \wedge \epsilon_{i_1+1} \wedge \cdots \wedge \epsilon_{i_2-1} \wedge A\epsilon_{i_2} \wedge \epsilon_{i_2+1} \wedge \cdots \wedge \epsilon_n,$$

which can be re-written as

$$\sum_{I = \{i_1, i_2\} \subset \{1,2,\ldots,n\}, 1 \le i_1 < i_2 \le n} \mathrm{sgn}(\sigma_I)A\epsilon_{i_1} \wedge A\epsilon_{i_2} \wedge \epsilon_J,$$

where $J$ is the complement of $I$. Since $\{\epsilon_{i_1} \wedge \epsilon_{i_2}\}$ is basis of $\wedge^2(V)$ and $\wedge^2(A)(\epsilon_{i_1} \wedge \epsilon_{i_2}) = A\epsilon_{i_1} \wedge A\epsilon_{i_2}$, Lemma 6.20 and Lemma 6.21 show that the coefficient of $x^{n-2}$ is exactly $\mathrm{tr}(\wedge^2(A))z$.

The proof of the general case is similar after Lemma 6.20 and Lemma 6.21, except that the notations are more complicate. We omit the details. □

**Exercise 6.23.** *For $A \in \mathrm{Mat}_{3\times 3}(F)$, check the identity*

$$\chi_A(x) = \det(xI_3 - A) = \sum_{k=0}^3 \mathrm{tr}(\wedge^k(A))(-1)^{3-k}x^{3-k},$$

*using an explicit calculations of $\wedge^k(A)$.*

Here is one application of Theorem 6.22 which was originally a homework problem.

**Proposition 6.24.** *Given two matrices $A, B \in \mathrm{Mat}_{n\times n}(F)$, we have*

$$\chi_{AB} = \chi_{BA}.$$

*Proof.* By Theorem 6.22, we have

$$\chi_{AB} = \sum_{k=1}^n (-1)^{n-k}\mathrm{tr}(\wedge^k(AB))x^{n-k}.$$

We have

$$\wedge^k(AB) = \wedge^k(A) \wedge^k (B).$$

Thus

$$\operatorname{tr}(\wedge^k(AB)) = \operatorname{tr}(\wedge^k(A)\wedge^k(B)) = \operatorname{tr}(\wedge^k(B)\wedge^k(A)) = \operatorname{tr}(\wedge^k(BA)).$$

This shows that $\chi_{AB} = \chi_{BA}$. $\square$

## 7. Symmetric Power

7.1. **Symmetric square.** Let $V, X$ be two vector spaces over a field $F$. A bilinear map $f : V \times V \to X$ is called symmetric if

$$f(\alpha,\beta) = f(\beta,\alpha), \forall \alpha,\beta \in V.$$

Let $\mathrm{Sym}(V\times V; X)$ be the set of all symmetric bilinear map $f : V\times V \to X$. Suppose $V$ has a basis $\{\alpha_1,\dots,\alpha_n\}$, then a symmetric bilinear map $f : V \times V \to X$ is uniquely determined by the values

$$f(\alpha_i,\alpha_j), 1 \le i \le j \le n.$$

**Definition 7.1.** *Let $V$ be a vector space over $F$. Then the symmetric square of $V$ is a pair $(\mathrm{Sym}^2(V),\sigma)$, where $\mathrm{Sym}^2(V)$ is a vector space and $\sigma : V\times V \to \mathrm{Sym}^2(V)$ is a symmetric bilinear map, such that for any other pair $(X, f)$ with a vector space $X$ and a symmetric bilinear map $f : V \times V \to X$, there is a unique linear map $\phi : \mathrm{Sym}^2(V) \to X$ such that $f = \phi\circ\sigma$:*

$$\begin{tikzcd} V\times V \arrow[rr, "\sigma"] \arrow[dr, "f"'] & & \mathrm{Sym}^2(V) \arrow[dl, dashed, "\phi"] \\ & X & \end{tikzcd}$$

**Proposition 7.2.** *The symmetric square $(\mathrm{Sym}^2(V),\sigma)$ exists and is unique up to a unique isomorphism. Moreover, $\dim_F(\mathrm{Sym}^2(V)) = n(n+1)/2$ if $\dim_F(V) = n$. In fact, if $\mathcal{B} = \{\alpha_1,\dots,\alpha_n\}$ is a basis of $V$, then $\{\sigma(\alpha_i,\alpha_j), 1\le i\le j\le n\}$ is a basis of $\mathrm{Sym}^2(V)$.*

The proof is similar to the other situations and we omit it here. We usually write $\sigma(\alpha,\beta) := \alpha\cdot\beta \in \mathrm{Sym}^2(V)$, and omit $\sigma$ from the notations.

