

HOMEWORK 12

Due date: Monday of Week 13

Exercises: 5.1, 5.2, 5.3, 5.6, 6.2, 6.3, 6.5, 6.6, 6.7, 6.10, 6.11, 8.1, 8.2, 8.6, 8.7, 8.8, 8.10, 8.12, 10.1, 10.2, 10.4, 11.4, 11.6, 11.8, 11.9, pages 72-76 of Artin's book

One important construction in group theory which is not covered in the textbook is *semidirect product*. We define it here. Given a group N , recall that $\text{Aut}(N)$ denotes the group of all automorphisms of N . It is consisting of all $f : N \rightarrow N$ such that f is an isomorphism. For example, if $N = \mathbb{Z}^+$, the map $f : N \rightarrow N$ defined by $f(x) = -x$ is an automorphism. The group structure on $\text{Aut}(N)$ is just composition.

Let H and N be two groups and let $\phi : H \rightarrow \text{Aut}(N)$ be a group homomorphism. In particular, for each $h \in H$, $\phi(h) : N \rightarrow N$ is an automorphism. We now define a group $N \rtimes_{\phi} H$, which is called the (outer) semidirect product of N with H with respect to ϕ . As a set, $N \rtimes_{\phi} H$ is just the Cartesian product of N with H , namely, as a set $N \rtimes_{\phi} H = \{(n, h) | n \in N, h \in H\}$. The group operation \bullet (product in the group) is defined by

$$(n_1, h_1) \bullet (n_2, h_2) = (n_1 \phi(h_1)(n_2), h_1 h_2), n_1, n_2 \in N, h_1, h_2 \in H.$$

Here recall that $\phi(h_1) : N \rightarrow N$ is an isomorphism, and thus $\phi(h_1)(n_2) \in N$. Note that if ϕ is the trivial homomorphism, namely, $\phi(h) = \text{id}_N$ for every $h \in H$, then $N \rtimes_{\phi} H$ is just the direct product $N \times H$. Thus semidirect product is a generalization of product.

Problem 1. Show that $N \rtimes_{\phi} H$ defined above is indeed a group. Moreover, consider the map $i_N : N \rightarrow N \rtimes_{\phi} H$ defined by $i_N(n) = (n, 1)$ and $i_H : H \rightarrow N \rtimes_{\phi} H$ defined by $i_H(h) = (1, h)$. Show that i_N, i_H are injective group homomorphisms. Furthermore, show that $i_N(N)$ is a normal subgroup of $N \rtimes_{\phi} H$.

One might ask how the group structure of $N \rtimes_{\phi} H$ depends on ϕ .

Problem 2. Let $f : H \rightarrow H$ be an automorphism and let $\phi_1 : H \rightarrow \text{Aut}(N)$ be a group homomorphism. Consider $\phi_2 = \phi_1 \circ f : H \rightarrow \text{Aut}(N)$. Show that $N \rtimes_{\phi_1} H \cong N \rtimes_{\phi_2} H$.

Let n be a positive integer and let C_n denote the cyclic group of order n . We can realize $C_n \cong \mathbb{Z}/n\mathbb{Z}$ with addition as the group operation.

Problem 3. Show that $\text{Aut}(C_n) = \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}$. Here recall that

$$(\mathbb{Z}/n\mathbb{Z})^{\times} = \{a \in \mathbb{Z}/n\mathbb{Z} : \text{there is an element } b \in \mathbb{Z}/n\mathbb{Z}, \text{ such that } ab = 1\}.$$

The group structure on $(\mathbb{Z}/n\mathbb{Z})^{\times}$ is multiplication.

If $n = 10$, this is Exercise 6.10 (a).

Problem 4. Let p, q be two primes.

- (1) If there exists a non-trivial group homomorphism $C_q \rightarrow \text{Aut}(C_p)$, show that $q|(p-1)$;
- (2) Suppose $q|(p-1)$. Determine all group homomorphisms $C_q \rightarrow \text{Aut}(C_p)$;
- (3) Suppose $q|(p-1)$. Let ϕ_1, ϕ_2 be two different nontrivial group homomorphisms $C_q \rightarrow \text{Aut}(C_p)$. Show that there exists an isomorphism $f : C_q \rightarrow C_q$ such that $\phi_2 = \phi_1 \circ f$.
- (4) Suppose $q|(p-1)$. Conclude that there are only two isomorphism classes $C_p \rtimes_{\phi} C_q$.

(This one might be hard. For part (3), you might need to use the following fact. The group $(\mathbb{Z}/p\mathbb{Z})^{\times}$ is a cyclic group. We will prove this later.)

We now consider a special case of semidirect product. Suppose that N and H are both subgroups of a group G with $N \cap H = \{1\}$. Moreover, suppose that for any $h \in H$ and $n \in N$, we have $hnh^{-1} \in N$. If this condition is satisfied, we say that H normalizes N . Then we define

$$\phi : H \rightarrow \text{Aut}(N)$$

by $\phi(h)(n) = hnh^{-1}$. Then we can form the semidirect product. $N \rtimes_{\phi} H$. In this case, we often drop ϕ from the notation, and write it as $N \rtimes H$.

Problem 5. Show that there is an injective homomorphism $N \rtimes H \rightarrow G$.

Hint: the map is just $(n, h) \rightarrow nh$.

We then identify $N \rtimes H$ as a subgroup of G . This is called the inner semidirect product of N and H .

Problem 6. Suppose that N, H are two subgroups of G . Show that $G = N \rtimes H$ if and only if the following conditions hold.

- (1) N is normal in G ;
- (2) $G = NH$;
- (3) $N \cap H = \{1\}$.

Compare this with Proposition 2.11.4, page 65.

Problem 7. Show that the quaternion group H defined in (2.4.5), page 47 of Artin's book is not a semidirect product of its two proper subgroups.