HOMEWORK 4

Due date: Monday of Week 5

Exercises: 2, 3, 4, pages 261-262, Hoffman-Kunze,
Exercises: 1, 2, 3, page 269.
Exercises: 5, 7, 8, 9, 12, 13, 14, 16, 17. pages 276-277.

Problem 1. *Consider the matrix*

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{Q}).$$

Find the Smith normal form of $xI_3 - A$, the invariant factors of A and the rational canonical form of A. Determine if A has a Jordan canonical form. If so, find its Jordan canonical form.

Problem 2. *Let F be a general field and $A \in \mathrm{Mat}_{n\times n}(F)$. Show that A is similar to A^t.*

Problem 3. *Let $\alpha = \sqrt[3]{2}$. Let $F = \{a + b\alpha + c\alpha^2 | a, b, c \in \mathbb{Q}\}$. We view F as a dimension 3 vector space over $\mathbb{Q}$. Let $\mathcal{B} = [1, \alpha, \alpha^2]$, which is an ordered basis of F over $\mathbb{Q}$. Given an element $x \in F$, we consider the linear operator $T_x : F \to F$ defined by $T_x(y) = xy$. Compute the matrix $A_x = [T_x]_{\mathcal{B}} \in \mathrm{Mat}_{3\times 3}(\mathbb{Q})$ for $x = a + b\alpha + c\alpha^2$. Show that T_x is a semi-simple operator when F is viewed as vector space over $\mathbb{Q}$.*

Problem 4. *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{Q})$ be a matrix such that $A^3 - 2I_n = 0$, where $I_n \in \mathrm{Mat}_{n\times n}(\mathbb{Q})$ is the identity matrix. Show that $3|n$ (3 divides n). Write $n = 3k$ for a positive integer k. Show that A is similar the matrix*

$$\begin{bmatrix} & & 2I_k \\ I_k & & \\ & I_k & \end{bmatrix}.$$

Problem 5. *Consider the matrix*

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{Q}).$$

Find a semisimple matrix $S \in \mathrm{Mat}_{4\times 4}(\mathbb{Q})$ and a nilpotent matrix $N \in \mathrm{Mat}_{4\times 4}(\mathbb{Q})$ such that $A = S + N$ and $SN = NS$.

Hint: you can repeat the proof of Theorem 13, page 267, in this special case. It is useful to notice that the characteristic polynomial χ_A of A is $(x^2 - 2)^2$. Moreover, $x^2 - 2$ is irreducible over $\mathbb{Q}$. Here a matrix S is called semi-simple if the linear operator defined by S is semi-simple, or equivalently, if the minimal polynomial μ_S of S is a product of irreducible polynomials of multiplicity one.

Problem 6. *Let $V = \mathrm{Mat}_{n\times n}(\mathbb{C})$. Define a map $(\ |\ ) : V \to V \times \mathbb{C}$ by*

$$(A|B) = \mathrm{tr}(AB^*).$$

Check that $(\ |\ )$ is an inner product on V.