HOMEWORK 7

Due date: Tuesday of Week 8

Exercises: 7.1, 7.2, 7.5, 7.8, 7.9, 8.6, page 439-440 of Artin's book; Exercises: 1.1, 1.2, 2.1, 2.3, page 472.

Problem 1. *Consider the abelian group*

$$A = C_{30} \oplus C_{49} \oplus C_{12} \oplus C_{25} \oplus C_{40}.$$

Find the invariant divisors of A. Here C_n denotes the cyclic groups of order n.

Problem 2. *Let R be a PID and M be a free R-module of rank m. Let N be a submodule of M. We know that N is a free module of rank n with $n \leq m$. Show that there exists a basis $\mathcal{B} = \{e_1, \ldots, e_m\}$ of M and non-zero elements $a_1, \ldots, a_n \in R$ such that:*

1. *the elements $a_1e_1, a_2e_2, \ldots, a_ne_n$ form a basis of N over R;*
2. *we have $a_i | a_{i+1}$ for $i = 1, \ldots, n-1$.*

The sequence of ideas $(a_1), \ldots, (a_n)$ is uniquely determined by the above conditions.

Problem 3. *Let $G = \mathrm{GL}_2(\mathbb{Q})$ and $H = \mathrm{GL}_2(\mathbb{Z})$. Determine the double coset*

$$H \backslash G / H.$$

Problem 4. *Let R be a commutative nonzero ring with 1. Let m, n be two positive integers. If there is an injective R-module homomorphism $f : R^m \to R^n$, show that $m \leq n$.*

Hint: By contradiction, suppose that $m > n$. Let $\iota : R^n \to R^m$ be the natural inclusion. Namely, $\iota(r_1, \ldots, r_n) = (r_1, \ldots, r_n, 0, \ldots, 0)$. Let $\phi = \iota \circ f : R^m \to R^m$, which is also injective. Let $\pi : R^m \to R$ be the projection to the last coordinate. Then $\pi \circ \phi = 0$. Let $\chi \in R[x]$ be the characteristic polynomial of ϕ. Then Cayley-Hamilton says that $\chi(\phi) = 0$. Think about why Cayley-Hamilton is still true (try to repeat the proof!) Assume that $\chi = X^k P$ for a polynomial $P = x^{m-k} + \cdots + c_0$ with $c_0 \neq 0$. Then $\chi(\phi) = 0$ implies that $\phi^k P(\phi) = 0$. Show that $P(\phi) = 0$ and thus $\pi \circ P(\phi) = 0$. Then get a contradiction.

Problem 5. *Let K/F be a field extension and $\alpha \in K$ is algebraic over F with $\deg(\alpha) = d$. Show that $\left\{1, \alpha, \ldots, \alpha^{d-1}\right\}$ is a basis of $F[\alpha]/F$.*

1. Linear operators and f.g. modules over PID

In this section, let F be a field and V be a finite dimensional vector space over F. Let $T : V \to V$ be a linear operator. We can view V as an $F[x]$-module by $f(x).v := f(T)v$ for any $f \in F[x]$.

Problem 6.
1. *Show that a subspace $W \subset V$ is T-invariant iff W is a submodule of V;*
2. *Show that V has a cyclic vector iff V can be generated by a single element as an $F[x]$-module.*

Recall that a subspace $W \subset V$ is called T-admissible if (1) W is T-invariant; and (2) if $f(T)\beta \in W$ for $\beta \in V, f \in F[x]$, then there exists a vector $\gamma \in W$ such that $f(T)\beta = f(T)\gamma$. See Section 7.2 of Hoffman-Kunze. The cyclic decomposition theorem (Theorem 7.3 and its corollary of Hoffman-Kunze) said that W is T-admissible iff there exists another T-invariant subspace W' such that $V = W \oplus W'$.

