

HOMEWORK 8

Due date: Nov 20, Monday of Week 13
 Exercises: 2, 4, 7, 8, 9, page 123
 Exercises: 1, 2, 3, 4, 5, 6, page 126-127

Let \mathbb{F}_p be the field with p -elements, where p is a prime number. Recall that they are constructed using equivalence classes. A different way to write this field is \mathbb{Z}/p .

Let F be a fixed field. Let $V_n = V_n(F)$ be the F -vector space of polynomials of degree $\leq n$. Then $\dim_F V_n = n + 1$ and thus $\dim_F V_n^* = n + 1$. Given $t \in F$, we have defined $L_t \in V_n^*$ by $L_t(f) = f(t)$ for $f \in V_n$. Lagrange interpolation says that if t_0, \dots, t_n are distinct points in F , then $\{L_{t_i} : 0 \leq i \leq n\}$ is a basis of V_n^* .

Problem 1. Let $F = \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ be the field of 5 elements. Consider $V_3^*(F)$ which has dimension 4 and thus $L_0, L_1, L_2, L_3, L_4 \in V_3^*$ are linearly dependent. Write L_4 as a linear combination of L_0, L_1, L_2, L_3 .

Problem 2. Let S be any subset of F . Show that the subset $\{L_s : s \in S\} \subset \text{Hom}_F(F[x], F)$ is linearly independent where L_s is viewed as a linear function on $F[x]$ defined by $L_s(f) = f(s)$, $f \in F[x]$.

The formal power series algebra $F[[x]]$ is written as F^∞ in the book. We also used the notation $F^\mathbb{N}$ to denote $F[[x]]$.

Problem 3. Given a formal power series $f = (f_0, f_1, \dots, f_n, \dots) \in F^\mathbb{N}$, we consider the map $\phi_f : F[x] \rightarrow F$ defined by

$$\phi_f(x^i) = f_i$$

or

$$\phi_f(a_0 + a_1x + a_2x^2 + \dots + a_nx^n) = a_0f_0 + a_1f_1 + \dots + a_nf_n.$$

- (1) Show that ϕ_f is linear and thus $\phi_f \in \text{Hom}_F(F[x], F)$.
- (2) Show that the map $\phi : F[[x]] \rightarrow \text{Hom}_F(F[x], F)$ is linear as F -vector spaces.
- (3) Show that ϕ is an isomorphism by explicitly constructing an inverse of ϕ .
- (4) Given an element $t \in F$. By part (3), we know that L_t must be of the form ϕ_f for some $f \in F[[x]]$. Describe f in terms of t .

(Comment: Even we cannot compare dimensions because both $F[x]$ and $F[[x]]$ are infinite dimensional as F -vector spaces, it should be clear that $F[[x]]$ is strictly larger than $F[x]$. Actually one can show that $F[x]$ and $F[[x]]$ are not isomorphic as F -vector spaces (You might search a proof of this online. But we won't show it in this course.) This problem shows that the dual of $F[x]$ is strictly larger than $F[x]$, which never happens in the finite dimension case.)

(Comment: In HW6, Problem 1, we know that $\text{Hom}(V, W) \cong W^n$ if $\dim V = n$. From this problem we know that $\text{Hom}(F[x], F) \cong F[[x]]$. Both isomorphisms is a special case of something you might learn in the future.)

You can do the above 3 problems after Wednesday's class.

- Problem 4.**
- (1) Find a nonzero polynomial $f \in \mathbb{F}_p[x]$ such that $f(a) = 0$ for any $a \in \mathbb{F}_p$.
 - (2) Let $f \in \mathbb{F}_p[x]$ be a nonzero polynomial such that $f(a) = 0$ for any $a \in \mathbb{F}_p$. Show that $\deg(f) \geq p$.
 - (3) Consider the set $I = \{f \in \mathbb{F}_p[x] : f(a) = 0, \forall a \in \mathbb{F}_p\}$. Show that I is ideal of $F[x]$.

Hint: (2) is a consequence of Lagrange interpolation. Check the proof of Theorem 3, page 126. The rest part should be easy.

Problem 5. Let $A \in \text{Mat}_{n \times n}(F)$ be a fixed non-zero polynomial. Consider the set

$$I = \{f \in F[x] : f(A) = 0\}.$$

Show that I is an ideal. Suppose that d is nonzero monic polynomial such that $I = dF[x]$. Show that $\deg(d) \leq n^2$.

You can do the above two problems after Friday's class.