HOMEWORK 15

This is the last HW. There is no need to submit it.

Exercises: 6.3, 6.6,7.1, 7.2, 7.3, 7.4, 7.7, 7.8, 7.9, 8.3, 8.4, 8.5. pages 223-225,

The above Exercises are very important. Please do them on your own.

Problem 1. *Let G be a group and N, H be two subgroups with N normal. Suppose that $G = N \rtimes H$. Show that there is an isomorphism $G/N \cong H$.*

Problem 2. *Let G be a group and N be a normal subgroup. Denote the quotient group G/N by H and denote the quotient map $G \to H = G/N$ by π. Suppose that there is a group homomorphism $s : H \to G$ such that $\pi \circ s = \mathrm{id}_H$. Show that s is injective and $G = N \rtimes s(H)$.*

Problem 3. *Let G_1, G_2 be two groups and we consider the direct product $G_1 \times G_2$. Let $p_1 : G_1 \times G_2 \to G_1$ be the projection map defined by $p_1(g_1, g_2) = g_1$. Similarly, let $p_2 : G_1 \times G_2$ be the map $p_2(g_1, g_2) = g_2$. Consider the triple $(G_1 \times G_2, p_1, p_2)$. Suppose that we are given another triple (H, f_1, f_2), where H is a group, $f_i : H \to G_i$ is a group homomorphism for $i = 1, 2$. Show that there is a unique group homomorphism $\phi : H \to G_1 \times G_2$ such that $f_i = p_i \circ \phi$ for $i = 1, 2$. In other words, we have the following commutative diagram*

$$\begin{tikzcd} & H \arrow[dl, "f_1"'] \arrow[d, dotted, "\phi"] \arrow[dr, "f_2"] & \\ G_1 & G_1 \times G_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & G_2 \end{tikzcd}$$

In other words, the direct product satisfies the same kind universal property as direct product of vector spaces, see section 2 of these notes. Since direct sum of finite number of vector spaces is the same as the direct product of finite number of vector spaces, one might ask if the same is true for groups. The answer is No.

Problem 4. *Let G_1, G_2 be two groups and we consider the direct product $G_1 \times G_2$. Let $\iota_1 : G_1 \to G_1 \times G_2$ be the map defined by $\iota_1(g_1) = (g_1, 1)$. Similarly, let $\iota_2 : G_2 \to G_1 \times G_2$ be the map $\iota_2(g_2) = (1, g_2)$. Consider the triple $(G_1 \times G_2, \iota_1, \iota_2)$. Suppose that we are given another triple (H, f_1, f_2), where H is a group, $f_i : G_i \to H$ is a group homomorphism for $i = 1, 2$. We ask if the following is true. Is there a unique group homomorphism $\phi : H \to G_1 \times G_2$ such that $f_i = \phi \circ \iota_i$ for $i = 1, 2$? In other words, is there a group homomorphism $\phi : H \to G$ such that the following diagram is commutative?*

$$\begin{tikzcd} G_1 \arrow[r, "\iota_1"] \arrow[dr, "f_1"'] & G_1 \times G_2 \arrow[d, dotted, "\phi"] & G_2 \arrow[l, "\iota_2"'] \arrow[dl, "f_2"] \\ & H & \end{tikzcd}$$

The answer is No.

Now the natural question arises. Given two groups G_1, G_2, is there a triple (G, ι_1, ι_2) with group homomorphisms $\iota_i : G_i \to G$ such that it is universal in the above sense? Namely, for any other triple (H, f_1, f_2), where H is a group, $f_i : G_i \to H$ is a group homomorphism for $i = 1, 2$. Then there a unique group homomorphism $\phi : H \to G_1 \times G_2$ such that $f_i = \phi \circ \iota_i$ for $i = 1, 2$. In other words, there is a group homomorphism $\phi : H \to G_1 \times G_2$ such that the following diagram is commutative.

$$\begin{tikzcd} G_1 \arrow[r, "\iota_1"] \arrow[dr, "f_1"'] & G \arrow[d, dotted, "\phi"] & G_2 \arrow[l, "\iota_2"'] \arrow[dl, "f_2"] \\ & H & \end{tikzcd}$$

The triple (G, ι_1, ι_2) indeed exists. It is called the free product of G_1 and G_2 and it is usually denoted by $G_1 * G_2$. The free product is much more complicate than direct product. Try to think about what the free product $C_2 * C_2$ is. Here C_2 is the group with 2 elements. At least how many elements are there? Try to find an answer online.