Given a linear map $T : V \to W$ of $F$-vector spaces, we naturally have a map $\mathrm{Sym}^2(T) : \mathrm{Sym}^2(V) \to \mathrm{Sym}^2(W)$, which satisfies

$$\mathrm{Sym}^2(T)(\alpha\cdot\beta) = T(\alpha)\cdot T(\beta).$$

If $T$ is invertible, then $\mathrm{Sym}^2(T)$ is also invertible. In fact, we still have $\mathrm{Sym}^2(T\circ S) = \mathrm{Sym}^2(T)\circ \mathrm{Sym}^2(S)$.

Let us work out one simple example. Suppose $V = F^2$, and $T : V \to V$ is given by a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Namely, $T\alpha = A\alpha$. Let $\{\epsilon_1,\epsilon_2\}$ be the standard basis of $V = F^2$ and we take

$$\mathcal{B} = \{e_1 = \epsilon_1\cdot\epsilon_1, e_2 = \epsilon_1\cdot\epsilon_2, e_3 = \epsilon_2\cdot\epsilon_2\}$$

as an ordered basis of $\mathrm{Sym}^2(V)$. We now compute $[\mathrm{Sym}^2(T)]_{\mathcal{B}}$. We have

$$\begin{aligned} \mathrm{Sym}^2(T)e_1 &= T\epsilon_1\cdot T\epsilon_1 = (a\epsilon_1 + c\epsilon_2)\cdot(a\epsilon_1 + c\epsilon_2) = a^2e_1 + 2ace_2 + c^2e_3, \\ \mathrm{Sym}^2(T)e_2 &= T\epsilon_1\cdot T\epsilon_2 = (a\epsilon_1 + c\epsilon_2)\cdot(b\epsilon_1 + d\epsilon_2) = abe_1 + (bc+ad)e_2 + cde_3, \\ \mathrm{Sym}^2(T)e_2 &= T\epsilon_2\cdot T\epsilon_2 = (b\epsilon_1 + d\epsilon_2)\cdot(b\epsilon_1 + d\epsilon_2) = b^2e_1 + (2bd)e_2 + d^2e_3. \end{aligned}$$

Thus

$$[\mathrm{Sym}^2(T)]_{\mathcal{B}} = \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & bc+ad & 2bd \\ c^2 & cd & d^2 \end{bmatrix}.$$

One find that the above calculation of product of two vectors is the same as calculations of polynomials. In fact, one can define polynomials using symmetric powers.

**Exercise 7.3.** *Suppose $F$ has characteristic zero. For $A \in \mathrm{Mat}_{2\times 2}(F)$, show that*

$$\mathrm{tr}(\mathrm{Sym}^2(A)) = \frac{1}{2}(\mathrm{tr}(A^2) + (\mathrm{tr}(A))^2).$$

*Is the same formula true for general $A \in \mathrm{Mat}_{n\times n}(F)$?*

In general, given a matrix $A \in \mathrm{Mat}_{n\times n}(F)$, we can view $A$ as a linear map $T_A : F^n \to F^n$. Then we can define $\mathrm{Sym}^2(A) \in \mathrm{Mat}_{N\times N}(F)$ with $N = n(n+1)/2 = \dim \mathrm{Sym}^2(F^n)$ as the matrix $[\mathrm{Sym}^2(T_A)]_{\mathcal{B}}$ for some ordered basis $\mathcal{B}$ of $\mathrm{Sym}^2(F^n)$. Of course, this depends on the choice of $\mathcal{B}$.

**Exercise 7.4.** *Compute $\det(\mathrm{Sym}^2(A))$ for $A \in \mathrm{Mat}_{2\times 2}(A)$. Make a guess for $\det(\mathrm{Sym}^2(A))$ for general $A \in \mathrm{Mat}_{n\times n}(F)$.*

7.2. **Symmetric power.** Let $V, X$ be two vector space over a field $F$ and let $r$ be a positive integer. Recall that we have defined the symmetric group $S_n$, whose elements are bijections $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. A multi-linear map $f : V^r = V \times V \times \cdots \times V \to X$ is call symmetric if

$$f(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots, \alpha_{\sigma(r)}) = f(\alpha_1, \dots, \alpha_r),$$

for any $\alpha_1, \dots, \alpha_r \in V$ and for any $\sigma \in S_r$.

