Svm 支持向量机

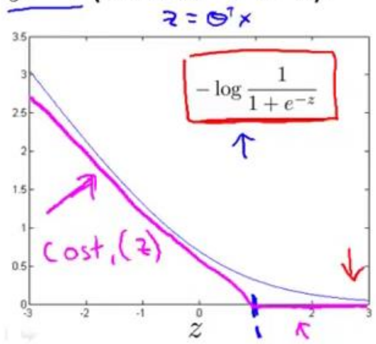
Alternative view of logistic regression

(x, y)

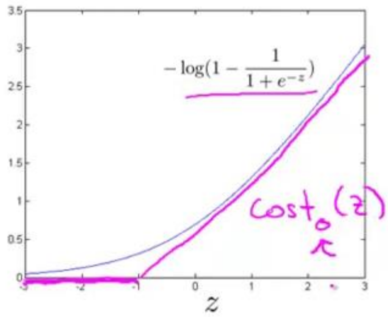
Cost of example: $-(y \log h_\theta(x) + (1-y)\log(1-h_\theta(x)))$ ←

$$= \boxed{-y \log \frac{1}{1+e^{-\theta^T x}}} - \boxed{(1-y)\log(1-\frac{1}{1+e^{-\theta^T x}})} \leftarrow$$

If $y=1$ (want $\theta^T x \gg 0$): $z = \theta^T x$



If $y=0$ (want $\theta^T x \ll 0$):



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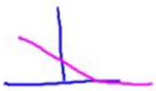
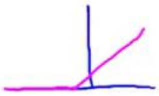
Support vector machine

Logistic regression:





$$\underbrace{\min_\theta \frac{1}{m}\left[\sum_{i=1}^m y^{(i)} \underbrace{\left(-\log h_\theta(x^{(i)})\right)}_{\text{cost}_1(\theta^T x^{(i)})} + (1-y^{(i)}) \underbrace{\left((-\log(1-h_\theta(x^{(i)}))\right)}_{\text{cost}_0(\theta^T x^{(i)})}\right]}_{A} + \underbrace{\frac{\lambda}{2m}\sum_{j=1}^n \theta_j^2}_{B}$$

Support vector machine:

$$\min_\theta \cancel{\frac{1}{m}}\, C \sum_{i=1}^m y^{(i)} \text{cost}_1(\theta^T x^{(i)}) + (1-y^{(i)}) \text{cost}_0(\theta^T x^{(i)}) + \frac{1\cancel{\lambda}}{2\cancel{m}} \sum_{i=0}^n \theta_j^2$$

$\min_u ((u-5)^2 + 1) \rightarrow u = 5$ $\times 10$

$\min_u 10(u-5)^2 + 10 \rightarrow u = 5$

$A + \lambda B$ ←

$\rightarrow C A + B$ ←

$C = \frac{1}{\lambda}$

$$\rightarrow \min_\theta C \sum_{i=1}^m \left[y^{(i)} cost_1(\theta^T x^{(i)}) + (1-y^{(i)}) cost_0(\theta^T x^{(i)})\right] + \frac{1}{2}\sum_{i=1}^n \theta_j^2$$

SVM hypothesis

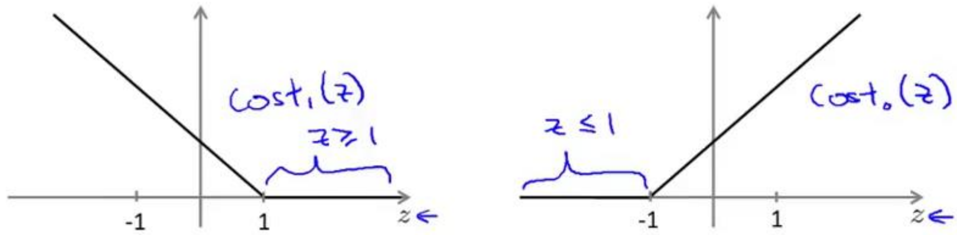
$$\rightarrow \min_\theta C \sum_{i=1}^m \left[y^{(i)} cost_1(\theta^T x^{(i)}) + (1-y^{(i)}) cost_0(\theta^T x^{(i)})\right] + \frac{1}{2}\sum_{i=1}^n \theta_j^2$$

