HOMEWORK 13

Due date: Monday of Week 14

Exercises: 12.4, M.1, M.2, M.9, M.10, M.14.pages 75-77.

Exercises: 7.7, 7.8, 7.9, 7.10, 8.1, 8.2, 8.4, 11.1, 11.2, 11.3, 11.5, 11.8, M.7, pages 191-194.

The following problem is about semi-direct product and it should be in last HW. But there were too many problems in last HW.

Problem 1. *Show that the quaternion group H defined in (2.4.5), page 47 of Artin's book is not a semidirect product of its two proper subgroups.*

Problem 2. *Determine the order of the group $\mathrm{GL}_n(\mathbb{F}_p)$, where p is a prime number.*

Hint: Consider the action of $\mathrm{GL}_n(\mathbb{F}_p)$ on $\mathbb{F}_p^n$ by left multiplication.

Problem 3. *Let G be a finite group, H be a subgroup of G. Let $C \subset G$ be a conjugacy class and suppose*

$$H \cap C = \coprod_{i=1}^{r} D_i,$$

where each D_i is a conjugacy class of H. Consider the set

$$X_i = \left\{(c, g) \in C \times G : g^{-1}cg \in D_i\right\}.$$

Express $|X_i|$ in terms of $|G|, |H|, |D_i|$.

Hint: Consider the group action $G \times X_i \to X_i$ defined by $x.(c, g) = (xcx^{-1}, xg)$.

Problem 4. *Let $G = D_4 = \left\{1, x, x^2, x^3, y, xy, x^2y, x^3y\right\}$ with $x^4 = 1 = y^2, yxy^{-1} = x^3$ and $H = \left\{1, x^2, y, x^2y\right\} \subset G$. Find all conjugacy classes C of G, and for each conjugacy class C of G, decompose $C \cap H$ into conjugacy classes of H.*

Problem 5. *Let $G = \mathrm{GL}_2(\mathbb{F}_p), H = \mathrm{SL}_2(\mathbb{F}_p) = \{g \in G : \det(g) = 1\}$. Let $C \subset G$ be the conjugacy class of the element $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Namely,*

$$C = \left\{gug^{-1} : g \in G\right\}.$$

Try to decompose $C \cap H$ into conjugacy classes of H.

The next several problems are about double cosets, and most of them could be in last HW.

Problem 6. *Let F be a field and let $B_n(F) \subset \mathrm{GL}_n(F)$ be the upper triangular subgroup.*

1. *Determine the double cosets $B_2(F)\backslash\mathrm{GL}_2(F)/B_2(F)$.*
2. *How about $B_n(F)\backslash\mathrm{GL}_n(F)/B_n(F)$?*

This problem might be hard. It is related to the UPL (upper triangular, permutation subgroup, and lower triangular subgroup) decomposition of a matrix, See HW 3, Problem 5 of last year. If you don't know how to do the general problem, try the case when $n = 2$ and $F = \mathbb{F}_2$ (or $\mathbb{F}_3$).

Let $G \times X \to X$ be an action of a group G on a set X. Recall that $G\backslash X$ denote the set of orbits.

Problem 7. *Let G be a group and H, K are subgroups of G. Show the following basic properties of double cosets.*

(1) *For $x \in G$, the double coset HxK is a union of right H-cosets and a union of left K-cosets. More precisely,*

$$HxK = \coprod_{Hxk \in H\backslash HxK} Hxk = \coprod_{hxK \in HxK/K} hxK.$$

(2) *Let G act on the left cosets G/K from the left by $x.(gK) = (xg)K$. See Section 6.8 of Artin. We restrict this action to H and consider the action*

$$H \times G/K \to G/K$$

defined by $(h, gK) = (hg)K$. Show that there is a bijection between the double coset $H\backslash G/K$ and the set of orbits $H\backslash(G/K)$. This explains that the notation is consistent. There is a similar statement when we switch the role of H and K.

(3) *Suppose that all groups are finite. For $x \in G$, show that*

$$|HxK| = [H : H \cap xKx^{-1}]|K| = [K : K \cap x^{-1}Hx]|H|.$$

(4) *Show that*

$$[G : H] = \sum_{HxK \in H\backslash G/K} [K : K \cap x^{-1}Hx]$$

and

$$[G : K] = \sum_{HxK \in H\backslash G/K} [H : H \cap xKx^{-1}].$$

(5) *Consider the group action of $(H \times K)$ on G defined by*

$$((h, k), g) = hgk^{-1}, (h, k) \in H \times K, g \in G.$$

Check that this is a group action and there is a bijection between $H\backslash G/K$ and the orbits of this action.

(6) *Let $G^{(h,k)} = \{g \in G : hgk = g\}$. Show that*

$$|H\backslash G/K| = \frac{1}{|H||K|} \sum_{(h,k) \in H \times K} |G^{(h,k)}|.$$

For the last one, use Ex. M.7, page 194 of Artin. The other parts are routine.

The next problem is covered in a previous class. You don't have to submit it. But if you cannot remember it or don't know how to do it at this moment, think about it.

Problem 8. *Let $n > 1$ be a positive integer and consider the group $\mathrm{SO}_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) : gg^t = I_n, \det(g) = 1\}$. Consider the subgroup H of $\mathrm{SO}_n(\mathbb{R})$ defined by*

$$H = \left\{ \begin{bmatrix} h & \\ & 1 \end{bmatrix}, h \in \mathrm{SO}_{n-1}(\mathbb{R}) \right\}.$$

Show that there is a bijection

$$G/H \cong S^{n-1},$$

where $S^{n-1} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\right\}$, which is the standard $(n-1)$-sphere. Similarly, we consider the group $\mathrm{SU}_n = \{g \in \mathrm{GL}_n(\mathbb{C}) : gg^* = I_n, \det(g) = 1\}$. We view SU_{n-1} as a subgroup of SU_n via the embedding

$$h \mapsto \begin{bmatrix} h & \\ & 1 \end{bmatrix}, h \in \mathrm{SU}_{n-1}.$$

Show that there is a bijection

$$\mathrm{SU}_n/\mathrm{SU}_{n-1} \cong S^{2n-1}.$$

The next problem is similar to problem 2. You don't have to submit it. But you are encouraged to do it.

Problem 9. *Let p be a prime number and n be a positive integer. Consider the group*

$$\mathrm{SO}_n(\mathbb{F}_p) = \left\{g \in \mathrm{GL}_n(\mathbb{F}_p) |\ gg^t = I_n, \det(g) = 1\right\}.$$

Compute the order of $\mathrm{SO}_n(\mathbb{F}_p)$.

The group $\mathrm{SO}_n(\mathbb{F}_p)$ is still the group which preserve a symmetric bilinear form on vector spaces over $\mathbb{F}_p$. But this time, this bilinear form is not an inner product. Inner product only defined on vector spaces over $\mathbb{R}$ or $\mathbb{C}$, while bilinear form can be defined over any fields.