There is also a notion called **free group**, which is defined as follows. Let S be any set. The free group over S is a pair $(\iota, G(S))$, where $G(S)$ is a group and $\iota : S \to G(S)$ is a map (between two sets) such that for any other pair (f, H), where H is a group and $f : S \to H$ is a map (between two sets), there is a unique group homomorphism $\phi : G(S) \to H$ such that the following diagram is commutative

$$\begin{tikzcd} S \arrow[rr, "\iota"] \arrow[dr] & & G(S) \arrow[dl, dotted, "\phi"] \\ & H & \end{tikzcd}$$

Free groups are covered in sections 7.9 and 7.10 of Artin's book. We don't have time to cover this. If you have time, read this part. If S is a singleton, (which means that $|S| = 1$), then $G(S) \cong \mathbb{Z}$. If $|S| > 1$, then $G(S)$ is not abelian anymore. Moreover, if $S = \{x_1, x_2\}$ contains 2 elements, then one can check that $G(S) = \mathbb{Z} * \mathbb{Z}$, which is the free product of two $\mathbb{Z}$, which is just the free group of a singleton. More precisely, if we write $S_1 = \{x_1\}, S_2 = \{S_2\}$, then $S = S_1 \coprod S_2$ (here $\coprod$ means disjoint union), then $G(S) = G(S_1) * G(S_2)$. This is true in general (namely without assuming that S_1, S_2 are singletons). These facts could be proved using the universal properties defined above. Try to check these facts without using the constructions of free groups and free products.

The next 2 problems are about double coset decomposition. It could be in HW12. Since they are complicate, it is more appropriate for you to do them in the summer break.

Problem 5. *Let F be a field and let $W = F^2$, the two dimensional vector space over F. After choosing the standard basis $\mathcal{B} = \{\epsilon_1, \epsilon_2\}$, we can identify $\mathrm{GL}(W)$ with $\mathrm{GL}_2(F)$. Let $\mathcal{B}' = \{e, f\}$ be another basis of W, where $e = \epsilon_1 + \epsilon_2, f = \epsilon_1 - \epsilon_2$. Consider the subgroup*

$$H = \{g \in \mathrm{GL}(W) : ge \in \mathrm{Span}\{e\}, gf = f\}.$$

Compute the double cosets

$$H \backslash \mathrm{GL}(W)/B,$$

where $B \subset \mathrm{GL}(W)$ is the upper triangular subgroup. If we want to make it coordinate free, then

$$B = \{g \in \mathrm{GL}(W) : g\epsilon_1 \in \mathrm{Span}\{\epsilon_1\}\}.$$

There is a higher dimensional version of this problem.

Let $J_2 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$. Recall that the symplectic group $\mathrm{Sp}_4(F)$ is defined by

$$\mathrm{Sp}_4(F) = \left\{ g \in \mathrm{GL}_4(F) | g^t \begin{bmatrix} & J_2 \\ -J_2 & \end{bmatrix} g = \begin{bmatrix} & J_2 \\ -J_2 & \end{bmatrix} \right\}.$$

Problem 6. *Let B be the upper triangular subgroup of $\mathrm{Sp}_4(F)$. Compute the double coset*

$$B \backslash \mathrm{Sp}_4(F)/B.$$

Apparently, this problem has a higher dimensional version.

