HOMEWORK 13

Due date: Monday of Week 14

Exercises: 12.4, M.1, M.2, M.9, M.10, M.14.pages 75-77.

Exercises: 7.7, 7.8, 7.9, 7.10, 8.1, 8.2, 8.4, 11.1, 11.2, 11.3, 11.5, 11.8, M.7, pages 191-194.

Problem 1. *Let G be a group (not necessarily finite) and $H < G$ be a subgroup of G. Recall that G/H denotes the set of all left H-cosets and $H\backslash G$ denotes the set of all right H-cosets. Show that $f : G/H \to H\backslash G$ defined by $f(gH) = Hg^{-1}$ is well-defined and defines a bijection between G/H and $H\backslash G$. In particular, the number of left cosets is the same is the number of right cosets.*

There are at least two elements ϕ_0, ϕ_1 in $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ defined by

$$\phi_0 = \mathrm{id}_{\mathbb{Z}/n\mathbb{Z}}; \phi_1(x) = -x, \forall x \in \mathbb{Z}/n\mathbb{Z}.$$

Consider the map

$$\phi : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$$

defined by $\phi(\overline{0}) = \phi_0, \phi(\overline{1}) = \phi_1$. It is clear that ϕ is a group homomorphism.

Problem 2. *Show that $\mathbb{Z}/n\mathbb{Z} \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$ is isomorphic to D_n, the dihedral group of order $2n$.*

Problem 3. *Determine the order of the group $\mathrm{GL}_n(\mathbb{F}_p)$, where p is a prime number.*

Hint: Consider the action of $\mathrm{GL}_n(\mathbb{F}_p)$ on $\mathbb{F}_p^n$ by left multiplication.
The next several problems are about double cosets, and most of them could be in last HW.

Problem 4. *Let F be a field and let $B_n(F) \subset \mathrm{GL}_n(F)$ be the upper triangular subgroup.*

1. *Determine the double cosets $B_2(F)\backslash\mathrm{GL}_2(F)/B_2(F)$.*
2. *How about $B_n(F)\backslash\mathrm{GL}_n(F)/B_n(F)$?*

This problem might be hard. It is related to the UPL (upper triangular, permutation subgroup, and lower triangular subgroup) decomposition of a matrix, See HW 3, Problem 5 of last year. If you don't know how to do the general problem, try the case when $n = 2$ and $F = \mathbb{F}_2$ (or $\mathbb{F}_3$).

Let $G \times X \to X$ be an action of a group G on a set X. Recall that $G\backslash X$ denote the set of orbits.

Problem 5. *Let G be a group and H, K are subgroups of G. Show the following basic properties of double cosets.*

1. *For $x \in G$, the double coset HxK is a union of right H-cosets and a union of left K-cosets. More precisely,*

$$HxK = \coprod_{Hxk \in H\backslash HxK} Hxk = \coprod_{hxK \in HxK/K} hxK.$$

2. *Let G act on the left cosets G/K from the left by $x.(gK) = (xg)K$. See Section 6.8 of Artin. We restrict this action to H and consider the action*

$$H \times G/K \to G/K$$

defined by $(h, gK) = (hg)K$. Show that there is a bijection between the double coset $H\backslash G/K$ and the set of orbits $H\backslash(G/K)$. This explains that the notation is consistent. There is a similar statement when we switch the role of H and K.

3. *Suppose that all groups are finite. For $x \in G$, show that*

$$|HxK| = [H : H \cap xKx^{-1}]|K| = [K : K \cap x^{-1}Hx]|H|.$$

(4) *Show that*
$$[G:H]=\sum_{HxK\in H\backslash G/K}[K:K\cap x^{-1}Hx]$$
and
$$[G:K]=\sum_{HxK\in H\backslash G/K}[H:H\cap xKx^{-1}].$$
(5) *Consider the group action of* $(H\times K)$ *on* G *defined by*
$$((h,k),g)=hgk^{-1},(h,k)\in H\times K,g\in G.$$
Check that this is a group action and there is a bijection between $H\backslash G/K$ *and the orbits of this action.*
(6) *Let* $G^{(h,k)}=\{g\in G:hgk=g\}$. *Show that*
$$|H\backslash G/K|=\frac{1}{|H||K|}\sum_{(h,k)\in H\times K}|G^{(h,k)}|.$$

For the last one, use Ex. M.7, page 194 of Artin. The other parts are routine.
The next problem is roughly talked in class.

