HOMEWORK 7

Due date: Nov 13, Monday of Week 12
Exercises: 1, 8, 9, 12, pages 105-107
Exercises: 1, 3, page 111
Exercises: 1, 3, 4, 6, 7, 8, pages 115-116.

You can do Problems 1 and 2 using what you learned from last week.

Problem 1. *Given two matrices $A, B \in \mathrm{Mat}_{m\times n}(F)$. Show that* $\mathrm{rank}(A+B) \leq \mathrm{rank}(A)+\mathrm{rank}(B)$.

Hint: Translate this problem to a problem on linear maps. You might find the following fact useful: Given two linear maps $T, S : V \to W$, then $\ker(T) \cap \ker(S) \subset \ker(T + S)$ (check this).

Problem 2. *Given two matrices $A, B \in \mathrm{Mat}_{n\times n}(F)$. If $AB = 0$, show that* $\mathrm{rank}(A)+\mathrm{rank}(B) \leq n$.

I know this follows from Sylverster's rank inequality (Problem 3, HW6). But it is a good exercise to prove this directly (without going through the whole proof of Sylverster, because this problem is much easier.) Hint: Think about what can you say about $\mathrm{Im}(T_B)$ and $\mathrm{Ker}(T_A)$. (As usual: T_A denotes the linear map $F^n \to F^n$ given by $T_A(X) = AX$. Here elements in F^n are viewed as column vectors.)

Exercise 16 page 107 gives a "coordinate free" definition of trace of a square matrix. Please keep in mind this assertion. In Chapter 5, we will see a "coordinate free" definition of determinant. You can find a proof in this link of Ex 16, page 107. But the proof given there uses heavy computations. In the following problem, we will have a more conceptual proof. Basically, we do Exercise 17, page 107 first and we show Ex 17 implies Ex 16 easily.

Problem 3. *Let F be a field and let $V = \mathrm{Mat}_{n\times n}(F)$, which is an F-vector space of dimension n^2. We consider the trace map $\mathrm{Tr} : V \to F$. Let W be the subspace of V which is spanned by the matrices of the form $AB - BA$ for $A, B \in V$. Then we know that $W \subset \ker(\mathrm{Tr})$. This space W is exactly the one in Ex 17, page 107.*

(1) *Show that $\mathrm{Tr} : V \to F$ is surjective and conclude that $\dim\ker(\mathrm{Tr}) = n^2 - 1$.*
(2) *Show that $\dim W = n^2 - 1$ by explicitly constructing enough linearly independent elements in W. (See the solution of Ex 17, page 107 given in the above link). Conclude that $W = \ker(\mathrm{Tr})$.*
(3) *Show that a linear functional $f : V \to F$ such that $f(AB) = f(BA)$ for all $A, B \in F$ is exactly an element in $\mathrm{Hom}_F(V/W, F)$. Conclude that such an f must be of the form $c\mathrm{Tr}$ for some $c \in F$.*

Hint for part (2) if you don't check the solution given online. We expect that $W = \mathrm{Ker}(\mathrm{tr})$. Try to find a natural basis of $\ker(\mathrm{tr})$ (if you still have no idea, try to think about the 2×2 matrices and then 3×3 matrices), and then try to show that they are indeed in W.

You should be able to do Problem 3 after Monday's class. Do the rest problems (except the last one) after the Friday's class.

Problem 4. *Given $V, W \in \mathrm{Vect}_F$ such that $\dim V, \dim W$ are finite. Let $T : W \to V$ be a linear operator.*

(1) *Given a linear functional $f \in W^*$ such that $f|_{\ker(T)} = 0$, show that there exists a linear functional $g \in V^*$ such that $g(T(\alpha)) = f(\alpha)$ for any $\alpha \in W$.*
(2) *If T is injective, conclude that T^t is surjective.*

Part (1) is a slightly variant/generalization of Ex 12, page 106.

Problem 5. *Let V, W be two finite dimensional vector spaces over F. Suppose $\dim_F V = n, \dim_F W = m$.*

(1) *Show that the map $\theta : \mathrm{Hom}_F(V, W) \to \mathrm{Hom}_F(W^*, V^*)$ defined by $\theta(T) = T^t$ is an isomorphism.*
(2) *Conclude that there is an isomorphism $\mathrm{Hom}(V, W) \to (V^*)^m$. Construct this isomorphism explicitly using Problem 1, HW6.*

(Comment: Part (1) is a generalization of Ex. 7, page 116.)

We can prove the assertion in part (2) of Problem 3 even we drop the condition that $\dim V$ is finite.

Problem 6. *Given two F-vector spaces V, W with $\dim_F W = m$ is finite. We don't require $\dim_F V$ is finite. Let $\{\beta_1, \ldots, \beta_m\}$ be a basis of W and let $S = \{f_1, \ldots, f_m\}$ be the dual basis of W^*. Consider the map*

$$\theta_S : \mathrm{Hom}(V, W) \to (V^*)^m,$$
$$\theta_S(T) = (T^t(f_1), \ldots, T^t(f_m)).$$

Show that θ_S is an isomorphism.

(Hint: The proof is not hard.) The above assertion is a slightly generalization of Ex.6 page 105. Actually Exercise 6 of page 105 gives an inverse map of the one defined above (for a specific choice of S). Try to explain this. Moreover, compare the result in Problem 6 with Problem 1 of HW6.

Problem 7. *Let V be a vector space over F and let $T : V \to V$ be a linear map. Suppose $\ker(T) = \ker(T^2)$. Show that $\ker(T^k) = \ker(T)$.*

Comment: This problem was supposed to be the last problem of HW5. But somehow there was a typo in that problem which made it much easier than that it was supposed to be. Now do this problem here. This problem has nothing to do with dual, double dual and transpose (the materials which were covered in the last 3 sections of Chapter 3). You can do this problem from what your learned from last week.