**Definition 7.5.** *Let $V$ be a vector space over $F$. Then the symmetric $r$-th power of $V$ is a pair $(\mathrm{Sym}^r(V), S)$, where $\mathrm{Sym}^r(V)$ is a vector space and $S : V^r \to \mathrm{Sym}^2(V)$ is a symmetric multi-linear map, such that for any other pair $(X, f)$ with a vector space $X$ and a symmetric multi-linear map $f : V^r \to X$, there is a unique linear map $\phi : \mathrm{Sym}^r(V) \to X$ such that $f = \phi \circ S$:*

$$\begin{tikzcd}
V^r \arrow[rr, "S"] \arrow[dr, "f"'] & & \mathrm{Sym}^r(V) \arrow[dl, dashed, "\phi"] \\
& X &
\end{tikzcd}$$

**Proposition 7.6.** *The symmetric power $(\mathrm{Sym}^r(V), S)$ exists and is unique up to unique isomorphism. Moreover, if $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ is a basis of $V$, then*

$$\{S(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}), 1 \le i_1 \le i_2 \le \cdots \le i_r \le n\}$$

*is a basis of $\mathrm{Sym}^r(V)$.*

**Exercise 7.7.** *If $\dim V = n$, what is $\dim \mathrm{Sym}^r(V)$? Answer: it is $\binom{n+r-1}{r}$.*

We usually write $S(\alpha_1, \dots, \alpha_r)$ as $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_r$ and omit $S$ from the notations.

If $T : V \to W$ is a linear map, we can construct a linear map $\mathrm{Sym}^r(T) : \mathrm{Sym}^r(V) \to \mathrm{Sym}^r(W)$. One can also check that $\mathrm{Sym}^r(T_1 \circ T_2) = \mathrm{Sym}^r(T_1) \circ \mathrm{Sym}^r(T_2)$ and $\mathrm{Sym}^r(\mathrm{id}_V) = \mathrm{id}_{\mathrm{Sym}^r(V)}$. A consequence of this is $\mathrm{Sym}^r(T)$ is invertible if $T$ is also invertible. Given a matrix $A \in \mathrm{Mat}_{m\times n}(F)$, we can define $\mathrm{Sym}^r(A)$ using linear maps after fixing a basis.

By convention, we define $\mathrm{Sym}^1(V) = V$ and $\mathrm{Sym}^1(T) = T$ for $T \in \mathrm{End}(V)$. Moreover, we define $\mathrm{Sym}^0(V) = F$ and $\mathrm{Sym}^0(T) = \mathrm{Id}_F \in \mathrm{Hom}_F(F, F)$ for $T \in \mathrm{End}(V)$.

**Exercise 7.8.** *Let $A \in \mathrm{Mat}_{2\times 2}(F)$, compute $\mathrm{Sym}^3(A)$ after fixing a basis of $\mathrm{Sym}^3(F^2)$. Assume the characteristic of $F$ is zero. Express $\mathrm{tr}(\mathrm{Sym}^3(F))$ in terms of $\mathrm{tr}(A), \mathrm{tr}(A^2)$ and $\mathrm{tr}(A^3)$. See Problem 7.3 for the $\mathrm{Sym}^2$ case.*

7.3. **Characteristic polynomial and symmetric power.** Recall that we use $F[[x]]$ to denote the formal power series ring over $F$. An interesting question is to determine invertible elements of $F[[x]]$, namely, the set $F[[x]]^\times = \{f \in F[[x]] | \exists g \in F[[x]] \text{ s.t. } fg = 1\}$. The set $F[[x]]^\times$ is called the units of $F[[x]]$. We have seen one nontrivial example: namely $(1 - x) \in F[[x]]^\times$ since

$$(1 - x)(1 + x + x^2 + x^3 + \dots) = 1.$$

In fact, one can show that if the constant term of $f \in F[[x]]$ is nonzero, then $f \in F[[x]]^\times$. We won't prove this result but we will give one example in the following.

