

HOMEWORK 4

Due date: Monday of Week 5

Exercises: 2, 3, 4, pages 261-262, Hoffman-Kunze,

Exercises: 1, 2, 3, page 269.

Exercises: 5, 7, 8, 9, 12, 13, 14, 16, 17. pages 276-277.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix} \in \text{Mat}_{3 \times 3}(\mathbb{Q}).$$

Find the Smith normal form of $xI_3 - A$, the invariant factors of A and the rational canonical form of A . Determine if A has a Jordan canonical form. If so, find its Jordan canonical form.

Problem 2. Consider the matrix

$$A = \begin{bmatrix} x-1 & & \\ & (x-1)^2 & \\ & & x-2 \end{bmatrix} \in \text{Mat}_{3 \times 3}(\mathbb{Q}[x]).$$

Find the Smith normal form of A .

Problem 3. Let F be a general field and $A \in \text{Mat}_{n \times n}(F)$. Show that A is similar to A^t .

Problem 4. Let $\alpha = \sqrt[3]{2}$. Let $F = \{a + b\alpha + c\alpha^2 \mid a, b, c \in \mathbb{Q}\}$. We view F as a dimension 3 vector space over \mathbb{Q} . Let $\mathcal{B} = [1, \alpha, \alpha^2]$, which is an ordered basis of F over \mathbb{Q} . Given an element $x \in F$, we consider the linear operator $T_x : F \rightarrow F$ defined by $T_x(y) = xy$. Compute the matrix $A_x = [T_x]_{\mathcal{B}} \in \text{Mat}_{3 \times 3}(\mathbb{Q})$ for $x = a + b\alpha + c\alpha^2$. Show that T_x is a semi-simple operator when F is viewed as vector space over \mathbb{Q} .

Problem 5. Let $A \in \text{Mat}_{n \times n}(\mathbb{Q})$ be a matrix such that $A^3 - 2I_n = 0$, where $I_n \in \text{Mat}_{n \times n}(\mathbb{Q})$ is the identity matrix. Show that $3 \mid n$ (3 divides n). Write $n = 3k$ for a positive integer k . Show that A is similar the matrix

$$\begin{bmatrix} & 2I_k \\ I_k & \\ & I_k \end{bmatrix}.$$

Problem 6. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \text{Mat}_{4 \times 4}(\mathbb{Q}).$$

Find a semisimple matrix $S \in \text{Mat}_{4 \times 4}(\mathbb{Q})$ and a nilpotent matrix $N \in \text{Mat}_{4 \times 4}(\mathbb{Q})$ such that $A = S + N$ and $SN = NS$.

Hint: you can repeat the proof of Theorem 13, page 267, in this special case. It is useful to notice that the characteristic polynomial χ_A of A is $(x^2 - 2)^2$. Moreover, $x^2 - 2$ is irreducible over \mathbb{Q} . Here a matrix S is called semi-simple if the linear operator defined by S is semi-simple, or equivalently, if the minimal polynomial μ_S of S is a product of irreducible polynomials of multiplicity one.

Problem 7. Let $V = \text{Mat}_{n \times n}(\mathbb{C})$. Define a map $(\mid) : V \rightarrow V \times \mathbb{C}$ by

$$(A \mid B) = (\text{tr}(AB^*),$$

Check that (\mid) is an inner product on V .

Let $T : V \rightarrow V$ be a linear operator. Assume that $\mu_T = q_1^{r_1} \dots q_k^{r_k}$ with q_i irreducible and pairwise distinct. To find a basis \mathcal{B} such that $[T]_{\mathcal{B}}$ is simpler, we usually do primary decomposition first and then apply cyclic decomposition to each component. More precisely, we have the primary decomposition

$$V = W_1 \oplus \dots \oplus W_k,$$

with $W_i = \ker(q_i(T)^{r_i})$. Let $T_i : W_i \rightarrow W_i$ be the restriction of T to W_i . Then we have $\mu_{T_i} = q_i^{r_i}$. To find a simpler form of T_i , we need to find the invariant factors of T_i . To do so, we need to do the primary decomposition first, find the matrix $A_i = [T]_{\mathcal{B}_i}$ of T_i explicitly and then find the Smith norm form of $xI - A_i$. This seems very complicate. Now a question is: is it possible to read the invariant factors of T_i directly from the invariant factors of T ? The answer is yes. Suppose that p_1, \dots, p_m are invariant factors of T . Then we have $p_1 = \mu_T = q_1^{r_1} \dots q_k^{r_k}$. Since $p_i | p_1$ for $i \geq 2$, we can assume that

$$\begin{aligned} p_1 &= q_1^{s_{11}} q_2^{s_{12}} \dots q_k^{s_{1k}}, \\ p_2 &= q_1^{s_{21}} q_2^{s_{22}} \dots q_k^{s_{2k}}, \\ &\dots \\ p_m &= q_1^{s_{m1}} q_2^{s_{m2}} \dots q_k^{s_{mk}}, \end{aligned}$$

where s_{ij} are non-negative integers with $s_{ij} \geq s_{i+1,j}$, and $s_{1j} = r_j$.

Problem 8. (1) Given $f, g \in F[x]$ and $\gcd(f, g) = 1$. Show that

$$F[x]/(fg) \cong F[x]/(f) \times F[x]/(g).$$

(2) Using the above and the uniqueness of cyclic decompositions, show that the invariant factors of T_j are

$$q_j^{s_{1j}}, q_j^{s_{2j}}, \dots, q_j^{s_{mj}}.$$

It is possible that many s_{ij} in the above sequence are zero and thus the corresponding term can be disregarded.

Problem 9. Let $T : V \rightarrow V$ be a linear operator such that its invariant factors are given by

$$(x-2)^4(x+1), (x-2)^2(x+1), (x-2).$$

Find the corresponding invariant factors of T_1 and T_2 , where $W_1 = \ker(T - 2I)^4$, $W_2 = \ker(T + I)$ and $T_i : W_i \rightarrow W_i$ is the corresponding linear operator defined by T_i . Moreover find the Jordan canonical form of T .