1. Projective space and projective linear group

The group theory is powerful because there are many natural group actions and we have the natural bijection $G/\mathrm{Stab}(x) \cong O_x$. Recall that one important application of these ideas is to prove the 3 Sylow Theorems. In order to use this, one needs to keep in mind several natural group actions. For example, given a group G and a subgroup H, we have the following natural group actions. (1) The group G acts on itself by left multiplication. (2) The group G acts on G by conjugation. (3)

The group G acts on the left coset space G/H by left multiplication. (4) The group G acts on the power set $\mathcal{P}(G)$ by left multiplication, where the power set $\mathcal{P}(G)$ is the set of all subsets of G. (5) The group G acts on $\mathcal{P}(G,k)$ by left multiplication, where $\mathcal{P}(G,k)$ is the subset of $\mathcal{P}(G)$ which is consisting of order k subsets of G. (6) The group $\mathrm{GL}_n(F)$ acts on F^n, where F is a field. (7) The group D_{2n} acts on a regular n-polygon. (8) The tetrahedral group T acts on a tetrahedron. (9) The octahedral group acts on a cube or an octahedron. (10) The icosahedral group acts on a dodecahedron or an icosahedron. In the following, we give another very useful group action.

Let F be a field and n be a positive integer. We introduce an equivalence relation on F^{n+1} as follows. For $\alpha, \beta \in F^{n+1}$, we say that $\alpha \sim \beta$ if there is an element $c \in F^\times$ such that $\beta = c\alpha$.

Problem 7. *Show that* $R = \left\{(\alpha,\beta) \in F^{n+1} \times F^{n+1} : \alpha \sim \beta\right\}$ *defines an equivalence relation on* F^{n+1}.

Denote $\mathbb{P}^n(F) := F^{n+1}/\sim$, which is the set of equivalence classes of the above equivalence relation. For example, $\mathbb{P}^1(F)$ is just the set of all lines which passes the origin. The set $\mathbb{P}^n(F)$ is called the n-dimensional projective space. Here we only view it as a set even it has more structures.

An element of $\mathbb{P}^n(F)$ is of the form $[\alpha]$, where $\alpha \in F^{n+1}$ and $[\alpha]$ denotes the equivalence class of α. Note that $[\alpha] = [\beta]$ if and only if $\beta = c\alpha$ for some $c \in F^\times$.

There is a group action of $\mathrm{GL}_{n+1}(F)$ on $\mathbb{P}^n(F)$ defined by

$$g.[\alpha] = [g\alpha], g \in \mathrm{GL}_{n+1}(F), \alpha \in F^{n+1}.$$

Consider the group $\mathrm{PGL}_{n+1}(F) = \mathrm{GL}_{n+1}(F)/Z$, where $Z = \{aI_{n+1}, a \in F^\times\}$ is the center of $\mathrm{GL}_{n+1}(F)$. The group $\mathrm{PGL}_{n+1}(F)$ is called the **projective general linear group** over F. An element of $\mathrm{PGL}_{n+1}(F)$ is of the form gZ, where $g \in \mathrm{GL}_{n+1}(F)$ and $gZ = hZ$ if and only if $g^{-1}h \in Z$, or $g = hz$ for some $z \in Z$. Similarly, we consider the group $\mathrm{PSL}_{n+1}(F) = \mathrm{SL}_{n+1}(F)/Z_0$, where $Z_0 = \left\{aI_{n+1} : a \in F^\times, a^{n+1} = 1\right\}$ is the center of $\mathrm{SL}_{n+1}(F)$.

Problem 8. *Show that the map* $\mathrm{PGL}_{n+1}(F) \times \mathbb{P}^n(F) \to \mathbb{P}^n(F)$ *defined by*

$$(gZ, [\alpha]) \mapsto [g\alpha]$$

is well-defined and it defines a group action of $\mathrm{PGL}_{n+1}(F)$ *on* $\mathbb{P}^n(F)$. *Similarly, show that there is a group action of* $\mathrm{PSL}_{n+1}(F)$ *on* $\mathbb{P}^n(F)$.

We next consider some very special cases.

Problem 9. *Let* $\mathbb{F}$ *be a finite field of order* q. *What is the cardinality of* $\mathbb{P}^n(F)$? *What is the cardinality of* $\mathrm{PGL}_n(\mathbb{F})$? *What is the cardinality of* $\mathrm{PSL}_2(\mathbb{F})$?

Here "cardinality" of a finite set means the number of elements in the finite set. For the second part and third part, think about how many elements are in the group $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{SL}_2(\mathbb{F})$.

