

HOMEWORK 3

Due date: Monday of Week 4
Exercises: 1, 2, 3, 6, 8, 10, 11, 12, 13, 15, 16, pages 250-251 of Hoffman-Kunze,

The next problem is not closely related to the materials of this week.

Problem 1. Given two matrices $A, B \in \text{Mat}_{n \times n}(\mathbb{C})$.

- (1) Show that $\chi_{AB} = \chi_{BA}$.
- (2) Suppose that $\deg(\mu_{AB}) > \deg(\mu_{BA})$, show that $\mu_{AB} = x\mu_{BA}$. Here $x\mu_{BA}$ denote the product of the polynomial x with the minimal polynomial μ_{BA} of BA .
- (3) Suppose that AB is diagonalizable, show that $(BA)^2$ is diagonalizable.
- (4) Give one example such that AB is diagonalizable but BA is not diagonalizable.
- (5) Let $\lambda \in \mathbb{C}, \lambda \neq 0$ and $r \in \mathbb{Z}$ be a positive integer. Show that $\dim \text{Ker}(\lambda I - AB)^r = \dim \text{Ker}(\lambda I - BA)^r$.

This problem was borrowed from [this link](#). For the first part, see HW 12 of last year. The second part follows from a direct computation. For (3), discuss μ_{BA} using (2). For (5), see the proof of Problem 1, HW 12 of last year.

Given a positive integer n , a partition λ of n is a sequence of decreasing positive numbers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $\sum_{i=1}^k \lambda_i = n$. We write $\lambda \vdash n$. Given a sequence of decreasing positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \dots \geq \lambda_k$, we also write $|\lambda| = \sum_{i=1}^k \lambda_i$. Thus $\lambda \vdash |\lambda|$. For example $(2, 2) \vdash 4, (2, 1, 1) \vdash 4$. Given a positive integer n , let $\mathcal{P}(n)$ be the set of all partitions of n . Let $P(n) = |\mathcal{P}(n)|$, which is the number of all partitions of n . For example,

$$\mathcal{P}(4) = \{(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\}.$$

Thus $P(4) = 5$.

Problem 2. (1) Compute $\mathcal{P}(n)$ and $|P(n)|$ for $n = 5, 6$.
(2) Show that $P(n)$ is the coefficient of x^n in the formal power series

$$\prod_{m=1}^{\infty} \left(\frac{1}{1 - x^m} \right) = (1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + x^9 + \dots) \dots$$

Recall that a matrix $A \in \text{Mat}_{n \times n}(\mathbb{C})$ is called nilpotent if $A^k = 0$ for some $k > 0$. We denote $\mathfrak{n}_n(\mathbb{C})$ the subset of nilpotent matrices in $\text{Mat}_{n \times n}(\mathbb{C})$. On $\mathfrak{n}_n(\mathbb{C})$, we define an equivalence relation R by similarity, namely, $R = \{(A, B) \in \mathfrak{n}_n(\mathbb{C}) \times \mathfrak{n}_n(\mathbb{C}) : A \text{ is similar with } B\}$. We consider the equivalence class $\mathfrak{n}_n(\mathbb{C})/R$. An element of $\mathfrak{n}_n(\mathbb{C})/R$ (which is an equivalence class) is called an conjugacy class of a nilpotent matrix. Recall that a typical element in $\mathfrak{n}_n(\mathbb{C})/R$ is of the form $\overline{A} = \{B \in \mathfrak{n}_n(\mathbb{C}) : B \text{ is similar with } A\}$ for some $A \in \mathfrak{n}_n(\mathbb{C})$. Given a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, we consider the nilpotent matrix

$$A_{\lambda} = \begin{bmatrix} A_{\lambda_1} & & \\ & \dots & \\ & & A_{\lambda_k} \end{bmatrix} \in \mathfrak{n}_n(\mathbb{C}),$$

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where for a positive integer m , A_m denotes the Jordan block with zero in the diagonal of size m , namely,

$$A_m = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 1 & 0 \end{bmatrix} \in \text{Mat}_{m \times m}(\mathbb{C}).$$

For example, for $\lambda = (3, 2) \vdash 5$, we have

$$A_\lambda = \begin{bmatrix} 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & & & 0 & 0 \\ & & & 1 & 0 \end{bmatrix}$$

Problem 3. Consider the map $\mathcal{P}(n) \rightarrow \mathfrak{n}_n(\mathbb{C})/R$ defined by $\lambda \mapsto A_\lambda$. Show that this map is a bijection.

