HOMEWORK 12

Due date: Monday of Week 13

Exercises: 5.1, 5.2, 5.3, 5.6, 6.2, 6.3, 6.5, 6.6, 6.7, 6.10, 6.11, 8.1, 8.2, 8.6, 8.7, 8.8, 8.10, 8.12, 10.1, 10.2, 10.4, 11.4, 11.6, 11.8, 11.9, pages 72-76 of Artin's book

One important construction in group theory which is not covered in the textbook is *semidirect product*. We consider its definition here. Given a group N, recall that $\mathrm{Aut}(N)$ denotes the *group* of automorphisms of N. Its elements are $f : N \to N$ such that f is an isomorphism. For example, if $N = \mathbb{Z}^+$, the map $f : N \to N$ defined by $f(x) = -x$ is an automorphism. The group structure on $\mathrm{Aut}(N)$ is just composition.

Let H and N be two groups and let $\phi : H \to \mathrm{Aut}(N)$ be a group homomorphism. In particular, for each $h \in H$, $\phi(h) : N \to N$ is an automorphism. We now define a group $N \rtimes_\phi H$, which is called the (outer) semidirect product of N with H with respect to ϕ. As a set, $N \rtimes_\phi H$ is just the Cartesian product of N with H, namely, as a set $N \rtimes_\phi H = \{(n, h) | n \in N, h \in H\}$. The group operation $\bullet$ (product in the group) is defined by

$$(n_1, h_1) \bullet (n_2, h_2) = (n_1\phi(h_1)(n_2), h_1h_2), n_1, n_2 \in N, h_1, h_2 \in H.$$

Here recall that $\phi(h_1) : N \to N$ is an isomorphism, and thus $\phi(h_1)(n_2) \in N$. Note that if ϕ is the trivial homomorphism, namely, $\phi(h) = \mathrm{id}_N$ for every $h \in H$, then $N \rtimes_\phi H$ is just the direct product $N \times H$. Thus semidirect product is a generalization of product.

Problem 1. *Show that $N \rtimes_\phi H$ defined above is indeed a group. Moreover, consider the map $i_N : N \to N \rtimes_\phi H$ defined by $i_N(n) = (n, 1)$ and $i_H : H \to N \rtimes_\phi H$ defined by $i_H(h) = (1, h)$. Show that i_N, i_H are injective group homomorphisms. Furthermore, show that $i_N(N)$ is a normal subgroup of $N \rtimes_\phi H$.*

One might ask how the group structure of $N \rtimes_\phi H$ depends on ϕ.

Problem 2. *Let $f : H \to H$ be an automorphism and let $\phi_1 : H \to \mathrm{Aut}(N)$ be a group homomorphism. Consider $\phi_2 = \phi_1 \circ f : H \to \mathrm{Aut}(N)$. Show that $N \rtimes_{\phi_1} H \cong N \rtimes_{\phi_2} H$.*

Let n be a positive integer and let C_n denote the cyclic group of order n. We can realize $C_n \cong \mathbb{Z}/n\mathbb{Z}$ with addition as the group operation.

Problem 3. *Show that $\mathrm{Aut}(C_n) = \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. Here recall that*

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{a \in \mathbb{Z}/n\mathbb{Z} : \text{ there is } b \in \mathbb{Z}/n\mathbb{Z}, \text{ such that } , ab = 1\}.$$

If $n = 10$, this is Exercise 6.10 (a).

There are at least two elements ϕ_0, ϕ_1 in $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ defined by

$$\phi_0 = \mathrm{id}_{\mathbb{Z}/n\mathbb{Z}}; \phi_1(x) = -x, \forall x \in \mathbb{Z}/n\mathbb{Z}.$$

Consider the map

$$\phi : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$$

defined by $\phi(\overline{0}) = \phi_0, \phi(\overline{1}) = \phi_1$. It is clear that ϕ is a group homomorphism.

Problem 4. *Show that $\mathbb{Z}/n\mathbb{Z} \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$ is isomorphic to D_n, the dihedral group of order $2n$.*

Problem 5. *Let p, q be two primes.*

1. *If there exists a non-trivial group homomorphism $C_q \to \mathrm{Aut}(C_p)$, show that $q | (p-1)$;*
2. *Suppose $q | (p-1)$. Determine all group homomorphisms $C_q \to \mathrm{Aut}(C_p)$;*
3. *Suppose $q | (p-1)$. Let ϕ_1, ϕ_2 be two different group homomorphisms $C_q \to \mathrm{Aut}(C_p)$. Show that there exists an isomorphism $f : C_q \to C_q$ such that $\phi_2 = \phi_1 \circ f$.*

(4) *Suppose* $q|(p-1)$. *Conclude that there are only two isomorphism classes* $C_p \rtimes_\phi C_q$.

We now consider a special case of semidirect product. Suppose that N and H are both subgroups of a group G with $N \cap H = \{1\}$. Moreover, suppose that for any $h \in H$ and $n \in N$, we have $hnh^{-1} \in N$. If this condition is satisfied, we say that H normalizes N. Then we define

$$\phi : H \to \mathrm{Aut}(N)$$

by $\phi(h)(n) = hnh^{-1}$. Then we can form the semidirect product. $N \rtimes_\phi H$. In this case, we often drop ϕ from the notation, and write it as $N \rtimes H$.

Problem 6. *Show that there is an injective homomorphism* $N \rtimes H \to G$.

Hint: the map is just $(n, h) \to nh$.

We then identify $N \rtimes H$ as a subgroup of G. This is called the inner semidirect product of N and H.

Problem 7. *Suppose that* N, H *are two subgroups of* G. *Show that* $G = N \rtimes H$ *if and only if the following conditions hold.*

(1) N *is normal in* G;
(2) $G = NH$;
(3) $N \cap H = \{1\}$.

Compare this with Proposition 2.11.4, page 65.