Hypothesis:

$$h_\theta(x) \begin{cases} 1 & \text{if } \theta^T x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Support Vector Machine

$$\min_\theta C \sum_{i=1}^{m} \left[y^{(i)} \underline{cost_1(\theta^T x^{(i)})} + (1 - y^{(i)}) \underline{cost_0(\theta^T x^{(i)})} \right] + \frac{1}{2} \sum_{i=1}^{n} \theta_j^2$$



If $\underline{y = 1}$, we want $\underline{\theta^T x \geq 1}$ (not just ≥ 0) $\theta^T x \geq \cancel{0}\ 1$

If $\underline{y = 0}$, we want $\underline{\theta^T x \leq -1}$ (not just < 0) $\theta^T x \leq \cancel{0}\ -1$

SVM Decision Boundary

$$\min_\theta C \underbrace{\boxed{\sum_{i=1}^{m} \left[y^{(i)} cost_1(\theta^T x^{(i)}) + (1 - y^{(i)}) cost_0(\theta^T x^{(i)}) \right]}}_{=0} + \frac{1}{2} \sum_{i=1}^{n} \theta_j^2$$

Whenever $y^{(i)} = 1$:

$\theta^T x^{(i)} \geq 1$

$$\min_\theta \cancel{C \times 0} + \frac{1}{2} \sum_{i=1}^{n} \theta_j^2$$

s.t. $\theta^T x^{(i)} \geq 1$ if $y^{(i)} = 1$

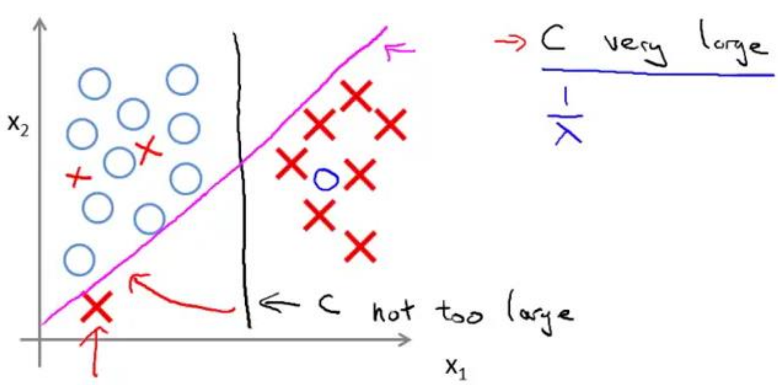
$\theta^T x^{(i)} \leq -1$ if $y^{(i)} = 0$.

Whenever $y^{(i)} = 0$:

