HOMEWORK 1

YOUR NAME

Due date: Monday of Week 5

Problem 1. *Let* $X = \mathbb{Z} \times (\mathbb{Z} - \{0\})$*. Consider the relation* R *on* X *defined by*

$$R = \{((a,b),(c,d)) \in X \times X : ad = bc\}.$$

Show that R *is an equivalence relation. (Actually* X/R *is the set of rational numbers. Think about why this is so.)*

Problem 2. *Show that the set* $F = \left\{a + b\sqrt{2} : a, b \in \mathbb{Q}\right\}$ *with the usual addition and multiplication is a field.*

Problem 3. *In this problem, we construct a field* $(F, +, \times)$ *which has 4 elements. Denote the 4 elements by* $\{0, 1, \alpha, \beta\}$*, where* 0 *is the identity of the abelian group* $(F, +)$*,* 1 *is the identity element of the abelian group* $(F - \{0\}, \times)$*,* α, β *are the other two elements in* F*. Fill the following tables of the binary operations and check that* F *together with these two binary operations is indeed a field. (You only need to write up the details that every element in* $F - \{0\}$ *has an inverse.)*

$+$	0	1	α	β
0				
1				
α				
β				

$\times$	0	1	α	β
0				
1				
α				
β				

Problem 4. *In this problem, we construct a field* $(F, +, \cdot)$ *which has 9 elements. Let* $\mathbb{F}_3 = \{0, 1, 2\}$ *be the finite field with 3 elements with the usual notation. This means that* 0 *is the additive identity ,* 1 *is the multiplicative identity and* $2 = 1 + 1$*. Let* α *be an extra element, which is not in* $\mathbb{F}_3$ *but* $\alpha^2 = 2$ *(where the 2 in the right side denotes the element " 2" in* $\mathbb{F}_3$*). Consider the set* $F = \{a\alpha + b | a, b \in \mathbb{F}_3\}$*. Define addition and multiplication on* F *in the usual sense. Namely,*

$$(a_1\alpha + b_1) + (a_2\alpha + b_2) = (a_1 + a_2)\alpha + (a_2 + b_2),$$

$$(a_1\alpha + b_1) \cdot (a_2\alpha + b_2) = (a_1b_2 + a_2b_1)\alpha + (2a_1a_2 + b_1b_2),$$

for $a_1, a_2, b_1, b_2 \in \mathbb{F}_3$*. Show that* $(F, +, \cdot)$ *is a field. This is a field with* 9 *elements.*

(Using similar method, try to construct a field with 25 elements.)