The following are some generalizations of the above terminology into more general modules. Let R be a general ring and let M be an R-module. A submodule N of M is called a **direct summand** of M if there exists another submodule N' of M such that $M = N \oplus N'$. This is a generalization that *there exists another T-invariant subspace W' such that $V = W \oplus W'$.*

A submodule N of M is called **pure** if for any $m \times n$ matrix $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathrm{Mat}_{m \times n}(R)$, and any element $Y = (y_1, \ldots, y_m)^t$ with $y_i \in N$, if there exist $X = (x_1, \ldots, x_n)^t$ with $x_i \in M$ such that

$$AX = Y,$$

then there exists $X' = (x'_1, \ldots, x'_n)^t$ with $x'_i \in N$ such that

$$AX' = Y.$$

The definition of pure submodule looks complicate. Here is a digression.

Problem 7. *Suppose that N is a pure submodule of M and there is a commutative diagram of R-modules*

$$\begin{array}{ccccc} & & R^n & \xrightarrow{f} & R^m \\ & & \downarrow u & & \downarrow v \\ 0 & \longrightarrow & N & \xrightarrow{i} & M \end{array}$$

Here m, n are positive integers and $i : N \to M$ denotes the inclusion. Show that there is homomorphism $\phi : R^m \to N$ such that $u = \phi \circ f$. (We don't require $v = i \circ \phi$.)

Let ϵ_i be the standard basis of R^n and e_j be the standard basis of R^m. Then $vf(\epsilon_i) = iu(\epsilon_i) \in N$. Or $v(\sum a_{ij}e_j) = \sum_j a_{ij}v(e_j) \in N$. By definition, there exists $x'_j \in N$ such that $\sum a_{ij}x'_j = \sum a_{ij}v(e_j)$. Define $\phi(e_j) = x'_j$. Then $\phi \circ f(\epsilon_i) = \sum a_{ij}x'_j = vf(\epsilon_i) = u(\epsilon_i)$. Thus $u = \phi \circ f$.

Let M be an R-module, a submodule $N < M$ is called **admissible** if for any $r \in R$ and $x \in M$ if $rx \in N$, then there exists an $n \in N$ such that $rx = rn$. This agrees with the notation defined in Hoffman-Kunze when $R = F[x]$ and $M = V$. Note that, a pure submodule is admissible (since pure requires a condition for any $m \times n$).

Problem 8. *Let R be a PID. Let M be an R-module and $N < M$ be a submodule. Show that N is a pure submodule iff it is an admissible submodule.*

Hint: You need to show any admissible submodule is pure. Use diagonalization. This is not hard.

Problem 9. *Let R be a ring and M be an R-module. Let N be a submodule of M. Consider the short exact sequence*

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{\pi} M/N \longrightarrow 0 \ .$$

Show that the following are equivalent

(1) *N is a direct summand of M;*
(2) *there exists a homomorphism $s \in \mathrm{Hom}_R(M, N)$ such that $s(x) = x$ for all $x \in N$ (namely, $s \circ i = \mathrm{id}_N$);*
(3) *for each R-module P, the sequence*

$$0 \to \mathrm{Hom}_R(M/N, P) \to \mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(N, P) \to 0$$

is exact;
(4) *there exists a homomorphism $u \in \mathrm{Hom}_R(M/N, M)$ such that $\pi \circ u = \mathrm{id}_{M/N}$;*
(5) *for each R-module P, the sequence*

$$0 \to \mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, M/N) \to 0$$

is exact.

This is called the splitting lemma. See Problems 3 and 4 of HW 5. Hint: Show (1) $\Longrightarrow$ [(2) $\Longleftrightarrow$ (3)] $\Longrightarrow$ [(4) $\Longleftrightarrow$ (5)] $\Longrightarrow$ (1).

Problem 10. *Let M be an R-module and let $N \subset M$ be a submodule. If N is a direct summand of M, show that N is a pure submodule.*

Problem 11. *Let R be a general ring and M be an R-module. Let $N < M$ be a pure submodule and X is a finitely presented R-modules. Show that the sequence the sequence*

$$0 \to \mathrm{Hom}_R(X, N) \to \mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(X, M/N) \to 0$$

is exact. As a consequence, show that if M/N is finitely presented, then N is a pure submodule of M iff it is a direct summand.

