HOMEWORK 3

Due date: Monday of Week 8, Oct. 16
Exercise: 5, 8(page 21);
Exercise: 1, 3, 6, 7, 10(page 27).
Note that Exercise 10, page 27 tells us that if $A \in \mathrm{Mat}_{m\times n}(F)$ with $n < m$, then A has no right inverse in the following sense: there is no $B \in \mathrm{Mat}_{n\times m}$ such that $AB = I_m$. Find an example of $A \in \mathrm{Mat}_{m\times n}(F)$ with $n < m$ such that A has a left inverse, namely, such that there exists a matrix $C \in \mathrm{Mat}_{n\times m}(F)$ with $CA = I_n$.

Problem 1. *Let F be a field. For any $A \in \mathrm{Mat}_{m\times n}(F)$ and $B \in \mathrm{Mat}_{n\times m}(F)$, show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.*

Here recall that for a square matrix $C = (c_{ij})_{1\le i,j\le n}$, $\mathrm{tr}(C) = \sum_{i=1}^{n} c_{ii}$.

A square matrix $A \in \mathrm{Mat}_{n\times n}(F)$ is called **nilpotent** if $A^k = 0$ for some integer $k > 0$. For example the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(F)$$

is nilpotent, because $A^3 = 0$.

Problem 2. *Let $B \in \mathrm{Mat}_{n\times n}(F)$ be a nilpotent matrix. Show that $I_n + B$ is invertible.*

Problem 3. *Let F be a field, $A \in \mathrm{Mat}_{m\times n}(F)$ and $B \in \mathrm{Mat}_{n\times m}(F)$. Show that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.*

(Hint: Use the identity $A(I_n - BA) = (I_m - AB)A$. This is Problem M.10, page 36 of Artin's book "Algebra", edition 2.)

Problem 4. *Let F be a field, $A, B \in \mathrm{Mat}_{n\times n}(F)$. If $AB = A + B$ show that $AB = BA$.*

(Hint: consider $(A - I_n)(B - I_n)$.)

A matrix $A \in \mathrm{Mat}_{n\times n}(F)$ is called a permutation matrix if each row and each column has only one nonzero entry and that non-zero entry is 1. For example,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(F)$$

is a permutation matrix. A matrix $A = (a_{ij})_{1\le i,j\le n} \in \mathrm{Mat}_{n\times n}(F)$ is called upper triangular if $a_{ij} = 0$ for all i, j with $i > j$, namely, if each entry below the main diagonal is zero. Similarly, a matrix $A = (a_{ij})_{1\le i,j\le n} \in \mathrm{Mat}_{n\times n}(F)$ is called lower triangular if $a_{ij} = 0$ for all i, j with $i < j$. For example

$$\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(F)$$

is upper triangular. Here an entry with $*$ means that the value of that entry is not important. Moreover, it is easy to see that if a matrix $A \in \mathrm{Mat}_{n\times n}(F)$ is a row reduced echelon matrix, then A is upper triangular.

Problem 5. (1) *Show that each permutation matrix $A \in \mathrm{Mat}_{n\times n}(F)$ is invertible.*

(2) *Let* $A \in \mathrm{Mat}_{n\times n}(F)$ *be a matrix such that one can obtain its row reduced echelon matrix using just the first two types element row operations. In other words, we can reduce* A *to row-reduced echelon matrix using E.R.O without interchanging any two rows. Show that one can write* $A = LU$*, where* $L \in \mathrm{Mat}_{n\times n}(F)$ *is a lower triangular matrix and* $U \in \mathrm{Mat}_{n\times n}(F)$ *is an upper triangular matrix.*

(3) *Show that each matrix* $A \in \mathrm{GL}_2(F)$ *can be written as a product* $A = LPU$*, where* $P \in \mathrm{Mat}_{2\times 2}(F)$ *is a permutation matrix,* $L \in \mathrm{Mat}_{2\times 2}(F)$ *is lower triangular, and* $U \in \mathrm{Mat}_{2\times 2}(F)$ *is upper triangular.*

(4) *Show that each matrix* $A \in \mathrm{GL}_3(F)$ *can be written as a product* $A = LPU$*, where* $P \in \mathrm{Mat}_{3\times 3}(F)$ *is a permutation matrix,* $L \in \mathrm{Mat}_{3\times 3}(F)$ *is lower triangular, and* $U \in \mathrm{Mat}_{3\times 3}(F)$ *is upper triangular.*

(Actually, in part (3) and (4), there is no need to assume the matrix A is invertible. But the argument might be a little bit simpler when we add this condition.) Think about how to do part (3) and (4) for general $n \times n$ matrix. Namely, for any $A \in \mathrm{GL}_n(F)$, show that it can be written as a product $A = LPU$, where $P \in \mathrm{Mat}_{n\times n}(F)$ is a permutation matrix, $L \in \mathrm{Mat}_{n\times n}(F)$ is a lower triangular matrix,and $U \in \mathrm{Mat}_{n\times n}(F)$ is an upper triangular matrix. There is no need to submit a proof of this. But it is helpful to keep in mind this assertion. This decomposition of a matrix $A \in \mathrm{GL}_n(F)$ is called the Bruhat decomposition.