

## HOMEWORK 12

Due date: Week 13

Exercises: 10.2, 10.7, 10.8, 10.9, 11.5, 12.3, 12.7 pages 509-511

For a solution of Ex 12.3, see [this link](#).

Let  $f \in F[x]$  be an irreducible polynomial of degree 5. It is known that  $G_f$  is a transitive subgroup of  $S_5$ . We classified transitive subgroups of  $S_4$  in last class. Transitive subgroups of  $S_5$  can also be classified, which are given by (up to conjugation)

- (1)  $C_5$ : generated by (12345);
- (2)  $D_{10}$ : generated by (12345) and (25)(34);
- (3)  $(\mathbb{Z}/5) \rtimes (\mathbb{Z}/5)^\times$ : generated by (12345) and (2354);
- (4)  $A_5$ ;
- (5)  $S_5$ .

Note that  $(\mathbb{Z}/5) \rtimes (\mathbb{Z}/5)^\times$  has a unique group structure because there is a unique nontrivial group homomorphism  $(\mathbb{Z}/5)^\times \rightarrow \text{Aut}(\mathbb{Z}/5)$  up to conjugation. See HW12 of 2024. Moreover, we know that  $|(\mathbb{Z}/5) \rtimes (\mathbb{Z}/5)^\times| = 20$ . Moreover, the subgroups of  $A_5$  are  $C_5, D_{10}$  and  $A_5$  itself.

See [this link](#) for the above classification. This is also Exercise 12.2, page 510.

We have seen in class that there exists an irreducible polynomial  $f \in \mathbb{Q}[x]$  such that  $G_f \cong S_5$ .

**Problem 1.** Consider the polynomial  $f = x^5 - 2 \in \mathbb{Q}[x]$ . Show that  $G_f \cong (\mathbb{Z}/5) \rtimes (\mathbb{Z}/5)^\times$ .

This should be easy by counting degree.

**Problem 2.** Let  $\zeta_{11} = \exp\left(\frac{2\pi\sqrt{-1}}{11}\right)$ . Let  $\alpha = \zeta_{11} + \zeta_{11}^{-1}$ . Find the minimal polynomial  $f$  of  $\alpha$  over  $\mathbb{Q}$  and show that  $G_f \cong C_5$ .

**Problem 3.** Consider the polynomial  $f = x^5 + ax + b$  with  $a \geq 0$ . Show that  $f$  has at least one complex root. Moreover, if  $f$  is irreducible, show that  $G_f$  cannot be  $C_5$ .

Hint: for the first part, use calculus. For the second part, notice that the complex conjugation is in the Galois group.

It is complicate to determine exactly what the Galois group of an irreducible quintic polynomial is. See [this article](#) for more details.

### 1. QUADRATIC RECIPROCITY LAW

Let  $p$  be an odd prime. Let  $\mathbb{F}_p^{\times,2} = \{x^2 : x \in \mathbb{F}_p^\times\}$ . For any  $x \in \mathbb{F}_p^\times$ , we define

$$\left(\frac{x}{p}\right) = \begin{cases} 1, & \text{if } x \in \mathbb{F}_p^{\times,2}, \\ -1, & \text{if } x \in \mathbb{F}_p^\times - \mathbb{F}_p^{\times,2}. \end{cases}$$

For any  $a \in \mathbb{Z}$ , if  $(a, p) = 1$ , we define

$$\left(\frac{a}{p}\right) := \left(\frac{\bar{a}}{p}\right),$$

where  $\bar{a}$  is the image of  $a$  under the natural map  $\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ .

**Problem 4.** Consider the map  $\chi_p : \mathbb{F}_p^\times \rightarrow \mathbb{C}^\times$  defined by

$$\chi_p(x) = \left(\frac{x}{p}\right).$$

Show that  $\chi_p$  is a nontrivial quadratic character. Show that  $\mathbb{F}_p^{\times,2}$  is the unique subgroup of  $\mathbb{F}_p^\times$  of index 2. Moreover, if  $\chi : \mathbb{F}_p^\times \rightarrow \mathbb{C}^\times$  is a nontrivial character, show that  $\chi = \chi_p$ .

Here are some terminologies used in the last problem. A character  $\chi : \mathbb{F}_p^\times \rightarrow \mathbb{C}^\times$  is just a group homomorphism (namely,  $\chi(ab) = \chi(a)\chi(b)$ ). A character is called quadratic if  $\chi(x)^2 = 1$  for any  $x \in \mathbb{F}_p^\times$ .

