HOMEWORK 10

Due date: Monday of Week 15

Exercises: 1 (a), (b), (c), 6, 10, 12, p.148-149;
Exercises: 4, 7, 10, 11, 12, 13. p.155-156.

Let K be a commutative ring with 1. An element $a \in K$ is called a **unit** if there exists an element $b \in K$ such that $ab = 1$. Denote by $K^\times$ the set of all units in K. For example, if K itself is a field, then $K^\times = \{x \in K : x \neq 0\}$. For the integer ring $\mathbb{Z}$, we have $\mathbb{Z}^\times = \{1, -1\}$. What is $F[x]^\times$ for a field F?

Consider $\mathrm{Mat}_{n\times n}(K)$ and the following 3 types "elementary matrices" in $\mathrm{Mat}_{n\times n}(K)$ which are defined as follows. A type I elementary matrix is obtained by multiplying an element $c \in K^\times$ to a row of I_n, which is denoted by $E_n(R_i \leftarrow cR_i)$. Here as usual, I_n is the identity matrix. A type II elementary matrix is obtained by adding cR_j to R_i of the identity matrix I_n for some $c \in K$, which is denoted by $E_n(R_i \leftarrow R_i + cR_j)$. A type III elementary matrix is obtained by switching two rows of I_n, which is denoted by $E_n(R_i \leftrightarrow R_j)$. One can also define elementary row operatations similarly and the only difference from what we learned in Chapter I (in that case K is a field) is: in the first kind elementary row operation, we require that c is a unit in K rather than any nonzero elements (we have seen that if F is a field, a unit in F is just a nonzero element in F. The elementary matrices defined here is actually the same as we defined in Chapter I if we generalize "$F^\times$" to "units".)

Similarly, we can define elementary column operations. We did not even talk about this when F is a field. The reason for it is: it is unnecessary using column operations for the purpose we did so far. But it is necessary to use elementary elementary column operations when K is not a field. For a matrix j, we denote by C_j its j-th column. We consider the following 3 types elementary column operations. Type 1, $e(C_i \leftarrow cC_i)$, replace C_i by cC_i with $c \in K^\times$; $e(C_i \leftarrow C_i + cC_j)$: replace C_i by $C_i + cC_j$; $e(C_i \leftrightarrow C_j)$: swap C_i and C_j. Then we can also define elementary matrices using elementary column operations applying to I_n: $E_n(C_i \leftarrow cC_i)$ $(c \in K^\times)$; $E_n(C_i \leftarrow C_i + cC_j)$; and $E(C_i \leftrightarrow C_j)$. For example $E_n(C_i \leftarrow C_i + cC_j)$ is the matrix obtained by adding cC_j of I_n to C_i.

Problem 1. *Show that the elementary matrices defined by elementary column operations can be also defined using elementary row operations.*

For example $E_n(C_i \leftarrow cC_i) = E_n(R_i \leftarrow cR_i)$. It is also easy to check the rest.

Problem 2. *Show that each elementary matrix in* $\mathrm{Mat}_{n\times n}(K)$ *is invertible. Write explicitly each type elementary matrices in* $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$.

Problem 3. *Given a matrix* $A \in \mathrm{Mat}_{m\times n}(K)$. *Let e be an elementary row operation and let E be the corresponding elementary matrix. Show that $e(A) = EA$. Similarly, if e is an elementary column operation, and E is the corresponding elementary matrix. Show that $e(A) = AE$.*

Just check this case by case. We proved a similar result for elementary row operations in Chapter 1.

If K is a field, we showed in Chapter 1 that every matrix can be reduced to Row-Reduced Echelon matrix using elementary row operations. If K is not a field, usually a matrix cannot be reduced to Row-Reduced Echelon matrix in the sense we defined in Chapter 1 using elementary row/column operations. Try to consider what "simple form" you can get for a matrix $\mathrm{Mat}_{m\times n}(K)$ if $K = \mathbb{Z}$ or $F[x]$ using just elementary row operation and elementary column operators. You can define what "simple" means. The next problem will give you some examples.

