

HOMEWORK 13

Due date: Monday of Week 14

Exercises: 12.4, M.1, M.2, M.9, M.10, M.14. pages 75-77.

Exercises: 7.7, 7.8, 7.9, 7.10, 8.1, 8.2, 8.4, 11.1, 11.2, 11.3, 11.5, 11.8, M.7, pages 191-194.

The following problem is about semi-direct product and it should be in last HW. But there were too many problems in last HW.

Problem 1. Show that the quaternion group H defined in (2.4.5), page 47 of Artin's book is not a semidirect product of its two proper subgroups.

Problem 2. Determine the order of the group $\mathrm{GL}_n(\mathbb{F}_p)$, where p is a prime number.

Hint: Consider the action of $\mathrm{GL}_n(\mathbb{F}_p)$ on \mathbb{F}_p^n by left multiplication.

Problem 3. Let G be a finite group, H be a subgroup of G . Let $C \subset G$ be a conjugacy class and suppose

$$H \cap C = \coprod_{i=1}^r D_i,$$

where each D_i is a conjugacy class of H . Consider the set

$$X_i = \{(c, g) \in C \times G : g^{-1}cg \in D_i\}.$$

Express $|X_i|$ in terms of $|G|, |H|, |D_i|$.

Hint: Consider the group action $G \times X_i \rightarrow X_i$ defined by $x.(c, g) = (xcx^{-1}, xg)$.

Problem 4. Let $G = D_4 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ with $x^4 = 1 = y^2, yxy^{-1} = x^3$ and $H = \{1, x^2, y, x^2y\} \subset G$. Find all conjugacy classes C of G , and for each conjugacy class C of G , decompose $C \cap H$ into conjugacy classes of H .

Problem 5. Let $G = \mathrm{GL}_2(\mathbb{F}_p)$, $H = \mathrm{SL}_2(\mathbb{F}_p) = \{g \in G : \det(g) = 1\}$. Let $C \subset G$ be the conjugacy class of the element $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Namely,

$$C = \{gug^{-1} : g \in G\}.$$

Try to decompose $C \cap H$ into conjugacy classes of H .

The next several problems are about double cosets, and most of them could be in last HW.

Problem 6. Let F be a field and let $B_n(F) \subset \mathrm{GL}_n(F)$ be the upper triangular subgroup.

- (1) Determine the double cosets $B_2(F) \backslash \mathrm{GL}_2(F) / B_2(F)$.
- (2) How about $B_n(F) \backslash \mathrm{GL}_n(F) / B_n(F)$?

This problem might be hard. It is related to the *UPL* (upper triangular, permutation subgroup, and lower triangular subgroup) decomposition of a matrix, See HW 3, Problem 5 of last year. If you don't know how to do the general problem, try the case when $n = 2$ and $F = \mathbb{F}_2$ (or \mathbb{F}_3).

Let $G \times X \rightarrow X$ be an action of a group G on a set X . Recall that $G \backslash X$ denote the set of orbits.

Problem 7. Let G be a group and H, K are subgroups of G . Show the following basic properties of double cosets.

- (1) For $x \in G$, the double coset HxK is a union of right H -cosets and a union of left K -cosets. More precisely,

$$HxK = \coprod_{Hxk \in H \backslash HxK} Hxk = \coprod_{hxK \in HxK/K} hxK.$$

- (2) Let G act on the left cosets G/K from the left by $x.(gK) = (xg)K$. See Section 6.8 of Artin. We restrict this action to H and consider the action

$$H \times G/K \rightarrow G/K$$

defined by $(h, gK) = (hg)K$. Show that there is a bijection between the double coset $H \backslash G/K$ and the set of orbits $H \backslash (G/K)$. This explains that the notation is consistent. There is a similar statement when we switch the role of H and K .

- (3) Suppose that all groups are finite. For $x \in G$, show that

$$|HxK| = [H : H \cap xKx^{-1}]|K| = [K : K \cap x^{-1}Hx]|H|.$$

- (4) Show that

$$[G : H] = \sum_{HxK \in H \backslash G/K} [K : K \cap x^{-1}Hx]$$

and

$$[G : K] = \sum_{HxK \in H \backslash G/K} [H : H \cap xKx^{-1}].$$

- (5) Consider the group action of $(H \times K)$ on G defined by

$$((h, k), g) = h g k^{-1}, (h, k) \in H \times K, g \in G.$$

Check that this is a group action and there is a bijection between $H \backslash G/K$ and the orbits of this action.

- (6) Let $G^{(h,k)} = \{g \in G : h g k = g\}$. Show that

$$|H \backslash G/K| = \frac{1}{|H||K|} \sum_{(h,k) \in H \times K} |G^{(h,k)}|.$$

For the last one, use Ex. M.7, page 194 of Artin. The other parts are routine.