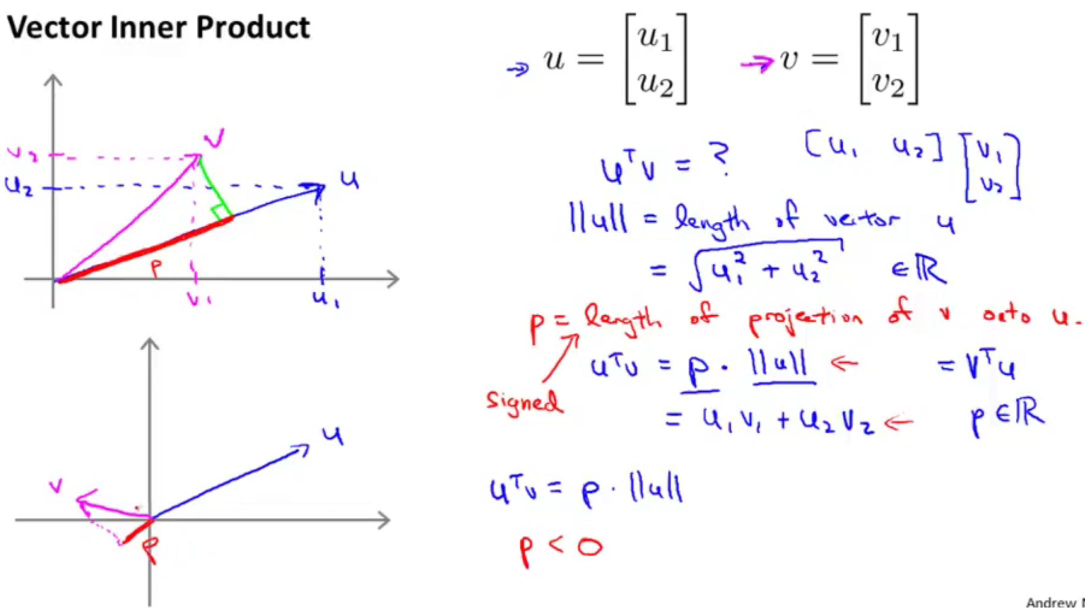
$\theta^T x^{(i)} \leq -1$

大间距分类器

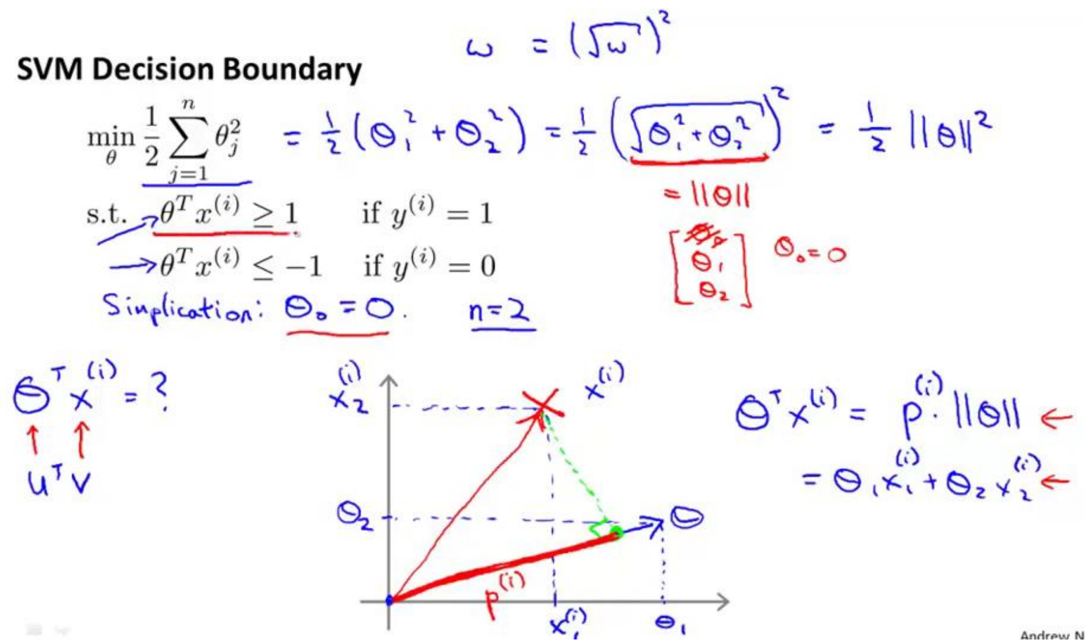
Large margin classifier in presence of outliers



数学原理



Vector Inner Product
$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$
$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
$u^T v = ?$
$[u_1 \quad u_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
$\|u\|$ = length of vector u
$= \sqrt{u_1^2 + u_2^2} \in \mathbb{R}$
p = length of projection of v onto u.
Signed
$u^T v = p \cdot \|u\|$
$= v^T u$
$= u_1 v_1 + u_2 v_2$
$p \in \mathbb{R}$
$u^T v = p \cdot \|u\|$
$p < 0$