Hint: This one might be hard. One only needs to show that $\mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(X, M/N)$ is surjective. Given $w \in \mathrm{Hom}_R(X, M/N)$, try to produce a commutative diagram

$$\begin{tikzcd}
& R^n \arrow[r, "f"] \arrow[d, "u"] & R^m \arrow[r] \arrow[d, "v"] & X \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & N \arrow[r, "i"] & M \arrow[r, "\pi"] & M/N \arrow[r] & 0
\end{tikzcd}$$

and use Problem 7 to get a hom $\phi : R^m \to N$. Then consider $\widetilde{w} \in \mathrm{Hom}_R(R^m, M)$ defined by $\widetilde{w} = v - \phi$.

Problem 12. (1) *Let R be a Noetherian ring and M be a finitely generated R-module. Show that a submodule $N < M$ is pure iff it is a direct summand.*

(2) *Let R be a PID and M be a finitely generated R-module. Show that a submodule $N < M$ is admissible iff it is a direct summand.*

This problem together with the structure theorem of finite generated modules over PID fully covers Theorem 3, page 233 of Hoffman-Kunze. In the general case, we have

$$\text{(direct summand submodules)} \subset \text{(pure submodules)} \subset \text{(admissible submodules)}.$$

See this link for an example of pure submodule which is not a direct summand.

2. Presentation of linear operator as $F[x]$-modules

This problem is from HW11, 2023. It is also Exercise 8.4, page 440 of Artin's book. Try it again. You don't have to submit your solution.

Let F be a field. We consider $K = F[x]$ and K^n. An element $u \in K^n$ will be considered as a column vector and thus it has the form

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

and each $u_i \in F[x]$ can be written as $u_i = u_{i0} + u_{i1}x + u_{i2}x^2 + \cdots + u_{ik}x^k$ with $u_{ij} \in F$. Since u_{ik} can be zero, we can take a k such that it works for all i, namely each u_i has its last term of the form $u_{ik}x^k$. Thus we can write u as

$$u = \begin{bmatrix} u_{10} \\ u_{20} \\ \vdots \\ u_{n0} \end{bmatrix} + \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix} x + \cdots + \begin{bmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{bmatrix} x^k.$$

Write

$$\mathbf{u}_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{nj} \end{bmatrix} \in F^n,$$

then we can write $u = \mathbf{u}_0 + x\mathbf{u}_1 + \cdots + x^k\mathbf{u}_k$. Here we write x^j in front of $\mathbf{u}_j$ (so that it looks like a scaler times a column vector). Thus an element in $K^n = F[x]^n$ can be viewed as a polynomial with coefficients in F^n.

Fix a matrix $A \in \mathrm{Mat}_{n\times n}(F)$. Note that as an element in $\mathrm{Mat}_{n\times n}(K)$, the matrix $xI_n - A$ defines a linear map $T_{(xI_n - A)} : K^n \to K^n$ defined by

$$T_{(xI_n - A)}u = (xI_n - A)u,$$

as usual. We now consider the map $\phi : K^n \to F^n$ defines as follows. Given an element

$$u = \mathbf{u}_0 + x\mathbf{u}_1 + \cdots + x^k\mathbf{u}_k \in K^n,$$

we define

$$\phi(u) = \mathbf{u}_0 + A\mathbf{u}_1 + \cdots + A^k\mathbf{u}_k \in F^n.$$

Namely, we just replace the symbol x by the matrix A. The notation should be clear.

Problem 13. (1) *Show that ϕ is surjective. (This should be trivial).*
(2) *Show that* $\mathrm{Im}(T_{(xI_n-A)}) \subset \ker(\phi)$. *(This is also trivial).*
(3) *Show that* $\ker(\phi) \subset \mathrm{Im}(T_{(xI_n-A)})$. *(It needs some work, but not very hard).*

The assertions of this problem say that the sequence

$$K^n \xrightarrow{T_{(xI_n-A)}} K^n \xrightarrow{\phi} F^n \longrightarrow 0$$

is exact (as K-modules), which gives a presentation of F^n as an $F[x]$-module.