Problem 4. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 6 \end{bmatrix} \in \mathrm{Mat}_{2\times 3}(\mathbb{Z}).$$

(1) *Using elementary row and elementary column operations (defined above for $K = \mathbb{Z}$, namely, in type I, c is only allowed in $\mathbb{Z}^\times = \{1, -1\}$) to reduce the matrix to*

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \in \mathrm{Mat}_{2\times 3}(\mathbb{Z}).$$

(2) *Find invertible matrices $P \in \mathrm{GL}_3(\mathbb{Z}), Q \in \mathrm{GL}_2(\mathbb{Z})$ such that $B = QAP$.*

(3) *Try to use both elementary row and elementary column operations to reduce $C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ to a simple matrix $C' \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$. Here it is up to you to decide what kind matrices are "simple". The answer depends on your own interpretation. Is $C \in \mathrm{GL}_2(\mathbb{Z})$?*

In the above Problem 4, we only considered matrices with coefficients in $\mathbb{Z}$. You can also try similar questions for matrices with coefficients in $F[x]$ for a field F. Try to make your own problems and solve them.

Also think about the following question. If $K = \mathbb{Z}$ or $F[x]$. Given $A \in \mathrm{GL}_n(K)$, is it possible to reduce A to the identity matrix using elementary row and column operations? Note that, if this is true, then it will imply that every matrix $A \in \mathrm{GL}_n(K)$ is still a product of elementary matrices. The answer is Yes if $K = \mathbb{Z}$ or $F[x]$. We will see how to do this in Section 7.4 if $K = F[x]$. When $K = \mathbb{Z}$, we will see how to do this in a future course (and see how it will be used to determine the structure of finitely generated abelian groups.) Please try some examples using 2×2 matrices if $K = \mathbb{Z}$. It is indeed easy. If K is more general, the question is related to something called K-theory. (The letter "K" in K-theory is not related to the letter "K" that we used to denote our ring). Hopefully you will learn something related to K-theory in the future.

The above problems are no hard at all. They are indeed linear algebra (but over a ring rather than a field).

Do the above problems after Monday's class.

Problem 5. *Let $V = F^2$ and $W = F^3$. Compute $\dim_F \mathrm{Alt}(V^3; F)$ and $\dim_F \mathrm{Alt}(W^2; F)$.*

You can do Problem 5 after Monday's class.

Recall that S_n denotes the symmetric group on n-elements. In other words, S_n consists of bijections $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. Recall that a matrix P is called permutation matrix if each matrix and each row of P has only one nonzero element and that nonzero element is 1. (We can view P as an element of $\mathrm{GL}_n(K)$ for any commutative ring K with identity because both 1 and 0 are defined in any such K. If it is confusing, you can view P as an element in $\mathrm{GL}_n(F)$ for a field F. If it is helpful, you might take F to be any field you are familiar with). Denote by Perm_n the set of all $n \times n$ permutation matrices. Also recall that, $\epsilon_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in F^n$ with 1 in the i-th position. For an element $\sigma \in S_n$, we consider the matrix P_σ defined by

$$P_\sigma = \begin{bmatrix} \epsilon_{\sigma(1)} \\ \epsilon_{\sigma(2)} \\ \cdots \\ \epsilon_{\sigma(n)} \end{bmatrix}.$$

For example, if $n = 3$, and $\sigma \in S_3$ is the element such that $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$, then

$$P_\sigma = \begin{bmatrix} \epsilon_{\sigma(1)} \\ \epsilon_{\sigma(2)} \\ \epsilon_{\sigma(3)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Problem 6. *Consider the map $\theta : S_n \to \mathrm{Perm}_n$ defined by $\theta(\sigma) = P_\sigma$. Show that*

(1) $\theta(\sigma\tau) = \theta(\sigma)\theta(\tau)$;
(2) θ *is a bijection;*
(3) $\det(P_\sigma) = \operatorname{sgn}(\sigma)$.

Some of these claims were proved in class. For (2), the following fact is useful, Let X, Y be two finite sets with the same cardinality, and let $f : X \to Y$ be an injective map. Then f must be bijective.

Problem 7. *Given* $x_1, \dots, x_n \in F$, *consider the matrix*

$$A(x_1, x_2, \dots, x_n) = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}.$$

Compute $\det(A(x_1, \dots, x_n))$.

This is a slight generalization of Ex 2, page 155. If you don't know how to do the above problem for general n, try the case when $n = 4$. Do the above two problems after Friday's class.

You don't have to do the next problem. If you have time, try to think about it for some small n (like $n = 3, 4$). Otherwise, ignore it. Like the authors said on page 162, our focus is not on explicit calculations of determinants of specific matrices.

Problem 8. *Consider the following* $n \times n$ *matrix with coefficients in* $\mathbb{Z}$:

$$A(n) = \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 1^3 & 2^3 & 3^3 & \dots & n^3 \\ 1^5 & 2^5 & 3^5 & \dots & n^5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1^{2n-1} & 2^{2n-1} & 3^{2n-1} & \dots & n^{2n-1} \end{bmatrix}.$$

Compute $\det(A(n))$.