SVM Decision Boundary
$\omega = (\sqrt{\omega})^2$
$\min_\theta \frac{1}{2} \sum_{j=1}^{n} \theta_j^2 = \frac{1}{2}(\theta_1^2 + \theta_2^2) = \frac{1}{2}\left(\sqrt{\theta_1^2 + \theta_2^2}\right)^2 = \frac{1}{2}\|\theta\|^2$
$= \|\theta\|$
s.t. $\theta^T x^{(i)} \geq 1$ if $y^{(i)} = 1$
$\theta^T x^{(i)} \leq -1$ if $y^{(i)} = 0$
Simplication: $\theta_0 = 0$. $n = 2$
$\theta_0 = 0$
$\theta^T x^{(i)} = ?$
$u^T v$
$\theta^T x^{(i)} = p^{(i)} \cdot \|\theta\|$
$= \theta_1 x_1^{(i)} + \theta_2 x_2^{(i)}$

SVM Decision Boundary

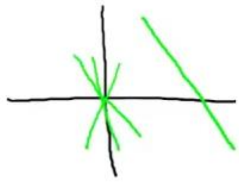
$$\min_\theta \frac{1}{2}\sum_{j=1}^{n}\theta_j^2 = \frac{1}{2}\|\theta\|^2$$

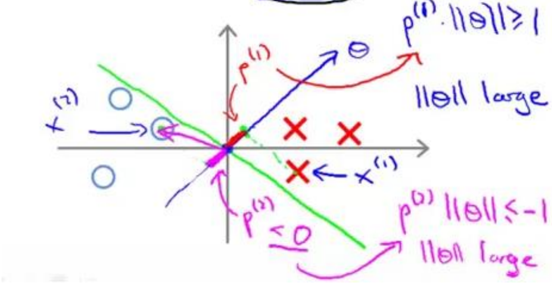
s.t. $\boxed{p^{(i)}\cdot\|\theta\| \geq 1}$ if $y^{(i)} = 1$

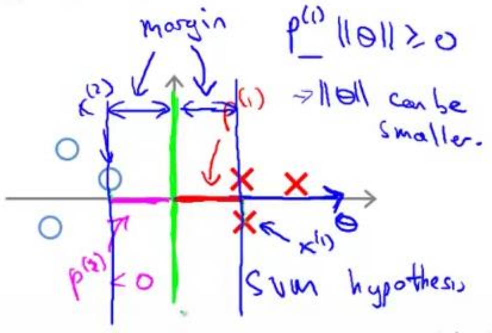
$p^{(i)}\cdot\|\theta\| \leq -1$ if $y^{(i)} = 1$

where $p^{(i)}$ is the projection of $x^{(i)}$ onto the vector θ.

Simplification: $\theta_0 = 0$







Kernels　核函数

Kernels and Similarity

$$f_1 = \text{similarity}(x, l^{(1)}) = \exp\left(-\frac{\boxed{\|x-l^{(1)}\|^2}}{2\sigma^2}\right) = \exp\left(-\frac{\sum_{j=1}^{n}(x_j - l_j^{(1)})^2}{2\sigma^2}\right)$$

If $x \approx l^{(1)}$:

$$f_1 \approx \exp\left(-\frac{0^2}{2\sigma^2}\right) \approx 1$$

$l^{(1)} \rightarrow f_1$

$l^{(2)} \rightarrow f_2$

$l^{(3)} \rightarrow f_3$.

x

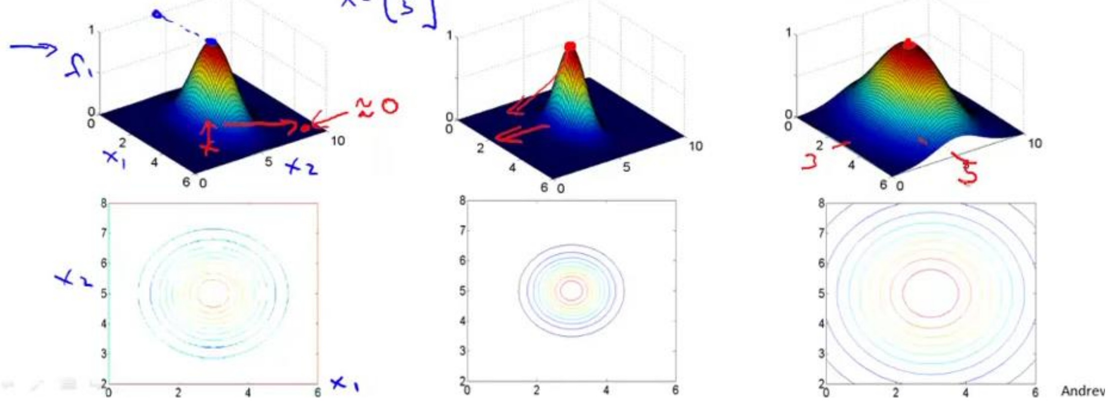
If x if far from $l^{(1)}$:

