

HOMEWORK 3

Due date: Monday of Week 8, Oct. 16

Exercise: 5, 8 (page 21);

Exercise: 1, 3, 6, 7, 10 (page 27).

Note that Exercise 10, page 27 tells us that if $A \in \text{Mat}_{m \times n}(F)$ with $n < m$, then A has no right inverse in the following sense: there is no $B \in \text{Mat}_{n \times m}$ such that $AB = I_m$. Find an example of $A \in \text{Mat}_{m \times n}(F)$ with $n < m$ such that A has a left inverse, namely, such that there exists a matrix $C \in \text{Mat}_{n \times m}(F)$ with $CA = I_n$.

Problem 1. Let F be a field. For any $A \in \text{Mat}_{m \times n}(F)$ and $B \in \text{Mat}_{n \times m}(F)$, show that $\text{tr}(AB) = \text{tr}(BA)$.

Here recall that for a square matrix $C = (c_{ij})_{1 \leq i, j \leq n}$, $\text{tr}(C) = \sum_{i=1}^n c_{ii}$.

A square matrix $A \in \text{Mat}_{n \times n}(F)$ is called **nilpotent** if $A^k = 0$ for some integer $k > 0$. For example the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \text{Mat}_{3 \times 3}(F)$$

is nilpotent, because $A^3 = 0$.

Problem 2. Let $B \in \text{Mat}_{n \times n}(F)$ be a nilpotent matrix. Show that $I_n + B$ is invertible.

Problem 3. Let F be a field, $A \in \text{Mat}_{m \times n}(F)$ and $B \in \text{Mat}_{n \times m}(F)$. Show that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

(Hint: Use the identity $A(I_n - BA) = (I_m - AB)A$. This is Problem M.10, page 36 of Artin's book "Algebra", edition 2.)

Problem 4. Let F be a field, $A, B \in \text{Mat}_{n \times n}(F)$. If $AB = A + B$ show that $AB = BA$.

(Hint: consider $(A - I_n)(B - I_n)$.)

A matrix $A \in \text{Mat}_{n \times n}(F)$ is called a permutation matrix if each row and each column has only one nonzero entry and that non-zero entry is 1. For example,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \text{Mat}_{3 \times 3}(F)$$

is a permutation matrix. A matrix $A = (a_{ij})_{1 \leq i, j \leq n} \in \text{Mat}_{n \times n}(F)$ is called upper triangular if $a_{ij} = 0$ for all i, j with $i > j$, namely, if each entry below the main diagonal is zero. Similarly, a matrix $A = (a_{ij})_{1 \leq i, j \leq n} \in \text{Mat}_{n \times n}(F)$ is called lower triangular if $a_{ij} = 0$ for all i, j with $i < j$. For example

$$\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \in \text{Mat}_{3 \times 3}(F)$$

is upper triangular. Here an entry with $*$ means that the value of that entry is not important. Moreover, it is easy to see that if a matrix $A \in \text{Mat}_{n \times n}(F)$ is a row reduced echelon matrix, then A is upper triangular.

Problem 5. (1) Show that each permutation matrix $A \in \text{Mat}_{n \times n}(F)$ is invertible.

- (2) *Let $A \in \text{Mat}_{n \times n}(F)$ be a matrix such that one can obtain its row reduced echelon matrix using just the first two types element row operations. In other words, we can obtain a row reduced echelon matrix from A using E.R.O without interchanging any two rows. Show that one can write $A = LU$, where $L \in \text{Mat}_{n \times n}(F)$ is a lower triangular matrix and $U \in \text{Mat}_{n \times n}(F)$ is an upper triangular matrix.*
- (3) *Show that each matrix $A \in \text{Mat}_{2 \times 2}(F)$ can be written as a product $A = LPU$, where $L \in \text{Mat}_{2 \times 2}(F)$ is a lower triangular matrix, $P \in \text{Mat}_{2 \times 2}(F)$ is a permutation matrix, and $U \in \text{Mat}_{2 \times 2}(F)$ is an upper triangular matrix.*
- (4) *Show that each matrix $A \in \text{Mat}_{3 \times 3}(F)$ can be written as a product $A = LPU$, where $L \in \text{Mat}_{3 \times 3}(F)$ is a lower triangular matrix, $P \in \text{Mat}_{3 \times 3}(F)$ is a permutation matrix, and $U \in \text{Mat}_{3 \times 3}(F)$ is an upper triangular matrix.*

Think about how to do the last part for general $n \times n$ matrix. Namely, for any $A \in \text{Mat}_{n \times n}(F)$, show that it can be written as a product $A = LPU$, where $L \in \text{Mat}_{n \times n}(F)$ is a lower triangular matrix, $P \in \text{Mat}_{n \times n}(F)$ is a permutation matrix, and $U \in \text{Mat}_{n \times n}(F)$ is an upper triangular matrix. There is no need to submit a proof of this. But you need to keep in mind this assertion. A decomposition of A into the product LPU is called a Bruhat decomposition of the matrix A .