HOMEWORK 8

Due date: Monday of Week 9

Exercises: 2, 4, 5, 6, 7, 9, 10, 11, pages 366-367
Exercises: 3, 6, 7, 8, 17, pages 373-375.

Let V be a finite dimensional inner product space over $\mathbb{R}$. Note that the normal operator T on V is more complicate because χ_T is not necessary a product of linear factors, and thus T is not (orthogonally) diagonalizable over $\mathbb{R}$ in general. Actually, we have seen that a normal linear operator T is orthogonally diagonalizable if and only if it is self-adjoint. But many other properties of normal vectors still hold. Actually, the following are equivalent

1. T is normal, i.e., $TT^* = T^*T$;
2. $(T\alpha|T\beta) = (\alpha|\beta)$, for all $\alpha, \beta \in V$;
3. $||T\alpha|| = ||\alpha||$, for every $\alpha \in V$;
4. T_1 commutes with T_2, where $T_1 = \frac{T+T^*}{2}, T_2 = \frac{T-T^*}{2}$;
5. $T^* = TU$ for some orthogonal operator $U \in \mathrm{End}(V)$, where U is said to be orthogonal if $UU^* = I$ and U is a linear operator on real vector space;
6. U commutes with N, where $T = UN$ is the polar decomposition of T with N non-negative and U orthogonal;
7. there exists a polynomial $f \in \mathbb{R}[x]$ such that $T^* = f(T)$.

Many of the above equivalences were proved in class; the proofs of the rest are similar to the complex case. Notice the difference between real and complex case. Over the complex field, for a normal operator T, there exists a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal. Over the real field $\mathbb{R}$, this statement is no longer true. The simplest form of $[T]_{\mathcal{B}}$ is given in Theorem 17 and 18.

Problem 1. *Let V be a finite dimensional inner product space over $\mathbb{R}$. Let $T \in \mathrm{End}(V)$.*

1. *Show that T is normal if and only if there exists a polynomial $f \in \mathbb{R}[x]$ such that $T^* = f(T)$.*
2. *Consider the matrix*
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbb{R}).$$
Chek that A is normal. Moreover, find a polynomial $f \in \mathbb{R}[x]$ such that $A^t = f(A)$.
3. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by the matrix A. Find an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^3$ such that $[T]_{\mathcal{B}}$ is of the form*
$$\begin{bmatrix} c & & \\ & a & -b \\ & b & a \end{bmatrix},$$
for $a, b, c \in \mathbb{R}$.

Hint for (1), translate this to a problem on matrix and view the corresponding matrix in $\mathrm{Mat}_{n\times n}(\mathbb{C})$, then use the conclusion in the complex case. Then show the corresponding polynomial is in fact in $\mathbb{R}[x]$. For (2), just go through the proof of (1) and specialize everything to this example. Note that part (1) gives a different proof of Theorem 19, page 354.

1. Various decompositions

It is almost the end of linear algebra part of this course. It is a good time to review what we have learned about various decompositions of matrices, which are important parts of linear algebra. I will remind you those decompositions in the following, and it is helpful to keep a record of a proof for each decomposition. We learned all of these in class or from HW problems.

1.1. **Bruhat decomposition.** Let F be a general field. Denote by B_n the subset of $\mathrm{GL}_n(F)$ consisting of upper triangular invertible matrices with entries in F. Let $W \subset \mathrm{GL}_n(F)$ be the subset of permutation matrices. The set W was denoted by P in Recall that a matrix $g \in \mathrm{GL}_n(F)$ is called a permutation matrix if in each row and each column of g, there is only one nonzero term and that nonzero term is 1. Also recall that, we have a map

$$S_n \to W$$

$$\sigma \mapsto g_\sigma = \left[e_{\sigma(1)}, \dots, e_{\sigma(n)}\right],$$

where S_n is the symmetric group on n-elements which consists of bijections $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$, e_i is the column vector whose only nonzero entry is 1 and it is at the i-th position. See HW 3 and HW 10 of last year. Recall that we have

$$g_{\sigma\tau} = g_\sigma g_\tau.$$

We also consider the special element $w_\ell \in W$ defined by

$$w_\ell = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}$$

Problem 2. *Let A be a upper triangular matrix. Show that $w_\ell A w_\ell$ is lower triangular.*