**Theorem 7.9.** *Suppose the characteristic of $F$ is zero. Given $A \in \mathrm{Mat}_{n\times n}(F)$, then $\det(I_n - xA) \in F[[x]]^{\times}$ and*

$$\det(I_n - xA)^{-1} = \sum_{r\geq 0} \mathrm{Tr}(\mathrm{Sym}^r(A))x^r.$$

One can compare this result with Theorem 6.22. We omit the proof of the general case. In the following remark, we prove the above theorem when $n = 2$.

*Remark* 7.10. Using Theorem 6.22, we have

$$\begin{aligned}\det(I_n - xA) &= \det(xI_n(x^{-1}I_n - A))\\ &= x^n\chi_A(x^{-1})\\ &= x^n\sum_{k\geq 0}(-1)^k\mathrm{tr}(\wedge^k(A))x^{-n+k}\\ &= \sum_{k=0}^{n}(-1)^k\mathrm{tr}(\wedge^k(A))x^k.\end{aligned}$$

Theorem 7.9 is equivalent to the identity

$$\left(\sum_{k=0}^{n}(-1)^k\mathrm{tr}(\wedge^k(A))x^k\right)\left(\sum_{r\geq 0}\mathrm{Tr}(\mathrm{Sym}^r(A))x^r\right) = 1 \tag{7.1}$$

in $F[[x]]$. Let us see the example when $n = 2$. Notice that $\wedge^1(A) = A$ and $\wedge^2(A) = \det(A)$. A simple simplification shows that (7.1) is equivalent to

$$1 + \sum_{r\geq 0}\left(\det(A)\mathrm{tr}(\mathrm{Sym}^r(A)) - \mathrm{tr}(A)\mathrm{tr}(\mathrm{Sym}^{r+1}(A)) + \mathrm{tr}(\mathrm{Sym}^{r+2}(A))\right)x^{r+2} = 1.$$

Thus Theorem 7.9 is equivalent to

$$\det(A)\mathrm{tr}(\mathrm{Sym}^r(A)) - \mathrm{tr}(A)\mathrm{tr}(\mathrm{Sym}^{r+1}(A)) + \mathrm{tr}(\mathrm{Sym}^{r+2}(A)) = 0, \forall r\geq 0. \tag{7.2}$$

To prove (7.2), we can assume that $F$ is algebraically closed, because the trace of a matrix is independent of the base field (this means, for a field $F$ and an algebraically closed field $\overline{F}$ with $F \subset \overline{F}$, and for a matrix $B \in \mathrm{Mat}_{r\times r}(F)$, if we view $B$ as a matrix in $\mathrm{Mat}_{r\times r}(\overline{F})$ which we denote by $B_{\overline{F}}$, we have $\mathrm{tr}(B) = \mathrm{tr}(B_{\overline{F}})$.). When $F$ is algebraically closed, $A$ can be triangulable and we may assume

$$A = \begin{bmatrix} a & * \\ 0 & b\end{bmatrix}.$$

We can find a basis such that $\mathrm{Sym}^r(A)$ is upper triangular and its main diagonal is

$$\mathrm{diag}(a^r, a^{r-1}b, a^{r-2}b^2, \ldots, ab^{r-1}, b^r).$$

Thus

$$\mathrm{tr}(\mathrm{Sym}^r(A)) = \sum_{i=0}^{r} a^{r-i}b^i.$$

Using this formula, it is easy to check (7.2). This gives a proof of Theorem 7.9 when $n = 2$. One can also see that Theorem 7.9 is indeed a result on symmetric polynomials.

**Exercise 7.11.** *Prove Theorem 7.9 for $n = 3$.*

## 8. Relationship between exterior/symmetric power and tensor product

In this section, we require that $V$ is a finite dimensional vector space over $F$.

**Lemma 8.1.** *There is a unique linear map $p : \otimes^r(V) \to \wedge^r(V)$ such that*

$$p(\alpha_1\otimes\cdots\otimes\alpha_r) = \alpha_1\wedge\cdots\wedge\alpha_r, \forall\alpha_1,\ldots,\alpha_r \in V.$$

*Similarly, there is a unique linear map $q : \otimes^r(V) \to \mathrm{Sym}^r(V)$ such that*

$$q(\alpha_1\otimes\cdots\otimes\alpha_r) = \alpha_1\cdot\alpha_2\cdot\cdots\cdot\alpha_r, \forall\alpha_1,\ldots,\alpha_r \in V.$$