Problem 10. *Let* $\mathbb{F}_2$ *be a finite field of* 2 *elements. Consider the action of* $\mathrm{PSL}_2(\mathbb{F}_2)$ *on* $\mathbb{P}^1(\mathbb{F}_2)$. *Show that*

$$\mathrm{PSL}_2(\mathbb{F}_2) \cong S_3.$$

Similarly, show that

$$\mathrm{PSL}_2(\mathbb{F}_3) \cong A_4.$$

Problem 11. *Let* $\mathbb{F}_4$ *be a finite field of* 4 *elements. Consider the action of* $\mathrm{PSL}_2(\mathbb{F}_4)$ *on* $\mathbb{P}^1(\mathbb{F}_4)$. *Show that*

$$\mathrm{PGL}_2(\mathbb{F}_4) \cong A_5.$$

There is indeed a finite field of 4 elements. See HW 2 of last year. The above two problems are indeed Exercises 8.2, 8.3, page 287 of Artin's book. Hint: The following fact will be useful. If H is a subgroup of S_n of index 2, then $H = A_n$. This Exercise 5.7, page 223. Actually, there is one more isomorphism

$$\mathrm{PSL}_2(\mathbb{F}_5) \cong A_5,$$

but this cannot be obtained directly from the action of $\mathrm{PSL}_2(\mathbb{F}_5)$ on $\mathbb{P}^1(\mathbb{F}_5)$. Because $|\mathbb{P}^1(\mathbb{F}_5)| = 6$ and thus this action only gives us an embedding

$$\mathrm{PSL}_2(\mathbb{F}_5) \to S_6,$$

from this it is not so direct to obtain $\mathrm{PSL}_2(\mathbb{F}_5) \cong A_5$. On the other hand, this embedding is related the icosahedral group.

Problem 12. *Let F be any field and let B be the upper triangular subgroup of $\mathrm{GL}_2(F)$. Using the group action of $\mathrm{GL}_2(F)$ on $\mathbb{P}^1(F)$, show that there is a bijective map*

$$\mathrm{GL}_2(F)/B \cong \mathbb{P}^1(F).$$

Conclude that $B\backslash\mathrm{GL}_2(F)/B$ is consisting of two elements.

Hint for the second part. The double coset $B\backslash\mathrm{GL}_2(F)/B$ can be viewed as the orbit space of the natural action of B on the space $\mathrm{GL}_2(F)/B$, which can be realized on the natural action of B on $\mathbb{P}^1(F)$ using the above bijection.

This double coset $B\backslash\mathrm{GL}_2(F)/B$ was computed in previous HWs, which we proved using elementary row operations from linear algebra. Here we give a group theoretic solution to this for the small group $\mathrm{GL}_2(F)$. This problem could be generalized to more general situations but it needs some terminologies that we lack at this moment.

2. Upper half plane

Let $i = \sqrt{-1}$. We consider the upper half plane $\mathcal{H} = \{x + yi |\ x, y \in \mathbb{R}, y > 0\}$ of the complex plane.

Problem 13. *Consider the map $\mathrm{SL}_2(\mathbb{R}) \times \mathcal{H} \to \mathcal{H}$ defined by*

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, z\right) \mapsto \frac{az+b}{cz+d}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}), z \in \mathcal{H}.$$

(1) *Show that the above map is well-defined and indeed defines a group action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$. Moreover, show that the center of $\mathrm{SL}_2(\mathbb{R})$ acts trivially on $\mathcal{H}$ and thus it defines a group action $\mathrm{PSL}_2(\mathbb{R})$ on $\mathcal{H}$.*
(2) *Find the stabilizer of $i \in \mathcal{H}$, namely, compute the group $\mathrm{Stab}(i) = \{g \in \mathrm{SL}_2(\mathbb{R}) | gi = i\}$.*
(3) *Find the orbit O_i of i. Explicate the bijection $\mathrm{SL}_2(\mathbb{R})/(\mathrm{Stab}(i)) \cong O_i$.*

The above action can also be explained using the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{C})$. Explain it.

This example of group action is important in number theory.