Proposition 1 (Bruhat decomposition)**.** *For any element $g \in \mathrm{GL}_n(F)$, there exists $b_1, b_2 \in B$ and $w \in W$ such that $g = b_1 w b_2$. The elements b_1, b_2 are not unique in general, but $w \in W$ is uniquely determined by g. In other words, we have the decomposition*

$$\mathrm{GL}_n(F) = \coprod_{w \in W} BwB,$$

where $\coprod$ denotes disjoint union (which means $BwB \cap Bw'B = \emptyset$ if $w \neq w'$).

The decomposition $g = b_1 w b_2$ is equivalent to the LUP decomposition, which was in HW 3 of last year. We did not check the uniqueness in our HW.

Problem 3.
1. *Show that the Bruhat decomposition in Proposition 1 is equivalent to the LUP decomposition in Problem 5, HW3 of last year.*
2. *Prove the above Bruhat decomposition for $n = 2, 3, 4$ by proving the LUP decomposition first. Also check the uniqueness part for $w \in W$ in the decomposition for $n = 2, 3, 4$.*
3. *Given $g \in \mathrm{GL}_n(F)$. Show that g has an LU decomposition (which means $g = g_1 g_2$ for g_1 lower triangular and g_2 upper triangular) if and only if all of its principle minors are all different from zero.*
4. *Given $g \in \mathrm{GL}_n(F)$. Find a condition on g such that g has a decomposition $g = b_1 w_\ell b_2$ for $b_1, b_2 \in B$.*

Part (3) is a result from our textbook (Lemma, page 326). You don't need to submit a solution of this but you should know how to prove it. Part (3) is here because it gives you a hint for (4).

1.2. **C-R decomposition.**

Proposition 2 (C-R decomposition)**.** *Let $A \in \mathrm{Mat}_{m\times n}(F)$ be a matrix of rank r. Then there exists a matrix $C \in \mathrm{Mat}_{m\times r}(F)$ and a matrix $R \in \mathrm{Mat}_{r\times n}(F)$ such that $A = CR$.*

A special case of the above decomposition is when A has rank 1, then $A = uv$ for $u \in \mathrm{Mat}_{m\times 1}(F)$ and $v \in \mathrm{Mat}_{1\times n}(F)$. If $m = n$, then from $A = uv$, we can get that $A^2 = \mathrm{tr}(A)A$. The existence of the above C-R (which means column-row) decomposition was given in HW 5 of last year. Here is another related fact. Let k be a position integer with $k < r$, then there does not exist matrices $C \in \mathrm{Mat}_{m\times k}(F), R \in \mathrm{Mat}_{k\times n}(F)$ such that $A = CR$.

Problem 4. *Let $A \in \mathrm{Mat}_{m\times n}(F)$ be a matrix of rank r and let $A = CR$ be a C-R decomposition with $C \in \mathrm{Mat}_{m\times r}$ and $R = \mathrm{Mat}_{r\times n}$. For any $P \in \mathrm{GL}_r(F)$, if we denote $C' = CP \in \mathrm{Mat}_{m\times r}, R' = P^{-1}R \in \mathrm{Mat}_{r\times n}$, we $A = C'R'$ is another C-R decomposition. The question is: do we know all C-R decomposition has the above form? In other words, suppose that $A = CR = C'R'$ with $C, C' \in \mathrm{Mat}_{m\times r}, R, R' \in \mathrm{Mat}_{r\times n}$ such that*

$$A = CR = C'R'.$$

Is there a matrix $P \in \mathrm{GL}_r(F)$ such that $C' = CP$ and $R' = P^{-1}R$? If so, prove it. If not, find a counter-example.