Given a matrix $A \in \text{Mat}_{n \times n}(F)$ (for simplicity, we assume that $F = \mathbb{C}$). Let c be an eigenvalue of A , we have defined algebraic multiplicity and geometric multiplicity of A at the eigenvalue c . We lack standard notations here. Recall that the geometric multiplicity of A at c is defined to be $\dim \text{Ker}(A - cI)$. Recall that if A and B are similar, then they have the same eigenvalues. Moreover, at each eigenvalue, the algebraic multiplicities (and geometric multiplicities) of A and B are the same. Conversely, even if A and B have the same algebraic and geometric multiplicities at each eigenvalue, it does not mean that A and B are similar. A notation which generalize the geometric multiplicity is to consider $\dim(A - cI)^r$ for every positive integer r .

Problem 4. Given a partition $\lambda \vdash n$.

- (1) Give two nilpotent matrices A, B such that A, B have the same geometric multiplicity, but A is not similar to B .
- (2) Determine $\dim \text{Ker}(A_\lambda - 0I)^r = \dim \text{Ker}(A_\lambda)^r$ in terms of λ (and r).
- (3) Suppose that A, B are two nilpotent matrices (so the only eigenvalue of A, B is 0) such that $\dim \text{Ker}(A)^r = \dim \text{Ker}(B)^r$ for every r . Is it true that A is similar to B ? In other words, if A is in Jordan canonical form, can the set $\{\dim \text{Ker}(A)^r : r \geq 1\}$ uniquely determine the corresponding partition of A ? If so, prove it. If not, give a counter example.

For (2) and (3), if it is hard, at least try the case when $n = 5$.

We consider the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ where p is a prime integer. For simplicity, we assume that $p > 2$. It is known that there is an element $\kappa \in \mathbb{F}_p^\times = \mathbb{F}_p - \{0\}$ such that $x^2 - \kappa = 0$ has no solution in \mathbb{F}_p . For example, if $p = 3$, we can take $\kappa = 2$; if $p = 5$, we can take $\kappa = 2$ or 3 . Such κ is not unique in general. But if κ_1, κ_2 are two such numbers, then $x^2 - \kappa_1\kappa_2^{-1} = 0$ has a solution in \mathbb{F}_p . For example, if $p = 5$, $\kappa_1 = 2, \kappa_2 = 3$, then $\kappa_1\kappa_2^{-1} = 4$ and $x^2 - 4 = 0$ has a solution in \mathbb{F}_p .

Problem 5. Fix an element $\kappa \in \mathbb{F}_p^\times$ such that $x^2 - \kappa = 0$ has no solution in \mathbb{F}_p . Show that any element $g \in \text{GL}_2(\mathbb{F}_p)$ is similar to one of the following type matrices

- (1) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, a \in \mathbb{F}_p^\times;$
- (2) $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}, a \in \mathbb{F}_p^\times;$
- (3) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, a, b \in \mathbb{F}_p^\times, a \neq b;$
- (4) $\begin{bmatrix} a & b\kappa \\ b & a \end{bmatrix}, a \in \mathbb{F}_p, b \in \mathbb{F}_p^\times.$

You don't have to submit solutions of the following problem.

- Problem 6.** (1) *Try to classify conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_p)$.*
 (2) *Try to classify conjugacy classes of $\mathrm{GL}_3(\mathbb{C})$ and $\mathrm{GL}_3(\mathbb{R})$.*