Problem 6. *Let* $n>1$ *be a positive integer and consider the group* $\mathrm{SO}_n(\mathbb{R})=\{g\in \mathrm{GL}_n(\mathbb{R}):gg^t=I_n,\det(g)=1\}$. *Consider the subgroup* H *of* $\mathrm{SO}_n(\mathbb{R})$ *defined by*
$$H=\left\{\begin{bmatrix}h&\\&1\end{bmatrix},h\in \mathrm{SO}_{n-1}(\mathbb{R})\right\}.$$
Show that there is a bijection
$$G/H\cong S^{n-1},$$
where $S^{n-1}=\left\{(x_1,\dots,x_n)\in\mathbb{R}^n:x_1^2+\cdots+x_n^2=1\right\}$, *which is the standard* $(n-1)$*-sphere. Similarly, we consider the group* $\mathrm{SU}_n=\{g\in\mathrm{GL}_n(\mathbb{C}):gg^*=I_n,\det(g)=1\}$. *We view* SU_{n-1} *as a subgroup of* SU_n *via the embedding*
$$h\mapsto\begin{bmatrix}h&\\&1\end{bmatrix},h\in\mathrm{SU}_{n-1}.$$
Show that there is a bijection
$$\mathrm{SU}_n/\mathrm{SU}_{n-1}\cong S^{2n-1}.$$

You don't have to submit solutions for the next several problems. They are for your summer break.

Problem 7. *Let* $p>2$ *be a prime number and* n *be a positive integer. Consider the group*
$$\mathrm{SO}_n(\mathbb{F}_p)=\left\{g\in\mathrm{GL}_n(\mathbb{F}_p)|\ gg^t=I_n,\det(g)=1\right\}.$$
Compute the order of $\mathrm{SO}_n(\mathbb{F}_p)$.

Hint: If you use the method the last problem, you need to know the order of the sets
$$X_n:=\left\{(x_1,\dots,x_n)\in\mathbb{F}_p^n:x_1^2+x_2^2+\cdots+x_n^2=1.\right\}$$
It is not easy to compute this. The answer is
$$|X_n|=\begin{cases}p^{n-1}+(-1)^{\frac{n-1}{2}\cdot\frac{p-1}{2}}p^{\frac{n-1}{2}} & 2\nmid n\\ p^{n-1}-(-1)^{\frac{n}{2}\cdot\frac{p-1}{2}}p^{\frac{n}{2}-1} & 2|n\end{cases}$$
This is Proposition 8.6.1, page 102 of Ireland-Rosen: A classical introduction to modern number theory (2nd edition). See also this link.

The group $\mathrm{SO}_n(\mathbb{F}_p)$ is still the group which preserve a symmetric bilinear form on vector spaces over $\mathbb{F}_p$. But this time, this bilinear form is not an inner product. Inner products are only defined on vector spaces over $\mathbb{R}$ or $\mathbb{C}$, while bilinear forms can be defined over any fields. There is also a way to defined $\mathrm{U}_n(\mathbb{F}_p)$ and $\mathrm{SU}_n(\mathbb{F}_p)$ and similarly one can ask how many elements are there in these groups.

The orthogonal groups over finite fields also depends on the symmetric bilinear form defined on that. Classifying symmetric bilinear forms over $\mathbb{F}_p$ is also an interesting question. We give one example below. Consider the group

$$\mathrm{SO}_{1,1}(\mathbb{F}_p) = \left\{ g \in \mathrm{GL}_2(\mathbb{F}_p) : g \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} g^t = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \right\}.$$

Problem 8. *Let $p > 2$ be a prime.*

(1) *Show that* $|\mathrm{SO}_{1,1}(\mathbb{F}_p)| = p - 1$.
(2) *Show that* $|\mathrm{SO}_2(\mathbb{F}_p)| = p - (-1)^{\frac{p-1}{2}}$.
(3) *In particular, if* $p \equiv 1 \bmod 4$ *(for example, if* $p = 5, 13, 17 \ldots$*), then* $|\mathrm{SO}_{1,1}(\mathbb{F}_p)| \cong |\mathrm{SO}_2(\mathbb{F}_p)|$. *Is it true that* $\mathrm{SO}_2(\mathbb{F}_p) \cong \mathrm{SO}_{1,1}(\mathbb{F}_p)$ *if* $p \equiv \bmod 4$*? If so, prove it.*

Try to generalize the last part to general n by considering the corresponding symmetric bilinear forms. Hint: Question: What is special for p with $p \equiv 1 \bmod 4$? Answer: the equation $x^2 + 1 = 0$ has a solution in $\mathbb{F}_p$.