HOMEWORK 6

Due date: Monday of Week 7

Exercises: 2, 6, 7, 10, 11, pages 308-311
Exercises: 2, 3, 8, 9, 11, 12, 13, 14, pages 317-318,

In this week, we focused on normal matrix (normal operators) on complex vector spaces. Similar results could be proved over real vector spaces once it is known that the corresponding eigenvalues are real numbers.

Problem 1. *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be symmetric, namely, $A = A^t$.*

(1) *Let $a \in \mathbb{C}$ be a root of $\chi_A = \det(xI_n - A)$ (namely, a is a complex eigenvalue of A). Show that $a \in \mathbb{R}$. (We showed in class that any eigenvalue of a self-adjoint linear operator is real. This is a special case of that. Please repeat the proof here).*
(2) *Show that there is an orthogonal matrix $P \in O_n(\mathbb{R})$ such that $P^T AP$ is diagonalizable.*

This is the corollary in page 314 of the textbook. But we did not cover it. Mimic the proof of the spectral decomposition of normal operators over $\mathbb{C}$.

Our treatment of the spectral decomposition theorem of normal operators over $\mathbb{C}$ was taken from Artin's book, and it is a little bit different from the treatment given in the textbook. In particular, Theorem 20, 21 were not covered in class. They are given in the following problem.

Problem 2.
(1) *Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $T \in \mathrm{End}(V)$. Show that there is an orthonormal basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is upper triangular. (This is Theorem 21, page 316.)*
(2) *Given a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Show that there is a unitary matrix $P \in \mathrm{U}(n)$ and an upper triangular matrix U such that $A = PUP^{-1}$. This is called the Schur decomposition of A.*
(3) *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be normal. If $A = PUP^{-1}$ is the Schur decomposition of A, then U is diagonal.*

Let $V = \mathrm{Mat}_{n\times n}(\mathbb{C})$ and define $(A|B)_{\mathbb{C}} = \mathrm{tr}(B^*A)$. We have seen that $(\ |\ )_{\mathbb{C}}$ define an inner product on V when we view V as a vector space over $\mathbb{C}$.

Problem 3.
(1) *Given $A \in V = \mathrm{Mat}_{n\times n}(\mathbb{C})$. Show that any eigenvalue of A^*A is non-negative. (The eigenvalues of A^*A are real because A^*A is self-adjoint).*
(2) *If every eigenvalue of $A^*A = 0$, then $A = 0$.*

(Hint: Since A^*A is self-adjoint (in particular, normal), there exists an orthonormal basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ of $\mathbb{C}^n$ such that each α_i is an eigenvector. Let λ_i be the corresponding eigenvalue, then $A^*A\alpha_i = \lambda_i\alpha_i$. Then consider $(A\alpha|A\alpha) = (\alpha|A^*A\alpha) = \dots$. Here $(\ |\ )$ is the standard inner product on $\mathbb{C}^n$, not the one on V. We just lack enough notations.)

This problem gives another way to check $\mathrm{tr}(A^*A) = (A|A)_{\mathbb{C}} = 0$ implies $A = 0$. We now view V as a vector space over $\mathbb{R}$ (and write it as $V_{/\mathbb{R}}$ to emphasize that it is a vector space over $\mathbb{R}$) and define $(A|B)_{\mathbb{R}} = \mathrm{Re}((A|B)_{\mathbb{C}})$. Then $(\ |\ )_{\mathbb{R}}$ defines an inner product on $V_{/\mathbb{R}}$. This is easy to check. (Check it!)

Problem 4. *Let $V_{/\mathbb{R}} = \mathrm{Mat}_{n\times n}(\mathbb{C})$ be the real vector space endowed with the inner product $(\ |\ )_{\mathbb{R}}$ defined above. Let*

$$W = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) | A = A^*\} \subset V.$$

(1) *Show that W is an $\mathbb{R}$ subspace of $V_{/\mathbb{R}}$ but not a $\mathbb{C}$-subspace of V. Thus we have the orthogonal decomposition $V_{/\mathbb{R}} = W \bigoplus W^{\perp}$. Here $W^{\perp}$ is of course defined w.r.t $(\ |\ )_{\mathbb{R}}$.*

(2) *Given $A \in V$, we write $A = A_1 + A_2$, with*

$$A_1 = \frac{A + A^*}{2}, A_2 = \frac{A - A^*}{2}.$$

Show that $A_1 \in W$ and $A_2 \in W^{\perp}$. Thus the orthogonal projection of $V_{/\mathbb{R}}$ to W associated to with the above orthogonal decomposition is $Proj_W(A) = \frac{1}{2}(A + A^*)$.

This problem might be helpful for solving Exercise 8, page 317.

Problem 5. *Given $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Suppose that $AA^* = A^2$. Show that A is self-adjoint.*

(Hint: Use the decomposition in Problem 4.)

Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Then the following are equivalent.

(1) A is normal, i.e., $AA^* = A^*A$;
(2) $(A\alpha|A\beta) = (\alpha|\beta)$, for any $\alpha, \beta \in \mathbb{C}^n$, where $(\ |\ )$ denotes the standard inner product on $\mathbb{C}^n$;
(3) $||A\alpha|| = ||A^*\alpha||$, for any $\alpha \in \mathbb{C}^n$;
(4) there exists a unitary matrix $P \in \mathrm{U}(n)$ and a diagonal matrix D such that $A = PDP^{-1}$;
(5) A_1 and A_2 commutes with $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2}$;
(6) there exists a polynomial $f \in \mathbb{C}[x]$ such that $A^* = f(A)$;
(7) let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, then $\mathrm{tr}(A^*A) = \sum_{j=1}^n |\lambda_j|^2$;
(8) $A^* = AP$ for some unitary matrix $P \in \mathrm{U}(n)$. (Actually, the condition $A^* = AP$ for $P \in \mathrm{U}(n)$ forces $AP = PA$.)

There are several other equivalent conditions, see this wiki-page if you can. Most of these were proved in classes and HW problems, see Ex 8, 13, page 317-318 for (5) (6). The above equivalences are for normal matrices. There is a similar version for normal operators. Try to translate the above equivalences to normal operators over $\mathbb{C}$. Normal operators over $\mathbb{R}$ is a little bit harder. We will learn them in next Chapter.

Problem 6. (1) *Prove the equivalence of (1) and (7).*
(2) *Prove the equivalence of (1) and (8).*

Hint: For the equivalence of (1) and (7) use Schur decomposition. This is easy. For the equivalence of (1) and (8), first show that the condition $A^* = AP$ for $P \in \mathrm{U}(n)$ implies $AP = PA$.