This is certain uniqueness of C-R decomposition. If you think this hard, try to consider some examples with small m, n, r, for example, when $m = n = 3$ and $r = 2$.

1.3. **Jordan decomposition.**

Proposition 3 (Jordan decomposition)**.** *Let $A \in \mathrm{Mat}_{n\times n}(F)$ be a matrix such that μ_A is a product of linear factors. There exists a unique diagonalizable matrix $D \in \mathrm{Mat}_{n\times n}(F)$ and a unique nilpotent matrix $N \in \mathrm{Mat}_{n\times n}(F)$ such that $DN = ND$ and $A = D + N$.*

Moreover, we know that such D, N are polynomials of A. This is Theorem 13, page 222.

Proposition 4 (Jordan decomposition, semisimple version)**.** *Let F be a field of characteristic zero. Let $A \in \mathrm{Mat}_{n\times n}(F)$ be a matrix. Then there exists a unique semi-simple matrix $S \in \mathrm{Mat}_{n\times n}(F)$ and a unique nilpotent matrix $N \in \mathrm{Mat}_{n\times n}(F)$ such that $SN = NS$ and $A = S + N$.*

This is Theorem 13, page 267.

1.4. **Iwasawa decomposition.** Let $F = \mathbb{R}$ or $\mathbb{C}$. We consider the group $\mathrm{GL}_n(F)$. We still let B_n be the upper triangular matrices in $\mathrm{GL}_n(F)$. Let $K_n = \mathrm{O}_n(\mathbb{R})$ if $F = \mathbb{R}$ and let $K_n = \mathrm{U}(n)$ if $F = \mathbb{C}$.

Proposition 5 (Iwasawa decomposition)**.** *We have $\mathrm{GL}_n(F) = B_n \cdot K_n$. In other words, for any $g \in \mathrm{GL}_n(F)$, there exists an element $b \in B_n$ and an element $k \in K_n$ such that $g = bk$.*

This is equivalent to Theorem 14 of page 305. Explain the equivalence between the above Proposition and Theorem 14 of page 305.

Problem 5. *Consider the matrix*

$$g = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 9 \\ 4 & 7 & 11 \end{bmatrix} \in \mathrm{GL}_3(\mathbb{R}).$$

Find a matrix $b \in B_3$ and $k \in \mathrm{O}_3(\mathbb{R})$ such that $g = bk$.

1.5. **Singular value decomposition, polar decomposition and Cartan decomposition.** Let $F = \mathbb{R}$ or $\mathbb{C}$. We consider the group $\mathrm{GL}_n(F)$. We still let A_n be the set of all diagonal matrices in $\mathrm{GL}_n(F)$. Let $K_n = \mathrm{O}_n(\mathbb{R})$ if $F = \mathbb{R}$ and let $K_n = \mathrm{U}(n)$ if $F = \mathbb{C}$.

Proposition 6 (Cartan decomposition)**.** *We have $\mathrm{GL}_n(F) = K_n \cdot A_n \cdot K_n$. In other words, for any $g \in \mathrm{GL}_n(F)$, there exists $k_1, k_2 \in K_n$ and $a \in A_n$ such that $g = k_1 a k_2$.*

This is just a slightly different way to say the singular value decomposition.

Proposition 7 (Polar decomposition)**.** *For any $g \in \mathrm{GL}_n(F)$, there exists a matrix $k \in K_n$ and a positive matrix p such that $g = kp$.*

This is Theorem 14, page 342. Singular value decomposition and polar decomposition are closely related.

1.6. **Schur decomposition.** Let $F = \mathbb{C}$ and let $K_n = \mathrm{U}(n)$. Let $B_n \subset \mathrm{GL}_n(F)$ be the subset consisting of upper triangular matrices.

Proposition 8 (Schur decomposition)**.** *For any $g \in \mathrm{GL}_n(F)$, there exists an element $k \in K_n$ and an element $b \in B_n$ such that $g = kbk^{-1}$.*

This is Theorem 21, page 316. See also HW 6.