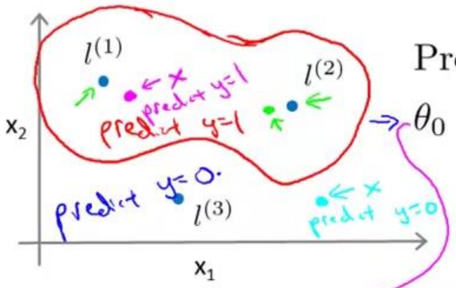
$$f_1 = \exp\left(-\frac{(\text{large number})^2}{2\sigma^2}\right) \approx 0.$$

Example:

$$l^{(1)} = \boxed{\begin{bmatrix}3\\5\end{bmatrix}}, \quad f_1 = \exp\left(-\frac{\|x-l^{(1)}\|^2}{2\boxed{\sigma^2}}\right)$$

$\sigma^2 = 1$ $\qquad x = \begin{bmatrix}3\\5\end{bmatrix}$ $\qquad \sigma^2 = 0.5$ $\qquad \sigma^2 = 3$





Predict "1" when

$$\theta_0 + \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3 \geq 0$$

$\theta_0 = -0.5, \quad \theta_1 = 1, \quad \theta_2 = 1, \quad \theta_3 = 0$

$f_1 \approx 1, \quad f_2 \approx 0, \; f_3 \approx 0.$

$\theta_0 + \theta_1 \times 1 + \theta_2 \times 0 + \theta_3 \times 0$

$= -0.5 + 1 \quad = 0.5 \geq 0$

$f_1, f_2, f_3 \approx 0$

$\theta_0 + \theta_1 f_1 + \dots \approx -0.5 < 0$

SVM with Kernels

Given $(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(m)}, y^{(m)})$,

choose $l^{(1)} = x^{(1)}, l^{(2)} = x^{(2)}, \ldots, l^{(m)} = x^{(m)}$.

Given example x:

$f_1 = \text{similarity}(x, l^{(1)})$ $x^{(i)}$

$f_2 = \text{similarity}(x, l^{(2)})$

$\ldots$

$f = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$ $f_0 = 1$

For training example $(x^{(i)}, y^{(i)})$:

$x^{(i)} \rightarrow$

$f_1^{(i)} = sim(x^{(i)}, l^{(1)})$

$f_2^{(i)} = sim(x^{(i)}, l^{(2)})$

$\vdots$

$f_i^{(i)} = sim(x^{(i)}, l^{(i)}) = exp(-\frac{0}{2\sigma^2}) = 1$ $x^{(i)}$

$f_m^{(i)} = sim(x^{(i)}, l^{(m)})$

$x^{(i)} \in \mathbb{R}^{n+1}$ (or $\mathbb{R}^n$)

$f^{(i)} = \begin{bmatrix} f_0^{(i)} \\ f_1^{(i)} \\ \vdots \\ f_m^{(i)} \end{bmatrix}$

$f_0^{(i)} = 1$

Andrew

SVM with Kernels

Hypothesis: Given x, compute features $f \in \mathbb{R}^{m+1}$ $\theta \in \mathbb{R}^{m+1}$

Predict "y=1" if $\theta^T f \geq 0$

$\theta_0 f_0 + \theta_1 f_1 + \cdots + \theta_m f_m$

Training: $n = m$

$$\min_\theta C \sum_{i=1}^{m} y^{(i)} cost_1(\theta^T f^{(i)}) + (1 - y^{(i)}) cost_0(\theta^T f^{(i)}) + \boxed{\frac{1}{2} \sum_{j=1}^{\cancel{n}\, m} \theta_j^2}$$

