HOMEWORK 14

Due date: Monday of Week 15,
Exercises: 2.1, 2.2, 2.3, 2.7, 2.13, 2.14, 2.17, 2.18, 3.2, 3.3, 5.5, 5.7, 5.11, 5.12, 6.1, 6.2, 6.4, 6.5 pages 221-223.

Problem 1. *Let G be a finite group with $|G| = n$. Let $\iota : G \to S_n$ be the embedding in Cayley's theorem (namely, $\iota : G \to \mathrm{Perm}(G)$ is determined by the left multiplication action $G \times G \to G$, $(g, x) \mapsto gx$).*

1. *Let $h \in G$ be an element of order d (so that $d|n$). Show that $\iota(h)$ is a product of n/d disjoint cycles of length d.*
2. *Suppose $n = 4m + 2$. Show that $\iota(h) \in A_n$ if and only if the order of h is odd.*

Hint: Let $H = \langle h \rangle \subset G$ and consider the right coset decomposition $G = \coprod Hg_i$. For (2), see Ex. 10.1, page 74. This problem was taken from a Quora answer here.

Problem 2. *Let C be a conjugacy class of S_n. Decompose $C \cap A_n$ as conjugacy classes of A_n.*

This is roughly Ex.5.11. For future references, here is a more precise statement. Suppose that $\sigma \in S_n$ has cycle lengths $k_1, \ldots, k_m$ with $k_1 + \cdots + k_m = 1$. This means that σ can be written as a product of m disjoint cycles of lengths $k_1, \ldots, k_m$. For example, for $\sigma = (123)(45)(6) \in S_6$, we have $m = 3$, $k_1 = 3, k_2 = 2, k_3 = 1$. Using this notation, we have $\mathrm{sign}(\sigma) = (-1)^{k_1-1+k_2-1+\cdots+k_m-1} = (-1)^{n-m}$. Thus σ is an even permutation (namely, $\sigma \in A_n$) if and only if $n-m$ is even. For example, $(123)(45)(6) \in S_6$ has signature -1. This is Ex 10.1, page 74. Now suppose that $n - m$ is even and so that $\sigma \in A_n$. We consider the S_n conjugacy class $C(\sigma) = \{g\sigma g^{-1} : g \in S_n\}$.

Problem 3.
1. *If all k_i are odd and distinct, then $C(\sigma)$ is the union of two A_n conjugacy classes and these two conjugacy classes have the same order.*
2. *Otherwise (which means, either one of k_i is even, or there are at least two k_i are the same), then $C(\sigma)$ is still a single A_n-conjugacy class.*

For example, in S_4, if $\sigma = (12)(34)$, then $C(\sigma)$ is a single A_4 conjugacy class; if $\sigma = (123)(4)$, then $C(\sigma)$ is the union of two different A_4 conjugacy classes. Actually, one can see that (123) and (132) are not conjugate in A_4.

Problem 4. *Given an element $p \in S_n$ with m_1 1-cycles, $\ldots$, m_n n-cycles. So $\sum_{i=1}^n im_i = n$. For example, for the cycle $p = (123)(45)(67)(89)$ of S_{10}, we have $m_1 = 1, m_2 = 3, m_3 = 1$ and $m_i = 0$ for $i \geq 4$. Determine how many elements are in the conjugacy class determined by p.*

Answer:

$$|C| = \frac{n!}{1^{m_1}m_1!2^{m_2}m_2! \ldots n^{m_n}m_n!} = \frac{n!}{\prod_{i=1}^n i^{m_i}m_i!}.$$

For example, in S_5, the conjugacy class contains $(12)(345)$ has $\frac{5!}{2^1\cdot 1!\cdot 3^1\cdot 1!} = 20$ elements, and the conjugacy class contains (12345) has $\frac{5!}{5^1} = 4! = 24$ elements.

Problem 5. *Let G be a finite group, H be a subgroup of G. Let $C \subset G$ be a conjugacy class and suppose*

$$H \cap C = \coprod_{i=1}^r D_i,$$

where each D_i is a conjugacy class of H. Consider the set

$$X_i = \left\{(c, g) \in C \times G : g^{-1}cg \in D_i\right\}.$$

Express $|X_i|$ in terms of $|G|, |H|, |D_i|$.

Hint: Consider the group action $G \times X_i \to X_i$ defined by $x.(c, g) = (xcx^{-1}, xg)$.

Problem 6. *Let $G = D_4 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ with $x^4 = 1 = y^2, yxy^{-1} = x^3$ and $H = \{1, x^2, y, x^2y\} \subset G$. Find all conjugacy classes C of G, and for each conjugacy class C of G, decompose $C \cap H$ into conjugacy classes of H.*

Problem 7. *Let $G = \mathrm{GL}_2(\mathbb{F}_p), H = \mathrm{SL}_2(\mathbb{F}_p) = \{g \in G : \det(g) = 1\}$. Let $C \subset G$ be the conjugacy class of the element $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Namely,*

$$C = \{gug^{-1} : g \in G\}.$$

Try to decompose $C \cap H$ into conjugacy classes of H.

Problem 8. *Let p be a prime number. Show that the cyclic group C_{p^n} for $n \geq 2$ is not a semi-direct product of two proper subgroups.*

Proposition 7.3.3, page 198, says that every group of order p^2 is abelian. In the following, we give some examples of non-abelian group of order p^3. We assume that $p > 2$. If $p = 2$, we have seen that the quaternion group is an order 2 non-abelian group.

The first one is called Heisenberg group of the field $\mathbb{F}_p$, and we temporarily denote it by $H(\mathbb{F}_p^2)$. (This looks like a wired notation, but it has generalizations). It is defined by

$$H(\mathbb{F}_p^2) = \left\{ \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} \in \mathrm{GL}_3(\mathbb{F}_p), x, y, z \in \mathbb{F}_p \right\}.$$

Its group structure is defined by matrix multiplication. The other group is temporarily denoted by G_p and it is defined by

$$G_p = \left\{ \begin{bmatrix} x & y \\ & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z}) : x \equiv 1 \bmod p, y \in \mathbb{Z}/p^2\mathbb{Z} \right\}.$$

Problem 9. *Show that $H(\mathbb{F}_p^2)$ and G_p are non-abelian group of order p^3. Moreover, show that they are indeed semidirect products of their own subgroups.*

It can be shown that these are the only two non-abelian groups of order p^3 up to isomorphism.

Problem 10. *Determine the class equations of $D_n, n \geq 3$, $\mathrm{GL}_2(\mathbb{F}_p), \mathrm{SL}_2(\mathbb{F}_p)$ and $\mathrm{PSL}_2(\mathbb{F}_p)$, where $\mathrm{PSL}_2(\mathbb{F}_p) = \mathrm{SL}_2(\mathbb{F}_p)/\{\pm I_2\}$.*

If you find this hard, at least try some examples for small p.