*Proof.* Consider the natural map $V^r \to \wedge^r(V)$, $(\alpha_1, \ldots, \alpha_r) \mapsto \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r$. This map is multi-linear by definition. Thus there is a unique linear map $p : \otimes^r(V) \to \wedge^r(V)$ such that

$$\begin{tikzcd} V^r \arrow[rr] \arrow[dr] & & \otimes^r(V) \arrow[dl, dashed, "p"] \\ & \wedge^r(V) & \end{tikzcd}$$

This map $p$ satisfies

$$p(\alpha_1 \otimes \cdots \otimes \alpha_r) = \alpha_1 \wedge \cdots \wedge \alpha_r, \forall \alpha_1, \ldots, \alpha_r \in V.$$

The linear map $q$ can be defined similarly. □

**Lemma 8.2.** *Assume* $\mathrm{Char}(F) = 0$. *Then there is a unique linear map* $i : \wedge^r(V) \to \otimes^r(V)$ *such that*

$$i(\alpha_1 \wedge \cdots \wedge \alpha_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(r)}.$$

*Similarly, there is a unique linear map* $\iota : \mathrm{Sym}^r(V) \to \otimes^r(V)$ *such that*

$$\iota(\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(r)}.$$

*Proof.* We consider the map $f : V^r \to \otimes^r(V)$ defined by

$$f(\alpha_1, \ldots, \alpha_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(r)}.$$

Then $f$ is multi-linear and alternating. Thus there is a unique map $i : \wedge^r(V) \to \otimes^r(V)$ such that the following diagram

$$\begin{tikzcd} V^r \arrow[rr, "\wedge^r"] \arrow[dr, "f"'] & & \wedge^r(V) \arrow[dl, dashed, "i"] \\ & \otimes^r(V) & \end{tikzcd}$$

is commutative. This linear map $i$ satisfying the required property. The linear map $\iota$ is defined similarly. □

*Remark* 8.3. The requirement $\mathrm{Char}(F) = 0$ is to ensure the factor $\frac{1}{r!}$ is well-defined in $F$.

**Lemma 8.4.** *Assume* $\mathrm{Char}(F) = 0$. *Let* $p, q, i, \iota$ *be the maps defined in the above two Lemmas. Then we have*

$$p \circ i = \mathrm{id}_{\wedge^r(V)}, \text{ and } q \circ \iota = \mathrm{id}_{\mathrm{Sym}^r(V)}.$$

*In particular,* $i$ *and* $\iota$ *are injective and* $p, q$ *are surjective.*

*Proof.* This follows from a direct computation. Given $\alpha_1, \ldots, \alpha_r \in V$, we have

$$\begin{aligned} p \circ i(\alpha_1 \wedge \cdots \wedge \alpha_r) &= p\left(\frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(r)}\right) \\ &= \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) \alpha_{\sigma(1)} \wedge \ldots \alpha_{\sigma(r)} \\ &= \frac{1}{r} \sum_{\sigma \in S_r} \alpha_1 \wedge \cdots \wedge \alpha_r \\ &= \alpha_1 \wedge \cdots \wedge \alpha_r. \end{aligned}$$

This shows that $p \circ i = \mathrm{id}_{\wedge^r(V)}$. In the above, we used the fact

$$\alpha_{\sigma(1)} \wedge \ldots \alpha_{\sigma(r)} = \mathrm{sgn}(\sigma) \alpha_1 \wedge \cdots \wedge \alpha_r.$$

The identity $q \circ \iota = \mathrm{id}_{\mathrm{Sym}^r(V)}$ can be checked similarly. □

Assume $\mathrm{Char}(F) = 0$. From Lemma 8.4, we see that $\wedge^r(V)$ can be viewed as a quotient space of $\otimes^r(V)$ via the map $p$, (since $\wedge^r(V)$ is isomorphic to $\otimes^r(V)/\mathrm{Ker}(p)$) and it can also be viewed as a subspace of $\otimes^r V$ via the map $i$ (since $\wedge^r(V)$ is isomorphic to $i(\wedge^r(V))$ which is subspace of $\otimes^r(V)$). Similarly, $\mathrm{Sym}^r(V)$ can also be viewed as a subspace and a quotient of $\otimes^r(V)$. In the following, we will view $\wedge^r(V)$ and $\mathrm{Sym}^r(V)$ as a subspace of $\otimes^r(V)$ and won't distinguish $\wedge^r(V)$ and its image $i(\wedge^r(V))$ in $\otimes^r(V)$. In many books (for example, [HK, section 5.7]), exterior (resp. symmetric) product are just defined as the image of $i$ (resp. $\iota$).