$\cancel{n} = m$

$\cancel{\theta^T x^{(i)}}$ $\theta^T f^{(i)}$

$\rightarrow \theta_0$

$\sum_j \theta_j^2 = \theta^T \theta \leftarrow \theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix}$ (ignore θ_0)

$\|\theta\|^2$

$\theta^T M \theta$

$m = 10{,}000$

SVM parameters:

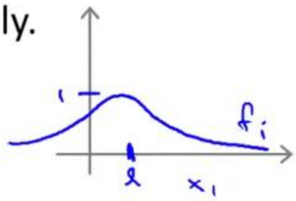
C ($= \frac{1}{\lambda}$). → Large C: Lower bias, high variance. (small λ)

→ Small C: Higher bias, low variance. (large λ)

σ^2 Large σ^2: Features f_i vary more smoothly.

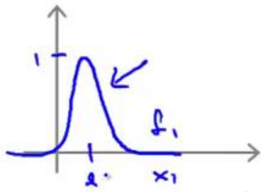
→ Higher bias, lower variance.

$\exp\left(-\frac{\|x-l^{(i)}\|^2}{2\sigma^2}\right)$



Small σ^2: Features f_i vary less smoothly.

Lower bias, higher variance.



线性核函数，高斯核函数

Use SVM software package (e.g. liblinear, libsvm, ...) to solve for parameters θ.

Need to specify:

→ Choice of parameter C.

Choice of kernel (similarity function):

E.g. No kernel ("linear kernel")

Predict "y = 1" if $\theta^T x \geq 0$

$\theta_0 + \theta_1 x_1 + \cdots + \theta_n x_n \geq 0$

→ n large, m small $x \in \mathbb{R}^{n+1}$

Gaussian kernel:

$$f_i = \exp\left(-\frac{||x - l^{(i)}||^2}{2\sigma^2}\right), \text{ where } l^{(i)} = x^{(i)}.$$

Need to choose σ^2.

$x \in \mathbb{R}^n$, n small and/or m large

Kernel (similarity) functions:

$x^{(i)}$ $\quad l^{(j)} = x^{(j)}$

```
function f = kernel(x1,x2)
```

$$f = \exp\left(-\frac{\|\mathbf{x1}-\mathbf{x2}\|^2}{2\sigma^2}\right)$$

```
return
```

$x \rightarrow f_1, f_2, \ldots, f_m$

Note: Do perform feature scaling before using the Gaussian kernel.

$\boxed{\|x-l\|^2}$ $\quad v = x - l \quad x \in \mathbb{R}^n$

$\|v\|^2 = v_1^2 + v_2^2 + \cdots + v_n^2$

$= (x_1 - l_1)^2 + (x_2 - l_2)^2 + \cdots + (x_n - l_n)^2$

1000 feet^2 $\quad$ 1-5 bedrooms

Other choices of kernel

Note: Not all similarity functions $\text{similarity}(x, l)$ make valid kernels. (Need to satisfy technical condition called "Mercer's Theorem" to make sure SVM packages' optimizations run correctly, and do not diverge).

Many off-the-shelf kernels available:

- Polynomial kernel: $k(x,l) = (x^T l)^2$ $\quad (x^T l + \text{constant})^{\text{degree}}$

  $(x^T l)^3, \quad (x^T l + 1)^3, \quad (x^T l + 5)^4$

- More esoteric: String kernel, chi-square kernel, histogram intersection kernel, ...

Logistic regression vs. SVMs

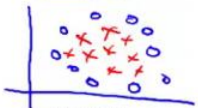
$n =$number of features ($x \in \mathbb{R}^{n+1}$), $m =$ number of training examples

If n is large (relative to m): (e.g. $n \geq m$, $n = 10{,}000$, $m = 10 \cdots 1000$)

Use logistic regression, or SVM without a kernel ("linear kernel")

If n is small, m is intermediate: ($n = 1 - 1000$, $m = 10 - 10{,}000$)

Use SVM with Gaussian kernel



If n is small, m is large: ($n = 1 - 1000$, $m = 50{,}000+$)

Create/add more features, then use logistic regression or SVM without a kernel

Neural network likely to work well for most of these settings, but may be slower to train.