1. $\int_a^b p(x;a,b)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{b-a}{b-a} = 1 \quad (a \neq b)$

So this distribution is normalized.

Mean: $\mu_x = \int x\,p(x;a,b)\,dx = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{2} \times \frac{b^2-a^2}{b-a} = \frac{a+b}{2}$

Variance: $Var(x) = \mu_{x^2} - (\mu_x)^2$

$$= \int_a^b x^2 p(x;a,b)\,dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{(a-b)^2}{12}$$

2. (a) Let event A = The neighbor has at least one boy.

event B = The neighbor has at least one girl.

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{4}} = \frac{2}{3}$$

(b) Let event A = We see one of his children run by, it is a boy.

Let event B = The neighbor has at least one girl.

For event A, since we already seen it is a boy, there could be only three possible event, Boy & Girl, Girl & Boy, Boy & Boy.

$P(A) = \frac{4}{2\times 3} = \frac{2}{3}$ $\quad P(AB) = \frac{2}{6} = \frac{1}{3}$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

3. $var[X+Y]$

$= Cov[X+Y, X+Y]$

$= cov[X,X], cov[X,Y] + cov[Y,X] + cov[Y,Y].$

$= Var[X] + var[Y] + 2cov[X,Y]$

4. $cov[x,y] = E[(x-E[x])(y-E[y])]$

$= E[xy - xE[y] - yE[x] + E[x]E[y]].$

$= E[xy] - E[x]E[y] - E[y]E[x] + E[x]E[y]$

$= E[xy] - E[x]E[y].$

Since x, y are independent. $E[xy] = E[x]E[y].$

So $cov[x,y] = E[x]E[y] - E[x]E[y] = 0.$

their covariance is zero.

5. (a) proof: since x, z are independent.

$$E(x+z) = \int (x+z)dF = \int x dF + \int z dF = E(x) + E(z).$$

(b) proof: $var[x+z] = var[x] + var[z] + 2cov[x,z].$

Since x, z are independent $cov[x,z]=0$

$var[x+z] = var[x] + var[z]$

6. apply $x_1=0, x_2=n$ on equation 1,

$\Rightarrow \tilde{f}(0) + \alpha \tilde{f}(n) = \tilde{f}(\tilde{f}(n))$.

apply $x_1=1, x_2=n-1$ on equation 1,

$\Rightarrow \tilde{f}(\alpha) + \alpha \tilde{f}(n-1) = \tilde{f}(\tilde{f}(n))$.

Guess $\tilde{f}(x) = \alpha x$

Let's verify it

$$\tilde{f}(\alpha x_1) + \alpha \tilde{f}(x_2) = \alpha^2 x_1 + \alpha^2 x_2 = \alpha^2(x_1+x_2)$$

$$\tilde{f}(\tilde{f}(x_1+x_2)) = \tilde{f}(\alpha(x_1+x_2)) = \alpha^2(x_1+x_2) = \tilde{f}(\alpha x_1) + \alpha \tilde{f}(x_2).$$

$\Rightarrow \tilde{f}(x) = \alpha x$ satisfies equation 1.

After prediction, G should be $G(x) = \alpha x + \varepsilon$

So, the prediction would be $G(\hat{x}) = \alpha \hat{x} + \varepsilon$, where α and ε could be calculated after testing all functions $f \in C$.

7. Suppose there exists such A. Let's consider two cases:
① each A_i contains only 1 point. In this case, since $\mathbb{R}$ is uncountable, A is uncountable, which is a contradiction. ② some A_i contains at least 2 points, say $A_i := \{x, y\}$. Then $\{x\}$ is a Borel Set which can not be written as a union of elements in A.

So for all 2 cases, Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ is not atomic.

8. (a) $\Omega = \{H, TH, TTH, \cdots, T\cdots TH\}$.

$$\mathcal{F} = \{\emptyset, \{H\}, \{TH\}, \{TTH\}, \cdots, \{T\cdots TH\}, \{H, TH\}, \{H, TTH\}, \cdots, \Omega\} = 2^{\Omega}$$

Proof: According to the defination of σ-algebra.

① $\emptyset \in \mathcal{F}$

② $\forall A \in \mathcal{F}$, $A \subset \Omega$. Then we have $A^c \subset \Omega$.