**Lemma 8.5.** *We have*

$$V \otimes V = \wedge^2(V) \oplus \mathrm{Sym}^2(V).$$

*Proof.* Consider the bilinear map $V^2 \to V \otimes V$ defined by $(\alpha, \beta) \mapsto \beta \otimes \alpha$. Using the universal property of tensor product, we get a linear map $\tau : V \otimes V \to V \otimes V$ such that $\tau(\alpha \otimes \beta) = \beta \otimes \alpha$ for all $\alpha, \beta \in V$. We consider the linear map $\tau^2 = \tau \circ \tau$. We have $\tau^2(\alpha \otimes \beta) = \tau(\beta \otimes \alpha) = \alpha \otimes \beta$. Thus $\tau^2 = \mathrm{id}_{V \otimes V}$. Thus the minimal polynomial $\mu_\tau$ of $\tau$ satisfies

$$\mu_\tau | (x^2 - 1).$$

Thus $\tau$ is diagonalizable and the only possible eigenvalues of $\tau$ are $\pm 1$. So we get

$$V \otimes V = \mathrm{Ker}(\tau - \mathrm{id}_{V \otimes V}) \oplus \mathrm{Ker}(\tau + \mathrm{id}_{V \otimes V}).$$

We claim that $\mathrm{Ker}(\tau - \mathrm{id}_{V \otimes V}) = \mathrm{Sym}^2(V)$ and $\mathrm{Ker}(\tau + \mathrm{id}_{V \otimes V}) = \wedge^2(V)$, and this will give the desired result. For an element

$$i(\alpha \wedge \beta) = \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha) \in i(\wedge^2(V)),$$

we have

$$\tau(i(\alpha \wedge \beta)) = \frac{1}{2}(-\alpha \otimes \beta + \beta \otimes \alpha) = -i(\alpha \wedge \beta).$$

Thus $\wedge^2(V) \subset \mathrm{Ker}(\tau + \mathrm{id}_{V \otimes V})$. Conversely, we consider a general element

$$\sum c_k \alpha_k \otimes \beta_k \in \mathrm{Ker}(\tau + \mathrm{id}_{V \otimes V}).$$

We have

$$\sum c_k \alpha_k \otimes \beta_k = -\sum c_k \beta_k \otimes \alpha_k.$$

We now consider $\gamma = \sum c_k \alpha_k \wedge \beta_k \in \wedge^2(V)$. We have

$$\begin{aligned} i(\gamma) &= \sum c_k \frac{1}{2}(\alpha_k \otimes \beta_k - \beta_k \otimes \alpha_k) \\ &= \frac{1}{2} \sum c_k \alpha_k \otimes \beta_k - \frac{1}{2} \sum c_k \beta_k \otimes \alpha_k \\ &= \sum c_k \alpha_k \otimes \beta_k. \end{aligned}$$

This shows that $\sum c_k \alpha_k \otimes \beta_k \in \wedge^2(V)$. Thus $\mathrm{Ker}(\tau + \mathrm{id}_{V \otimes V}) \subset \wedge^2(V)$ and hence $\mathrm{Ker}(\tau + \mathrm{id}_{V \otimes V}) = \wedge^2(V)$. It is similar to show $\mathrm{Ker}(\tau - \mathrm{id}_{V \otimes V}) = \mathrm{Sym}^2(V)$. □

*Remark* 8.6. An alternating but actually easier proof of the above lemma is: one first show that $\wedge^2(V) \cap \mathrm{Sym}^2(V) = \{0\}$, and then the dimension counting will give us the above decomposition.

Decompositions of $\otimes^r(V) = V^{\otimes r}$ for $r \geq 3$ is much harder.

**Exercise 8.7.** *Show that* $\mathrm{Sym}^3(V) \cap \wedge^3(V) = \{0\}$ *and thus* $\mathrm{Sym}^3(V) \oplus \wedge^3(V)$ *can be viewed as a subspace of* $\otimes^3(V) = V \otimes V \otimes V$. *Show that* $\otimes^3(V)$ *is in general larger than* $\mathrm{Sym}^3(V) \oplus \wedge^3(V)$. *What is missing in the decomposition* $V \otimes V \otimes V = \mathrm{Sym}^3(V) \oplus \wedge^3(V) \oplus (\textit{something})$*? Try to figure this "something" out when* $\dim V = 2, 3$.

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