HOMEWORK 5

Due date: Oct 30, Monday of Week 10
Exercise: 6, 7, page 66;
Exercise: 5, 11, 12, 13, page 73-74;

In the following, we fix a field F.

Problem 1. *Let $A \in \mathrm{Mat}_{m\times n}(F)$ be a matrix of rank 1.*

(1) *Show that there is a matrix $u \in \mathrm{Mat}_{m\times 1}(F)$ and a matrix $v \in \mathrm{Mat}_{1\times n}(F)$ such that $A = uv$. (Here uv means the matrix product of u and v).*
(2) *If $m = n$, show that $A^2 = \mathrm{tr}(A)A$.*

The next problem is a generalization of the above one.

Problem 2. *Let $A \in \mathrm{Mat}_{m\times n}(F)$ be a matrix of rank k. Show that there is a matrix $C \in \mathrm{Mat}_{m\times k}(F), R \in \mathrm{Mat}_{k\times n}$ such that $A = CR$.*

(Hint: use elementary row operations. Here one can take R to be the nonzero rows of the RRE matrix of A. What can you say about C? For an integer p with $p < \mathrm{rank}(A)$, is it possible to find a matrices $C_1 \in \mathrm{Mat}_{m\times p}(F)$ and $R_1 \in \mathrm{Mat}_{p\times n}(F)$ such that $A = C_1R_1$? We will go back to this question later.)

Problem 3. *Let F be a field of characteristic 0. We view elements of F^3 as column vectors. Consider the matrix*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

and the linear operator $T : F^3 \to F^3$ given by $T(\alpha) = A\alpha$. Compute $\ker(T)$ and $\ker(T^2)$. Moreover, show that $\ker(T^k) = \ker(T)$ for any positive integer k. Here $T^2 = T \circ T$, and $T^k = T \circ T \circ \cdots \circ T$ (k-times).