Since $\mathcal{F}$ contains all subset of Ω, $A^c \in \mathcal{F}$.

③ Given $\overset{\infty}{\underset{i=1}{A_i}} \in \mathcal{F}$, $\overset{\infty}{\underset{i=1}{A_i}} \subset \Omega$. $\bigcup_{i=1}^{\infty} A_i \subset \Omega$, so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

(b) $\mathcal{F}$ is uncountable.

There is a bijection between natrual numbers and elements in Ω.

ω	H	TH	TTH	$\cdots$	$\overbrace{T\cdots T}^{i\ T's}H$
$\mathbb{N}$	0	1	2	$\cdots$	i

Suppose $\mathcal{F}$ is countable. If we find a bijection between $\mathbb{N}$ and $2^{\mathbb{N}}$, there should be a bijection between $\mathbb{N}$ and 2^{Ω}.

$2^{\mathbb{N}} \backslash \mathbb{N}$	0	1	2	3	4 $\cdots$
$2^{\mathbb{N}}_0$	0	1	0	0	$\cdots$
$2^{\mathbb{N}}_1$	1	0	0	0	$\cdots$
$2^{\mathbb{N}}_2$	0	0	1	1	$\cdots$
$2^{\mathbb{N}}_3$	1	0	0	1	$\cdots$
$\vdots$					

Suppose the table on the left is the bijection table, where $2^{\mathbb{N}}_i$ means ith element in $2^{\mathbb{N}}$. In this table, ith row, jth column $= 1$ means $j \in 2^{\mathbb{N}}_i$, else $j \notin 2^{\mathbb{N}}_i$. Take diagonal elements as a sequence, flip each element i to $1-i$, we'll get a new sequence, whose ith element is different from the ith element in the ith row.

While from the defination of this table, all sequences Should be included, where is a contradiction.

So 2^{Ω} is uncountable.

(c) Let $w_i = \underbrace{TT\cdots T}_{i\ T's}H$, $P(w_i) = \frac{1}{2^{i+1}}$

$w_0 = H$, $P(w_0) = \frac{1}{2}$.

For $\mathcal{F}$, $\forall A_i \in \mathcal{F}$, A_i is the Union of some w_j's in Ω.

$0 \leq P(A_i) = \sum_j P(w_j) \leq 1$ $\quad P(\phi)=0$. $P(\Omega)=1$

Justify: ① Since $P(w_i) > 0$, $P(A_i) \geq 0$.

② $P(\phi) = 0$

③ $P(\bigcup_{i=0}^{\infty} w_i) = \sum_{i=0}^{\infty} P(w_i)$ (which is the defination of our P)

So P is probability measure.

(d). Let $X(w_i) = \sum_{k=0}^{i} P(w_i)$

$\lim_{x\to+\infty} P(X\leq x) = \lim_{n\to+\infty} \sum_{i=0}^{n} P(w_i) = \lim_{n\to+\infty} \frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = \lim_{n\to+\infty} (1-\frac{1}{2^n}) = 1$

$E(x) = \sum_i p_i w_i = \sum_i (P(w_i))^2 = \lim_{n\to+\infty} (\frac{1}{2})^2+(\frac{1}{2^2})^2+(\frac{1}{2^3})^2+\cdots+(\frac{1}{2^n})^2$

$= \lim_{n\to+\infty} \frac{\frac{1}{4}\times(1-\frac{1}{2^{2n}})}{1-\frac{1}{4}} = \frac{1}{3}$

9. $\liminf_{n\to+\infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n\geq k} A_n.$

Let $B_k = \bigcap_{n\geq k} A_n \Rightarrow \liminf_{n\to+\infty} A_n = \bigcup_{k=1}^{\infty} B_k$

Since B_k is increasing sequence. and $B_k \subset A_n$. $\forall n \geq k$.
$(B_k = B_{k+1} \cap A_k)$

$P(B_k) \leq \inf_{n\geq k} P(A_n)$

$\Rightarrow P(\liminf_{n\to+\infty} A_n) = P(\bigcup_{k=1}^{\infty} B_k) = \lim_{n\to+\infty} P(B_n) \leq \lim_{n\to+\infty} \inf_{k\geq n} P(A_n) = \liminf_{n\to+\infty} P(A_n)$