**Theorem 1.1** (Quadratic reciprocity law). *Let  $p, q$  be two distinct odd primes. Then*

- (1)  $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$ ;
- (2)  $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$ ;
- (3)  $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$ .

We proved the above (1) and (3) in class. We recall the proof of (3) in the following. Denote  $\zeta_p = e^{2\pi\sqrt{-1}/p}$ . Recall that

$$\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong \mathbb{F}_p^\times,$$

and the isomorphism is given as follows. For  $k \in \mathbb{F}_p^\times$ , define  $\sigma_k \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$  by  $\sigma_k(\zeta_p) = \zeta_p^k$ . Then the isomorphism is given by  $\sigma_k \mapsto k$ . It is indeed well-defined and is an isomorphism. Moreover, consider

$$S = \sum_{a \in \mathbb{F}_p^\times} \left(\frac{a}{p}\right) \zeta_p^a.$$

This is called a Gauss sum. We computed that  $S^2 = \left(\frac{-1}{p}\right)p$ . Consider the field  $L = \mathbb{Q}(S)$ . Since  $S^2 \in \mathbb{Q}$ , we know that  $[L : \mathbb{Q}] \leq 2$ . But  $S \notin \mathbb{Q}$  and thus  $[L : \mathbb{Q}] = 2$ . Moreover, we have  $S \in \mathbb{Q}(\zeta_p)$  by definition. Thus we have the tower fields

$$\mathbb{Q} \subset L = \mathbb{Q}(S) \subset K := \mathbb{Q}(\zeta_p).$$

Thus we have the exact sequence

$$1 \rightarrow \text{Gal}(K/L) \rightarrow \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Gal}(L/\mathbb{Q}) \rightarrow 1.$$

Denote the projection map  $\text{Gal}(K/\mathbb{Q}) \rightarrow \text{Gal}(L/\mathbb{Q})$  by  $\pi$ . Thus we have  $[K : L] = \frac{p-1}{2}$  and  $\text{Gal}(K/L) \cong \mathbb{F}_p^{\times,2}$ . Consider the element  $\sigma_q \in \text{Gal}(K/\mathbb{Q})$  defined by  $\sigma_q(\zeta_p) = \zeta_p^q$ . By the description of the isomorphism  $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong \mathbb{F}_p^\times$ , we see that

$$q \in \mathbb{F}_p^{\times,2} \text{ iff } \sigma_q \in \text{Gal}(K/L) \text{ iff } \sigma_q \in \text{Ker}(\pi) \text{ iff } \sigma_q|_L = 1 \text{ iff } \sigma_q(S) = S.$$

This implies that  $\sigma_q(S) = \left(\frac{q}{p}\right)S$ . The above (3) follows from this equation easily. The proof of (2) is simpler. It is given in the following.

**Problem 5.** *Let  $\zeta = e^{2\pi\sqrt{-1}/8}$  and  $\tau = \zeta + \zeta^{-1}$ .*

- (1) *Show that  $\tau^2 = 2$ .*
- (2) *Show that there exists an element  $\alpha \in \mathbb{Z}[\zeta]$  such that  $\tau^p - \left(\frac{2}{p}\right)\tau = p\alpha$ .*
- (3) *Show that there exists an element  $\beta \in \mathbb{Z}[\zeta]$  such that  $\tau^p - (\zeta^p + \zeta^{-p}) = p\beta$ .*
- (4) *If  $p \equiv \pm 1 \pmod{8}$ , show that  $\zeta^p + \zeta^{-p} = \tau$  and  $\left(\frac{2}{p}\right) = 1$ .*
- (5) *If  $p \equiv \pm 3 \pmod{8}$ , show that  $\zeta^p + \zeta^{-p} = -\tau$  and  $\left(\frac{2}{p}\right) = -1$ .*

The last two parts are equivalent to  $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$ .

**Problem 6.** *Let  $p = 1093$ , which is known to be a prime number. Determine whether 516 is in  $\mathbb{F}_p^{\times,2}$  or not.*

**Problem 7.** *Let  $x$  be an integer. Show that any prime divisor of  $x^4 - x^2 + 1$  is congruent to 1 modulo 12.*

Hint: Use quadratic reciprocity law and the identities  $(x^2 - 1)^2 = (x^4 - x^2 + 1) - x^2$  and  $(2x^2 - 1)^2 = 4(x^4 - x^2 + 1) - 3$ .

**Problem 8.** (1) *Determine all prime numbers  $p$  such that  $3 \in \mathbb{F}_p^{\times,2}$ .*

(2) *Determine all prime numbers  $p$  such that  $7 \in \mathbb{F}_p^{\times,2}$ .*