

## HOMEWORK 12

Due date: Monday of Week 13

Exercises: 5.1, 5.2, 5.3, 5.6, 6.2, 6.3, 6.5, 6.6, 6.7, 6.10, 6.11, 8.1, 8.2, 8.6, 8.7, 8.8, 8.10, 8.12, 10.1, 10.2, 10.4, 11.4, 11.6, 11.8, 11.9, pages 72-76 of Artin's book

One important construction in group theory which is not covered in the textbook is *semidirect product*. We define it here. Given a group  $N$ , recall that  $\text{Aut}(N)$  denotes the group of all automorphisms of  $N$ . It is consisting of all  $f : N \rightarrow N$  such that  $f$  is an isomorphism. For example, if  $N = \mathbb{Z}^+$ , the map  $f : N \rightarrow N$  defined by  $f(x) = -x$  is an automorphism. The group structure on  $\text{Aut}(N)$  is just composition.

Let  $H$  and  $N$  be two groups and let  $\phi : H \rightarrow \text{Aut}(N)$  be a group homomorphism. In particular, for each  $h \in H$ ,  $\phi(h) : N \rightarrow N$  is an automorphism. We now define a group  $N \rtimes_{\phi} H$ , which is called the (outer) semidirect product of  $N$  with  $H$  with respect to  $\phi$ . As a set,  $N \rtimes_{\phi} H$  is just the Cartesian product of  $N$  with  $H$ , namely, as a set  $N \rtimes_{\phi} H = \{(n, h) | n \in N, h \in H\}$ . The group operation  $\bullet$  (product in the group) is defined by

$$(n_1, h_1) \bullet (n_2, h_2) = (n_1 \phi(h_1)(n_2), h_1 h_2), n_1, n_2 \in N, h_1, h_2 \in H.$$

Here recall that  $\phi(h_1) : N \rightarrow N$  is an isomorphism, and thus  $\phi(h_1)(n_2) \in N$ . Note that if  $\phi$  is the trivial homomorphism, namely,  $\phi(h) = \text{id}_N$  for every  $h \in H$ , then  $N \rtimes_{\phi} H$  is just the direct product  $N \times H$ . Thus semidirect product is a generalization of product.

**Problem 1.** Show that  $N \rtimes_{\phi} H$  defined above is indeed a group. Moreover, consider the map  $i_N : N \rightarrow N \rtimes_{\phi} H$  defined by  $i_N(n) = (n, 1)$  and  $i_H : H \rightarrow N \rtimes_{\phi} H$  defined by  $i_H(h) = (1, h)$ . Show that  $i_N, i_H$  are injective group homomorphisms. Furthermore, show that  $i_N(N)$  is a normal subgroup of  $N \rtimes_{\phi} H$ .

One might ask how the group structure of  $N \rtimes_{\phi} H$  depends on  $\phi$ .

**Problem 2.** Let  $f : H \rightarrow H$  be an automorphism and let  $\phi_1 : H \rightarrow \text{Aut}(N)$  be a group homomorphism. Consider  $\phi_2 = \phi_1 \circ f : H \rightarrow \text{Aut}(N)$ . Show that  $N \rtimes_{\phi_1} H \cong N \rtimes_{\phi_2} H$ .

Let  $n$  be a positive integer and let  $C_n$  denote the cyclic group of order  $n$ . We can realize  $C_n \cong \mathbb{Z}/n\mathbb{Z}$  with addition as the group operation.

**Problem 3.** Show that  $\text{Aut}(C_n) = \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}$ . Here recall that

$$(\mathbb{Z}/n\mathbb{Z})^{\times} = \{a \in \mathbb{Z}/n\mathbb{Z} : \text{there is an element } b \in \mathbb{Z}/n\mathbb{Z}, \text{ such that } ab = 1\}.$$

The group structure on  $(\mathbb{Z}/n\mathbb{Z})^{\times}$  is multiplication.

If  $n = 10$ , this is Exercise 6.10 (a).

**Problem 4.** Let  $p, q$  be two primes.

- (1) If there exists a non-trivial group homomorphism  $C_q \rightarrow \text{Aut}(C_p)$ , show that  $q|(p-1)$ ;
- (2) Suppose  $q|(p-1)$ . Determine all group homomorphisms  $C_q \rightarrow \text{Aut}(C_p)$ ;
- (3) Suppose  $q|(p-1)$ . Let  $\phi_1, \phi_2$  be two different group homomorphisms  $C_q \rightarrow \text{Aut}(C_p)$ . Show that there exists an isomorphism  $f : C_q \rightarrow C_q$  such that  $\phi_2 = \phi_1 \circ f$ .
- (4) Suppose  $q|(p-1)$ . Conclude that there are only two isomorphism classes  $C_p \rtimes_{\phi} C_q$ .

We now consider a special case of semidirect product. Suppose that  $N$  and  $H$  are both subgroups of a group  $G$  with  $N \cap H = \{1\}$ . Moreover, suppose that for any  $h \in H$  and  $n \in N$ , we have  $hnh^{-1} \in N$ . If this condition is satisfied, we say that  $H$  normalizes  $N$ . Then we define

$$\phi : H \rightarrow \text{Aut}(N)$$

by  $\phi(h)(n) = hnh^{-1}$ . Then we can form the semidirect product.  $N \rtimes_{\phi} H$ . In this case, we often drop  $\phi$  from the notation, and write it as  $N \rtimes H$ .

**Problem 5.** *Show that there is an injective homomorphism  $N \rtimes H \rightarrow G$ .*

Hint: the map is just  $(n, h) \rightarrow nh$ .

We then identify  $N \rtimes H$  as a subgroup of  $G$ . This is called the inner semidirect product of  $N$  and  $H$ .

**Problem 6.** *Suppose that  $N, H$  are two subgroups of  $G$ . Show that  $G = N \rtimes H$  if and only if the following conditions hold.*

- (1)  $N$  is normal in  $G$ ;
- (2)  $G = NH$ ;
- (3)  $N \cap H = \{1\}$ .

Compare this with Proposition 2.11.4, page 65.

**Problem 7.** *Show that the quaternion group  $H$  defined in (2.4.5), page 47 of Artin's book is not a semidirect product of its two proper subgroups.*